# PARTIAL ORDERING.

* **Partial Order Relations:-**

A relation R on a set A is called a partial order relation iff R is

i) reflexive relation:-

i.e. $aRa \ \forall a \in A$. i.e $(a,a) \in R$, $\forall a \in A$

ii) Antisymmetric relation:-

if $aRb$ and $bRa$ then $a=b$.

i.e. $(a,b) \in R, (b,a) \in R \Rightarrow a=b$, $a,b \in A$.

iii) Transitive relation

$aRb, bRc \Rightarrow aRc$.

$(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$, $a,b,c \in A$.

Ex:- Let N be a set of natural numbers. R is a relation defined on set N.

$R = \{(a,b) ; a \text{ divides } b \text{ or } a|b\}$.

$aRb$ iff $a|b$.

a is related with b iff a divides b.

i) $\forall a \in N$ $a|a \Rightarrow (a,a) \in R$ hence R is ~~set~~ reflexive relation

ii) $a,b \in N$ and if $a|b$ and $b|a$ then $a=b$ hence R is antisymmetric

iii) $a,b,c \in N$ if $a|b$ and $b|c$ then $a|c$ i.e $(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$ hence R is transitive relation.

Being reflexive, antisymm. & transitive, R is a partial order relation.

Ex:- $S = \{1, 2, 3\}$.

$P(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

R is a relation defined on set P(S).

$ARB$ iff $A \subseteq B$.

Then R is partial ordering relation on P(S).

i) R is reflexive.

Every set is subset of itself hence $A \subseteq A$.

$\Rightarrow ARA$, $\forall A \in P(S)$. $\therefore$ R is reflexive relation

ii) R is antisymmetric

Let $ARB$ & $BRA$

$\Rightarrow A \subseteq B$ and $B \subseteq A \Rightarrow A=B$ i.e R is antisymm.

iii) R is transitive.

Let $ARB$ & $BRC \Rightarrow A \subseteq B$ & $B \subseteq C \Rightarrow A \subseteq C$

$\Rightarrow ARC$. $\therefore$ R is transitive.

Being reflexive, antisymm. & transitive relation, R is a partial ordering relation.

Eg:- The relation '$\leq$' is a partial order relation on the set of real numbers

Eg The relation '$\geq$' is a partial order relation on set of real numbers.

## * Partially Ordered Set or Poset

⇒ If A is any non empty set and R is a partial ordered relation on set A, then the ordered pair (A,R) is called partially ordered set or poset.

Eg:- 1) S is a non empty set and P(S) is powerset of S the relation ⊆ is partial order relation on set P(S). Hence (P(S), ⊆) is known as poset.

2) aRb iff a|b (a divides b) is a partial order relation on set of natural nos. set. Hence (N, R) is poset.

## *. Inverse Order & Dual of Poset:-

→ If a relation R is a partially ordered relation on a set A then the inverse relation $R^{-1}$ is also partially ordered relation on the set A.

Hence $(A, R^{-1})$ is a poset., known as dual of poset (A, R).

Remark:- The relation '≤' is a partial order relation on the set of real nos. Hence generally '≤' is used to denoted relation on a set A as partial order relation and (A, ≤) is poset.

Ex:- Draw the digraph for the following relation and determine whether the relation is reflexive, symm., transitive & antisymm.

A = {1, 2, 3, 4, 5, 6, 7, 8} and let xRy whenever y is divisible by x.

Soln:- A = {1, 2, 3, 4, 5, 6, 7, 8}

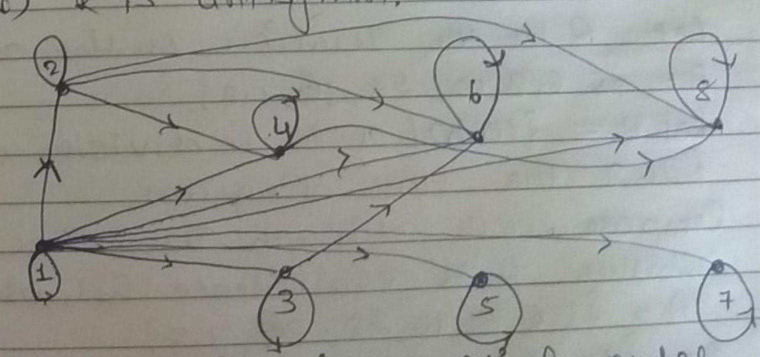
R = { (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (2,2), (2,4), (2,6), (2,8), (3,3), (3,6), (4,4), (4,8), (5,5), (6,6), (7,7), (8,8) }

i) R is reflexive as ∀a∈A, (a,a) ∈ R.

ii) R is not symm. as (4,2) ∈ R but (2,4) ∉ R.

iii) R is transitive.

iv) R is antisymm.




Hence R is partial order relation and A is partial order set.

## *. Hass Diagrams:-

→ Poset can be represented by diagraphs. A simpler way of representing poset is Hass diagram.

- Method to find Hass diagram.

From digraph,

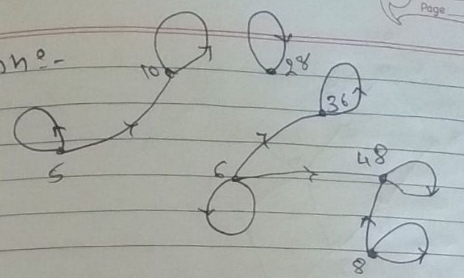
1. Omit loops as relation is reflexive on poset.
2. All arrows that appears on the edges are omitted.
3. Eliminate all edges that are implied by transitive relation.
4. An arc pointing upward is drawn from a to b if $a \neq b$ and $aRb$.

**Ex1:-** Let R be the relation on the set A.
$A = \{5, 6, 8, 10, 28, 36, 48\}$.
Let $R = \{(a,b) \mid a$ is a division of $b\}$.
Draw the Hass diagram.
Compare with digraph. Determine whether R is equivalence relation.

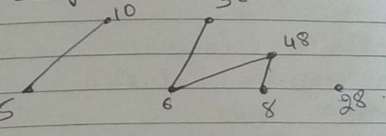
**Soln:-** $A = \{5, 6, 8, 10, 28, 36, 48\}$

$R = \{(5,5), (6,6), (8,8), (10,10), (28,28), (36,36), (48,48), (5,10), (6,36), (6,48), (8,48)\}$.

Diagraph:-




Hass diagram:-




R is reflexive, but R is not symmetric relation. Hence R is not equivalence relation.

**Ex2-** Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 24\}$ be ordered by the relation x divides y. Show that the relation is partial ordering and draw the Hass diagram.

$\therefore$ $R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,9), (1,12),$
$(2,18), (2,2), (2,4), (2,6), (2,8), (2,12), (1,18), (1,24), (2,18),$
$(2,24), (3,3), (3,6), (3,9), (3,12), (3,18), (3,24), (4,4),$
$(4,8), (4,12), (4,24), (5,5), (6,6), (6,12), (6,18), (6,24),$
$(7,7), (8,8), (8,24), (9,9), (9,18), (12,12), (12,24), (24,24)\}$

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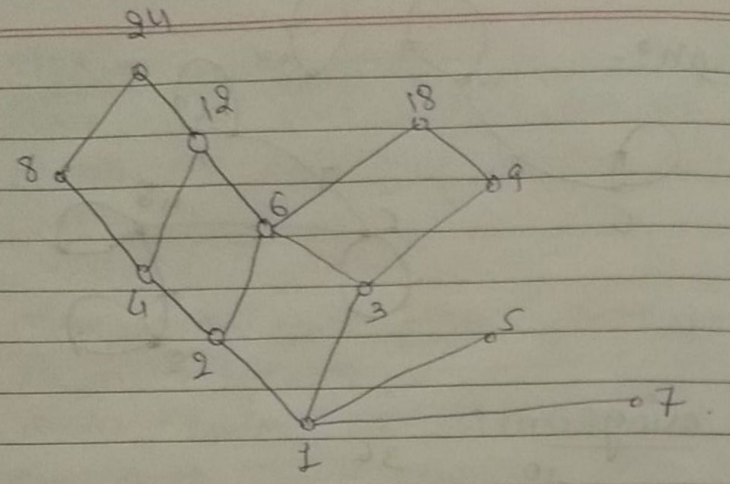



Hasse diagram

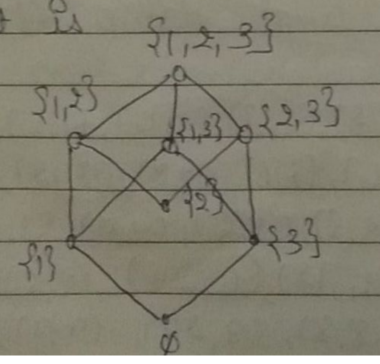
1.) R is reflexive relation.
as $\forall a \in A$, $aRa$.

2) R is antisymm. as if $a|b$ & $b|a$ then $a=b$.

3) R is transitive as if $a|b$ & $b|c \Rightarrow a|c$

Hence R is partial ordering relation.

Ex:- Let $S = \{1,2,3\}$.
$(P(S), \subseteq)$ is a poset & Hass diagram of poset is

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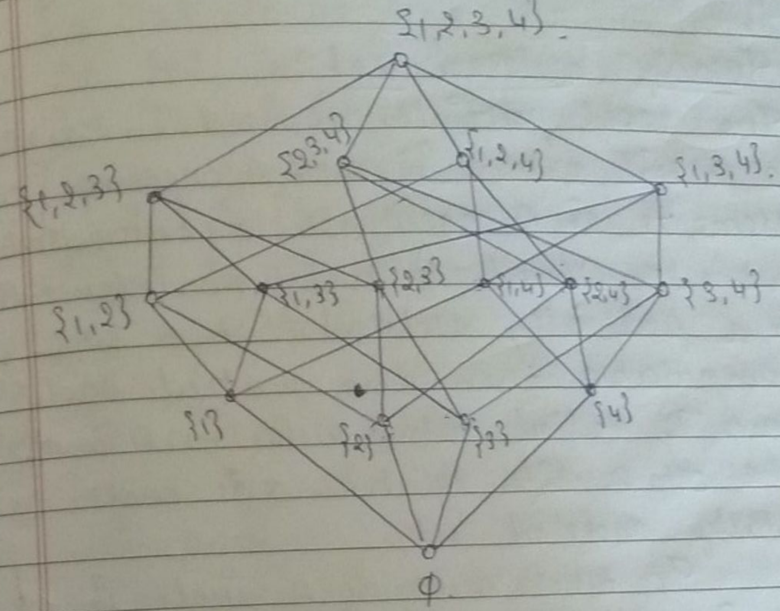
Ex:- $S = \{1,2,3,4\}$.
$P(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\},$
$\{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$.



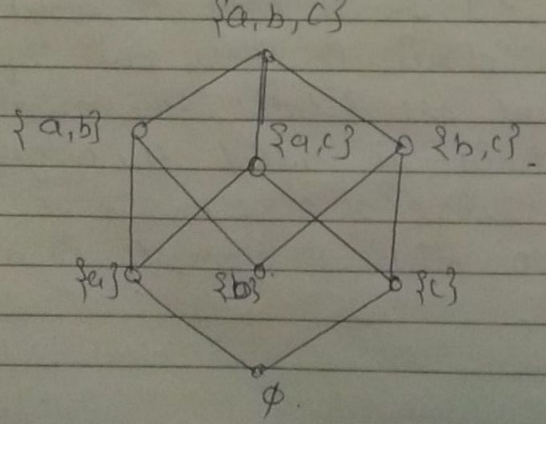

* Chains and Antichains :-

→ Let $(A, \leq)$ be a poset. A subset of A is known as chain if every pair of elements in the subset are related. The number of elements in a chain is called the length of the chain.

A Subset of A is known as antichain if no two distinct elements in a subset are related.

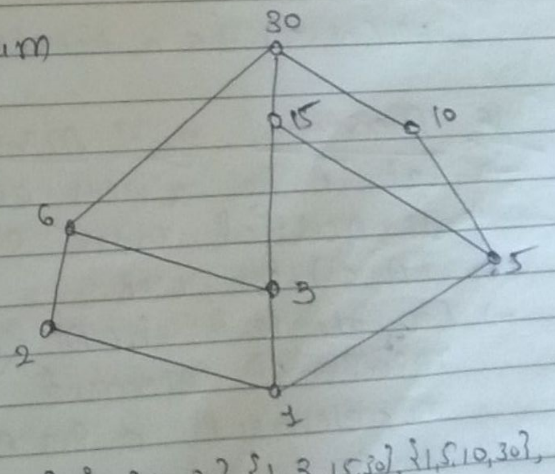
**Remark:** Two elements a & b in a partially ordered set are said to be not comparable if $a \not\leq b$ (a is not related with b)

* **Totally ordered Set:**
The word partially is used in defining a partial order in a set A because some elements in A need not be comparable. On the otherhand if every two elements in a partially ordered set A are comparable, then the partial order on A is called total order on A and the set A with relation $\leq$ is known as totally ordered set.
In otherwords, if A itself is a chain the poset $(A, \leq)$ is called a totally ordered set or linearly ordered set.




**Ex:-** $S = \{a,b,c\}$ Poset $(P(S), \subseteq)$
Hass diagram
$\{\phi, \{a\}, \{a,b\}, \{a,b,c\}\}$, $\{\phi, \{a\}, \{a,c\}, \{a,b,c\}\}$,
$\{\phi, \{b\}, \{a,b\}, \{a,b,c\}\}$, $\{\phi, \{b\}, \{b,c\}, \{a,b,c\}\}$,
$\{\phi, \{c\}, \{a,c\}, \{a,b,c\}\}$, $\{\phi, \{c\}, \{b,c\}, \{a,b,c\}\}$,
$\{\phi, \{a\}, \{a,b\}\}$, $\{\{a\}, \{a,c\}, \{a,b,c\}\}$
are examples of chains.
$\{\{a\}, \{b\}\}$, $\{\{a\}, \{c\}\}$, $\{\{b\}, \{c\}\}$, $\{\{a,b\}, \{a,c\}\}$,
$\{\{b,c\}, \{a,c\}\}$, $\{\{a,b\}, \{b,c\}\}$ are the examples of antichains.

**Ex:-** $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ [Factors of 30].
$aRb$ iff $a|b$ (a divides b) then $(A, R)$ is poset.
- Hass diagram

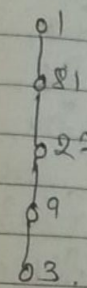



- Chains:
$\{1,3,15,30\}$, $\{1,5,10,30\}$,
$\{1,2,6,30\}$, $\{1,3,6,30\}$, $\{1,2,10,30\}$, $\{1,3,15,30\}$, $\{1,3,15\}$, $\{1,3,6\}$,
$\{1,5,15,30\}$, $\{2,6,30\}$, $\{1,2,6\}$, $\{3,15,30\}$,
$\{1,5,15\}$, $\{1,5,10\}$ etc.

- Antichains:
$\{2,3\}, \{2,5\}, \{3,5\}, \{6,10\}, \{6,15\}, \{10,15\}$.

* **Totally Ordered Relation**

Ex:- $A = \{3, 9, 27, 81, \ldots\}$
and $aRb$ iff $a|b$, then
$(A, R)$ is a totally ordered relation




- A itself is a chain.

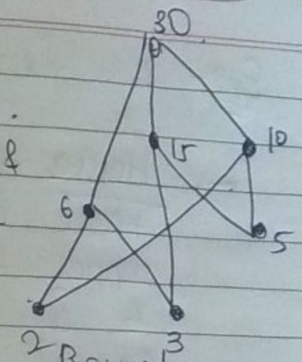
Ex:- N is the set of natural numbers and R is a relation defined as $aRb$ iff $a \leq b$.
then N is a chain and hence N is a total ordered set with $\leq$.
i.e $(N, \leq)$ is a totally ordered set.

* **Maximal and Minimal Elements**:-

→ Let A be a non empty set & '$\leq$' is partial order relation on A.
$(A, \leq)$ is poset.

→ An element $b \in A$ is known as minimal element of A if there is no element c in A such that $c \leq b$.

Ex:- $A = \{2, 3, 5, 6, 10, 15, 30\}$.
Hass diagram of poset.
maximal element is 30 &
minimal elements are
2, 3, 5.




* **Upper Bounds & Lower Bounds**:-

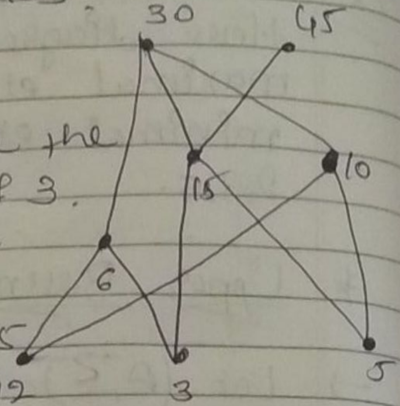
→ Let $(A, \leq)$ be a poset for elements $a, b \in A$, an element $c \in A$ is called upper bound of a & b if $a \leq c$ & $b \leq c$.

- c is known as least upper bound (lub) of a & b if c is an upper bound of a, b and if there is no other upper bound d of a & b such that $d \leq c$. lub is also known as supremum

- Similarly an element e is said to be a lower bound of a & b if $e \leq a$ & $e \leq b$, and e is known as greatest lower bound (glb) of a & b if there is no other lower bound f of a & b such that $e \leq f$. glb is also known as infimum

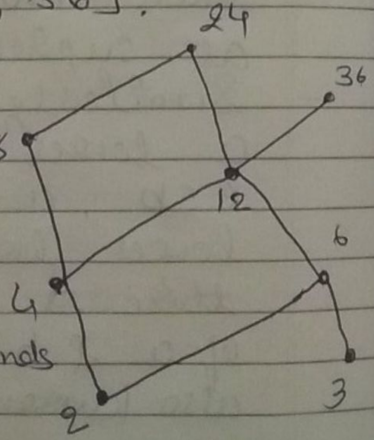
Ex2- $A = \{2, 3, 5, 6, 10, 15, 30, 45\}$.

→ Hasse diagram :-




Here 6 & 30 are the upper bounds of 2 & 3. 6 is least upper bound of 2 & 3.
Similarly 15, 30 & 45 are the upper bounds of 3 & 5. 15 is the lub. of 3 & 5.
Also 10 & 30 are upper bounds of 2 and 5. 10 is the lub of 2 & 5.
Similarly, 15, 3, 5 are the lower bounds of 30 & 45 in which 15 is the greatest lower bound.

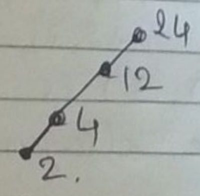
Ex:- $\{2, 3, 4, 6, 8, 12, 24, 36\}$.




→ Upper bounds of 2 & 3 are 6, 12 and 24, 36 in which 6 is the least upper bound.
Similarly lower bounds of 24 & 36 are 12, 4, 6, 2, 3 in which 12 is the greatest.

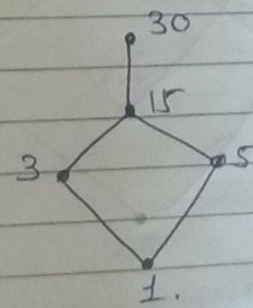
Ex2- Draw the Hasse diagram of the following sets under the partial ordering relation 'divides' and indicate those which are chains:

1. $A = \{2, 4, 12, 24\}$
Hasse diagram




A itself is a chain. Hence $(A, \leq)$ is totally ordered set or linearly ordered set.

2. $S = \{1, 3, 5, 15, 30\}$.
Hasse diagram.




Chains are
$\{1, 3, 15, 30\}$ & $\{1, 5, 15, 30\}$.

## * Lattice :-

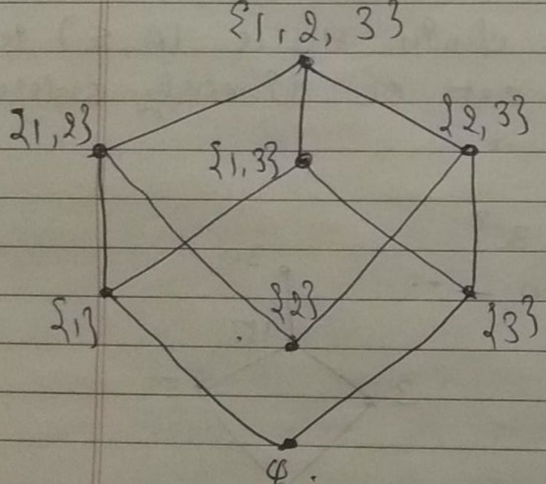
→ A lattice is a poset $(A, \leq)$ in which every subset $\{a, b\}$ of $A$, has a least upper bound and a greatest lower bound.

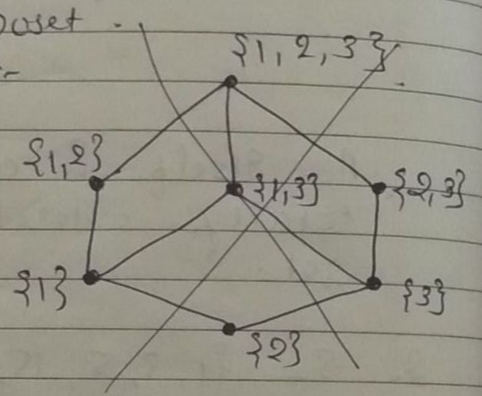
Ex:- $A = \{1, 2, 3\}$.

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

$(P(A), \leq)$ is a poset.

Hasse diagram :-

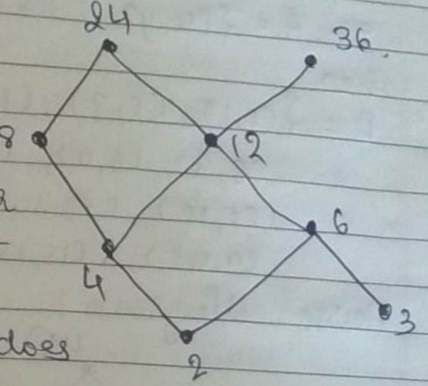



Here every pair of elements has lub & glb. Hence $(P(A), \subseteq)$ is a ~~lattice~~ lattice.

Ex:- $A = \{2, 3, 4, 6, 8, 12, 24, 36\}$.

Poset $(A, \leq)$




→ The poset is not lattice since the pair $\{2, 3\}$ does not have glb & also $\{24, 36\}$ does not have lub.

## * Lattice Operators:-

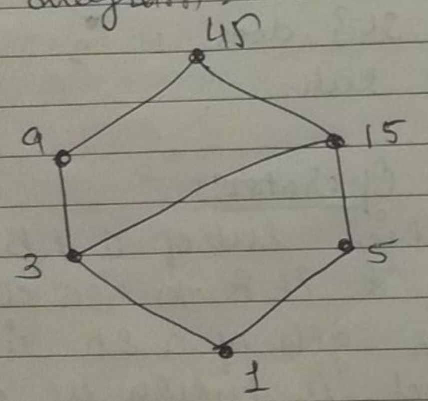
In lattice lub of a & b is denoted by $a \vee b$ & it is known as a join b. Similarly glb of a & b is denoted by $a \wedge b$ and is known as a meet b.

Ex:- Let A be set of factor of positive integer m and relation is divisibility on A. i.e. $R = \{(x,y) \mid x, y \in A, x \text{ divides } y\}$. For $m = 45$ show that poset $(A, \leq)$ is lattice. Draw Hasse diagram and give join & meet for the lattice.

Sol$^n$:- A is the set of divisor of 45.

∴ $A = \{1, 3, 5, 9, 15, 45\}$.
∴ $R = \{(x, y) : x|y,\ x, y \in A\}$.
then
$A = \{(1,1), (1,3), (1,5), (1,9), (1,15), (1,45),$
$(3,3), (3,9), (3,15), (3,45), (5,5),$
$(5,15), (5,30), (5,45), (9,9),$
$(9,45), (15,15), (15,45), (45,45)\}$

Hasse diagram :-




Every pair of elements of A has glb & lub. Hence $(A, \leq)$ is lattice

Join of a & b is $a \vee b$ (lub).
And meet of a & b is $a \wedge b$ (glb).

Table for join & meet.

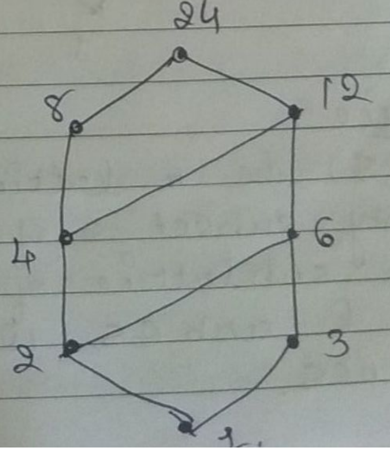
| $\vee$ | 1 | 3 | 5 | 9 | 15 | 45 |
|---|---|---|---|---|---|---|
| 1 | 1 | 3 | 5 | 9 | 15 | 45 |
| 3 | 3 | 3 | 15 | 9 | 15 | 45 |
| 5 | 5 | 15 | 5 | 45 | 15 | 45 |
| 9 | 9 | 9 | 45 | 9 | 45 | 45 |
| 15 | 15 | 15 | 15 | 45 | 15 | 45 |
| 45 | 45 | 45 | 45 | 45 | 45 | 45 |

| $\wedge$ | 1 | 3 | 5 | 9 | 15 | 45 |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 3 | 1 | 3 | 3 | 3 |
| 5 | 1 | 1 | 5 | 1 | 5 | 5 |
| 9 | 1 | 3 | 1 | 9 | 3 | 9 |
| 15 | 1 | 3 | 5 | 3 | 15 | 15 |
| 45 | 1 | 3 | 5 | 9 | 15 | 45 |

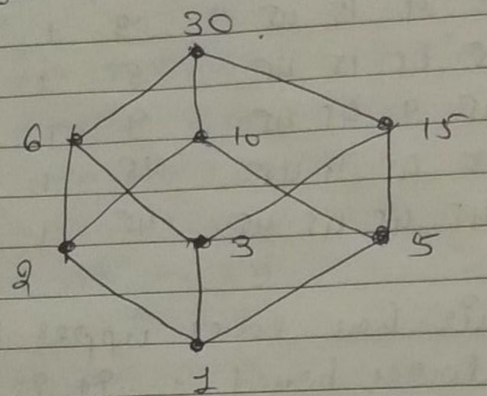
→ Every pair has least upper bound & greatest lower bound, so it is lattice.

Ex:- Let n be a positive integer $S_n$, be the set of all divisors of n. Let D denote the relation of 'division'. Draw the diagram of lattices for:

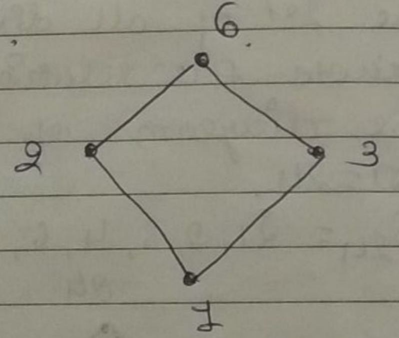
1) $n = 24$.
∴ $S_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$.



2) $n = 30$.
$S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.



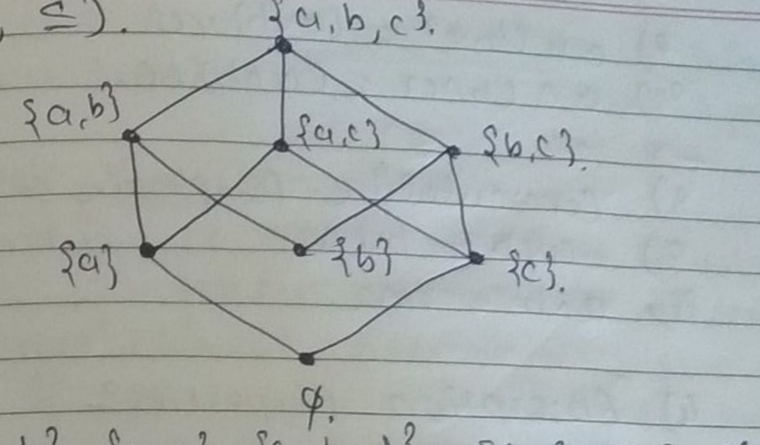

3) $n = 6$.
$S_6 = \{1, 2, 3, 6\}$.




*. **Sublattice :-**

→ Let $(A, \leq)$ be a lattice. A non-empty subset $S$ of $A$ is called a sublattice of $A$, if $a \vee b \in S$ & $a \wedge b \in S$ whenever $a \in S$ & $b \in S$.

Eg:- $(P(S), \subseteq)$.




$\{\{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$, $\{b, \{a,b\}, \{b,c\}, \{a,b,c\}\}$, $\{c\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$ are sublattice of $(P(S), \subseteq)$

However, $A_1 = \{\phi, \{a\}, \{c\}, \{a,b,c\}\}$ is not a sublattice as $a \vee c \notin A_1$.

also $A_2 = \{\phi, \{a,b\}, \{a,c\}, \{a,b,c\}\}$ is not a sublattice as $\{a,b\} \cap \{a,c\} = a \notin A_2$

* **Properties of Lattices :-**

Let $(A, \leq)$ be a lattice then

1) Idempotent properties.
   i) $a \vee a = a$.
   ii) $a \wedge a = a$.

2) Associative properties:-

i) $a \vee (b \vee c) = (a \vee b) \vee c$
ii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

3) Commutative properties :-
i) $a \vee b = b \vee a$
ii) $a \wedge b = b \wedge a$

4) Absorption properties :-
i) $a \vee (a \wedge b) = a$
ii) $a \wedge (a \vee b) = a$

* <u>Types</u> <u>of</u> <u>Lattices</u> :-

• <u>Distributive</u> <u>Lattice</u> :-
A lattice $(A, \leq)$ is called distributive lattice if for any elements $a, b, c \in A$
i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

<u>Remark</u> :- If $(A, \leq)$ does not satisfy distributive property $(A, \leq)$ is known as non-distributive lattice.

<u>e.g.</u> $(P(S), \subseteq)$ is a distributive lattice as $A \cup B$ (i.e. A union B) and $A \cap B$. (or join & meet resp.) satisfy the distributive property.

* <u>Bounded</u> <u>Lattice</u> :-
A lattice $(A, \leq)$ is said to be bounded lattice if greatest element is ~~denoted~~ & least element are in A.
If the greatest element is denoted by 1 & least element is denoted by 0 then $\forall\ a \in A$

$0 \leq a \leq 1$.

also,

$a \vee 0 = a$ $\quad$ $a \wedge 0 = 0$

$a \vee 1 = 1$ $\quad$ $a \wedge 1 = a$

<u>Ex</u> :- $(P(S), \subseteq)$ is a bounded lattice as $\phi \in P(S)$ & $S \in P(S)$. Where S is the greatest element & $\phi$ is the least element of $P(S)$.
also $\forall\ A \in P(S)$.

$\phi \subseteq A \subseteq S$.

also $A \cup \phi = A$ $\quad$ $A \cap \phi = \phi$.

$A \cup S = S$. $\quad$ $A \cap S = A$.

## * Complemented Lattice :-

→ Let $(A, \leq)$ be a bounded lattice. The greatest element of A is say 1 and the least element of A is 0. An element $a' \in A$ is known as complement of $a \in A$.
if i) $a \vee a' = 1$
ii) $a \wedge a' = 0$.

A lattice is known as complemented lattice if it is bounded and every element of A has complement.

Eg. $S = \{1, 2, 3\}$.
$(P(S), \subseteq)$ is a complemented lattice as $(P(S), \subseteq)$ is bounded and every element of P(S) has complement.

Remark:- 1) Empty set $\phi$ is the least element of P(S).
2) S is the greatest element of P(S)
also $\{a\}' = \{b, c\}$.
$\{b\}' = \{a, c\}$
$\{c\}' = \{a, b\}$
$\{a, b\}' = \{c\}$

also, $\{a, b, c\}' = \phi$.
and $\phi' = S$.

## * Modular lattice :-

A lattice $(A, \leq)$ is known as modular lattice if $\forall\ a, b, c \in A,\ a \leq c$
$\Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$. (Modular Equation).

## * Boolean Lattice :-

A lattice $(A, \leq)$ is known as Boolean lattice if
1) $(A, \leq)$ is complemented lattice
2) $(A, \leq)$ is distributive lattice.