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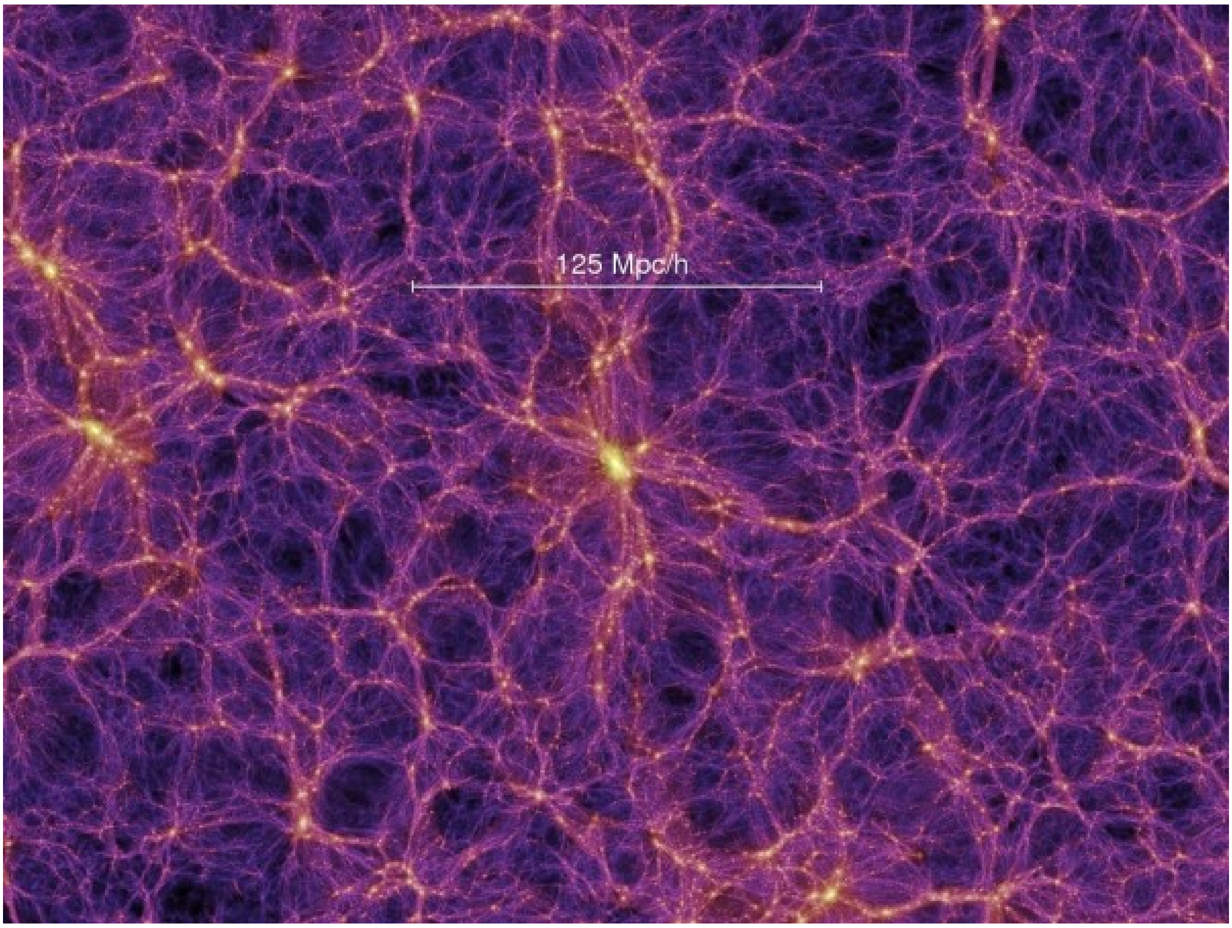

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Paper :	Computational Methods (AA 609)
Assignment No :	Lab 2
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Abstract

This assignment discusses some algorithms to solve partial differential equations. Required theoretical points and justifications needed in the assignment are given within the code file.

Contents

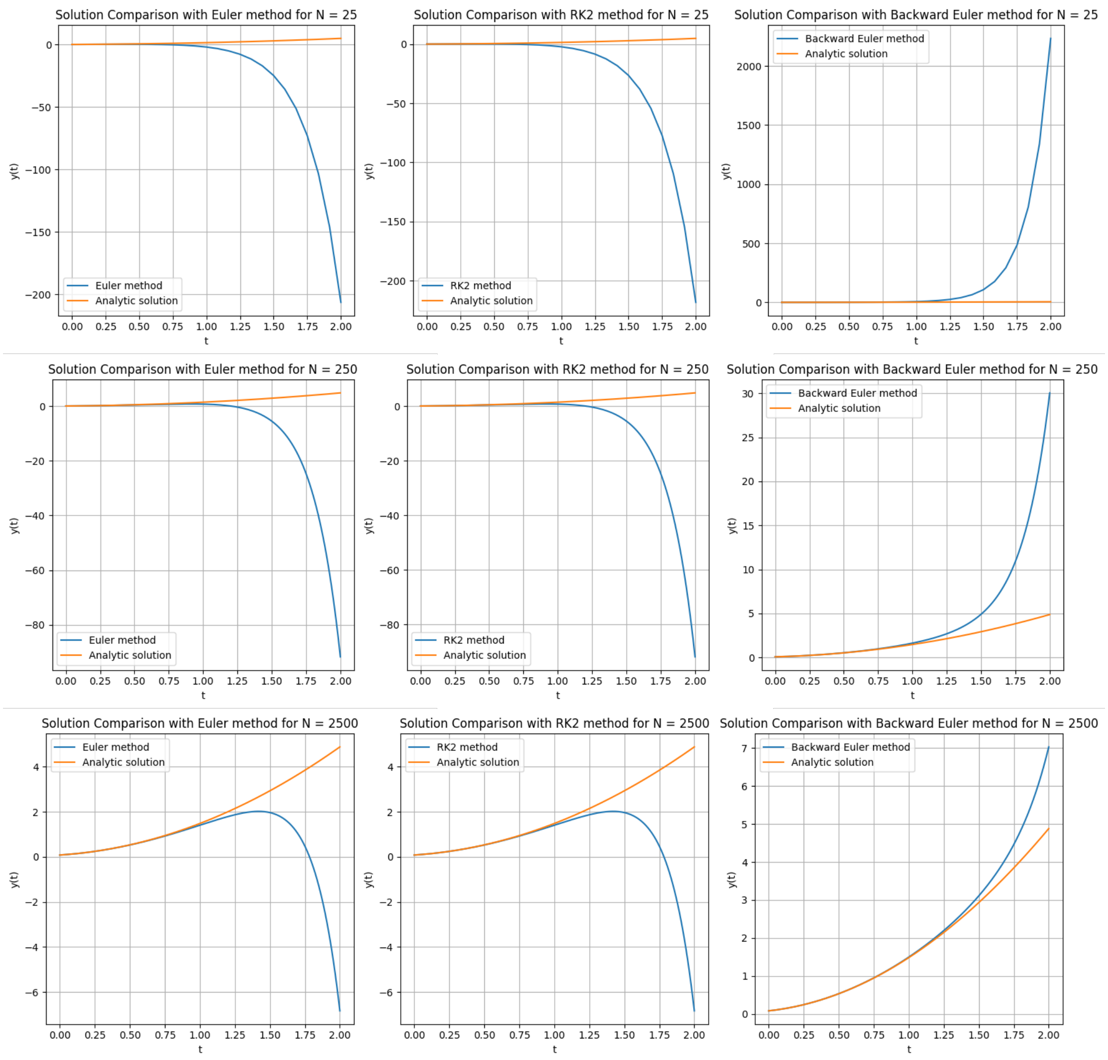
1 Problem 1

1.1 Stiff ODEs and Astrochemical Reactions

Stiff equations are differential equations where certain components vary much more rapidly than others. Numerical methods may fail to accurately capture these rapid variations, leading to inaccurate results. Stiffness can cause numerical instability and difficulties in choosing appropriate time steps, making it challenging for methods to provide reliable and accurate solutions. Consider an element X whose concentration $y(t)$ evolves with time according to the following differential equation:

$$\boxed{\frac{dy}{dt} = 5y - \frac{5t^2}{y}, \quad y(0) = 25}$$

This equation describes the rate of change of the concentration y with respect to time t. The initial condition is given by $y(0) = 25$. The solution for this equation has plots as shown:



2 Problem 2

2.1 1D Advection Equation

Advection equation encapsulates transport mechanism for a given physican quantity u. The 1D advection equation describes the transport of a quantity u in one dimension as a function of space x and time t:

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0}$$

The initial function to work upon is given by:

$$u(x, t = 0) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

2.2 Numerical Solution Schemes

2.2.1 FTCS (Forward Time, Centered Space)

The FTCS scheme is a simple method that updates each point using neighboring values in time and space. The algorithm is as follows:

$$\boxed{u_i^{n+1} = u_i^n - \frac{c\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)}$$

2.2.2 Upwind

The Upwind scheme considers values upwind (against the flow) to account for advection. The algorithm is:

$$\boxed{u_i^{n+1} = u_i^n - \frac{c\Delta t}{\Delta x}(u_i^n - u_{i-1}^n)}$$

The index of i changes accordingly for directional motion and hence need to take into account.

2.2.3 Lax Method

The Lax Method combines upwind and downwind values in a weighted manner. The algorithm is:

$$\boxed{u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{c\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)}$$

2.2.4 MacCormack Method

The MacCormack Method is a two-step predictor-corrector scheme. The predictor step estimates the solution, and the corrector step refines it for better accuracy. The predictor step is:

$$\boxed{\tilde{u}_i^{n+1} = u_i^n - \frac{c\Delta t}{\Delta x}(u_{i+1}^n - u_i^n)}$$

The corrector step is:

$$\boxed{u_i^{n+1} = \frac{1}{2}(\tilde{u}_i^{n+1} + u_i^n) - \frac{c\Delta t}{2\Delta x}(\tilde{u}_i^{n+1} - u_i^n)}$$

The predictor step allows for an initial estimation of the next time step, while the corrector step refines this estimate to provide better accuracy. This two-step approach enhances stability and reduces numerical errors.

2.2.5 Lax-Wendroff Scheme

The Lax-Wendroff Scheme improves accuracy using a Taylor series expansion. The algorithm is:

$$\boxed{u_i^{n+1} = u_i^n - \frac{c\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) + \frac{c^2\Delta t^2}{2\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)}$$

CFL Condition in Lax-Wendroff Scheme The CFL (Courant-Friedrichs-Lewy) condition ensures stability. For Lax-Wendroff, it's given by CFL $= \frac{c \cdot \Delta t}{\Delta x} \leq 1$, where c is the wave speed, Δt is the time step, and Δx is the spatial step Violating this condition makes change go faster that advection speed c which is not a physical problem and hence, instability arises. The results are in given notebook.

3 Reference

- Lecture notes and online resources for theory.
- Python code for plots.
- Google for cover page.