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Study of Black-Scholes Model and its Applications

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Abstract

The aim of this paper is to study the Black-Scholes option pricing model. We discuss some definitions and different derivations, which are useful for further development of Black-Scholes formula and Black-Scholes partial differential equation. As an application, we obtain the solution of the Black-Scholes equation and it is represented graphically by Maple software.

Keywords: Option price, Black-Scholes, European call option, Volatility, Maple.

1. Introduction

Finance is one of the most rapidly changing and fastest growing areas in the corporate business world. Financial options are widely used in the field of finance. During last decades, the valuation of option contracts has been topic of active research. There are various type of mathematical models for pricing different kind of options. In the year 1955, Paul Samuelson, wrote an unpublished paper entitled "Brownian Motion in the Stock Market". During the same year, Richard Kruizenga, cited Bachelier's work in his dissertation entitled "Put and Call Options: A Theoretical and Market Analysis" [5]. The research picked up in the 1960′s. Furthermore, many mathematician such as Sprenkle (1961), Ayes (1963), A. James Boness (1964), Samuelson (1965), Baumol, Malkiel and Quandt (1966) and Chen (1970) etc. work on the valuation of options has been expressed in terms of warrants (a warrant is an option is a liability of a corporation) and all produced valuation formulas of the same general form but their formulas were not complete [3]. In the year 1973, Fischer Black and Myron Scholes develop the original option pricing formula and it is published in the paper entitled, "The Pricing of Options and Corporate Liabilities", in the journal of Political Economy [3]. In the same year Black and Scholes transformed the option pricing problem into the new partial differential equation (PDE) with variable coefficients. The main conceptual idea of Black and Scholes lies in the construction of a riskless portfolio taking positions in bonds (cash), option, and the underlying stock. The Black-Scholes equation governs the price of the option over time. Now a days, the Black-Scholes partial differential equation is used in financial engineering because of its simplicity and clarity to obtain

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the price option calls. In our survey study, we discuss some basic definitions and derivations which are very useful in further development of Black-Scholes model. Also, we use the Maple software to obtain the solution of Black-Scholes equation for varying the parameters and these are represented graphically.

We organize this paper as follows: The section 2, is devoted for some definitions and derivations which are useful for the development of Black-Scholes model. In section 3, we derive the Black-Scholes formula and the Black-Scholes partial differential equation. We obtain the solution of Black-Scholes equation for valuing an option using Maple software in the last section.

In the next section we discuss some definitions which are useful in financial mathematics.

2. Definitions in Financial Mathematics

2.1: Option

A security giving the right to buy or sell an asset, subject to certain conditions, within a specified period of time is called as an option [8].

There are two types of options which are defined as follows:

(i) Call Option: An option which grants its holder the right to buy the underlying asset at a strike price at some moment in the future is called as call option.

(ii) Put Option: An option which grants its holder the right to sell the underlying asset at a strike price at some moment in the future is called as put option.

2.2: Expiration Date

The date on which an option right or warrant expires, and becomes worthless if not exercised is called an expiration date. There are two different types of options with respect to expiration:

(i) European Option: An option which cannot be exercised until the expiration date is called an European option.

(ii) American Option: An option which can be exercised at any time up to and including the expiration date is called as an American option [8].

2.3: Risk-Less Interest Rate

The annual interest rate of bonds or other ”risk-free” investments, is called as the risk-less interest rate. It is denoted by r [8].

2.4: Volatility

A measure for variation of price of a financial instrument over time is called volatility [1].

There are two important types of volatility as follows:

(i) Implied Volatility: Volatility derived from the market price of a market traded derivative is called implied volatility [8].

(ii) Historic Volatility: Volatility derived from time series of past market prices is called historic volatility.

2.5: Strike Price

The predetermined price of an underlying asset is called as strike price [8].

2.6: Hedge

A transaction that reduces the risk of an investment is called as hedge [8].

2.7: Portfolio

Any collection of financial assets such as stocks, bonds and cash equivalents held by an investment institution or company is called portfolio [8].

2.8: Geometric Brownian Motion

A continuous time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion is called geometric Brownian motion [6, 8].

2.9: Black-Scholes Model

A mathematical formula designed to price an option as a function of certain variables generally stock price, striking price, volatility, time to expiration, dividends to be paid and the current risk-free interest rate for pricing European Options on stocks, developed by Fisher Black, Myron Scholes and Robert Merton [3].

2.10: Stochastic Differential Equation

Let (Ω, F, P) be a probability space and let X_t, t ε R_+ be a stochastic process $X : \Omega \times R_t \rightarrow R$. Moreover, assume that $a(X_t, t) : \Omega \times R \times R_+ \rightarrow R$ and $b(X_t, t) : \Omega \times R \times R_+ \rightarrow R$ are stochastically integrable functions of t ε R_+. Then the equation

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t \tag{1}$$

is called stochastic differential equation.

Note that (1) has to be understood as a symbolic notation of the stochastic integral equation

$$X_t = X_0 + \int_0^t a(X_s, s)ds + \int_0^t b(X_s, s)dW_s \tag{2}$$

The functions $a(X_t, t)$ and $b(X_t, t)$ are referred to as the drift term and the diffusion term respectively.

2.11: Itoˆ Process

A stochastic process X_t satisfying equation

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t$$

is called a stochastic differential equation.

Note that (1) has to be understood as a symbolic notation of the stochastic integral equation

$$X_t = X_0 + \int_0^t a(X_s, s)ds + \int_0^t b(X_s, s)dW_s$$

The functions $a(X_t, t)$ and $b(X_t, t)$ are referred to as the drift term and the diffusion term respectively.

2.11: Itoˆ Process

A stochastic process X_t satisfying equation

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t$$

is said to be an Itoˆ process.

2.12: Itoˆ Integral

Assume that $b = b(t)$ is a stochastically integrable function in the sense that there exists a sequence b_n, n ε N of simple processes such that

$$\lim_{n\to\infty} E\left(\int_0^T (b(t) - b_n(t))^2 dt\right) = 0$$

Then, the Itoˆ integral of b is defined as

$$\int_0^T b(t)dW_t = \lim_{n\to\infty} \int_0^T b_n(t)dW_t$$

2.13: Ito'sˆ Lemma

Let X_t, t ε R_+, be an Itoˆ process $X : \Omega \times R_+ \rightarrow R$ and $f := C^2(R \times R_+ \times R_+)$. Then, the stochastic process $f_t := f(X_t, t)$ is also an Itoˆ process which satisfies

$$df_t = \left(\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial t^2}\right)dt + \frac{\partial f}{\partial x}dW_t \tag{3}$$

Proof: By Taylor's series the expansion of **f** ($X_{t+\Delta t}$, $t + \Delta t$) about (X_t, t) is given as follows

$$f(X_{t+\Delta t,}t+\Delta t) = f(X_t,t) + \frac{\partial f}{\partial t}(X_t,t)(\Delta t) + \frac{\partial f}{\partial x}(X_t,t)(X_{t+\Delta t} - X_t) + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(X_t,t)(\Delta t)^2$$
$$+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X_{t,}t)(X_{t+\Delta t},t - X_t)^2 + \frac{\partial^2 f}{\partial x \partial t}(X_t,t)(\Delta t)(X_{t+\Delta t} - X_t) + O(\Delta t)^2$$
$$+O(\Delta t)(X_{t+\Delta t} - X_t)^2 + O((X_{t+\Delta t} - X_t)^3)$$

Taking, limit as $\Delta t \rightarrow 0$ gives

$$df_t = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dX_t^{\,2} + O((dt)^2) + O(dt(dX_t)^2) + O((dX_t)^3) \tag{4}$$

Consider X_t is an Ito'sˆ process and $dW_t^2 = dt$, then from equation (4), we get

$$dX_t^2 = (adt + bdW_t)^2 = a^2(dt)^2 + 2abdtdW_t + b^2 dW_t^2$$
$$= b^2 dt + O((dt)^{3/2}) \tag{5}$$

From equations (4) and (5), we get

$$df_t = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(adt + bdW_t) + \frac{1}{2}b^2\frac{\partial^2 f}{\partial x^2}dt$$

$$df_t = \left(\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial x^2}\right)dt + b\frac{\partial f}{\partial x}dW_t \tag{6}$$

where W_t is a Wiener process.

3. Black-Scholes Mathematical Model

In this section first we discuss some basic assumptions underlying the Black-Scholes model of calculating options pricing. The most significant is that volatility, a measure of how much a stock can be expected to move in the near-term, is a constant over time. The Black-Scholes model also assumes stocks move in a manner referred to as an random walk; at any given moment, they are as likely to move up as they are to move down. These assumptions are combined with the principle that options pricing should provide no immediate gain to either seller or buyer. The exact six assumptions of the Black-Scholes models are as follows:

(i) stock pays no dividends, (ii) option can only be exercised upon expiration, (iii) market direction cannot be predicted, hence "Random Walk", (iv) no commissions are charged in the transaction, (v) interest rates remain constant (vi) stock returns are normally distributed, thus volatility is constant over time.

(i) Black-Scholes Option Pricing Formula (1973)

The Black-Scholes option pricing formula is developed in the year 1973, prices the European put or call options on a stock that does not pay a dividend or make other distributions. The formula assumes the underlying stock price follows a geometric Brownian motion with constant volatility. It is historically significant as the original option pricing formula introduced in their landmark option pricing model [2].

Black-Scholes state the formula for a call price C and put price P as follows:

$$C_{call} = S\phi(d_1) - Xe^{-rT}\phi(d_2) \tag{7}$$

$$P_{put} = Xe^{-rT}\phi(-d_2) - S\phi(-d_1) \tag{8}$$

where $d_1 = \dfrac{\log(S/X) + (r + \sigma^2/2)}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}$

C_{call} – the price for an option call, P_{put} - the price for an option put, S - the current option price of the stock, X - the strike price of an underlying stock, r - the annualized risk-free interest rate, continuously compounded, T - the time in year until the expiration of the option, σ - implied volatility of an underlying stock, Φ(.) - the standard normal cumulative distribution function.

Here, Φ(d) denotes the standard normal cumulative distribution function and it is defined as follows:

$$\phi(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-x^2/2}dx \tag{9}$$

Note that log denotes the natural logarithm.

(ii) Black-Scholes Option Pricing Model Using Stochastic Differential Equation

We obtain the derivation of the Black-Scholes partial differential equation using the stochastic differential equation and Ito'sˆ Lemma. We first consider the price process for assets. It is obtain by a stochastic differential equation based on a geometric Brownian motion [8]. The change or price difference in the asset prices is assumed to be a Markov process. The return is, the change in the price divided by its original value. It is well discussed in the Samuelson model [8] and there are two contributions to the return which are deterministic contribution and Stochastic contribution. If ν is the average rate of growth of the asset price, then the deterministic contribution in time dt is νdt. If σ is the volatility related to the standard deviation of the returns and dX a sample from a normal distribution, then the contribution is assumed to be σdX. Therefore, the resulting equation can be written in the following form:

$$\frac{dS}{S} = \nu dt + \sigma dX \tag{10}$$

The equation (10) is called the stochastic differential equation. The normal distribution used in (10) is a Wiener process with the following properties:

$$(i)\ E(dX) = 0,$$

$$(ii)\ E(dX)^2 = dt$$

Now, σ is proportional to $Var(dS)$, the expectation and variance are calculated as follows:

$$E(dS) = E(\nu S\,dt + \sigma S\,dX) = E(\nu S\,dt) + E(\sigma S\,dX) = \nu S\,dt\ Var(dS)$$

$$= E(dS^2) - [E(dS)]^2$$

$$= E[(\nu S\,dt + \sigma S\,dX)^2] - (\nu S\,dt)^2$$

$$= \sigma^2 S^2 dX^2$$

Since $E(S^2 dXdt) = 0$, the standard deviation is the square-root of the variance.

Therefore, σ is proportional to

$$\frac{\sqrt{Var(ds)}}{S}$$

This is the Stochastic model.

Black-Scholes Partial Differential Equation

Consider a general option value $V(S, t)$. Therefore, from Taylor's theorem we have the following series expansion for $V(S, t)$:

$$\delta V = V_s \delta S + V_t \delta t + \frac{1}{2!} V_{ss} \delta S^2 + \frac{1}{2!} 2 V_{st} \delta S \delta t + \frac{1}{2!} V_{tt} \delta t^2 + \dots \tag{11}$$

We substitute $dS = vS\,dt + \sigma S\,dX$ in its discrete form, that is $\delta S = vS\,\delta t + \sigma S\,\delta X$ in equation (11), we get

$$\delta V = V_s (vS\delta t + \sigma S \delta X) + V_t \delta t + \frac{1}{2!} V_{ss} (vS\delta t + \sigma S\delta X)^2 + \frac{1}{2!} 2V_{st} (vS\delta t + \sigma S\delta X)\delta t + \frac{1}{2!} V_{tt} \delta t^2 + \dots \tag{12}$$

After, cancelling all insignificant terms in equation (12), we get

$$\delta V \approx V_s (vS\delta t + \sigma S\delta X) + V_t \delta t + \frac{1}{2!} V_{ss} (vS\delta t + \sigma S\delta X)^2 \tag{13}$$

Therefore, by taking the limits $\delta S \to 0$, $\delta X^2 \to \delta t$ as $\delta t \to 0$ the above equation can be written in the following form:

$$dV = V_s (vSdt + \sigma SdX) + V_t dt + \frac{1}{2!} V_{ss} (vSdt + \sigma SdX)^2 \tag{14}$$

From equation (10), we have

$$dS^2 = (vSdt + \sigma SdX)^2 = (v^2 S^2 dt^2 + 2\sigma v S^2 dXdt + \sigma^2 S^2 dX^2) \tag{15}$$

Therefore, by applying Ito'sˆ Lemma 2.13 and assuming that $dX^2 \to dt$ as $dt \to 0$, then from equation (15), we get

$$dS^2 \to \sigma^2 S^2 dt \tag{16}$$

Putting equation (16) into equation (14), we get

$$dV = \frac{\partial V}{\partial S}(vSdt + \sigma SdX) + \frac{\partial V}{\partial t} dt + \frac{1}{2!} \frac{\partial^2 V}{\partial S^2} (\sigma^2 S^2 dt) \tag{17}$$

Now, rearranging the terms in above equation, we get

$$dV = \sigma S \frac{\partial V}{\partial S} dX + (vS \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}) dt \tag{18}$$

This is the random walk process for $V(S, t)$. By setting up a portfolio consisting of one option with value $V(S, t)$ and a number $-\Delta$ of the underlying asset. The value of this portfolio will be

$$\Pi = V - \Delta S \tag{19}$$

Therefore, the change in the portfolio is

$$d\Pi = dV - \Delta dS \tag{20}$$

Now, combining equations (10), (18) and (20), we get

$$d\Pi = \sigma S\left(\frac{\partial V}{\partial S} - \Delta\right)dX + \left(\nu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \nu\Delta S\right)dt \tag{21}$$

To eliminate the main contribution of randomness, we choose

$$\Delta = \frac{\partial V}{\partial S} \tag{22}$$

Now, the Δ is chosen in equation (22) such that the portfolio (19) will be deterministic i.e. it is instantaneously risk free. The change in an instantaneously risk free portfolio should equal to the exponential growth of placing money in the bank.
Therefore, putting the value of Δ in equation (21) and after simplification, we get

$$d\Pi = r\Pi\, dt = \left(\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)dt \tag{23}$$

Finally, after substituting the value of Π from equation (20) into equation (23) and dividing it by dt, we obtain the equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{24}$$

This is the partial differential equation with variable coefficients, is called the Black-Scholes equation for valuing an option with value $V(S, t)$. This is second order partial differential equation in S - space and first order in time. Thus from its extensions and variants, it plays the major role in the option pricing theory. We have derived the Black-Scholes equation for the value of an option, we must consider initial and boundary conditions. However, option problems have final conditions. The boundary and the final conditions make the difference between American and European style as well as between put and call and other types of options.

The initial condition at time $t = 0$ can be derived from the definition of call option. If at expiration $S > K$ the call option will be worth $S - K$ because the buyer of the option can buy the stock for K and immediately sell it for S . If at expiration $S < K$ the option will not be exercised and it will expire worthless. At $t = 0$ the value of the option is known for certain to be the payoff, it is mathematically expressed as follows:

$$V(S, 0) = max\ (S - K, 0) \tag{25}$$

This is the initial condition of our differential equation.
In order to find boundary conditions we consider the value of V when $S = 0$ and as $S \to \infty$: If $S = 0$ then it is easy to see from stochastic differential equation that $dS = 0$ and therefore, S will never change. If $S = 0$ then from equation (24) the payo must be 0. Consequently, when $S = 0$, we have, the following boundary condition:

$$V(0, t) = 0 \tag{26}$$

Now when $S \to \infty$ it become more and more likely, the option will be exercised and the payoff will be $S - K$. The exercise price becomes less and less important as $S \to \infty$, so the value of the option is equivalent to $V(S, t)$ to S . Therefore, we have the following right boundary condition:

$$V(S, t) = S \text{ as } S \to \infty \tag{27}$$

Now, combining the equations (24) and (25) − (27), we have the following general Black-Scholes initial boundary value problem (BS-IBVP):

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0, \quad (S, t) \in R_+ \times (0, T) \tag{28}$$

$$\text{initial condition: } V(S, 0) = max(S - K, 0), \; S \in R_+ \tag{29}$$

$$\text{boundary conditions: } V(0, t) = 0, \; V(S, t) = S \text{ as } S \to \infty, \; t \in [0, T] \tag{30}$$

where $V(S, t)$ − the price for an option

S − the current option price of the stock

K − the strike price of the option

r − the annualized risk-free interest rate, continuously compounded

t − the time in year generally use now t = 0, at expiry t = T

σ − volatility of an underlying asset.

In particular, we must consider final and boundary conditions in case of European Call $C(S, t)$ with exercised price E ($K = E$) and expiry date T . The final condition at time $t = T$ can be derived from the definition of call option. If at expiration $S > E$ the call option will be worth $S - E$ because the buyer of the option can buy the stock for E and immediately sell it for S. If at expiration $S < E$ the option will not be exercised and it will expire worthless. At $t = T$ the value of the option is known for certain to be the payoff, is called initial condition and it is mathematically expressed as follows:

$$C(S, T) = max(S - E, 0) \tag{31}$$

This is the final condition of our differential equation. In order to find boundary conditions we consider the value of V when S = 0 and as $S \to \infty$: If S = 0 then it is easy to see from stochastic differential equation that dS = 0 and therefore, S will never change. If S = 0 then the value of the call option equals 0. Consequently, when S = 0, we have the following boundary condition:

$$C(0, t) = 0 \tag{32}$$

Now for $S \to \infty$ it become more and more likely, the option will be exercised and the payo will be $S - E$. The exercise price becomes less and less important as $S \to \infty$, so the value of the option is equivalent to $C(S, t) = S - Ee^{-r(T-t)}$. This is the right boundary condition. Therefore, we have the following boundary condition:

$$C(S, t) = S - Ee^{-r(T-t)} \text{ as } S \to \infty \tag{33}$$

Now, combining the equations (24) and (31) − (33), we have the following Black-Scholes European call option initial (final) boundary value problem (BSECOP-FBVP):

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0, \quad (S, t) \in R_+ \times (0, T) \tag{34}$$

$$\text{initial condition: } C(S, 0) = max(S - E, 0), \; S \in R_+ \tag{35}$$

$$\text{boundary conditions: } C(0, t) = 0, \, , C(S, T) = S - Ee^{-r(T-t)}, \text{ as } S \to \infty, \; t \in [0, T] \tag{36}$$

4. Applications

As an application of Black-Scholes model, we obtain the solution of Black-Scholes equation using the Maple software. In our test problem we chosen the parameters as follows:

$$E = 50,\ T = 0.5,\ r = 0.10,\ \sigma = 0.40$$

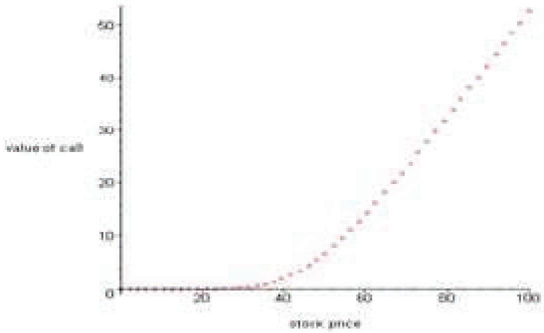

Figure 4.1: The solution for a European call with parameters:

$$E = 50,\ T = 0.5,\ r = 0.10,\ \sigma = 0.40$$

Let us examine how this result changes by changing the parameters.

(i) Increasing the risk-free interest rate

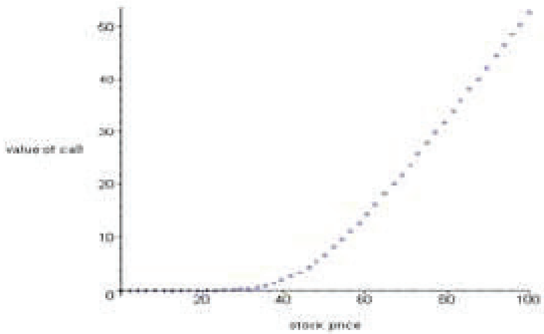

Figure 4.2: The solution for a European call with parameters:

$$E = 50,\ T = 0.5,\ r = 0.11,\ \sigma = 0.40$$

(ii) Increasing the stock volatility

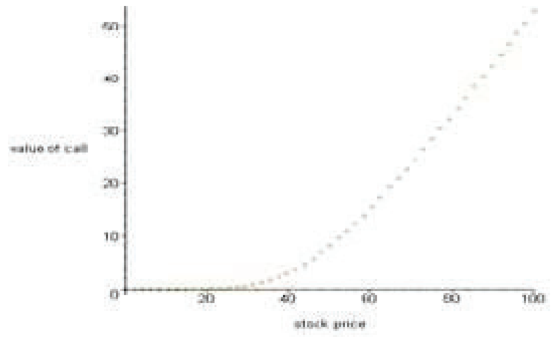

Figure 4.3: The solution for a European call with parameters:

$$E = 50,\ T = 0.5,\ r = 0.10,\ \sigma = 0.50$$

Conclusions

(i) We study the Black-Scholes formula and the Black-Scholes partial differential equation.
(ii) The Black -Scholes partial differential equation is very much useful in the financial engineering.
(iii) We obtain the call option values of an underlying asset by Black-Scholes equation and these are simulated by Maple software.
(iv) The call option price change by varying the parameters.

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