Assignment-4 (Constrained Optimization)

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1 Lagrangian Function for constrained optimization

In constraint optimization we use *Lagrangian function*, which is a scalar function defined as follows:

$$\mathcal{L}(x, \lambda) = f(x) + h(x)^T \lambda \tag{1}$$

Here, $f(x)$ is the objective function, $h(x)$ is the equality constrain functions and λ is the Lagrangian multiplier.

This function helps us convert an constrained optimization problem to a constrained optimization with more number of design variables. While solving the optimization problems numerically now we can use unconstrained optimizer with slight modifications. Also taking the gradient of the lagrangian w.r.t to the design variables and λ and setting it to zero, gives us the first-order optimality conditions for the equality constraints.

$$\begin{aligned} \nabla f(x^*) &= -J_h(x^*)^T \lambda \\ h(x^*) &= 0 \end{aligned} \tag{2}$$

1.1 4.1: Constrained Area Maximization

In this problem we will maximize the area of a rectangle constrained to a given perimeter of the rectangle and show that it happens to be a square. So we will minimize the negative of the area to maximize the area.

$$\begin{aligned} \min_{x,y} \quad & f(x, y) = -xy \\ \text{subject to:} \quad & h(x, y) = 2x + 2y - p = 0 \end{aligned} \tag{3}$$

where x, y are the sides of the rectangle and the design variables. Further p is the fixed perimeter of the rectangle.

We construct the Lagrangian as shown in the 4,

$$\mathcal{L}(x, y, \lambda) = -xy + \lambda(2x + 2y - p) \tag{4}$$

Using the first order optimality conditions we get,

$$\begin{aligned}\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial x} &= -y + \lambda(2) = 0 \\ \frac{\partial \mathcal{L}(x,y,\lambda)}{\partial y} &= -x + \lambda(2) = 0 \\ \frac{\partial \mathcal{L}(x,y,\lambda)}{\partial \lambda} &= (2x + 2y - p) = 0\end{aligned} \tag{5}$$

Using the three equations shown in 5 we get,

$$\begin{aligned}8\lambda &= p \\ \lambda &= \frac{p}{8} \\ Or, \quad x = y &= \frac{p}{4}\end{aligned} \tag{6}$$

Solving for x and y we get Eq. 6 that both x and y are equal. Hence the square has the maximum area given a perimeter.

Here, the Lagrange multiplier gives us the sensitivity of the objective with the given constraint. In this case Lagrangian multiplier represents the change in area to respect the perimeter constraint.

Now, if we solve the problem of minimizing the perimeter given the optimum area obtained above. The optimization problem becomes. 20

$$\begin{aligned}\min_{x,y} \quad & f(x,y) = 2(x+y) \\ \text{subject to:} \quad & h(x,y) = xy - \frac{p^2}{16} = 0\end{aligned} \tag{7}$$

Applying the first order optimality conditions, we get

$$\begin{aligned}\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial x} &= 2 + \lambda y = 0 \\ \frac{\partial \mathcal{L}(x,y,\lambda)}{\partial y} &= 2 + \lambda x = 0 \\ \frac{\partial \mathcal{L}(x,y,\lambda)}{\partial \lambda} &= (xy - \frac{p^2}{16}) = 0\end{aligned} \tag{8}$$

From set of equations 8, we get that $x = y = \frac{p}{4}$, hence we get the same solution as before.

2 KKT conditions

When solving the optimization problem with inequality constraint, we introduce *sack variables* $\mathbf{s}$ and lagrange multiplier for the inequality constraint σ. Lagrange multiplier is zero (which means that the constraint is inactive), or the slack variable is zero (which means that the constraint is active). Putting everything together condition for the first order of optimality gives KKT conditions.

$$\begin{aligned} \nabla f + J_h^T \lambda + J_g^T \sigma &= 0 \\ h &= 0 \\ g + s \odot s &= 0 \\ \sigma \odot s &= 0 \\ \sigma &\geq 0 \end{aligned} \tag{9}$$

2.1 4.2 Cantilevered Beam with H -section

We will assume the following substitution for the simplification.

$$\begin{aligned} u &= 2b, \\ v &= h, \\ c1 &= \frac{Plh}{2}, \\ c2 &= \sigma_{yield}, \\ c3 &= \frac{1.5P}{h}, \\ c4 &= \tau_{yield} \end{aligned} \tag{10}$$

The optimization problem for minimizing the area of a cantilevered beam, is posed as shown below

$$\begin{aligned} I = pt_w + qt_b^2 + rt_b \\ \min_{t_b, t_w} \quad & f(tb, tw) = ut_b + vt_w \\ \text{subject to:} \quad & g1 = \frac{c1}{I} - c1 \leq 0 \\ & g2 = \frac{c3}{t_w} - c4 \leq 0 \end{aligned} \tag{11}$$

The corresponding values for the constants are as follows:

$$\begin{aligned} u = 250mm, & v = 250mm, \\ c1 = 12500, & c2 = 200, \\ c3 = 600, c4 = 116, & p = \frac{250^3}{12}, \\ q = \frac{125}{6}, & r = \frac{250^2 125}{2} \end{aligned} \tag{12}$$

Now we construct the lagrangian for the inequality constraint as shown below:

$$\mathcal{L}(t_b, t_w, \sigma_1, \sigma_2, s_1, s_2) = ut_b + vt_w + \sigma_1(\frac{c1}{I} - c2) + \sigma_2(\frac{c3}{t_w} - c4) \tag{13}$$

Assumption	Meaning	t_b	t_w	σ_1	σ_2	s_1	s_2	point
$s_1 = 0$ $s_2 = 0$	g1 active, g2 active	-1.724	5.172	5.82e-05	166.670	0	0	x^*
$\sigma_1 = 0$ $\sigma_2 = 0$	g1 inactive, g2 inactive	-	-	-	-	-	-	-
$s_1 = 0$ $\sigma_2 = 0$	g1 active, g2 inactive	imaginary	-	-	-	-	-	-
$\sigma_1 = 0$ $s_2 = 0$	g1 inactive, g2 active	-	-	-	-	-	-	-

Using KKT conditions analytically we get.

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial t_b} = u - \frac{c1\sigma_1(3qt_b^2 + r)}{(qt_b^3 + rt_b + pt_w)^2} = 0 \\
\frac{\partial \mathcal{L}}{\partial t_w} = v - \frac{c3\sigma_2}{t_w^2} - \frac{c1\sigma_1 p}{(qt_b^3 + rt_b + pt_w)^2} = 0 \\
\frac{\partial \mathcal{L}}{\partial \sigma_1} = \frac{c1}{(qt_b^3 + rt_b + pt_w)^2} - c2 + s_1^2 = 0 \\
\frac{\partial \mathcal{L}}{\partial t_b} = s_2^2 - c4 + \frac{c3}{t_w} = 0 \\
\frac{\partial \mathcal{L}}{\partial s_1} = 2\sigma_1 s_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial s_2} = 2\sigma_2 s_2 = 0
\end{aligned}
\tag{14}
$$

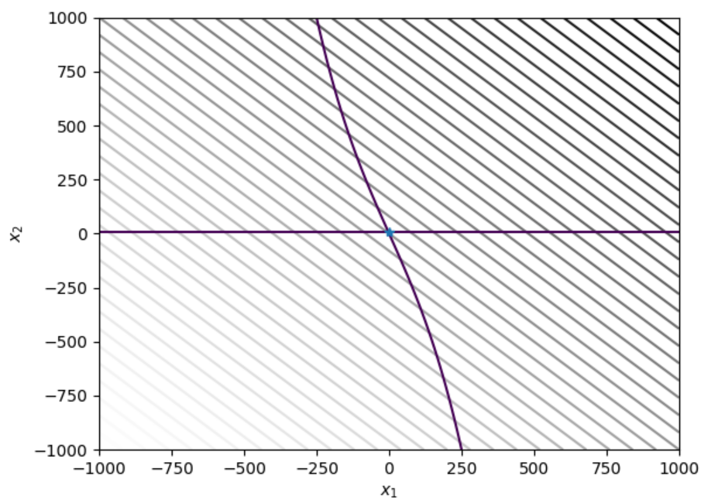

Figure 1: Plot showing the equality constraints are the objective with the optimal point which is located at the intersection of the two constraints

For the case σ_1 and $\sigma_2 = 0$, we get $u = 0$ and $v = 0$ so this solution is not possible. For the case $s_1 = 0$ and $\sigma_2 = 0$ we obtain imaginary solutions for t_b and t_w . In the final case where $g1$ and $g2$ are inactive and active respectively we get $u = 0$ which has no solutions.

3 Penalty Methods

3.1 4.3 Using interior and Exterior penalty method on example problem

The to be solve is formulated as shown below:

$$\begin{aligned} \min_{x,y} \quad & f(x1, x2) = x1 + 2x2 \\ \text{subject to:} \quad & g(x, y) = \frac{x1^2}{4} - x_2^2 - 1 = 0 \end{aligned} \tag{15}$$

While using exterior penalty method the penalty is applied as shown below:

The gradient inside the feasible region is [1,2] The function inside the feasible region is $x_1 + 2x_2$

The gradient outside the feasible region is

$$[1 + \mu * (0.5 * x_1) * (0.25 * x_1^2 + x_2^2 - 1), 2 + \mu * (2 * x_2) * (0.25 * x_1^2 + x_2^2 - 1)] \tag{16}$$

The function outside the feasible region is $x_1 + 2x_2 + \mu * 0.5 * (\frac{1}{4}x_1^2 - x_2^2 + 1)^2$

While using interior penalty method the penalty is applied as shown below:

The gradient inside the feasible region is

$$[1 - \mu * (-0.5 * x_1) * (1/(-0.25 * x_1^2 - x_2^2 + 1)), 2 - \mu * (-2 * x_2) * (1/(-0.25 * x_1^2 - x_2^2 + 1))] \tag{17}$$

The function inside the feasible region is

$$x_1 + 2 * x_2 - \mu * \log(-0.25 * x_1^2 - x_2^2 + 1) \tag{18}$$

The gradient outside the feasible region is

[1,2]

The function outside the feasible region is :

$$x_1 + 2 * x_2 \tag{19}$$

The result of Exterior point is shown below:

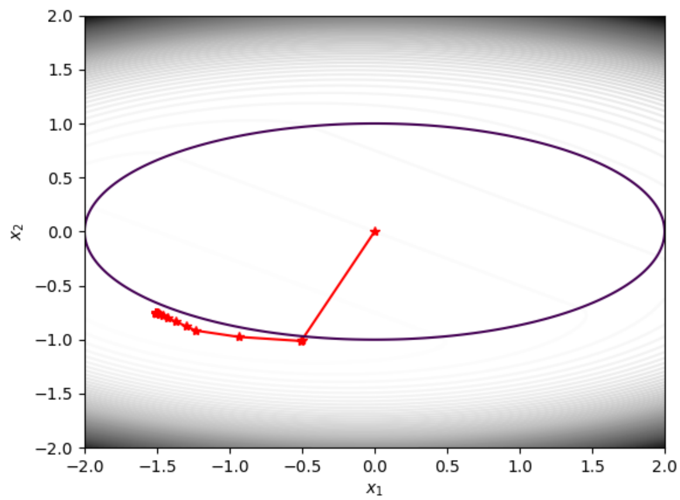

Figure 2: The path of convergence of the optimizer starting at (0,0)

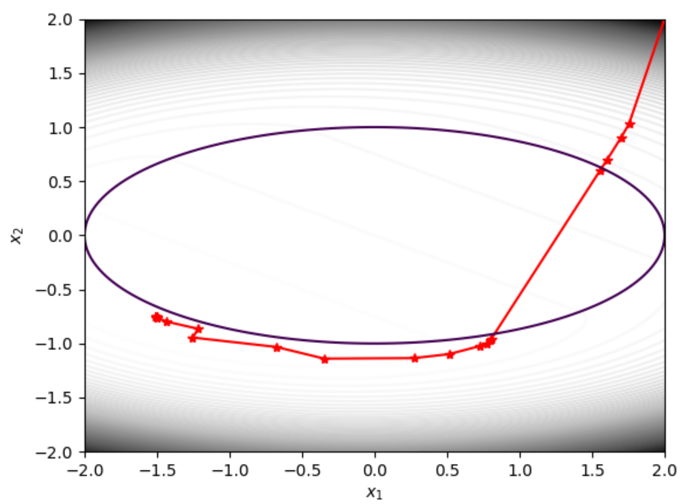

Figure 3: The path of convergence of the optimizer starting at (2,2)

starting point	optimal	closeness (L2)	feasibility
0,0	-1.505,-0.75	4.620290109	infeasible
2,2	-1.505, -0.7526	4.620290112	infeasible
3,1	,-1.505, -0.7526	4.62029	infeasible

Table 1: Feasibility and Closeness using L_2 norm for the exterior penalty methods

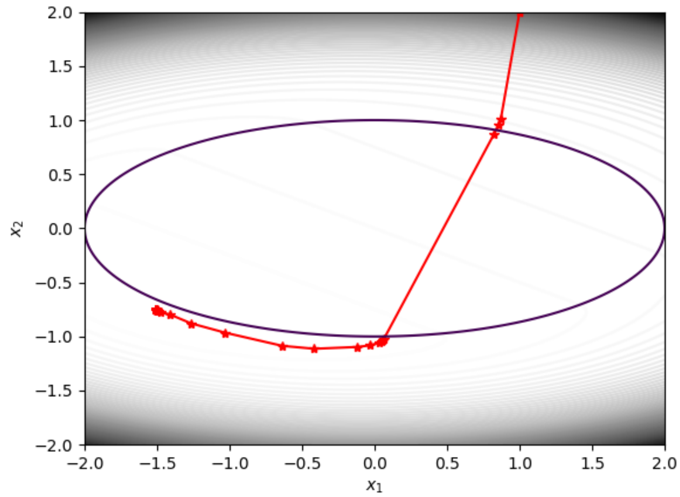

Figure 4: The path of convergence of the optimizer starting at (1,2)

The results for Interior penalty is shown below:

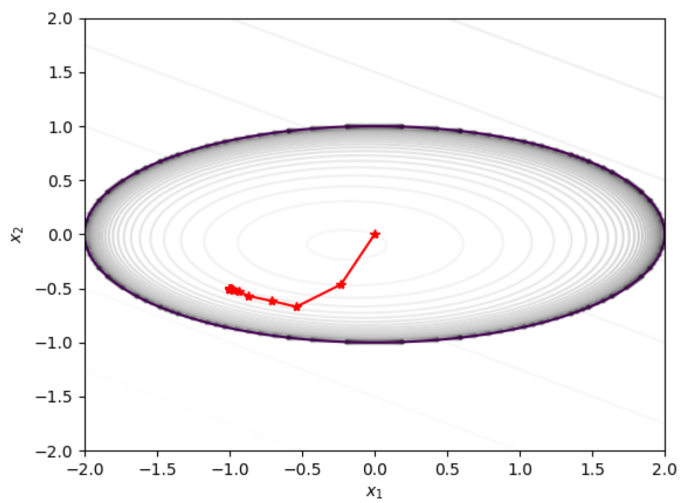

Figure 5: The path of convergence of the optimizer starting at (0,0)

starting point	optimal	closeness (L2)	feasibility
0,0	-0.9999999762171207, -0.500	4.111	feasible
1,0	-0.9999999775054349, -0.50000	4.111	feasible
-9.0,0.5	-1.0000000251072345, -0.5	4.111	feasible

Table 2: Feasibility and Closeness using L_2 norm for the interior penalty methods

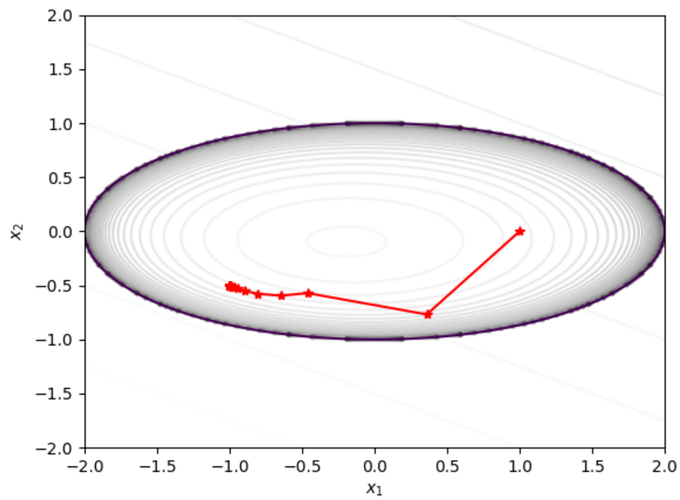

Figure 6: The path of convergence of the optimizer starting at (2,2)

3.2 4.3 Using exterior penalty method on beam problem

Results for Beam Problem with intial point [100,500]

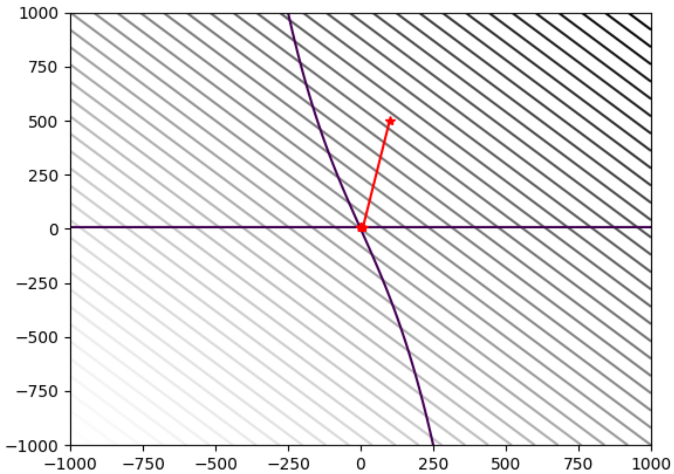

Figure 7: The path of convergence of the optimizer starting for beam using exterior penalty methods

4 SQP

Results for example problem with initial point [1,2]

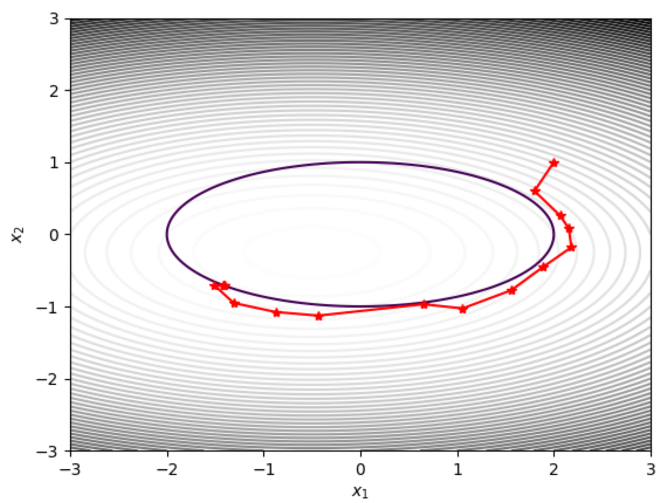

Figure 8: The path of convergence of the optimizer starting from [1,2], using SQP with bracketing line search

The merit function for the above line search is shown below:

$$x_1 + 2 * x_2 + \mu * |(0.25 * x_1^2 + x_2^2 - 1)| \tag{20}$$

The other problem that were tried with SQP is beam problem, and it robustness is displayed starting from different points.

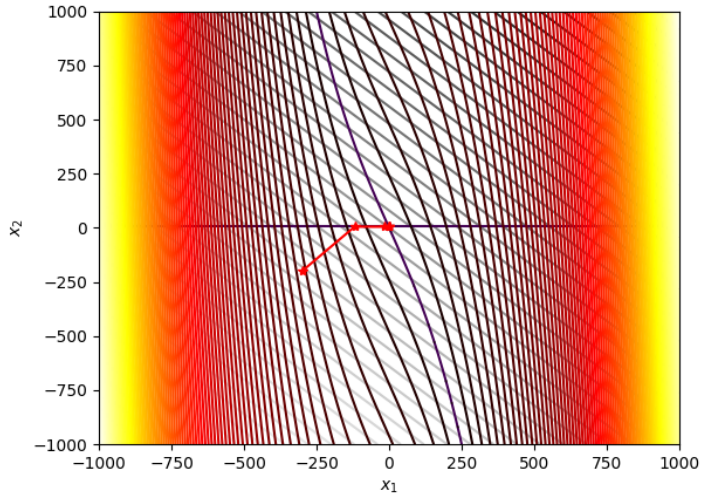

Figure 9: The path of convergence of the optimizer starting from [-300,-200], using SQP with backtracking line search for beam problem

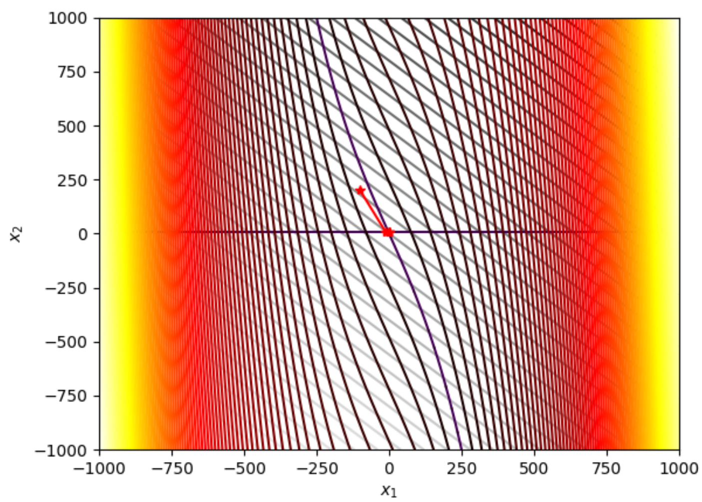

Figure 10: The path of convergence of the optimizer starting from [-100,200], using SQP with backtracking line search for beam problem

the Lagrangian in the beam problem optimization is shown in the contour as hot.

The computational efficiency of the SQP w.r.t to other available optimizers is shown below:

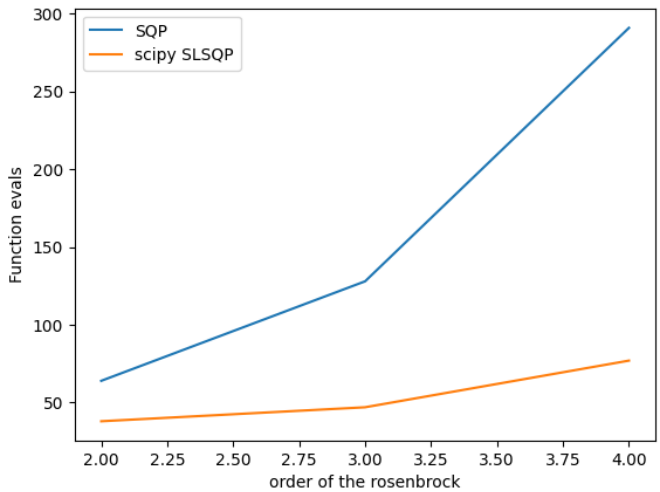

Figure 11: Computation cost vs order of rosenbrock for assessing the robustness