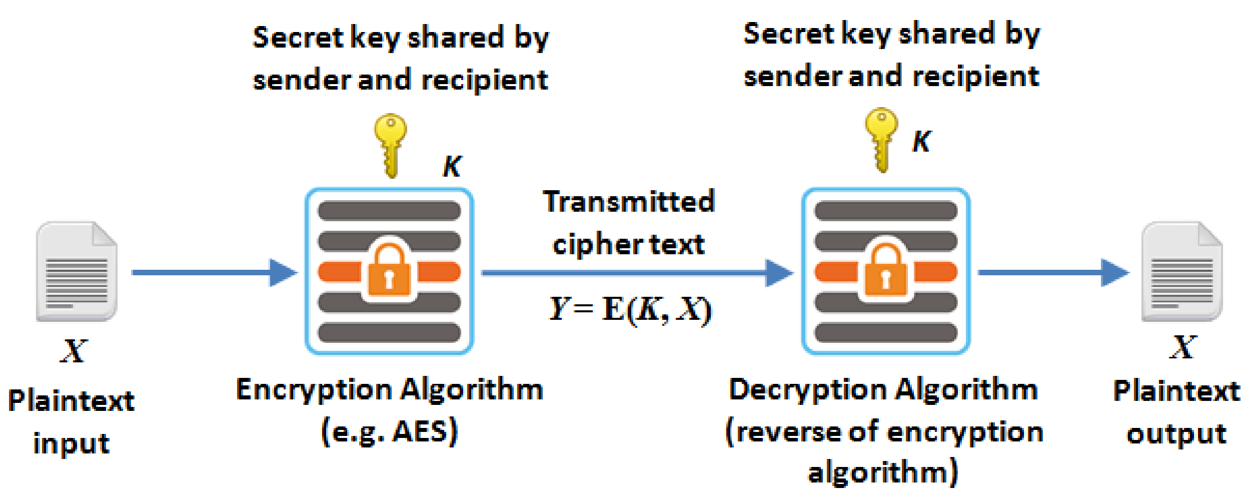
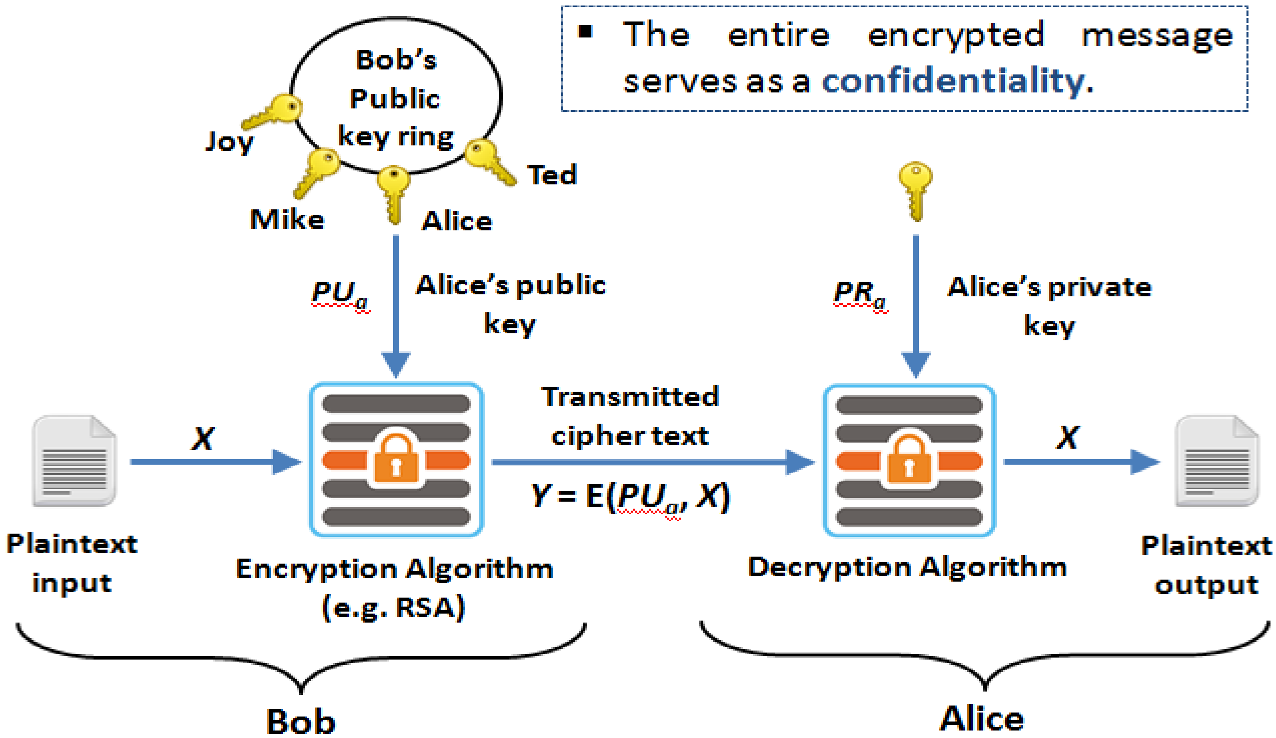
Outline

- Public Key Cryptosystems with Applications
- Requirements and Cryptanalysis
- RSA algorithm
- RSA computational aspects and security
- Diffie-Hillman Key Exchange algorithm
- Man-in-Middle attack

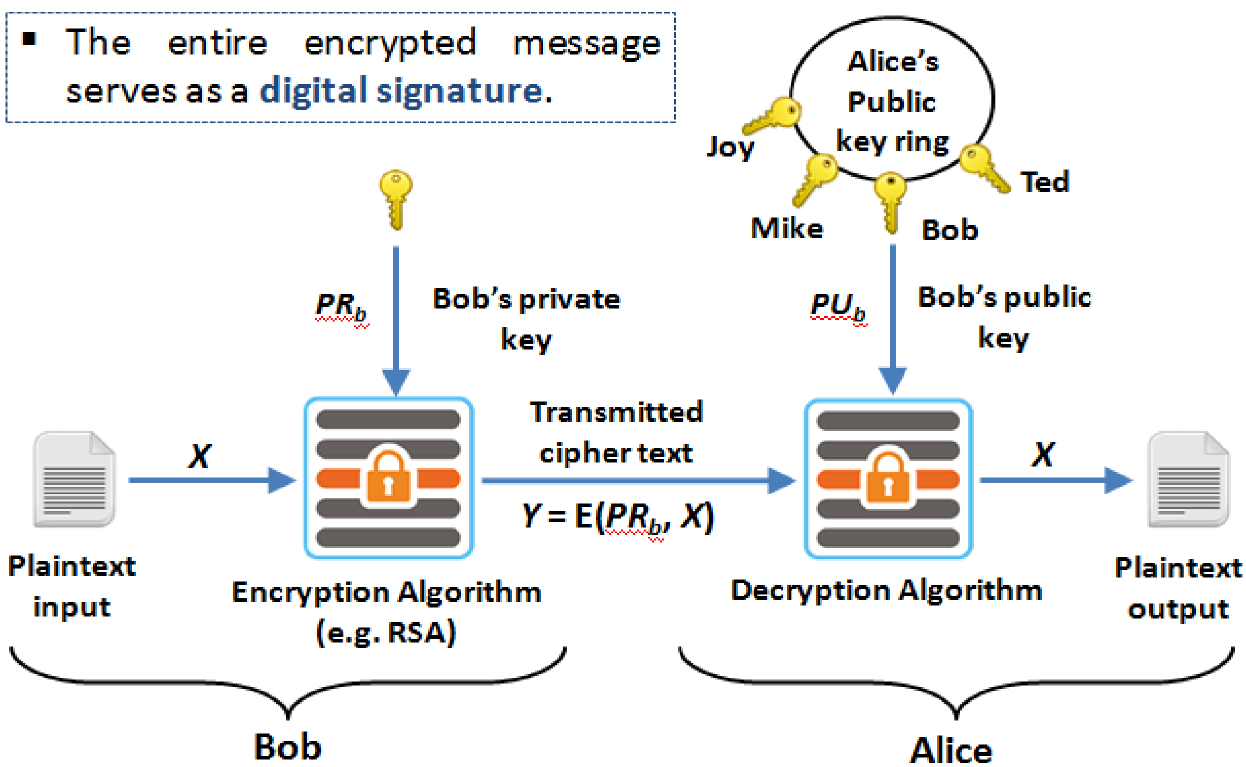
Symmetric key Encryption



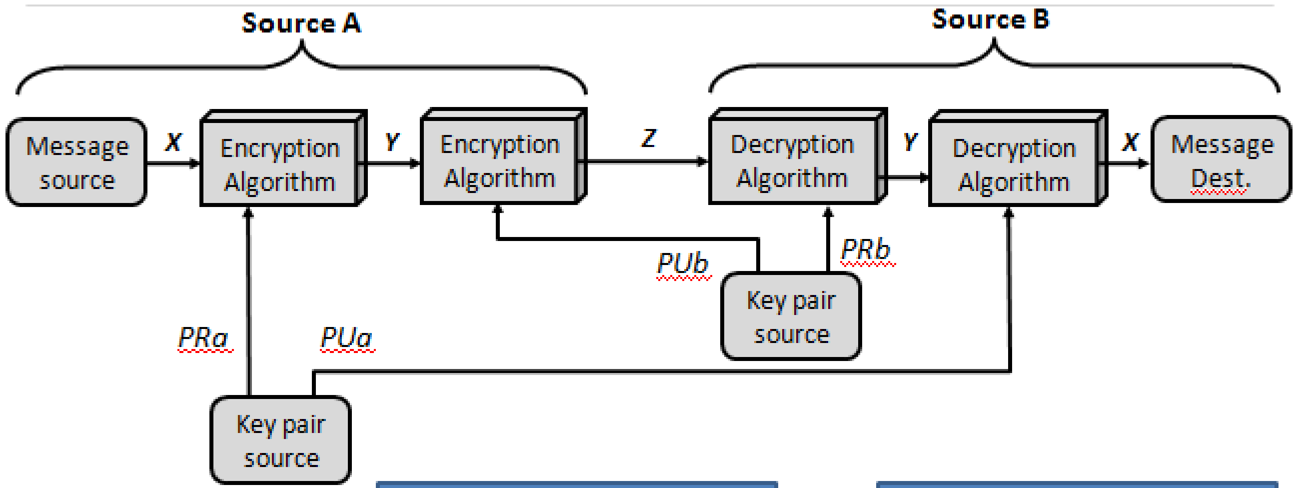
Asymmetric key Encryption with Public Key



Asymmetric key Encryption with Private Key



Authentication and Confidentiality



$Z = E(PU_b, E(PR_a, X))$

$X = D(PU_a, D(PR_B, Z))$

Applications for Public-Key Cryptosystems

- **Encryption/decryption:** The sender encrypts a message with the recipient's public key.
- **Digital signature:** The sender "signs" a message with its private key. Signing is achieved by a cryptographic algorithm applied to the message or to a small block of data that is a function of the message.
- **Key exchange:** Two sides cooperate to exchange a session key. Several different approaches are possible, involving the private key(s) of one or both parties.

RSA Algorithm

- **RSA** is a block cipher in which the Plaintext and Ciphertext are represented as integers between **0** and **n-1** for some n.
- Large messages can be broken up into a number of blocks.
- Each block would then be represented by an integer.

Step-1: Generate Public key and Private key

Step-2: Encrypt message using Public key

Step-3: Decrypt message using Private key

Step-1: Generate Public key and Private key

- Select two large prime numbers: **p** and **q**
- Calculate modulus : $n = p * q$
- Calculate Euler's totient function : $\phi(n) = (p-1) * (q-1)$
- Select **e** such that **e** is **relatively prime** to $\phi(n)$ and $1 < e < \phi(n)$

Two numbers are relatively prime if they have no common factors other than 1.

- Determine **d** such that $d * e \equiv 1 \pmod{\phi(n)}$
- Publickey : **PU = { e, n }**
- Privatekey : **PR = { d, n }**

Step-1: Generate Public key and Private key

- Select two large prime numbers: p = 3 and q = 11
- Calculate modulus : $n = p * q$, $n = 33$
- Calculate Euler's totient function : $\phi(n) = (p-1) * (q-1)$

  $\phi(n) = (3 - 1) * (11 - 1) = 20$
- Select e such that e is relatively prime to $\phi(n)$ and $1 < e < \phi(n)$
- We have several choices for e : 7, 11, 13, 17, 19 Let's take e = 7
- Determine d such that $d * e \equiv 1 \pmod{\phi(n)}$
- $? * 7 \equiv 1 \pmod{20}$, $3 * 7 \equiv 1 \pmod{20}$
- Public key : PU = { e, n } , PU = { 7, 33 }
- Private key : PR = { d, n }, PR = { 3, 33 }

> - This is equivalent to finding d which satisfies $de = 1 + j.\phi(n)$ where j is any integer.
> - We can rewrite this as $d = (1 + j.\ \phi(n)) / e$

Step-2 : Encrypt Message

- Encryption Using Public key: $C = M^e \bmod n$

Ciphertext　Input Message　Publickey

PU = { e, n } , PU = { 7, 33 }

For message M = 14

$C = 14^7 \bmod 33$

$C = [(14^1 \bmod 33) \times (14^2 \bmod 33) \times (14^4 \bmod 33)] \bmod 33$

$C = (14 \times 31 \times 4) \bmod 33 = 1736 \bmod 33$

$C = 20$

Step-3 : Decrypt Message

- Encryption Using Public key: $M = C^d \bmod n$

Plaintext Message

Cipher Message

Privatekey

PR = { d, n } , PR = { 3, 33 }

For Ciphertext C = 20

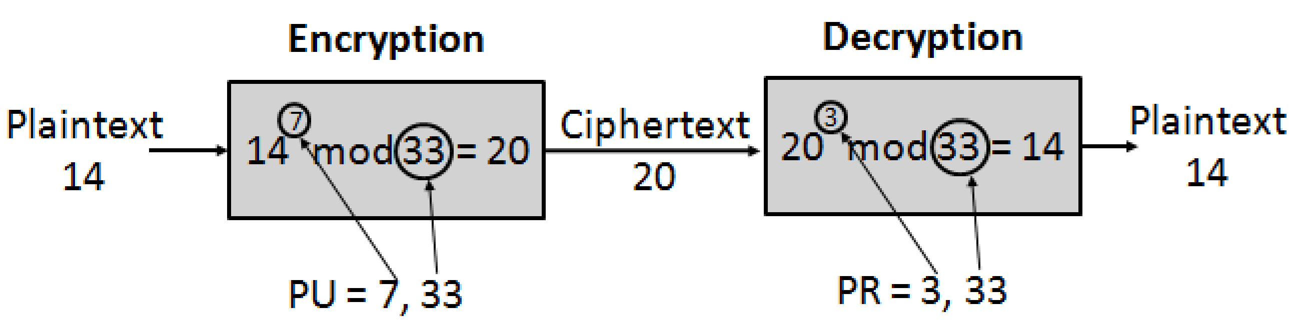
$M = 20^3 \bmod 33$

$M = [(20^1 \bmod 33) \times (20^2 \bmod 33)] \bmod 33$

$M = (20 \times 4) \bmod 33 = 80 \bmod 33$

$M = 14$

Example RSA Algorithm



RSA Example

- Find n, ϕ(n), e, d for p=7 and q= 19 then demonstrate encryption and decryption for M = 6

n = p * q = 7 * 19 = 133

ϕ(n) = (p – 1) * (q – 1) = 108

Finding e relatively prime to 108
e = 2 => GCD(2, 108) = 2 (no)
e = 3 => GCD(3, 108) = 3 (no)
e = 5 => GCD(5, 108) = 1 (Yes)

- Finding d such that (d * e) mod ϕ(n) = 1
- We can rewrite this as d = (1 + j . ϕ(n)) / e

j = 0 => d = 1 / 5 = 0.2 ← integer ? (no)
j = 1 => d = 109 / 5 = 21.8 ← integer ? (no)
j = 2 => d = 217 / 5 = 43.4 ← integer ? (no)
j = 3 => d = 325 / 5 = 65 integer ? (yes)

Public key :
PU = { e, n } = {5, 133}
Private key :
PR = { d, n } = {65, 133}

RSA Example – cont...

- Encryption:

$C = M^e \bmod n$

PU = { e, n } , PU = { 5, 133 }

For message M = 6

$C = 6^5 \bmod 133$
C = 7776 mod 33
C = 62

- Decryption:

$M = C^d \bmod n$

PR = { d, n } , PU = { 65, 133 }

For C = 62

$M = 62^{65} \bmod 133$
M = 2666 mod 33
M = 6

RSA Example

- P and Q are two prime numbers. P=7, and Q=17. Take public key E=5. If plain text value is 10, then what will be cipher text value according to RSA algorithm?
- n = 119
- ϕ(n) = 96
- e = 5
- d = 77
- PU = { 5, 119 }
- PR = {77, 119}
- C = 10^5 mod 119 => C = 40

Diffie-Hellman key Exchange

- The purpose of the **Diffie-Hellman** algorithm is to enable two users to securely exchange a key that can be used for subsequent encryption of message.
- This algorithm depends for its effectiveness on the difficulty of computing **discrete logarithms**.

Primitive root

- Let p be a prime number
- Then a is a primitive root for p, if the powers of a modulo p generates all integers from 1 to $p-1$ in some permutation.

$$a \; mod \; p, a^2 mod \; p, \ldots, a^{p-1} mod \; p$$

- Example: p = 7 then primitive root is 3 because powers of 3 mod 7 generates all the integers from 1 to 6

$$\begin{aligned}
3^1 &= 3 \equiv 3 \; (mod \; 7) \\
3^2 &= 9 \equiv 2 \; (mod \; 7) \\
3^3 &= 27 \equiv 6 \; (mod \; 7) \\
3^4 &= 81 \equiv 4 \; (mod \; 7) \\
3^5 &= 243 \equiv 5 \; (mod \; 7) \\
3^6 &= 729 \equiv 1 \; (mod \; 7)
\end{aligned}$$

Discrete Logarithm

- For any integer b and a primitive root a of prime number p, we can find a unique exponent i such that

$$b = \; a^i (mod \; p) \;\; where \; 0 \leq i \leq (p-1)$$

- The exponent i is referred as the discrete logarithm of b for the base a, mod p. It expressed as below.

$$b\mathrm{dlog}_{a,p}(b)$$

Diffie-Hellman Key Exchange – Cont...

- User A and User B agree on two large prime numbers q and α. User A and User B can use insecure channel to agree on them.
- User A selects a random integer $X_A < q$ and calculates Y_A

Diffie-Hellman Key Exchange – Cont...

Global Public Elements	
q	prime number
α	$\alpha < q$ and α is primitive root of q

User A Key Generation	
Select private X_A	$X_A < q$
Calculate public Y_A	$Y_A = \alpha^{XA} \bmod q$

User B Key Generation	
Select private X_B	$X_B < q$
Calculate public Y_B	$Y_B = \alpha^{XB} \bmod q$

Diffie-Hellman Key Exchange – Cont...

User A Key Generation

Select private X_A $\quad X_A < q$

Calculate public Y_A $\quad Y_A = \alpha^{XA} \bmod q$

User B Key Generation

Select private X_B $\quad X_B < q$

Calculate public Y_B $\quad Y_B = \alpha^{XB} \bmod q$

Calculation of Secret Key by User A

$K =$

Calculation of Secret Key by User b

$K =$

Diffie-Hellman Key Exchange – Cont...

User A Key Generation

Private X_A $\quad X_A < q$, $\quad$ Public Y_A $\quad Y_A = \alpha^{XA} \bmod q$

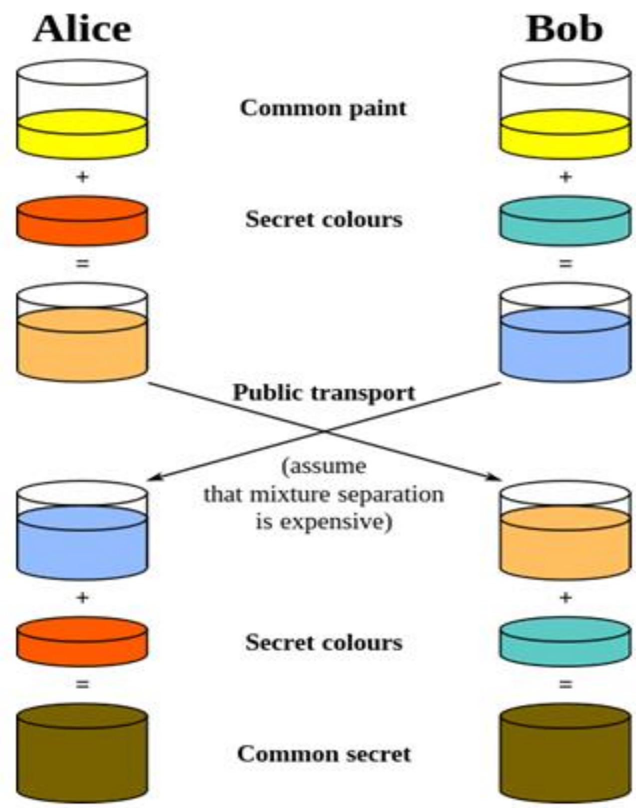
User B Key Generation

Private X_B $\quad X_B < q$, $\quad$ Public Y_B $\quad Y_B = \alpha^{XB} \bmod q$

Secret Key by User A : $K = (Y_B)^{XA} \bmod q$

Secret Key by User B : $K = (Y_A)^{XB} \bmod q$

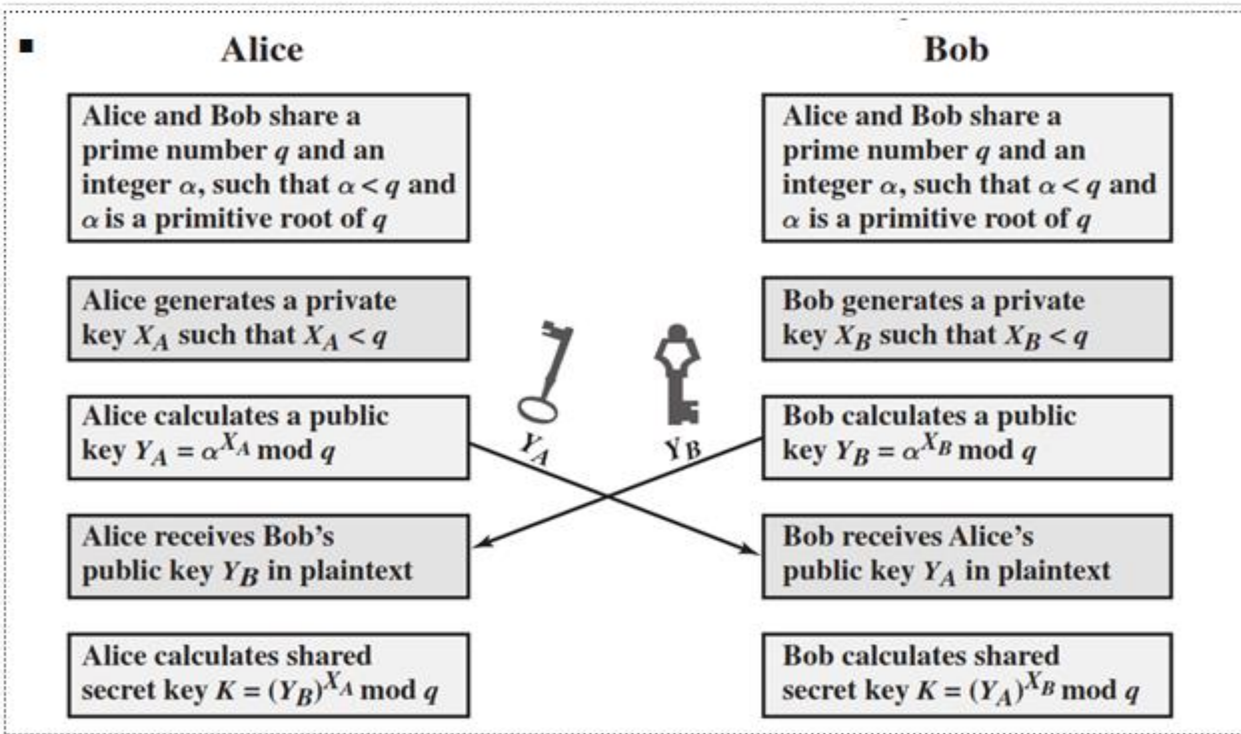
$$K = (Y_B)^{XA} \bmod q$$
$$K = \left(\alpha^{XB} \bmod q\right)^{XA} \bmod q$$
$$K = \left(\alpha^{XB}\right)^{XA} \bmod q$$
$$K = \alpha^{XB\,XA} \bmod q$$
$$K = \left(\alpha^{XA}\right)^{XB} \bmod q$$
$$K = \left(\alpha^{XA} \bmod q\right)^{XB} \bmod q$$
$$K = (Y_A)^{XB} \bmod q$$



Diffie-Hellman Key Exchange Example

- Alice and bob agrees on a prime number $q = 23$
- $\alpha = 5$ as primitive root of q
- Alice selects a private integer $X_A = 6$
- Alice computes $Y_A = \alpha^{XA} \bmod q$ => $Y_A = 5^6 \bmod 23 = 8$
- Bob selects a private integer $X_B = 15$
- Bob computes $Y_B = \alpha^{XB} \bmod q$ => $Y_B = 5^{15} \bmod 23 = 19$
- Alice sends Y_A to Bob and Bob sends Y_B to Alice
- Alice computes key $K = (Y_B)^{XA} \bmod q$ => $K = (19)^6 \, mod \, 23$
- $K = 2$
- Bob computes key $K = (Y_A)^{XB} \bmod q$=> $K = (8)^{15} \, mod \, 23$
- $K = 2$

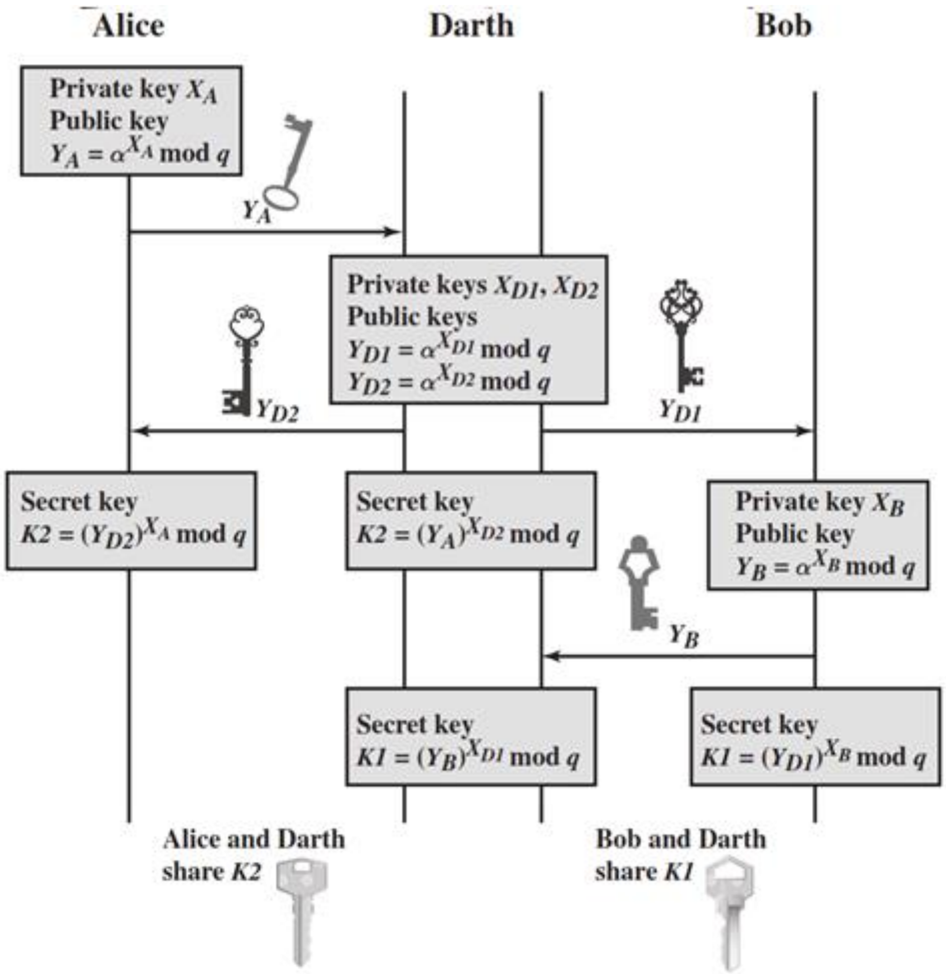
Diffie-Hellman Key Exchange Illustration



Man in the middle attack

- Suppose Alice and Bob wish to exchange keys, and Darth is the adversary.

1. Darth prepares for the attack by generating two random private keys X_{D1} and X_{D2} and then computes corresponding public keys Y_{D1} and Y_{D2}.
2. Alice transmits Y_A to Bob.
3. Darth intercepts Y_A and transmits Y_{D1} to Bob. Darth also calculates $K_2=(Y_A)^{XD2}$ **mod q.**
4. Bob receives Y_{D1} and calculates $K_1 = (Y_{D1})^{XB}$ **mod q.**
5. Bob transmits Y_B to Alice.
6. Darth intercepts Y_B and transmits Y_{D2} to Alice. Darth calculates $K_1=(Y_B)^{XD1}$ **mod q.**
7. Alice receives Y_{D2} and calculates $K_2 = (Y_{D2})^{XA}$ **mod q.**



Alice
Darth
Bob
Private key X_A
Public key
$Y_A = \alpha^{X_A} \bmod q$
Y_A
Private keys X_{D1}, X_{D2}
Public keys
$Y_{D1} = \alpha^{X_{D1}} \bmod q$
$Y_{D2} = \alpha^{X_{D2}} \bmod q$
Y_{D2}
Y_{D1}
Secret key
$K2 = (Y_{D2})^{X_A} \bmod q$
Secret key
$K2 = (Y_A)^{X_{D2}} \bmod q$
Private key X_B
Public key
$Y_B = \alpha^{X_B} \bmod q$
Y_B
Secret key
$K1 = (Y_B)^{X_{D1}} \bmod q$
Secret key
$K1 = (Y_{D1})^{X_B} \bmod q$
Alice and Darth
share $K2$
Bob and Darth
share $K1$