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Section 0 - Notation used in MAST10006 Calculus 2

Standard Abbreviations

1. such that or given that: $|$
2. therefore: $\therefore$
3. for all: $\forall$
4. there exists: $\exists$
5. equivalent to: $\equiv$
6. that is: $i.e$
7. approximate: $\approx$

Standard Notation for Sets of Numbers

1. natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$
2. integers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$
3. rational numbers: $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$
4. real numbers: $\mathbb{R}$ (rational numbers plus irrational numbers)
5. complex numbers: $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R},\ i^2 = -1\}$
6. $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ (xy plane)
7. $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ (3 dimensional space)

Standard Notation for Intervals

1. element of: $\in$
   so $a \in X$ means "a is an element of the set X "

2. open interval: (a, b)
   so $x \in (0, 1)$ means "$0 < x < 1$"

3. closed interval: $[a, b]$
   so $x \in [0, 1]$ means "$0 \leq x \leq 1$"

4. partial open and closed interval: $(a, b]$ or $[a, b)$
   so $x \in [0, 1)$ means "$0 \leq x < 1$"

5. not including: $\setminus$
   so $x \in \mathbb{R} \setminus \{0\}$ means "x is any real number excluding 0".
   Alternatively, we could write $(-\infty, 0) \cup (0, \infty)$ where $\cup$ means the "union of the two intervals".

More Standard Notation

1. natural logarithm: $\log x$
   base 10 logarithm: $\log_{10} x$

   Alternative notations for natural logarithms used in textbooks: $\log_e x, \ln x$

2. inverse trigonometric functions: $\arcsin x$, $\arctan x$ etc

   Alternative notations used in textbooks: $\sin^{-1} x$, $\tan^{-1} x$ etc

3. implies: $\Rightarrow$
   so $p \Rightarrow q$ means "p implies q"

4. if and only if (iff): $\Leftrightarrow$ (means both $\Leftarrow$ and $\Rightarrow$)
   so $p \Leftrightarrow q$ means "p implies q" AND "q implies p"

5. approaches: $\rightarrow$
   so $f(x) \rightarrow 1$ as $x \rightarrow 0$ means "$f(x)$ approaches 1 as x approaches 0"

Greek Alphabet

α	alpha	ν	nu
β	beta	ξ	xi
γ	gamma	o	omicron
δ	delta	π	pi
ϵ or ε	epsilon	ρ	rho
ζ	zeta	σ	sigma
η	eta	τ	tau
θ	theta	υ	upsilon
ι	iota	ϕ	phi
κ	kappa	χ	chi
λ	lambda	ψ	psi
μ	mu	ω	omega

Limits

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function.

We say that f has the limit L as x approaches a,

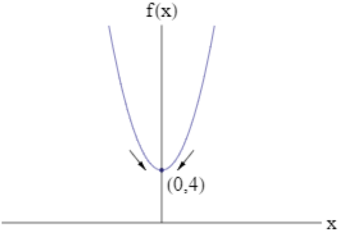
$$\lim_{x \to a} f(x) = L,$$

if $f(x)$ gets arbitrarily close to L whenever x is close enough to a (but $x \neq a$).

Note:

1. L must be finite.
2. The limit can exist even if f is undefined at $x = a$.

Example 1: If $f(x) = x^2 + 4$, evaluate $\lim_{x \to 0} f(x)$.



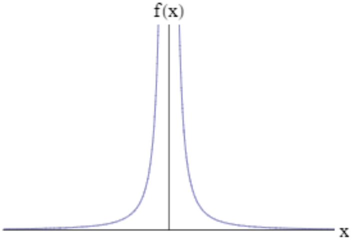
Solution:

$f(x)$ approaches 4 when x approaches 0 from above or below.

Therefore $\lim_{x \to 0} f(x)$ exists and

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(x^2 + 4\right) = 4.$$

Example 2: If $f(x) = \dfrac{1}{x^2}$, evaluate $\lim_{x \to 0} f(x)$.

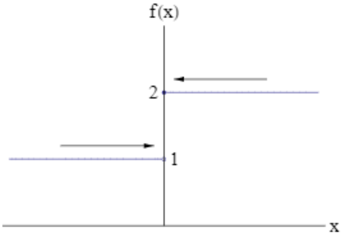


Solution:

As x approaches 0 from above or below $f(x)$ gets larger, growing without bound. Since $f(x)$ does not approach a finite number

$$\lim_{x \to 0} \frac{1}{x^2} \text{ does not exist.}$$

Example 3: If $f(x) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0 \end{cases}$, evaluate $\lim_{x \to 0} f(x)$.



Solution:

$\lim_{x \to 0} f(x)$ does not exist since

• $f(x)$ approaches 2 when x approaches 0 from above ($f(x) \to 2$ as $x \to 0^+$).

• $f(x)$ approaches 1 when x approaches 0 from below ($f(x) \to 1$ as $x \to 0^-$).

We can describe this behaviour in terms of one-sided limits.
We write

$$\lim_{x \to 0^-} f(x) = 1 \qquad \text{(left hand limit)}$$

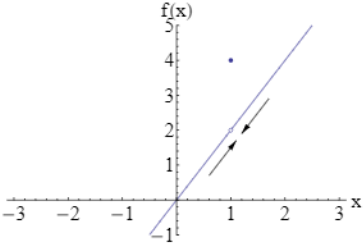
$$\lim_{x \to 0^+} f(x) = 2 \qquad \text{(right hand limit)}$$

Theorem:

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

That is, the limit exists if and only if the left and right hand limits exist and are equal.

Example 4: If $f(x) = \begin{cases} 2x & x \neq 1 \\ 4 & x = 1 \end{cases}$, evaluate $\lim_{x\to 1} f(x)$.



Solution:

Even though $f(1) = 4$, $\lim_{x\to 1} f(x) \neq 4$. As x approaches 1 from above and below, $f(x)$ approaches 2.

$$\Rightarrow \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 2$$

and $\lim_{x\to 1} f(x) = 2.$

Limit Laws

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. If $c \in \mathbb{R}$ is a constant and $\lim_{x\to a} f(x), \lim_{x\to a} g(x)$ exist, then

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$

2. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x).$

3. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x).$

4. $\lim_{x\to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0.$

5. $\lim_{x\to a} c = c.$

6. $\lim_{x\to a} x = a.$

Example 5: Use the limit laws to evaluate $\lim\limits_{x\to 2} \dfrac{x^3+2x^2-1}{5-3x}$.

Solution:

$$
\begin{aligned}
\lim_{x\to 2} \frac{x^3+2x^2-1}{5-3x} &= \frac{\lim\limits_{x\to 2} \left(x^3+2x^2-1\right)}{\lim\limits_{x\to 2} (5-3x)} && \text{(Law 4)} \\
&= \frac{\lim\limits_{x\to 2} x^3 + 2\lim\limits_{x\to 2} x^2 + \lim\limits_{x\to 2}(-1)}{\lim\limits_{x\to 2} 5 - 3\lim\limits_{x\to 2} x} && \text{(Law 1,2)} \\
&= \frac{(\lim\limits_{x\to 2} x)^3 + 2(\lim\limits_{x\to 2} x)^2 - 1}{5-3(2)} && \text{(Law 3,5,6)} \\
&= -(2^3+2(2)^2-1) && \text{(Law 6)} \\
&= -15.
\end{aligned}
$$

Limits as x Approaches Infinity

We say that f has the limit L as x approaches positive infinity,

$$\lim_{x\to\infty} f(x) = L,$$

if $f(x)$ gets arbitrarily close to L whenever x is sufficiently large and positive.
We say that f has the limit M as x approaches negative infinity:

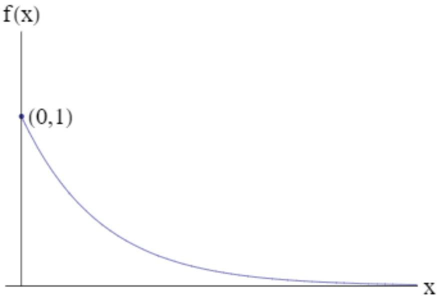
$$\lim_{x\to-\infty} f(x) = M$$

if $f(x)$ gets arbitrarily close to M whenever x is sufficiently large and negative.

Note:

1. L and M must be finite.
2. The same limit laws apply.

Example 6: If $f(x) = e^{-x}$, evaluate $\lim_{x \to \infty} f(x)$.



Solution:

As x grows larger and larger, $f(x)$ approaches 0. We write

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^{-x} = 0$$

or $f(x) \to 0$ as $x \to \infty$.

Evaluating Limits with Indeterminate Forms

Example 7: Evaluate $\lim_{x\to 2} \dfrac{x^2-4}{x-2}$.

Solution:

$$\left(\text{At } x = 2 \text{ has form } \frac{0}{0}\right)$$

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2-4}{x-2} &= \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)} \\
&= \lim_{x\to 2} (x+2) \\
&= 4.
\end{aligned}$$

Example 8: Evaluate $\lim\limits_{x\to\infty} \dfrac{3x^2 - 2x + 3}{x^2 + 4x + 4}$.

Solution:

$$\left(\text{As } x \to \infty \text{ has form } \frac{\infty}{\infty}\right)$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x^2 - 2x + 3}{x^2 + 4x + 4} &= \lim_{x\to\infty} \frac{x^2\left(3 - \frac{2}{x} + \frac{3}{x^2}\right)}{x^2\left(1 + \frac{4}{x} + \frac{4}{x^2}\right)} \\
&= \lim_{x\to\infty} \frac{3 - \frac{2}{x} + \frac{3}{x^2}}{1 + \frac{4}{x} + \frac{4}{x^2}} \\
&= \frac{3}{1} \qquad \text{since } \lim_{x\to\infty} \frac{1}{x} = 0, \lim_{x\to\infty} \frac{1}{x^2} = 0 \\
&= 3.
\end{aligned}$$

Example 9: Evaluate $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$.

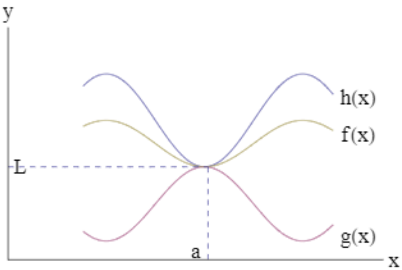
Solution:

(As $x \to \infty$ has form $\infty - \infty$)

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right) &= \lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\frac{\left(\sqrt{x^2+1}+x\right)}{\left(\sqrt{x^2+1}+x\right)} \\
&= \lim_{x\to\infty}\frac{x^2+1+x\sqrt{x^2+1}-x\sqrt{x^2+1}-x^2}{\sqrt{x^2+1}+x} \\
&= \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x} \\
&= 0.
\end{aligned}$$

Sandwich Theorem:

If $g(x) \leq f(x) \leq h(x)$ when x is near a (but $x \neq a$), and
$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$
then
$$\lim_{x \to a} f(x) = L.$$



Note:

Sandwich Theorem works for limits approaching infinity.

Example 10: Evaluate $\lim_{x\to 0}\left[x^2 \sin\left(\frac{1}{x}\right)\right]$.

Solution:

Since $-1 \le \sin\left(\frac{1}{x}\right) \le 1, \quad x \neq 0,$ then

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2, \quad x \neq 0,$$

Now $\lim_{x\to 0}(-x^2) = 0$ and $\lim_{x\to 0} x^2 = 0$

so $\lim_{x\to 0}\left[x^2 \sin\left(\frac{1}{x}\right)\right] = 0$ by Sandwich Theorem.

Example 11: Evaluate $\lim_{x \to 0} \left[x \sin\left(\frac{1}{x}\right) \right]$.

Solution:

Since $-1 \le \sin\left(\frac{1}{x}\right) \le 1, \;\; x \neq 0,$ then

$$-|x| \le x \sin\left(\frac{1}{x}\right) \le |x|, \;\; x \neq 0,$$

Now $\lim_{x \to 0}(-|x|) = 0$ and $\lim_{x \to 0} |x| = 0$

so $\lim_{x \to 0} \left[x \sin\left(\frac{1}{x}\right) \right] = 0$ by Sandwich Theorem.

Continuity

1. A function f is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.
2. A function f is left continuous (continuous from the left) at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$.
3. A function f is right continuous (continuous from the right) at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.

At the endpoints of a domain, we cannot take both left and right hand limits, so we use the appropriate limit to test continuity.

Example 12: Is $f(x) = \sqrt{x}$ continuous in its domain?

Solution:

$f(x) = \sqrt{x}$ has domain $[0, \infty)$.

Since f is right continuous at the endpoint $x = 0$ and continuous for $(0, \infty)$, we say it is continuous at every point in its domain.

Example 13: Let $f(x) = \begin{cases} 2x & x \neq 1 \\ 4 & x = 1. \end{cases}$
Is f continuous at $x = 1$?

Solution:

Since $\lim_{x \to 1} f(x) = 2 \neq f(1) = 4$,

f is not continuous at $x = 1$.

Example 14: Let $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 4 & x = 2. \end{cases}$

Is f continuous at $x = 2$?

Solution:

Since $\displaystyle\lim_{x \to 2} \frac{x^2-4}{x-2} = 4 = f(2)$,

f is continuous at $x = 2$.

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be real-valued functions.

Continuity Theorem 1:

If the functions f and g are continuous at a and $c \in \mathbb{R}$ is a constant, then the following functions are continuous at a:

1. $f + g$,
2. cf,
3. fg,
4. $\dfrac{f}{g}$ if $g(a) \neq 0$.

Continuity Theorem 2:

If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at $x = a$.

[Recall that $(g \circ f)(x) = g(f(x))$.]

Continuity Theorem 3:

The following function types are continuous at every point in their domains: polynomials, trigonometric functions, exponentials, logarithms, nth root functions, hyperbolic functions.

Example 15: Let $f(x) = \dfrac{\log x + \sin x}{\sqrt{x^2 - 1}}$. For which values of x is f continuous?

Solution:

- $\log x$ is continuous for $x > 0$.

- $\sin x$ is continuous for $x \in \mathbb{R}$.

$\Rightarrow \log x + \sin x$ is continuous for $x > 0$.

- $x^2 - 1$ is continuous for $x \in \mathbb{R}$.

- $\sqrt{z}$ is continuous for $z \geq 0$.

$\Rightarrow \sqrt{x^2 - 1}$ is a composition of continuous functions so is continuous for $x^2 - 1 \geq 0 \Rightarrow |x| \geq 1$.

- f is a quotient of continuous functions, so it is continuous for $x \geq 1$, except when

$$x^2 - 1 = 0 \Rightarrow x = \pm 1.$$

$\Rightarrow f$ is continuous for $x \in (1, \infty)$.

Example 16: $f(x) = \begin{cases} x^3 - cx + 8, & x \leq 1 \\ x^2 + 2cx + 2, & x > 1. \end{cases}$

For which values of c is f continuous on $(-\infty, \infty)$? Explain.

Solution:

- Since both branches of f are polynomials

f is continuous for $x < 1$ and $x > 1$ regardless of c.

- For continuity at $x = 1$, need

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1).$$

- $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \left(x^3 - cx + 8\right)$
  $= 1 - c + 8 = 9 - c.$

- $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left(x^2 + 2cx + 2\right)$
  $= 1 + 2c + 2 = 3 + 2c.$

- $f(1) = 9 - c.$
- f is continuous at $x = 1$ if

$$9 - c = 3 + 2c$$
$$\Rightarrow 3c = 6$$
$$\Rightarrow c = 2.$$

Theorem:

If f is continuous at b and $\lim_{x\to a} g(x) = b$ then

$$\lim_{x\to a} f[g(x)] = f\left[\lim_{x\to a} g(x)\right] = f(b).$$

Example 17: Evaluate $\lim_{x\to 0} \log\left(7 + 3x - x^2\right)$.

Solution:

$$\begin{aligned} \lim_{x\to 0} \log\left(7 + 3x - x^2\right) &= \log\left[\lim_{x\to 0}\left(7 + 3x - x^2\right)\right] \\ &\quad \text{by continuity of } \log z \text{ for } z > 0 \\ &= \log\left[7 + 0 + 0\right] \\ &= \log 7. \end{aligned}$$

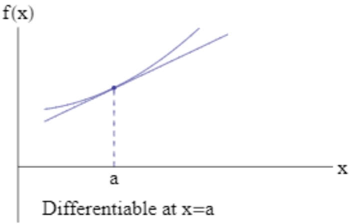
Differentiability

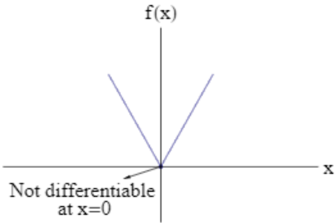
Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. The derivative of f at $x = a$ is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The function f is differentiable at $x = a$ if this limit exists.

Geometrically, this means that the graph $y = f(x)$ has a *tangent line* at $x = a$ which gives a good approximation to the graph near $x = a$.





L'Hôpital's Rule

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be differentiable functions near $x = a$, and $g'(x) \neq 0$ at all points x near a with $x \neq a$. If

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

if the limit involving the derivatives exists.

Example 18: Evaluate $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$. $\left(\dfrac{0}{0}\right)$

Solution:

$$
\begin{aligned}
\lim_{x \to 0} \frac{\sin x}{x} &= \lim_{x \to 0} \frac{\cos x}{1} && \text{by L'Hôpital's Rule} \\
&= \cos\left(\lim_{x \to 0} x\right) \\
& \qquad \text{by continuity of } \cos x \text{ for all } x \\
&= \cos(0) \\
&= 1.
\end{aligned}
$$

Example 19: Evaluate $\lim\limits_{x\to\infty} \dfrac{3x^2 - 2x + 3}{x^2 + 4x + 4}$. $\left(\dfrac{\infty}{\infty}\right)$

Solution:

$$
\begin{aligned}
\lim_{x\to\infty} \frac{3x^2 - 2x + 3}{x^2 + 4x + 4} &= \lim_{x\to\infty} \frac{6x - 2}{2x + 4} && \text{by L'Hôpital's Rule} \quad \left(\frac{\infty}{\infty}\right) \\
&= \lim_{x\to\infty} \frac{6}{2} && \text{by L'Hôpital's Rule} \\
&= 3.
\end{aligned}
$$

Example 20: Evaluate $\lim\limits_{x\to\infty}\left[x^{-\frac{1}{3}}\log x\right]$. $\quad (0\cdot\infty)$

Solution:

$$\begin{aligned}
\lim_{x\to\infty} x^{-\frac{1}{3}}\log x &= \lim_{x\to\infty}\frac{\log x}{x^{\frac{1}{3}}} \qquad \left(\frac{\infty}{\infty}\right)\\
&= \lim_{x\to\infty}\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{3}x^{-\frac{2}{3}}\right)} \qquad \text{by L'Hôpital's Rule}\\
&= \lim_{x\to\infty}\frac{3}{x^{\frac{1}{3}}}\\
&= 0.
\end{aligned}$$

Example 21: Evaluate $\lim\limits_{x \to 0} \dfrac{\cos x}{x}$.

Solution:

Since the indeterminate form is $\left(\dfrac{1}{0}\right)$, we cannot use L'Hôpital's Rule.

So $\lim\limits_{x \to 0} \dfrac{\cos x}{x}$ does not exist.

Sequences

A sequence is a function $f : \mathbb{N} \to \mathbb{R}$.
It can be thought of as an ordered list of real numbers

$$a_1, a_2, a_3, a_4, \ldots, a_n \ldots$$

Thus, $f(n) = a_n$.
The sequence is denoted by $\{a_n\}$, where a_n is the n^{th} term.

Example

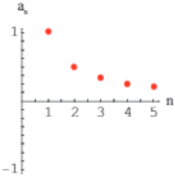
$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \quad \Longrightarrow \quad a_n = \frac{1}{n}$$

Example

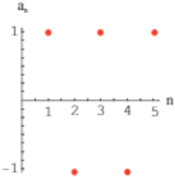
$$1, -1, 1, -1, 1, -1, \ldots \quad \Longrightarrow \quad a_n = (-1)^{n-1}$$

The graph of a sequence $\{a_n\}$ can be plotted on a set of axes with n on the x-axis and a_n on the y-axis.

Example: $a_n = \frac{1}{n}$



Example: $a_n = (-1)^{n-1}$



Limits of Sequences

A sequence $\{a_n\}$ has the limit L if a_n approaches L as n approaches infinity. Note, that L must be finite.

We write

$$\lim_{n\to\infty} a_n = L,$$

$$\text{or } \lim_{n\to\infty} \{a_n\} = L,$$

$$\text{or } a_n \to L \text{ as } n \to \infty.$$

If the limit exists we say that the sequence converges. Otherwise, we say that the sequence diverges.

Example 22: Determine whether the following sequences converge or diverge: (a) $\left\{\frac{1}{n}\right\}$ (b) $\left\{(-1)^{n-1}\right\}$ (c) $\{n\}$

Solution:

(a) $\left\{\frac{1}{n}\right\}$ converges to 0.

(b) $\left\{(-1)^{n-1}\right\}$ oscillates so it diverges.

(c) $\{n\}$ diverges to infinity.

The only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that n is a natural number whereas x is a real number.

Theorem:

Let $f : \mathbb{R} \to \mathbb{R}$ be a real function and $\{a_n\}$ be a sequence of real numbers such that $a_n = f(n)$. If

$$\lim_{x\to\infty} f(x) = L, \quad \text{then} \quad \lim_{n\to\infty} a_n = L.$$

This means that we can use the techniques for evaluating limits of functions to evaluate limits of sequences.
Note that

$$\lim_{n\to\infty} a_n = L \quad \not\Rightarrow \quad \lim_{x\to\infty} f(x) = L.$$

eg. $a_n = \sin(2\pi n)$, $f(x) = \sin(2\pi x)$.

Theorem:
Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Suppose that $c \in \mathbb{R}$ is a constant and $\lim_{n\to\infty} a_n$, $\lim_{n\to\infty} b_n$ exist. Then

1. $\lim_{n\to\infty} [a_n + b_n] = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.
2. $\lim_{n\to\infty} [ca_n] = c \lim_{n\to\infty} a_n$.
3. $\lim_{n\to\infty} [a_n b_n] = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.
4. $\lim_{n\to\infty} \left[\frac{a_n}{b_n}\right] = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ provided $\lim_{n\to\infty} b_n \neq 0$.

Sandwich Theorem:

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers.

If $a_n \leq c_n \leq b_n$ for all $n > N$ for some N, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$$

then

$$\lim_{n\to\infty} c_n = L.$$

The Factorial Function

The factorial function $n!$ $(n = 0, 1, 2, ...)$ is defined by

$$n! = n(n-1)!\ , \quad 0! = 1$$

or

$$n! = n \times (n-1) \times (n-2) \times ... \times 3 \times 2 \times 1$$

Therefore

$$1! = 1$$
$$2! = 2 \times 1 = 2$$
$$3! = 3 \times 2 \times 1 = 6$$
$$4! = 4 \times 3 \times 2 \times 1 = 24$$

Example

$$(2n+2)! = (2n+2) \times (2n+1) \times (2n) \times (2n-1) \times ... \times 3 \times 2 \times 1$$

or

$$(2n+2)! = (2n+2) \times (2n+1) \times (2n)!$$

Standard Limits

1. $\lim_{n\to\infty} \frac{1}{n^p} = 0, \quad (p > 0).$
2. $\lim_{n\to\infty} r^n = 0, \quad (|r| < 1).$
3. $\lim_{n\to\infty} a^{\frac{1}{n}} = 1, \quad (a > 0).$
4. $\lim_{n\to\infty} \frac{a^n}{n!} = 0, \quad a \in \mathbb{R}.$
5. $\lim_{n\to\infty} \frac{\log n}{n^p} = 0, \quad (p > 0).$
6. $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = \mathrm{e}^a, \quad a \in \mathbb{R}.$
7. $\lim_{n\to\infty} \frac{n^p}{a^n} = 0, \quad (p \in \mathbb{R}, a > 1).$

Example 23: Evaluate $\lim_{n\to\infty}\left[\left(\frac{n-2}{n}\right)^n + \frac{4n^2}{3^n}\right]$.

Solution:

$$\lim_{n\to\infty}\left[\left(\frac{n-2}{n}\right)^n + \frac{4n^2}{3^n}\right] = \lim_{n\to\infty}\left(\frac{n-2}{n}\right)^n + \lim_{n\to\infty}\frac{4n^2}{3^n}$$

$$= \lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n + 4\lim_{n\to\infty}\frac{n^2}{3^n}$$

$$= e^{-2} + 4\times 0$$

by standard limits 6, 7

$$= e^{-2}.$$

Example 24: Find the limit of the sequence with terms

$$a_n = \frac{3^n + 2}{4^n + 2^n}, \quad n \geq 1.$$

Solution:

$$\lim_{n\to\infty} \frac{3^n + 2}{4^n + 2^n} = \lim_{n\to\infty} \frac{\dfrac{3^n}{4^n} + \dfrac{2}{4^n}}{1 + \dfrac{2^n}{4^n}}$$

$$= \frac{\lim\limits_{n\to\infty} \left(\frac{3}{4}\right)^n + 2 \lim\limits_{n\to\infty} \left(\frac{1}{4}\right)^n}{1 + \lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^n}$$

$$= 0$$

by standard limit 2.

Example 25: Prove Standard Limit 5.

Solution:

Since

$$\lim_{x\to\infty} \frac{\log x}{x^p} = \lim_{x\to\infty} \frac{1/x}{px^{p-1}} \qquad \text{(by L'Hôpital's Rule } \left(\frac{\infty}{\infty}\right) \text{ if } p > 0)$$

$$= \lim_{x\to\infty} \frac{1}{px^p}$$

$$= 0,$$

therefore

$$\lim_{n\to\infty} \frac{\log n}{n^p} = \lim_{x\to\infty} \frac{\log x}{x^p} = 0.$$

Example 26: Evaluate $\lim\limits_{n\to\infty} [\log(3n-2) - \log n]$.

Solution:

Since

$$\begin{aligned}
\lim_{x\to\infty} [\log(3x-2) - \log x] &= \lim_{x\to\infty} \log\left(\frac{3x-2}{x}\right) \\
&= \lim_{x\to\infty} \log\left(3 - \frac{2}{x}\right) \\
&= \log\left(3 - \lim_{x\to\infty} \frac{2}{x}\right) \quad \text{(by continuity of } \log z\text{)} \\
&= \log 3,
\end{aligned}$$

therefore

$$\lim_{n\to\infty} [\log(3n-2) - \log n] = \log 3.$$

Example 27: Evaluate $\lim\limits_{n\to\infty} \dfrac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}}$.

Solution:

$$\text{Since } 0 \le \sin^2\left(\frac{n\pi}{3}\right) \le 1 \text{ then}$$
$$1 \le 1+\sin^2\left(\frac{n\pi}{3}\right) \le 2 \text{ and}$$
$$\frac{1}{\sqrt{n}} \le \frac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}} \le \frac{2}{\sqrt{n}}.$$

Now $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0$ and $\lim\limits_{n\to\infty} \dfrac{2}{\sqrt{n}} = 0$ (by standard limit 1)

so $\lim\limits_{n\to\infty} \dfrac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}} = 0$ by Sandwich Theorem.

Adding Infinitely Many Numbers

Starting with any sequence $\{a_n\}$, adding the a_n's together in order gives a new sequence $\{s_n\}$:

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1 + a_2, \\ s_3 &= a_1 + a_2 + a_3, \\ \vdots \quad &\quad \vdots \qquad \vdots \qquad \vdots \end{aligned}$$

The sequence $\{s_n\}$ may or may not converge. If it does converge, we call

$$S = \lim_{n\to\infty} s_n = \lim_{n\to\infty}(a_1 + a_2 + \ldots + a_n)$$

the sum of the a_n's.

Example 28: Find the sum S of $a_n = \left(\frac{1}{2}\right)^n, n \geq 1$.

Solution:

Since $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{4}$, $a_3 = \frac{1}{8}$, …, then

$$
\begin{aligned}
s_1 &= \frac{1}{2}, \\
s_2 &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}, \\
s_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}, \\
s_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}, \\
\vdots \quad & \quad \vdots \qquad\qquad\qquad\quad \vdots \qquad \vdots \\
s_n &= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.
\end{aligned}
$$

Standard limit 2 gives $S = 1 - \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 1$.

Series

The series with terms a_n is denoted by

$$\sum_{n=1}^{\infty} a_n.$$

If the sum S of the series is finite, we say that the series converges. Otherwise we say that the series diverges.

Example

The sequence $\{n\} = 1, 2, 3, 4, \ldots$.

The series $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 \ldots$

The sequence and series both diverge to infinity, so the sum does not exist.

Application: Decimals

The decimal representation of a number is actually a series.

Example

The sequence $\left\{\frac{1}{10^n}\right\} = 0.1, 0.01, 0.001, \ldots.$

The series $\sum_{n=1}^{\infty} \frac{1}{10^n} = 0.1 + 0.01 + 0.001 + \ldots = 0.11111111\ldots$

The sequence converges to 0 while the series converges to $\frac{1}{9}$.

In General

For a number $x \in (0,1)$ with decimal digits $d_1,\ d_2,\ d_3,\ d_4,\ \ldots$

$$x = 0.d_1 d_2 d_3 d_4 \ldots \; = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

Geometric Series

A geometric series has the form

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$$

The series converges if $|r| < 1$ and diverges if $|r| \geq 1$.

If $|r| < 1$, we have

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

Example 29: What does the series

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\dots$$

converge to?

Solution:

Geometric series with $a = 1$ and $r = \frac{1}{2}$, so

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-1/2} = 2$$

Note:

From example 28, $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = 1.$

Harmonic p Series

A harmonic p series has the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

The series converges if $p > 1$ and diverges if $p \leq 1$.

Note:

For the case $p > 1$, we don't usually know how to find the sum, we just know it exists.

Convergence Tests for Series

Theorem:

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Note:

If $\lim_{n\to\infty} a_n = 0$, we CANNOT use the Divergence Test to determine the behaviour of $\sum_{n=1}^{\infty} a_n$.

Example 30: Does the series $\sum_{n=1}^{\infty} \frac{n+1}{n}$ converge?

Solution:
Since

$$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1 \neq 0$$

(by standard limit 1)

then

$\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges by the Divergence test.

Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive term series.

1. If $a_n \leq b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $a_n \geq b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

To apply the comparison test we compare a given series to a harmonic p series or geometric series.

Example 31: Does $\sum_{n=1}^{\infty} \frac{7}{2n^2+4n+3}$ converge or diverge?

Solution:

For large n, $\frac{7}{2n^2+4n+3} \approx \frac{7}{2n^2}$. So expect convergence.

Now $\frac{7}{2n^2+4n+3} \leq \frac{7}{2n^2}$ for $n \geq 1$.

Since $\sum_{n=1}^{\infty} \frac{7}{2n^2}$ is a multiple of a harmonic p series ($p = 2$) it converges.

So $\sum_{n=1}^{\infty} \frac{7}{2n^2+4n+3}$ converges by the comparison test.

Example 32: Does $\sum_{n=1}^{\infty} \frac{n^2+4}{n^3+1}$ converge or diverge?

Solution:

For large n, $\frac{n^2+4}{n^3+1} \approx \frac{n^2}{n^3} = \frac{1}{n}$. So expect divergence.

Now

$$\begin{aligned}\frac{n^2+4}{n^3+1} &\geq \frac{n^2}{n^3+1} \\ &\geq \frac{n^2}{n^3+n^3} \\ &= \frac{1}{2n}\end{aligned}$$

for $n \geq 1$.

Since $\sum_{n=1}^{\infty} \frac{1}{2n}$ is a multiple of a harmonic p series ($p = 1$) it diverges.

So $\sum_{n=1}^{\infty} \frac{n^2+4}{n^3+1}$ diverges by the comparison test.

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a positive term series and

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

1. If $L < 1$, $\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, the ratio test is inconclusive.

The ratio test is particularly useful if a_n contains an exponential or factorial function of n.

Example 33: Does $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ converge or diverge?

Solution:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \left(\frac{10^{n+1}}{(n+1)!} \times \frac{n!}{10^n} \right) \\
&= \lim_{n\to\infty} \frac{10^n\ 10\ n!}{(n+1)\ n!\ 10^n} \\
&= \lim_{n\to\infty} \frac{10}{n+1} \\
&= \lim_{n\to\infty} \frac{\frac{10}{n}}{1+\frac{1}{n}} \\
&= 0
\end{aligned}$$

by standard limit 1.

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ converges by the ratio test.

Example 34: Does $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$ converge or diverge?

Solution:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \left(\frac{(2n+2)!}{(n+1)!\,(n+1)!} \times \frac{n!\,n!}{(2n)!} \right) \\
&= \lim_{n\to\infty} \frac{(2n+2)\,(2n+1)\,(2n)!\,n!\,n!}{(n+1)\,n!\,(n+1)\,n!\,(2n)!} \\
&= \lim_{n\to\infty} \frac{(2n+2)\,(2n+1)}{(n+1)\,(n+1)} \\
&= \lim_{n\to\infty} \left(\frac{4n+2}{n+1} \right) \\
&= \lim_{n\to\infty} \left(\frac{4+\frac{2}{n}}{1+\frac{1}{n}} \right) \\
&= 4
\end{aligned}$$

by standard limit 1.

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1, \quad \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$ diverges by the ratio test.

Section 2: Hyperbolic Functions

Even Functions

A function $f(x)$ is an even function if

$$f(x) = f(-x)$$

Example

$f(x) = \cos x$ and $f(x) = x^2$

Odd Functions

A function $f(x)$ is an odd function if

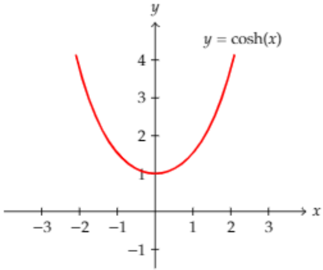
$$f(x) = -f(-x)$$

Example

$f(x) = \sin x$ and $f(x) = x^3$

We define the hyperbolic cosine function:

$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right), \quad x \in \mathbb{R}$$

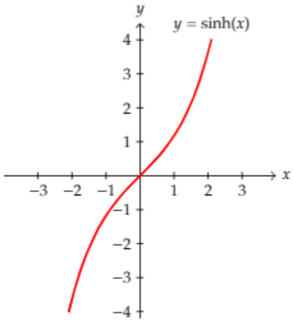


Properties

- $\cosh 0 = \frac{1}{2}(1+1) = 1$
- $\cosh(-x) = \frac{1}{2}\left(e^{-x} + e^{x}\right) = \cosh x$ (even function).

We define the hyperbolic sine function:

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right), \quad x \in \mathbb{R}$$

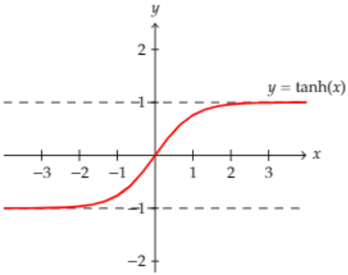


Properties

- $\sinh 0 = \frac{1}{2}(1 - 1) = 0.$
- $\sinh(-x) = \frac{1}{2}\left(e^{-x} - e^{x}\right) = -\sinh x$ (odd function).

We define the hyperbolic tangent function:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{1}{2}\left(e^x - e^{-x}\right)}{\frac{1}{2}\left(e^x + e^{-x}\right)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad x \in \mathbb{R}.$$



Properties

- $\tanh 0 = \dfrac{\sinh 0}{\cosh 0} = \dfrac{0}{1} = 0.$

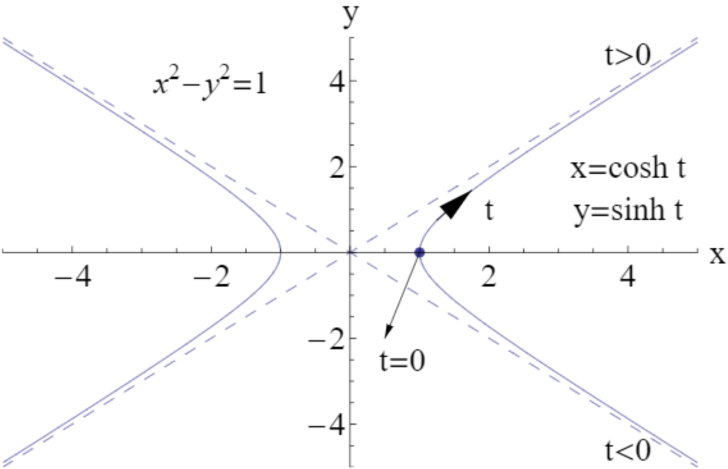
- $\tanh(-x) = \dfrac{\sinh(-x)}{\cosh(-x)} = \dfrac{-\sinh(x)}{\cosh(x)} = -\tanh x$ (odd function).

Why call them hyperbolic functions?

Let $x = \cosh t$ and $y = \sinh t$ then

$$
\begin{aligned}
x^2 - y^2 &= \cosh^2 t - \sinh^2 t \\
&= \left[\frac{1}{2}\left(e^t + e^{-t}\right)\right]^2 - \left[\frac{1}{2}\left(e^t - e^{-t}\right)\right]^2 \\
&= \frac{1}{4}\left[e^{2t} + 2 + e^{-2t} - \left(e^{2t} - 2 + e^{-2t}\right)\right] \\
&= \frac{1}{4} \cdot 4 = 1.
\end{aligned}
$$

So $(x, y) = (\cosh t, \sinh t)$ denotes a point on the hyperbola $x^2 - y^2 = 1$. Since $x \geq 1$, the right hand branch of the hyperbola can be parametrized by $x = \cosh t, y = \sinh t, t \in \mathbb{R}$.



Example 1: Write $(\cosh x - \sinh x)^7$ in terms of exponentials.

Solution:

$$\begin{aligned}
\bullet \cosh x - \sinh x &= \frac{1}{2}\left(e^x + e^{-x}\right) - \frac{1}{2}\left(e^x - e^{-x}\right) \\
&= \frac{1}{2}\left(e^x + e^{-x} - e^x + e^{-x}\right) \\
&= \frac{1}{2} \cdot 2e^{-x} \\
&= e^{-x}.
\end{aligned}$$

$$\bullet (\cosh x - \sinh x)^7 = \left(e^{-x}\right)^7 = e^{-7x}.$$

Example 2: If $\cosh x = \frac{13}{12}$ and $x < 0$ find $\sinh x$ and $\tanh x$.

Solution:

$$\begin{aligned}
\text{Now } \cosh^2 x - \sinh^2 x &= 1, \text{ so} \\
\sinh^2 x &= \cosh^2 x - 1 \\
\Rightarrow \sinh^2 x &= \left(\frac{13}{12}\right)^2 - 1 = \frac{169}{144} - 1 = \frac{25}{144} \\
\Rightarrow \sinh x &= \pm\sqrt{\frac{25}{144}} = \pm\frac{5}{12}.
\end{aligned}$$

Since $x < 0$ then $\sinh x < 0$ so we choose the negative solution.

$$\begin{aligned}
\Rightarrow \sinh x &= -\frac{5}{12} \\
\text{and } \tanh x &= \frac{\sinh x}{\cosh x} \\
&= \frac{(-5/12)}{(13/12)} \\
&= -\frac{5}{13}.
\end{aligned}$$

Example 3: Write $\sinh(2\log x)$ as an algebraic expression in x.

Solution:

$$\begin{aligned}\sinh(2\log x) &= \frac{1}{2}\left(e^{2\log x} - e^{-2\log x}\right)\\ &= \frac{1}{2}\left(e^{\log x^2} - e^{\log x^{-2}}\right)\\ &= \frac{1}{2}\left(x^2 - x^{-2}\right).\end{aligned}$$

Example 4: Write $\cosh^3 x$ in terms of the functions $\cosh nx$ for integers n.

Solution:

$$
\begin{aligned}
\cosh^3 x &= \left[\frac{1}{2}(e^x + e^{-x})\right]^3 \\
&= \frac{1}{8}\left(e^{3x} + 3e^x + 3e^{-x} + e^{-3x}\right) \\
&= \frac{1}{8}\left(e^{3x} + e^{-3x}\right) + \frac{3}{8}(e^x + e^{-x}) \\
&= \frac{1}{4}\cosh 3x + \frac{3}{4}\cosh x.
\end{aligned}
$$

Addition Formulae

$$
\begin{aligned}
\sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y \\
\cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y \\
\sinh(x-y) &= \sinh x \cosh y - \cosh x \sinh y \\
\cosh(x-y) &= \cosh x \cosh y - \sinh x \sinh y
\end{aligned}
$$

Example 5: Prove the $\sinh(x+y)$ addition formula.

Solution:

Now $\sinh x \cosh y + \cosh x \sinh y$

$$
\begin{aligned}
&= \frac{1}{2}\left(e^x - e^{-x}\right)\frac{1}{2}\left(e^y + e^{-y}\right) + \frac{1}{2}\left(e^x + e^{-x}\right)\frac{1}{2}\left(e^y - e^{-y}\right) \\
&= \frac{1}{4}\left(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y} + e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}\right) \\
&= \frac{1}{4}\left(2e^{x+y} - 2e^{-x-y}\right) \\
&= \frac{1}{2}\left(e^{x+y} - e^{-x-y}\right) \\
&= \sinh(x+y).
\end{aligned}
$$

Double Angle Formulae

$$\boxed{\begin{aligned} \sinh 2x &= 2\sinh x \cosh x \\ \cosh 2x &= \cosh^2 x + \sinh^2 x \\ \cosh 2x &= 2\cosh^2 x - 1 \\ \cosh 2x &= 2\sinh^2 x + 1 \end{aligned}}$$

Example 6: Prove $\sinh 2x$ double angle formula.

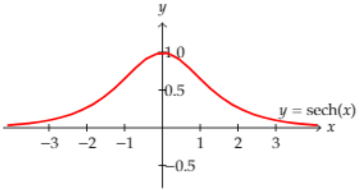
Solution:

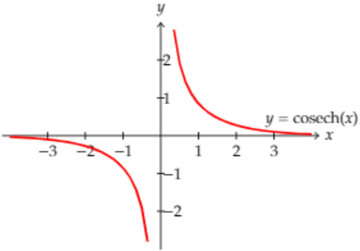
$$\begin{aligned}
\sinh 2x &= \sinh(x+x) \\
&= \sinh x \cosh x + \cosh x \sinh x \\
&= 2 \sinh x \cosh x.
\end{aligned}$$

Reciprocal Hyperbolic Functions

We define the three reciprocal hyperbolic functions:

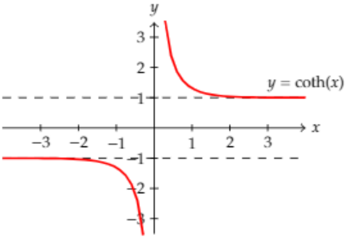
$$\operatorname{sech} x = \frac{1}{\cosh x}, \quad x \in \mathbb{R} \qquad\qquad \operatorname{cosech} x = \frac{1}{\sinh x}, x \in \mathbb{R} \setminus \{0\}$$





Reciprocal Hyperbolic Functions

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}\,,$$
$$x \in \mathbb{R} \setminus \{0\}$$



Basic Identities

$$\boxed{\begin{aligned} \cosh^2 x - \sinh^2 x &= 1 \\ \coth^2 x - 1 &= \operatorname{cosech}^2 x \\ 1 - \tanh^2 x &= \operatorname{sech}^2 x \end{aligned}}$$

Derivatives of Hyperbolic Functions

$$
\begin{aligned}
\frac{d}{dx}(\cosh x) &= \sinh x, \quad x \in \mathbb{R} \\
\frac{d}{dx}(\sinh x) &= \cosh x, \quad x \in \mathbb{R} \\
\frac{d}{dx}(\tanh x) &= \operatorname{sech}^2 x, \quad x \in \mathbb{R} \\
\frac{d}{dx}(\operatorname{sech} x) &= -\operatorname{sech} x \tanh x, \quad x \in \mathbb{R} \\
\frac{d}{dx}(\operatorname{cosech} x) &= -\operatorname{cosech} x \coth x, \quad x \in \mathbb{R} \setminus \{0\} \\
\frac{d}{dx}(\coth x) &= -\operatorname{cosech}^2 x, \quad x \in \mathbb{R} \setminus \{0\}
\end{aligned}
$$

Example 7: Prove that $\dfrac{d\,(\cosh x)}{dx} = \sinh x.$

Solution:

$$\begin{aligned}
\frac{d}{dx}(\cosh x) &= \frac{d}{dx}\left[\frac{1}{2}\left(e^x + e^{-x}\right)\right] \\
&= \frac{1}{2}\left(e^x - e^{-x}\right) \\
&= \sinh x.
\end{aligned}$$

Example 8: Prove that $\dfrac{d(\tanh x)}{dx} = \operatorname{sech}^2 x$.

Solution:

$$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)$$

$$= \frac{\cosh x \frac{d}{dx}(\sinh x) - \sinh x \frac{d}{dx}(\cosh x)}{\cosh^2 x}$$

(quotient rule)

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$

$$= \frac{1}{\cosh^2 x}$$

$$= \operatorname{sech}^2 x.$$

Example 9: Let $y = \sqrt{\sinh 6x}$, $x > 0$. Find $\dfrac{dy}{dx}$.

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2} \cdot \frac{1}{\sqrt{\sinh 6x}} \cdot \frac{d}{dx}(\sinh 6x) \quad \text{(chain rule)} \\
&= \frac{6\cosh 6x}{2\sqrt{\sinh 6x}} \\
&= \frac{3\cosh 6x}{\sqrt{\sinh 6x}}.
\end{aligned}$$

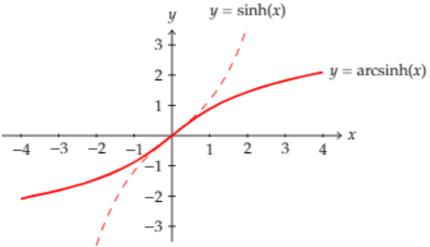
Inverses of Hyperbolic Functions

We define three inverse hyperbolic functions.

1. Inverse hyperbolic sine function: $\operatorname{arcsinh} x$

Since $\sinh x$ is a 1-1 function

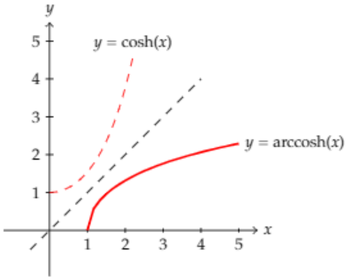
$$\begin{aligned}
\text{domain } \operatorname{arcsinh} x &= \text{range } \sinh x = \mathbb{R}. \\
\text{range } \operatorname{arcsinh} x &= \text{domain } \sinh x = \mathbb{R}. \\
\operatorname{arcsinh}(\sinh x) &= x, \quad x \in \mathbb{R}. \\
\sinh(\operatorname{arcsinh} x) &= x, \quad x \in \mathbb{R}.
\end{aligned}$$



2. Inverse hyperbolic cosine function: $\operatorname{arccosh} x$

Restrict domain of $\cosh x$ to be $[0, \infty)$ to give a 1-1 function. Then

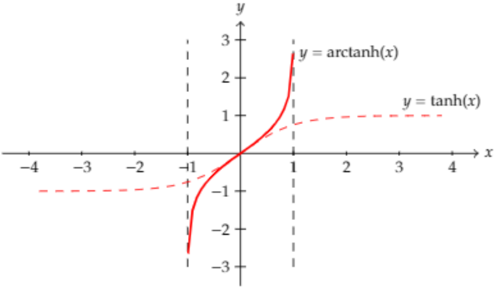
$$\begin{aligned}
\text{domain } \operatorname{arccosh} x &= \text{range } \cosh x = [1, \infty). \\
\text{range } \operatorname{arccosh} x &= \text{restricted domain } \cosh x = [0, \infty). \\
\cosh(\operatorname{arccosh} x) &= x, \quad x \geq 1. \\
\operatorname{arccosh}(\cosh x) &= x, \quad x \geq 0.
\end{aligned}$$



3. Inverse hyperbolic tangent function: $\operatorname{arctanh} x$

Since $\tanh x$ is a 1-1 function

$$
\begin{aligned}
\text{domain } \operatorname{arctanh} x &= \text{range } \tanh x = (-1, 1). \\
\text{range } \operatorname{arctanh} x &= \text{domain } \tanh x = \mathbb{R}. \\
\tanh(\operatorname{arctanh} x) &= x, \quad -1 < x < 1. \\
\operatorname{arctanh}(\tanh x) &= x, \quad x \in \mathbb{R}.
\end{aligned}
$$



The inverse hyperbolic functions can be expressed in terms of logarithms.

$$\begin{array}{|lcll|}\hline \operatorname{arcsinh} x & = & \log\left(x + \sqrt{x^2+1}\right), & x \in \mathbb{R} \\ \operatorname{arccosh} x & = & \log\left(x + \sqrt{x^2-1}\right), & x \geq 1 \\ \operatorname{arctanh} x & = & \frac{1}{2}\log\left(\frac{1+x}{1-x}\right), & -1 < x < 1 \\ \hline \end{array}$$

We can also define 3 more inverse hyperbolic functions $\operatorname{arcsech} x$, $\operatorname{arccosech} x$, $\operatorname{arccoth} x$ in terms of logarithms.

Example 10: Proof of $\operatorname{arcsinh} x$ relation.

Solution:

$$\text{Let } y = \operatorname{arcsinh} x \Rightarrow x = \sinh y$$

$$\Rightarrow x = \frac{1}{2}\left(e^{y} - e^{-y}\right)$$

$$\Rightarrow 2x = e^{y} - e^{-y} \quad (\times e^{y})$$

$$\text{So } 2xe^{y} = e^{2y} - 1$$

$$\Rightarrow e^{2y} - 2xe^{y} - 1 = 0.$$

Solving gives :

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

We require $e^y > 0$ so we take the

positive root as $\sqrt{x^2+1} > x$

$$\Rightarrow e^y = x + \sqrt{x^2+1}$$

$$\Rightarrow y = \log\left(x + \sqrt{x^2+1}\right).$$

Example 11: Find the exact value of $\sinh[\operatorname{arccosh} 3]$.

Solution:

$$
\begin{aligned}
\sinh[\operatorname{arccosh} 3] &= \sinh\left[\log(3+\sqrt{8})\right] \\
&= \frac{1}{2}\left[e^{\log(3+\sqrt{8})} - e^{-\log(3+\sqrt{8})}\right] \\
&= \frac{1}{2}\left[3+\sqrt{8} - \frac{1}{3+\sqrt{8}}\right] \\
&= \frac{1}{2}\left[3+\sqrt{8} - \frac{3-\sqrt{8}}{(3+\sqrt{8})(3-\sqrt{8})}\right] \\
&= \frac{1}{2}\left[3+\sqrt{8} - (3-\sqrt{8})\right] \\
&= \sqrt{8}.
\end{aligned}
$$

Example 12: Express $\cosh(\operatorname{arctanh} x)$ as an algebraic function of x for $-1 < x < 1$.

Solution:

Let $y = \operatorname{arctanh} x \Rightarrow x = \tanh y$.

Now $1 - \tanh^2 y = \operatorname{sech}^2 y$

$$\Rightarrow \operatorname{sech}^2 y = 1 - x^2$$

$$\Rightarrow \frac{1}{\cosh^2 y} = 1 - x^2$$

$$\Rightarrow \cosh^2 y = \frac{1}{1 - x^2}$$

$$\Rightarrow \cosh y = \pm \frac{1}{\sqrt{1 - x^2}}.$$

Since $\cosh y > 0$ for all y,

we require the positive root, and

$$\cosh(\operatorname{arctanh} x) = \frac{1}{\sqrt{1 - x^2}}.$$

Derivatives

$$\boxed{\begin{aligned} \frac{d}{dx}(\operatorname{arcsinh} x) &= \frac{1}{\sqrt{x^2+1}} \quad (x \in \mathbb{R}) \\ \frac{d}{dx}(\operatorname{arccosh} x) &= \frac{1}{\sqrt{x^2-1}} \quad (x > 1) \\ \frac{d}{dx}(\operatorname{arctanh} x) &= \frac{1}{1-x^2} \quad (-1 < x < 1) \end{aligned}}$$

Each formula is derived using implicit differentiation or by differentiating the logarithm defintion of each function.

Example 13: Prove that $\frac{d}{dx}(\operatorname{arcsinh} x) = \frac{1}{\sqrt{x^2+1}}$.

Solution:

Let $y = \operatorname{arcsinh} x \Rightarrow x = \sinh y$.

Differentiate both sides with respect to x

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sinh y)$$

$$\Rightarrow 1 = \cosh y \frac{dy}{dx} \quad \text{(chain rule)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}.$$

Now $\cosh^2 y - \sinh^2 y = 1$

$$\Rightarrow \cosh^2 y = 1 + \sinh^2 y = 1 + x^2$$

$$\Rightarrow \cosh y = \pm\sqrt{1 + x^2}.$$

Since $\cosh y > 0$ for all y,
we require the positive root, and

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + x^2}}, \quad x \in \mathbb{R}.$$

Example 14: Find $\frac{d}{dx}(\operatorname{arctanh} 2x \cosh 3x)$.

Solution:

$$\begin{aligned}
\frac{d}{dx}(\operatorname{arctanh} 2x \cosh 3x) &= \frac{d}{dx}(\operatorname{arctanh} 2x) \cosh 3x \\
&+ \operatorname{arctanh} 2x \frac{d}{dx}(\cosh 3x)
\end{aligned}$$

(product rule)

$$= \frac{2 \cosh 3x}{1-(2x)^2} + 3 \operatorname{arctanh} 2x \sinh 3x.$$

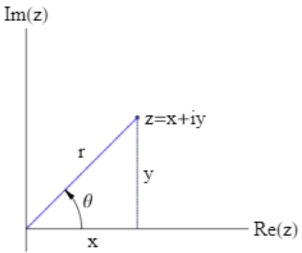
Section 3: Complex Numbers

The Cartesian form of a complex number is

$$\boxed{z = x + iy} \quad \text{where} \quad x, y \in \mathbb{R}$$

and

- $x = Re(z)$ is the real part of z,
- $y = Im(z)$ is the imaginary part of z,
- $i^2 = -1$.



The complex number can be written as

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

where

- $r = |z| = \sqrt{x^2 + y^2}$
- $\tan\theta = \dfrac{y}{x}$

Note:

The angle θ is not unique – only defined up to multiples of 2π. We choose θ such that $-\pi < \theta \leq \pi$ and call this angle the principal argument of z.

The Complex Exponential

We define the complex exponential using Euler's formula

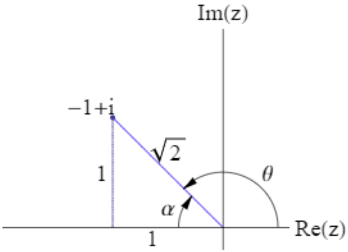
$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

for $\theta \in \mathbb{R}$.

We can then write the polar form of a complex number as

$$\boxed{z = re^{i\theta}}$$

Example 1: Write $z = -1 + i$ in polar form.



Solution:

- $r = \sqrt{1+1} = \sqrt{2}$
- $\tan\alpha = 1 \quad \Rightarrow \alpha = \dfrac{\pi}{4}$
- $\theta = \pi - \alpha = \dfrac{3\pi}{4}$

So $z = \sqrt{2}e^{\frac{3\pi i}{4}}$.

Properties of the Complex Exponential

1. $e^{i0} = 1$
2. $e^{i\theta} \cdot e^{i\phi} = e^{i(\theta+\phi)}$

Property 1

$$e^{i0} = \cos 0 + i \sin 0 = 1.$$

Property 2

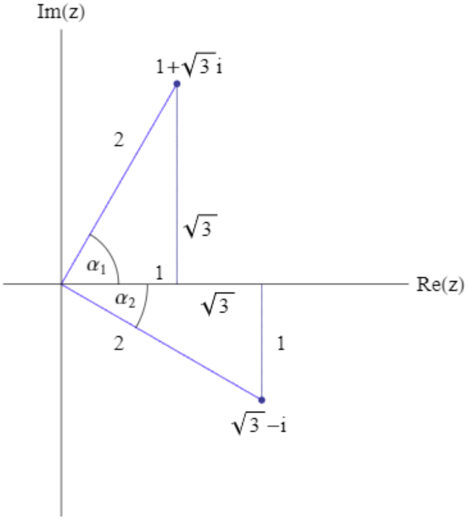
$$
\begin{aligned}
e^{i\theta} \cdot e^{i\phi} &= (\cos\theta + i\sin\theta)\Big(\cos\phi + i\sin\phi\Big) \\
&= \cos\theta\cos\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi - \sin\theta\sin\phi \\
&= \Big(\cos\theta\cos\phi - \sin\theta\sin\phi\Big) + i\Big(\cos\theta\sin\phi + \sin\theta\cos\phi\Big) \\
&= \cos\Big(\theta + \phi\Big) + i\sin(\theta + \phi) \\
&= e^{i(\theta+\phi)}.
\end{aligned}
$$

Products and Division in Polar Form

If $z = r_1 e^{i\theta}$ and $w = r_2 e^{i\phi}$ then

$$\boxed{\begin{aligned} zw &= r_1 r_2 e^{i(\theta+\phi)} \\ \frac{z}{w} &= \frac{r_1}{r_2} e^{i(\theta-\phi)} \end{aligned}}$$

Example 2: Using the complex exponential, simplify $(\sqrt{3}-i)(1+\sqrt{3}i)$ and $\dfrac{\sqrt{3}-i}{1+\sqrt{3}i}$.



Solution:

- For $z_1 = 1 + \sqrt{3}i: \quad r_1 = 2, \quad \theta_1 = \alpha_1 = \dfrac{\pi}{3}$

$\Rightarrow z_1 = 2e^{\frac{i\pi}{3}}$

- For $z_2 = \sqrt{3} - i: \quad r_2 = 2, \quad \alpha_2 = \dfrac{\pi}{6}, \quad \theta_2 = -\alpha_2 = \dfrac{-\pi}{6}$

$\Rightarrow z_2 = 2e^{-\frac{i\pi}{6}}$

$$
\begin{aligned}
\text{(a) } (\sqrt{3} - i)(1 + \sqrt{3}i) &= 2e^{-\frac{i\pi}{6}} \cdot 2e^{\frac{i\pi}{3}} \\
&= 4e^{i\left(\frac{\pi}{3} - \frac{\pi}{6}\right)} \\
&= 4e^{\frac{i\pi}{6}} \qquad \text{(polar form)} \\
&= 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \\
&= 4\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) \\
&= 2\sqrt{3} + 2i \qquad \text{(Cartesian form).}
\end{aligned}
$$

(b) $$\frac{\sqrt{3}-i}{1+\sqrt{3}i} = \frac{2e^{\frac{-i\pi}{6}}}{2e^{\frac{i\pi}{3}}}$$

$$= e^{i\left(-\frac{\pi}{6}-\frac{\pi}{3}\right)}$$

$$= e^{\frac{-3\pi i}{6}}$$

$$= e^{\frac{-\pi i}{2}} \qquad \text{(polar form)}$$

$$= \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$$

$$= -i \qquad \text{(Cartesian form).}$$

De Moivre's Theorem:

If $z = re^{i\theta}$ and n is a positive integer then

$$\boxed{z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta}}$$

Example 3: Evaluate $\left(1+\sqrt{3}i\right)^{15}$.

Solution:

$$\begin{aligned}
\left(1+\sqrt{3}i\right)^{15} &= \left(2e^{\frac{i\pi}{3}}\right)^{15} \\
&= 2^{15}e^{\frac{15\pi i}{3}} \qquad \text{(De Moivre's Theorem)} \\
&= 2^{15}e^{5\pi i} \\
&= 2^{15}\left(\cos 5\pi + i\sin 5\pi\right) \\
&= -2^{15}.
\end{aligned}$$

Exponential Form of $\sin\theta$ and $\cos\theta$

$$\text{Now } e^{i\theta} = \cos\theta + i\sin\theta \qquad (1)$$

$$\Rightarrow e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

$$\Rightarrow e^{-i\theta} = \cos\theta - i\sin\theta \qquad (2)$$

Equation (1) + (2) gives

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\Rightarrow \boxed{\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)}$$

Equation (1) – (2) gives

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

$$\Rightarrow \boxed{\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)}$$

Note:

These formulae give a connection between the hyperbolic and trigonometric functions.

$$\begin{aligned}
\cosh(i\theta) &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \cos\theta \\
\sinh(i\theta) &= \frac{1}{2}\left(e^{i\theta} - e^{-i\theta}\right) = i\sin\theta
\end{aligned}$$

Example 4: Express $\sin^5\theta$ as a sum of sines of multiples of θ.

Solution:

$$\begin{aligned}
\sin^5\theta &= \left[\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)\right]^5 \\
&= \frac{1}{32i^5}\left[e^{5i\theta}-5e^{3i\theta}+10e^{i\theta}-10e^{-i\theta}+5e^{-3i\theta}-e^{-5i\theta}\right] \\
&= \frac{1}{32i}\left[\left(e^{5i\theta}-e^{-5i\theta}\right)-5\left(e^{3i\theta}-e^{-3i\theta}\right)+10\left(e^{i\theta}-e^{-i\theta}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{16}\Big[\frac{1}{2i}\left(e^{5i\theta} - e^{-5i\theta}\right) - \frac{5}{2i}\left(e^{3i\theta} - e^{-3i\theta}\right) + \frac{10}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\Big] \\
&= \frac{1}{16}\left[\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right].
\end{aligned}
$$

Differentiation Via the Complex Exponential

If $z = x + yi$ where $x, y \in \mathbb{R}$ then we define

$$e^z = e^x\, e^{iy} = e^x(\cos y + i \sin y).$$

Show that $\dfrac{d}{dt}\left(e^{kt}\right) = ke^{kt}$ when $k = a + bi \in \mathbb{C}$.

$$\begin{aligned}
\frac{d}{dt}\left[e^{(a+bi)t}\right] &= \frac{d}{dt}\left[e^{at}e^{ibt}\right] \\
&= \frac{d}{dt}\left[e^{at}\left(\cos bt + i \sin bt\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= ae^{at}\left(\cos bt + i\sin bt\right) + e^{at}\left(-b\sin bt + bi\cos bt\right) \\
&= ae^{at}\left(\cos bt + i\sin bt\right) + e^{at}\left(bi^2\sin bt + bi\cos bt\right) \\
&= ae^{at}\left(\cos bt + i\sin bt\right) + bie^{at}\left(\cos bt + i\sin bt\right) \\
&= (a+bi)e^{at}\left(\cos bt + i\sin bt\right) \\
&= (a+bi)e^{at}e^{ibt} \\
&= (a+bi)e^{(a+ib)t}.
\end{aligned}
$$

Example 5: Using the complex exponential, find $\dfrac{d}{dt}\left(e^t \cos 5t\right)$.

Solution:

$$
\begin{aligned}
\text{Now } e^t \cos 5t &= e^t Re\left(e^{5it}\right) \\
&= Re\left(e^t \cdot e^{5it}\right) \\
&= Re\left(e^{(1+5i)t}\right) \\
\Rightarrow \frac{d}{dt}\left(e^t \cos 5t\right) &= \frac{d}{dt}\left[Re\left(e^{(1+5i)t}\right)\right] \\
&= Re\left[\frac{d}{dt}\left(e^{(1+5i)t}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= Re\left[(1+5i)\,e^{(1+5i)t}\right] \\
&= Re\left[(1+5i)\,e^{t}\,(\cos 5t + i\sin 5t)\right] \\
&= Re\left[e^{t}\,(\cos 5t + i\sin 5t + 5i\cos 5t - 5\sin 5t)\right] \\
&= e^{t}\cos 5t - 5e^{t}\sin 5t.
\end{aligned}
$$

Example 6: Find $\frac{d^{56}}{dt^{56}}\left(e^{-t}\sin t\right)$.

Solution:

$$\begin{aligned}
\text{Now } e^{-t}\sin t &= e^{-t} Im\left(e^{it}\right) \\
&= Im\left(e^{-t}\cdot e^{it}\right) \\
&= Im\left(e^{(-1+i)t}\right) \\
\Rightarrow \frac{d^{56}}{dt^{56}}\left(e^{-t}\sin t\right) &= \frac{d^{56}}{dt^{56}}\left[Im\left(e^{(-1+i)t}\right)\right] \\
&= Im\left[\frac{d^{56}}{dt^{56}}\left(e^{(-1+i)t}\right)\right] \\
&= Im\left[(-1+i)^{56}e^{(-1+i)t}\right]
\end{aligned}$$

$$
\begin{aligned}
\text{Now } (-1+i)^{56} &= \left(\sqrt{2}e^{\frac{3\pi i}{4}}\right)^{56} \\
&= \left(\sqrt{2}\right)^{56} e^{\frac{168\pi i}{4}} \\
&= 2^{28} e^{42\pi i} \\
&= 2^{28}\left(\cos 42\pi + i \sin 42\pi\right) \\
&= 2^{28}
\end{aligned}
$$

$$
\begin{aligned}
\text{Hence } \frac{d^{56}}{dt^{56}}\left(e^{-t}\sin t\right) &= Im\left[2^{28}e^{(-1+i)t}\right] \\
&= 2^{28}e^{-t}\sin t.
\end{aligned}
$$

Note:

Example 6 also gives the answer to $\dfrac{d^{56}}{dt^{56}}\left(e^{-t}\cos t\right)$.

$$
\begin{aligned}
\text{Since } e^{-t}\cos t &= Re\left(e^{(-1+i)t}\right) \text{ then} \\
\frac{d^{56}}{dt^{56}}\left(e^{-t}\cos t\right) &= Re\left[\frac{d^{56}}{dt^{56}}\left(e^{(-1+i)t}\right)\right] \\
&= Re\left[2^{28}e^{(-1+i)t}\right] \\
&= 2^{28}e^{-t}\cos t.
\end{aligned}
$$

Integration Via the Complex Exponential

Since $\dfrac{d}{dx}\left(e^{kx}\right) = k\,e^{kx}$ if $k = a + bi$ $(a, b \in \mathbb{R})$, then

$$\begin{aligned} \int k\,e^{kx}\,dx &= e^{kx} + C \\ \Rightarrow \int e^{kx}\,dx &= \frac{1}{k}e^{kx} + D \end{aligned}$$

Example 7: Evaluate $\int e^{3x} \sin 2x \, dx$.

Solution:

$$\begin{aligned}
\text{Now } e^{3x} \sin 2x &= e^{3x} Im\left[e^{2ix}\right] \\
&= Im\left[e^{3x} \cdot e^{2ix}\right] \\
&= Im\left[e^{(3+2i)x}\right] \\
\text{So } \int e^{3x} \sin 2x \, dx &= \int Im\left[e^{(3+2i)x}\right] dx \\
&= Im\left[\int e^{(3+2i)x} dx\right] \\
&= Im\left[\frac{1}{3+2i} e^{(3+2i)x} + c + di\right]
\end{aligned}$$

$$
\begin{aligned}
&= Im\left[\frac{3-2i}{(3+2i)(3-2i)}e^{3x}\cdot e^{2ix}+c+di\right] \\
&= Im\left[\frac{(3-2i)}{13}e^{3x}\left(\cos 2x+i\sin 2x\right)+c+di\right] \\
&= Im\Big[\frac{e^{3x}}{13}\Big(3\cos 2x+3i\sin 2x-2i\cos 2x \\
&\qquad\qquad +2\sin 2x\Big)+c+di\Big] \\
&= \frac{e^{3x}}{13}\left(3\sin 2x-2\cos 2x\right)+d.
\end{aligned}
$$

Note:

Example 7 also gives the answer to $\int e^{3x}\cos 2x\,dx$.

$$\text{Since } e^{3x}\cos 2x = Re\left[e^{(3+2i)x}\right] \text{ then}$$

$$\begin{aligned}
\int e^{3x}\cos 2x\,dx &= Re\left[\int e^{(3+2i)x}\,dx\right] \\
&= Re\Big[\frac{e^{3x}}{13}\Big(3\cos 2x + 3i\sin 2x - 2i\cos 2x \\
&\qquad\qquad +2\sin 2x\Big) + c + di\Big] \\
&= \frac{e^{3x}}{13}\left(3\cos 2x + 2\sin 2x\right) + c.
\end{aligned}$$

Derivative Substitutions

To evaluate

$$\int f[g(x)]g'(x)dx$$

put $u = g(x) \Rightarrow \dfrac{du}{dx} = g'(x)$.

Then integral becomes

$$\begin{aligned} &= \int f(u)\frac{du}{dx}\,dx \\ &= \int f(u)\,du \end{aligned}$$

Example 1: Evaluate $\int (6x^2 + 10)\sinh(x^3 + 5x - 2)dx$.

Solution:

$$\text{Put } u = x^3 + 5x - 2 \Rightarrow \frac{du}{dx} = 3x^2 + 5$$

$$\begin{aligned}
\int (6x^2 + 10)\sinh(x^3 + 5x - 2)dx &= \int 2\frac{du}{dx}\sinh u\,dx \\
&= \int 2\sinh u\,du \\
&= 2\cosh u + c \\
&= 2\cosh(x^3 + 5x - 2) + c.
\end{aligned}$$

Example 2: Evaluate $\int \frac{\operatorname{cosech}^2(3x)}{10 - 2\coth(3x)} dx$.

Solution:

$$\text{Put } u = 10 - 2\coth(3x) \Rightarrow \frac{du}{dx} = 6\operatorname{cosech}^2 3x$$

$$\begin{aligned}
\int \frac{\operatorname{cosech}^2(3x)}{10 - 2\coth(3x)} dx &= \int \frac{\frac{1}{6}\frac{du}{dx}}{u} dx \\
&= \frac{1}{6}\int \frac{1}{u} du \\
&= \frac{1}{6}\log|u| + c \\
&= \frac{1}{6}\log|10 - 2\coth(3x)| + c.
\end{aligned}$$

Trigonometric and Hyperbolic Substitutions

We can use trigonometric and hyperbolic substitutions to integrate expressions containing

$$\sqrt{a^2 - x^2}, \quad \sqrt{a^2 + x^2}, \quad \sqrt{x^2 - a^2},$$

where a is a positive real number.

Method:

Put $x = g(\theta)$. Then

$$\int f(x)\, dx = \int f[g(\theta)]g'(\theta)\, d\theta$$

Integrand	Substitution
$\sqrt{a^2 - x^2}, \quad \dfrac{1}{\sqrt{a^2 - x^2}}$	$x = a \sin\theta$
$\sqrt{a^2 + x^2}, \quad \dfrac{1}{\sqrt{a^2 + x^2}}$	$x = a \sinh\theta$
$\sqrt{x^2 - a^2}, \quad \dfrac{1}{\sqrt{x^2 - a^2}}$	$x = a \cosh\theta$
$\dfrac{1}{a^2 + x^2}$	$x = a \tan\theta$

Example 3: Evaluate $\int \frac{1}{\sqrt{x^2+25}}\,dx$ using a substitution.

Solution:

$$\text{Put } x = 5\sinh\theta \Rightarrow \frac{dx}{d\theta} = 5\cosh\theta$$

$$\begin{aligned} \text{and } x^2+25 &= 25\sinh^2\theta + 25 = 25\cosh^2\theta \\ \Rightarrow \sqrt{x^2+25} &= 5\cosh\theta \end{aligned}$$

So
$$\begin{aligned}\int \frac{1}{\sqrt{x^2+25}}dx &= \int \frac{1}{5\cosh\theta}\cdot 5\cosh\theta\, d\theta \\ &= \int 1\, d\theta \\ &= \theta + c \\ &= \operatorname{arcsinh}\left(\frac{x}{5}\right) + c.\end{aligned}$$

Example 4: Evaluate $\int \frac{1}{x^2+2}\,dx$ using a substitution.

Solution:

$$\text{Put } x = \sqrt{2}\tan\theta \Rightarrow \frac{dx}{d\theta} = \sqrt{2}\sec^2\theta$$

$$\text{and } x^2+2 = 2\tan^2\theta + 2 = 2\sec^2\theta.$$

$$\begin{aligned}
\text{So } \int \frac{1}{x^2+2}dx &= \int \frac{1}{2\sec^2\theta}\cdot\sqrt{2}\sec^2\theta\,d\theta \\
&= \int \frac{\sqrt{2}}{2}\,d\theta \\
&= \frac{\sqrt{2}}{2}\theta + c \\
&= \frac{\sqrt{2}}{2}\arctan\left(\frac{x}{\sqrt{2}}\right) + c.
\end{aligned}$$

Example 5: Evaluate $\displaystyle\int \sqrt{9-4x^2}\,dx$ if $|x| \leq \frac{3}{2}$.

Solution:

$$\int \sqrt{9-4x^2}dx = 2\int \sqrt{\frac{9}{4}-x^2dx}$$

$$\text{Put } x = \frac{3}{2}\sin\theta \Rightarrow \frac{dx}{d\theta} = \frac{3}{2}\cos\theta$$

$$\text{and } \frac{9}{4} - x^2 = \frac{9}{4} - \frac{9}{4}\sin^2\theta = \frac{9}{4}\cos^2\theta$$

If $\cos\theta \geq 0$ or $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$ then

$$\sqrt{\frac{9}{4}-x^2} = \frac{3}{2}\cos\theta$$

$$
\begin{aligned}
\text{So } \int \sqrt{9-4x^2}\,dx &= 2\int \frac{3}{2}\cos\theta \cdot \frac{3}{2}\cos\theta\,d\theta \\
&= \frac{9}{2}\int \cos^2\theta\,d\theta \\
&= \frac{9}{4}\int \cos 2\theta + 1\,d\theta \\
&= \frac{9}{4}\left(\frac{1}{2}\sin 2\theta + \theta\right) + c \\
&= \frac{9}{4}\Big(\underbrace{\sin\theta}_{\swarrow\ \frac{2x}{3},}\ \underbrace{\cos\theta}_{\downarrow\ \frac{2}{3}\sqrt{\frac{9}{4}-x^2},} + \underbrace{\theta}_{\searrow\ \arcsin\left(\frac{2x}{3}\right)}\Big) + c \\
&= x\sqrt{\frac{9}{4}-x^2} + \frac{9}{4}\arcsin\left(\frac{2x}{3}\right) + c.
\end{aligned}
$$

Example 6: Evaluate $\int \left(x^2-1\right)^{\frac{3}{2}}\,dx$ if $x \geq 1$.

Solution:

$$\text{Put } x = \cosh\theta \Rightarrow \frac{dx}{d\theta} = \sinh\theta$$

$$\text{and } x^2 - 1 = \cosh^2\theta - 1 = \sinh^2\theta$$

If $\sinh\theta \geq 0$ or $\theta \geq 0$ then

$$\left(x^2-1\right)^{\frac{3}{2}} = \sinh^3\theta$$

$$\text{So } \int \left(x^2-1\right)^{\frac{3}{2}}\,dx = \int \sinh^3\theta \cdot \sinh\theta\,d\theta$$

$$= \int \sinh^4\theta\,d\theta.$$

Powers of Hyperbolic Functions

Consider the integral:

$$\int \sinh^m x \cosh^n x \, dx$$

where m, n are integers (≥ 0).

- If m or n is odd, create a "derivative" substitution.

- If m and n are even, use double angle formulae.

Example 7: Evaluate $\int \sinh^4 x \, dx$.

Solution:

$$\begin{aligned}
\int \sinh^4 x \, dx &= \int \left(\sinh^2 x\right)^2 dx \\
&= \int \left(\frac{\cosh 2x - 1}{2}\right)^2 dx \\
&= \frac{1}{4} \int \underbrace{\cosh^2 2x}_{\overset{\|}{\frac{1}{2}(\cosh 4x + 1)}} - 2\cosh 2x + 1 \, dx \\
&= \frac{1}{4} \int \frac{1}{2} \cosh 4x - 2\cosh 2x + \frac{3}{2} \, dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\left[\frac{1}{8}\sinh 4x - \sinh 2x + \frac{3}{2}x\right] + c \\
&= \frac{1}{32}\sinh 4x - \frac{1}{4}\sinh 2x + \frac{3}{8}x + c.
\end{aligned}
$$

Finish Example 6:

$$
\begin{aligned}
& \int \left(x^2-1\right)^{\frac{3}{2}}\, dx \\
= & \int \sinh^4\theta\, d\theta \\
= & \frac{1}{32}\sinh 4\theta - \frac{1}{4}\sinh 2\theta + \frac{3}{8}\theta + c \\
= & \frac{1}{16}\sinh 2\theta \cosh 2\theta - \frac{1}{2}\sinh\theta\cosh\theta + \frac{3}{8}\theta + c \\
= & \frac{1}{8}\sinh\theta\cosh\theta(2\cosh^2\theta - 1) - \frac{1}{2}\sinh\theta\cosh\theta + \frac{3}{8}\theta + c \\
= & \frac{x}{8}\sqrt{x^2-1}\,(2x^2-1) - \frac{x}{2}\sqrt{x^2-1} + \frac{3}{8}\operatorname{arccosh} x + c
\end{aligned}
$$

Example 8: Evaluate $\int \sinh^5 x \cosh^6 x \, dx$.

Solution:

$$\begin{aligned}
\int \sinh^5 x \cosh^6 x \, dx &= \int \sinh x (\sinh^2 x)^2 \cosh^6 x \, dx \\
&= \int \sinh x (\cosh^2 x - 1)^2 \cosh^6 x \, dx \\
\text{put } u = \cosh x &\Rightarrow \frac{du}{dx} = \sinh x \\
&= \int \left(u^2 - 1\right)^2 u^6 \, du \\
&= \int u^{10} - 2u^8 + u^6 \, du
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{11}u^{11} - \frac{2}{9}u^9 + \frac{1}{7}u^7 + c \\
&= \frac{1}{11}\cosh^{11}x - \frac{2}{9}\cosh^9 x + \frac{1}{7}\cosh^7 x + c.
\end{aligned}
$$

Example 9: Evaluate $I = \int \sinh^5 x \cosh^7 x \, dx$.

Solution:

Method 1

$$\begin{aligned} I &= \int \sinh x \left(\sinh^2 x\right)^2 \cosh^7 x \, dx \\ &= \int \sinh x \left(\cosh^2 x - 1\right)^2 \cosh^7 x \, dx \\ \text{put } u &= \cosh x \Rightarrow \frac{du}{dx} = \sinh x \\ &= \int \left(u^2 - 1\right)^2 u^7 \, du \quad \text{etc.} \end{aligned}$$

Method 2

$$
\begin{aligned}
I &= \int \sinh^5 x \left(\cosh^2 x\right)^3 \cosh x \, dx \\
&= \int \sinh^5 x \left(1 + \sinh^2 x\right)^3 \cosh x \, dx \\
\text{put } u &= \sinh x \Rightarrow \frac{du}{dx} = \cosh x \\
&= \int u^5 \left(1 + u^2\right)^3 du \quad \text{etc.}
\end{aligned}
$$

Partial Fractions

Let $f(x)$ and $g(x)$ be polynomials. If $g(x)$ can be factorized then

$$\frac{f(x)}{g(x)} \quad \begin{array}{l} \longrightarrow \text{ degree } n \\ \longrightarrow \text{ degree } d \end{array}$$

can be written as the sum of partial fractions.

Case 1: $n < d$

1. Factorize $g(x)$ over the real numbers.
2. Write down partial fraction expansion.
3. Find unknown coefficients

$$A, A_1, A_2, \ldots, A_r, B, B_1, B_2, \ldots, B_r.$$

Denominator Factor	Partial Fraction Expansion
$(x-a)$	$\frac{A}{x-a}$
$(x-a)^r$	$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_r}{(x-a)^r}$
(x^2+bx+c)	$\frac{Ax+B}{x^2+bx+c}$
$(x^2+bx+c)^r$	$\frac{A_1x+B_1}{x^2+bx+c}+\frac{A_2x+B_2}{(x^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(x^2+bx+c)^r}$

Example 10: Evaluate $\int \frac{4}{x^2(x+2)}\,dx \quad (x \neq 0, -2)$.

Solution:

$$
\begin{aligned}
\frac{4}{x^2(x+2)} &\equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} \\
&= \frac{Ax(x+2) + B(x+2) + Cx^2}{x^2(x+2)} \\
&= \frac{Ax^2 + 2Ax + Bx + 2B + Cx^2}{x^2(x+2)}
\end{aligned}
$$

Equate coefficients in numerator:

$$x^2: \quad A + C = 0 \tag{1}$$

$$x: \quad 2A + B = 0 \tag{2}$$

$$x^0: \quad 2B = 4 \tag{3}$$

Solving equations $(1) \to (3)$ gives

$$A = -1, \quad B = 2, \quad C = 1$$

$$\Rightarrow \frac{4}{x^2(x+2)} = -\frac{1}{x} + \frac{2}{x^2} + \frac{1}{x+2}$$

So $$\int \frac{4}{x^2(x+2)}\,dx = \int -\frac{1}{x} + \frac{2}{x^2} + \frac{1}{x+2}\,dx$$
$$= -\log|x| - \frac{2}{x} + \log|x+2| + c.$$

Example 11: Evaluate $\int \frac{4x}{(x^2+4)(x-2)}\,dx \quad (x \neq 2)$.

Solution:

$$\begin{aligned}
\frac{4x}{(x^2+4)(x-2)} &\equiv \frac{Ax+B}{x^2+4} + \frac{C}{x-2} \\
&= \frac{(Ax+B)(x-2)+C(x^2+4)}{(x^2+4)(x-2)} \\
&= \frac{Ax^2-2Ax+Bx-2B+Cx^2+4C}{(x^2+4)(x-2)}
\end{aligned}$$

Equate coefficients in numerator:

$$x^2 : \quad A + C = 0 \tag{1}$$

$$x : \quad -2A + B = 4 \tag{2}$$

$$x^0 : \quad -2B + 4C = 0 \tag{3}$$

Solving equations $(1) \to (3)$ gives

$$A = -1, \quad B = 2, \quad C = 1$$

$$\Rightarrow \frac{4x}{(x^2+4)(x-2)} = \frac{2-x}{x^2+4} + \frac{1}{x-2}$$

$$\text{So} \int \frac{4x}{(x^2+4)(x-2)}\,dx$$

$$= \int \frac{2}{x^2+4} - \frac{x}{x^2+4} + \frac{1}{x-2}\,dx$$

$$= \arctan\left(\frac{x}{2}\right) - \frac{1}{2}\log(x^2+4) + \log|x-2| + c.$$

Example 12: Decompose $\dfrac{25x^3}{(x^2+1)^2(x+2)}$ into partial fractions for $x \neq -2$.

Solution:

$$\frac{25x^3}{(x^2+1)^2(x+2)} \equiv \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x+2}$$

Find that

$$A = 8, \quad B = 9, \quad C = -10, \quad D = -5, \quad E = -8.$$

Case 2: $n \geq d$

Use long division, then apply case 1.

Example 13: Find $\displaystyle\int \frac{5x^4 + 13x^3 + 6x^2 + 4}{x^3 + 2x^2}\,dx \quad (x \neq 0, -2).$

Solution:

$$
\begin{array}{r|llll}
 & 5x & & +3 & \\
\hline
x^3 + 2x^2 & 5x^4 & +13x^3 & +6x^2 & +4 \\
 & 5x^4 & +10x^3 & & \\
\cline{2-3}
 & & 3x^3 & +6x^2 & +4 \\
 & & 3x^3 & +6x^2 & \\
\cline{3-5}
 & & & & 4 \quad \longleftarrow \text{Remainder}
\end{array}
$$

$$
\Rightarrow \frac{5x^4 + 13x^3 + 6x^2 + 4}{x^3 + 2x^2} = 5x + 3 + \frac{4}{x^3 + 2x^2}
$$

So $\displaystyle\int \frac{5x^4 + 13x^3 + 6x^2 + 4}{x^3 + 2x^2}\,dx$

$$= \int 5x + 3 + \frac{4}{x^3 + 2x^2}\,dx$$

$$= \int 5x + 3 - \frac{1}{x} + \frac{2}{x^2} + \frac{1}{x+2}\,dx$$

(from Example 11)

$$= \frac{5}{2}x^2 + 3x - \log|x| - \frac{2}{x} + \log|x+2| + c.$$

Integration by Parts

The product rule for differentiation is

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

Integrate

$$\int \frac{d}{dx}(uv)\,dx = \int \left(\frac{du}{dx}v + u\frac{dv}{dx}\right) dx$$

$$\Rightarrow uv = \int \frac{du}{dx}v\,dx + \int u\frac{dv}{dx}\,dx$$

$$\Rightarrow \quad \boxed{\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx}$$

Example 14: Evaluate $\int x^2 \log x \, dx \qquad (x > 0)$.

Solution:

$$\text{Let } u = \log x, \qquad \frac{dv}{dx} = x^2$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{x}, \qquad v = \frac{1}{3}x^3$$

$$\begin{aligned}
\text{So } \int x^2 \log x \, dx &= \log x \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \cdot \frac{1}{x} dx \\
&= \frac{1}{3}x^3 \log x - \frac{1}{3}\int x^2 \, dx \\
&= \frac{1}{3}x^3 \log x - \frac{1}{9}x^3 + c.
\end{aligned}$$

Example 15: Evaluate $\int xe^{5x}\,dx$.

Solution:

$$\text{Let } u = x, \qquad \frac{dv}{dx} = e^{5x}$$

$$\Rightarrow \frac{du}{dx} = 1, \qquad v = \frac{1}{5}e^{5x}$$

$$\begin{aligned}\text{So } \int xe^{5x}\,dx &= x \cdot \frac{1}{5}e^{5x} - \int \frac{1}{5}e^{5x} \cdot 1 \cdot dx \\ &= \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x} + c.\end{aligned}$$

Example 16: Evaluate $\int x^2 \sinh x \, dx$.

Solution:

$$\text{Let } u = x^2, \qquad \frac{dv}{dx} = \sinh x$$

$$\Rightarrow \frac{du}{dx} = 2x, \qquad v = \cosh x$$

$$\begin{aligned} \text{So } \int x^2 \sinh x \, dx &= x^2 \cosh x - \int \cosh x \cdot 2x \, dx \\ &= x^2 \cosh x - 2 \int x \cosh x \cdot \, dx \end{aligned}$$

$$\text{Let } u = x, \qquad \frac{dv}{dx} = \cosh x$$

$$\Rightarrow \frac{du}{dx} = 1, \qquad v = \sinh x$$

$$\begin{aligned}
\text{So } \int x \cosh x \, dx &= x \sinh x - \int \sinh x \cdot 1 \, dx \\
&= x \sinh x - \cosh x + c.
\end{aligned}$$

$$\begin{aligned}
\text{Then } \int x^2 \sinh x \, dx &= x^2 \cosh x - 2\,(x \sinh x - \cosh x + c) \\
&= x^2 \cosh x - 2x \sinh x + 2 \cosh x + d.
\end{aligned}$$

Example 17: Evaluate $\displaystyle\int \log x \, dx \qquad (x > 0)$.

Solution:

$$\int \log x \, dx = \int 1 \cdot \log x \, dx$$

$$\text{Let } u = \log x, \qquad \frac{dv}{dx} = 1$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{x}, \qquad v = x$$

So
$$\begin{aligned}\int \log x \, dx &= \log x \cdot x - \int x \cdot \frac{1}{x} \, dx \\ &= x \log x - \int 1 \, dx \\ &= x \log x - x + c.\end{aligned}$$

Note:
This technique can also be used to integrate inverse trigonometric functions and inverse hyperbolic functions.

Example 18: Evaluate $\int e^{3x} \sin 2x \, dx$.

Solution:

$$\text{Let } u = e^{3x}, \qquad \frac{dv}{dx} = \sin 2x$$

$$\Rightarrow \frac{du}{dx} = 3e^{3x}, \qquad v = -\frac{1}{2}\cos 2x$$

$$\begin{aligned} \text{So } \int e^{3x} \sin 2x \, dx &= e^{3x} \cdot -\frac{1}{2}\cos 2x - \int -\frac{1}{2}\cos 2x \cdot 3e^{3x} dx \\ &= -\frac{1}{2}e^{3x}\cos 2x + \frac{3}{2}\int e^{3x}\cos 2x \, dx \end{aligned}$$

$$\text{Let } u = e^{3x}, \qquad \frac{dv}{dx} = \cos 2x$$

$$\Rightarrow \frac{du}{dx} = 3e^{3x}, \qquad v = \frac{1}{2}\sin 2x$$

$$\begin{aligned} \text{So } \int e^{3x}\cos 2x\,dx &= e^{3x}\cdot\frac{1}{2}\sin 2x - \int \frac{1}{2}\sin 2x \cdot 3e^{3x}dx \\ &= \frac{1}{2}e^{3x}\sin 2x - \frac{3}{2}\int e^{3x}\sin 2x\,dx \end{aligned}$$

$$\Rightarrow \int e^{3x} \sin 2x \, dx = -\frac{1}{2} e^{3x} \cos 2x + \frac{3}{4} e^{3x} \sin 2x - \frac{9}{4} \int e^{3x} \sin 2x \, dx$$

$$\Rightarrow \quad \frac{13}{4} \int e^{3x} \sin 2x \, dx = -\frac{1}{2} e^{3x} \cos 2x + \frac{3}{4} e^{3x} \sin 2x + c \qquad \left(\times \frac{4}{13} \right)$$

$$\Rightarrow \quad \int e^{3x} \sin 2x \, dx = -\frac{2}{13} e^{3x} \cos 2x + \frac{3}{13} e^{3x} \sin 2x + d.$$

Ordinary Differential Equations

(1) An equation of the form

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0$$

is an ordinary differential equation (o.d.e) of order n.

Example 1: What order is $3\frac{d^4y}{dx^4} = \left(\frac{dy}{dx}\right)^2 + 2x^2y$?

Solution:

Fourth order

(2) A solution of an o.d.e is a function $y(x)$ that satisfies the o.d.e for all x in some interval.

Example 2: Verify that $y(x) = x^2 + \frac{2}{x}$ is a solution of $\frac{dy}{dx} + \frac{y}{x} = 3x$ for all $x \in \mathbb{R} \setminus \{0\}$.

Solution:

If $y(x) = x^2 + \frac{2}{x} \qquad (x \neq 0)$

$$\begin{aligned}
\Rightarrow \frac{dy}{dx} + \frac{y}{x} &= 2x - \frac{2}{x^2} + \frac{1}{x}\left(x^2 + \frac{2}{x}\right) \\
&= 2x - \frac{2}{x^2} + x + \frac{2}{x^2} \\
&= 3x \\
\Rightarrow y(x) &= x^2 + \frac{2}{x} \text{ is a solution for } x \neq 0.
\end{aligned}$$

First Order O.D.E's

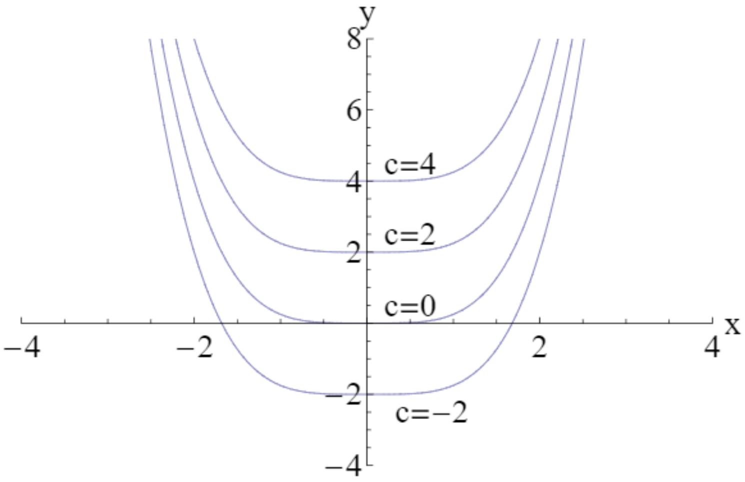
The general form of a first order o.d.e. is $\frac{dy}{dx} = f(x, y)$.

Example 3: Solve $\frac{dy}{dx} = x^3$.

Solution:

$$\begin{aligned} y &= \int x^3\,dx \\ \Rightarrow y(x) &= \frac{1}{4}x^4 + c. \end{aligned}$$

This is the general solution where $c \in \mathbb{R}$ is an arbitrary constant.
Each value of c corresponds to a different solution of the o.d.e.



y
8
6
c=4
4
c=2
2
c=0
−4
−2
2
4
x
−2
c=−2
−4

Initial value problem for a first order o.d.e.

Solve $\dfrac{dy}{dx} = f(x, y)$ subject to the condition $y(x_0) = y_0$.

Example 4: Solve $\dfrac{dy}{dx} = x^3$ given $y(0) = 2$.

Solution:

Now $y(x) = \dfrac{1}{4}x^4 + c$ (general solution).

The solution satisfying $y(0) = 2$ is:

$$y(0) = 2 = 0 + c \quad \Rightarrow c = 2$$

$$\text{So } y(x) = \frac{1}{4}x^4 + 2.$$

Separable O.D.E'S

A separable o.d.e has the form:

$$\frac{dy}{dx} = \mathcal{M}(x)\mathcal{N}(y), \quad (\mathcal{M}(x) \neq 0, \ \ \mathcal{N}(y) \neq 0)$$

To solve:

$$\begin{aligned}
& \frac{dy}{dx} && = \mathcal{M}(x)\mathcal{N}(y) \\
\Rightarrow \quad & \frac{1}{\mathcal{N}(y)}\frac{dy}{dx} && = \mathcal{M}(x) \\
\Rightarrow \quad & \int \frac{1}{\mathcal{N}(y)}\frac{dy}{dx}\, dx && = \int \mathcal{M}(x) dx \\
\Rightarrow \quad & \int \frac{1}{\mathcal{N}(y)} dy && = \int \mathcal{M}(x)\, dx
\end{aligned}$$

Example 5: Solve $\dfrac{dy}{dx} = \dfrac{y}{1+x} \qquad (x \neq -1)$.

Solution:

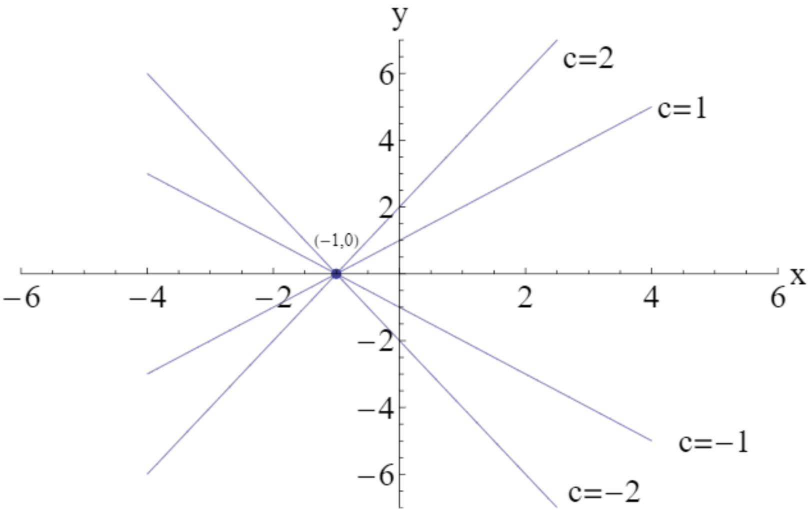
$$\frac{dy}{dx} = \left(\frac{1}{1+x}\right) \cdot (y) \qquad \text{(separable)}$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{1+x} \qquad (y \neq 0)$$

$$\Rightarrow \quad \int \frac{1}{y}\frac{dy}{dx}dx = \int \frac{1}{1+x}dx$$

$$\Rightarrow \quad \int \frac{1}{y}dy = \int \frac{1}{1+x}dx$$

$$\begin{aligned}
\Rightarrow \log|y| &= \log|1+x| + c_1 \\
\Rightarrow e^{\log|y|} &= e^{(\log|1+x|+c_1)} \\
\Rightarrow |y| &= e^{\log|1+x|}e^{c_1} \\
\Rightarrow |y| &= c_2|1+x| \\
\Rightarrow y &= \pm c_2(1+x) \\
\Rightarrow y(x) &= c(1+x) \qquad \text{(general solution)}
\end{aligned}$$


y
x
c=2
c=1
c=−1
c=−2
(−1,0)
−6
−4
−2
2
4
6
6
4
2
−2
−4
−6

Example 6: Solve $\dfrac{dy}{dx} = \dfrac{1}{2y\sqrt{1-x^2}}$ if $y(0) = 3$.

Solution:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{2y} \qquad \text{(separable)}$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \int 2y\frac{dy}{dx}\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx$$

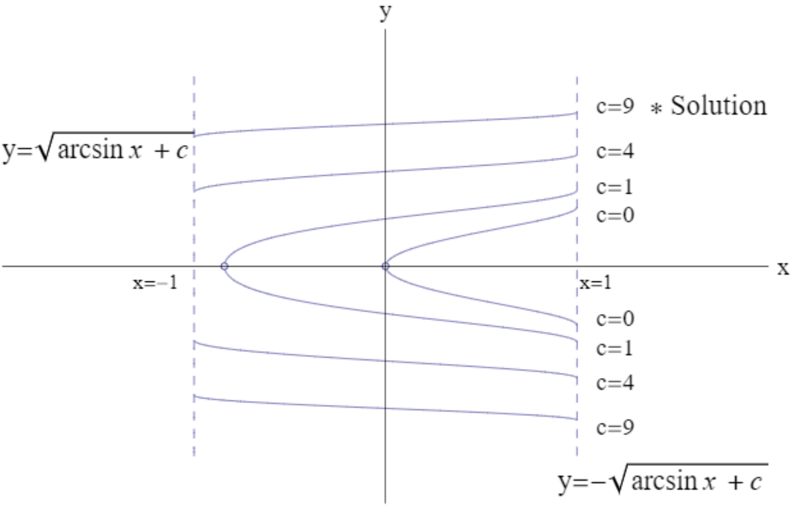
$$\Rightarrow \quad \int 2y\,dy = \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\Rightarrow \quad y^2 = \arcsin x + c \qquad \text{(general solution)}$$

The solution satisfying $y(0) = 3$ is:

$$\begin{aligned} 3^2 &= \arcsin 0 + c \\ \Rightarrow c &= 9 \\ \text{So } y^2 &= \arcsin x + 9 \\ \Rightarrow y(x) &= \sqrt{\arcsin x + 9} \end{aligned}$$

Need '+' sign since $y(0) = 3$.



Note:

Solution: valid for $-1 < x < 1, y \neq 0$.

Linear First Order O.D.E's

Example 7: Solve $x\dfrac{dy}{dx} + y = 1.$

Solution:

$$\begin{aligned}
x\frac{dy}{dx} + y &= 1 \\
\Rightarrow \frac{d}{dx}(yx) &= 1 \\
\Rightarrow yx &= \int 1\, dx \\
\Rightarrow yx &= x + c \\
\Rightarrow y(x) &= 1 + \frac{c}{x}.
\end{aligned}$$

A linear first order o.d.e has the form:

$$\frac{dy}{dx} + \mathcal{P}(x)y = \mathcal{Q}(x)$$

To solve:

Multiply d.e. by $\mathcal{I}(x)$

$$\mathcal{I}(x)\frac{dy}{dx} + \mathcal{P}(x)\mathcal{I}(x)y = \mathcal{Q}(x)\mathcal{I}(x)$$

If the left side can be written as the derivative of $y(x)\mathcal{I}(x)$, then

$$\begin{aligned} \frac{d}{dx}\left[y(x)\mathcal{I}(x)\right] &= \mathcal{Q}(x)\mathcal{I}(x) \\ \Rightarrow y(x)\mathcal{I}(x) &= \int \mathcal{Q}(x)\mathcal{I}(x)\,dx \end{aligned}$$

$$\Rightarrow y(x) = \frac{1}{\mathcal{I}(x)} \int Q(x)I(x)\, dx.$$

Aim:

Find an integrating factor $\mathcal{I}$ so the left side will be the derivative of $y\mathcal{I}$. Then

$$\begin{aligned}
\frac{d}{dx}(y\mathcal{I}) &\equiv \mathcal{I}\frac{dy}{dx} + \mathcal{P}\mathcal{I}y \\
\Rightarrow \quad \frac{dy}{dx}\mathcal{I} + y\frac{d\mathcal{I}}{dx} &= \mathcal{I}\frac{dy}{dx} + \mathcal{P}\mathcal{I}y \\
\Rightarrow \quad y\frac{d\mathcal{I}}{dx} &= \mathcal{P}\mathcal{I}y
\end{aligned}$$

Since $y(x) \neq 0$,

$$\Rightarrow \quad \boxed{\frac{d\mathcal{I}}{dx} = \mathcal{P}\mathcal{I}} \qquad \text{(separable)}$$

$$
\begin{aligned}
\Rightarrow \frac{1}{\mathcal{I}}\frac{d\mathcal{I}}{dx} &= \mathcal{P} \\
\Rightarrow \int \frac{1}{\mathcal{I}} d\mathcal{I} &= \int \mathcal{P} dx \\
\Rightarrow \log|\mathcal{I}| &= \int \mathcal{P}\, dx + c \\
\Rightarrow |\mathcal{I}| &= e^{\int \mathcal{P} dx + c} \\
&= e^{\int \mathcal{P} dx} \cdot e^{c} \\
\Rightarrow \mathcal{I} &= \underbrace{\pm e^{c}}_{\text{constant}} \cdot e^{\int \mathcal{P} dx}
\end{aligned}
$$

So one integrating factor is

$$\boxed{\mathcal{I}(x) = e^{\int \mathcal{P} dx}}$$

Note:
Since we only need one integrating factor $\mathcal{I}$, we can neglect the '$+c$' and modulus signs when calculating $\mathcal{I}$.

Example 8: Find the general solution of

$$\frac{dy}{dx} + \frac{y}{x} = \sin x \qquad (x \neq 0).$$

Solution:

$$\left(\mathcal{P}(x) = \frac{1}{x}, Q(x) = \sin x, \text{ linear } \right)$$

- Find one integrating factor:

$$\begin{aligned} \mathcal{I}(x) &= e^{\int \mathcal{P}\, dx} \\ &= e^{\int \frac{1}{x}\, dx} \\ &= e^{\log x} \\ &= x \end{aligned}$$

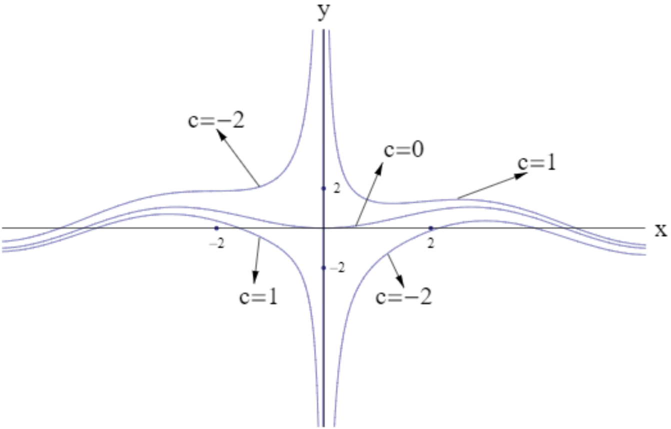
- Multiply d.e. by $\mathcal{I}$:

$$\begin{aligned}
\underbrace{x\frac{dy}{dx}+y}_{\parallel} &= x\sin x \\
\frac{d}{dx}(yx) &= x\sin x \\
\Rightarrow yx &= \int x\sin x\,dx
\end{aligned}$$

$\int$ by parts: $u = x, \qquad \dfrac{dv}{dx} = \sin x$

$$\Rightarrow \frac{du}{dx} = 1, \qquad v = -\cos x$$

$$
\begin{aligned}
\Rightarrow yx &= -x\cos x - \int -\cos x\,dx \\
&= -x\cos x + \sin x + c \\
\Rightarrow y(x) &= -\cos x + \frac{1}{x}\sin x + \frac{c}{x}.
\end{aligned}
$$



$$y(x) = -\cos x + \frac{1}{x}\sin x + \frac{c}{x}$$

Example 9: Solve $\frac{1}{2}\frac{dy}{dx} - xy = x$ if $y(0) = -3$.

Solution:

- Write in standard form:

$$\frac{dy}{dx} - 2xy = 2x$$

$(\mathcal{P}(x) = -2x, Q(x) = 2x,$ linear)

- Find one integrating factor:

$$\begin{aligned} \mathcal{I}(x) &= e^{\int \mathcal{P} dx} \\ &= e^{\int -2x\, dx} \\ &= e^{-x^2} \end{aligned}$$

- Multiply o.d.e. in standard form by $\mathcal{I}$:

$$
\begin{aligned}
e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2}y &= 2xe^{-x^2} \\
\frac{d}{dx}\left(ye^{-x^2}\right) &= 2xe^{-x^2} \\
\Rightarrow ye^{-x^2} &= \int 2xe^{-x^2}dx \\
\text{put } u = -x^2 \Rightarrow \frac{du}{dx} &= -2x \\
\Rightarrow ye^{-x^2} &= -e^{-x^2} + c \\
\Rightarrow y(x) &= -1 + ce^{x^2} \qquad \text{(general solution)}
\end{aligned}
$$

- Apply condition $y(0) = -3$

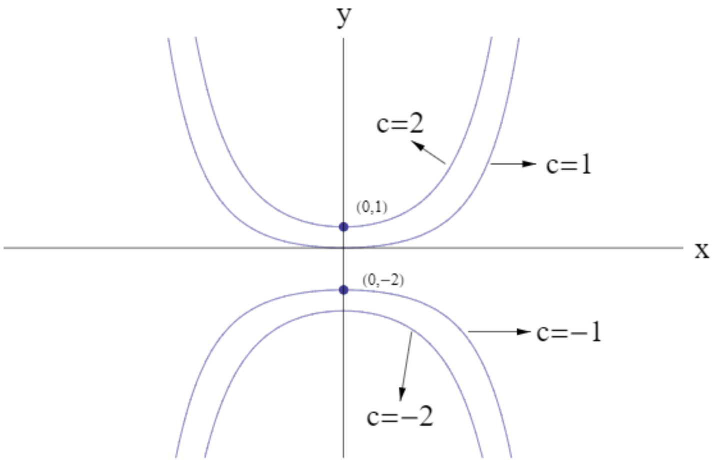
$$\Rightarrow -3 = -1 + c \Rightarrow c = -2$$

so $y(x) = -1 - 2e^{x^2}$

Note:

d.e. is also separable as

$$\frac{1}{2}\frac{dy}{dx} = x + xy = x(1 + y)$$



$$y = -1 + ce^{x^2}$$

Other First Order O.D.E's

Sometimes it is possible to make a substitution to reduce a general first order o.d.e to a separable or linear o.d.e.

- A homogeneous type o.d.e has the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Substituting $u = \frac{y}{x}$ reduces the o.d.e to a separable o.d.e.

- Bernoulli's equation has the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Substituting $u = y^{1-n}$ reduces the o.d.e. to a linear o.d.e.

Example 10: Solve the homogeneous type differential equation

$$\frac{dy}{dx} = \frac{y}{x} + \cos^2\left(\frac{y}{x}\right) \quad \text{by substituting} \quad u = \frac{y}{x}.$$

Solution:

- Put $u = \dfrac{y}{x}$ and write d.e. in terms of u and x.

$$\Rightarrow y = xu, \qquad \frac{dy}{dx} = u + x\frac{du}{dx}$$

so d.e. becomes

$$
\begin{aligned}
u + x\frac{du}{dx} &= u + \cos^2 u \\
\Rightarrow x\frac{du}{dx} &= \cos^2 u \qquad \text{(separable)}
\end{aligned}
$$

- Solve for $u(x)$

$$\Rightarrow \quad \frac{1}{\cos^2 u}\frac{du}{dx} = \frac{1}{x}$$

$$\Rightarrow \quad \int \sec^2 u\, du = \int \frac{1}{x}\, dx$$

$$\Rightarrow \quad \tan u = \log|x| + c$$

- Put $u = \dfrac{y}{x}$ and write solution as $y(x)$

$$\Rightarrow \quad \tan\left(\frac{y}{x}\right) = \log|x| + c$$

$$\Rightarrow \quad y(x) = x \arctan\left(\log|x| + c\right).$$

Example 11: Solve the Bernoulli equation

$$\frac{dy}{dx} + y = e^{3x}y^4 \quad \text{by substituting} \quad u = y^{-3}.$$

Solution:

- Put $u = y^{-3}$ and write d.e. in terms of u and x.

$$\Rightarrow y = u^{-1/3}, \qquad \frac{dy}{dx} = -\frac{1}{3}u^{-4/3}\frac{du}{dx}$$

so d.e. becomes

$$
\begin{aligned}
-\frac{1}{3}u^{-4/3}\frac{du}{dx} + u^{-1/3} &= e^{3x}u^{-4/3} \\
\Rightarrow \frac{du}{dx} - 3u &= -3e^{3x} \qquad \text{(linear)}
\end{aligned}
$$

- Solve for $u(x)$

$$
\mathcal{I}(x) = e^{\int -3\,dx} = e^{-3x}
$$

$$
(\times \mathcal{I}) \qquad e^{-3x}\frac{du}{dx} - 3ue^{-3x} = -3
$$

$$\frac{d}{dx}\left(ue^{-3x}\right) = -3$$

$$\Rightarrow \quad ue^{-3x} = -3x + c$$

$$\Rightarrow \quad u(x) = (-3x + c)\, e^{3x}$$

- Put $u = y^{-3}$ and write solution as $y(x)$

$$\Rightarrow \quad y^{-3} = -3xe^{3x} + ce^{3x}$$

$$\Rightarrow \quad y(x) = \left(-3xe^{3x} + ce^{3x}\right)^{-\frac{1}{3}}.$$

Population Models

Malthus (Doomsday) model.

Rate of growth is proportional to the population $p(t)$ at time t.

$$\begin{aligned} \frac{dp}{dt} &\propto p \\ \Rightarrow \frac{dp}{dt} &= kp \qquad \text{(separable/linear)} \end{aligned}$$

where k is the net birth rate.

If the initial population is $p(0) = p_0$, then the general solution is

$$p(t) = p_0 e^{kt}$$

Note:
The Doomsday model is unrealistic since if

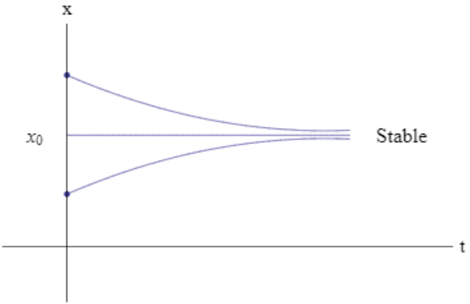
- $k > 0 -$ unbounded exponential growth.
- $k < 0 -$ population dies out.
- $k = 0 -$ population stays constant.

Equilibrium Solutions

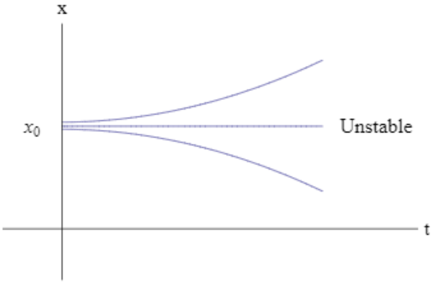
1. An equilibrium solution is a solution that does not change with time.

$$\text{i.e. } \frac{dx}{dt} = 0 \quad \Rightarrow \quad x(t) = x_0$$

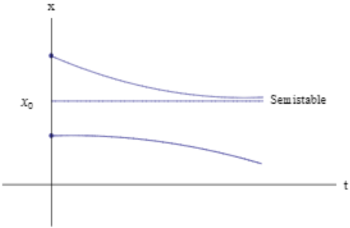
2. Stable equilibrium — solutions that start nearby move closer as t increases.

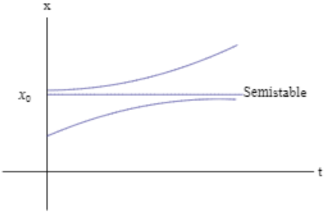


3. Unstable equilibrium — solutions that start nearby move further away as t increases.



4. Semistable equilibrium — on one side of x_0 solutions that start nearby move closer as t increases whereas on the other side of x_0 solutions move further away as t increases.





5. Phase plots:

If $\frac{dx}{dt} = f(x)$, a plot of $\frac{dx}{dt}$ as a function of x will give the equilibrium solutions and the behaviour of solutions close by.

Doomsday model with harvesting.

Remove some of the population at a constant rate.

$$\frac{dp}{dt} = kp - h, \quad h > 0$$

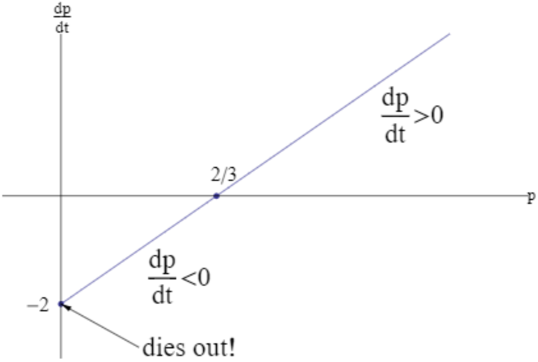
Example 12: $\dfrac{dp}{dt} = 3p - 2 \qquad (k = 3, h = 2)$

Solution:

- Equilibrium solutions

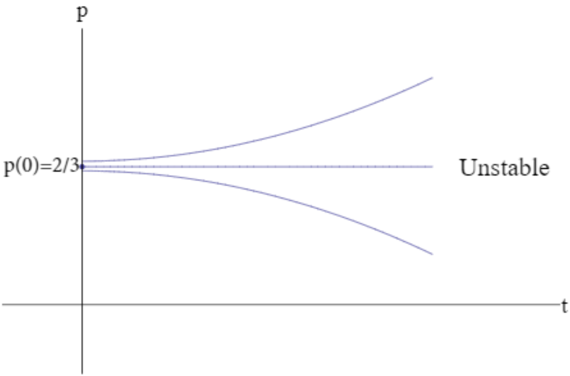
$$\frac{dp}{dt} = 0 \text{ when } p = \frac{2}{3}$$

- Phase plot



- If $0 \le p < \frac{2}{3}$, $\quad \frac{dp}{dt} < 0 \quad$ so p decreases over time.

- If $p > \frac{2}{3}$, $\quad \frac{dp}{dt} > 0 \quad$ so p increases over time.



Note:

Solving $\dfrac{dp}{dt} = 3p - 2$ with $p(0) = p_0$ gives $p(t) = \dfrac{2}{3} + \left(p_0 - \dfrac{2}{3}\right)e^{3t}$ which agrees with predicted behaviour.

Logistic model.

Include “competition” term in Malthus’ model since overcrowding, disease, lack of food and natural resources will cause more deaths.

$$\frac{dp}{dt} = \underbrace{kp}_{\text{net birth rate}} - \underbrace{\frac{k}{a}p^2}_{\text{competition term}} = kp\left(1 - \frac{p}{a}\right)$$

where $a > 0$ is the carrying capacity.

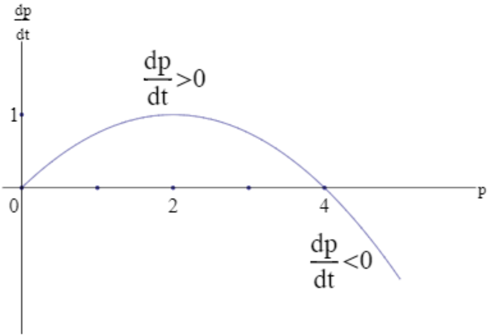
Example 13: $\dfrac{dp}{dt} = p\left(1 - \dfrac{p}{4}\right) \qquad (k = 1, a = 4)$

Solution:

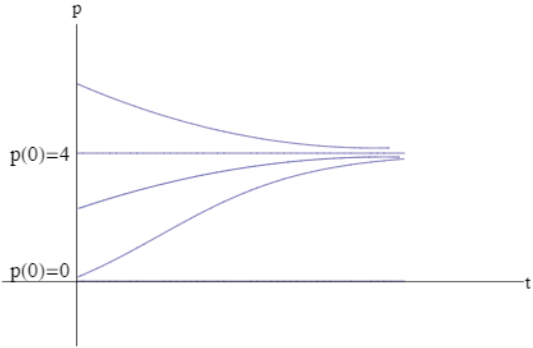
- Equilibrium solutions

$$\frac{dp}{dt} = 0 \text{ when } p = 0, 4$$

- Phase plot



- If $0 < p < 4$, $\dfrac{dp}{dt} > 0$ so p increases with time.
- If $p > 4$, $\dfrac{dp}{dt} < 0$ so p decreases with time.



So $p(0) = 4$ is stable and $p(0) = 0$ is unstable.

- Exact solution

$$\frac{dp}{dt} = \frac{p}{4}(4-p) \qquad \text{(separable)}$$

$$\Rightarrow \int \underbrace{\frac{4}{p(4-p)}}_{\text{partial fractions}} \, dp = \int 1 \, dt$$

$$\Rightarrow \int \frac{1}{p} + \frac{1}{4-p} \, dp = t + c$$

$$\Rightarrow \log|p| - \log|4-p| = t + c$$

$$\Rightarrow \log\left|\frac{p}{4-p}\right| = t + c$$

$$\Rightarrow \frac{p}{4-p} = \pm e^t \cdot e^c = Ae^t$$

Suppose $p(0) = 1$

then $A = \dfrac{1}{3}$ and

$$\frac{p}{4-p} = \frac{1}{3}e^t$$

$$\Rightarrow \quad 3p = (4-p)e^t$$

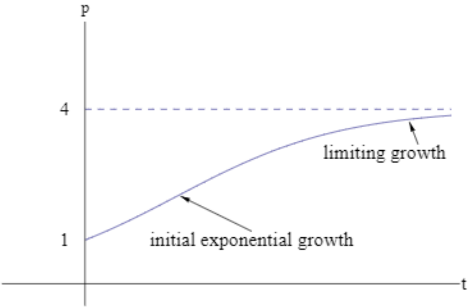
$$\Rightarrow \quad p(3+e^t) = 4e^t$$

$$\Rightarrow \quad p = \frac{4e^t}{3+e^t} \quad \left(\times \frac{e^{-t}}{e^{-t}}\right)$$

$$\Rightarrow \quad p(t) = \frac{4}{3e^{-t}+1}.$$

Note:

As $t \to \infty \quad e^{-t} \to 0$ and $p(t) \to 4$ as expected.



Note:

Logistic model accurately predicts

- population in a limited space (e.g. bacteria culture)
- population of USA from 1790-1950.

Logistic model with harvesting.

Remove some of the population at constant rate:

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{a}\right) - h, \quad h > 0, a > 0$$

Example 14:

$$\frac{dp}{dt} = p\left(1 - \frac{p}{4}\right) - \frac{3}{4} \qquad \left(a = 4, k = 1, h = \frac{3}{4}\right)$$

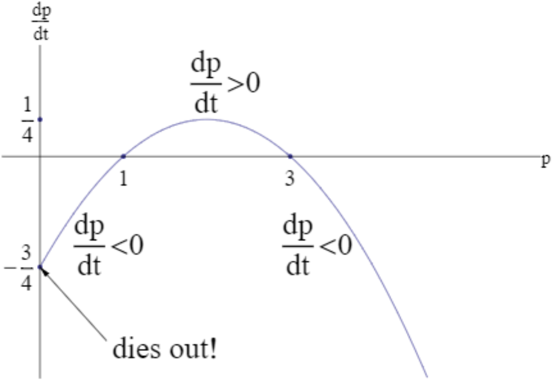
Solution:

$$\begin{aligned}
\frac{dp}{dt} &= p - \frac{p^2}{4} - \frac{3}{4} \\
&= -\frac{1}{4}\left(p^2 - 4p + 3\right) \\
&= -\frac{1}{4}(p-3)(p-1)
\end{aligned}$$

- Equilibrium solutions

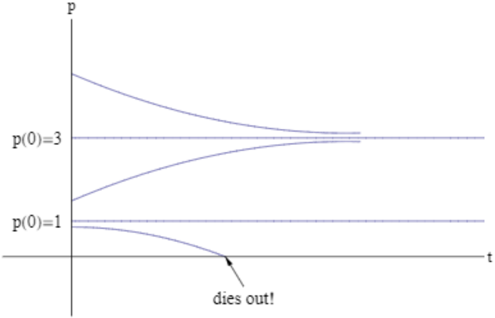
$$\frac{dp}{dt} = 0 \text{ when } p = 1, 3$$

- Phase plot



- If $0 < p < 1$ or $p > 3$, then $\dfrac{dp}{dt} < 0$ and p decreases with time.

- If $1 < p < 3$, then $\dfrac{dp}{dt} > 0$ and p increases with time.



So $p(0) = 3$ is stable and $p(0) = 1$ is unstable.

Find the time taken until the population dies out if $p(0) = \dfrac{1}{2}$.

$$\frac{dp}{dt} = -\frac{1}{4}(p-3)(p-1) \quad \text{(separable)}$$

$$\Rightarrow \int \frac{-4}{(p-3)(p-1)} dp = \int 1\, dt$$

Using partial fractions

$$\Rightarrow \int \frac{-2}{p-3} + \frac{2}{p-1}\, dp = \int 1\, dt$$

$$\Rightarrow -2\log|p-3| + 2\log|p-1| = t + c$$

$$\Rightarrow 2\log\left|\frac{p-1}{p-3}\right| = t + c.$$

Since $p(0) = \frac{1}{2}$

$$c = 2\log\left|\frac{-1/2}{-5/2}\right| = 2\log\frac{1}{5}$$

$$\Rightarrow t = 2\log\left|\frac{p-1}{p-3}\right| - 2\log\frac{1}{5}$$

Population dies out when $p(t) = 0$, so

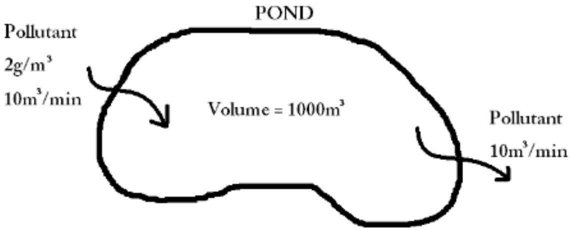
$$t = 2\log\frac{1}{3} - 2\log\frac{1}{5} = 2\log\frac{5}{3}.$$

Mixing Problems

Example 15: Effluent (pollutant concentration $2g/m^3$) flows into a pond (volume $1000m^3$, initially $100g$ pollutant) at a rate of $10m^3/min$. The pollutant mixes quickly and uniformly with pond water and flows out of pond at a rate of $10m^3/min$.

Find the concentration of pollutant in the pond at any time.

Solution:



Let $x(t)$ be the amount (grams) of pollutant in pond at time t minutes. Then $\frac{x}{V}$ is the concentration of pollutant in pond $(grams/m^3)$, where $V(t)$ is the volume of the pond (m^3) at time t.

$$\boxed{\frac{dx}{dt} = \text{rate pollutant flows in - rate pollutant flows out}}$$

- $$\begin{aligned} \text{Rate in} &= \begin{pmatrix}\text{concentration}\\ \text{in}\end{pmatrix} \times \begin{pmatrix}\text{flow}\\ \text{rate in}\end{pmatrix} \\ &= 2(g/m^3) \times 10(m^3/min) \\ &= 20(g/min) \end{aligned}$$

$$\bullet\ \text{Rate out} = \begin{pmatrix}\text{concentration}\\ \text{out}\end{pmatrix} \times \begin{pmatrix}\text{flow}\\ \text{rate out}\end{pmatrix}$$

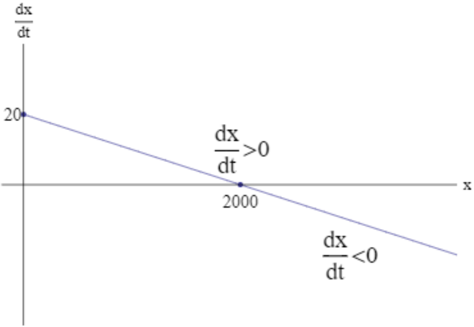
$$= \frac{x}{1000}(g/m^3) \times 10(m^3/min)$$

$$= \frac{x}{100}(g/min)$$

$$\boxed{\frac{dx}{dt} = 20 - \frac{x}{100}}$$

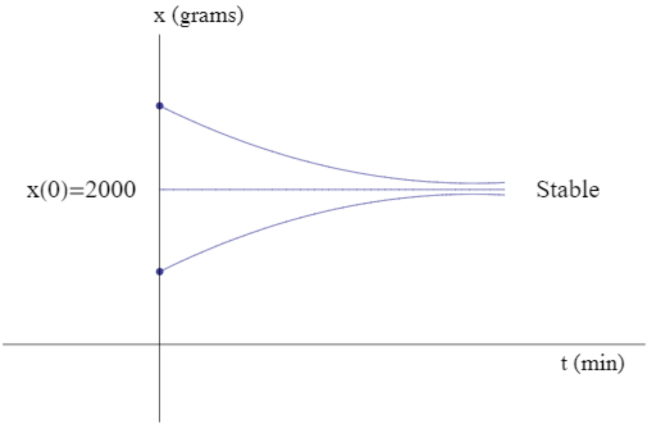
- Equilibrium solutions + Phase plot

$$\frac{dx}{dt} = 0 \quad \text{when} \quad x = 2000$$



- If $0 \leq x < 2000$, $\dfrac{dx}{dt} > 0$ so x increases with time.
- If $x > 2000$ $\dfrac{dx}{dt} < 0$ so x decreases with time.

Expect solution (for all initial conditions) to look like



- Exact solution

$$\frac{dx}{dt} + \frac{x}{100} = 20 \qquad \text{(Linear/Separable)}$$

$$\mathcal{I}(t) = e^{\frac{t}{100}}$$

$$(\times \mathcal{I}) \quad \frac{d}{dt}\left(xe^{\frac{t}{100}}\right) = 20e^{\frac{t}{100}}$$

$$\Rightarrow xe^{\frac{t}{100}} = 2000e^{\frac{t}{100}} + c$$

$$\Rightarrow x(t) = 2000 + ce^{\frac{-t}{100}}$$

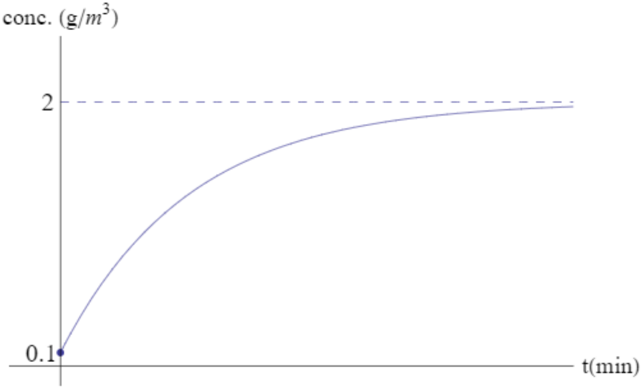
Now $x(0) = 100 \quad \Rightarrow c = -1900$

$$\Rightarrow x(t) = 2000 - 1900e^{\frac{-t}{100}}$$

$$\text{Concentration} = C(t) = \frac{\text{amount}}{\text{volume}} = \frac{x(t)}{1000}$$

$$\Rightarrow \ \mathrm{C(t)} = 2 - \frac{19}{10}e^{\frac{-t}{100}}$$

As $t \to \infty$, concentration $\to 2g/m^3$.



Definitions

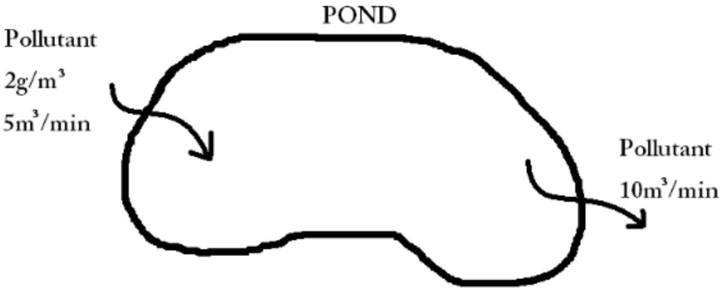
1. Transient solution: terms decaying to 0 as $t \to \infty$.
2. Steady state solution: terms NOT decaying to 0 as $t \to \infty$.

So the concentration solution can be classified as follows.

$$\begin{aligned} \text{steady state solution} &: \quad 2 \\ \text{transient solution} &: \quad -\frac{19}{10}e^{\frac{-t}{100}} \end{aligned}$$

Example 16: Find the concentration of pollutant in pond if input flow rate is decreased to $5m^3/min$.

Solution:



Let $V(t)$ be volume in pond (m^3) at time t minutes.

$$\begin{array}{cccccc} V(t) = & 1000 & + & 5t & - & 10t \\ & \downarrow & & \downarrow & & \downarrow \\ & \text{Volume} & & \text{Volume} & & \text{Volume} \\ & (t=0) & & \text{`in'} & & \text{`out'} \end{array}$$

$$\Rightarrow V(t) = 1000 - 5t \qquad (0 \leq t \leq 200)$$

- $$\begin{aligned} \frac{dx}{dt} &= \text{rate in} - \text{rate out} \\ &= 2(g/m^3) \times 5(m^3/min) - \frac{x}{V}(g/m^3) \times 10(m^3/min) \\ &= 10 - \frac{10x}{1000-5t} \end{aligned}$$

So $$\boxed{\frac{dx}{dt} + \frac{10x}{1000-5t} = 10} \qquad 0 \le t < 200$$

(Linear)

Note:
Since $\frac{dx}{dt} = f(x,t)$ there are no equilibrium solutions.

$$\begin{aligned}
\mathcal{I}(t) &= e^{\int \frac{10}{1000-5t}\,dt} \\
&= e^{-2\log(1000-5t)} \\
&= (1000-5t)^{-2}
\end{aligned}$$

$$(\times \mathcal{I}) \quad \frac{d}{dt}\left[\frac{x}{(1000-5t)^2}\right] = \frac{10}{(1000-5t)^2}$$

$$\Rightarrow \frac{x}{(1000-5t)^2} = \frac{2}{1000-5t} + c$$

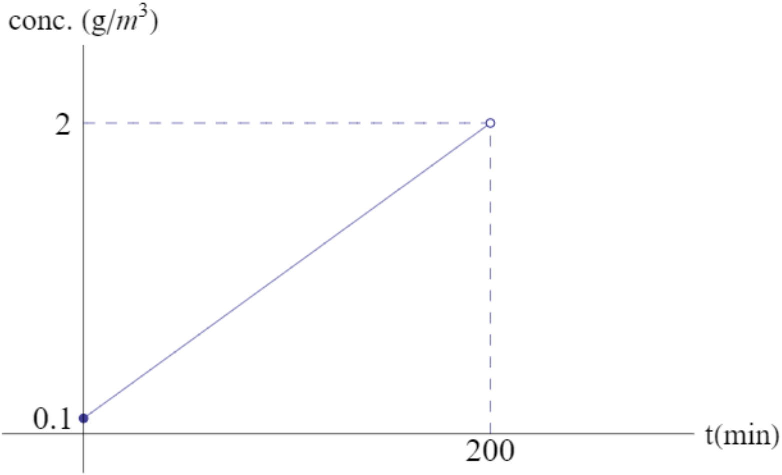
$$\Rightarrow x(t) = 2\,(1000-5t) + c(1000-5t)^2$$

Now $x(0) = 100 \quad \Rightarrow \quad c = -\frac{19}{10000}$

$$\text{concentration } = C(t) = \frac{\text{amount}}{\text{volume}} = \frac{x(t)}{1000-5t}$$

$$\Rightarrow C(t) = 2 - \frac{19}{10000}\,(1000-5t)$$

$$= 0.1 + 0.0095t.$$


conc. (g/m^3)
2
0.1
200
t(min)

Section 6: Second Order Differential Equations

A second order o.d.e. has the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

The general form of a linear second order o.d.e. is

$$\frac{d^2y}{dx^2} + \mathcal{P}(x)\frac{dy}{dx} + \mathcal{Q}(x)y = \mathcal{R}(x)$$

- If $\mathcal{R}(x) = 0$, the o.d.e. is homogeneous (H).
- If $\mathcal{R}(x) \neq 0$, the o.d.e. is inhomogeneous (IH).

Theorem:

The general solution of

$$y'' + \mathcal{P}(x)y' + \mathcal{Q}(x)y = \mathcal{R}(x)$$

is the function y given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_{\mathcal{P}}(x),$$

where

- $y_1(x), y_2(x)$ are linearly independent solutions of the homogeneous o.d.e.,
- $y_{\mathcal{H}}(x) = c_1 y_1(x) + c_2 y_2(x)$ is the general solution of the homogeneous o.d.e. (called the “homogeneous solution”, GS(H)),
- $y_{\mathcal{P}}(x)$ is a solution of the inhomogeneous o.d.e. (called a “particular solution”, PS(IH)),
- $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

Definition:
Two functions y_1 and y_2 are linearly independent if

$$y_2(x) \neq \text{ constant } \cdot y_1(x)$$

for all x where they are both defined.

Example 1: Are $y_1(x) = x^2, \quad y_2(x) = 2x^2$ linearly independent?

Solution: As $y_2(x) = 2y_1(x)$ for all $x \in \mathbb{R}$, y_1 and y_2 are NOT linearly independent.

Example 2: Are $y_1(x) = e^{2x}, \quad y_2(x) = xe^{2x}$ linearly independent?

Solution: As $y_2(x) = xy_1(x)$ for all $x \in \mathbb{R}$, y_1 and y_2 are linearly independent.

Homogeneous 2^{nd} Order Linear O.D.E's with Constant Coefficients

General form:

$$ay'' + by' + cy = 0$$

where a, b, c are constants.

To solve for $y(x)$:

$$\begin{aligned} \text{Try } y(x) &= e^{\lambda x} \\ \Rightarrow y'(x) &= \lambda e^{\lambda x}, \quad y''(x) = \lambda^2 e^{\lambda x} \end{aligned}$$

$$\text{so } \left(a\lambda^2 + b\lambda + c\right) \underbrace{e^{\lambda x}}_{\neq 0} = 0$$

$$\Rightarrow \boxed{a\lambda^2 + b\lambda + c = 0}$$ Characteristic Equation

$$\Rightarrow \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Case 1: $b^2 - 4ac > 0$

- 2 distinct real values λ_1, λ_2
- 2 linearly independent solutions

$$e^{\lambda_1 x}, \quad e^{\lambda_2 x}$$

- General Solution:

$$\boxed{y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}}$$

Example 3: Solve $y'' + 7y' + 12y = 0$ for $y(x)$.

Solution:

$$\begin{aligned}
\text{Try } y(x) &= e^{\lambda x} \\
\Rightarrow y'(x) &= \lambda e^{\lambda x}, \quad y''(x) = \lambda^2 e^{\lambda x}
\end{aligned}$$

$$\text{so } \left(\lambda^2 + 7\lambda + 12\right) \underbrace{e^{\lambda x}}_{\neq 0} = 0$$

$$\Rightarrow \quad \lambda^2 + 7\lambda + 12 = 0 \qquad \text{(Characteristic equation)}$$

$$\Rightarrow \quad (\lambda + 4)(\lambda + 3) = 0$$

$$\Rightarrow \quad \lambda = -4, -3$$

So e^{-4x}, e^{-3x} are linearly independent solutions.

General Solution:

$$y(x) = Ae^{-4x} + Be^{-3x}.$$

Case 2: $b^2 - 4ac = 0$

- 1 real value $\lambda = \dfrac{-b}{2a}$

- 1 solution is $e^{\lambda x}$

- 2^{nd} linearly independent solution is $xe^{\lambda x}$ (found using variation of parameters — not in syllabus).

- General Solution:

$$\boxed{y(x) = Ae^{\lambda x} + Bxe^{\lambda x}}$$

We now verify that $xe^{\lambda x}$ is a solution:

If $y(x) = xe^{\lambda x}$, then

$$
\begin{aligned}
y'(x) &= (\lambda x + 1)\, e^{\lambda x}, \\
y''(x) &= \left(\lambda^2 x + 2\lambda\right) e^{\lambda x}.
\end{aligned}
$$

So $ay'' + by' + cy$

$$
\begin{aligned}
&= a\left(\lambda^2 x + 2\lambda\right) e^{\lambda x} + b\,(\lambda x + 1)\, e^{\lambda x} + cxe^{\lambda x} \\
&= xe^{\lambda x} \underbrace{\left(a\lambda^2 + b\lambda + c\right)}_{=0} + \underbrace{(2\lambda a + b)}_{=0} e^{\lambda x} \\
&= 0
\end{aligned}
$$

So $y(x) = xe^{\lambda x}$ is a solution.

Example 4: Solve $y'' + 2y' + y = 0$ for $y(x)$.

Solution:

$$\text{Try } y(x) = e^{\lambda x}$$

$$\Rightarrow y'(x) = \lambda e^{\lambda x}, \quad y''(x) = \lambda^2 e^{\lambda x}$$

$$\text{so } \left(\lambda^2 + 2\lambda + 1\right) \underbrace{e^{\lambda x}}_{\neq 0} = 0$$

$$\Rightarrow \quad \lambda^2 + 2\lambda + 1 = 0 \qquad (\text{ Characteristic equation })$$

$$\Rightarrow \quad (\lambda + 1)^2 = 0$$

$$\Rightarrow \quad \lambda = -1$$

So e^{-x} is one solution. A second linearly independent solution is xe^{-x}.

General Solution:

$$y(x) = Ae^{-x} + Bxe^{-x}.$$

Case 3: $b^2 - 4ac < 0$

- 2 complex conjugate values

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta$$

- 2 complex linearly independent solutions

$$e^{(\alpha+i\beta)x}, \quad e^{(\alpha-i\beta)x}$$

- General Solution:

$$\begin{aligned} y(x) &= C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} \qquad \text{where } C_1, C_2 \in \mathbb{C} \\ &= C_1 e^{\alpha x} (\cos\beta x + i \sin\beta x) + C_2 e^{\alpha x} (\cos\beta x - i \sin\beta x) \\ &= \underbrace{(C_1 + C_2)}_{A} e^{\alpha x} \cos\beta x + \underbrace{(C_1 i - C_2 i)}_{B} e^{\alpha x} \sin\beta x \end{aligned}$$

Put $A = C_1 + C_2$ and $B = (C_1 - C_2)i$. If $C_1 = \overline{C_2}$, then $A, B \in \mathbb{R}$.

- 2 real linearly independent solutions

$$e^{\alpha x} \cos \beta x, \quad e^{\alpha x} \sin \beta x$$

Real General Solution:

$$\boxed{y(x) = Ae^{\alpha x} \cos \beta x + Be^{\alpha x} \sin \beta x}$$

Example 5: Solve $y'' - 4y' + 13y = 0$ for $y(x)$ if $y(0) = -1$ and $y'(0) = 2$.

(Note: This is called an initial value problem for a second order o.d.e.)

Solution:

$$\begin{aligned} \text{Try } y(x) &= e^{\lambda x} \\ \Rightarrow y'(x) &= \lambda e^{\lambda x}, \quad y''(x) = \lambda^2 e^{\lambda x}. \end{aligned}$$

$$\text{So } \left(\lambda^2 - 4\lambda + 13\right) \underbrace{e^{\lambda x}}_{\neq 0} = 0$$

$$\Rightarrow \quad \lambda^2 - 4\lambda + 13 = 0 \qquad \text{(Characteristic equation)}$$

$$\Rightarrow \quad \lambda = 2 \pm 3i$$

So $e^{(2+3i)x}, e^{(2-3i)x}$ are linearly independent solutions.

General Solution:

$$y(x) = C_1 e^{(2+3i)x} + C_2 e^{(2-3i)x}$$

$$\Rightarrow y(x) = Ae^{2x}\cos 3x + Be^{2x}\sin 3x$$

- $y(0) = -1 \quad \Rightarrow A = -1$

Now $y'(x) = 2Ae^{2x}\cos 3x - 3Ae^{2x}\sin 3x + 2Be^{2x}\sin 3x + 3Be^{2x}\cos 3x$

- $y'(0) = 2 \quad \Rightarrow 2A + 3B = 2 \quad \Rightarrow \quad B = \dfrac{4}{3}$

Solution satisfying the initial conditions is
$y(x) = -e^{2x}\cos 3x + \frac{4}{3}e^{2x}\sin 3x.$

Inhomogeneous 2^{nd} Order Linear O.D.E's with Constant Coefficients

General form:

$$ay'' + by' + cy = \mathcal{R}(x)$$

where a, b, c are constants.

Example 6: Solve $y'' + 2y' - 8y = \mathcal{R}(x)$ where

(a) $\mathcal{R}(x) = 1 - 8x^2$

(b) $\mathcal{R}(x) = e^{3x}$

(c) $\mathcal{R}(x) = 85 \cos x$

(d) $\mathcal{R}(x) = 3 - 24x^2 + 7e^{3x}$.

Solution:

Step 1: Find the general solution of $y'' + 2y' - 8y = 0$.

$$\text{Try } y_{\mathcal{H}}(x) = e^{\lambda x}$$

$$\Rightarrow \left(\lambda^2 + 2\lambda - 8\right)\underbrace{e^{\lambda x}}_{\neq 0} = 0$$

$$\Rightarrow \quad \lambda^2 + 2\lambda - 8 = 0 \qquad \text{(characteristic equation)}$$

$$\Rightarrow \quad (\lambda + 4)(\lambda - 2) = 0$$

$$\Rightarrow \quad \lambda = -4, 2$$

$$GS(H): \quad y_{\mathcal{H}}(x) = Ae^{-4x} + Be^{2x}$$

Step 2: Find a particular solution of $y'' + 2y' - 8y = \mathcal{R}(x)$.

(a) $\mathcal{R}(x) = 1 - 8x^2$: $\quad y'' + 2y' - 8y = 1 - 8x^2$

$$\text{Try} \quad y_{\mathcal{P}}(x) = ax^2 + bx + c$$

$$\Rightarrow \quad y'_{\mathcal{P}}(x) = 2ax + b$$

$$\Rightarrow \quad y''_{\mathcal{P}}(x) = 2a$$

$$\text{So} \quad 2a + 2(2ax + b) - 8(ax^2 + bx + c) = 1 - 8x^2$$

$$2a + 2b - 8c + (4a - 8b)x - 8ax^2 = 1 - 8x^2$$

Equate coefficients:

$$x^2: \quad -8a = -8 \quad \Rightarrow a = 1$$

$$x^1: \quad 4a - 8b = 0 \quad \Rightarrow b = \frac{a}{2} = \frac{1}{2}$$

$$x^0: \quad 2a + 2b - 8c = 1 \quad \Rightarrow -8c = -2 \quad \Rightarrow c = \frac{1}{4}$$

So $y_{\mathcal{P}}(x) = \quad x^2 + \frac{1}{2}x + \frac{1}{4}$ is a $PS(IH)$

$$GS(IH): \quad y(x) = Ae^{-4x} + Be^{2x} + x^2 + \frac{1}{2}x + \frac{1}{4}.$$

(b) $\mathcal{R}(x) = e^{3x}$: $\quad y'' + 2y' - 8y = e^{3x}$

$$\left(e^{3x} \text{ is NOT part of } GS(H)\right)$$

$$\text{Try} \quad y_{\mathcal{P}}(x) = ae^{3x}$$

$$\Rightarrow \quad y'_{\mathcal{P}}(x) = 3ae^{3x}$$

$$\Rightarrow \quad y''_{\mathcal{P}}(x) = 9ae^{3x}$$

$$\text{So} \quad (9a + 6a - 8a)e^{3x} = e^{3x}$$

$$\Rightarrow 7a = 1$$

$$\Rightarrow a = \frac{1}{7}$$

So $\quad y_{\mathcal{P}}(x) = \dfrac{1}{7}e^{3x}$ is a $PS(IH)$.

$$GS(IH): \quad y(x) = Ae^{-4x} + Be^{2x} + \frac{1}{7}e^{3x}.$$

(c) $\mathcal{R}(x) = 85\cos x$: $\quad y'' + 2y' - 8y = 85\cos x$

Try $\quad y_{\mathcal{P}}(x) = a\cos x + b\sin x$

$$\Rightarrow \quad y_{\mathcal{P}}'(x) = -a\sin x + b\cos x$$

$$\Rightarrow \quad y_{\mathcal{P}}''(x) = -a\cos x - b\sin x$$

So $\quad (-a\cos x - b\sin x) + 2(-a\sin x + b\cos x)$

$$-8(a\cos x + b\sin x) = 85\cos x$$

$$\Rightarrow (-9a + 2b)\cos x + (-9b - 2a)\sin x = 85\cos x$$

Equate coefficients:

$$\left.\begin{aligned} \sin x : -9b - 2a &= 0 \\ \cos x : -9a + 2b &= 85 \end{aligned}\right\} \Rightarrow a = -9, \quad b = 2$$

So $\quad y_{\mathcal{P}}(x) = -9\cos x + 2\sin x \quad$ is a $PS(IH)$.

$$GS(IH): \quad y(x) = Ae^{-4x} + Be^{2x} - 9\cos x + 2\sin x.$$

Superposition of Particular Solutions

Theorem:
A particular solution (PS) of

$$ay'' + by' + cy = c_1\mathcal{R}_1(x) + c_2\mathcal{R}_2(x)$$

is $y_{\mathcal{P}}(x) = c_1 y_1(x) + c_2 y_2(x)$ where

- $y_1(x)$ is a particular solution of $ay'' + by' + cy = \mathcal{R}_1(x)$
- $y_2(x)$ is a particular solution of $ay'' + by' + cy = \mathcal{R}_2(x)$
- a, b, c, c_1, c_2 are constants.

Example 6 (d): $\mathcal{R}(x) = 3 - 24x^2 + 7e^{3x}$.

Solution:

Now $\mathcal{R}(x) = 3\left(1 - 8x^2\right) + 7\left(e^{3x}\right)$

From parts (a) and (b), the PS is

$$y_{\mathcal{P}}(x) = 3\left(x^2 + \frac{x}{2} + \frac{1}{4}\right) + 7\left(\frac{1}{7}e^{3x}\right)$$

$$\Rightarrow GS(IH) : y(x) = Ae^{-4x} + Be^{2x} + 3x^2 + \frac{3x}{2} + \frac{3}{4} + e^{3x}.$$

Example 7: Solve $y'' - y = e^x$.

Solution:

$GS(H) : y_{\mathcal{H}}(x) = Ae^x + Be^{-x}$

Since e^x is a part of $GS(H)$, try

$$\begin{aligned} y_{\mathcal{P}}(x) &= axe^x \\ \Rightarrow y'_{\mathcal{P}}(x) &= ae^x + axe^x \\ \Rightarrow y''_{\mathcal{P}}(x) &= 2ae^x + axe^x \end{aligned}$$

So $\quad 2ae^x + axe^x - axe^x = e^x$

$$\Rightarrow \quad a = \frac{1}{2}$$

$$\Rightarrow \quad y_{\mathcal{P}}(x) = \frac{1}{2}xe^x$$

$$GS(IH) : y(x) = Ae^x + Be^{-x} + \frac{1}{2}xe^x.$$

Example 8: Solve $y'' + 2y' + y = e^{-x}$.

Solution:

$GS(H) : y_{\mathcal{H}}(x) = (A + Bx)\, e^{-x}$

Since xe^{-x} and e^{-x} are part of $GS(H)$, try

$$y_{\mathcal{P}}(x) = ax^2 e^{-x}$$

Substitute $y_{\mathcal{P}}, y'_{\mathcal{P}}, y''_{\mathcal{P}}$ into d.e.

$$\left(2a - 4ax + ax^2\right) e^{-x} + 2\left(2ax - ax^2\right) e^{-x} + ax^2 e^{-x} = e^{-x}$$

$$\begin{aligned}
&\Rightarrow \quad 2ae^{-x} = e^{-x} \\
&\Rightarrow \quad a = \frac{1}{2} \\
&\Rightarrow \quad y_{\mathcal{P}}(x) = \frac{1}{2}x^2e^{-x}
\end{aligned}$$

$$GS(IH) : y(x) = (A + Bx)\, e^{-x} + \frac{1}{2}x^2e^{-x}.$$

Example 9: Solve $y'' + 49y = 28\sin 7t$.

Solution:

$GS(H): y_{\mathcal{H}}(t) = A\cos 7t + B\sin 7t$

Since $\sin 7t$ is part of $GS(H)$, try

$$y_{\mathcal{P}}(t) = at\cos 7t + bt\sin 7t$$

Substitute $y_{\mathcal{P}}, y'_{\mathcal{P}}, y''_{\mathcal{P}}$ into d.e.

$$(-14a\sin 7t + 14b\cos 7t - 49at\cos 7t - 49bt\sin 7t)$$

$$+49\,(at\cos 7t + bt\sin 7t) = 28\sin 7t$$

$$\Rightarrow -14a\sin 7t + 14b\cos 7t = 28\sin 7t$$

Equate coefficients:

$$\sin 7t: \qquad -14a = 28 \quad \Rightarrow a = -2$$

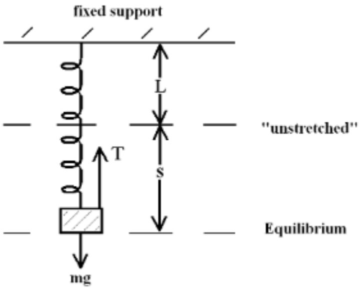
$$\cos 7t: \qquad 14b = 0 \quad \Rightarrow b = 0$$

$$\text{So } y_{\mathcal{P}}(t) = -2t\cos 7t$$

$$GS(IH): y(t) = A\cos 7t + B\sin 7t - 2t\cos 7t.$$

Springs - Free Vibrations

An object (mass m kg) stretches a spring (natural length L m) hanging from a fixed support by s m.



The forces are:

- gravitational force = mg $\qquad (g = 9.8 \text{ m/s}^2)$

- restoring force in spring (from Hooke's Law)

$$T \quad = \quad k \cdot \text{extension} \qquad\qquad (k > 0)$$

spring constant

At equilibrium, forces balance so:

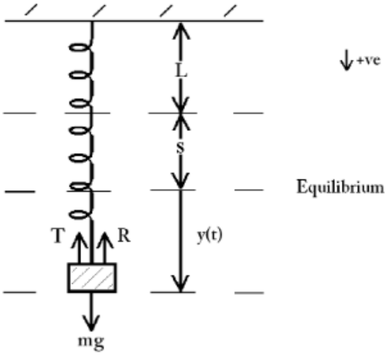
$$mg = T$$

$$\Rightarrow \boxed{mg = ks}$$

Suppose the mass is set in motion. Let y be the displacement of the object from the equilibrium position ($y = 0$) at any time t.

Assume

- downward direction is positive
- spring is stretched below equilibrium
- mass is moving down (so damping is upwards)



Extra force:

- damping force is proportional to velocity

$$R = \underset{\substack{\nwarrow \\ \text{damping constant}}}{\beta}\dot{y} \qquad (\beta \geq 0)$$

Using Newton's Law $\quad (F = ma)$

$$m\ddot{y} = mg - T - R$$

$$\Rightarrow \quad m\ddot{y} = mg - k(s + y) - \beta\dot{y}$$
$$\Rightarrow \quad m\ddot{y} = -ky - \beta\dot{y}$$

$$\Rightarrow \boxed{m\ddot{y} + \beta\dot{y} + ky = 0}$$

"Equation of motion"

To solve, try $y(t) = e^{\lambda t}$

$$\Rightarrow \quad m\lambda^2 + \beta\lambda + k = 0$$

$$\Rightarrow \quad \lambda = \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m}$$

- If $\beta = 0$: $\quad \lambda = \pm ib$ simple harmonic motion
- If $0 < \beta^2 < 4km$: $\quad \lambda = a \pm ib$ underdamped, weak damping
- If $\beta^2 = 4km$: $\quad \lambda = a, a$ critical damping
- If $\beta^2 > 4km$: $\quad \lambda = a, b$ overdamped, strong damping

Example 10: A $\frac{40}{49}$ kg mass stretches a spring hanging from a fixed support by 0.2m. The mass is released from the equilibrium position with a downward velocity of 3m/s. Find the position of the mass $y(t)$ below equilbrium if the damping constant β is:

(a) 0 (b) $\frac{160}{49}$ (c) $\frac{80}{7}$ (d) $\frac{2000}{49}$

Solution:

The spring constant is:

$$k = \frac{mg}{s} = \frac{\frac{40}{49} \cdot (9.8)}{0.2} = 40$$

The equation of motion is:

$$\frac{40}{49}\ddot{y} + \beta\dot{y} + 40y = 0$$

$$\Rightarrow \ddot{y} + \frac{49}{40}\beta\dot{y} + 49y = 0$$

The initial conditions are:

$$y(0) = 0, \quad \dot{y}(0) = 3$$

(a) $\beta = 0 : \quad \ddot{y} + 49y = 0$

$$\text{Try } y(t) = e^{\lambda t} \quad \Rightarrow \quad \lambda^2 + 49 = 0$$

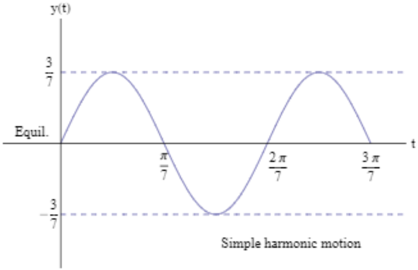
$$\Rightarrow \quad \lambda = \pm 7i$$

$$GS : y(t) = A\cos 7t + B\sin 7t$$

$$\text{Now } y(0) = 0, \quad \dot{y}(0) = 3$$

$$\Rightarrow A = 0, \quad B = \frac{3}{7}$$

$$\Rightarrow y(t) = \frac{3}{7}\sin 7t$$



(b) $\beta = \frac{160}{49}$: $\quad \ddot{y} + 4\dot{y} + 49y = 0$

Try $y(t) = e^{\lambda t} \quad \Rightarrow \quad \lambda^2 + 4\lambda + 49 = 0$

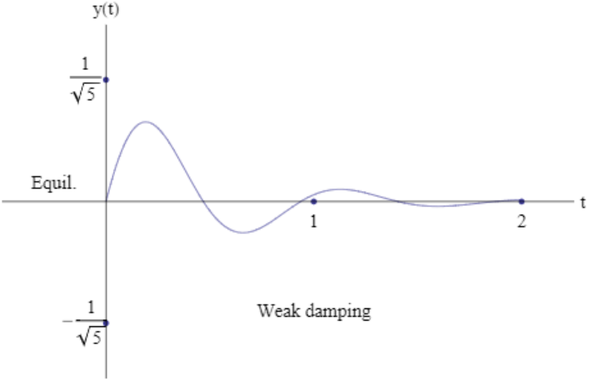
$$\Rightarrow \quad \lambda = -2 \pm 3\sqrt{5}i$$

$$GS : y(t) = Ae^{-2t}\cos(3\sqrt{5}t) + Be^{-2t}\sin(3\sqrt{5}t)$$

$$\text{Now } y(0) = 0, \quad \dot{y}(0) = 3$$

$$\Rightarrow A = 0, \quad B = \frac{1}{\sqrt{5}}$$

$$\Rightarrow y(t) = \frac{1}{\sqrt{5}} e^{-2t} \sin(3\sqrt{5}t)$$



(c) $\beta = \frac{80}{7}$: $\quad \ddot{y} + 14\dot{y} + 49y = 0$

$$\text{Try } y(t) = e^{\lambda t} \quad \Rightarrow \quad \lambda^2 + 14\lambda + 49 = 0$$

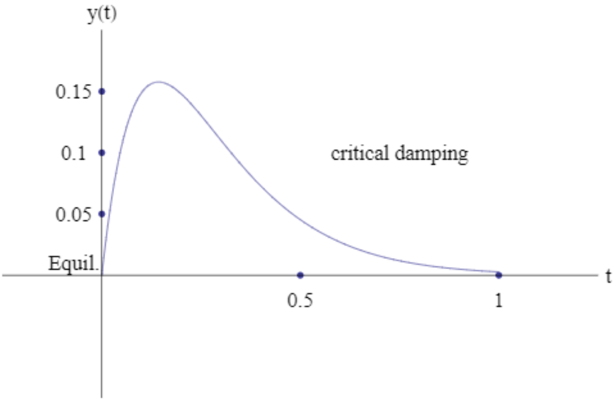
$$\Rightarrow \quad \lambda = -7$$

$$GS : y(t) = (A + Bt)e^{-7t}$$

$$\text{Now } y(0) = 0, \quad \dot{y}(0) = 3$$

$$\Rightarrow A = 0, \quad B = 3$$

$$\Rightarrow y(t) = 3te^{-7t}$$


y(t)
0.15
0.1
0.05
Equil.
critical damping
t
0.5
1

(d) $\beta = \frac{2000}{49}$: $\quad \ddot{y} + 50\dot{y} + 49y = 0$

$$\text{Try } y(t) = e^{\lambda t} \quad \Rightarrow \quad \lambda^2 + 50\lambda + 49 = 0$$

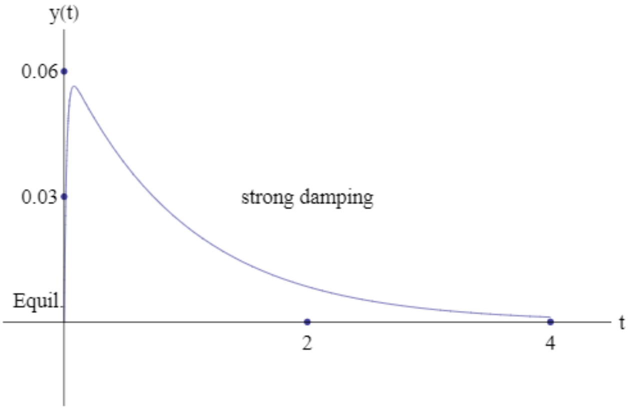
$$\Rightarrow \quad \lambda = -1, -49$$

$$GS : y(t) = Ae^{-t} + Be^{-49t}$$

$$\text{Now } y(0) = 0, \quad \dot{y}(0) = 3$$

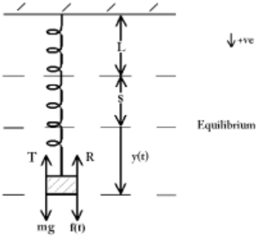
$$\Rightarrow A = \frac{1}{16}, \quad B = \frac{-1}{16}$$

$$\Rightarrow y(t) = \frac{1}{16}\left(e^{-t} - e^{-49t}\right)$$


y(t)
0.06
0.03
strong damping
Equil.
t
2
4

Springs - Forced Vibrations

If an external downwards force $f(t)$ is applied to the spring-mass system, the forces acting on the mass are:



The equation of motion is:

$$
\begin{aligned}
& m\ddot{y} = mg - T - R + f \\
\Rightarrow\quad & m\ddot{y} = mg - k(s+y) - \beta\dot{y} + f \\
\Rightarrow\quad & m\ddot{y} + \beta\dot{y} + ky = f.
\end{aligned}
$$

Example 11: Apply an external downwards force $f(t) = \frac{160}{7}\sin 7t$ in Example 10.

Solution:

The equation of motion is:

$$\frac{40}{49}\ddot{y} + \beta\dot{y} + 40y = \frac{160}{7}\sin 7t$$

$$\Rightarrow \quad \ddot{y} + \frac{49}{40}\beta\dot{y} + 49y = 28\sin 7t$$

The initial conditions are:

$$y(0) = 0, \quad \dot{y}(0) = 3$$

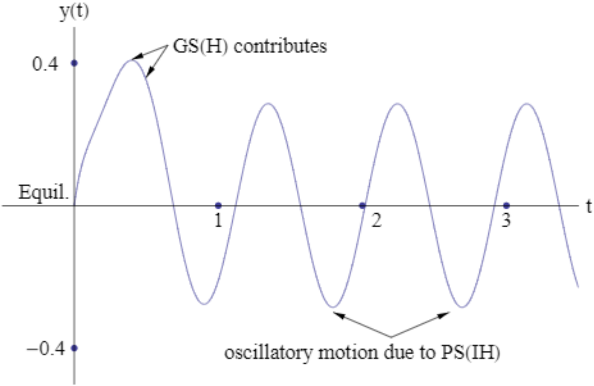
$$(a) \quad \beta = \frac{80}{7} : \quad \ddot{y} + 14\dot{y} + 49y = 28 \sin 7t$$

$$GS(IH) : \quad y(t) = (A + Bt)\, e^{-7t} - \frac{2}{7} \cos 7t$$

Now $y(0) = 0, \quad \dot{y}(0) = 3$

$$\Rightarrow \quad A = \frac{2}{7}, \quad B = 5$$

$$\Rightarrow \quad y(t) = \underbrace{\left(\frac{2}{7} + 5t\right) e^{-7t}}_{\text{transient solution}} \quad - \quad \underbrace{\frac{2}{7} \cos 7t}_{\text{steady state solution}}$$


y(t)
GS(H) contributes
0.4
Equil.
1
2
3
t
−0.4
oscillatory motion due to PS(IH)

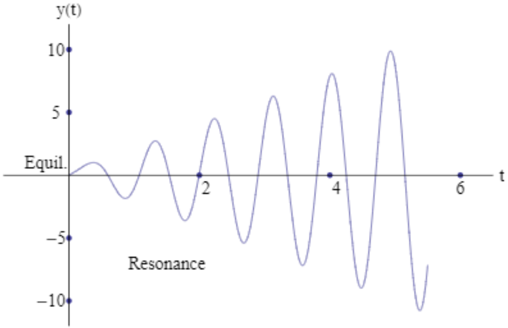
$$(b) \quad \beta = 0 : \quad \ddot{y} + 49y = 28 \sin 7t$$

$$GS(IH) : \quad y(t) = A \cos 7t + B \sin 7t - 2t \cos 7t$$

Now $y(0) = 0, \quad \dot{y}(0) = 3$

$$\Rightarrow \quad A = 0, \quad B = \frac{5}{7}$$

$$\Rightarrow \quad y(t) = \frac{5}{7} \sin 7t - 2t \cos 7t.$$



As $t \to \infty$ amplitude of oscillation increases without bound. Spring will break!

Note:
Resonance occurs when the external force f is part of $GS(H)$. The $PS(IH)$ will grow without bound as $t \to \infty$.

Section 7: Functions of Two Variables

Example:

The temperature T at a point on the Earth's surface at a given time depends on the latitude x and the longitude y. We think of T being a function of the variables x, y and write $T = f(x, y)$.

In general

A function of two variables is a mapping f that assigns a unique real number $z = f(x, y)$ to each pair of real numbers (x, y) in some subset D of the xy plane $\mathbb{R}^2$. We also write

$$f : D \to \mathbb{R}$$

where D is called the domain of f.

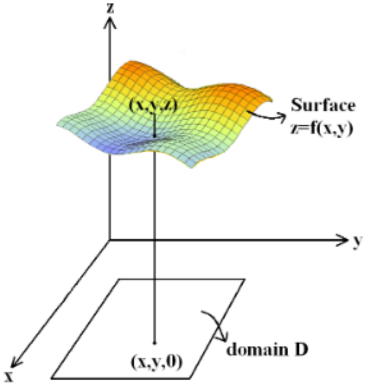
Example:

If $f(x, y) = x^2 + y^3$ then $f(2, 1) = 4 + 1 = 5$.

We can represent the function f by its graph in $\mathbb{R}^3$. The graph of f is:

$$\Big\{(x, y, z) : (x, y) \in D \text{ and } z = f(x, y)\Big\}.$$

This is a surface lying directly above the domain D. The x and y axes lie in the horizontal plane and the z axis is vertical.



Equations of a Plane

The Cartesian form of a plane is

$$ax + by + cz = d$$

where a, b, c, d are real constants.

Note that a normal vector to the plane is $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

A plane passing through (x_0, y_0, z_0) can be written in the form

$$z - z_0 = \alpha(x - x_0) + \beta(y - y_0)$$

where α, β are constants.

Example 1: The plane $4x + 3y + z = 2$ can be written as $z = 2 - 4x - 3y$, so is the graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 2 - 4x - 3y$. Sketch the plane.

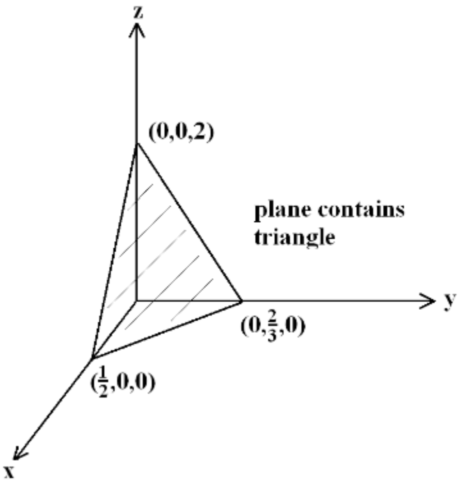
Solution:

$$x - \text{intercept:} \left(\frac{1}{2}, 0, 0\right)$$

$$y - \text{intercept:} \left(0, \frac{2}{3}, 0\right)$$

$$z - \text{intercept:} (0, 0, 2)$$

A normal to plane is $4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.


z
(0,0,2)
plane contains
triangle
y
(0,2/3,0)
(1/2,0,0)
x

Level Curves

A curve on the surface $z = f(x, y)$ for which z is a constant is a contour.

The same curve drawn in the xy plane is a level curve.

So a level curve of f has the form

$$\{(x, y) : f(x, y) = c\}$$

where $c \in \mathbb{R}$ is a constant.

Sketching Functions of Two Variables

The key steps in a drawing a graph of a function of two variables $z = f(x,y)$ are:

1. Draw the x,y,z axes.
   For right handed axes: the positive x axis is towards you, the positive y axis points to the right, and the positive z axis points upward.
2. Draw the $y - z$ cross section.
3. Draw some level curves.
4. Draw the $x - z$ cross section.
5. Label any x,y,z intercepts and key points.

Example 2: Find the level curves of $z = \sqrt{1 - x^2 - y^2}$.
Hence identify the surface and sketch it.

Solution:

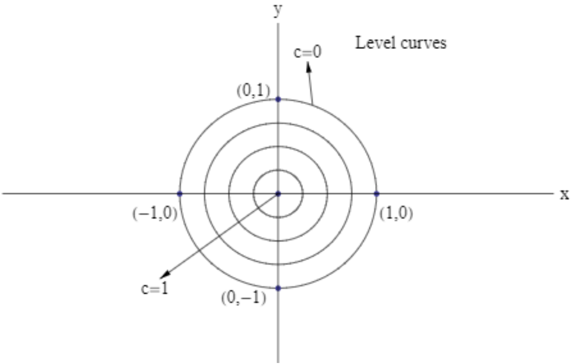
$$
\begin{aligned}
\text{Let } z = c \in \mathbb{R} \quad &\Rightarrow \quad c = \sqrt{1 - x^2 - y^2} \qquad (c \geq 0) \\
&\Rightarrow \quad 1 - x^2 - y^2 = c^2 \\
&\Rightarrow \quad x^2 + y^2 = 1 - c^2
\end{aligned}
$$

Now $1 - c^2 \geq 0 \Rightarrow c^2 \leq 1 \Rightarrow |c| \leq 1$.

Combining restrictions gives $0 \leq c \leq 1$ so

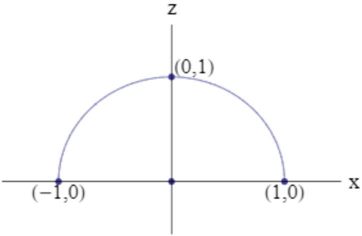
$$0 \leq 1 - c^2 \leq 1.$$

So the level curves are circles centred at $(0, 0)$ with radius $0 \leq r \leq 1$.

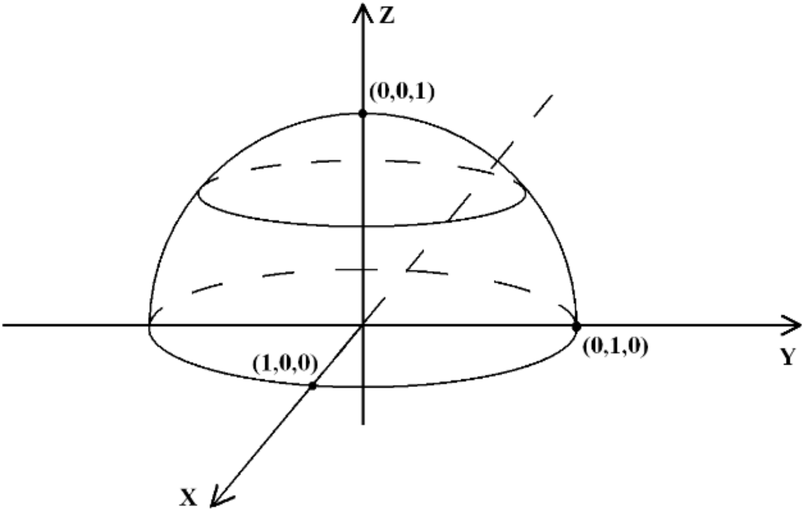


Consider cross sections (slices) to help sketch graph.

- If $y = 0$ (xz plane) $z = \sqrt{1 - x^2}$
- If $x = 0$ (yz plane) $z = \sqrt{1 - y^2}$



Surface is a hemisphere radius 1, centre at $(0, 0, 0)$ for $z \geq 0$.

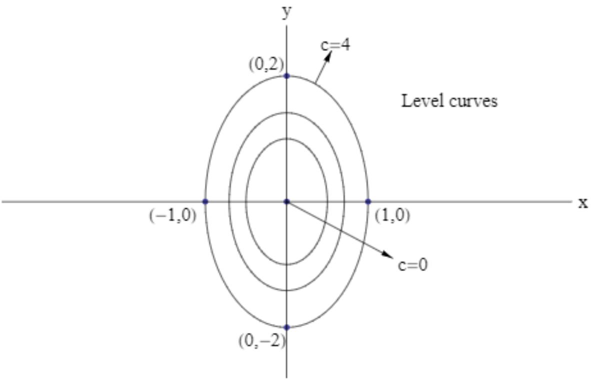


Example 3: Sketch the graph of $z = 4x^2 + y^2$.

Solution:

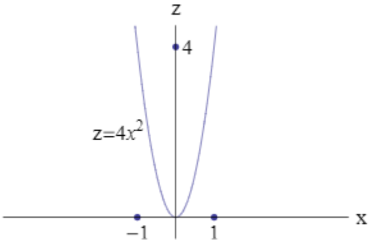
$$\text{Let } z = c \in \mathbb{R} \Rightarrow 4x^2 + y^2 = c$$

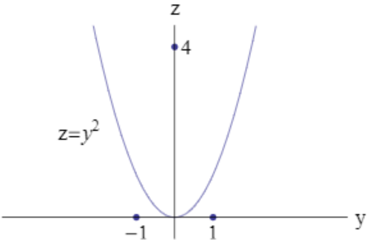
If $c \geq 0$, the level curves are ellipses



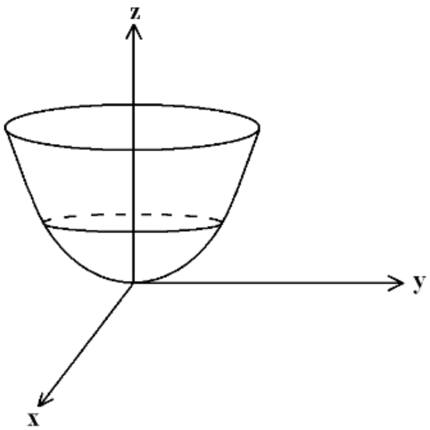
Consider cross sections (slices) to help sketch graph.

- If $y = 0$ (xz plane) $z = 4x^2$
- If $x = 0$ (yz plane) $z = y^2$





The surface is an elliptic paraboloid (parabolic bowl).



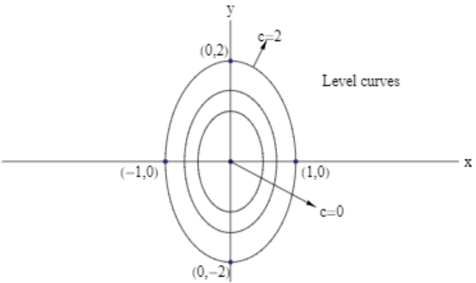
Example 4: Sketch the graph of $z = \sqrt{4x^2 + y^2}$.

Solution:

$$\text{Let } z = c \in \mathbb{R} \quad \Rightarrow \quad \sqrt{4x^2 + y^2} = c \qquad (c \geq 0)$$

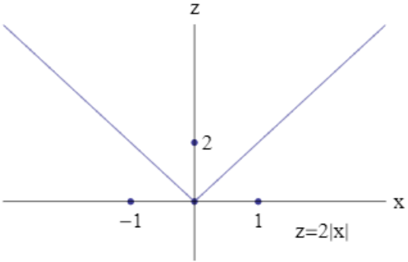
$$\Rightarrow \quad 4x^2 + y^2 = c^2$$

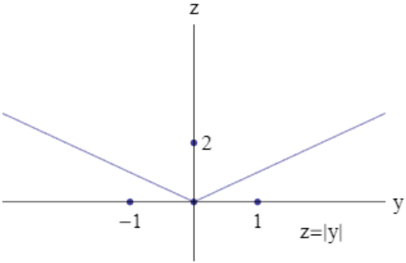
If $c \geq 0$, level curves are ellipses



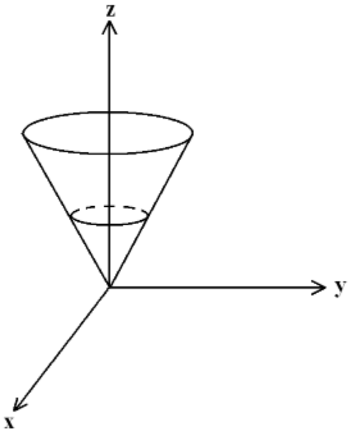
Cross sections

- If $y = 0$ (xz plane) $z = \sqrt{4x^2} = 2|x|$
- If $x = 0$ (yz plane) $z = \sqrt{y^2} = |y|$





The surface is an elliptic cone.



Limits

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function.

We say f has the limit L as (x, y) approaches (x_0, y_0)

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if when (x, y) approaches (x_0, y_0) along ANY path in the domain, $f(x, y)$ gets arbitrarily close to L.

Note:

1. L must be finite.
2. The limit can exist if f is undefined at (x_0, y_0).
3. The usual limit laws apply.

Continuity

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function.

f is continuous at $(x, y) = (x_0, y_0)$ if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

Note:
The continuity theorems for functions of one variable can be generalised to functions of two variables.

Example 5: Let $f(x, y) = x^2 + y^2$. For which values of x and y is f continuous?

Solution:

Since f is a polynomial in x and y,

it is continuous for all $(x, y) \in \mathbb{R}^2$.

Example 6: Evaluate $\lim_{(x,y)\to(2,1)} \log(1 + x^2 + 2y^2)$.

Solution:

$$\begin{aligned}
\lim_{(x,y)\to(2,1)} \log(1 + x^2 + 2y^2) &= \log\left[\lim_{(x,y)\to(2,1)} (1 + x^2 + 2y^2)\right] \\
&\qquad \text{(continuity of } \log\text{)} \\
&= \log(1 + 4 + 2) \\
&= \log 7.
\end{aligned}$$

First Order Partial Derivatives

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function.

The first order partial derivatives of f with respect to the first and second variables (say x and y) are defined by the limits:

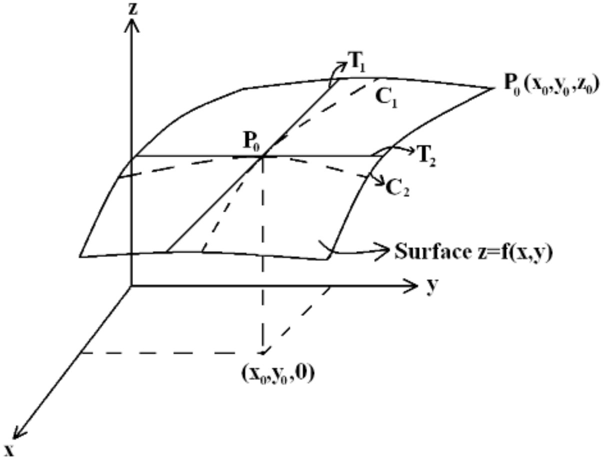
$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Note:

- $\dfrac{\partial f}{\partial x}$ measures the rate of change of f with respect to x when y is held constant, and
- $\dfrac{\partial f}{\partial y}$ measures the rate of change of f with respect to y when x is held constant.

Geometric Interpretation of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$



Let C_1 be the curve where the vertical plane $y = y_0$ intersects the surface. Then $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ gives the slope of the tangent to C_1 at (x_0, y_0, z_0).

Let C_2 be the curve where the vertical plane $x = x_0$ intersects the surface. The $\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}$ gives the slope of the tangent to C_2 at (x_0, y_0, z_0).

- T_1 and T_2 are the tangent lines to C_1 and C_2.

Example 7: Let $f(x,y) = xy^2$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ from first principles.

Solution:

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \lim_{h\to 0} \frac{(x+h)y^2 - xy^2}{h} \\
&= \lim_{h\to 0} \frac{hy^2}{h} \\
&= \lim_{h\to 0} y^2 \\
&= y^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \lim_{h\to 0} \frac{x(y+h)^2 - xy^2}{h} \\
&= \lim_{h\to 0} \frac{x(y^2 + 2yh + h^2) - xy^2}{h} \\
&= \lim_{h\to 0} \frac{2xyh + xh^2}{h} \\
&= \lim_{h\to 0} (2xy + xh) \\
&= 2xy
\end{aligned}
$$

Example 8: Let $f(x, y) = 3x^3y^2 + 3xy^4$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution:

- $$\begin{aligned} \frac{\partial f}{\partial x} &= 3y^2 \underbrace{\frac{\partial}{\partial x}\left(x^3\right)}_{=3x^2} + 3y^4 \underbrace{\frac{\partial}{\partial x}(x)}_{=1} \\ &= 9x^2y^2 + 3y^4 \end{aligned}$$

- $$\begin{aligned} \frac{\partial f}{\partial y} &= 3x^3 \underbrace{\frac{\partial}{\partial y}\left(y^2\right)}_{=2y} + 3x \underbrace{\frac{\partial}{\partial y}\left(y^4\right)}_{=4y^3} \\ &= 6x^3y + 12xy^3 \end{aligned}$$

Example 9: Let $f(x,y) = y \log x + x \tanh 3y$. Find f_x, f_y at $(1,0)$.

Solution:

- $f_x(x,y) \;=\; \dfrac{y}{x} + \tanh 3y$
- $f_y(x,y) \;=\; \log x + 3x \operatorname{sech}^2 3y$

Evaluating at $(1,0)$ gives

$$\begin{aligned} f_x(1,0) &= \frac{0}{1} + \tanh 0 = 0 \\ f_y(1,0) &= \log 1 + 3 \operatorname{sech}^2 0 = 3. \end{aligned}$$

Second Order Partial Derivatives

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function.

The second order partial derivatives of f with respect to the first and second variables (say x and y) are defined by:

- $$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$
- $$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$
- $$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$
- $$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

Theorem:

If the second order partial derivatives of f exist and are continuous then $f_{xy} = f_{yx}$.

Example 10: Find the second order partial derivatives of $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 + 2x^3y^2 - 3y^4$.

Solution:

- $f_x(x, y) = 2x + 6x^2y^2$
- $f_y(x, y) = 4x^3y - 12y^3$
- $f_{xx}(x, y) = \dfrac{\partial}{\partial x}\left(2x + 6x^2y^2\right) = 2 + 12xy^2$
- $f_{yy}(x, y) = \dfrac{\partial}{\partial y}\left(4x^3y - 12y^3\right) = 4x^3 - 36y^2$

- $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\left(2x + 6x^2y^2\right) = 12x^2y$

- $f_{yx}(x,y) = \dfrac{\partial}{\partial x}\left(4x^3y - 12y^3\right) = 12x^2y$

Note:
$f_{xy} = f_{yx}$ as expected since polynomials are continuous for all $(x,y) \in \mathbb{R}^2$.

Example 11: Find the second order partial derivatives of $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x\sin(x + 2y)$.

Solution:

- $f_x(x, y) = \sin(x + 2y) + x\cos(x + 2y)$
- $f_y(x, y) = 2x\cos(x + 2y)$
- $$\begin{aligned} f_{xx}(x, y) &= \frac{\partial}{\partial x}\left[\sin(x + 2y) + x\cos(x + 2y)\right] \\ &= 2\cos(x + 2y) - x\sin(x + 2y) \end{aligned}$$
- $$\begin{aligned} f_{yy}(x, y) &= \frac{\partial}{\partial y}\left[2x\cos(x + 2y)\right] \\ &= -4x\sin(x + 2y) \end{aligned}$$

- $$\begin{aligned} f_{xy}(x,y) &= \frac{\partial}{\partial y}\left[\sin(x+2y) + x\cos(x+2y)\right] \\ &= 2\cos(x+2y) - 2x\sin(x+2y) \end{aligned}$$

- $$\begin{aligned} f_{yx}(x,y) &= \frac{\partial}{\partial x}\left[2x\cos(x+2y)\right] \\ &= 2\cos(x+2y) - 2x\sin(x+2y) \end{aligned}$$

Note:

$f_{xy} = f_{yx}$ as expected since trigonometric functions and polynomials are continuous for all $(x,y) \in \mathbb{R}^2$.

Tangent Planes and Differentiability

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function.

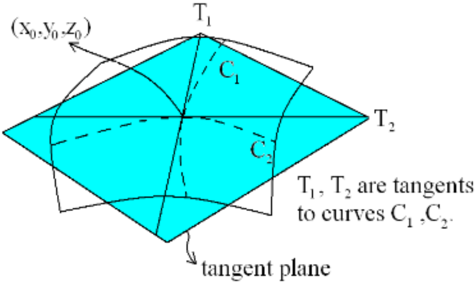
We say that f is differentiable at (x_0, y_0) if the tangent lines

to all curves on the surface $z = f(x, y)$ passing through (x_0, y_0, z_0)

form a plane, called the tangent plane at (x_0, y_0, z_0).

(This holds if f_x and f_y exist and are continuous near (x_0, y_0).)

Then, near (x_0, y_0, z_0) the surface can be well-approximated by the tangent plane.



The tangent line T_1 has equation ($y = y_0$ fixed):

$$z - z_0 = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} (x - x_0)$$

The tangent line T_2 has equation ($x = x_0$ fixed):

$$z - z_0 = \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} (y - y_0)$$

Since a plane passing through (x_0, y_0, z_0) has the form

$$z - z_0 = \alpha(x - x_0) + \beta(y - y_0)$$

the tangent plane has equation

$$\boxed{z - z_0 = \frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)}(y - y_0).}$$

Example 12: Find the equation of the tangent plane to the surface $z = f(x, y) = 2x^2 + y^2$ at $(1, 1, 3)$.

Solution:

- $(x_0, y_0, z_0) = (1, 1, 3)$
- $\dfrac{\partial f}{\partial x} = 4x \quad \Rightarrow \left.\dfrac{\partial f}{\partial x}\right|_{(1,1)} = 4$
- $\dfrac{\partial f}{\partial y} = 2y \quad \Rightarrow \left.\dfrac{\partial f}{\partial y}\right|_{(1,1)} = 2$

The equation of the tangent plane is

$$z - 3 = \left.\frac{\partial f}{\partial x}\right|_{(1,1)}(x-1) + \left.\frac{\partial f}{\partial y}\right|_{(1,1)}(y-1)$$

$$\Rightarrow \quad z - 3 = 4x - 4 + 2y - 2$$

$$\Rightarrow \quad z = 4x + 2y - 3 \quad \text{or} \quad 4x + 2y - z = 3.$$

Linear Approximations

If f is differentiable at (x_0, y_0) we can approximate $z = f(x, y)$ by its tangent plane at (x_0, y_0).

This linear approximation of $f(x, y)$ near (x_0, y_0) is:

$$\boxed{f(x,y) \approx f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y - y_0)}$$

Let $\Delta x = x - x_0, \quad \Delta y = y - y_0, \quad \Delta f = z - z_0 = f(x, y) - f(x_0, y_0)$.

Then the approximate change in $f(x, y)$ for given small changes in x and y is:

$$\boxed{\Delta f \approx \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}\Delta x + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}\Delta y}$$

Example 13: Let $z = f(x,y) = x^2 + 3xy - y^2$. If x changes from 2 to 2.05 and y changes from 3 to 2.96, estimate the change in z.

Solution:

- $f_x(x,y) = 2x + 3y$
- $f_y(x,y) = 3x - 2y$

Now $(x_0, y_0) = (2,3)$ – initial values

- $f_x(2,3) = 13$
- $f_y(2,3) = 0$

$$\begin{aligned} \text{Also } \Delta x &= 2.05 - 2 = 0.05 \quad (x \uparrow) \\ \Delta y &= 2.96 - 3 = -0.04 \quad (y \downarrow) \end{aligned}$$

$$\text{So } \Delta z \approx f_x(2,3)\Delta x + f_y(2,3)\Delta y$$

$$= 13(0.05) + 0(-0.04)$$

$$= 0.65$$

Note:
The actual change in f is

$$\Delta f = f(2.05, 2.96) - f(2,3)$$

$$= 13.6449 - 13$$

$$= 0.6449$$

Example 14: Find the linear approximation of $f(x, y) = xe^{xy}$ at $(1, 0)$. Hence, approximate $f(1.1, -0.1)$.

Solution:

- $f_x(x, y) = e^{xy} + xye^{xy} \quad \Rightarrow f_x(1, 0) = 1$
- $f_y(x, y) = x^2e^{xy} \quad \Rightarrow f_y(1, 0) = 1$

Linearizing f at $(1, 0)$ gives

$$f(x, y) \approx f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$$

$$\Rightarrow f(x,y) \approx 1 + x - 1 + y = x + y$$

So $f(1.1, -0.1) \approx 1.1 - 0.1 = 1.$

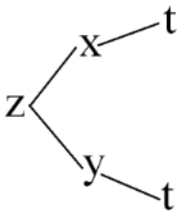
Note:
The actual value is

$$(1.1)e^{-0.11} \approx 0.98542$$

Chain Rule

1. If $z = f(x, y)$ and $x = g(t)$, $y = h(t)$ are differentiable functions, then $z = f(g(t), h(t))$ is a function of t, and

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$



Example 15: If $z = x^2 - y^2$, $x = \sin t$, $y = \cos t$. Find $\dfrac{dz}{dt}$ at $t = \dfrac{\pi}{6}$.

Solution:

- Use chain rule.

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (2x)(\cos t) + (-2y)(-\sin t) \end{aligned}$$

At $t = \dfrac{\pi}{6}$ $\quad x = \sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}, \quad y = \cos\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$

So $\left.\dfrac{dz}{dt}\right|_{\frac{\pi}{6}} = 2 \cdot \dfrac{1}{2} \cdot \dfrac{\sqrt{3}}{2} + 2 \cdot \dfrac{\sqrt{3}}{2} \cdot \dfrac{1}{2} = \sqrt{3}$

- check by substitution

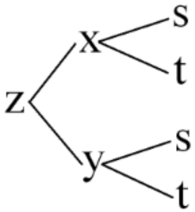
$$z = x^2 - y^2 = \sin^2 t - \cos^2 t = -\cos(2t)$$

$$\Rightarrow \frac{dz}{dt} = 2\sin(2t)$$

$$\Rightarrow \left.\frac{dz}{dt}\right|_{\frac{\pi}{6}} = 2\sin\left(\frac{\pi}{3}\right) = \sqrt{3}$$

2. If $z = f(x, y)$ and $x = g(s, t)$, $y = h(s, t)$ are differentiable functions, then z is a function of s and t with

$$\boxed{\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\[1em] \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \end{aligned}}$$



Example 16: If $z = e^x \sinh y$, $x = st^2$, $y = s^2 t$. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

Solution:

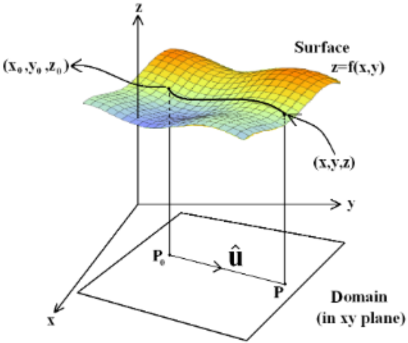
- $$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= e^x \sinh y\, t^2 + e^x \cosh y\, 2st \\ &= t^2 e^{st^2} \sinh(s^2 t) + 2st e^{st^2} \cosh(s^2 t) \end{aligned}$$

- $$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= e^x \sinh y\, 2st + e^x \cosh y\, s^2 \\ &= 2st e^{st^2} \sinh(s^2 t) + s^2 e^{st^2} \cosh(s^2 t) \end{aligned}$$

Directional Derivatives

Let $\hat{\mathbf{u}} = (u_1, u_2)$ be a unit vector in the xy-plane (so $u_1^2 + u_2^2 = 1$). The rate of change of f at $P_0 = (x_0, y_0)$ in the direction $\hat{\mathbf{u}}$ is the directional derivative $D_{\hat{\mathbf{u}}} f \big|_{P_0}$.

Geometrically this represents the slope of the surface $z = f(x, y)$ above the point P_0 in the direction $\hat{\mathbf{u}}$.



The straight line starting at $P_0 = (x_0, y_0)$ with velocity $\hat{\mathbf{u}} = (u_1, u_2)$ has parametric equations:

$$x = x_0 + tu_1, \quad y = y_0 + tu_2.$$

Hence,

$$\begin{aligned} D_{\hat{\mathbf{u}}} f &= \text{rate of change of } f \text{ along the straight line} \\ &= \frac{d}{dt} f(x_0 + tu_1, y_0 + tu_2) \\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \qquad \text{by the chain rule} \\ &= \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2. \end{aligned}$$

We can also write this as a dot product

$$D_{\hat{\mathbf{u}}} f \Big|_{P_0} = \left(\frac{\partial f}{\partial x}\Big|_{P_0}, \frac{\partial f}{\partial y}\Big|_{P_0} \right) \cdot (u_1, u_2).$$

Gradient Vectors

If $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function, we can define the gradient of f to be the vector

$$\boxed{\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)}$$

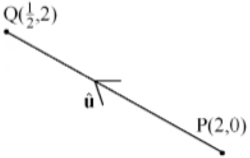
Then the directional derivative of f at the point $P_0(x_0, y_0)$ in the direction $\hat{\mathbf{u}}$ is the dot product

$$\boxed{D_{\hat{\mathbf{u}}} f \Big|_{P_0} = \nabla f\Big|_{P_0} \cdot \hat{\mathbf{u}}}$$

Example 17: Find the directional derivative of $f(x,y) = xe^y$ at $(2,0)$ in the direction from $(2,0)$ towards $\left(\frac{1}{2}, 2\right)$.

Solution:

- direction $\hat{\mathbf{u}}$



$$
\begin{aligned}
\mathbf{u} &= \vec{PQ} = \left(\frac{-3}{2}, 2\right) \\
\|\mathbf{u}\| &= \sqrt{\frac{9}{4} + 4} = \sqrt{\frac{25}{4}} = \frac{5}{2} \\
\hat{\mathbf{u}} &= \frac{2}{5}\left(\frac{-3}{2}, 2\right) = \left(\frac{-3}{5}, \frac{4}{5}\right)
\end{aligned}
$$

- ∇f at $(2,0)$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = e^y\mathbf{i} + xe^y\mathbf{j}$$

$$\Rightarrow \nabla f(2,0) = \mathbf{i} + 2\mathbf{j}$$

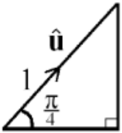
- $D_{\hat{\mathbf{u}}} f$ at $(2,0)$

$$\begin{aligned} D_{\hat{\mathbf{u}}} f(2,0) &= \nabla f(2,0) \cdot \hat{\mathbf{u}} \\ &= (1,2) \cdot \left(-\frac{3}{5}, \frac{4}{5}\right) \\ &= -\frac{3}{5} + \frac{8}{5} \\ &= 1. \end{aligned}$$

Example 18: Find the directional derivative of $f(x,y) = \arcsin\left(\dfrac{x}{y}\right)$ at $(1,2)$ in the direction $\dfrac{\pi}{4}$ anticlockwise from the positive x axis.

Solution:

- direction $\hat{\mathbf{u}}$



$$
\begin{aligned}
a &= \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \\
b &= \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \\
\text{so } \hat{\mathbf{u}} &= \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}
\end{aligned}
$$

- ∇f at $(1,2)$

$$\frac{\partial f}{\partial x} = \frac{1}{y\sqrt{1-\frac{x^2}{y^2}}} \qquad \frac{\partial f}{\partial y} = \frac{-x}{y^2\sqrt{1-\frac{x^2}{y^2}}}$$

At $(1,2)$ $\sqrt{1-\dfrac{x^2}{y^2}} = \sqrt{1-\dfrac{1}{4}} = \sqrt{\dfrac{3}{4}} = \dfrac{\sqrt{3}}{2}$

$$\text{So } \left.\frac{\partial f}{\partial x}\right|_{(1,2)} = \frac{1}{2\cdot\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,2)} = \frac{-1}{4\cdot\frac{\sqrt{3}}{2}} = \frac{-1}{2\sqrt{3}}$$

- $D_{\hat{\mathbf{u}}} f$ at $(1,2)$

$$\begin{aligned} D_{\hat{\mathbf{u}}} f(1,2) &= \boldsymbol{\nabla} f(1,2) \cdot \hat{\mathbf{u}} \\ &= \left(\frac{1}{\sqrt{3}}, \frac{-1}{2\sqrt{3}}\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{6}} - \frac{1}{2\sqrt{6}} \\ &= \frac{1}{2\sqrt{6}}. \end{aligned}$$

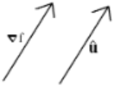
Properties of ∇f and $D_{\hat{\mathbf{u}}} f$

$$\begin{aligned} D_{\hat{\mathbf{u}}} f &= \nabla f \cdot \hat{\mathbf{u}} \\ &= \|\nabla f\| \, \|\hat{\mathbf{u}}\| \cos\theta \\ &= \|\nabla f\| \cos\theta \end{aligned}$$

where θ is the angle between ∇f and $\hat{\mathbf{u}}$, and $\|\mathbf{v}\|$ denotes the length of a vector $\mathbf{v}$.

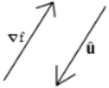
So for fixed ∇f:

- $D_{\hat{\mathbf{u}}}f$ is maximum when $\cos\theta = 1$ so $\theta = 0$



$\Rightarrow f$ increases most rapidly along $\boldsymbol{\nabla} f$.

- $D_{\hat{\mathbf{u}}}f$ is minimum when $\cos\theta = -1$ so $\theta = \pi$



$\Rightarrow f$ decreases most rapidly along $-\boldsymbol{\nabla} f$.

- $D_{\hat{\mathbf{u}}} f = 0$ when $\cos\theta = 0$ so $\theta = \dfrac{\pi}{2}$ and $\nabla f \perp \hat{\mathbf{u}}$.

But $D_{\hat{\mathbf{u}}} f = 0$, whenever $\hat{\mathbf{u}}$ is tangent to a level curve of f (where f = constant).

$$\Rightarrow \nabla f \perp \text{ level curves of } f$$

Example 19: Let $f(x, y) = 4x^2 + y^2$.
(a) Find ∇f at $(1, 0)$ and $(0, 2)$.
(b) Show that ∇f is perpendicular to the level curves, by sketching ∇f at these points and the level curves of f.

Solution:
(a)

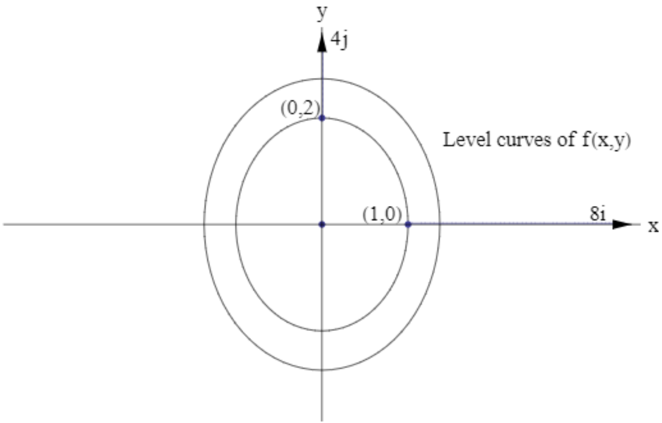
$$\text{Now } \frac{\partial f}{\partial x} = 8x, \qquad \frac{\partial f}{\partial y} = 2y$$

$$\text{So} \quad \nabla f = 8x\mathbf{i} + 2y\mathbf{j}$$

$$\text{At } (1, 0), \quad \nabla f(1, 0) = 8\mathbf{i}$$

$$\text{At } (0, 2), \quad \nabla f(1, 0) = 4\mathbf{j}.$$

(b)



Example 20: In what direction does $f(x, y) = xe^y$

(i) increase (ii) decrease

most rapidly at $(2, 0)$?

Solution:

From Example 17

$$\nabla f(2, 0) = \mathbf{i} + 2\mathbf{j}$$

$$\Rightarrow ||\nabla f(2, 0)|| = \sqrt{5}$$

So a unit vector in direction of $\nabla f(2, 0)$ is

$$\hat{\mathbf{u}} = \frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$$

The direction of most rapid

- increase is $\nabla f(2,0)$ so $\hat{\mathbf{u}} = \frac{1}{\sqrt{5}}(1,2)$
- decrease is $-\nabla f(2,0)$ so $\hat{\mathbf{u}} = -\frac{1}{\sqrt{5}}(1,2)$.

Stationary Points

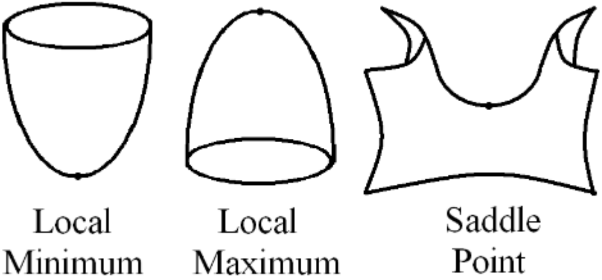
A stationary point of f is a point (x_0, y_0) at which

$$\nabla f = \mathbf{0}$$

So $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$ simultaneously at (x_0, y_0).

Geometrically, this means that the tangent plane to the graph $z = f(x, y)$ at (x_0, y_0) is horizontal, i.e. parallel to the xy-plane.

Three important types of stationary points are

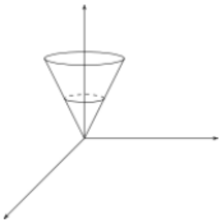


A function f has a

1. local maximum at (x_0, y_0) if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) in an open disk centred at (x_0, y_0),
2. local minimum at (x_0, y_0) if $f(x, y) \geq f(x_0, y_0)$ for all (x, y) in an open disk centred at (x_0, y_0),
3. saddle point at (x_0, y_0) if (x_0, y_0) is a stationary point, and there are points near (x_0, y_0) with $f(x, y) > f(x_0, y_0)$ and other points near (x_0, y_0) with $f(x, y) < f(x_0, y_0)$.

Note: Any local maximum or minimum of f will occur at a critical point (x_0, y_0) such that

1. $\nabla f(x_0, y_0) = \mathbf{0}$ <u>or</u>
2. $\dfrac{\partial f}{\partial x}$ and/or $\dfrac{\partial f}{\partial y}$ do not exist at (x_0, y_0).



$z = \sqrt{x^2 + y^2}$. Minimum at $(0, 0)$ BUT ∇f does not exist at $(0, 0)$.

Second Derivative Test

If $\nabla f(x_0, y_0) = \mathbf{0}$ and the second partial derivatives of f are continuous on an open disk centred at (x_0, y_0), consider the Hessian function

$$H(x, y) = f_{xx}f_{yy} - (f_{xy})^2$$

evaluated at (x_0, y_0).

Then (x_0, y_0) is a

1. local minimum if $H(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$.
2. local maximum if $H(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$.
3. saddle point if $H(x_0, y_0) < 0$.

Note: Test is inconclusive if $H(x_0, y_0) = 0$.

Example 21: Find and classify the stationary points of $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$.

Solution:

• Stationary points

$$\bullet \quad f_x(x, y) = 3x^2 + 6x = 0 \quad \Rightarrow \quad 3x(x + 2) = 0$$

$$\Rightarrow \quad x = 0 \text{ or } x = -2 \qquad (1)$$

$$\bullet \quad f_y(x, y) = 3y^2 - 6y = 0 \quad \Rightarrow \quad 3y(y - 2) = 0$$

$$\Rightarrow \quad y = 0 \text{ or } y = 2 \qquad (2)$$

Combining (1) and (2) gives 4 points

$$(0, 0), \quad (-2, 0), \quad (-2, 2), \quad (0, 2)$$

- Classify stationary points

$$f_{xx}(x,y) = 6x + 6, \quad f_{yy}(x,y) = 6y - 6, \quad f_{xy}(x,y) = 0$$

$$\begin{aligned} \text{So } H(x,y) &= f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 \\ &= (6x+6)(6y-6) - 0 \\ &= 36(x+1)(y-1) \end{aligned}$$

- $H(0,0) = -36 < 0$, therefore $(0,0)$ is a saddle point

- $H(-2,0) = 36 > 0$ and $f_{xx}(-2,0) = -6 < 0$, therefore f has a local maximum at $(-2,0)$

- $H(-2,2) = -36 < 0$, therefore $(-2,2)$ is a saddle point

- $H(0,2) = 36 > 0$ and $f_{xx}(0,2) = 6 > 0$, therefore f has a local minimum at $(0,2)$

Example 22: Find and classify the stationary points of $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = y \sin x$.

Solution:

• Stationary points

- $f_x(x,y) \;=\; y\cos x = 0$ if $y = 0$ or $\cos x = 0$

$$\Rightarrow \quad y = 0 \text{ or } x = (m + \tfrac{1}{2})\pi \qquad (m \in \mathbb{Z}) \qquad (1)$$

- $f_y(x,y) \;=\; \sin x = 0$ if $x = n\pi \qquad (n \in \mathbb{Z}) \qquad (2)$

Combining (1) and (2) gives stationary points at $(n\pi, 0)$ where n is an integer

- Classify stationary points

$$
\begin{aligned}
f_{xx}(x,y) &= -y\sin x\\
f_{yy}(x,y) &= 0\\
f_{xy}(x,y) &= \cos x
\end{aligned}
$$

$$
\begin{aligned}
\text{So } H(x,y) &= f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2\\
&= (-y\sin x)(0) - \cos^2 x\\
&= -\cos^2 x
\end{aligned}
$$

$$
\text{So } H(n\pi,0) = -\cos^2(n\pi) = -1 < 0.
$$

Saddle points at $(n\pi, 0)$ where n is an integer.

Partial Integration

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function over a domain D in $\mathbb{R}^2$.

The partial indefinite integrals of f with respect to the first and second variables (say x and y) are denoted by:

$\int f(x,y)\,dx$ and $\int f(x,y)\,dy$.

- $\int f(x,y)\,dx$ is evaluated by holding y fixed and integrating with respect to x.

- $\int f(x,y)\,dy$ is evaluated by holding x fixed and integrating with respect to y.

Example 23: Evaluate $\int (3x^2y + 12y^2x^3)\, dx$.

Solution:

Hold y fixed and $\int$ with respect to x.

$$\int (3x^2y + 12y^2x^3)\, dx \;=\; x^3y + 3y^2x^4 + \underbrace{c(y)}_{\nearrow} .$$

constant of integration depends on y

Example 24: Evaluate $\int_0^1 (3x^2y + 12y^2x^3)\,dy$.

Solution:
Hold x fixed and $\int$ with respect to y.

$$\begin{aligned} \int_0^1 (3x^2y + 12y^2x^3)\,dy &= \left[\frac{3}{2}x^2y^2 + 4y^3x^3\right]_{y=0}^{y=1} \\ &= \frac{3}{2}x^2 + 4x^3. \end{aligned}$$

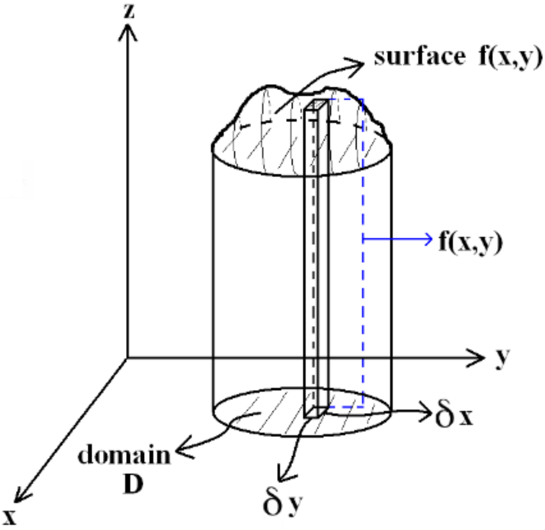
Double Integrals

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function over a domain D in $\mathbb{R}^2$.

We can evaluate the double integral:

$$\boxed{\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dx\,dy}$$

$\iint_D f(x,y)\,dA$ is the volume under the surface $z = f(x,y)$ that lies above the domain D in the xy plane, if $f(x,y) \geq 0$ in D.


z
surface f(x,y)
f(x,y)
y
δx
domain
D
δy
x

$$\text{Volume of thin rod} = \underbrace{(\text{Area base})}_{\substack{\| \\ \underbrace{\delta A}_{\substack{\| \\ \delta x \delta y}}}} \cdot \underbrace{(\text{height})}_{\substack{\| \\ f(x,y)}}$$

The double integral is defined as the limit of sums of the volumes of the rods:

$$\begin{aligned} \iint_D f(x,y)\,dA &= \iint_D f(x,y)\,dx\,dy \\ &= \lim_{\delta x \to 0} \lim_{\delta y \to 0} \sum_{i=1}^{n} [f(x,y)\delta x \delta y]_i \end{aligned}$$

Note:

If $f(x,y) = 1$ then

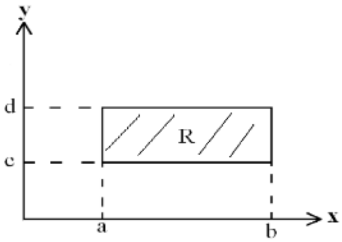
$$\iint_D dA = \iint_D dx\,dy$$

gives the area of the domain D.

Double Integrals Over Rectangular Domains

Definitions

1. $R = [a, b] \times [c, d]$ is a rectangular domain defined by
   $a \le x \le b, \quad c \le y \le d.$



2. $\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$ means integrate with respect to x first and then integrate with respect to y.

Fubini's Theorem:

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function over the domain $R = [a,b] \times [c,d]$. Then

$$\begin{aligned} \iint_R f(x,y)dA &= \int_c^d \int_a^b f(x,y)\,dx\,dy \\ &= \int_a^b \int_c^d f(x,y)\,dy\,dx \end{aligned}$$

So order of integration is NOT important.

Example 25: Evaluate $\iint_R (x^2 + y^2)\,dx\,dy$ if $R = [-1,1] \times [0,1]$.

Solution:

In R: $-1 \le x \le 1, \quad 0 \le y \le 1$

- Integrate with respect to x first

$$\begin{aligned}
\iint_R (x^2 + y^2)\,dx\,dy &= \int_0^1 \int_{-1}^1 (x^2 + y^2)\,dx\,dy \\
&= \int_0^1 \left[\frac{1}{3}x^3 + y^2 x\right]_{x=-1}^{x=1} dy \\
&= \int_0^1 \left[\left(\frac{1}{3} + y^2\right) - \left(-\frac{1}{3} - y^2\right)\right] dy
\end{aligned}$$

$$\begin{aligned} &= \int_0^1 (\frac{2}{3} + 2y^2)\, dy \\ &= \left[\frac{2}{3}y + \frac{2}{3}y^3\right]_{y=0}^{y=1} \\ &= \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \end{aligned}$$

- Integrate with respect to y first

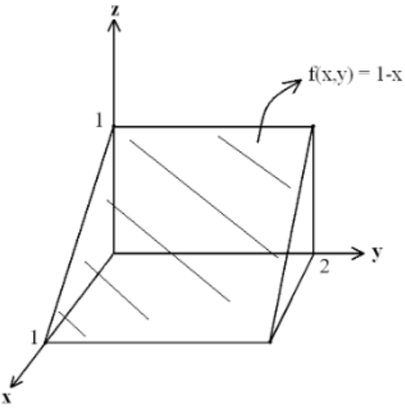
$$\begin{aligned} \iint_R (x^2 + y^2)\, dy\, dx &= \int_{-1}^1 \int_0^1 (x^2 + y^2)\, dy\, dx \\ &= \int_{-1}^1 \left[x^2 y + \frac{1}{3}y^3\right]_{y=0}^{y=1} dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{-1}^{1} (x^2 + \frac{1}{3})\, dx \\
&= \left[\frac{1}{3}x^3 + \frac{1}{3}x\right]_{x=-1}^{x=1} \\
&= \left(\frac{1}{3} + \frac{1}{3}\right) - \left(-\frac{1}{3} - \frac{1}{3}\right) \\
&= \frac{4}{3}.
\end{aligned}
$$

Note:

As expected, the order of integration is not important since polynomials are continuous for all $(x, y) \in \mathbb{R}^2$.

Example 26: Using double integrals, find the volume of the wedge shown below.



Solution:

The domain in the xy plane is $R = [0,1] \times [0,2]$ or
$0 \leq x \leq 1, \quad 0 \leq y \leq 2.$

$$
\begin{aligned}
\text{Volume} &= \iint_R f(x,y)\,dx\,dy \\
&= \int_0^2 \int_0^1 1 - x\,dx\,dy \\
&= \int_0^2 \left[x - \frac{1}{2}x^2\right]_{x=0}^{x=1} dy \\
&= \int_0^2 \frac{1}{2}\,dy \\
&= \left[\frac{1}{2}y\right]_{y=0}^{y=2} \\
&= 1\ (\text{units})^3.
\end{aligned}
$$