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A.Y.2019-20: EVEN SEMESTER
3140708: DISCRETE MATHEMATICS
Assignment 2: Functions

1. Determine whether f is a function from Z to R if
(a) $f(n) = \pm n$ (b) $f(n) = \sqrt{n^2 + 1}$ (c) $f(n) = 1/(n^2 - 4)$

2. Determine whether f is a function from the set of all bit strings to the set of integers if
(a) $f(S)$ is the position of a 0 bit in S.
(b) $f(S)$ is the number of a 1 bits in S.
(c) $f(S)$ is the smallest integer i such that the ith bit of S is 1 and $f(S) = 0$ when S is the empty string, the string with no bits.

3. Find the domain and range of each of the following functions that assigns:
(a) to each nonnegative integer its last digit
(b) the next largest integer to a positive integer
(c) to a bit string the number of one bits in the string
(d) to each bit string the number of ones minus the number of zeros
(e) to each bit string twice the number of zeros in that string
(f) to each positive integer the largest perfect square not exceeding this integer
(g) the number of bits left over when a bit string is split into bytes

4. Find the domain and range of each of the following functions that assigns to:

(a) each pair of positive integers the first integer of the pair
(b) each positive integer its largest decimal digit
(c) a bit string the longest string of ones in the string
(d) each positive integer the largest integer not exceeding the square root of the integer
(e) each pair of positive integers the maximum of these two integers
(f) each positive integer the number of the digits that do not appear as decimal digits of the integer
(g) a bit string the number of times the block 11 appears
(h) a bit string the numerical position of the first 1 in the string and the value 0 to a bit string consisting of all 0s.

5. Find the values:

$(a)\ \left\lceil \left\lfloor \frac{1}{2} \right\rfloor + \left\lceil \frac{1}{2} \right\rceil + \frac{1}{2} \right\rceil$ $\quad (b)\ \left\lfloor \frac{1}{2} + \left\lceil \frac{3}{2} \right\rceil \right\rfloor$ $\quad (c)\ \left\lfloor \frac{1}{2} \cdot \left\lfloor \frac{5}{2} \right\rfloor \right\rfloor$

6. Determine whether each of these functions from Z to Z is one-to-one and onto.

$(a)\ f(n) = n - 1$ $\quad (b)\ f(n) = n^2 + 1$ $\quad (c)\ f(n) = n^3$ $\quad (d)\ f(n) = \lceil {}^{n}/_{2} \rceil$

7. Determine whether $f: Z \times Z \to Z$ is one-to-one and onto if

$(a)\quad f(m,n) = 2\,m - n$ $\qquad (b)\quad f(m,n) = m^2 - n^2$ $\qquad (c)\quad f(m,n) = |m| - |n|$

$(d)\quad f(m,n) = m + n$ $\qquad (e)\quad f(m,n) = m$ $\qquad (f)\quad f(m,n) = |n|$

8. Determine whether each of these functions is a bijection from $\boldsymbol{R}$ **to** $\boldsymbol{R}$.

$(a)\ f(x) = -3x + 4 \qquad (b)\ f(x) = 3x^2 + 7 \qquad (c)\ f(x) = \dfrac{x+1}{x+2} \qquad (d)\ f(x) = \dfrac{x^2+1}{x^2+2}$

9. Show that the function $f: \boldsymbol{R} \to \boldsymbol{R}$ defined by $f(x) = e^x$ is not invertible. Modify the domain or codomain of f so that it becomes invertible.

10. Show that the function $f: \boldsymbol{R} \to \boldsymbol{R}^{+} \cup \{0\}$ defined by $f(x) = |x|$ is not invertible. Modify the domain or codomain of f so that it becomes invertible.

11. Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from $\boldsymbol{R}$ to $\boldsymbol{R}$.

12. If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer.

13. If f and $f \circ g$ are onto, does it follow that g is onto? Justify your answer.

14. Suppose that g is a function from A to B and f is a function from B to C.
(a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.
(b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

15. Show that the function $f(x) = ax + b$ from $\boldsymbol{R}$ to $\boldsymbol{R}$ is invertible, where a and b are constants, with $a \neq 0$, and find the inverse of f.

16. Let $f: \boldsymbol{R} \to \boldsymbol{R}$ be defined by $f(x) = x^2$. Find
(a) $f^{-1}(\{1\})$ (b) $f^{-1}(\{x \mid 0 < x < 1\})$ (c) $f^{-1}(\{x \mid x > 4\})$.

17. Let $g(x) = \lfloor x \rfloor$. Find
(a) $g^{-1}(\{0\})$ (b) $g^{-1}(\{-1, 0, 1\})$ (c) $g^{-1}(\{x \mid 0 < x < 1\})$.

18. Let $f\colon A \to B$ and $S, T \subseteq A$. Show that

$$(a)\ f(S \cup T) = f(S) \cup f(T) \qquad (b)\ f(S \cap T) \subseteq f(S) \cap f(T).$$

Give an example to show that the inclusion in part (b) may be proper.

19. Let $f\colon A \to B$ and $S, T \subseteq B$. Show that

$(a)\ f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$ $\qquad$ $(b)\ f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

$(c)\ f^{-1}(\bar{S}) = \overline{f^{-1}(S)}.$

20. In asynchronous transfer mode (ATM) (a communications protocol used on backbone networks), data are organized into cells of 53 bytes. How many ATM cells can be transmitted in 10 seconds over a link operating at the following rates?

(a) 128 kilobits per second (b) 300 kilobits per second (c) 1 megabit per second

21. Draw graphs of each of these functions.

(a) $f(x) = \lceil x \rceil + \lfloor x/2 \rfloor$ (b) $f(x) = \lceil 1/x \rceil$ (c) $f(x) = \lfloor 2x + 1 \rfloor$

(d) $f(x) = \lceil x/2 \rceil \lfloor x/2 \rfloor$ (e) $f(x) = \lfloor 2\lceil x/2 \rceil + 1/2 \rfloor$ (f) $f(x) = \lfloor x^2 \rfloor$

Use separate sheet of paper to draw the graphs.

For any real number x, it is possible to find an integer n such that

$$n \leq x < n + 1.$$

In this case, $\lfloor x \rfloor = n$, and $\lceil x \rceil = n + 1$. It is clear that $\lfloor x \rfloor \leq x \leq \lceil x \rceil$. It is also clear that we can express x as $x = n + \varepsilon$, where $0 \leq \varepsilon < 1$. Use this fact to prove the elementary results in the examples below.

22. Let x be a real number and m, n are integers. Prove that

(a) $\lceil x \rceil - \lfloor x \rfloor = \begin{cases} 1, & if\ x\ is\ not\ an\ ineger \\ 0, & if\ x\ is\ an\ ineger \end{cases}.$

(b) $\lceil x + m \rceil = \lceil x \rceil + m.$

(c) $x < n$ if and only if $\lfloor x \rfloor < n$. (d) $n < x$ if and only if $n < \lceil x \rceil$.

(e) $x \leq n$ if and only if $\lceil x \rceil \leq n$. (f) $n \leq x$ if and only if $n \leq \lfloor x \rfloor$.

23. Prove or disprove each of these statements about the floor and ceiling functions. Let x and y be real numbers.

(a) $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$

(b) $\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = 0 \; or \; 1$

(c) $\lfloor 2x \rfloor = 2\lfloor x \rfloor$

(d) $\lceil xy \rceil = \lceil x \rceil \lceil y \rceil$

(e) $\left\lceil \frac{x}{2} \right\rceil = \left\lfloor \frac{x+1}{2} \right\rfloor$

(f) $\left\lceil \frac{\lceil x/2 \rceil}{2} \right\rceil = \left\lceil \frac{x}{4} \right\rceil$

(g) $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

24. Prove that if x is a positive real number, then

(a) $\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \lfloor \sqrt{x} \rfloor$ $\qquad$ (b) $\left\lceil \sqrt{\lceil x \rceil} \right\rceil = \lceil \sqrt{x} \rceil$.

In these examples, square roots of floor and ceiling functions are involved. So, instead of assuming $x = n + \varepsilon$, we will represent x as $x = n^2 + m + \varepsilon$, where n^2 is the nearest perfect square less than or equal to x. For example, 12.2 can be written as $12 = 9 + 3 + \varepsilon$.

Proof of (a):

Let $x = n^2 + m + \varepsilon$. Then $\lfloor x \rfloor = n^2 + m$.

So, $n \leq \sqrt{\lfloor x \rfloor} < n + 1. \quad (Why?)$

$\therefore \; L.H.S. = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = n.$

Also, $n \leq \sqrt{x} < n + 1. \quad (Why?)$

$\therefore R.H.S. = \lfloor \sqrt{x} \rfloor = n.$

Thus, (a) is proved.

In a similar way, try to prove the second.

25. Let a and b be real numbers with $a < b$. Use the floor and/or ceiling functions to express the number of integers n that satisfy the inequality $a \leq n \leq b$.

26. Suppose that f is an invertible function from Y to Z and g is an invertible function from X to Y. Show that the inverse of the composition $f \circ g$ is given by $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

27. Let S be subset of a universal set U. The characteristic function f_S of S is the function from U to the set $\{0, 1\}$ such that $f_S(x) = 1$ if x belongs to S and $f_S(x) = 0$ if x does not belong to S. Let A and B be sets. Show that for all x,

$(a)\ f_{A \cap B}(x) = f_A(x) \cdot f_B(x)$ $(b)\ f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$

$(c)\ f_{\bar{A}}(x) = 1 - f_A(x)$ $(d)\ f_{A \oplus B}(x) = f_A(x) + f_B(x) - 2f_A(x) \cdot f_B(x)$