

### PART-I RELATIONS

#### ❖ INTRODUCTION

- ✓ Much of mathematics is about finding a pattern – a recognizable link between quantities that change. In our daily life, we come across many patterns that characterize relations such as brother and sister, father and son, teacher and student.
- ✓ In mathematics also, we come across many relations such as number m is less than number n, line l is parallel to line m, set A is a subset of set B. In all these, we notice that a relation involves pairs of objects in certain order.
- ✓ In this Chapter, we will learn how to link pairs of objects from two sets and then introduce relations between the two objects in the pair.

#### ❖ RELATION

- ✓ Let A and B be non-empty sets. A binary relation or simply relation from A to B is a subset of  $A \times B$ .
- ✓ Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from B. That is, for each pair  $x \in A$  and  $y \in B$ , exactly one of the following is true:
  1.  $(x, y) \in R$ ; we then say x is R-related to y, written  $x R y$ .
  2.  $(x, y) \notin R$ ; we then say x is not R-related to y.
- ✓ Note: If R is a relation from a set A to itself, that is, if R is a subset of  $A^2 = A \times A$ , then we say that R is a relation on A.

#### ❖ DOMAIN

- ✓ Let R be a binary relation from A to B. Then the domain is denoted and defined as,

$$D_R = \{x \in A : (x, y) \in R\}$$

#### ❖ RANGE

- ✓ Let R be a binary relation from A to B. Then the Range is denoted and defined as,

$$R_R = \{y \in B : (x, y) \in R\}$$

#### Example:

1. A familiar example is relation “greater than” for real numbers. This relation is denoted by  $>$ . In fact,  $>$  should be considered as the name of a set whose elements are ordered pairs.

$$> = \{(x, y) : x \text{ and } y \text{ are real numbers and } x > y\}$$

2. The definition of relation permits any set of ordered pairs to define a relation like following

$$S = \{(2, 4), (1, 3), (x, 6), (\text{veer}, *)\}$$

- For this,  $D_R = \{2, 1, x, \text{veer}\}$

## **Unit-3 >> Relations, Partial ordering & Recurrence relation**

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- $R_R = \{4,3,6,*\}$

### **❖ UNIVERSAL RELATION**

- ✓ Let A and B be two non-empty sets. Then  $A \times B$ , subset of itself, is called universal relation from A to B.
- ✓ Let  $A = \{1,2\}$  and  $B = \{x,y\}$ . Then the relation presented by the set  $A \times B = \{\{1,x\}, \{1,y\}, \{2,x\}, \{2,y\}\}$  is called universal relation.

### **❖ VOID RELATION**

- ✓ Let A and B be two non-empty sets. Then the empty set  $\{\emptyset\} \subset A \times B$  is called void (null) relation from A to B.
- ✓ Let  $A = \{1,2\}$  and  $B = \{x,y\}$ . Then the relation presented by the set  $\{\emptyset\}$  is called void (null) relation.

### **❖ UNION, INTERSECTION, AND COMPLEMENT OPERATIONS ON RELATIONS**

- ✓ Let  $A = \{1,2,3,4\}$  and  
 $R = \{(x,y) : \text{for } x, y \in A, x - y \text{ is an integral multiple of } 2\} = \{(1,3), (3,1), (2,4), (4,2)\}$   
 $S = \{(x,y) : \text{for } x, y \in A, x - y \text{ is an integral multiple of } 3\} = \{(1,4), (4,1)\}$
- ✓ Then,  $R \cup S = \{(1,3), (3,1), (2,4), (4,2), (1,4), (4,1)\}$  and  $R \cap S = \emptyset$
- ✓ For complement let,  $A = \{a, b\}$  and  $B = \{1,2\}$ , then  $A \times B = \{\{a, 2\}, \{b, 2\}, \{a, 1\}, \{b, 1\}\}$ . Also let  $R = \{\{a, 1\}, \{a, 2\}, \{b, 2\}\}$ . Then complement relation for R is  $\{\{b, 1\}\}$ .

### **❖ PROPERTIES OF BINARY RELATIONS IN A SET**

- ✓ Reflexive: A binary relation R in a set A is said to be reflexive if, for every  $x \in A, (x, x) \in R$ .
- ✓ Symmetric: A binary relation R in a set A is said to be Symmetric if, for every  $x, y \in A$ , Whenever  $(x, y) \in R$ , then  $(y, x) \in R$ .
- ✓ Transitive: A binary relation R in a set A is said to be trasitive if, for every  $x, y, z \in A$ , Whenever  $(x, y) \in R$  and  $(y, z) \in R$ , then  $(x, z) \in R$ .
- ✓ Anti-symmetric: A binary relation R in a set A is said to be Anti-Symmetric if, for every  $x, y \in A$ , Whenever  $(x, y) \in R$  and  $(y, x) \in R$ , then  $x = y$ .
- ✓ If relations R and S both are reflexive, then  $R \cup S$  and  $R \cap S$  are also reflexive.
- ✓ If relations R and S are reflexive, symmetric and transitive, then  $R \cap S$  is also reflexive, symmetric and transitive.

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### ❖ METHOD-1: BASIC EXAMPLES ON RELATIONS

H	1	Define the following terms with example:  Binary relation. Domain. Range. Universal relation. Void relation. Union. Intersection. Complement relation. Reflexive. Symmetric. Transitive. Anti-symmetric.	
H	2	Let $A = \{(1,2), (2,4), (3,3)\}$ and $B = \{(1,3), (2,4), (4,2)\}$ . Find $A \cup B$ , $A \cap B$ , $D(A)$ , $D(B)$ , $D(A \cup B)$ , $R(A)$ , $R(B)$ and $R(A \cap B)$ .	
C	3	What are the ranges of the relations $S = \{(x, x^2) : x \in N\}$ and $T = \{(x, 2x) : x \in N\}$ where $N = \{0, 1, 2, 3, \dots\}$ ? Also find $S \cap T$ and $S \cup T$ .	
C	4	Let $L$ denotes the relation "less than or equal to" and $D$ denotes the relation "divides" where $x D y$ means "x divides y" defined on a set $\{1, 2, 3, 6\}$ . Write $L$ and $D$ as a sets, and find $L \cap D$ .	
H	5	Give an example of a relation which is (a) neither reflexive nor irreflexive, (b) both symmetric and antisymmetric, (c) reflexive but not symmetric or transitive, (d) symmetric but not reflexive or transitive transitive but not reflexive or symmetric.	
H	6	Show whether the following relations are transitive: $R_1 = \{(1,1)\}$ , $R_2 = \{(1,2), (2,2)\}$ , $R_3 = \{(1,2), (2,3), (1,3), (2,1)\}$	
C	7	Let $L$ denotes the relation "less than or equal to" and $D$ denotes the relation "divides" where $x D y$ means "x divides y" defined on a set $\{1, 2, 3, 6\}$ . Show that both $L$ & $D$ are reflexive, antisymmetric and transitive.	
H	8	Given $S = \{1, 2, 3, 4, \dots, 10\}$ and a relation $R$ on $S$ where $R = \{(x, y) : x + y = 10\}$ what are the properties of the relation $R$ ?	

### ❖ RELATION MATRIX

- ✓ A relation  $R$  from a set  $A$  to a set  $B$  can be represented by a matrix called relation matrix of  $R$ .
- ✓ The relation matrix of  $R$  can be represented by constructing a table whose columns are successive elements of  $A$  & rows are successive elements of  $B$ . i.e. if  $(x_i, y_j) \in R$ , then we enter 1 in  $i^{\text{th}}$  row and  $j^{\text{th}}$  column. Similarly if  $(x_i, y_j) \notin R$ , then we enter 0 in  $i^{\text{th}}$  row and  $j^{\text{th}}$  column.
- ✓ Let  $A = \{x_1, x_2, x_3\}$  and  $B = \{y_1, y_2\}$ . Also, Let  $R = \{(x_1, y_1), (x_2, y_1), (x_3, y_2), (x_2, y_2)\}$ , Then the table representation looks like,

	$y_1$	$y_2$
$x_1$	1	0
$x_2$	1	1
$x_3$	0	1

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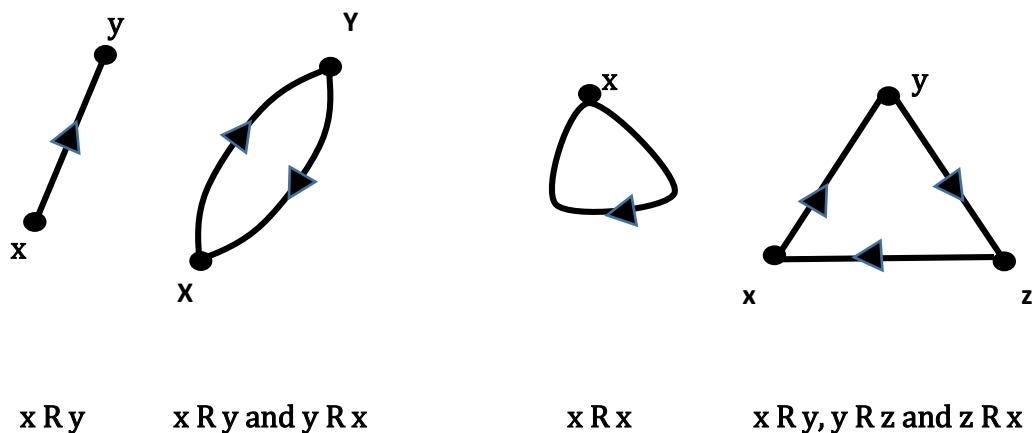
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- ✓ Hence the relation matrix is

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

### ❖ GRAPH OF RELATION

- ✓ A relation can also be represented pictorially by drawing its graph.
- ✓ Let  $R$  be a relation in  $A = \{x_1, x_2, \dots, x_m\}$ . The elements of  $A$  are represented by points of circles called nodes. The nodes may also be called vertices.
- ✓ Now If  $(x_i, x_j) \in R$ , then we connect nodes  $x_i$  and  $x_j$  by an arc and put an arrow on the arc in the direction from  $x_i$  to  $x_j$ . Thus, when all the nodes corresponding to the ordered pairs in  $R$  are connected by arcs with proper arrows, we get a graph of the relation  $R$ .
- ✓ If  $(x_i, x_j) \in R$  and  $(x_j, x_i) \in R$ , then we draw two arcs between  $x_i$  and  $x_j$ . Such an arc is called a loop.



### ❖ METHOD-2: EXAMPLES ON RELATION MATRIX AND GRAPH OF RELATION

C	1	Let $X = \{1, 2, 3, 4\}$ and $R = \{(x, y) : x > y\}$ . Draw the graph of $R$ and give its matrix.	
C	2	Let $A = \{a, b, c\}$ and denote the subsets of $A$ by $B_0, B_1, \dots, B_7$ given below: $B_0 = \emptyset, B_1 = \{c\}, B_2 = \{b\}, B_3 = \{b, c\}, B_4 = \{a\}, B_5 = \{a, c\}, B_6 = \{a, b\}, B_7 = \{a, b, c\}$ . If $R$ is the relation of proper subset on these subsets, then give the matrix of the relation.	
H	3	Let $X = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 4), (4, 1), (4, 4), (2, 2), (2, 3), (3, 2), (3, 3)\}$ . Write matrix of $R$ and sketch its graph.	

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C	4	<p>Determine the properties of the relations given by the graphs given below. And also write corresponding relation matrices.</p>   
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### ❖ PARTITION AND COVERING OF A SET

- ✓ Let  $S$  be a given set and  $A = \{A_1, A_2, A_3, \dots, A_m\}$  where each  $A_i ; i = 1, 2, \dots, m$  is a subset of  $S$  and

$$\bigcup_{i=1}^m A_i = S$$

Then the set  $A$  is called a covering of  $S$ , and the sets  $A_1, A_2, \dots, A_m$  are said to be covers of  $S$ .

- ✓ Also If the elements of  $A$ , which are subsets of  $S$ , are mutually disjoint, then  $A$  is called a partition of  $S$ , and sets  $A_1, A_2, \dots, A_m$  are called the blocks of the partition.
- ✓ e.g. Let  $S = \{x, y, z\}$  and consider the following collections of subsets of  $S$ 
  - $A = \{\{x, y\}, \{y, z\}\}$  is a covering of  $S$  and  $B = \{\{x\}, \{y, z\}\}$  is a partition of  $S$ .
  - $C = \{\{x, y, z\}\}$  is a partition of  $S$  and  $D = \{\{x\}, \{y\}, \{z\}\}$  is a partition of  $S$ .
  - $F = \{\{x\}, \{x, y\}, \{x, z\}\}$  is a covering of  $S$ .

**Note:** Every partition is a cover, but a cover may not be partition.

### ❖ EQUIVALENCE RELATION

- ✓ A relation  $R$  on  $A$  is called an equivalence relation if it is reflexive, symmetric and transitive.

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### **❖ EQUIVALENCE CLASS**

- ✓ Let R be an Equivalence relation on a set A. For any  $x \in A$ , the set  $[x]_R \subseteq A$  given by  $[x]_R = \{y : y \in A \text{ and } x R y\}$ , is called an R-Equivalence class generated by  $x \in A$ .

### **❖ PROPERTIES OF EQUIVALENCE CLASS**

- ✓ For any  $x \in A$ , we have  $x R x$  because R is reflexive; therefore  $x \in [x]_R$ .
- ✓ Let  $y \in A$  be any element such that  $x R y$ , then we have  $[x]_R = [y]_R$ .
- ✓ If  $(x, y) \notin R$ , then  $[x]_R \neq [y]_R$ . Because if  $[x]_R = [y]_R$ . Then there exist at least one  $z \in [x]_R$  and  $z \in [y]_R$ , that give  $x R z$  and  $y R z$ , i.e.  $x R y$ , which contradicts to  $(x, y) \notin R$

**Example:**

- ✓ Let  $A = \{1, 2, \dots, 7\}$  and  $R = \{(x, y); x - y \text{ is divisible by } 3\}$ .
  - For any  $a \in A$ ,  $a - a$  is divisible by 3, hence,  $a R a$ . so R is reflexive.
  - For any  $a, b \in A$ , if  $a R b$ . so,  $a - b$  is divisible by 3. Then  $b - a$  is also divisible by 3 therefore  $b R a$ . So R is symmetric.
  - For any  $a, b, c \in A$ , if  $a R b$  and  $b R c$ , so,  $a - b$  and  $b - c$  are divisible by 3. Then  $a - c = (a - b) + (b - c)$  is also divisible by 3. So,  $a R c$  and R is transitive.
  - Hence, R is an equivalence relation on A.

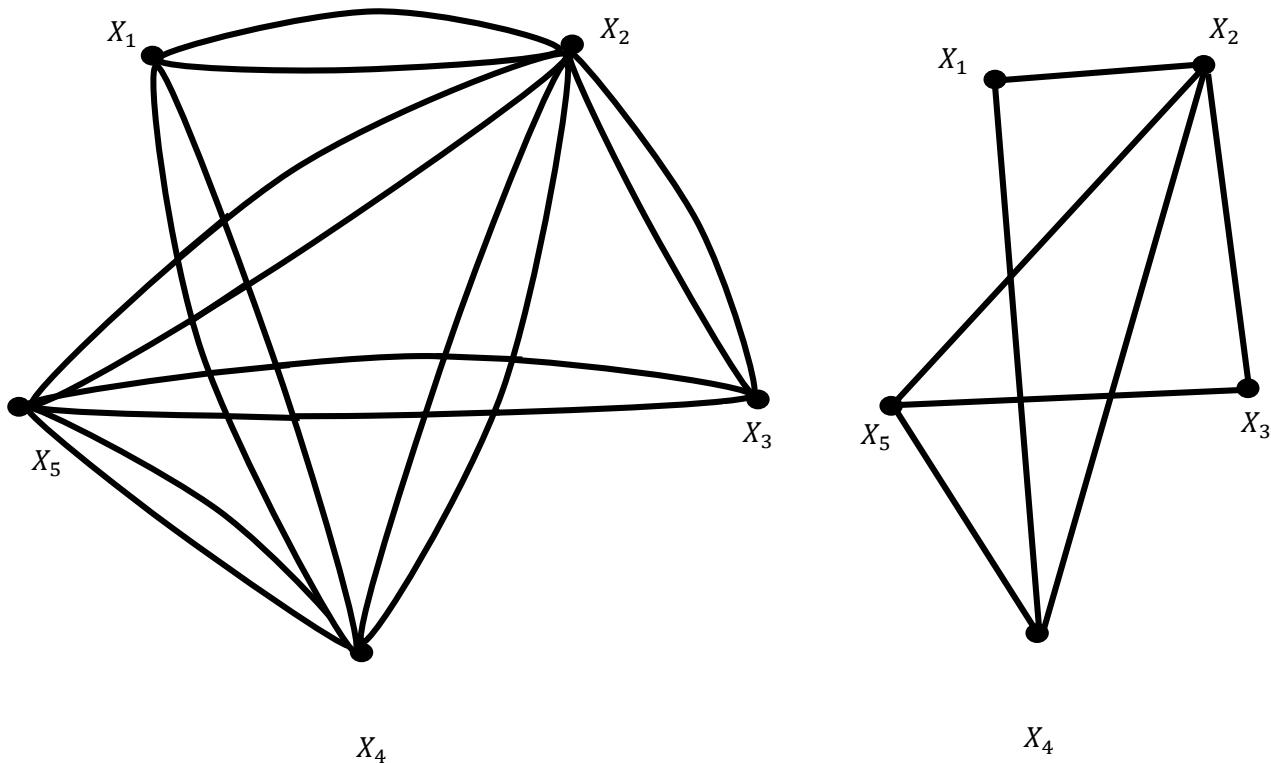
### **❖ COMPATIBILITY CLASS**

- ✓ A relation R in A is said to be compatibility relation if it is reflexive and symmetric.

**Note: Every equivalence class is compatibility class. Reverse is not true.**

### **❖ MAXIMUM COMPATIBILITY BLOCK**

- ✓ Let A be a set and R be a compatibility relation on A. Then a subset  $C \subseteq A$  is called a maximum compatibility block if any element of C is compatible to every other element of C and no element of  $A - C$  is compatible to all the elements of C.
  - Let  $R = \{(x, y); x, y \in X \text{ and } x \text{ and } y \text{ contain some common letter}\}$  be relation on X where  $X = \{\text{ball, bed, dog, let, egg}\}$ .
  - Then R is a compatibility relation and is denoted by  $\approx$ . Also note that R is not equivalence relation. If we denote the element of X by  $x_1, x_2, x_3, x_4, x_5$ , then the graph is as shown here in figure (1).
- ✓ Since relation is compatibility relation, it is not necessary to draw loops at each element nor it is necessary to draw both  $x R y$  and  $y R x$ . So we can simplify graph as shown in figure (2).



- ✓ The relation matrix here is symmetric and has its diagonal elements unity. Therefore it is sufficient to give only the elements of the lower triangular part only as shown as below.

$x_2$	1			
$x_3$	0	1		
$x_4$	1	1	0	
$x_5$	0	1	1	1
	$x_1$	$x_2$	$x_3$	$x_4$

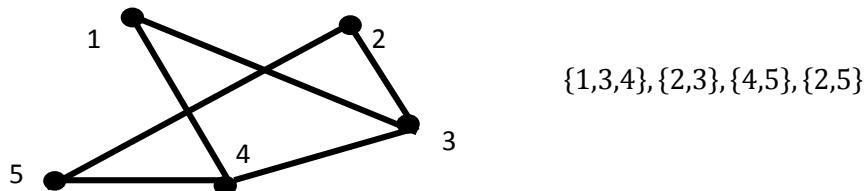
- ✓ It is clear that the subsets  $\{x_1, x_2, x_4\}, \{x_2, x_3, x_5\}, \{x_2, x_4, x_5\}$  are maximal compatibility blocks.

#### ❖ HOW TO FIND MAXIMAL COMPATIBILITY BLOCK AND MATRIX

- ✓ For this first we draw a simplified graph of the compatibility relation and pick from this graph the largest complete polygons. i.e. a polygon in which any vertex is connected to every other vertex. In addition to this any element which is related only to itself forms a maximal compatibility block. Similarly any two elements which are compatible to one another but to no other elements also form a maximal compatibility block. For example triangle.
- ✓ The maximal compatibility blocks of a compatibility relation R with simple graph as given below are:

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- ✓ Also the matrix for this compatibility relation is

<b>2</b>	0				
<b>3</b>	1	1			
<b>4</b>	1	0	1		
<b>5</b>	0	1	0	1	
	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	

### ❖ COMPOSITE RELATION

- ✓ Let R be a relation from A to B and S be a relation from B to C. Then a relation written as  $R \circ S$  is called a composite relation of R and S where

$$R \circ S = \{(x, z) : \text{for } x \in A \text{ and } z \in C \text{ there exist } y \in B \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}$$

- ✓ The operation of obtaining  $R \circ S$  is called composition of relations.

### ❖ CONVERSE OF A RELATION

- ✓ Given a relation R from A to B, a relation  $\bar{R}$  from B to A is called the converse of R, where the ordered pairs of  $\bar{R}$  are obtained by interchanging the members in each of the ordered pairs of R. This means, for  $x \in A$  and  $y \in B$ , that  $x R y \Leftrightarrow y \bar{R} x$ .

### ❖ TRANSITIVE CLOSURE OF A RELATION

- ✓ Let R be relation in a finite set A. The relation  $R = R \cup R^2 \cup R^3 \cup \dots$  in A is called the transitive closure of R in A.

### ❖ METHOD-3: EXAMPLES ON COVERING AND EQUIVALENCE RELATION

H	1	Define with examples: Partition, Covering, Equivalence relation, Equivalence class, Compatibility relation, Maximum compatibility block, Composite relation, Converse of a relation, Transitive closure.	
C	2	Let $X = \{1, 2, \dots, 7\}$ and $R = \{(x, y) ; x - y \text{ is divisible by } 3\}$ . Show that R is an equivalence relation. Draw the graph of R.	

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H	3	Let $X = \{1, 2, \dots, 7\}$ and $R = \{(x, y); x - y \text{ is even}\}$ . Show that $R$ is an equivalence relation.																																					
H	4	Let $Z = A_1 \cup A_2 \cup A_3$ where $A_1 = \{\dots, 1, 4, 7 \dots\}, A_2 = \{\dots, 2, 5, 8 \dots\}, A_3 = \{\dots, 3, 6, 9 \dots\}$ . Then Define equivalence relation whose equivalence classes are $A_1, A_2, A_3$ .																																					
C	5	Let $Z$ be the set of integers and $R$ be the relation called "congruence modulo 3" defined by $R = \{(x, y); x - y \text{ is divisible by } 3\}$ . Determine the equivalence classes generated by the elements of $Z$ .																																					
H	6	Prove that relation "congruence modulo $m$ " given by $R = \{(x, y); x - y \text{ is divisible by } m\}$ over the set of positive integers is an equivalence relation.																																					
H	7	Let $S$ be the set of all statement functions in $n$ variables and let $R$ be the relation given by $R = \{(x, y); x \Leftrightarrow y\}$ . Discuss the equivalence classes generated by the elements of $S$ .																																					
C	8	Let $X = \{a, b, c, d, e\}$ and let $C = \{\{a, b\}, \{c\}, \{d, e\}\}$ . Show that the partition $C$ defines an equivalence relation on $X$ .																																					
H	9	Let $R$ denote a relation on the set of ordered pairs of positive integers such that $(x, y) R (u, v)$ iff $xv = yu$ . Show that $R$ is an equivalence relation.																																					
C	10	Two equivalence relations $R$ and $S$ are given by their relation matrices $M_R$ and $M_S$ . Show that $R \circ S$ is not an equivalence relation. $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$																																					
C	11	Let $R = \{(x, y); x, y \in X \text{ and } x \text{ and } y \text{ contain some common letter}\}$ be relation on $X$ where $X = \{\text{ball, bed, dog, let, egg}\}$ . Then show that $R$ is a compatibility relation.																																					
C	12	Let the compatibility relation on a set $\{x_1, x_2, x_3, \dots, x_6\}$ be given by the following matrix. Draw the graph. Also find maximal compatibility blocks of the relation. <table border="1" style="margin-left: auto; margin-right: auto;"> <tr> <td><math>x_2</math></td> <td colspan="2">1</td> <td colspan="3"></td> </tr> <tr> <td><math>x_3</math></td> <td>1</td> <td>1</td> <td colspan="3"></td> </tr> <tr> <td><math>x_4</math></td> <td>0</td> <td>0</td> <td>1</td> <td colspan="2"></td> </tr> <tr> <td><math>x_5</math></td> <td>0</td> <td>0</td> <td>1</td> <td>1</td> <td></td> </tr> <tr> <td><math>x_6</math></td> <td>1</td> <td>0</td> <td>1</td> <td>0</td> <td>1</td> </tr> <tr> <td></td> <td><math>x_1</math></td> <td><math>x_2</math></td> <td><math>x_3</math></td> <td><math>x_4</math></td> <td><math>x_5</math></td> </tr> </table>	$x_2$	1					$x_3$	1	1				$x_4$	0	0	1			$x_5$	0	0	1	1		$x_6$	1	0	1	0	1		$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	
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	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$																																		
H	13	Let $R = \{(1, 2), (3, 4), (2, 2)\}$ and $s = \{(4, 2), (2, 5), (3, 1), (1, 3)\}$ . Find $R \circ S, S \circ R, R \circ (S \circ R), (R \circ S) \circ R, R \circ R, S \circ S$ , AND $R \circ R \circ R$ .																																					
C	14	Let $R$ and $S$ be two relations on a set of positive integers $A$ , where $R = \{(x, 2x); x \in A\}$ and $S = \{(x, 7x); x \in A\}$ . Find $R \circ S, R \circ R, R \circ R \circ R$ , and $R \circ S \circ R$ .																																					

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C	15	Let $R = \{(1,2), (3,4), (2,2)\}$ and $s = \{(4,2), (2,5), (3,1), (1,3)\}$ be relations defined on the set $A = \{1,2,3,4,5\}$ , obtain the relation matrices for $R \circ S$ and $S \circ R$ .	
C	16	Given the relation matrix $M_R$ of a relation $r$ on the set $\{a, b, c\}$ , find the relation matrices of $\tilde{R}$ , $R^2 = R \circ R$ , $R^3 = R \circ R \circ R$ , and $R \circ \tilde{R}$ . $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	
H	17	Given the relation matrices $M_R$ and $M_S$ , find $M_{R \circ S}$ , $M_{\tilde{R}}$ , $M_{\tilde{S}}$ , $M_{\tilde{R} \circ \tilde{S}}$ , and show that $M_{\tilde{R} \circ \tilde{S}} = M_{\tilde{S} \circ \tilde{R}}$ . $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad M_S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$	

## PART-II PARTIAL ORDERING

### ❖ PARTIAL ORDERING

- ✓ A binary relation  $R$  in a set  $A$  is called a partial order relation or partial ordering in  $P$  iff  $R$  is reflexive, antisymmetric, and transitive. Also it is denoted by  $\leq$  and the ordered pair  $(P, \leq)$  is called partial order set or poset.

### ❖ TOTALLY ORDERED SET (CHAIN)

- ✓ Let  $(P, \leq)$  be a partially ordered set. If for every  $x, y \in P$  we have either  $x \leq y$  or  $y \leq x$ , then  $\leq$  is called a simple ordering or linear ordering on  $P$ , and  $(P, \leq)$  is called a totally ordered or simply ordered set or a chain.

**Example:** The simplest totally order set is  $I_n = \{1, 2, 3, \dots, n\}$  with natural ordering "less than or equal to".

### ❖ FREQUENTLY USED PARTIALLY ORDERED RELATIONS:-

- ✓ Let  $P$  be the set of real numbers. The relation  $\leq$  (less than or equal to) is a partial ordering on  $P$ .
- ✓ The relation inclusion  $\subseteq$  on  $P = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$  is a partial ordering.
- ✓ Let  $P = \{2, 3, 6, 8\}$  and  $\leq$  be a relation "divides" then  $(P, \leq)$  is a poset.

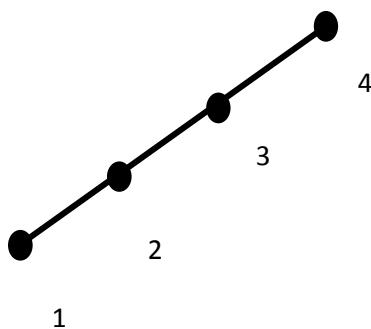
### ❖ REPRESENTATION

- ✓ In a Poset  $(P, \leq)$ , an element  $y \in P$  is said to cover an element  $x \in P$  if  $x < y$  and if there does not exist an element  $z \in P$  such that  $x \leq z$  and  $z \leq y$ .

## Unit-3 >> Relations, Partial ordering & Recurrence relation

### ❖ HASSE DIAGRAM

- ✓ A poset  $(P, \leq)$  can be represented by means of a diagram known as a Hasse diagram or partial ordered set diagram. In such a diagram, each element is represented by a small circle or a dot. The circle for  $x \in P$  drawn below the circle for  $y \in P$  if  $x < y$ , and a line is drawn between  $x$  and  $y$ . If  $x < y$ , but  $y$  does not cover  $x$ , then  $x$  and  $y$  are not connected directly by a single line. However, they are connected through one or more elements of  $P$ .
- ✓ Let  $P = \{1, 2, 3, 4\}$  and  $\leq$  be a relation “less than or equal to” then the Hasse diagram is as follows:



### ❖ LEAST AND GREATEST MEMBER:-

- ✓ Let  $(P, \leq)$  be a poset. If there exist an element  $y \in P$  such that  $y \leq x$  for all  $x \in P$ , then  $y$  is called the least member in  $P$  relative to the partial ordering  $\leq$ . Similarly, if there exists an element  $y \in P$  such that  $x \leq y$  for all  $x \in P$ , then  $y$  is called greatest member in  $P$  relative to  $\leq$ .

Note:-For any poset least and greatest member, if it exists, is unique.it may happen that the least or the greatest member does not exist. The least member is usually denoted by 0 and greatest by 1.

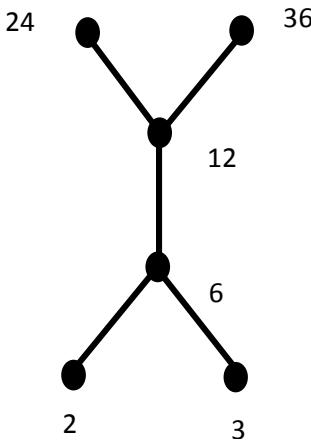
### ❖ MINIMAL AND MAXIMAL MEMBER:-

- ✓ Let  $(P, \leq)$  be a poset. If there does not exist an element  $x \in P$  such that  $x < y$  for  $y \in P$ , then  $y$  is called the minimal member in  $P$  relative to the partial ordering  $\leq$ . If there does not exist an element  $x \in P$  such that  $y < x$  for  $y \in P$ , then  $y$  is called the maximal member in  $P$  relative to the partial ordering  $\leq$ .

**Example:** Let  $P = \{1, 2, 3, 4\}$  and  $\leq$  be a relation “less than or equal to”.

- ✓ Then here least member is 1 and greatest member is 4.

**Example:** Let  $P = \{2, 3, 6, 12, 24, 36\}$  and the relation  $\leq$  be for divide.



- ✓ Then there are two minimal members 2 and 3. Also there are two maximal members 24 & 36.

#### ❖ UPPER BOUND AND LOWER BOUND

- ✓ Let  $(P, \leq)$  be a poset and let  $A \subseteq P$ . Any element  $x \in P$  is an upper bound for  $A$  if for all  $a \in A, a \leq x$ . Similarly, any element  $x \in P$  is a lower bound for  $A$  if for all  $a \in A, x \leq a$ .

#### ❖ LEAST UPPER BOUND AND GREATEST LOWER BOUND

- ✓ Let  $(P, \leq)$  be a partially ordered set and let  $A \subseteq P$ . An element  $x \in P$  is a least upper bound or supremum, for  $A$  if  $x$  is an upper bound for  $A$  and  $x \leq y$  where  $y$  is any upper bound for  $A$ . Similarly, An element  $x \in P$  is a greatest lower bound or infimum, for  $A$  if  $x$  is a lower bound for  $A$  and  $y \leq x$  where  $y$  is any lower bound for  $A$ .

**Note:-Least upper bound is denoted by “LUB” or “sup.”, and greatest lower bound is denoted by “GLB” or “inf.”. Also both are unique if exist.**

**Example:** Let  $P = \{2, 3, 6, 12, 24, 36\}$  and the relation  $\leq$  be for divide. Let  $A = \{6, 12\}$ .

- ✓ Then lower bounds are 2, 3, and 6. But the greatest lower bound is 6.
- ✓ Similarly upper bounds are 24, 36. But the least upper bound is 24.

#### ❖ WELL-ORDERED POSET

- ✓ A Partially ordered set is called well-ordered if every nonempty subset of it has a least member.

**Example:** Simplest well order set is  $I_n = \{1, 2, \dots, n\}$  with natural ordering “less than or equal to”.

#### ❖ METHOD-4: EXAMPLES ON PARTIAL ORDERED SET

H	1	Define the following terms with example:  Partial ordered relation, Partial ordered set (Poset), Simple (linear) ordering relation, Totally ordered set (chain or simply ordered set), Least member, Greatest member, Minimal member, Maximal member, Upper bound, Lower bound, Least upper bound (supremum), Greatest lower bound (infimum), Well-ordered set.	
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## Unit-3 >> Relations, Partial ordering & Recurrence relation

C	2	Show that the relation $\subseteq$ on a set $p(A)$ , i.e. power set of $A = \{a, b, c\}$ , is poset.	
H	3	Show that $(P(A), \subset)$ is a poset.	
C	4	Let $A = \{2, 3, 6, 12, 24, 36\}$ and the relation $\leq$ be such that $x \leq y$ if $x$ divides $y$ . Draw the Hasse diagram of $(A, \leq)$ .	
H	5	Let $\subseteq$ be a relation on a set $p(A)$ , i.e. power set of $A$ . Then draw Hasse diagram for $A$ . (1) $A = \{a\}$ (2) $A = \{a, b\}$ (3) $A = \{a, b, c\}$	
H	6	Draw the Hasse diagram of the following sets under the partial ordering relation "divides," and indicate those which are totally ordered. (1) $\{2, 6, 24\}$ (2) $\{3, 5, 15\}$ (3) $\{1, 2, 3, 6, 12\}$ (4) $\{3, 9, 27, 54\}$	
C	7	Give an example of a set $A$ such that $(p(A), \subseteq)$ is a totally ordered set.	
H	8	Give a relation which is both a partial ordered relation and an equivalence relation on a set.	
C	9	Hasse diagram of a poset $(P, R)$ , where $P = \{x_1, x_2, x_3, x_4, x_5\}$ is given below: Find which of the following are true:	
		<pre> graph TD     x1((x1)) --- x2((x2))     x1 --- x3((x3))     x2 --- x4((x4))     x3 --- x5((x5))   </pre>	
		(1) $x_1 R x_2$ (2) $x_4 R x_1$ (3) $x_3 R x_5$ (4) $x_2 R x_5$ (5) $x_1 R x_1$ (6) $x_2 R x_3$ (7) $x_4 R x_5$	
C	10	For above Poset given in example 9. Find least and greatest member in $P$ if exists. Also find minimal and maximal elements. Find upper and lower bounds. Find LUB and GLB if exists.	
C		Let $A = \{2, 3, 6, 12, 24, 36\}$ and the relation $\leq$ be such that $x \leq y$ if $x$ divides $y$ . then find least and greatest member in $P$ if exists. Also find minimal and maximal elements. Find upper and lower bounds. Find LUB and GLB if exists. For following sets: 1. $\{2, 3, 6\}$ 2. $\{2, 3\}$ 3. $\{12, 6\}$ 4. $\{24, 36\}$ 5. $\{3, 12, 24\}$	

### ❖ LATTICE

- ✓ A lattice is a poset  $(L, \leq)$  in which every pair of elements  $a, b \in L$  has a greatest lower bound and least upper bound.
- ✓ The greatest lower bound of a subset  $\{a, b\} \subseteq L$  is call meet and denoted by  $GLB\{a, b\}, a * b, a \wedge b$  or  $a \cdot b$ . The least upper bound is call join and denoted by  $LUB\{a, b\}, a \oplus b, a \vee b, a + b$ .

## **Unit-3 >> Relations, Partial ordering & Recurrence relation**

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- ✓ Example:-Let S be any set and P(S) be its power set. The partially ordered set  $(P(S), \subseteq)$  is a lattice In which the meet and join are the same as the operations  $\cap$  and  $\cup$  respectivey.

### **❖ PROPERTIES OF LATTICES**

- ✓ Let  $(L, \leq)$  be a lattice and  $*$  and  $\oplus$  betwo binary operations meet and join. Then for  $, b, c \in L$ , we have

- $a * a = a$  and  $a \oplus a = a$  (Idempotent)
- $a * b = b * a$  and  $a \oplus b = b \oplus a$  (Commutative)
- $(a * b) * c = a * (b * c)$ and  $(a \oplus b) \oplus c = a \oplus (b \oplus c)$  (Associative)
- $a * (a \oplus b) = a$  and  $a \oplus (a * b) = a$  (Absorption)

### **❖ COMPLETE LATTICE**

- ✓ A lattice is called complete if each of its nonempty subsets has a least upper bound and a greatest lower bound.

**Note:-Every finite lattice must be complete.**

- ✓ Example:-Let S be any set and P(S) be its power set. The partially ordered set  $(P(S), \subseteq)$  is a lattice In which the meet and join are the same as the operations  $\cap$  and  $\cup$  respectivey.

### **❖ DISTRIBUTIVE LATTICE**

- ✓ A lattice  $(L, *, \oplus)$  is called a distributive lattice if for any  $a, b, c \in L$ ,

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$

- ✓ Example:-Let S be any set and P(S) be its power set. The partially ordered set  $(P(S), \subseteq)$  is a lattice In which the meet and join are the same as the operations  $\cap$  and  $\cup$  respectivey.

### **❖ MODULAR LATTICE**

- ✓ Lattice is said to be modular if  $a \leq c \Rightarrow a \oplus (b * c) = (a \oplus b) * c$ .

- ✓ Example:-Let S be any set and P(S) be its power set. The partially ordered set  $(P(S), \subseteq)$  is a lattice In which the meet and join are the same as the operations  $\cap$  and  $\cup$  respectivey.

### **❖ BOUNDED LATTICE**

- ✓ A lattice is said to be bounded if it has both least and greatest elements ,i.e. 0 and 1.

- ✓ Example:-Let S be any set and P(S) be its power set. The partially ordered set  $(P(S), \subseteq)$  is a lattice In which the meet and join are the same as the operations  $\cap$  and  $\cup$  respectivey. And least element is  $\phi$  and greatest element is S.

### **❖ COMPLEMENT**

- ✓ In a bounded Lattice  $(L, *, \oplus, 0, 1)$ , an element  $b \in L$  is called a complement of an element  $a \in L$  if  $a * b = 0$  and  $a \oplus b = 1$

## Unit-3 >> Relations, Partial ordering & Recurrence relation

### ❖ COMPLEMENTED LATTICE

- ✓ A lattice  $(L, *, \oplus)$  is said to be a complemented lattice if every element of  $L$  has at least one complement.
- ✓ Example:-Let  $S$  be any set and  $P(S)$  be its power set. The partially ordered set  $(P(S), \subseteq)$  is a lattice In which the meet and join are the same as the operations  $\cap$  and  $\cup$  respectivey.

### ❖ BOOLEAN LATTICE

- ✓ A Boolean Lattice (Boolean algebra) is a complemented, distributive lattice.
- ✓ Example:-Let  $S$  be any set and  $P(S)$  be its power set. The partially ordered set  $(P(S), \subseteq)$  is a lattice In which the meet and join are the same as the operations  $\cap$  and  $\cup$  respectivey. and least element is  $\phi$  and greatest element is  $S$ .

### ❖ PSEUDO BOOLEAN LATTICES

- ✓ A bounded lattice  $(L, \leq)$  is called pseudo-Boolean if for all  $a, b \in L$ , there exists  $c \in L$  such that  $a \wedge x \leq b \Leftrightarrow x \leq c, \forall x \in L$ .
- ✓ If such element  $c$  exists, then it is unique and will be denoted by  $b : a$ .
- ✓ Example:- Let  $S$  be any set and  $P(S)$  be its power set. The partially ordered set  $(P(S), \subseteq)$  is a lattice In which the meet and join are the same as the operations  $\cap$  and  $\cup$  respectively, and least element is  $\phi$  and greatest element is  $S$ .

### ❖ METHOD-5: BASIC EXAMPLES ON LATTICES

H	1	Define With example: Lattice, complete Lattice, Distributive lattice, Modular lattice, bounded lattice, Complement element, complemented lattice, Boolean lattice(algebra), pseudo Boolean lattice(algebra).	
C	2	Let $S$ be any set and $P(S)$ be its power set. In which the meet and join are the same as the operations $\cap$ and $\cup$ respectivey.then show that the partially ordered set $(P(S), \subseteq)$ is a lattice.	
H	3	Let $A = \{a, b, c\}$ . Then show that the poset $(p(A), \subseteq)$ is a complete lattice.	
C	4	Let $S$ be any set and $P(S)$ be its power set. In which the meet and join are the same as the operations $\cap$ and $\cup$ respectivey. Then show that The partially ordered set $(P(S), \subseteq)$ is a modular and distributive lattice.	
H	5	Let $S$ be any set and $P(S)$ be its power set. In which the meet and join are the same as the operations $\cap$ and $\cup$ respectivey. Then show that the partially ordered set $(P(S), \subseteq)$ is complemented lattice.	

## **Unit-3 >> Relations, Partial ordering & Recurrence relation**

C	6	Let S be any set and P(S) be its power set. In which the meet and join are the same as the operations $\cap$ and $\cup$ respectively. Then show that The partially ordered set $(P(S), \subseteq)$ is a boolean lattice (algebra).	
H	7	Let $S = \{1,2,3\}$ be any set and P(S) be its power set. In which the meet and join are the same as the operations $\cap$ and $\cup$ respectively. Then show that The partially ordered set $(P(S), \subseteq)$ is Lattice, complete Lattice, Distributive lattice, Modular lattice, bounded lattice, Complement element, complemented lattice, Boolean lattice (algebra).	

### **PART-III RECURSION RELATION**

#### **❖ RECURSION**

- ✓ Suppose  $n$  is a natural number. We often defines  $n!$  as  $n! = n \times n - 1 \times n - 2 \times \dots \times 1$ . Sometimes, it is difficult to define a computation explicitly and it is easy to define it in terms of itself, that is, recursively.
- ✓ Recursion is an elegant and powerful problem solving technique, used extensively in both discrete mathematics and computer science. We can use recursion to define sequence, functions, sets, algorithms and many more.

#### **❖ RECURSIVELY DEFINED SEQUENCES**

- ✓ A sequence is a discrete structure to represent ordered list of elements. Sequence is a function whose domain is some infinite set of integers and whose range is some set. We use the notation  $a_n$  to denote the image of integer  $n$  and  $a_n$  is called as term of sequence. The notation  $\{a_n\}$  is used to denote the sequence.
- ✓ Consider the following instructions for generating a sequence:
  - Start with 1
  - Given a term of sequence, get the next term by adding two to it.
- ✓ If we generate terms of sequence with these two rules, we obtain  $1, 3, 5, 7, \dots$
- ✓ If we denote the sequence as  $a_1, a_2, a_3, a_4\dots$
- ✓ Then we can rephrase instruction as,  $a_n = a_{n-1} + 2$  for  $n \geq 2$  and  $a_1 = 1$
- ✓ The equation  $a_n = a_{n-1} + 2$  is an example of recurrence relation and  $a_1 = 1$  is an initial condition.

#### **❖ RECURSIVELY DEFINED SETS**

- ✓ Sets can be defined recursively. Recursive definition of sets have two parts, a basis step and a recursive step.
- ✓ In the basis step, primitive elements are specified. In the recursive step, rules for generating new elements in the set from those already known to be in the set are provided.

## **Unit-3 >> Relations, Partial ordering & Recurrence relation**

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- ✓ Consider an example offset of multiples of 3. Let A denote this set. The recursive definition to define A is as:

1.  $3 \in A$
2. if  $x \in A$  then  $x + 3 \in A$

- ✓ We can ensure elements of A as  $\{3, 6, 9, 12, 15, \dots\}$

### **❖ RECURSIVELY DEFINED FUNCTION**

- ✓ Let  $x \in A$  and  $B = \{x, x + 1, x + 2, \dots\}$ . The recursive definition of a function f with domain B consists of following steps: for  $K \geq 1$ ,

#### **1. Basis Step**

- A few initial values of the function  $f(x), f(x + 1), \dots, f(x + k - 1)$  are specified. An equation that specifies such initial values is called an initial condition

#### **2. Recursive step**

- A formula to compute  $f(n)$  from the  $k$  preceding functional values  $f(n - a), f(n - 2), \dots, f(n - k)$  is made. Such a formula is called recursive formula or recurrence relation.

- ✓ The factorial function f is defined by  $f(n) = n!$ ;  $f(0) = 1$ . F is recursively defined as  $f(0) = 1$  (initial value) and  $f(n) = n * f(n - 1), n \geq 1$  (recurrence relation).

### **❖ RECURRENCE RELATION**

- ✓ We have defined sequence, sets, and functions recursively. In recursive definition, the recursive step is used to find new terms from the existing terms. This is defined in terms of a formula called as recurrence relation.

- ✓ A recurrence relation for the sequence  $\{a_n\}$  is an equation that expresses  $a_n$  in terms of one or more preceding terms of the sequence, like,  $a_0, a_1, \dots, a_{n-1}$ , for  $n \geq n_0$ . Here  $n_0$  is used to define initial condition and is a nonnegative integer. A sequence is called as a solution of a recurrence relation if its terms satisfy the recurrence relation.

### **❖ SOLVING RECURRENCE RELATIONS**

- ✓ Solving recurrence relation for a function f means finding an explicit formula for  $f(n)$ . The following are methods for this.

### **❖ INTELLIGENT GUESSWORK**

- ✓ This approach is based on four steps:

1. Compute first few values of the recurrence.
2. Look for regularity in computed values.
3. Guess some suitable general formula.
4. Prove the guessed formula to be correct (using mathematical induction)

## **Unit-3 >> Relations, Partial ordering & Recurrence relation**

### **❖ SUBSTITUTION METHOD**

- ✓ Simple class of recurrence relations can be solved by substitution method. The substitution method is also called as iterative method which is used for finding formula for recurrence relation. Solving recurrence relation by iteration means finding an explicit formula for  $f(n)$ . This is done in two steps:
  1. Apply the recurrence formula iteratively and look for a pattern to predict an explicit formula using initial conditions:
  2. Use induction to prove the predicted relation.

### **❖ METHOD-6: EXAMPLES ON RECURRENCE RELATIONS**

C	1	<p>Compute first four terms of the following recurrence sequences:</p> $a_n = na_{n-1} + n^2 a_{n-2}; n \geq 2 \quad a_n = 1; n = 0 \quad a_n = 1; n = 1$ $a_n = a_{n-1}^2 \quad n \geq 2 \quad a_n = 2; n = 1$ $a_n = a_{n-1} - a_{n-3}; n \geq 3 \quad a_n = 1; n = 0 \quad a_n = 2; n = 1 \quad a_n = 0; n = 2$ $a_n = 6a_{n-1}; n \geq 1 \quad a_n = 2; n = 0$ $a_n = 2^n + 5(3)^n; n \geq 0$	
C	2	<p>Give recurrence definition of the following sequences:</p> <p>(a) 1,5,5<sup>2</sup>,5<sup>3</sup>,5<sup>4</sup>,.....      (b) 5,3,1,-1,-3,.....</p> <p>(c) 16,8,4,2,1,1,1,...      (d) 1,3,7,15,31,63,....</p> <p>(e) 0,1,0,4,0,16,.....</p>	
C	3	<p>Solve the following recurrence relation by intelligent guesswork</p> $t_n = 2; n = 1 \text{ and } t_{n+1} = t_n + 3; n \geq 2.$	
C	4	<p>Predict formula for following recurrence relation by substitution method.</p> $t_n = 1; n = 2 \text{ and } t_n = 2t_{n-1} + 1; n > 2$	
H	5	<p>Using iterative(substitution) method, predict a solution for the following recurrence relation</p> $t_n = 1; n = 0 \text{ and } t_n = 2t_{n-1}; n \geq 1$	
H	6	<p>Using iterative(substitution) method, predict a solution for the following recurrence relation</p> $t_n = 1; n = 0 \text{ and } t_n = t_{n-1} + n; n \geq 1$	
H	7	<p>Using iterative(substitution) method, predict a solution for following recurrence relation</p> $t_n = 0; n = 0 \text{ and } t_n = t_{n-1} + 4n; n \geq 1$	

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