MAST10007 Linear Algebra Summer Exam 2017 

(a) $610 + x_4 = 450 + x_1 \Rightarrow x_1 - x_4 = 160$

$x_1 + 400 = 640 + x_2 \Rightarrow x_1 - x_2 = 240$

$x_2 + 600 = x_3 \Rightarrow -x_2 + x_3 = 600$

$x_3 = 520 + x_4 \Rightarrow x_3 - x_4 = 520$

(b) (i) The augmented matrix on the LHS corresponds to the linear system in (a). There is no leading entry for $x_4$. Set $x_4 = t$, $t \in \mathbb{R}$. Back substitution then gives

$$x_3 = 520 + t$$

$$x_2 = -80 + t$$

$$x_1 = 160 + t$$

Hence the general solution is

$$\{(160+t, -80+t, 520+t, t) \mid t \in \mathbb{R}\}$$

(ii) $x_4 = 80$.

(c) As an augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & k & k \end{array}\right] \underset{R_3 - 2R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & k-2 & k-4 \end{array}\right]$$

Unique solution provided there are three leading entries and thus $k \neq 2$. In this circumstance

$$x_3 = \frac{k-4}{k-2}.$$

(a) (i) $BC = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 13 \end{bmatrix}$

(ii) $C^T B^T B C = (BC)^T BC = \begin{bmatrix} 5 & 1 & 13 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 13 \end{bmatrix} = 25 + 1 + 169$

$= 195.$

(iii) $ABC = \begin{bmatrix} 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 13 \end{bmatrix} = 5 + 3 + 13 = 21$

Hence $(ABC)^{-1} = \frac{1}{21}$

(b) (i) For $A_{m \times n} B_{p \times q}$ to be well defined we require $n = p$

For $B_{p \times q} A_{m \times n}$ to be well defined we require $q = m$

For $A_{m \times n} B_{p \times q} - B_{p \times q} A_{m \times n}$ to be well defined, in addition we require

$m = p, \quad q = n$

Thus

$q = m = p = n$

so both matrices must be square of the same size.

3(a)(i)
$$A = \begin{array}{c} a \\ b \\ c \end{array}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(ii) The entry in row 1, column 1 of $A^4$ is the number of walks from vertex a back to itself in exactly 4 steps.

In exactly 4 steps, there are 3 possibilities:

(a) (a → b → c)

or (a → b → c) (a) or $(a)^4$

Hence the entry will be 3.

b(i) Consider
$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -2 & 0 & 0 & 1 \end{array}\right] R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right] R_2 \leftrightarrow R_3 \sim \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1+2R_3 \\ R_2-R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] R_1+R_2 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

Hence
$$B^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(ii)
$$C^{-1} = \frac{1}{\alpha\delta - \beta\gamma}\begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}, \quad \tilde{C}^{-1} = \frac{1}{\gamma\beta - \delta\alpha}\begin{bmatrix} \beta & -\delta \\ -\alpha & \gamma \end{bmatrix}$$
$$= \frac{1}{\alpha\delta - \beta\gamma}\begin{bmatrix} -\beta & \delta \\ \alpha & -\gamma \end{bmatrix}$$

Hence the two columns of $C^{-1}$ should be swapped.

Q. 4

4.a) i) $\underset{\sim}{u} \times \underset{\sim}{v}$, let $\underset{\sim}{u} = (u_1, u_2, u_3)$

$\underset{\sim}{v} = (v_1, v_2, v_3)$

~~A~~ $\underset{\sim}{u} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

(where $|A| = \det(A)$")

ii) $\underset{\sim}{u} \cdot (\underset{\sim}{u} \times \underset{\sim}{v}) = \underset{\sim}{u} \cdot \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ (from (i))

$$= \underset{\sim}{u} \cdot \left( \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

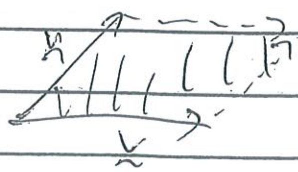
$$= u_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - u_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + u_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0$$ (two identical rows ~~A~~ a row of zeros $\Rightarrow$ det =

$\underset{\sim}{u}$ and $(\underset{\sim}{u} \times \underset{\sim}{v})$

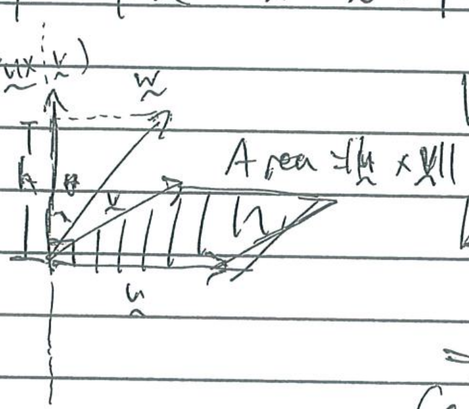
$\therefore$ perpendicular w.r.t. the dot product.

iii) Area of parallelogram $= \|\underset{\sim}{u} \times \underset{\sim}{v}\|$



## Q. 4

iii) cont.) $|w \cdot (u \times v)| = \|w\| \|u \times v\| \cos\theta$.




Volume = height x area

but $\frac{h}{\|w\|} = \cos\theta$ (see diagram

$\Rightarrow h = \|w\| \cos\theta$

(and Area $= \|u \times v\|$ ~~from~~ )

$$|w \cdot (u \times v)| = \|w\| \cos\theta \; \|u \times v\|$$

$$= h \times \text{Area}$$

$$= \text{Volume.}$$

$\therefore$ Volume of parallelepiped formed by $u, v, w$ $= |w \cdot (u \times v)|$

b) Area of $\Delta = \frac{1}{2}$ Area of parallelogram

$$= \frac{1}{2} \|u \times v\|$$

$$= \frac{1}{2} \|(1-0, 2-0, 0-0) \times (1-0, 2-0, 3-0)\|$$

$$= \frac{1}{2} \left| \det \begin{bmatrix} i & j & k \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} \right| = \frac{1}{2} \|(6-0, -(3-0), (2-2)\|$$

$$= \frac{1}{2}\sqrt{6^2 + 9 + 0} = \frac{1}{2}\sqrt{36+9} = \frac{1}{2}\sqrt{45}$$

Q.4 ~~& 5~~

4.b) (cont'd) $= \frac{1}{2}\sqrt{9}\sqrt{5} = \frac{3}{2}\sqrt{5}$

$\therefore$ Area is $\frac{3}{2}\sqrt{5}$.

5. a) $\{ a(1,1,0,-1) + b(2,0,2,0) + c(0,2,-2,-2) \mid a,b,c \in$

$= \text{span}\{(1,1,0,-1), (2,0,2,0), (0,2,-2,-2)\}$ (clearly)

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -2 \\ -1 & 0 & -2 \end{bmatrix} \overset{R_4+R_2}{\sim} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_2-R_1}{\sim} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R_3+R_2}{\sim} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ rank $= 2$, first & second columns are lin. indep., third column depend on first and second columns.

$\Rightarrow$ basis $= \{(1,1,0,-1), (2,0,2,0)\}$

and dimension $= 2$

b) ~~Suppose general vectors~~ Suppose general vectors $\underset{\sim}{u}$, and $\underset{\sim}{v}$ are in $\mathbb{R}^3$:
$\underset{\sim}{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, $\underset{\sim}{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$

belong in $S$. That is
$$\left.\begin{aligned} &v_1 + 2v_2 + v_3 = 0, \\ &v_1 - v_2 + v_3 = 0 \end{aligned}\right] ①$$
$$\left.\begin{aligned} &u_1 + 2u_2 + u_3 = 0 \\ \text{and } &u_1 - u_2 + u_3 = 0. \end{aligned}\right] ②$$

so that $\underset{\sim}{u}, \underset{\sim}{v} \in S$.

Q. 5

$\underset{\sim}{u} + \underset{\sim}{v} = (u_1, u_2, u_3) + (v_1, v_2, v_3)$

$= (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

But since $\underset{\sim}{u}, \underset{\sim}{v} \in S$.

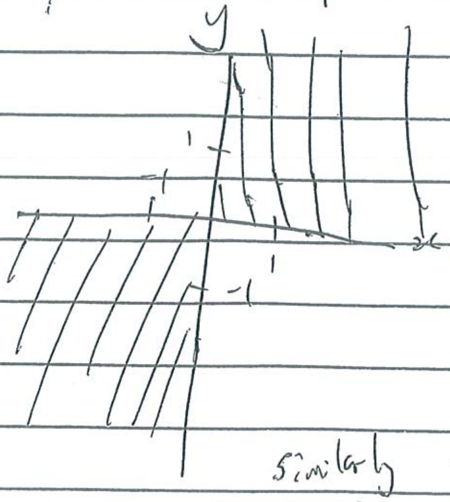
$\Rightarrow (u_1 + v_1) + 2(u_2 + v_2) + (u_3 + v_3)$
$= (u_1 + 2u_2 + u_3) + (v_1 + 2v_2 + v_3)$
$\geq 0 + 0 = 0$ (from ①)

and $(u_1 + v_1) - (u_2 + v_2) + (u_3 + v_3)$
$= (u_1 - u_2 + u_3) + (v_1 - v_2 + v_3)$
$= 0 + 0 = 0$ (from ②)

$\therefore (\underset{\sim}{u} + \underset{\sim}{v}) \in S$.

$\therefore$ S is c. u. v. a.

c) $\{(x, y) \mid x, y \geq 0\} \cup \{(x, y) \mid x, y \leq 0\} = R$.




take $(1, 0) \in R$ $(1, 0 \geq 0)$
and $(0, -1) \in R$ $(0, -1 \leq 0)$

$(1, 0) + (0, -1) = (1, -1)$

$1 \geq 0$ but $-1 \ngeq 0$

similarly $-1 \leq 0$ but $+1 \nleq 0$

$\therefore (1, -1) \notin R$ $\therefore$ not. c. u. v. a.

## Q. 4

(a) $u = (u_1, u_2, u_3)$ $v = (v_1, v_2, v_3)$

i) $u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

ii) $u \cdot (u \times v) = u \cdot \left( \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \right)$

$= u \cdot \left( \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k \right)$

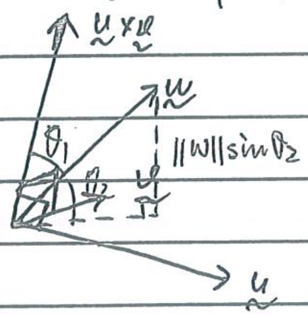
$= (u_1, u_2, u_3) \cdot (u_2v_3 - u_3v_2, -u_1v_3 + u_3v_1, u_1v_2 - u_2v_1)$

$= u_1u_2v_3 - u_1u_3v_2 - u_2u_1v_3 + u_2u_3v_1 + u_3u_1v_2 - u_3u_2v_1$

$= 0$

$\therefore$ $u$ is perpendicular to $u \times v$

iii)




$\cancel{|w \cdot (u \times v)|}$

$\cancel{= \|w\| \|u \times v\| \cos\theta_1}$

volume = (height) times (base)

= (height) times $\|u \times v\|$

from the graph we can see that

$\theta_1 + \theta_2 = \pi/2$ and height $= \|w\| \sin\theta_2$

$\therefore |\cos\theta_1| = \sin\theta_2$ assume $\sin\theta_2 \geq 0$

$\therefore$ volume = (height) times $\|u \times v\|$

$= \|w\| \|u \times w\| \sin\theta_2$

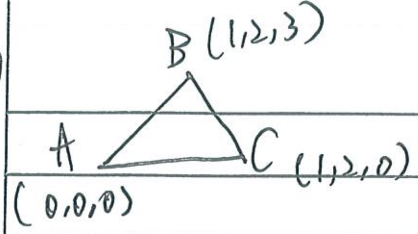
$= \|w\| \|u \times w\| |\cos\theta_1|$

since $\theta_1$ is the angle between $u \times v$ and $w$

we have volume $= |w \cdot (u \times w)|$

Q. 4

(b)




$\overrightarrow{AB} = (1,2,3)$

$\overrightarrow{AC} = (1,2,0)$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 2 & 3 \\ 1 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & 0 \end{vmatrix} \underset{\sim}{i} - \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} \underset{\sim}{j} + \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} \underset{\sim}{k}$$

$$= (-6, 3, 0)$$

Area of the triangle $= \frac{1}{2} \| \overrightarrow{AB} \times \overrightarrow{AC} \|$

$$= \frac{1}{2} \sqrt{(-6)^2 + 3^2 + 0^2}$$

$$= \frac{1}{2} \sqrt{36+9}$$

$$= \frac{1}{2} \sqrt{45} = \frac{3}{2} \sqrt{5}$$

Q. 5

(a) let $S = \{a(1,1,0,-1) + b(2,0,2,0) + c(0,2,-2,-2) : a,b,c \in \mathbb{R}\}$

$\therefore S = \text{Span}\{(1,1,0,-1),(2,0,2,0),(0,2,-2,-2)\}$

represent the ~~set~~ 3 vectors in matrix form to compute the rank

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -2 \\ -1 & 0 & -2 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ \\ R_4 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & -2 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_2 \\ R_4 + R_2 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ only 1st & 2nd column have leading entry in RE form.

$\therefore$ Basis for $S$ is

$\{(1,1,0,-1),(2,0,2,0)\}$

and it has dimension $= 2$

(b) we set $\underset{\sim}{v_1} = (x_1, y_1, z_1) \in S$, ~~$x_1, y_1, z_1 \in \mathbb{R}$~~

$x_1 + 2y_1 + z_1 = 0 \quad x_1 - y_1 + z_1 = 0$

$\underset{\sim}{v_2} = (x_2, y_2, z_2) \in S$,

$x_2 + 2y_2 + z_2 = 0 \quad x_2 - y_2 + z_2 = 0$

$\underset{\sim}{v_1} + \underset{\sim}{v_2} = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$(x_1 + x_2) + 2(y_1 + y_2) + z_1 + z_2 = (x_1 + 2y_1 + z_1) + (x_2 + 2y_2 + z_2)$

$= 0 + 0 = 0$

$(x_1 + x_2) - (y_1 + y_2) + (z_1 + z_2) = (x_1 - y_1 + z_1) + (x_2 - y_2 + z_2)$

$= 0 + 0$

$\therefore \underset{\sim}{v_1} + \underset{\sim}{v_2} \in S$

$\therefore$ the set is closed under vector addition

Q. 5

Denote the subset as $S$

(c) let $\underset{\sim}{v_1} = (5,4)$ $5,4 \geq 0$

$\therefore \underset{\sim}{v_1} \in$ ~~Subspace Subset~~ $S$

$\underset{\sim}{v_2} = (-3,-5)$ $-3,-5 \leq 0$

$\therefore \underset{\sim}{v_2} \in S$

$\underset{\sim}{v_1} + \underset{\sim}{v_2} = (2,-1)$ $2 \geq 0$ $-1 < 0$

$\therefore \underset{\sim}{v_1} + \underset{\sim}{v_2} \notin S$

$\therefore$ the counter-example exists.

$\therefore$ the subset is not closed under vector addition

2. (a) dim (row space of A)

= number of leading entries of RE form of A
= 3

(b) dim (solution space of A)

= number of columns with no leading entries of A in RE form

= 2

(c) Column space of A

$$= \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$$

(d) We see

$$[\underset{\sim}{a}_1 \ \underset{\sim}{a}_2 \ \underset{\sim}{a}_4 \ \underset{\sim}{b}_1] \sim \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It follows that

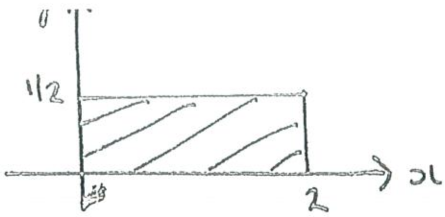
$$7\underset{\sim}{a}_1 - \underset{\sim}{a}_2 - \frac{1}{3}\underset{\sim}{a}_4 = \underset{\sim}{b}_1$$

(e) We have $[A \,|\, \underset{\sim}{b}_1] = \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & 0 & 7 \\ 0 & 2 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

With the unknowns denoted $x_1, \ldots, x_5$, there is no leading entry for $x_5$ nor $x_3$. We set $x_5 = t$, $x_3 = s$, $t, s \in \mathbb{R}$. Back substitution gives $3x_4 = -1 \Rightarrow x_4 = -1/3$, $2x_2 = -s - t - 2 \Rightarrow x_2 = -\frac{1}{2}s - \frac{1}{2}t - 1$, $x_1 = 2s + 7$. Hence the solution set is $\{(2s+7, -\frac{1}{2}s - \frac{1}{2}t - 1, s, -1/3, t) \mid t, s \in \mathbb{R}\}$

(f) The linear system corresponding to the augmented matrix $[A \,|\, \underset{\sim}{b}_2]$ is inconsistent. Hence $\underset{\sim}{b}_2$ cannot be expressed as a linear combination of the columns of A

(a) (i)




(ii) We read off that

$$T\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix},\quad T\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\1/2\end{bmatrix}$$

(iii) Because the area of the image is unchanged, and the order of the axes is unchanged.

(b) (i) Reflection in the line $y=x$.

(ii) $$S_1\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x-y\\y\end{bmatrix}=\begin{bmatrix}1&-1\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

and hence

$$A_{S_1}=\begin{bmatrix}1&-1\\0&1\end{bmatrix}$$

We recognise this as the standard matrix for a shear by $-1$ units in the $x$-direction.

(iii) $T_1$ applied 5 times is a rotation by $\pi$ anti-clockwise. The corresponding standard matrix is

$$\begin{bmatrix}\cos\pi&-\sin\pi\\\sin\pi&\cos\pi\end{bmatrix}=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}.$$

3(a) (i) $\underset{\sim}{w}_1 = \frac{1}{\sqrt{3}}(1,1,1)$ $\quad\quad$ $\underset{\sim}{w}_2 = \frac{1}{\sqrt{2}}(0,1,-1)$

With $\underset{\sim}{x} = (1,1,1)$

$$\underset{\sim}{w}_1 \cdot \underset{\sim}{x} = \frac{1}{\sqrt{3}}3 \qquad \underset{\sim}{w}_1 \cdot \underset{\sim}{w}_2 = 0$$

Hence
$$T(1,1,1) = (1,1,1)$$

(ii) $$(1,1,1)\times(0,1,-1) = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= -2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$$

Geometrically, $T$ is an orthogonal projection onto the plane spanned by $\underset{\sim}{w}_1$ and $\underset{\sim}{w}_2$. All vectors perpendicular to this plane will map to $\underset{\sim}{0}$ and thus give the kernel. The direction perpendicular to the plane is $\underset{\sim}{w}_1 \times \underset{\sim}{w}_2$ and so Ker $T$, is equal to its span.

(b) (i) $T(1,0,0) = \frac{1}{3}(1,1,1)$

$$T(0,1,0) = \frac{1}{3}(1,1,1) + \frac{1}{2}(0,1,-1) = \frac{1}{6}(2,5,-1)$$

$$T(0,0,1) = \frac{1}{3}(1,1,1) - \frac{1}{2}(0,1,-1) = \frac{1}{6}(2,-1,5)$$

Putting these vectors down the column of a matrix in order gives $A_T$

(ii) Since $A_T\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ the vector

$(-2,1,1)$ belongs to the solution set of $A_T$.

9. (a) Because there are 3 vectors, each an element of $\mathbb{R}^3$, and the 3 vectors are linearly independent.

(b) $[\underset{\sim}{x}]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \underset{\sim}{x} = 2\underset{\sim}{c}_1 + \underset{\sim}{c}_2 + \underset{\sim}{c}_3$

$= (2,0,-2) + (1,0,1) + (0,1,0)$

$= (3,1,-1)$

(c) $P_{S,C} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}, \quad P_{C,S} = P_{S,C}^{-1}$

Consider

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] R_2 + R_1$$

$R_2 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \tfrac{1}{2}R_2 \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] R_1 -$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

Hence

$$P_{C,S} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

(d) $[\underset{\sim}{x}]_{\mathcal{E}} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Hence $\underset{\sim}{x} = \frac{1}{2}\underset{\sim}{c}_1 + \frac{1}{2}\underset{\sim}{c}_2$.

(e) From (d) we read off from the first column in $P_{\mathcal{E},\mathcal{B}}$ that

$[\underset{\sim}{b}_1]_{\mathcal{E}} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Hence

$\underset{\sim}{b}_1 = -\frac{1}{2}\underset{\sim}{c}_1 + \frac{1}{2}\underset{\sim}{c}_2$

$= -\frac{1}{2}(1,0,-1) + \frac{1}{2}(1,0,1)$

$= (0,0,1)$.

0 (a) In the formula

$$u_1 v_1 + u_2 v_2 + u_3 v_3$$

we substitute

$$u_3 = (u_1 - u_2)$$
$$v_3 = (v_1 - v_2)$$

This gives

$$u_1 v_1 + u_2 v_2 + (u_1 - u_2)(v_1 - v_2)$$
$$= u_1 v_1 + u_2 v_2 + u_1 v_1 - u_1 v_2 - u_2 v_1 + u_2 v_2$$
$$= 2u_1 v_1 - u_1 v_2 - u_2 v_1 + 2u_2 v_2 .$$

which agrees with the expanded form of

$$[u_1, u_2]\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} v \\ v \end{bmatrix}$$

(b) From the construction of $\langle (u_1, u_2), (v_1, v_2) \rangle$ we have

$$\langle (u_1, u_2), (u_1, u_2) \rangle = u_1^2 + u_2^2 + u_3^2 \quad \text{with} \quad u_3 = u_1 - u_2$$

Hence $\langle (u_1, u_2), (u_1, u_2) \rangle \geq 0$

and $\langle (u_1, u_2), (u_1, u_2) \rangle = 0$ when $u_1 = u_2 = u_3 = 0$

Since $u_3 = u_1 - u_2$ this can be reduced to $u_1 = u_2 = 0$.

(c) Let the sought vector be $(x, y)$. We have

$$\langle (1,1), (x,y) \rangle = 2x - y - x + 2y = (x+y).$$

Hence for $\langle (1,1), (x,y) \rangle = 0$, require $x = -y = t \in \mathbb{R}$

so $(x,y) = (t, -t)$. For normalisation

$$\langle (t,-t), (t,-t) \rangle = 2t^2 + t^2 + t^2 + 2t^2 = 6t^2 \Rightarrow t = \frac{1}{\sqrt{6}}$$

Hence $(x,y) = \frac{1}{\sqrt{6}}(1,-1)$.

1.(a) Define $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$

The unknowns $\begin{bmatrix} a \\ b \end{bmatrix}$ in the line of best fit satisfy

$$A^T A \begin{bmatrix} a \\ b \end{bmatrix} = A^T \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Now $A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 9 & 29 \end{bmatrix}$

$$A^T \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 16 \end{bmatrix}$$

Thus

$$\begin{bmatrix} 3 & 9 \\ 9 & 29 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 16 \end{bmatrix}$$

Linear system $\left[\begin{array}{cc|c} 3 & 9 & 5 \\ 9 & 29 & 16 \end{array}\right]_{R_2 - 3R_1} \sim \left[\begin{array}{cc|c} 3 & 9 & 5 \\ 0 & 2 & 1 \end{array}\right]$

$\Rightarrow$ $b = \frac{1}{2}$

$3a + 9 \times \frac{1}{2} = 5 \Rightarrow a = \frac{1}{6}$

Thus the line of best fit is $Y = \frac{1}{6} + \frac{1}{2} X$

(b) Substituting $X = 6$ gives $Y = 3\frac{1}{6}$

To the nearest integer, 3kg are predicted as the weight loss.

2, (a) (i) Let the two vectors be denoted $\underset{\sim}{b}_1$ and $\underset{\sim}{b}_2$. By inspection

$$\underset{\sim}{b}_1 \cdot \underset{\sim}{b}_1 = 1, \quad \underset{\sim}{b}_2 \cdot \underset{\sim}{b}_2 = 1, \quad \underset{\sim}{b}_1 \cdot \underset{\sim}{b}_2 = 0$$

Hence $\mathcal{B}$ is an orthonormal set.

(ii)
$$A = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

(iii) Since $\mathcal{B}$ is an orthonormal set, $Q^{-1} = Q^T$ so we have

$$AQ = QD$$

This is the matrix form of the eigenvalue equation and tells us that

$\lambda = 1$ is an eigenvalue, normalised eigenvector $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 2$ is an eigenvalue, normalised eigenvector $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(i) The lines $y = -x$, and $y = x$.

(ii) The line $y = -x$ is left unchanged, implying eigenvalue $\lambda = 1$.
The line $y = x$ is flipped, implying eigenvalue $\lambda = -1$.