Recurrence Relations

Recurrence Relations

A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n is terms of one or more of the previous terms of the sequence, namely, a_0, a_1, …, a_{n-1}, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a **solution** of a recurrence relation if it terms satisfy the recurrence relation.

Recurrence Relations

Example:

Consider the recurrence relation

$a_n = 2a_{n-1} – a_{n-2}$ for $n = 2, 3, 4, …$

Is the sequence $\{a_n\}$ with $a_n=3n$ a solution of this recurrence relation?

For $n \geq 2$ we see that

$2a_{n-1} – a_{n-2} = 2(3(n – 1)) – 3(n – 2) = 3n = a_n$.

Therefore, $\{a_n\}$ with $a_n=3n$ is a solution of the recurrence relation.

Recurrence Relations

Is the sequence $\{a_n\}$ with $a_n=5$ a solution of the same recurrence relation?

For $n \geq 2$ we see that

$2a_{n-1} – a_{n-2} = 2\cdot5 - 5 = 5 = a_n$.

Therefore, $\{a_n\}$ with $a_n=5$ is also a solution of the recurrence relation.

Recurrence Relations

In other words, a recurrence relation is like a recursively defined sequence, but **without specifying any initial values (initial conditions)**.

Therefore, the same recurrence relation can have (and usually has) **multiple solutions**.

If **both** the initial conditions and the recurrence relation are specified, then the sequence is **uniquely** determined.

Modeling with Recurrence Relations

Example: (Compound Interest)

Someone deposits Rs 10,000/- in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years?

Solution:

Let P_n denote the amount in the account after n years.

How can we determine P_n on the basis of P_{n-1}?

Modeling with Recurrence Relations

We can derive the following **recurrence relation**:

$P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}$.

The initial condition is $P_0 = 10{,}000$.

Then we have:

$P_1 = 1.05P_0$

$P_2 = 1.05P_1 = (1.05)^2 P_0$

$P_3 = 1.05P_2 = (1.05)^3 P_0$

…

$P_n = 1.05P_{n-1} = (1.05)^n P_0$

We now have a **formula** to calculate P_n for any natural number n and can avoid the iteration.

Modeling with Recurrence Relations

Let us use this formula to find P_{30} under the initial condition $P_0 = 10,000$:

$P_{30} = (1.05)^{30} \cdot 10,000 = 43,219.42$

After 30 years, the account contains Rs 43,219.42.

Modeling with Recurrence Relations

Another example:

Let a_n denote the number of bit strings of length n that do not have two consecutive 0s ("valid strings"). Find a recurrence relation and give initial conditions for the sequence $\{a_n\}$.

Solution:

Idea: The number of valid strings equals the number of valid strings ending with a 0 plus the number of valid strings ending with a 1.

Modeling with Recurrence Relations

Let us assume that $n \geq 3$, so that the string contains at least 3 bits.

Let us further assume that we know the number a_{n-1} of valid strings of length $(n - 1)$.

Then how many valid strings of length n are there, if the string ends with a 1?

There are a_{n-1} such strings, namely the set of valid strings of length $(n - 1)$ with a 1 appended to them.

Note: Whenever we append a 1 to a valid string, that string remains valid.

Modeling with Recurrence Relations

Now we need to know: How many valid strings of length n are there, if the string ends with a **0**?

Valid strings of length n ending with a 0 **must have a 1 as their (n – 1)st bit** (otherwise they would end with 00 and would not be valid).

And what is the number of valid strings of length (n – 1) that end with a 1?

We already know that there are a_{n-1} strings of length n that end with a 1.

Therefore, there are a_{n-2} strings of length (n – 1) that end with a 1.

Modeling with Recurrence Relations

So there are a_{n-2} valid strings of length n that end with a 0 (all valid strings of length (n – 2) with 10 appended to them).

As we said before, the number of valid strings is the number of valid strings ending with a 0 plus the number of valid strings ending with a 1.

That gives us the following **recurrence relation**:

$a_n = a_{n-1} + a_{n-2}$

Modeling with Recurrence Relations

What are the **initial conditions**?

$a_1 = 2$ (0 and 1)

$a_2 = 3$ (01, 10, and 11)

$a_3 = a_2 + a_1 = 3 + 2 = 5$

$a_4 = a_3 + a_2 = 5 + 3 = 8$

$a_5 = a_4 + a_3 = 8 + 5 = 13$

…

This sequence satisfies the same recurrence relation as the **Fibonacci sequence**.

Since $a_1 = f_3$ and $a_2 = f_4$, we have $a_n = f_{n+2}$.

Solving Recurrence Relations

In general, we would prefer to have an **explicit formula** to compute the value of a_n rather than conducting n iterations.

For one class of recurrence relations, we can obtain such formulas in a systematic way.

Those are the recurrence relations that express the terms of a sequence as **linear combinations** of previous terms.

Linear recurrences

1. Linear homogeneous recurrences
2. Linear non-homogeneous recurrences

There are three method of solving Recurrence Relation

1. Iteration
2. Characteristic roots and
3. Generating Functions

Solving Recurrence Relations

Definition: A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form:

$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$,

Where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

A sequence satisfying such a recurrence relation is uniquely determined by

the recurrence relation and the k initial conditions

$a_0 = C_0, a_1 = C_1, a_2 = C_2, \ldots, a_{k-1} = C_{k-1}$.

- it is *linear* because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of n.
- it is *homogeneous* because no terms occur that are not multiples of the a_js. Each coefficient is a constant.
- the *degree* is k because a_n is expressed in terms of the previous k terms of the sequence.

Examples of Linear Homogeneous Recurrence Relations

- $P_n = (1.11)P_{n-1}$ linear homogeneous recurrence relation of degree one
- $f_n = f_{n-1} + f_{n-2}$ linear homogeneous recurrence relation of degree two
- $a_n = a_{n-1} + a_{n-2}^2$ not linear
- $H_n = 2H_{n-1} + 1$ not homogeneous
- $B_n = nB_{n-1}$ coefficients are not constants

Solving Recurrence Relations

Examples:

The recurrence relation $P_n = (1.05)P_{n-1}$
is a linear homogeneous recurrence relation of **degree one**.

The recurrence relation $f_n = f_{n-1} + f_{n-2}$
is a linear homogeneous recurrence relation of **degree two**.

The recurrence relation $a_n = a_{n-5}$
is a linear homogeneous recurrence relation of **degree five**.

Solving Recurrence Relations

Basically, when solving such recurrence relations,

we try to find solutions of the form $a_n = r^n$, where r is a constant.

$a_n = r^n$ is a solution of the recurrence relation
$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ if and only if
$r^n = c_1r^{n-1} + c_2r^{n-2} + \ldots + c_kr^{n-k}$.

Divide this equation by r^{n-k} and subtract the right-hand side from the left:

$r^k - c_1r^{k-1} - c_2r^{k-2} - \ldots - c_{k-1}r - c_k = 0$

This is called the **characteristic equation** of the recurrence relation.

Solving Recurrence Relations

The solutions of this equation are called the **characteristic roots** of the recurrence relation.

Distinct Roots

Let us consider linear homogeneous recurrence relations of **degree two**.

Theorem: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2.

Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Solving Recurrence Relations

Example: What is the solution of the recurrence relation

$a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Solution: The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$.

Its roots are $r = 2$ and $r = -1$.

Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if:

$a_n = \alpha_1 2^n + \alpha_2(-1)^n$ for some constants α_1 and α_2.

Solving Recurrence Relations

Given the equation $a_n = \alpha_1 2^n + \alpha_2(-1)^n$ and the initial conditions $a_0 = 2$ and $a_1 = 7$, it follows that

$a_0 = 2 = \alpha_1 + \alpha_2$

$a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1)$

Solving these two equations gives us
$\alpha_1 = 3$ and $\alpha_2 = -1$.

Therefore, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with

$a_n = 3 \cdot 2^n - (-1)^n$.

Solving Recurrence Relations

Example: Give an explicit formula for the Fibonacci numbers.

Solution: The Fibonacci numbers satisfy the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 0$ and $f_1 = 1$.

The characteristic equation is $r^2 – r – 1 = 0$.

Its roots are

Solving Recurrence Relations

Therefore, the Fibonacci numbers are given by

for some constants α_1 and α_2.

We can determine values for these constants so that the sequence meets the conditions $f_0 = 0$ and $f_1 = 1$:

Solving Recurrence Relations

The unique solution to this system of two equations and two variables is

So finally we obtained an explicit formula for the Fibonacci numbers:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Theorem: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2.

Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Solving Recurrence Relations

<u>But what happens if the characteristic equation has only one root?</u>

How can we then match our equation with the initial conditions a_0 and a_1 ?

Theorem: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0 which is repeated two times.

A sequence $\{a_n\}$ is a solution of the recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Example

- Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 2, 2, 2, 5, 5, and 9 . What is the form of the general solution?

$$a_n = \left(\alpha_1 (2)^n + \alpha_2 n(2)^n + \alpha_3 n^2 (2)^n\right) + \left(\alpha_3 (5)^n + \alpha_4 n(5)^n\right) + \alpha_5 (9)^n$$

Solving Recurrence Relations

Example: What is the solution of the recurrence relation

$a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution: The only root of $r^2 - 6r + 9 = 0$ is $r_0 = 3$.

Hence, the solution to the recurrence relation is

$a_n = \alpha_1 3^n + \alpha_2 n3^n$ for some constants α_1 and α_2.

To match the initial condition, we need

$a_0 = 1 = \alpha_1$

$a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3$

Solving these equations yields $\alpha_1 = 1$ and $\alpha_2 = 1$.

Consequently, the overall solution is given by

$a_n = 3^n + n3^n$.

Example 8. Find the solution to the recurrence relation

$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

Sol :

$r^3 + 3r^2 + 3r + 1 = 0$ has a single root $r_0 = -1$ of multiplicity three.

$\therefore a_n = (\alpha_1+\alpha_2 n+\alpha_3 n^2)\, r_0^{\,n} = (\alpha_1+\alpha_2 n+\alpha_3 n^2)(-1)^n$

initial conditions are given $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

$$a_0 = \alpha_1 = 1$$

$$a_1 = (\alpha_1+\alpha_2+\alpha_3) \cdot (-1) = -2$$

$$a_2 = \alpha_1+2\alpha_2+4\alpha_3 = -1$$

$\therefore \alpha_1 = 1,\ \alpha_2 = 3,\ \alpha_3 = -2 \Rightarrow a_n = (1+3n-2n^2) \cdot (-1)^n$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

The General form of **Linear nonhomogeneous recurrence relation with constant coefficients is**

$a_n = c_1 a_{n-1} + c2 a_{n-2} + \cdot\cdot\cdot + c_k a_{n-k} + F(n)$

The recurrence relation $a_n = 3a_{n-1} + 2n$ is an example of a **Linear nonhomogeneous recurrence relation with constant coefficients.**

- Every solution of a linear nonhomogeneous recurrence relation is the sum of
  - a particular relation and
  - a solution to the associated linear homogeneous recurrence relation

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

The following are linear nonhomogeneous recurrence relations with constant coefficients:

$a_n = a_{n-1} + 2^n$,

$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$,

$a_n = 3a_{n-1} + n3^n$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$

where the following are the associated linear homogeneous recurrence relations, respectively:

$a_n = a_{n-1}$,

$a_n = a_{n-1} + a_{n-2}$,

$a_n = 3a_{n-1}$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$

Solving Linear NonHomogeneous Recurrences (1)

- **Theorem** :

If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \mathbf{+ f(n)}$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$ where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

Solving Linear NonHomogeneous Recurrences (2)

- There is no general method for solving such relations.
- However, we can solve them for special cases
- In particular, if f(n) is
  - a polynomial function
  - exponential function, or
  - the product of a polynomial and exponential functions,

then there is a general solution

Example. Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1=3$?

Sol :

associated homogeneous recurrence relation is

$a_n = 3a_{n-1}$

Characteristic equation: $r - 3 = 0 \Rightarrow r = 3 \Rightarrow a_n^{(h)} = \alpha \times 3^n$.

{particular solution}

$\because F(n) = 2n$

$\therefore$ Let $a_n^{(p)} = cn+d$, where $c, d \in \mathbf{R}$.

If $a_n^{(p)} = cn+d$ is a solution to $a_n = 3a_{n-1} + 2n$,

then $cn+d = 3(c(n-1)+d)+2n$

$cn+d = 3cn - 3c + 3d + 2n$

$\Rightarrow 2cn-3c + 2d + 2n = 0$

$\Rightarrow (2c+2)n + (2d-3c) = 0n+0$

Example continued

$\therefore$ By comparing coefficients of *n* and constant

We get $2c+2 = 0$, and $2d-3c = 0$

$\Rightarrow\ c = -1,\ d = -3/2$

$\Rightarrow\ a_n^{(p)} = -n - 3/2$

$\Rightarrow a_n = a_n^{(h)} + a_n^{(p)} = \alpha \times 3^n - n - 3/2$

Given that $a_1 = 3$

If $a_1 = \alpha \times 3 - 1 - 3/2 = 3 \quad \Rightarrow \alpha = 11/6$

$\Rightarrow\ a_n = (11/6) \times 3^n - n - 3/2$

Example . Find all solutions of the recurrence relation

$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.

Sol :

{associated homogeneous recurrence relation is

$a_n = 5a_{n-1} - 6a_{n-2}$ }

Characteristic equation: $r^2 - 5r + 6 = 0$

$\Rightarrow r_1 = 3, r_2 = 2$

$\Rightarrow a_n^{(h)} = \alpha_1 \times 3^n + \alpha_2 \times 2^n$.

{particular solution}

$\because F(n) = 7^n$ $\therefore$ Let $a_n^{(p)} = c \cdot 7^n$, where $c \in \mathbf{R}$.

If $a_n^{(p)} = c \cdot 7^n$ is a solution to $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$,

then $c \cdot 7^n = 5c \cdot 7^{n-1} - 6c \cdot 7^{n-2} + 7^n$

continue

$$c \cdot 7^n = 5c \cdot 7^{n-1} - 6c \cdot 7^{n-2} + 7^n$$

$$\Rightarrow c \cdot 7^2 = 5c \cdot 7^1 - 6c + 7^2$$

$$\Rightarrow 49c = 35c - 6c + 49$$

$$\Rightarrow c = 49/20$$

$$\Rightarrow a_n^{(p)} = (49/20) \cdot 7^n$$

$$\Rightarrow a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = \alpha_1 \times 3^n + \alpha_2 \times 2^n + (49/20) \cdot 7^n$$

Example 11. What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, and $F(n) = (n^2+1)3^n$.

Sol :

The associated linear homogeneous recurrence relation is $a_n = 6a_{n-1} - 9a_{n-2}$.

characteristic equation: $r^2 - 6r + 9 = 0 \Rightarrow r = 3$ (Multiple root)

$F(n) = 3^n$, and 3 is a root $\Rightarrow a_n^{(p)} = p_0 n^2 3^n$

$F(n) = n3^n$, and 3 is a root $\Rightarrow a_n^{(p)} = n^2(p_1 n + p_0)\, 3^n$

$F(n) = n^2 2^n$, and 2 is not a root $\Rightarrow a_n^{(p)} = (p_2 n^2 + p_1 n + p_0) 2^n$

$F(n) = (n^2+1)3^n$,

and 3 is a root $\Rightarrow a_n^{(p)} = n^2 (p_2 n^2 + p_1 n + p_0)\, 3^n$

Generating Functions

Def 1. The generating function for the sequence $a_0, a_1, a_2, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \ldots + a_n x^n + \ldots$$

$$= \sum_{k=0}^{\infty} a_k x^k$$

Example 1. Find the generating functions for the sequences $\{a_k\}$ with

(1) $a_k = 3$

(2) $a_k = k+1$

(3) $a_k = 2^k$

Sol :

(1) $G(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} 3x^k$

(2) $G(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} (k+1)x^k$

(3) $G(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} 2^k x^k$

Example 2. What is the generating function for the sequence 1,1,1,1,1,1 ?

Sol : $a_0\ a_1\ a_2\ a_3\ a_4\ a_5\ a_6$

$$G(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= 1 + x + x^2 + \ldots + x^5$$ (expansion)

$$= \frac{x^6 - 1}{x - 1}$$ (closed form)

Using Generating Functions to solve Recurrence Relations.

Example 16.

Solving the recurrence relation $a_k = 3a_{k-1}$ for $k=1,2,3,\ldots$ and initial condition $a_0 = 2$.

Sol :

$$r - 3 = 0 \;\Rightarrow r = 3 \;\Rightarrow\; a_n = \alpha \cdot 3^n$$

$$\because a_0 = 2 = \alpha$$

$$\therefore a_n = 2 \cdot 3^n$$

Another Method to solve recurrence relation is using Generating Function

Let $G(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{k=0}^{\infty} a_k x^k$

be the generating function for $\{a_k\}$.

First note that $a_k = 3a_{k-1}$

$$\Rightarrow \sum_{k=1}^{\infty} a_k x^k = 3\sum_{k=1}^{\infty} a_{k-1} x^k = 3x\sum_{k=1}^{\infty} a_{k-1} x^{k-1} = 3x\sum_{k=0}^{\infty} a_k x^k$$

$$\Rightarrow G(x) \underline{- a_0} = 3x \cdot G(x)$$

$$\because a_0 = 2 \Rightarrow G(x) - 3x \cdot G(x) = G(x)(1-3x) = 2$$

$$\therefore G(x) = \frac{2}{1-3x} = 2 \cdot \sum_{k=0}^{\infty} (3x)^k = \sum_{k=0}^{\infty} 2 \cdot 3^k \cdot x^k$$

$$\therefore a_k = 2 \cdot 3^k$$

Example 17

Solving $a_k = 8a_{k-1} + 10^{k-1}$ for $k = 1,2,3,\ldots$ and initial condition $a_0 = 1$ and $a_1 = 9$

Sol :

Let $G(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{k=0}^{\infty} a_k x^k$

be the generating function for $\{a_k\}$.

$$G(x) - 1 = \sum_{k=1}^{\infty} a_k x^k = \sum_{k=1}^{\infty} (8a_{k-1} + 10^{k-1}) x^k$$

$$= 8\sum_{k=1}^{\infty} a_{k-1} x^k + \sum_{k=1}^{\infty} 10^{k-1} x^k = 8x\sum_{k=0}^{\infty} a_k x^k + x\sum_{k=0}^{\infty} 10^k x^k$$

$$= 8xG(x) + \frac{x}{1-10x}$$

$$(1-8x)G(x)=1+\frac{x}{1-10x}=\frac{1-9x}{1-10x}$$

$$G(x)=\frac{1-9x}{(1-10x)(1-8x)}=\frac{1}{2}\left(\frac{1}{1-10x}+\frac{1}{1-8x}\right)$$

$$=\frac{1}{2}\left(\sum_{k=0}^{\infty}10^k x^k+\sum_{k=0}^{\infty}8^k x^k\right)=\sum_{k=0}^{\infty}\frac{1}{2}(10^k+8^k)x^k$$

$\therefore a_k = (10^k + 8^k)/2$