# Relations:

**Ordered Pair:-** Let $A$ & $B$ be two sets. Let $a \in A$ and $b \in B$. Then $(a,b)$ denotes ordered pair and $a$ is known as first co-ordinate of the ordered pair $(a,b)$ and $b$ as second co-ordinate of the ordered pair $(a,b)$.

**Cartesian Product:**

- Let $A$ and $B$ be two sets $A \times B$ is called cartesian product of $A$ and $B$.
- $A \times B$ is a set of all distinct ordered pairs, in which first co-ordinate of ordered pair is from set $A$ and second co-ordinate is from set $B$.

$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$.

Let $A = \{1,2,3\}$, $B = \{a,b\}$.

then $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$.

$= \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$.

$(1,2) \notin A \times B$.

as $2 \notin B$.

$(a,b) \notin A \times B$.

as $a \notin A$.

$(b,3) \notin A \times B$.

as $b \notin A$ and $3 \notin B$.

$\Rightarrow (a,b) \notin A\times B$, either $a \notin A$ or $b \notin B$ or both.

Hence $A\times B \neq B\times A$.

So, Cartesian product is non commutative, if A and B are non empty and different sets. i.e. $A \neq \phi$, $B \neq \phi$, and $A \neq B$.

Similarly, $A\times B\times C$ is a set of all distinct ordered triple

$A\times B\times C = \{(a,b,c) : a\in A \;\&\; b\in B$ and $c\in C\}$.

this way,

$A_1\times A_2 \times A_3 \dots \times A_n = \{(a_1, a_2, a_3, \dots, a_n) : a_1\in A_1, a_2 \in A_2, \dots a_i\in A_i, \dots a_n \in A_n\}$

is set of ordered n-tuples.

**Remark**:- 1.) If A contains 3 elements and B contains 4 ~~elements~~ elements then $A\times B$ contains $3\times 4 = 12$, element.

i.e $|A\times B| = m.n$

if $|A| = m$ and $|B| = n$.

2) $A\times B\times C = \{(a,b,c) ; a\in A, b\in B \;\&\; c\in C\}$

$A\times(B\times C) = \{(a,(b,c)) ; a\in A, (b,c) \in (B\times C)\}$

$(A\times B)\times C = \{((a,b),c) ; (a,b)\in A\times B, c\in C\}$.

All these $A\times B\times C$, $(A\times B)\times C$, $A\times(B\times C)$ are different.

Normally product of A, B, C, we define as $A\times B\times C$.

**Thm**:- If A, B, C are sets then

i) $A\times(B\cap C) = (A\times B)\cap(A\times C)$.

ii) $(A\cap B)\times C = (A\times C)\cap(B\times C)$.

iii) $A\times(B\cup C) = (A\times B)\cup(A\times C)$.

iv) $(A\cup B)\times C = (A\times C)\cup(B\times C)$.

**proof**:- Let $(a,b)\in A\times(B\cap C)$.

$\Leftrightarrow a\in A$ and $b\in(B\cap C)$

$\Leftrightarrow a\in A$ and $(b\in B$ and $b\in C)$

$\Leftrightarrow (a\in A$ and $b\in B)$ and $(a\in A$ and $b\in C)$.

$\Leftrightarrow (a,b)\in(A\times B)$ and $(a,b)\in(A\times C)$

$\Leftrightarrow (a,b)\in(A\times B)\cap(A\times C)$.

$\Rightarrow A\times(B\cap C) = (A\times B)\cap(A\times C)$.

Students are advised to prove the other three results, as an exercise.

**Remarks**:- 1) If $\mathbb{R}$ is the set of real numbers then $\mathbb{R}\times\mathbb{R}$ we represent by $\mathbb{R}^2$ and $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ by $\mathbb{R}^3$.

2) $\mathbb{R}\times\mathbb{R}$ is called the Euclidean plane.

* **Relation or Binary Relation:-**

→ Let S be a set of family members

$S = \{f, m, b_1, b_2, s_1\}$.

Number of elements of $S = 5$ (also known as cardinality of S and denoted by $n(S)$ or $|S|$).

$f \to$ father, $m \to$ mother, $b_1, b_2 \to$ brothers, $s_1 \to$ sister than $S \times S$, the cartesian product is a set of ordered pair.

$$S \times S = \begin{bmatrix} (f,f) & (f,m) & (f,b_1) & (f,b_2) & (f,s_1) \\ (m,f) & (m,m) & (m,b_1) & (m,b_2) & (m,s_1) \\ (b_1,f) & (b_1,m) & (b_1,b_1) & (b_1,b_2) & (b_1,s_1) \\ (b_2,f) & (b_2,m) & (b_2,b_1) & (b_2,b_2) & (b_2,s_1) \\ (s_1,f) & (s_1,m) & (s_1,b_1) & (s_1,b_2) & (s_1,s_1) \end{bmatrix}$$

Number of elements in $S \times S$ is $5 \times 5 = 25$ i.e. cardinality of $S \times S$ is 25.

Now let R be a relation on the set S say R be father relation.

then $R \subset A \times B$.

$R = \{(f, b_1), (f, b_2), (f, s_1)\}$

f is father of $b_1$, denoted by $f R b$ or $(f, b) \in R$.

f is father of $b_2$, f is father of $s_1$.

i.e. $R \subset S \times S$.

**Remarks:-** 1) Relation is always defined on set.

2) Binary relation is a subset of cartesian product of two sets. Both sets may be same or different. If R is defined from set A to set B then, $R \subset A \times B$.

3) The sets A and B are also known as set of departure and the set of destination of R respect.

Ex:- (1) A is the set of natural number N. The set $\{(x, y) : x = y\}$ means that x & y are so related that $x = y$.

then $R = \{(1,1), (2,2), (3,3), \ldots (n,n) \ldots\}$.

(2) $A = N$.

$R = \{(x, y) : y = x^2\}$.

then $R = \{(1,1), (2,4), (3,9), (4,16), (5,25) \ldots \ldots (n, n^2) \ldots\}$.

(3) $A = \{1, 2, 3, \ldots, 19, 20\}$.

$R = \{(x, y) : x = 3y\}$.

then $R = \{(3,1), (6,2), (9,3), (12,4), (15,5), (18,6)\}$.

(4) $A = \{2, 3, 5, 6\}$.

$R = \{(x, y) : x | y$ (x divides y)$\}$.

then $R = \{(2,2), (2,6), (3,3), (3,6), (5,5), (6,6)\}$.

**Result 1:** If $A, B, C$ are three sets and $A \subseteq B$ then $A \times C \subseteq B \times C$.

**Proof:** Let $(x, y) \in A \times C \Rightarrow x \in A$ and $y \in C$.
$\Rightarrow x \in B$ and $y \in C$ (as $A \subseteq B$)
$\Rightarrow (x, y) \in B \times C$.
$\Rightarrow A \times C \subseteq B \times C$.

**Result - 2:** $(A-B) \times C = (A \times C) - (B \times C)$
Let $(x, y) \in (A-B) \times C$
$\Leftrightarrow x \in (A-B)$ and $y \in C$
$\Leftrightarrow (x \in A$ and $x \notin B)$ and $y \in C$.
$\Leftrightarrow (x \in A$ and $y \in C)$ and $(x \notin B$ & $y \in C)$.
$\Leftrightarrow (x, y) \in A \times C$ and $(x, y) \notin (B \times C)$
$\Leftrightarrow (x, y) \in (A \times C) - (B \times C)$
$\Rightarrow (A-B) \times C = (A \times C) - (B \times C)$.

* **Domain and Range of a Relation:**
→ Let $A$ and $B$ be two sets and $R$ be a relation from $A$ to $B$.
i.e $R \subset A \times B$.
and $R = \{(a, b) : a \in A$ & $b \in B\}$.
The set of all first coordinates of $R$ is known as domain of $R$ and the set of all second coordinates of $R$ is known as range of $R$.
Thus domain of $R = \{a : (a, b) \in R\}$.
and range of $R = \{b : (a, b) \in R\}$.

* **Matrix Representation of a Relation:**
→ Let $A = \{a_1, a_2, \ldots, a_n\}$
& $B = \{b_1, b_2, \ldots, b_m\}$.
$A$ and $B$ are finite sets containing $n$ and $m$ elements respectively. Then $A \times B$ will contain $n \times m$ elements.
→ Let $R$ be a relation from $A$ to $B$. then $R \subseteq A \times B$.
To represent $R$ as matrix $n \times m$ known as $M_R$ [matrix of relation] $= [m_{ij}]$.
We take, $m_{ij} = 1$ if $a_i R b_j$ or $(a_i, b_j) \in R$.
$= 0$ if $a_i \not R b_j$ or $(a_i, b_j) \notin R$.

**Ex:-** $A = \{1, 2, 3, 4\}$, $B = \{1, 4, 6, 8, 9\}$.
$aRb$ if and only if $b = a^2$ then find the relation matrix $M_R$.

**Sol:-** $R = \{(1,1), (2,4), (3,9)\}$.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 4 & 6 & 8 & 9 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 5} \end{array}$$

Ex:- $A = \{1, 2, 3, 4, 6\} = B$.
$aRb$ if and only if $a|b$ then find the relation matrix $M_R$.

Solⁿ:- $R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (2,4), (2,6), (3,6), (2,2), (3,3), (4,4), (6,6)\}$.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5\times 5} \end{array}$$

Ex:- $A = \{1, 2, 3, 5, 6, 10, 15, 30\} = B$,
$aRb$ iff $a|b$. Find relation matrix.
$R = \{(1,1), (1,2), (1,3), (1,5), (1,6), (1,10), (1,15), (1,30), (3,6), (3,15), (3,30), (2,6), (2,10), (2,30), (5,10), (5,15), (5,30), (6,30), (10,30), (15,30)\} \cup \{(2,2), (3,3), (5,5), (6,6), (10,10), (15,15), (30,30)\}$

Solⁿ:

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 5 \\ 6 \\ 10 \\ 15 \\ 30 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 5 & 6 & 10 & 15 & 30 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

* **Relation Matrix Operations:**

→ Relation matrix is known as Boolean matrix as the entries are either 0 or 1.

→ Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m\times n$ then $A+B = [a_{ij} + b_{ij}]$.
$= [c_{ij}]$
where $c_{ij} = 1$ if $a_{ij} = 1$ or $b_{ij} = 1$.
and $c_{ij} = 0$ if $a_{ij} = 0$ and $b_{ij} = 0$.

also if $A = [a_{ij}]_{m\times n}$, $B = [b_{jk}]_{n\times k}$
be Boolean matrix then, $A.B = [a_{ij} b_{jk}]$.
$= [d_{ik}]$
and $d_{ik} = 1$ if $a_{ij} = 1$ and $b_{jk} = 1$.
$= 0$ if $a_{ij} = 0$ or $b_{jk} = 0$.

e.g. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$A+B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Q. $A \cdot B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $A \cdot B = \cancel{\begin{bmatrix} 1+0+0 & 1+0+0 & 1+0+0 \\ 0+0+0 & 0+1+0 & 0+1+0 \\ 1+0+1 & 1+1+0 & 1+1+0 \end{bmatrix}}$

* Properties of Relation matrix :

→ Let $R_1$ be a relation from A to B and $R_2$ be a relation from B to C. Then the relation matrices satisfy the following properties.

$M_{R_1 \cdot R_2} = M_{R_1} \cdot M_{R_2}$.

$M_{R^{-1}} =$ Transpose of $M_R$.

$\cancel{(M_{R_1 \cdot R_2})^{-1}} =$

$M_{(R_1 \cdot R_2)^{-1}} = M_{R_2^{-1}} \cdot M_{R_1^{-1}}$

* Graphical Representation of a Relation (Diagraph)

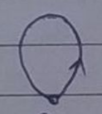
→ Let A be a finite set and R be a relation defined on A, then R can be represented by graph.

→ The elements of set A are represented by circles or points. These elements are known as nodes or vertices.

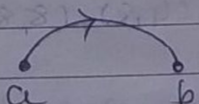
If aRa then, this can be represented by loop around a. If aRb, then, this can be shown by an arc from a to b with an arrow from a to b.

i.e. $a \to b$. These arcs and loops are known as edges of the graph.

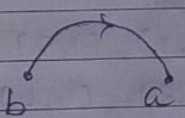
→ This graphical representation is known as directed graphs or diagraph of R.



aRa.

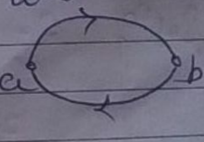


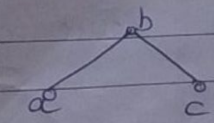

aRb.



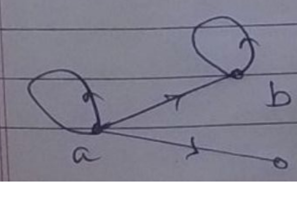

bRa

aRb ∧ bRa

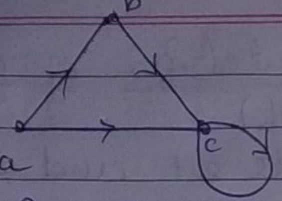


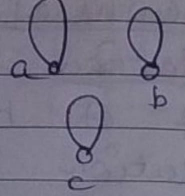

aRb ∧ bRc



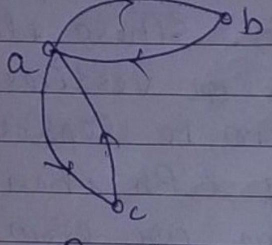

⇒ aRa ∧ bRb ∧ aRb ∧ aRc.



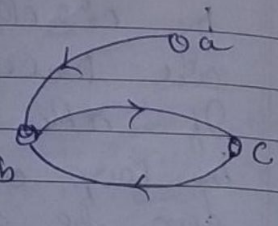

$cRc \wedge aRb \wedge bRc \wedge aRc.$




$\Rightarrow aRa \wedge bRb \wedge cRc.$




$aRb \wedge bRa \wedge aRc \wedge cRa$

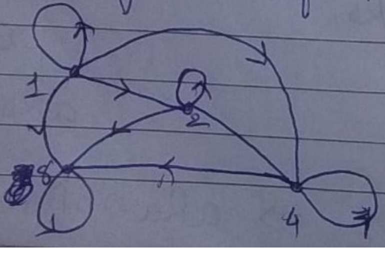



$aRb \wedge bRc \wedge cRb.$

Ex:- $A = \{1, 2, 4, 8\}$.
$aRb$ iff $a|b$ ($a$ divides $b$) then find the diagraph of relation.

Solⁿ:- $R = \{(1,1), (1,2), (1,4), (1,8), (2,2), (2,4), (2,8), (4,4), (4,8), (8,8)\}$.

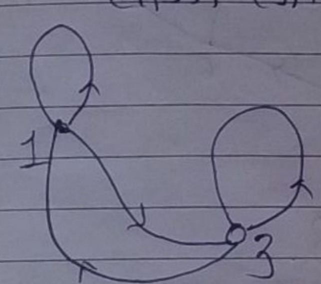
Hence diagraph of R is the following

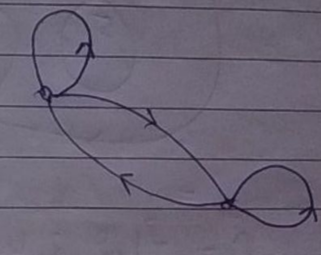





Ex:- $A = \{1, 2, 3, 4\}$.
If $R = \{(a,b) \mid (a-b)$ is an integral multiple of $2\}$.
then find the diagraph of relation

Solⁿ:- $R = \{(1,1), (2,2), (3,3), (4,4), (2,4), (4,2), (1,3), (3,1)\}$.

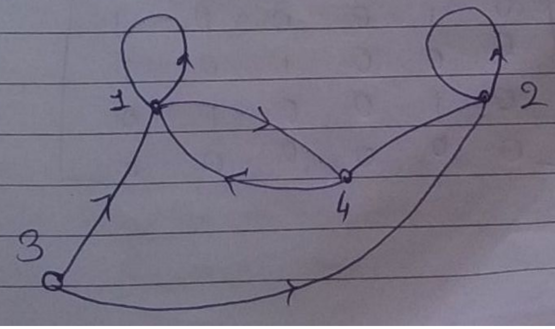


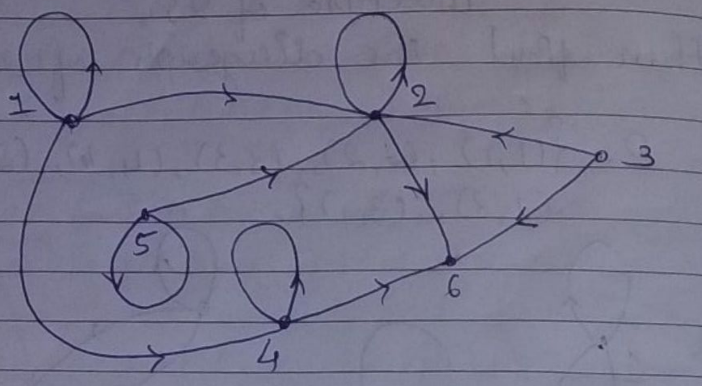
Ex:- $A = \{1,2,3,4\}$ & $M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Draw the diagraph of R.

Solⁿ:- $R = \{(1,1), (1,4), (2,2), (2,4), (3,1), (3,2), (4,1)\}$.



Ex:- Find the relation determined by the diagraph and give its matrix.




Solⁿ:- $A = \{1,2,3,4,5,6\}$.

$R = \{(1,1), (1,2), (2,2), (1,4), (3,2), (3,6), (4,6), (4,4), (5,5), (5,2), (2,6)\}$.

and the relation matrix.

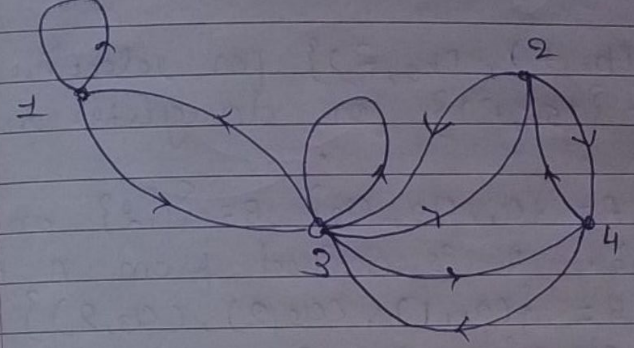
$$M_R = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Ex:- $A = \{1,2,3,4\}$ the relation matrix is

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Draw its diagraph.

Solⁿ:- $R = \{(1,1), (1,3), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,2), (4,3)\}$.




*. Inverse Relation (or Converse Relation).

→ If $R$ is a relation from $A$ to $B$ then inverse relation of $R$, denoted by $R^{-1}$ is a relation $B$ to $A$.

$R^{-1} = \{(y,x) : y \in B, x \in A \text{ and } (x,y) \in R\}$.

If $(x,y) \in R$, i.e. $xRy$.

then $yR^{-1}x$ or $(y,x) \in R^{-1}$.

– The inverse relation is also known as

converse relation and hence sometimes it is denoted by $R^c$.

**Ex:-** I) If $R$ is father relation on set $S$ then
$R = \{(f,b_1), (f,b_2), (f,s_1)\}$.

then inverse relation of $R$ can be 'son relation' or 'daughter relation', so,

$R_1^{-1} = \{(b_1,f), (b_2,f)\}$ for relation son
& $R_2^{-1} = \{(s_1,f)\}$ for daughter relation.

2) If $A = \{a_1, a_2, a_3\}$, $B = \{1,2\}$ and relation $R$ is defined from $A$ to $B$
and $R = \{(a_1,1), (a_2,2), (a_3,2)\}$
then relation $R^{-1}$ is defined from $B$ to $A$.
and $R^{-1} = \{(1,a_1), (2,a_2), (2,a_3)\}$.

**Results:-** If $R_1, R_2$ and $R$ are relations from $A$ to $B$ then
i) $(R^{-1})^{-1} = R$.
ii) $(R_1 \cup R_2)^{-1} = R_2^{-1} \cup R_1^{-1}$
iii) $(R_1 \cap R_2)^{-1} = R_2^{-1} \cap R_1^{-1}$.

## * Complement of a Relation:-

→ Let $A$ and $B$ be two non empty sets. A relation $R$ from $A$ to $B$ is defined as
$R = \{(a,b) : a \in A \ \& \ b \in B\}$.
then $R'$ (complement of $R$) is defined as
$R' = \{(a,b) : (a,b) \notin R\}$.
i.e. $aR'b$ iff $a \not R b$.

**Results:-** 1) $(R_1 \cup R_2)' = R_2' \cap R_1'$
2) $(R_1 \cap R_2)' = R_2' \cup R_1'$ } De Morgan's Laws.

→ If relation is defined between two elements i.e. $R \subset A \times B$ then it is called binary relation. If relation is defined on set $A$ for one element then $R$ is known as unary relation e.g. negation of an element is a unary relation.

→ If relation is defined between three element i.e. $R \subset A \times B \times C$ then relation is called ternary relation.

**Remark:-** If $R \subset A \times B$ then $R^{-1} \subset B \times A$.

## * Composite of Binary Relations :

**Defn.** Let $R_1$ be a relation from A to B and $R_2$ a relation from B to C. The composite relation from A to C, denoted by $R_1 . R_2$ (or $R_1 R_2$) is defined as

$R_1 . R_2 = \{(a,c) \mid a \in A$ and $c \in C$ and $b \in B$ such that $(a,b) \in R_1$ and $(b,c) \in R_2\}$.

**Ex:- 1)** $A = \{1, 2, 3, 4, 5\}$.
$R_1 = \{(1,2), (1,3), (2,4), (3,5)\}$.
$R_2 = \{(3,2), (4,5), (5,1)\}$
→ then $R_1 . R_2 = \{(1,2), (2,5), (3,1)\}$
also $R_2 . R_1 = \{(3,4), (5,2), (5,3)\}$.
Clearly $R_1 . R_2 \neq R_2 . R_1$.

## * Relation in a set :-

→ Let R be a relation from A to itself, then R is a relation in A.
i.e $R \subseteq A \times A$.

## * Identity Relation in a Set :-

→ Let A be a set. The relation $I_A$ defined by, $I_A = \{(a,b) : a \in A, b \in A, a = b\}$ is called the identity relation. Thus identity relation in a set A is the set of ordered pair (a,b) of $A \times A$ for which $a = b$.

Ex:- $A = \{1, 2, 3, 4, 5, 6, 7\}$.
→ ∴ $I_A = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7)\}$

## * Universal Relation in a set :-

→ Let A be any set and R be any relation on set A, then $R \subseteq A \times A$.
→ If $R = A \times A$, then the relation R is known as universal relation.

## * Void Relation (Empty Relation) :

→ A relation R on a set A is called void relation. If no element of set A is related to any element of A.
Hence $R = \phi$.

## * Types of Relation :-

**(1) Reflexive Relation :-**
⇒ Let R be a relation in a set A, i.e. $R \subset A \times A$ then R is known as

. Reflexive relation if $\forall a \in A$
$(a,a) \in R$.
i.e. every element is related with itself.

**Ex:-** If A is a set of all straight lines in 2-D plane & R is a relation,
$R = \{(a,b) : a \text{ is parallel to } b\}$.
then R is reflexive relation as every straight to itself.

**Remark:-** 1) If R is reflexive on set A.
i.e. $a \in A$, $aRa$ reflexive
i) In matrix of relation, $(M_R)$, all the diagonal elements will be 1.
ii) Digraph of reflexive relation will have loop for every element of A.

2) If R is not reflexive, then R is known as irreflexive if $\forall a \in A$.
$(a,a) \notin R$ or $a \not R a$.
hence all the diagonal entries in $M_R$ will be zero and digraph of irreflexive relation will have no loop for any element of A.

**Ex:-** I is the set of integers and R is a relation.
$R = \{(a,b) : a|b \text{ (a divides b)}\}$.
then R is reflexive relation as every element divide itself.

**Ex:-** If I is a set of integers and R is a relation
$R = \{(a,b) ; 5 | a-b \text{ (5 divides } (a-b))\}$.
then R is reflexive relation as
[$5 | a-a$ or $5|0$] zero is divisible by every non zero number].

**Ex:-** If A is set of all straight lines in 2-D plane & R is a relation.
$R = \{(a,b) : a \text{ is perpendicular to } b\}$.
then R is not reflexive relation as no straight line is perpendicular to itself. This is an example of irreflexive relation.

## (2)* Symmetric Relation:-
→ Let R be a relation in a set A i.e. $R \subset A \times A$. Then the relation R is said to be symmetric relation.

If $(a,b) \in R \Rightarrow (b,a) \in R$.
i.e. if a is related with b with the relation R then b should be related with a with the same relation R.

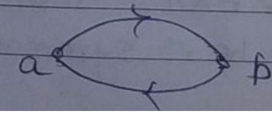
$aRb \Rightarrow bRa$

In other way $R = R^{-1}$ then relation is symmetric.

Ex:- A is a set of all straight lines in 2-D plane and R is a relation.
$R = \{(a,b) ; a \text{ is parallel to } b\}$.
then R is symmetric relation as if a is parallel to b then b is also parallel to a.
i.e. $(a,b) \in R \Rightarrow (b,a) \in R$.

Ex:- A is a set of all straight lines in 2-D plane & R is a relation.
$R = \{(a,b) ; a \text{ is perpendicular to } b\}$
then R is symmetric relation as if a is perpendicular to b then b is also perpendicular to a.

⇒ Digraph of a symmetric relation:-
$aRb \Rightarrow bRa$



Remarks:- Matrix of a relation $M_R$ will have same entries for $ij$th value and $ji$th value i.e. if $a_{ij}=1$ then $a_{ji}=1$.
also if $a_{ij}=0$ then $a_{ji}=0$.
in short transpose of relation matrix $(M_R)^T$ is same as relation matrix $(M_R)$.
i.e. $(M_R)^T = M_R$.
Also, $R = R^{-1}$, then R is symm. relation

* Compatible Relation:-
→ A relation R on a set S is said to be compatible if it is reflexive and symmetric.

Remark. If R is a compatible relation its relation matrix is symmetric with the diagonal elements being 1.

(3) * Antisymmetric Relation:-
→ Let R be a relation in a set A i.e. $R \subset A \times A$. The relation R is said to be antisymmetric relation: If $(a,b) \in R$ and $(b,a) \in R \Rightarrow a = b$.
i.e. if a is related with b with relation R and also b is related with

a with the same relation R then a and b are same.

Ex:-1) $A \subseteq B$, $B \subseteq A \Rightarrow A = B$.
So 'subset' relation is antisymmetric relation.

Ex:-2) $a \le b$, $b \le a \Rightarrow a = b$.
So, 'less than or equal to' relation is antisymmetric relation.

Ex:-3) As ex.2 $a \ge b$, $b \ge a \Rightarrow a = b$.
'greater than or equal to' relation is antisymmetric relation.

Ex:-4) N is a set of natural numbers.
$R = \{(a,b) : a|b$, a divides b$\}$.
then R is antisymmetric relation.
i.e. if $a|b$ and $b|a \Rightarrow a = b$.
But the same relation is not antisymmetric on the set of integers
$4|{-4}$ also $-4|4$ but $4 \neq -4$.
i.e $\{(1,1),(2,2),(3,3),(4,4),\ldots\}$ for N
$\{(1,1),(1,-1),(-1,-1),(-1,1),\ldots\}$ for I

(4) * Transitive Relation:-
→ Let R be a relation on set A i.e. $R \subset A \times A$ then R is called transitive relation on set A if
$(a,b) \in R$, $(b,c) \in R \Rightarrow (a,c) \in R$.
i.e if a is related with b and b is related with c and this shows a is related with c with the same relation then R is called transitive relation.
i.e $aRb$, $bRc \Rightarrow aRc$

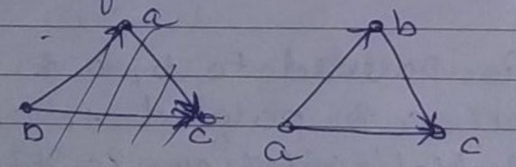
Ex-1 $A \subseteq B$, $B \subseteq C \Rightarrow A \subseteq C$
So 'subset' relation is transitive relation.

Ex-2 a is parallel to b, b is parallel to c, then a is parallel to c.
'parallel' relation is transitive on the set of straight line in 2-D plane.

Ex-3 Let I be the set of integers and R is a relation defined on set I.
$R = \{(a,b) : a|b\}$.
then R is transitive relation as if
$a|b$ & $b|c \Rightarrow a|c$

i.e. $(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$.
or $aRb, bRc \Rightarrow aRc$

**Ex-4)** R is a relation on the set of integers.
$R = \{(a,b) \text{ iff } 5|a-b\}$.
$aRb \Leftrightarrow 5|a-b$.
then R is transitive relation as $aRb, bRc$
$\Rightarrow 5|a-b$ and $5|b-c$.
$\Rightarrow 5|(a-b+b-c)$.
$\Rightarrow 5|a-c$
$\Rightarrow aRc$
hence $aRb, bRc \Rightarrow aRc$.
So, R is transitive relation.
$\Rightarrow$ Digraph of transitive relation




## (5) * Transitive Closure :

**Defn:** The transitive closure of a relation R is the smallest transitive relation containing R.
Transitive closure of R is denoted by $R^*$.

$\Rightarrow$ To find transitive closure of R.
$\rightarrow$ Let A be a set and cardinality of A is $n$, i.e. $|A| = n$.
and let R be a relation on A. Then,
$R^* = R \cup R^2 \cup R^3 \cup \dots \cup R^n$.
where $R^2 = R \circ R$
$R^3 = R.R.R$ etc.

**Ex2.** If $A = \{1,2,3,4,5\}$.
and $R = \{(1,2),(3,4),(4,5),(4,1),(1,1)\}$.
find its transitive closure.

**Solⁿ.** Let $R^*$ the transitive closure of R.
Where $R = \{(1,2),(3,4),(4,5),(4,1),(1,1)\}$.
then $R^* = R \cup R^2 \cup R^3 \cup \dots \cup R^n$.
$R.R = R^2 = \{(3,5),(3,1),(4,2),(4,1),(1,1),(1,2)\}$.
$R.(R.R) = R^3 = \{(3,2),(3,1),(4,1),(4,2),(1,1),(1,2)\}$
$R^4 = \{(3,1),(3,2),(4,1),(4,2),(1,1),(1,2)\}$
$R^5 = \{(3,1),(3,2),(4,1),(4,2),(1,1),(1,2)\}$
Hence $R^3 = R^4 = R^5$.
So $R^* = R \cup R^2 \cup R^3$
$\therefore R^* = \{(1,2),(3,4),(4,5),(4,1),(1,1),(3,5),(3,1),(4,2),(3,2)\}$.

## * Warshall's Algorithm :-

→ To find the transitive closure of a relation R, sometimes the method of computing $R^2, R^3, \ldots$ is inefficient for large number of set. Warshall's algorithm is an efficient method for finding transitive closure of a relation R.

→ Let R be a relation on a set $A = \{a_1, a_2, \ldots, a_n\}$.

→ If $x, x_1, x_2, \ldots, x_n, y$ is a path in R then the vertices other than $x, y$ are known as interior vertices and $xRx_1, x_1Rx_2, \ldots, x_nRy$.

→ $x_1, x_2, \ldots, x_n$ are interior vertices of the path. For $1 \le i \le n$, define a boolean matrix $W_k$ has 1 in position $i, j$ if and only if there is a path from $a_i$ to $a_j$ in R whose interior vertices, if any are from the set $\{a_1, a_2, \ldots, a_k\}$.

→ Since any vertex must come from the set $\{a_1, a_2, \ldots, a_n\}$, it follows that the matrix $W_n$ has a 1 in position $i, j$ if and only if some path in R connects $a_i$ with $a_j$.

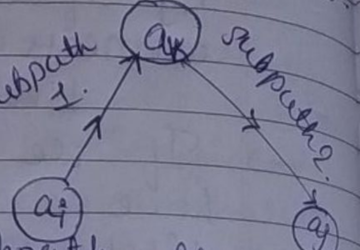
Hence $W_n = M_R^*$.

→ If we define $W_0 = M_R$, then we will have a sequence $W_0, W_1, \ldots, W_n$ whose first term is $M_R$ and last term is $M_R^*$.

→ Warshall's algorithm gives a procedure to compute each matrix $W_k$ from the previous matrix $W_{k-1}$. Beginning with the matrix of relation R, we proceed one step at a time, until we ~~such~~ reach the matrix of $R^*$, in n steps. The matrices $W_k$, being different from powers of the matrix $M_R$, result in a considerable saving of steps in the computation of the transitive closure of relation R.

→ Suppose $W_{k-1} = [u_{ij}]$ & $W_k = [v_{ij}]$, If $v_{ij} = 1$, there is a path from $a_i$ to $a_j$ whose interior vertices come from the set $\{a_1, a_2, \ldots, a_k\}$. If $a_k$ is not an interior vertex of this path, then all the interior vertices must come actually from $\{a_1, a_2, \ldots, a_{k-1}\}$ hence $u_{ij} = 1$.

→ If $a_k$ is an interior vertex of the path, then we must have the situation.




→ Since there is a subpath from $a_i$ to $a_k$ whose interior vertices come from $\{a_1, a_2, \ldots, a_{k-1}\}$ we must have $u_{ik} = 1$, similarly $u_{kj} = 1$.

Hence $v_{ij} = 1$ if and only if

i) $u_{ij} = 1$

ii) $u_{ik} = 1$ and $u_{kj} = 1$.

This is the basis of Warshall's algorithm. If $W_{k-1}$ has 1 in position $i,j$ then by (i) $W_k$ will have 1 in position $i,j$. A new 1 can be added in position $i,j$ of $W_k$ if and only if column $k$ of $W_{k-1}$ has 1 in position $i$, and row $k$ of $W_{k-1}$ has 1 in position $j$.

Thus we have the following procedure for computing $W_k$ from $W_{k-1}$.

**Step-1.** Transfer to $W_k$, all the 1's in $W_{k-1}$.

**Step-2.** List the locations $p_1, p_2, \ldots$ in column $k$ of $W_{k-1}$ where the entry is 1, and locations $q_1, q_2, \ldots$ in row $k$ of $W_{k-1}$ where the entry is 1.

**Step-3** Put 1's in all the position $p_i, q_j$ of $W_k$ (if they are not already there).

**Ex:-** Use Warshall's algorithm to find transitive closure of R where,

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } A = \{1, 2, 3\}.$$

**Solⁿ:-** Use Warshall's algorithm to find transitive closure of R where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } A = \{1, 2, 3\}.$$

**Solⁿ:-** 
$$W_0 = M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

and $n = 3$.

First we find $W_1$, so that $k = 1$, $W_0$ has 1's in location 1 of column 1 and 3 of column 1.

i.e (1,1) and (3,1).
and 1 of row 1 and 3 of row 1.
i.e (1,1), (1,3).

$p_1 : (1,1)$ $\quad p_2 = (3,1)$

$q_1 : (1,1)$ $\quad q_2 = (1,3)$.

Therefore add (1,1), (3,3), (1,3) in $W_1$. Thus $W_1$ is just $W_0$ with new 1's in the position (1,1), (3,3), (1,3). (If 1's are not already there.)

$$W_1 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 1 \end{array}$$

Now to compute $W_2$, so that $k = 2$, consider $2^{nd}$ column and $2^{nd}$ row.

$p_1 : (2,2)$ $\quad p_2 : (3,2)$.
$q_1 : (2,2)$

Therefore add (2,2), (3,2) in $W_1$ which are already 1 in $W_1$.

$p_i$ : column
$q_i$ : row.

Hence $W_2 = W_1 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 1 \end{array}$

$p_1 : (1,3)$ $\quad p_2 : (3,3)$
$q_1 : (3,1)$ $\quad q_2 : (3,2)$, $\quad q_3 : (3,3)$.

Therefore add (1,1), (1,2), (1,3), (3,1), (3,2), (3,3) in $W_2$ (If 1's are not already there).

Hence,

$$W_3 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 1 \end{array}$$

Hence $R^* = W_3$ is the transitive closure of R.

i.e. $R^* = \{(1,1), (1,2), (1,3), (2,2), (3,1), (3,2), (3,3)\}$.

Ex 2:- Find the transitive closure of R by Warshall's algorithm. where $A = \{1, 2, 3, 4, 5, 6\}$ and $R = \{(x-y) : |x-y| = 2\}$.

Solⁿ:- $A = \{1,2,3,4,5,6\}$.

$R = \{(1,3), (3,1), (2,4), (4,2), (4,6), (6,4), (3,5), (5,3)\}$.

$$W_0 = M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$n=6$.

First we find $W_1$, so that $k=1$.
$p_1 : (3,1)$
$q_1 : (1,3)$
add $(p_i, q_j)$ i.e $(3,3)$ in $W_0$ to get $W_1$.

$$W_1 = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Now we compute $W_2$, so that $k=2$.
$p_1 : (4,2)$
$q_1 : (2,4)$

So add path $(4,4)$ in $W_1$ to get $W_2$.

$$W_2 = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Now compute $W_3$, so that $k=3$.
$p_1 : (1,3)$, $p_2 : (3,3)$, $p_3 : (5,3)$.
$q_1 : (3,1)$, $q_2 : (3,3)$, $q_3 : (3,5)$.

$\therefore$ add path $(p_i, q_j)$ i.e $(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5)$ in $W_2$ to get $W_3$.

$$W_3 = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Now, for $W_4$, $k=4$.

$p_1 : (2,4)$ $p_2 : (4,4)$ $p_3 : (6,4)$
$q_1 : (4,2)$ $q_2 : (4,4)$ $q_3 : (4,6)$

∴ add path $(p_i, q_i)$ i.e $(2,2), (2,4), (2,6)$ $(4,2), (4,4), (4,6), (6,2), (6,4), (6,6)$ in $W_3$ to get $W_4$.

$$W_4 = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 & 1 \\ 3 & 1 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 1 & 0 & 1 \\ 5 & 1 & 0 & 1 & 0 & 1 & 0 \\ 6 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}$$

Now for $W_5$; $k=5$.
$p_1 : (1,5)$, $p_2 : (3,5)$, $p_3 : (5,5)$
$q_1 : (5,1)$, $q_2 : (5,3)$, $q_3 : (5,5)$.

∴ add path $(p_i, q_i)$ i.e $(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5)$, in $W_4$ to get $W_5$.

Here $W_4 = W_5$.
Now for $W_6$, $k=6$
$p_1 : (2,6)$ $p_2 : (4,6)$ $p_3 : (6,6)$
$q_1 : (6,2)$ $q_2 : (6,4)$ $q_3 : (6,6)$.

∴ add path $(p_i, q_i)$ i.e $(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)$ in $W_5$ to get $W_6$. and $W_5 = W_6$. Hence $W_6$ is $R^*$ (i.e. transitive closure of R).

∴ $R^* = \{(1,1), (1,3), (1,5), (2,2), (2,4), (2,6), (3,1), (3,3), (3,5), (4,2), (4,4), (4,6), (5,1), (5,3), (5,5), (6,2), (6,4), (6,6)\}$

Ex:- Find the transitive closure of the relation R on $A = \{1,2,3,4\}$ defined by $R = \{(1,2), (1,3), (1,4), (2,1), (2,3), (3,4), (3,2), (4,2), (4,3)\}$.

Sol^n:-

$$M_R = W_0 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 0 & 1 & 1 & 0 \end{array} \quad n=4.$$

First we find $W_1$, so that $k=1$.
$p_1 : (2,1)$.
$q_1 : (1,2)$, $q_2 : (1,3)$, $q_3 : (1,4)$
∴ add $(p_i, q_i)$ i.e. $(2,2), (2,3), (2,4)$ in $W_0$ to get $W_1$.

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
W_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}
\end{array}
$$

Now we compute $W_2$, so that $k = 2$

$p_1 : (1,2)$ $\quad p_2 : (2,2)$ $\quad p_3 : (3,2)$ $\quad p_4 : (4,2)$

$q_1 : (2,1)$ $\quad q_2 : (2,2)$ $\quad q_3 : (2,3)$ $\quad q_4 : (2,4)$.

$\therefore$ add paths $(p_i, q_j)$, i.e. (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4) in $W_1$ to get $W_2$.

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
W_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}
\end{array}
$$

All positions of $W_2$ are 1, hence $W_2 = R^*$.

Hence $R^* = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$

**Ex:-** Let $R = \{(1,4), (2,1), (2,5), (2,4), (4,3), (5,3), (3,2)\}$

use Warshall's algorithm to find the matrix of transitive closure where $A = \{1, 2, 3, 4, 5\}$.

**Sol^n.**

$$
W_0 = M_R =
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}
\end{array}
\quad n = 5.
$$

First we find $W_1$, so that $k = 1$.

$W_0$ has 1's in location 2 of column 1, i.e. (2,1) and location 4 of row 1, i.e. (1,4).

$p_1 : (2,1)$

$q_1 : (1,4)$.

add $(p_1, q_1)$ i.e. (2,4) in $W_1$.

Thus $W_1$ is just $W_0$ with a new 1 in position (2,4) (1 is already at (2,4) position).

Hence,

$$W_0 = W_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Now we compute $k=2$, column 2 has 1 in $3^{rd}$ location, i.e. (3, 2). Row 2 have 1 in $4^{th}$, $5^{th}$ & $6^{th}$ location.
i.e. (2,1), (2,4), (2,5).
$p_1$ : (3, 2)
$q_1$ : (2,1), (2,4), (2,5).
Hence $(p_i, q_i)$ i.e (3,1), (3, 4), (3,5) paths are added to $W_1$ to get $W_2$, i.e. $W_2$ has 1 in location (3,1), (3,4), and (3,5).

$$W_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Now we compute $k = 3$.
$p_1$: (4, 3), $p_2$: (5,3).
$q_1$: (3,1), $q_2$: (3,2), $q_3$: (3,4), $q_4$: (3,5).

Hence $(p_i, q_i)$ i.e (4,1), (4,2), (4,4), (4,5), (5,1), (5,2), (5,4), (5,5) paths are added to $W_2$ to get $W_3$.

$$W_3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Now we compute $k=4$.
$p_1$: (1,4), $p_2$: (2,4), $p_3$: (3,4), $P_4$: (4,4), $P_5$: (5,4).
$q_1$: (4,1), $q_2$: (4,2), $q_3$: (4,4), $q_5$: (4,5).

Hence $(p_i, q_i)$ i.e (1,1), (1,2), (1,4), (1,5), (2,1), (2,2), (2,4), (2,5), (3,1), (3,2), (3,4), (3,5), (4,1), (4,2), (4,4), (4,5), (5,1), (5,2), (5,4), (5,5) paths are added to $W_3$ to get $W_4$.

$$W_4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Now we compute $k=5$.
$p_1: (1,5)$, $p_2: (2,5)$, $p_3: (3,5)$, $p_4: (4,5)$, $p_5: (5,5)$.
$q_1: (5,1)$ $q_2: (5,2)$, $q_3: (5,4)$, $q_4: (5,5)$.

Hence $(p_i, q_j)$ i.e. (1,1), (1,2), (1,4), (1,5), (2,1), (2,2), (2,4), (2,5), (3,1), (3,2), (3,4), (3,5), (4,1), (4,2), (4,4), (4,5), (5,1), (5,2), (5,4), (5,5) add to $W_4$ to get $W_5$.

$$W_5 = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \hline \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$W_5 = W_4 = R^* . M_{R^*}$.

Hence $R^* = \{(1,1), (1,2), (1,4), (1,5), (2,1), (2,2), (2,4), (2,5), (3,1), (3,2), (3,4), (3,5), (4,1), (4,2), (4,4), (4,5), (5,1), (5,2), (5,4), (5,5)\}$

**Ex:-** Let R be a relation on set $A = \{1, 2, 3, 4, 5\}$ and
~~R = {(1,1), (1,2), (1,3), (1,4), (3,1), (3,2),~~
~~(5,1), (5,2), (5,3), (5,4), (5,5)}~~
$R = \{(1,2), (3,4), (4,5), (4,1), (1,1)\}$ find transitive closure for R using Warshall's algorithm.

**Sol^n:-**

$$M_R = W_0 = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \hline \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad n=5$$

First we find $W_1$, so that $k=1$.
$W_0$ has 1 in location 1 of column 1 i.e. (1,1) and location 4 of column 1 i.e. (4,1) also 1 in location 1 of row 1, i.e. (1,1) and location 2 in row 1, i.e. (1,2).

$p_1: (1,1)$ $\quad p_2: (4,1)$
$q_1: (1,1)$ $\quad q_2: (1,2)$
and $(p_i, q_j)$ i.e. (1,1), (1,2), (4,1), (4,2) in $W_0$. Hence,

$$W_1 = W_0 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Now compute $W_2$, so that $k=2$, $W_1$ has 1 in location 1 and 4 of column 2.
i.e (1,2)
$p_1$: (1,2), $p_2$: (4,2).
$q_i$: —
Hence $W_2 = W_1$.

$$W_2 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Now compute $W_3$, so that $k=3$.
$p_i$: —
$q_i$: (3,4)
$W_2 = W_3$.

$$W_3 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Now compute $W_4$, so that $k=4$.
$p_1$: (3,4).
$q_1$: (4,1), $q_2$: (4,2), $q_3$: (4,5).
and $(p_i, q_i)$, i.e (3,1), (3,2), (3,5).

$$W_4 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$p_i$: (3,5), (4,5).
$q_i$: —
$W_4 = W_5$.
Hence $W_4$ is the transitive closure of the relation matrix $M_R$.

Hence $R^* = \{(1,1), (1,2), (3,1), (3,2), (3,5), (4,1), (4,2), (4,5)\}$

**Ex:-** Let $R$ be a relation on set $A = \{1, 2, 3, 4, 5\}$ and
$R = \{(1,1), (1,2), (1,3), (1,4), (3,1), (3,2), (5,1), (5,2), (5,3), (5,4), (5,5)\}$.
Find transitive closure for $R$ using Warshall's algorithm.

**Sol$^n$:-**

$$M_R = W_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \quad n = 5$$

First we find $W_1$, so that $k = 1$.
$p_1: (1,1)$, $p_2: (3,1)$, $p_3: (5,1)$.
$q_1: (1,1)$, $q_2: (1,2)$, $q_3: (1,3)$, $q_4: (1,4)$.
∴ add path $(p_i, q_i)$, i.e. $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(3,1)$, $(3,2)$, $(3,3)$, $(3,4)$, $(5,1)$, $(5,2)$, $(5,3)$, $(5,4)$ in $W_0$ to get $W_1$.

$$W_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Now compute $W_2$, so that $k = 2$.
$p_1: (1,2)$, $p_2: (3,2)$, $p_3: (5,2)$.
$q_1: -$
Hence no path is added to $W_1$.
Therefore $W_1 = W_2$

$$W_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

For $W_3$, $k = 3$.
$p_1: (1,3)$, $p_2: (3,3)$, $p_3: (5,3)$.
$q_1: (3,1)$, $q_2: (3,2)$, $q_3: (3,3)$, $q_4: (3,4)$.
Therefore add path $(p_i, q_i)$ i.e. $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(3,1)$, $(3,2)$, $(3,3)$, $(3,4)$, $(5,1)$, $(5,2)$, $(5,3)$, $(5,4)$.

Again $W_2 = W_3$.

$$W_3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

For $W_4$, $k=4$.
$p_1: (1,4)$, $p_2: (3,4)$, $p_3: (5,4)$.
$q_1$: —
Hence no path is added to $W_4$.
Therefore $W_3 = W_4$.

$$W_4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

For $W_5$, $k=5$.
$p_1: (5,5)$.
$q_1: (5,1)$, $q_2: (5,2)$, $q_3: (5,3)$, $q_4: (5,4)$, $q_5: (5,5)$.

Therefore add path $(p_i, q_j)$, i.e.
$(5,1)$, $(5,2)$, $(5,3)$, $(5,4)$, $(5,5)$,
in $W_4$ to get $W_5$.
Again $W_4 = W_5$ therefore $W_5 = R^*$.

and $R^* = \{(1,1), (1,2), (1,3), (1,4), (3,1), (3,2), (3,3), (3,4), (5,1), (5,2), (5,3), (5,4), (5,5)\}$



Ex: Find the transitive closure of R by Warshall's algorithm $A = \{$set of positive integers $\le 10\}$ and $R = \{(a,b) \mid a$ divides $b\}$.

Soln: $A = \{1,2,3,4,5,6,7,8,9,10\}$.
$R = \{(1,2), (1,1), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), (1,10), (2,2), (2,4), (2,6), (2,8), (2,10), (3,3), (3,6), (3,9), (4,4), (4,8), (5,5), (5,10), (6,6), (7,7), (8,8), (9,9), (10,10)\}$.

$$W_0 = M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array}\begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The relation R itself is transitive relation on the set of positive integers. Hence $R = R^*$, and $M_R$ is the matrix of transitive closure.

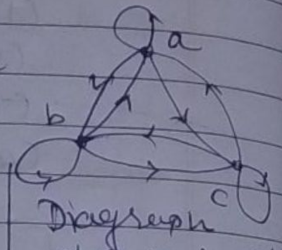
(6) ✱ **Equivalence Relation:-**

→ Let R be a relation in a set A. Then R is an equivalence relation on A iff,

i) R is reflexive relation
i.e. $\forall a \in A \Rightarrow aRa$.
or $(a,a) \in R$.

ii) R is symmetric relation
i.e. $(a,b) \in R \Rightarrow (b,a) \in R$.
or $aRb \Rightarrow bRa$

iii) R is transitive relation :
i.e $aRb$ and $bRc \Rightarrow aRc$.
or $(a,b) \in R$ & $(b,c) \in R \Rightarrow (a,c) \in R$



Diagraph of Equivalence Rel<sup></sup>

Ex:-1). Let A be the set of all straight lines in 2-D plane and R be the relation on A defined as aRb iff a is parallel to b.

i) aRa for every $a \in A$ since every straight line is parallel to itself.
So R is reflexive

ii) $aRb \Rightarrow bRa$, since if a line is parallel to line b then b is also parallel to a, hence R is reflexive.

iii) $aRb$, $bRc \Rightarrow aRc$ since if a line is parallel to line b, b line is parallel to line c then a is also parallel to c. hence R is transitive.

Since R is reflexive, symmetric and transitive, therefore R is an equivalence relation.

Ex:-2). Let A be a set of all triangles in 2-D plane. Let R be the relation on A defined as for $a, b \in A$.
aRb iff a is congruent to b. or
$a \equiv b$.

i) Every triangle is congruent to itself,
So aRa, $\forall a \in A$ hence R is reflexive

ii) $aRb \Rightarrow bRa$, since if triangle a is congruent to b then b is also congruent to a and this shows that R is symm.

iii) $aRb$ and $bRc \Rightarrow aRc$, as if triangle a is congruent to b and triangle b is congruent to c then triangle a is congruent to c. Thus R is transitive.

Since R is reflexive, symmetric and transitive, therefore R is an equivalence relation.

Ex:-3) Let $I$ be the set of all integers.
Let $R$ be a relation of $I$ defined as $aRb$ iff $a-b$ is divisible by 5. or $5|a-b$.
i.e $R = \{(a,b) ; 5|a-b\}$

i) $\forall a \in I$, $5|a-a$
as zero is divisible by 5.
$\Rightarrow (a,a) \in R$ or $aRa$.
hence $R$ is reflexive.

ii) If $a,b \in I$ and $5|a-b$.
then 5 also divides $b-a$
i.e $5|b-a \Rightarrow bRa$ or $(b,a) \in R$.
hence $R$ is symmetric.

iii) If $a,b,c \in I$
and $5|a-b$ also $5|b-c$
then $5|a-b+b-c$ i.e $5|a-c$.
hence if $aRb, bRc \Rightarrow aRc$.
hence $R$ is transitive. Being reflexive, symmetric and transitive, $R$ is an equivalence relation.

Note:- The same relation $R$, $aRb$ iff $5|a-b$ is not equivalence relation on the set of natural numbers $N$.
As $0 \notin N$ so $R$ is not reflexive.
Also if $5|a-b$ i.e. $a-b \in N$.
then $b-a \notin N$
Hence $R$ is not symmetric relation.
But $R$ is transitive, as if $5|a-b$ & $5|b-c \Rightarrow 5|(a-b)+(b-c)$ i.e $5|a-c$.
So, $R$ is transitive.

Ex:- If $A$ is the set of all points in a plane, the relation 'at the same distance from the origin as' is an equivalence relation.

Solⁿ: i) Since the point $x$ is at the same distance from origin as $x$, i.e itself.
$\therefore xRx, \forall x \in R$ or $(x,x) \in R, \forall x \in R$
Thus the relation is reflexive.

ii) If $xRy$ or $(x,y) \in R$.
$\Rightarrow$ The point $x$ is at the same distance from origine as $y$.
$\Rightarrow$ $y$ is at the same distance from origin as $x$.
Thus $(x,y) \in R \Rightarrow (y,x) \in R$.
Thus the relation is symmetric.

iii) If $x$ & $y$ points and $y$ and $z$ points are at same distance from origin.
$(x,y) \in R, (y,z) \in R \Rightarrow (x,z) \in R$.

i.e. $x$ & $z$ are at same distance from origin. Thus R is transitive relation. Being reflexive, symm. and transitive relation, R is an equivalence relation.

Ex:- If I be the set of all integers and if the relation R be defined over the set I by aRb iff a-b is an even integers, where $a, b \in I$, show that R is an equivalence relation.

Soln:- i) Since $x-x=0$ and 0 is an even integer $\Rightarrow xRx$.
Hence R is reflexive relation.

ii) If $x-y$ is an even integer then $y-x$ is also an even integer.
Hence $xRy \Rightarrow yRx$. Hence R is symm.

iii) If $xRy$, $yRz$
$\Rightarrow$ $x-y$ is an even integer & also $y-z$ is an even integer.
$\Rightarrow$ $(x-y)+(y-z)$ is even integer.
$\Rightarrow$ $x-z$ is an even integer
$\Rightarrow$ $xRz$.
Hence R is transitive relation. Being reflexive, symm & transitive R is an equivalence relation.

Ex:- Let $X=\{1,2,\ldots,7\}$ & $R=\{(x,y) \mid x-y$ is divisible by $3\}$. Show that R is an equivalence relation. Draw graph of R.

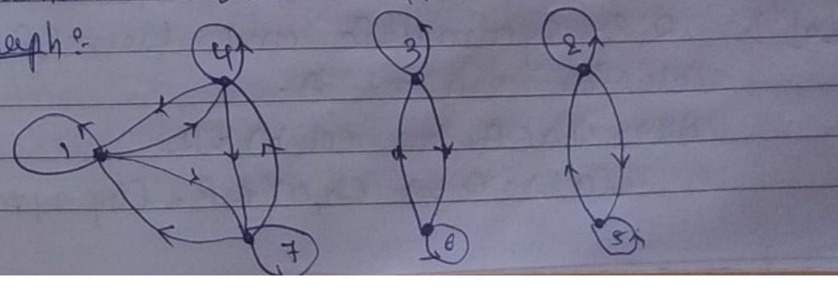
Soln. $X=\{1,2,3,4,5,6,7\}$.
$R=\{(x,y) ; x-y=3k, k \in I\}$.

i) $xRx, \forall x \in X$.
as $x-x$, i.e. 0 is always divisible by any nonzero finite number and hence divisible by 3.
Hence $x-x=3.m$, $m \in I$. Hence $xRx$.
$\therefore$ R is reflexive relation.

ii) If $xRy \Rightarrow x-y=3k, k \in I$.
$\Rightarrow y-x=3(-k)$. where $-k \in I$.
$\Rightarrow yRx$. $\therefore$ R is symmetric relation.

iii) If $xRy$ & $yRz$.
$\Rightarrow x-y=3k_1$ & $y-z=3k_2$.
$\Rightarrow (x-y)+(y-z)=3(k_1+k_2)$.
$\Rightarrow x-z=3(k_1+k_2)$.
$\Rightarrow xRz$. [as if $k_1 \in I$, $k_2 \in I$, $\Rightarrow k_1+k_2 \in I$].
Being reflexive, symmetric and transitive relation, R is an equivalence relation

• Graph:-



## * Some more Examples of equivalence relation :-

1) S is the set of all real numbers then $aRb$ iff $a = \pm b$.
2) I is the set of all integers ; $n > 1$ is a fixed integer then $aRb$ iff $a-b$ is a multiple of n.
3) I is the set of all integer then $aRb$ iff $a+b$ is an even integer.
4) S is any non empty set then $aRb$ iff $a = b$.
5) N is the set of natural numbers and $S = \{(a,b) ; a,b \in N\}$ then $(a,b) R (c,d)$ iff $ad = bc$.
6) $S = \{(a,b) ; a,b \in I\}$ and $(a,b) R (c,d)$ iff $a+d = b+c$.

Ex:- Let R be a symmetric and transitive relation on set A. show that if for every a in A there exist b such that $(a,b)$ is in R, then R is equivalence relation.

Soln:- R is symmetric and transitive relation on set A.
Also, $\forall a \in A \Rightarrow (a,b) \in R$.
$(a,b) \in R \Rightarrow (b,a) \in R$. (by symm.)
also $(a,b) \in R$, $(b,a) \in R \Rightarrow (a,a) \in R$.
hence $\forall a \in R$, $(a,a) \in R$.
hence R is a reflexive relation.
Hence R is reflexive, symm. and transitive relation. ∴ R is equivalence relation.

## * Properties of Equivalence Relation :

1) If R and S are equivalence relations on a set A, then $R \cap S$ is also an equivalence relation set on set A.

Proof :- To prove that $R \cap S$ is an equivalence relation, we have to prove that,
i) $R \cap S$ is reflexive relation
ii) $R \cap S$ is symmetric relation
iii) $R \cap S$ is transitive relation

i) As R is reflexive,
then $(a,a) \in R$, $\forall a \in A$.
also S is reflexive.
then $(a,a) \in S$, $\forall a \in A$.
$\Rightarrow (a,a) \in R \cap S$, $\forall a \in A$.
Hence $R \cap S$ is reflexive.

ii) R is symmetric.
then $(a,b) \in R \Rightarrow (b,a) \in R$.
also S is symmetric.
hence $(a,b) \in S \Rightarrow (b,a) \in S$.

i.e. $(a,b) \in R \cap S \Rightarrow (b,a) \in R \cap S$.

Thus $R \cap S$ is symm. relation.

iii) R & S are transitive relations

hence $(a,b) \in R$, $(b,c) \in R \Rightarrow (a,c) \in R$ &

$(a,b) \in S$, $(b,c) \in S \Rightarrow (a,c) \in S$.

i.e. $(a,b) \in R \cap S$, $(b,c) \in R \cap S$,

$\Rightarrow (a,c) \in R \cap S$.

Hence $R \cap S$ is transitive relation. Being reflexive, symmetric & transitive relation, $R \cap S$ is an equivalence relation.

(2) If R and S are equivalence relations then $R \cup S$ need not be an equivalence relation.

- Counter Example:-

Let $A = \{1, 2, 3\}$.

$R = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$.

$S = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$.

then R is equivalence & also S is equivalence relation but $R \cup S$ is not equivalence relation,

as $(1,2) \in R \cup S$ & $(2,3) \in R \cup S$, but $(1,3) \notin R \cup S$.

i.e. $R \cup S$ is not transitive relation.

(3) If R & S are equivalence relations then $R \cup S$ is an equivalence relation iff $R \subset S$ or $S \subset R$, that is one is contained in other.

* Equivalence Classes:-

Let A be a non empty set and R be an equivalence relation on set A. Then equivalence class of any element $a \in A$ is denoted by $[a]$ or $\bar{a}$ or $A_a$ and $[a]$ is set of all those elements of A which are related to a.

i.e. $[a] = \{x ; (x,a) \in R$ or $xRa\}$.

Ex:- Let I be the set of all integers and R is a relation.

$R = \{(a,b) ; 5 \mid a-b\}$, then

R is an equivalence relation on set I, further if $a \in I$. Then equivalence class of A is set of all those elements of I which are related with a.

i.e. $[a] = \{x ; (x,a) \in R$ or $5 \mid x-a\}$.

hence

equivalence class zero.

i.e. $[0] = \{x : 5 \mid x-0\}$.

hence $[0] = \{\dots, -10, -5, 0, 5, 10, 15, \dots\}$.

Ily, $[1] = \{x : 5 | x-1$ or $\frac{x-1}{5} = y$ say
i.e. $x = 5y+1\}$.
hence $[1] = \{\ldots, -9, -4, 1, 6, 11, 16, \ldots\}$
$[2] = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$
$[3] = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$
$[4] = \{\ldots, -6, -1, 4, 9, 14, \ldots\}$
$[5] = \{\ldots, -10, -5, 0, 5, 10, 15, \ldots\}$.

From the above example it is clear that $4 \in [4]$, $3 \in [3]$ etc. also two classes are either identical i.e. $[0]$ & $[5]$ or disjoint e.g. $[1]$ & $[2]$.

* Properties of equivalence classes:

→ Let $A$ be a non empty set & $R$ be an equivalence relation on set $A$.
$a, b \in A$.
Then i) $a \in [a]$
i.e. equivalence classes are non empty.
ii) if $b \in [a]$ then $[b] = [a]$
iii) $aRb$ iff $[a] = [b]$, i.e. $(a,b) \in R$ iff $[a] = [b]$.
iv) Either $[a] = [b]$ or $[a] \cap [b] = \phi$.
i.e two equivalence classes are either identical or disjoint.

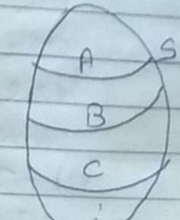
proof: i) As $R$ is equivalence relation on set $A$.
⇒ $R$ is also reflexive.
⇒ $aRa$, $\forall a \in A$.
⇒ $a \in [a]$.
as $[a] = \{x ; xRa, x \in A\}$.

ii) If $b \in [a] \Rightarrow bRa$
to show that $[b] = [a]$.
We have to show that $[b] \subseteq [a]$ & $[a] \subseteq [b]$.
→ for $[b] \subseteq [a]$.
Let $x \in [b] \Rightarrow xRb$.
also $bRa$ is given
So $xRb, bRa \Rightarrow xRa$. (by transitive prop)
⇒ $x \in [a]$.
Hence $x \in [b] \Rightarrow x \in [a] \Rightarrow [b] \subseteq [a]$. —— (1)
→ Now let $y \in [a]$.
⇒ $yRa$.
also $bRa \Rightarrow aRb$. (as R is symm.)
also $yRa, aRb \Rightarrow yRb$. (as R is transitive)
So, $y \in [a] \Rightarrow y \in [b]$.
⇒ $[a] \subseteq [b]$. —— (2).
by equations (1) & (2)
$[a] = [b]$

iii) Suppose $[a]=[b]$, then to P.T. $aRb$,
since $R$ is reflexive.
$\Rightarrow aRa \Rightarrow a\in[a]$. also $[a]=[b]$.
$\Rightarrow aRb$.
Thus $[a]=[b] \Rightarrow aRb$.
$\rightarrow$ Conversely to prove $aRb \Rightarrow [a]=[b]$.
1st $[a]\subseteq[b]$ Let $x\in[a] \Rightarrow xRa$.
$xRa, aRb \Rightarrow xRb$. (by transitive prop)
$\Rightarrow x\in[b]$
$\Rightarrow [a]\subseteq[b]$ ——— (3).
Also, if $y\in[b] \Rightarrow yRb$
also $aRb \Rightarrow bRa$ (by symmetric prop)
and $yRb, bRa \Rightarrow yRa$ (by transitive prop)
$\Rightarrow y\in[a]$
$\Rightarrow [b]\subseteq[a]$. ——— (4)
From equations (3) & (4), $[a]=[b]$.

iv) Either $[a]=[b]$ or $[a]\cap[b]=\phi$.
if no element is common to $[a]$ & $[b]$
then $[a]\cap[b]=\phi$ so nothing to prove.
Let one element is common to $[a]\cap[b]$
i.e $x\in[a]\cap[b]$
$\Rightarrow x\in[a]$ & $x\in[b]$
$\Rightarrow xRa$ and $xRb$
$\Rightarrow aRx$ & $xRb$. (as $R$ is symm.)
$\Rightarrow aRb$. [as $R$ is transitive]
$\Rightarrow [a]=[b]$. (from above statement (iii))
Hence two classes are either identical or disjoint. Hence to show equality of two classes, it is enough to show that one element is common to both the classes.

## * Partitions:-




$\rightarrow$ Let $S$ be a non empty set
A set $P=\{A,B,C,\ldots\}$ of non empty subsets of $S$ will be called a partition of $S$ if
i) $A\cup B\cup C\cup\ldots=S$, i.e. the set $S$ is the union of the sets of $P$ and
ii) The intersection of every pair of distinct subsets of $S\in P$ is the null set. i.e. if $A$ and $B\in P$, then either $A=B$ or $A\cap B=\phi$.

Ex:- Let $I$ be the set of all integers and $R=\{(a,b): 5|a-b\}$ is an equivalence relation on the set $I$.
Soln:- Consider the set of all integers and $R=\{(a,b): 5|a-b\}$ is an equivalence

relation on the set I.

Consider the set of five equivalence classes $[0], [1], [2], [3], [4]$ where

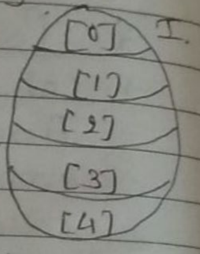
$[0] = \{\dots, -10, -5, 0, 5, 10, 15, \dots\}$

$[1] = \{\dots, -9, -4, 1, 6, 11, 16, \dots\}$

$[2] = \{\dots, -8, -3, 2, 7, 12, 17, \dots\}$

$[3] = \{\dots, -7, -2, 3, 8, 13, 18, \dots\}$

$[4] = \{\dots, -6, -1, 4, 9, 14, 19, \dots\}$



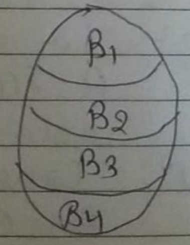

Here i) $[0], [1], [2], [3]$ & $[4]$ are non empty sets.

ii) The sets $[0], [1], [2], [3], [4]$ are pairwise disjoint.

iii) $I = [0] \cup [1] \cup [2] \cup [3] \cup [4]$.

Hence $\{[0], [1], [2], [3], [4]\}$ is a partition of I.

Ex:- $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and its subsets $B_1 = \{1, 3\}$, $B_2 = \{7, 8, 10\}$, $B_3 = \{2, 5, 6\}$, $B_4 = \{4, 9\}$.




Sol:- The set $P = \{B_1, B_2, B_3, B_4\}$ is such that

i) $B_1, B_2, B_3, B_4$ are all non empty subsets of S.

ii) $B_1 \cup B_2 \cup B_3 \cup B_4 = S$ and

iii) For any sets $B_i, B_j$ either $B_i = B_j$ or $B_i \cap B_j = \phi$.

Hence the set $\{B_1, B_2, B_3, B_4\}$ is a partition of S.

* **Relation Induced by a Partition:**

→ Corresponding to any partition of a set S, we can define a relation R on S by the requirement that $xRy$ iff $x$ & $y$ belong to the same subset of S belonging to the partition. The relation R is then said to be induced by the partition.

Ex:- Consider the subsets

$A = \{1, 4, 7, \dots, 25\}$

$B = \{2, 5, 8, \dots, 23\}$.

$C = \{3, 6, 9, \dots, 24\}$.

& $S = \{1, 2, 3, \dots, 25\}$.

then $A \cup B \cup C = S$, $A \cap B = \phi$, $B \cap C = \phi$, $A \cap C = \phi$ and $A, B, C$ are non empty sets.

Hence $\{A, B, C\}$ is partition of S.

If R be the relation induced by this partition, then we have $xRy$

iff x & y belong to the same subset - A, B, C.

* **Refinement of partitions:-**

→ Let $A_1$ and $A_2$ be partitions of a non empty set A. Then $A_2$ is called refinement of $A_1$ if every block (element) of $A_2$ is contained in a block of $A_1$.

Let $A = \{1, 2, 3, 4\}$

and let $A_1 = \{\{1\}, \{2,3\}, \{4\}\}$.

and $A_2 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$.

Then $A_2$ is a refinement of $A_1$.

Let $R_1$ and $R_2$ be the equivalence relations induced by $A_1$ and $A_2$ resp.

Then the following theorem relates $R_1$ & $R_2$.

**Thm:-** Let $A_1$ and $A_2$ be partition of a non empty set A and let $R_1$ & $R_2$ be the equivalence relation induced by $A_1$ & $A_2$ resp. Then $A_2$ refines $A_1$ iff $R_2 \subseteq R_1$.

**Proof:-** Let $A_2$ be a refinement of A.

We have to prove that $R_2 \subseteq R_1$. Let $aR_2b$ then $\exists$ some block $T_j' \in A_2$ such that $a, b \in T_j'$. Since $A_2$ refines $A_1$, $T_j' \subseteq T_j$ for same block $T_j \in A_1$. Hence $a, b \in T_j$ which implies $aR_1b$. Hence $R_2 \subseteq R_1$.

Conversely, let $R_2 \subseteq R_1$. We have to prove that $A_2$ is refinement of $A_1$. Let $T_j' \in A_2$ and $a \in T_j'$. Then $T_j' = \{a\}_{R_2}$. Let $x \in R_2 a$ and hence $xR_1a$ since $R_2 \subseteq R_1$. This shows that $\{a\}_{R_2} \subseteq \{a\}_{R_1}$. If $\{a\}_R$ is a block $T_j$ then $T_j' \subseteq T_j$ which indicates that $A_2$ is the refinement of $A_1$.

* **Product And Sum Of Partitions.**

Let S be a non empty set and $A_1$ and $A_2$ are partitions of S then $A_1 \cdot A_2$ consists of the set of intersection of every element of $A_1$ ~~which~~ with every element of $A_2$. Omitting the empty intersections.

**Example:-** $S = \{1, 2, 3, 4, 5\}$.

$A_1 = \{\{1,2\}, \{3\}, \{4,5\}\}$

$A_2 = \{\{1,2,3\}, \{4\}, \{5\}\}$

then $A_1 \cdot A_2 = \{\{1,2\},\{3\},\{4\},\{5\}\}$.

* **Product of Partitions**

→ $A_1 \cdot A_2$ denote product of partition $A_1$ & $A_2$ and let $A_1 \cdot A_2$ be denoted by $A'$ then $A'$ refines both $A_1$ and $A_2$.

If $T$ refines both $A_1$ & $A_2$, then $T$ refines $A'$.

**Result 1**: Let $R_1$ & $R_2$ be the equivalence relations induced by partitions $A_1$ and $A_2$ resp. Then the relation $R = R_1 \cap R_2$ induced the product partition $A_1 \cdot A_2$.

* **Sum of Partitions:-**

→ Let $A_1$ and $A_2$ be partitions of a nonempty set $S$. The sum of $A_1$ & $A_2$ denoted by $A_1 + A_2$ is a partition of $S$ such that both $A_1$ and $A_2$ refines $A_1 + A_2$. Also if $A'$ is a partition on $S$ such that $A_1$ and $A_2$ refines $A'$ then also $A_1 + A_2$ refines $A'$.

**Result 2**. Let $R_1$ & $R_2$ be equivalence relations on a non empty set $S$ induced by the partitions $A_1$ and $A_2$. Let $R = (R_1 \cup R_2)^+$ the transitive closure of $R_1 \cup R_2$. Then $R$ is an equivalence relation on $S$ and induced by the partition $A_1 + A_2$.

* **Remark:-**

→ The product and the sum of two partitions always exist and is unique.

* **Fundamental Theorem on Equivalence Relations:-**

→ An equivalence relation $R$ on a nonempty set $S$ determines a partition of $S$ and conversely, a partition of $S$ defines an equivalence relation on $S$.

**Proof**: Let $R$ be an equivalence relation on $S$. Let $A$ be the set of equivalence classes of $S$ with respect to $R$
i.e let $A = \{[a];\ a \in S\}$
where $[a] = \{x : x \in S \text{ and } xRa\}$.
Now $R$ is an equivalence relation.
Therefore $\forall a \in S$, we have $aRa$. Hence

$a \in [a]$ and thus $[a] \neq \phi$.

Further every element $a$ of $S$ is an element of equivalence class $[a]$ in $A$. From this we conclude that $S = \bigcup_{a \in S} [a]$.

Finally, if $[a]$ and $[b]$ are two equivalence classes then either $[a] = [b]$ or $[a] \cap [b] = \phi$.

Hence $A$ is a partition of $S$. Thus an equivalence relation $R$ in $S$ decomposes the set into equivalence classes, any two of which are either identical or mutually disjoint.

Conversely, let $P = [T_a, T_b, T_c, \ldots]$ be any partition of $S$. If $p, q \in S$, let us define a relation $R$ on $S$ by $pRq$ iff there is a $T_i$ in the partition such that $p, q \in T_i$.

Now $S = T_a \cup T_b \cup T_c \cup \ldots$ therefore $\forall x \in S$, $\exists$ $T_i \in P$ such that $x \in T_i$.

Hence $x \in T_i$, $x \in T_i$ means $xRx$. Thus $\forall x \in S$, we have $xRx$, and thus $R$ is reflexive.

Again if we have $xRy$, then $\exists$ $T_i \in P$ such that $x \in T_i$ and $y \in T_i$.
But $x \in T_i$ and ($y \in T_i \Rightarrow y \in T_i$) and $x \in T_i$. $\Rightarrow yRx$.

Therefore $R$ is symmetric.

Finally, suppose $xRy$, $yRz$. Then by the def$^n$ of $R$, $\exists$ subsets $T_j$ & $T_k$ (not necessarily distinct) such that $x, y \in T_j$ and $y, z \in T_k$, since $y \in T_j$ and also $y \in T_k$, therefore $T_j \cap T_k \neq \phi$. But $T_j$ and $T_k$ belong to a partition of $S$. Therefore $T_j \cap T_k \neq \phi$ implies $T_j = T_k$.

Now, $T_j = T_k$ implies $x, z \in T_j$ and consequently we have $xRz$.

Thus $R$ is transitive. Since $R$ is reflexive, symmetric and transitive, therefore $R$ is an equivalence relation.

* **Quotient Set**:- Let $S$ be any nonempty set and let $R$ be an equivalence relation defined on set $S$. The set of mutually disjoint equivalence classes in which $S$ is partitioned relatively to the equivalence relation $R$, is said to be Quotient set of $S$ for the equivalence relation & is denoted by $S/R$ or $\bar{S}$.

Ex:- The Quotient set of $I$, for the equivalence relation $R$ $\ni$ $R = \{(a,b) : 5 | a-b\}$ then $I/R = \{I_0, I_1, I_2, I_3, I_4\}$
or $I/R = \{[0], [1], [2], [3], [4]\}$.