Solutions to Assignment 1 : CS6510 - Applied Machine Learning

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Question 1

Part 1

Since X and Y are independent, $P(X) \times P(Y) = P(X \cap Y)$. $P(\bar{X}) = 1 - P(X)$. Due to this:

$$\begin{aligned} & P(\bar{X} \cap Y) = P(Y) - P(X \cap Y) \quad \text{(from Set Theory)} \\ \implies & P(\bar{X} \cap Y) = P(Y) - P(Y) \times P(X) = P(Y)(1 - P(X)) = P(Y) \times P(\bar{X}) \end{aligned}$$

Hence $\bar{X}$ and Y are independent.

Part 2

Listing down the information from the question:

$$\begin{aligned} P(C1 = H) &= 0.5 & P(C1 = T) &= 0.5 \\ P(C2 = H|C1 = H) &= 0.7 & P(C2 = H|C1 = T) &= 0.5 \end{aligned}$$

From Bayes' Theorem: $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$. We need to find: $P(C1 = T \cap C2 = H)$.

$$\begin{aligned} & P(C1 = T \cap C2 = H) = P(C2 = H) \times P(C1 = T|C2 = H) \quad \text{(from Conditional probability)} \\ \implies & P(C1 = T \cap C2 = H) = P(C2 = H) \times \frac{P(C2 = H|C1 = T)P(C1 = T)}{P(C2 = H)} \quad \text{(from Bayes' Theorem)} \\ \implies & P(C1 = T \cap C2 = H) = P(C1 = T|C2 = H)P(C1 = T) = 0.5 \times 0.5 = \boxed{0.25} \end{aligned}$$

Question 2

Part a

A set S is a vector space if the following conditions hold:

- if $u, v \in S$, then $u + v \in S$.
- if $u \in S$, then $\alpha u \in S \quad \forall \alpha$.

$\nexists u, v \in S = \emptyset$, hence the first condition holds [`False` $\implies$ `True` is `True`]. Similarly, $\nexists u \in S = \emptyset$, hence the second condition holds for the same reason above. Hence the empty set is a vector space.

Part b and c

Let M^{-1} be of the form: $I + \alpha(\mathbf{u}\mathbf{v}^T)$. We know that: $MM^{-1} = I$. Applying this we get:

$$\begin{aligned}
MM^{-1} &= I \\
(I + \mathbf{u}\mathbf{v}^T)(I + \alpha(\mathbf{u}\mathbf{v}^T)) &= I \\
\implies I + \mathbf{u}\mathbf{v}^T(1+\alpha) + \alpha\mathbf{u}(\mathbf{v}^T\mathbf{u})\mathbf{v}^T &= I \\
\implies \mathbf{u}\mathbf{v}^T\left((1+\alpha) + \alpha(\mathbf{v}^T\mathbf{u})\right) &= \mathbf{0} \\
\implies \alpha &= \frac{-1}{1+\mathbf{v}^T\mathbf{u}}
\end{aligned}$$

Since there is an α, we can tell that our assumption is true, and hence the inverse of the matrix M is of the form $I + \alpha(\mathbf{u}\mathbf{v}^T)$.

Part d and e

If M is singular: then $\det M = 0$. We know that the eigenvalues of $M = I + \mathbf{u}\mathbf{v}^T$ are of the form: $1 + \lambda_i$, where λ_is are the eigenvalues of $\mathbf{u}\mathbf{v}^T$. We also know that the determinant of the matrix is the product of its eigenvalues.

$$\begin{aligned}
\det M = 0 &\implies \prod_{i=1}^{n}(1+\lambda_i) = 0 \\
&\implies \exists \lambda_k \text{ such that } (1+\lambda_k) = 0
\end{aligned}$$

This means that at least one of the eigenvalues of $\mathbf{u}\mathbf{v}^T$ is exactly -1. From the expression for α, we know that this definitely happens when $\mathbf{v}^T\mathbf{u} = -1$, in which case, α is undefined.

The Null space of M is defined as: $M\mathbf{x} = \mathbf{0}$. This means that:

$$\begin{aligned}
(I + \mathbf{u}\mathbf{v}^T)\mathbf{x} &= \mathbf{0} \\
\mathbf{x} + \mathbf{u}\mathbf{v}^T\mathbf{x} &= \mathbf{0}
\end{aligned}$$

Recall that if M is singular, then -1 is an eigenvalue of $\mathbf{u}\mathbf{v}^T$ and hence: $\mathbf{u}\mathbf{v}^T\mathbf{x} = -\mathbf{x}$.
Resuming from where we left off: consider $\mathbf{x} = \mathbf{u}$. $\mathbf{u} - \mathbf{u} = 0$ (since $\mathbf{v}^T\mathbf{u} = -1$).

Question 3

Part a

$$A = \begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix}$$

The eigenvalues of A are given by the roots to the equation $\det(A - \lambda I) = 0$. We get:
$\lambda^2 - 3\lambda + 2 = 0 \implies \lambda = 1, 2$. The eigenvectors are solutions to the equation:

$$A\mathbf{x} = \lambda_i \mathbf{x}$$

For $\lambda = 1$:

$$\begin{aligned}
A\mathbf{x} = \mathbf{x} &\implies \begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \\
\mathbf{x} &= a\begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} \quad \forall a
\end{aligned}$$

For $\lambda = 2$:

$$A\mathbf{x} = 2\mathbf{x} \implies \begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$$
$$\mathbf{x} = a \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \forall a$$

Part b

Let $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$. Solving the system:

$$AU = U\Lambda$$
$$\implies u_3 = 3\frac{u_1}{2} \text{ and } u_4 = 2u_2$$

U is the matrix of the eigenvectors of A. For simplicity, consider: $u_1 = 2\gamma, u_2 = \zeta \implies u_3 = 3\gamma, u_4 = 2\zeta$. Hence:

$$U = \begin{bmatrix} 2\gamma & \zeta \\ 3\gamma & 2\zeta \end{bmatrix}$$

Part c

For an invertible 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse is:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This can be found using the transpose of the co-factor matrix (adjoint). In our case:

$$U^{-1} = \frac{1}{\gamma\zeta} \begin{bmatrix} \zeta & -\zeta \\ -3\gamma & 2\gamma \end{bmatrix}$$
$$U\Lambda U^{-1} = \begin{bmatrix} 2\gamma & 2\zeta \\ 3\gamma & 4\zeta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2\zeta & -\zeta \\ -3\gamma & 2\gamma \end{bmatrix} \frac{1}{\gamma\zeta}$$
$$\implies U\Lambda U^{-1} = \begin{bmatrix} 2\gamma & 2\zeta \\ 3\gamma & 4\zeta \end{bmatrix} \begin{bmatrix} 2\zeta & -\zeta \\ -3\gamma & 2\gamma \end{bmatrix} \frac{1}{\gamma\zeta} = \frac{1}{\gamma\zeta} \begin{bmatrix} -2\gamma\zeta & 2\gamma\zeta \\ -6\gamma\zeta & 5\gamma\zeta \end{bmatrix}$$
$$\implies U\Lambda U^{-1} = \begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix} = A$$

Verified.