Network Graphs Part 2

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Tellegen's Theorem

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Tellegen's Theorem

Consider an arbitrary lumped network whose graph G has b branches and n_t nodes. Suppose that to each branch of the graph we assign arbitrarily a branch voltage v_k and a branch current j_k for $k = 1, 2, \ldots, b$ and suppose that they are measured with respect to arbitrarily picked associated reference directions. If the branch voltages $v_1, v_2, \ldots, v_b$ satisfy all the constraints imposed by KVL and if the branch currents $j_1, j_2, \ldots, j_b$ satisfy all the constraints imposed by KCL, then

$$\sum_{k=1}^{b} v_k j_k = 0$$

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Remarks

- It is of crucial importance to realize that the branch voltages $v_1, v_2, ..., v_b$ are picked arbitrarily subject only to the KVL constraints. Similarly, the branch currents $j_1, j_2, ..., j_b$ are picked arbitrarily subject only to the KCL constraints.

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- Suppose $\hat{v}_1, \hat{v}_2, ..., \hat{v}_b$ and $\hat{j}_1, \hat{j}_2, ..., \hat{j}_b$ are other sets of arbitrarily selected branch voltages and branch currents that obey the same KVL constraints and the same KCL constraints.

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Then

$$\sum_{k=1}^{b} \hat{v}_k \hat{j}_k = 0$$

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$$v_k j_k = (e_\alpha - e_\beta) j_{\alpha\beta} \tag{1}$$

$v_k j_k$ can also be written in terms of $j_{\beta\alpha}$,the current from node β to node α.

$$v_k j_k = (e_\beta - e_\alpha) j_{\beta\alpha} \tag{2}$$

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Adding the equations (1) and (2), we obtain

$$v_k j_k = \frac{1}{2}[(e_\alpha - e_\beta)j_{\alpha\beta} + (e_\beta - e_\alpha)j_{\beta\alpha}] \tag{3}$$

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By **KCL**, each one of these sums is zero, hence,

$$\sum_{k=1}^{b} v_k j_k = 0$$

Conservation of Energy

- From Tellegen's theorem we get

$$\sum_{k=1}^{b} v_k(t) j_k(t) = 0 \text{ for all } t$$

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- Suppose the network has several independent sources; separating in the sum the sources from other branches, we can conclude that **the sum of the power delivered by the independent sources to the network is equal to the sum of the power absorbed by all the other branches of the network.**
- This means, for lumped circuits , **KCL** and **KVL** imply **conservation of energy**

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For **linear time-invariant RLC** circuits

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- The power delivered by the sources is the rate at which energy is absorbed by the network.
- The energy is
  - dissipated in the resistors at the rate $R_k j_k^2(t)$ for the k'th resistor in the form of heat.
  - stored as magnetic energy in inductors $\frac{1}{2} L_k j_k^2(t)$
  - stored as electric energy in capacitors $\frac{1}{2} C_k v_k^2(t)$

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$$\sum_{k=1}^{b} v_k j_k = \sum_{k=1}^{b} \hat{v}_k \hat{j}_k = 0 \text{ (expressions of the conservation of energy)}$$

and

$$\sum_{k=1}^{b} \hat{v}_k j_k = \sum_{k=1}^{b} v_k \hat{j}_k = 0 \text{ (do not have an energy interpretation)}$$

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Since V_1 and J_1 is the source voltage and associated current, $\frac{1}{2} V_1 \bar{J}_1$ is the complex power delivered to branch 1 and $-\frac{1}{2} V_1 \bar{J}_1$ is the complex power delivered by the source to the rest of the network.

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$$-\frac{1}{2}V_1\bar{J}_1 = \sum_{k=2}^{b} \frac{1}{2} V_k \bar{J}_k$$

Conservation of Complex Power

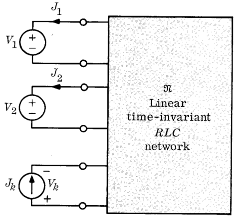
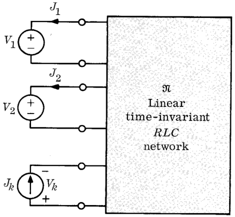


Figure 1: Theorem of conservation of complex power

Theorem

Consider a linear time-invariant network which is in the sinusoidal steady state and which is driven by several independent sources that are at the same frequency.



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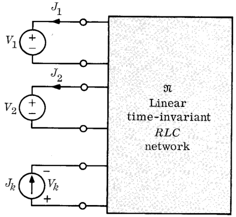

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The Real Part and Phase of Driving-point Impedances

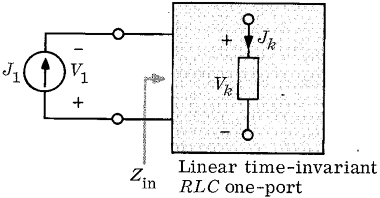

Figure 2: Properties of driving-point impedance $Z_{in}(j\omega)$.

The conservation-of-complex-power theorem can be used to derive many important properties of driving-point impedances.

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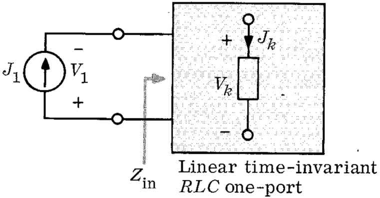

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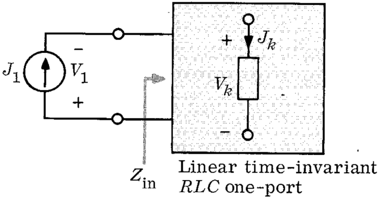

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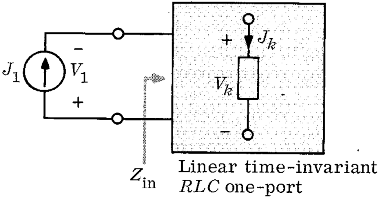

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With reference to the Figure 2, let the driving point impedence of the network $\mathfrak{N}$ is Z_{in}. $\mathfrak{N}$ is a **linear time-invariant one-port network**. Then,

$$V_1 = -J_1 Z_{in}(j\omega)$$

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Let the branches inside $\mathfrak{N}$ be numbered from 2 to b, and let the branch phasor currents and branch impedances be denoted by J_k and Z_k, $k = 2, 3, ..., b$.

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Let the branches inside $\mathfrak{N}$ be numbered from 2 to b, and let the branch phasor currents and branch impedances be denoted by J_k and Z_k, $k = 2, 3, ..., b$. Using Tellegen's theorem, we get the complex power P as

$$P = -\frac{1}{2}V_1\bar{J_1} = \frac{1}{2}Z_{in}(j\omega)\left|J_1\right|^2 = \frac{1}{2}\sum_{k=2}^{b} V_k\bar{J_k} = \frac{1}{2}\sum_{k=2}^{b} Z_k(j\omega)\left|J_k\right|^2 \qquad (6)$$

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The average power delivered by the source is given by P_{av} which is

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Note: All these impedances are evaluated at the same angular frequency ω, which is the angular frequency ofthe source.

The Real Part and Phase of Driving-point Impedances

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- Consequently, Z_{in} is also a positive real number.
- In this case Z_{in} is independent of angular frequency ω.

Remark

The input impedance of a resistive network made of positive resistances is a positive resistance.

Case 2

- RL networks have either positive resistances or positive inductances resulting in Z_k being either a positive number or a purely imaginary number of the form $j\omega L_k$ where $\omega > 0$ and $L_k > 0$

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- Equivalently,

$$0 \leq \measuredangle Z_{in}(j\omega) \leq 90^\circ \text{ for all } \omega \geq 0$$

Remark

At any positive angular frequency ω, the driving point impedance of a linear time-invariant RL network made of positive resistances and positive inductances has a phase angle between 0 and 90°

Case 3

- RC networks made of branches all having either positive resistances or positive capacitances.
- A similar reasoning shows that at any positive angular frequency ω, the driving—point impedance of a linear time—invariant RC network made of positive resistances and positive capacitances has a phase angle between 0 and 90°.

The Real Part and Phase of Driving-point Impedances

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- Lossless networks made of capacitors, inductors (including coupled inductors), and/or ideal transformers.

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- Previously we have learnt that coupled inductors can be replaced by inductors without coupling and an ideal transformer.
- Thus in $\mathfrak{N}$ we can assume that all are positive inductances, capacitances or ideal transformer windings.
- Ideal transformers neither dissipate nor store energy, the sum $\sum_k V_k \bar{J}_k$ over all branches with ideal transformers is 0, resulting in ideal transformers contributing nothing i the sum in (6)

Case 4 contd.

- Other terms are either of form $j\omega L_k \left|J_k\right|^2$ or $\frac{1}{j\omega C_k}\left|J_k\right|^2$ which are purely imagiary.

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- Equivalently,

$$\measuredangle Z_{in}(j\omega) = \pm 90^\circ \text{ for all } \omega$$

Remark

At any angular frequency ω, the driving point impedance of a linear time—invariant network made of inductors (coupled or uncoupled), capacitors, and ideal transformers is purely imaginary; i.e., it has a phase angle of either $+90^\circ$ or -90°.

Case 5

- RLC networks with ideal transformers having all branches with either positive resistances, positive inductances, positive capacitances, and/or ideal transformer windings.

The Real Part and Phase of Driving-point Impedances

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- Each term $Z_k \left|J_k\right|^2$ is either positive number or an imaginary number. Hence, Z_{in}, is a complex number with a real part that is larger than or equal to zero and an imaginary part that may be of either sign.

Case 5 contd.

- Then we can say,

$$Re[Z_{in}(j\omega)] \geq 0 \text{ for all } \omega \tag{8}$$

- Equivalently,

$$-90^\circ \leq \measuredangle Z_{in}(j\omega) \leq 90^\circ \text{ for all } \omega$$

Remark

At any angular frequency ω, the driving-point impedance of a linear time-invariant RLC network (which may include ideal transformers) has a non-negative real part; equivalently, it has a phase angle between -90° and $+90^\circ$. that is,

Driving-point Impedance, Power Dissipated, and Energy Stored

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$$P = \frac{1}{2} Z_{in}(j\omega) \left|J_1\right|^2 = \frac{1}{2} \sum_{m=2}^{b} Z_m(j\omega) \left|J_m\right|^2$$

$$= \frac{1}{2} \sum_i R_i \left|J_i\right|^2 + \frac{1}{2} \sum_k j\omega L_k \left|J_k\right|^2 + \frac{1}{2} \sum_l \frac{1}{j\omega C_l} \left|J_l\right|^2$$

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Exhibiting the real and imaginary part of P, we get

$$P = \frac{1}{2} \sum_i R_i \left|J_i\right|^2 + 2j\omega \left(\sum_k \frac{1}{4} L_k \left|J_k\right|^2 - \sum_l \frac{1}{4} \frac{1}{\omega^2 C_l} \left|J_l\right|^2 \right) \qquad (9)$$

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- In the sinusoidal steady state the average (over one period) of $R_i j_i^2(t)$ is
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$$\frac{1}{4} L_k \left| J_k \right|^2 \text{ and } \frac{1}{4} C_l \left| V_l \right|^2 = \frac{1}{4} \frac{1}{\omega^2 C_l} \left| J_l \right|^2$$

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- The first term in (9) is the average power(P_{av}) dissipated in $\mathfrak{N}$.
- The two terms in parenthesis are respectively average magnetic energy stored($\mathcal{E}_M$) and the average electric energy stored($\mathcal{E}_E$).

Driving-point Impedance, Power Dissipated, and Energy Stored

Now we can rewrite (9) as

$$Z_{in}(j\omega) = \frac{2P_{av} + 4j\omega(\mathcal{E}_M - \mathcal{E}_E)}{|J_1|^2} \tag{10}$$

- $|J_1|$ is the peak amplitude of the sinusoidal input current.

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Theorem

Given a linear time-invariant RLC network driven by a sinusoidal current source of one ampere peak amplitude and given that the network is in the sinusoidal steady state, the driving-point impedance seen by the source has a real part that is equal to twice the average power dissipated and an imaginary part that is 4ω times the difference between the average magnetic energy stored and the average electrical energy stored.