Solutions

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1 Question 2.1

1.1 Part (a)

For proving Abelian group, we need to prove the following properties:

1.1.1 Closure

Consider two elements, a and b which are part of $\mathbb{R}\backslash\{-1\}$, then:

$$a \star b = ab + a + b$$

Since, a and b are real numbers, ab, a and b will also be real numbers, and thus, their sum will also be real. We only need to check if $a * b$ can result in -1.

Let's assume that for a and b not equal to -1, $a \star b$ is -1. Then:

$$ab + a + b = -1$$

$$a(b+1) + b = -1$$

$$a(b+1) = -(b+1)$$

$$a = -1$$

The above step can be performed, since, $b \neq 1$

But, we know that $a \neq 1$, so, $ab + a + b \neq 1$.

Hence, we have proven the closure property.

1.1.2 Associativity

Consider three elements, a, b and c which are part of $\mathbb{R}\backslash\{-1\}$, then:

$$a \star (b \star c) = a \star (bc + b + c)$$

$$a \star (b \star c) = a(bc + b + c) + a + (bc + b + c)$$

$$a \star (b \star c) = abc + ab + ac + a + bc + b + c$$

$$a \star (b \star c) = abc + ac + bc + c + ab + a + b$$

$$a \star (b \star c) = (ab + a + b)c + (ab + a + b) + c$$

$$a \star (b \star c) = (a \star b)c + (a \star b) + c$$

$$a \star (b \star c) = (a \star b) \star c$$

Hence, we have proven the associativity property.

1.1.3 Neutral Element

Consider two elements, a and e which are part of $\mathbb{R} \backslash \{-1\}$, such that:

$a \star e = a$ and $e \star a = a$

We need to check if such an element e exists or not.

$$a \star e = ae + a + e$$

$$a = ae + a + e$$

$$ae + e = 0$$

$$(a + 1)e = 0$$

Since, $a \neq 1$

$$e = 0$$

We also know that {0} is a part of $\mathbb{R} \backslash \{-1\}$, so, we have found that such an element exists. Now, let's verify if $e \star a = a$ is true or not.

$$e \star a = ea + a + e$$

$$a = ea + a + e$$

$$ea + e = 0$$

$$(a + 1)e = 0$$

$$e = 0$$

We have reached at the same element.

Hence, we have proved that a neutral element exists.

1.1.4 Inverse Element

Consider two elements, a and b, and the neutral element, e, all lying in $\mathbb{R}\backslash\{-1\}$. We have to check if the following condition can be fulfilled for any b or not, given a.

$$a \star b = e$$

$$ab + a + b = e$$

Since we know that $e = 0$

$$ab + a + b = 0$$

$$a(b + 1) = -b$$

Since $b \neq 1$

$$a = \frac{-b}{b + 1}$$

For a real number b, the above a will also be a real number. We need to check if a can be -1 or not. Let's assume that $a = -1$, then,

$$-1 = \frac{-b}{b + 1}$$

$$b + 1 = b$$

$$1 = 0$$

The above statement is clearly false, which means that $a \neq -1$. Thus, a also lies in $R\backslash${-1}

Hence, we have seen that for any b lying in $\mathbb{R}\backslash\{-1\}$, an inverse element $a = \frac{-b}{b+1}$ lying in $\mathbb{R}\backslash\{-1\}$ also exists.

1.1.5 Commutativity

Consider two elements a and b lying in $\mathbb{R}\backslash\{-1\}$, then

$$a \star b = ab + a + b$$

$$a \star b = ba + b + a$$

$$a \star b = b \star a$$

This proves the commutative property as well.

Since all the above properties hold true, we can $(\mathbb{R}\backslash\{-1\}, \star)$ is an Abelian group.

1.2 Part (b)

$$3 \star x \star x = 15$$

$$(3 \star x) \star x = 15$$

$$(3x + x + 3) \star x = 15$$

$$(4x + 3) \star x = 15$$

$$(4x + 3)x + (4x + 3) + x = 15$$

$$4x^2 + 3x + 4x + 3 + x = 15$$

$$4x^2 + 8x + 3 = 15$$

$$4x^2 + 8x - 12 = 0$$

$$x^2 + 2x - 3 = 0$$

$$(x - 2)(x - 1) = 0$$

$$x = 2, x = 1$$

2 Question 2.2

2.1 Part (a)

We have to show that $(\mathbb{Z}_n, \oplus)$ is a group. In order to do that, we have to verify that some properties exist for $(\mathbb{Z}_n, \oplus)$

2.1.1 Closure

Consider two elements $\bar{a}, \bar{b}$ which are part of $\mathbb{Z}_n$. Then:

$$\bar{a} \oplus \bar{b} = \overline{a + b}$$

$$\bar{a} \oplus \bar{b} = \{x \in \mathbb{Z} | \exists c \in \mathbb{Z} : x - (a + b) = nc\}$$

We now have to prove that $\overline{a + b}$ is present in $\mathbb{Z}_n$

We already know that a and b are at least 0, so $a + b \geq 0$. We now need to check for the upperbound, i.e., $\overline{a + b} \in \mathbb{Z}_n$

Let's assume the contrary. Assume that, $\overline{a + b} > \overline{n - 1}$

We can write this as:

$$\overline{a+b} = \overline{n-1+k}$$

Where, $k > 0$
Let's simplify the RHS

$$\overline{n-1+k} = \{x \in \mathbb{Z} | \exists c \in \mathbb{Z} : x - (n-1+k) = nc\}$$

$$\overline{n-1+k} = \{x \in \mathbb{Z} | \exists c \in \mathbb{Z} : x - n + 1 - k = nc\}$$

$$\overline{n-1+k} = \{x \in \mathbb{Z} | \exists c \in \mathbb{Z} : x + 1 - k = nc + n\}$$

$$\overline{n-1+k} = \{x \in \mathbb{Z} | \exists c \in \mathbb{Z} : x + 1 - k = n(c+1)\}$$

$$\overline{n-1+k} = \{x \in \mathbb{Z} | \exists c \in \mathbb{Z} : x - (k-1) = n(c+1)\}$$

Let's assume that $k = n\alpha + \beta$, or $k = \beta \mod n$, then

$$\overline{n-1+k} = \{x \in \mathbb{Z} | \exists c \in \mathbb{Z} : x - (n\alpha + \beta - 1) = n(c+1)\}$$

$$\overline{n-1+k} = \{x \in \mathbb{Z} | \exists c \in \mathbb{Z} : x - n\alpha - (\beta - 1) = n(c+1)\}$$

$$\overline{n-1+k} = \{x \in \mathbb{Z} | \exists c \in \mathbb{Z} : x - (\beta - 1) = n(c+1+\alpha)\}$$

Let's replace $c+1+\alpha$ with γ, then

$$\overline{n-1+k} = \{x \in \mathbb{Z} | \exists c \in \mathbb{Z} : x - (\beta - 1) = n\gamma\}$$

$$\overline{n-1+k} = \overline{\beta - 1}$$

Since, $\beta < n$, thus, $\beta - 1 < n - 1$
Thus, $\overline{\beta - 1} \in \mathbb{Z}_n$ and hence, $\overline{a+b} \in \mathbb{Z}_n$
We have proven the closure property. Let's now move to the next property.

2.1.2 Associativity

Consider three elements $\bar{a}$,$\bar{b}$ and $\bar{c} \in \mathbb{Z}_n$, then:

$$\bar{a} \oplus (\bar{b} \oplus \bar{c})$$

$$= \bar{a} \oplus (\overline{b+c})$$

$$= \overline{a + \overline{b+c}}$$

$$= \overline{a + \{x \in \mathbb{Z} | \exists m \in \mathbb{Z} : (x - (b+c) = nm)\}}$$

$$= \overline{a + \{nm + (b + c)\}}$$

$$= \{x \in \mathbb{Z} | \exists m' \in \mathbb{Z} : (x - (a + nm + (b + c)) = nm')\}$$

$$= \{nm' + a + nm + b + c\}$$

Similarly:

$$(\bar{a} \oplus \bar{b}) \oplus \bar{c}$$

$$= \overline{\overline{a + b} + c}$$

$$= \overline{\{x \in \mathbb{Z} | \exists m \in \mathbb{Z} : (x - (a + b) = nm)\} + c}$$

$$= \overline{\{nm + a + b\} + c}$$

$$= \{x \in \mathbb{Z} | \exists m' \in \mathbb{Z} : (x - (c + nm + (a + b)) = nm')\}$$

$$= \{nm' + a + b + c + nm\}$$

Thus:

$$\bar{a} \oplus (\bar{b} \oplus \bar{c}) = (\bar{a} \oplus \bar{b}) \oplus \bar{c}$$

This proves the associativity property.

2.1.3 Neutral Element

Consider two elements a and e in $\mathbb{Z}_n$. We need to check if it's possible for the following to exist:

$$a \oplus e = e \oplus a = a$$

Let's first check for $a \oplus e = a$

$$a \oplus e = a$$

$$\overline{a + e} = a$$

$$\{x \in \mathbb{Z} | \exists k \in \mathbb{Z} : (x - (a + e) = nk)\} = a$$

$$\{nk + a + e\} = a$$

This can hold true if $e = \{-nk | \exists k \in \mathbb{Z}\} = \bar{0}$

Similarly, we can show that $e \oplus a = a$ if $e = \bar{0}$

Note that we can also write $e = -\bar{0}$ but since our $k \in \mathbb{Z}$, it will automatically cover negative elements as well, and thus, $\bar{0} = -\bar{0}$

2.1.4 Inverse Element

Consider three elements a, b and e in $\mathbb{Z}_n$. We need to check if it's possible for the following to exist:

$$a \oplus b = b \oplus a = e$$

$$\overline{a+b} = \overline{b+a} = \bar{0}$$

Let's first focus on $\overline{a+b} = \bar{0}$

$$\overline{a+b} = \bar{0}$$

$$\{x \in \mathbb{Z} | \exists k \in \mathbb{Z} : (x - (a+b) = nk)\} = \bar{0}$$

$$\{nk + a + b\} = \{nk\}$$

This is true if $a = -b$
Similarly, $\overline{b+a} = \bar{0}$ if $a = -b$

2.1.5 Commutative Property

Let's consider two elements a and b in $\mathbb{Z}_n$. We need to verify if the following property holds true or not.

$$a \oplus b = b \oplus a$$

$$\overline{a+b} = \overline{b+a}$$

$$\{x \in \mathbb{Z} | \exists k \in \mathbb{Z} : (x - (a+b) = nk)\} = \{x \in \mathbb{Z} | \exists k \in \mathbb{Z} : (x - (b+a) = nk)\}$$

$$\{nk + a + b\} = \{nk + b + a\}$$

which we know is true.
Thus, commutative property also holds true.
Thus, we can say that $(\mathbb{Z}_n, \oplus)$ is an abelian group.

2.2 Part (b)

$$\mathbb{Z}_5 \backslash \{\bar{0}\} = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$$

$$\bar{1} = \{5a + 1\}$$
$$\bar{2} = \{5a + 2\}$$
$$\bar{3} = \{5a + 3\}$$
$$\bar{4} = \{5a + 4\}$$

Where, $a \in \mathbb{Z}$
Times table of the elements of $\mathbb{Z}_5 \backslash \{\bar{0}\}$ under $\otimes$

$$\bar{1} \otimes \bar{1} = \overline{1 \times 1} = \bar{1} = \{5a + 1\}$$
$$\bar{1} \otimes \bar{2} = \overline{1 \times 2} = \bar{2} = \{5a + 2\}$$
$$\bar{1} \otimes \bar{3} = \overline{1 \times 3} = \bar{3} = \{5a + 3\}$$
$$\bar{1} \otimes \bar{4} = \overline{1 \times 4} = \bar{4} = \{5a + 4\}$$
$$\bar{2} \otimes \bar{1} = \overline{2 \times 1} = \bar{2} = \{5a + 2\}$$
$$\bar{2} \otimes \bar{2} = \overline{2 \times 2} = \bar{4} = \{5a + 4\}$$
$$\bar{2} \otimes \bar{3} = \overline{2 \times 3} = \bar{6} = \{5a + 6\} = \{5(a + 1) + 1\} = \{5a + 1\} = \bar{1}$$
$$\bar{2} \otimes \bar{4} = \overline{2 \times 4} = \bar{8} = \{5a + 3\} = \{5(a + 1) + 3\} = \{5a + 3\} = \bar{3}$$
$$\bar{3} \otimes \bar{1} = \overline{3 \times 1} = \bar{3} = \{5a + 3\}$$
$$\bar{3} \otimes \bar{2} = \overline{3 \times 2} = \bar{6} = \{5a + 6\} = \{5(a + 1) + 1\} = \{5a + 1\} = \bar{1}$$
$$\bar{3} \otimes \bar{3} = \overline{3 \times 3} = \bar{9} = \{5a + 9\} = \{5(a + 1) + 4\} = \{5a + 4\} = \bar{4}$$
$$\bar{3} \otimes \bar{4} = \overline{3 \times 4} = 1\bar{2} = \{5a + 12\} = \{5(a + 2) + 2\} = \{5a + 2\} = \bar{2}$$
$$\bar{4} \otimes \bar{1} = \overline{4 \times 1} = \bar{4} = \{5a + 4\}$$
$$\bar{4} \otimes \bar{2} = \overline{4 \times 2} = \bar{8} = \{5a + 8\} = \{5(a + 1) + 3\} = \{5a + 3\} = \bar{3}$$
$$\bar{4} \otimes \bar{3} = \overline{4 \times 3} = 1\bar{2} = \{5a + 12\} = \{5(a + 2) + 2\} = \{5a + 2\} = \bar{2}$$
$$\bar{4} \otimes \bar{4} = \overline{4 \times 4} = 1\bar{6} = \{5a + 16\} = \{5(a + 3) + 1\} = \{5a + 1\} = \bar{1}$$

Thus, $\mathbb{Z}_5 \backslash \{\bar{0}\}$ is closed under $\otimes$
The neutral element is $\bar{1}$
The inverses for the elements in $\mathbb{Z}_5$ are:

$$\bar{1}^{-1} = \bar{1}$$
$$\bar{2}^{-1} = \bar{3}$$
$$\bar{3}^{-1} = \bar{2}$$
$$\bar{4}^{-1} = \bar{4}$$

Based on the times table, we can show that the following property exist:

1. Closure
2. Neutral element
3. Inverse element
4. Commutativity

Let's check for associativity as well. We will assume three elements a, b and c exist in $\mathbb{Z}_5 \backslash \{\bar{0}\}$, then:

$$a \otimes (b \otimes c) = \overline{a \times \overline{b \times c}}$$

$$= \overline{a \times \{5k + bc\}}$$

$$= \{5k' + a(5k + bc)\}$$

$$= \{5(ka + k') + abc\}$$

$$= \{5k'' + abc\}$$

Similarly,

$$(a \otimes b) \otimes c = \overline{\overline{a \times b} \times c}$$

$$= \overline{\{5k + ab\} \times c}$$

$$= \{5k' + (5k + ab)c\}$$

$$= \{5k' + 5kc + abc\}$$

$$= \{5(kc + k') + abc\}$$

$$= \{5k'' + abc\}$$

We can replace $kc + k'$ and $ka + k'$ with k'' since $k \in \mathbb{Z}$. Hence, we can conclude the associativity. This shows that $\mathbb{Z}_5 \backslash \{\bar{0}\}$ is an Albeian group.

2.3 Part (c)

Let's define the elements in $(\mathbb{Z}_8 \backslash \{\bar{0}\})$

$$\bar{1} = \{8a + 1\}$$
$$\bar{2} = \{8a + 2\}$$
$$\bar{3} = \{8a + 3\}$$
$$\bar{4} = \{8a + 4\}$$
$$\bar{5} = \{8a + 5\}$$
$$\bar{6} = \{8a + 6\}$$
$$\bar{7} = \{8a + 7\}$$

Now, if we consider $\bar{2} \otimes \bar{4}$, then we get:

$$\bar{2} \otimes \bar{4} = \overline{2 \times 4}$$

$$= \overline{8}$$

But that's not a part of $\mathbb{Z}_8 \backslash \{\bar{0}\}$ and that's why, $(\mathbb{Z}_8 \backslash \{\bar{0}\}, \otimes)$ does not follow closure property, and hence is not a group.

2.4 Part (d)

In order for $(\mathbb{Z}_n \backslash \{\bar{0}\})$ to be a group, it should follow these properties:

2.4.1 Closure

Consider two elements $a, b \in \mathbb{Z}_n \backslash \{\bar{0}\}$, then:

$$\bar{a} \otimes \bar{b} = \overline{a \times b}$$

$$= kn + ab$$

Where, $k \in \mathbb{Z}$

Let's assume that $ab = \alpha n + \beta$, where $\beta < n$ and $\alpha, \beta \in \mathbb{Z}$

$$\bar{a} \otimes \bar{b}$$

$$= kn + \alpha n + \beta$$

$$= (k + \alpha)n + \beta$$

$$= k'n + \beta$$

The above can fail only if $\beta = 0$, in which case, $\bar{a} \otimes \bar{b} = \bar{0} \notin \mathbb{Z}_n \backslash \{\bar{0}\}$

If $\beta = 0$, then, $ab = \alpha n$

Now, in order for $(\mathbb{Z}_n \backslash \{\bar{0}\}, \otimes)$ to be a group, the closure property should be satisfied, which in turn means that $ab \mod n \neq 0$ or ab and n are relatively prime.

But, if n is not prime, then there will exist at least one pair $(a, b) < n$, such that, $ab = n$. This means that at least for one pair (a, b), the closure property will fail.

Thus, for $(\mathbb{Z}_n \backslash \{\bar{0}\}, \otimes)$ to be a group, $n \in \mathbb{N} \backslash \{0\}$ should be prime.

But, in order to prove that if this is a sufficient condition, we will need to prove the rest of the properties as well.

2.4.2 Associativity

Consider three elements, $\bar{a}, \bar{b}, \bar{c} \in \mathbb{Z}_n \backslash \{\bar{0}\}$, then:

$$\bar{a} \otimes (\bar{b} \otimes \bar{c})$$

$$= \bar{a} \otimes \overline{b \times c}$$

$$= \overline{a \times \overline{b \times c}}$$

$$= \overline{a \times (nk + bc)}$$

$$= nk' + a(nk + bc)$$

$$= n(ak + k') + abc$$

$$= nk'' + abc$$

Similarly:

$$(\bar{a} \otimes \bar{b}) \otimes \bar{c})$$

$$= \overline{a \times b} \otimes \bar{c}$$

$$= \overline{\overline{a \times b} \times c}$$

$$= \overline{(nk + ab) \times c}$$

$$= nk' + (nk + ab)c$$

$$= n(k' + ck) + abc$$

$$= nk'' + abc$$

This proves the associativity.

2.4.3 Neutral Element

Consider two elements $\bar{a}, \bar{e} \in \mathbb{Z}_n \backslash \{\bar{0}\}$. We need to check if the following can exist or not:

$$\bar{a} \otimes \bar{e} = \bar{e} \otimes \bar{a} = \bar{a}$$

$$\bar{a} \otimes \bar{e}$$

$$= \overline{a \times e}$$

$$= \{nk + ae\}$$

For this to be equal to $\bar{a}$, $\bar{e} = \bar{1}$
So, $\bar{1}$ is the neutral element.

2.4.4 Inverse Element

Consider two elements $\bar{a}, \bar{b} \in \mathbb{Z}_n \backslash \{\bar{0}\}$. We need to check if the following can exist:

$$\bar{a} \otimes \bar{b} = \bar{b} \otimes \bar{a} = \bar{e}$$

$$\bar{a} \otimes \bar{b} = \overline{a \times b}$$

$$\overline{a \times b} = \bar{1}$$

$$\{nk + ab\} = \{nk + 1\}$$

This can happen if $ab = 1 \mod n$

$$ab = 1 \mod n$$

$$ab = k'n + 1$$

$$b = \frac{k'n + 1}{a}$$

Where, $k' \in \mathbb{Z}$
Thus, we have shown that for every $\bar{a}$, an inverse element exists.
All the above properties show that $(\mathbb{Z}_n \backslash \{\bar{0}\}, \otimes)$ is a group if $n \in \mathbb{N} \backslash \{0\}$ is a prime number.

3 Question 2.3

As in the previous questions, we will verify the properties.

3.1 Closure

Consider two matrices $A, B \in \mathbb{R}^{3\times3}$ then:

$$A = \begin{bmatrix} 1 & x_a & z_a \\ 0 & 1 & y_a \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & x_b & z_b \\ 0 & 1 & y_b \\ 0 & 0 & 1 \end{bmatrix}$$

$$A \cdot B$$

$$= \begin{bmatrix} 1 & x_a + x_b & z_b + x_a y_b + z_a \\ 0 & 1 & y_a + y_b \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & x_c & z_c \\ 0 & 1 & y_c \\ 0 & 0 & 1 \end{bmatrix}$$

Where,

$$x_c = x_a + x_b \in \mathbb{R}^{3\times3}$$

$$y_c = y_a + y_b \in \mathbb{R}^{3\times3}$$

$$z_c = z_b + x_a y_b + z_a \in \mathbb{R}^{3\times3}$$

Thus, $A \cdot B \in \mathcal{G}$
Hence, closure property is proved.

3.2 Associativity

Consider three elements $A, B, C \in \mathcal{G}$, then:

$$A \cdot (B \cdot C)$$

$$= A \cdot \left(\begin{bmatrix} 1 & x_b & z_b \\ 0 & 1 & y_b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x_c & z_c \\ 0 & 1 & y_c \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= A \cdot \begin{bmatrix} 1 & x_b + x_c & z_b + x_b y_c + z_c \\ 0 & 1 & y_c + y_b \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & x_a & z_a \\ 0 & 1 & y_a \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x_b + x_c & z_b + x_b y_c + z_c \\ 0 & 1 & y_c + y_b \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & x_a + x_b + x_c & z_b + x_b y_c + z_c + x_a y_c + x_a y_b + z_a \\ 0 & 1 & y_a + y_b + y_c \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly,

$$(A \cdot B) \cdot C$$

$$= \left(\begin{bmatrix} 1 & x_a & z_a \\ 0 & 1 & y_a \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x_b & z_b \\ 0 & 1 & y_b \\ 0 & 0 & 1 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 & x_c & z_c \\ 0 & 1 & y_c \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & x_a + x_b & z_b + x_a y_b + z_a \\ 0 & 1 & y_a + y_b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x_c & z_c \\ 0 & 1 & y_c \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & x_a + x_b + x_c & z_c + x_a y_c + x_b y_c + z_b + x_a y_b + z_a \\ 0 & 1 & y_a + y_b + y_c \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, we can see that $A \cdot (B \cdot C) = (A \cdot B) \cdot C$

3.3 Neutral Element

Consider two elements $A, E \in \mathcal{G}$. We need to check if the following can exist or not:

$$A \cdot E = A$$

$$\begin{bmatrix} 1 & x_a & z_a \\ 0 & 1 & y_a \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x_e & z_e \\ 0 & 1 & y_e \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x_a & z_a \\ 0 & 1 & y_a \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & x_a + x_e & z_a + x_a y_e + z_e \\ 0 & 1 & y_a + y_e \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x_a & z_a \\ 0 & 1 & y_a \\ 0 & 0 & 1 \end{bmatrix}$$

The above means that:

$$x_a + x_e = x_a$$
$$x_e = 0$$

$$y_a + y_e = y_a$$
$$y_e = 0$$

$$z_a + x_a y_e + z_e = z_a$$
$$z_a + 0 + z_e = z_a$$
$$z_e = 0$$

Thus, the neutral element is:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3.4 Inverse Element

Consider two elements $A, B \in \mathcal{G}$. We need to check if the following is possible or not.

$$A \cdot B = E$$

$$\begin{bmatrix} 1 & x_a & z_a \\ 0 & 1 & y_a \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x_b & z_b \\ 0 & 1 & y_b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & x_a + x_b & z_a + x_a y_b + z_b \\ 0 & 1 & y_a + y_b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This means that:

$$x_b = -x_a$$

$$y_b = -y_a$$

$$z_a + x_a y_b + z_b = 0$$

$$z_a - x_a y_a + z_b = 0$$

$$z_b = x_a y_a - z_a$$

Thus, we saw that an inverse element exists for every $A \in \mathcal{G}$

3.5 Commutative Property

The above properties were enough to show that $(\mathcal{G}, \cdot)$ is a group. Let's check if it is an Abelian group or not.

Consider two elements $A, B \in \mathcal{G}$, then:

$$A \cdot B$$

$$= \begin{bmatrix} 1 & x_a + x_b & z_a + x_a y_b + z_b \\ 0 & 1 & y_a + y_b \\ 0 & 0 & 1 \end{bmatrix}$$

$$B \cdot A$$

$$= \begin{bmatrix} 1 & x_b + x_a & z_b + x_b y_a + z_a \\ 0 & 1 & y_b + y_a \\ 0 & 0 & 1 \end{bmatrix}$$

As we can see, for the above two matrices to be equal, $x_a y_b = x_b y_a$ or $\frac{x_a}{y_a} = \frac{x_b}{y_b}$, but, since this won't be true for all the matrices, we can say that commutative property does not hold in general and thus, $(\mathcal{G}, \cdot)$ is not an Abelian group.

4 Question 2.4

4.1 Part (a)

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}_{3\times 2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}_{3\times 3}$$

As the number of columns of matrix 1 and number of rows matrix 2 don't match, the matrix multiplication is not possible.

4.2 Part (b)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times 3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}_{3\times 3}$$
$$= \begin{bmatrix} 4 & 3 & 5 \\ 10 & 9 & 11 \\ 16 & 15 & 17 \end{bmatrix}$$

4.3 Part (c)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}_{3\times 3} \begin{bmatrix} 1 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times 3}$$
$$= \begin{bmatrix} 5 & 6 & 9 \\ 11 & 13 & 15 \\ 8 & 9 & 12 \end{bmatrix}$$

4.4 Part (d)

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix}_{2\times 4} \begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix}_{4\times 2}$$
$$= \begin{bmatrix} 14 & 5 \\ -21 & 2 \end{bmatrix}$$

4.5 Part (e)

$$\begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix}_{4\times 2} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix}_{2\times 4}$$
$$\begin{bmatrix} 12 & 3 & -3 & -12 \\ -3 & 1 & 2 & 6 \\ 6 & 5 & 1 & 0 \\ 13 & 12 & 3 & 2 \end{bmatrix}$$

5 Question 2.5

5.1 Part (a)

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 5 & -7 & -5 \\ 2 & -1 & 1 & 3 \\ 5 & 2 & -4 & 2 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 6 \end{bmatrix}$$

The augmented matrix is as follows:

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 2 & -1 & 1 & 3 & 4 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right]$$

$R_2 \rightsquigarrow R_2 - 2R_1$, $R_3 \rightsquigarrow R_3 - 2R_3$ and $R_4 \rightsquigarrow R_4 - 5R_1$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & -3 & 3 & 5 & 2 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right]$$

$R_3 \rightsquigarrow R_3 + R_2$ and $R_4 \rightsquigarrow R_4 + R_2$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & -4 & 4 & -3 \end{array}\right]$$

$R_3 \rightsquigarrow -\frac{1}{2}R_3$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -4 & 4 & -3 \end{array}\right]$$

$R_4 \rightsquigarrow R_4 + 4R_3$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Because of the last row of the augmented matrix, we can see that the ranks of augmented matrix (4) and coefficient matrix (3) don't match and thus, no solution exists for this matrix equation.

Solution space, $\mathcal{S} = \phi$

5.2 Part (b)

The augmented matrix is as follows:

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -3 & 0 & 6 \\ 2 & -1 & 0 & 1 & -1 & 5 \\ -1 & 2 & 0 & -2 & -1 & -1 \end{array}\right]$$

We can drop the 3rd column, since its coefficients are anyways 0.
Thus, the revised augmented matrix is:

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 1 & 1 & -3 & 0 & 6 \\ 2 & -1 & 1 & -1 & 5 \\ -1 & 2 & -2 & -1 & -1 \end{array}\right]$$

$R_2 \rightsquigarrow R_2 - R_1$, $R_3 \rightsquigarrow R_3 - 2R_1$ and $R_4 \rightsquigarrow R_4 + R_1$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 2 & -3 & -1 & 3 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 1 & -2 & 0 & 2 \end{array}\right]$$

$R_4 \rightsquigarrow R_4 - R_3$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 2 & -3 & -1 & 3 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & -3 & 3 & 3 \end{array}\right]$$

$R_4 \rightsquigarrow -\frac{1}{3}R_4$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 2 & -3 & -1 & 3 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right]$$

$R_2 \rightsquigarrow R_2 - 2R_3$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 0 & -5 & 5 & 5 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right]$$

$R_2 \rightsquigarrow -\frac{1}{5}R_2$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right]$$

$R_4 \rightsquigarrow R_4 - R_2$, $R_3 \leftrightarrow R_2$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \rightsquigarrow R_2 - R_3$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \rightsquigarrow R_1 + R_2$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = x_4 + 3$$

$$x_2 = 2x_4$$

$$x_3 = x_4$$

$$x_4 = \lambda \in \mathbb{R}$$

Thus, the solution: $\mathcal{S} = \{(\lambda + 3, 2\lambda, \gamma, \lambda, \lambda) | \lambda, \gamma \in \mathbb{R}\}$

6 Question 2.6

The augmented matrix is:

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$R_3 \rightsquigarrow R_3 - R2$

$$\rightsquigarrow \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \end{array}\right]$$

$R_3 \rightsquigarrow -R_3$

$$\rightsquigarrow \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

$R_1 \rightsquigarrow R_1 - R_3$, $R_2 \rightsquigarrow R_2 - R_3$

$$\rightsquigarrow \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

The above can be transformed to:

$$x_2 = 1 - x_6$$
$$x_4 = -2 - x_6$$
$$x_5 = 1 + x_6$$

Thus, the solution space is:

$$\mathcal{S} = \{(\alpha, 1-\gamma, \beta, -2-\gamma, 1+\gamma, \gamma) \mid (\alpha, \beta, \gamma) \in \mathbb{R}\}$$

7 Question 2.7

$$Ax = 12x$$

$$\implies (A - 12)x = 0$$

$$\implies \begin{bmatrix} -6 & -8 & -9 \\ -6 & -12 & -3 \\ -12 & -4 & -12 \end{bmatrix} x = 0$$

One solution of this can be $x = 0$ but since we know $\sum x = 1$, thus, the above can't be true.
So, let's find the non-trivial solution.
The augmented matrix is:

$$\left[\begin{array}{ccc|c} -6 & -8 & -9 & 0 \\ -6 & -12 & -3 & 0 \\ -12 & -4 & -12 & 0 \end{array}\right]$$

$R_3 \rightsquigarrow -\frac{1}{2}R_3$, $R_2 \rightsquigarrow -R_2$ and $R_1 \rightsquigarrow -R_1$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 6 & 8 & 9 & 0 \\ 6 & 12 & 3 & 0 \\ 6 & 2 & 6 & 0 \end{array}\right]$$

$R_3 \rightsquigarrow R_3 - R_1$, $R_2 \rightsquigarrow R_2 - R_1$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 6 & 8 & 9 & 0 \\ 0 & 4 & -6 & 0 \\ 0 & -6 & -3 & 0 \end{array}\right]$$

$R_2 \rightsquigarrow \frac{1}{2}R_2$, $R_3 \rightsquigarrow -\frac{1}{3}R_3$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 6 & 8 & 9 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

$R_3 \rightsquigarrow R_3 - R_2$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 6 & 8 & 9 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

$R_3 \rightsquigarrow \frac{1}{4}R_3$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 6 & 8 & 9 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$R_1 \rightsquigarrow R_1 - 9R_3$, $R_2 \rightsquigarrow R_2 + 3R_3$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 6 & 8 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$R_2 \rightsquigarrow \frac{1}{2}R_2$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 6 & 8 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$R_1 \rightsquigarrow R_1 - 8R_2$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 6 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$R_1 \rightsquigarrow \frac{1}{6}R_1$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The above results in $x_1 = x_2 = x_3 = 0$ but, we know that $\sum x = 1$. Since both equations contradict, we can see that the given equation does not have any solution.

8 Question 2.8

8.1 Part (a)

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

The augmented matrix is:

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 3 & 4 & 5 & 0 & 1 & 0 \\ 4 & 5 & 6 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \rightsquigarrow R_3 - R_2$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 3 & 4 & 5 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & -1 & 1 \end{array}\right]$$

$R_2 \rightsquigarrow R_2 - R_1$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 & -1 & 1 \end{array}\right]$$

$R_3 \rightsquigarrow R_3 - R_2$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

As the rank of the left matrix is 2, the inverse does not exist.

8.2 Part (b)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

The augmented matrix is:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_4 \rightsquigarrow R_4 - R_2$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

$R_4 \leftrightarrow R_1$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$R_4 \rightsquigarrow R_4 - R_1$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$R_3 \leftrightarrow R_4$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$R_2 \rightsquigarrow R_2 - R_3$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$R_4 \rightsquigarrow R_4 - R_2 - R_1$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -2 \end{array}\right]$$

Thus,

$$\mathbf{A}^{-1} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -2 \end{bmatrix}$$

9 Question 2.9

In order to check if a set is a subspace of $\mathbb{R}^3$ we need to prove that scalar product and vector sum lie in $\mathbb{R}^3$ as well.

9.1 Option (a)

$$A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) | \lambda, \mu \in \mathbb{R}\}$$

Consider $\alpha \in \mathbb{R}$, then, for $\mathbf{x} \in A$:

$$\alpha \mathbf{x}$$

$$= \{(\alpha\lambda, \alpha\lambda + \alpha\mu^3, \alpha\lambda - \alpha\mu^3)\}$$

$$= \{(\alpha\lambda, \alpha\lambda + (\alpha^{\frac{1}{3}}\mu)^3, \alpha\lambda - (\alpha^{\frac{1}{3}}\mu)^3)\}$$

$$= \{\lambda', \lambda' + \mu'^3, \lambda' - \mu^3\}$$

$$\in A$$

Consider, $\mathbf{x}, \mathbf{y} \in A$, then:

$$\mathbf{x} + \mathbf{y}$$

$$= \{\lambda_a, \lambda_a + \mu_a^3, \lambda_a - \mu_a^3\} + \{\lambda_b, \lambda_b + \mu_b^3, \lambda_b - \mu_b^3\}$$

$$= \{\lambda_a + \lambda_b, \lambda_a + \lambda_b + \mu_a^3 + \mu_b^3, \lambda_a + \lambda_b - \mu_a^3 - \mu_b^3\}$$

The above cannot be expressed in the general form of A, since $\mu_a^3 + \mu_b^3 \neq \mu_c^3$ (generally).
Thus, $\mathbf{x} + \mathbf{y} \notin A$
Thus, A is not a subspace of $\mathbb{R}^3$

9.2 Option (b)

$$B = \{(\lambda^2, -\lambda^2, 0) \,|\lambda \in \mathbb{R}\}$$

Consider $\alpha \in \mathbb{R}$ and $\mathbf{x} \in B$, then:

$$\alpha\mathbf{x}$$

$$= \{(\alpha\lambda^2, -\alpha\lambda^2, 0)\}$$

$$= \left\{\left(\left(\alpha^{\frac{1}{2}}\lambda\right)^2, -\left(\alpha^{\frac{1}{2}}\lambda\right)^2, 0\right)\right\}$$

$$\in B$$

Consider, $\mathbf{x}, \mathbf{y} \in B$:

$$\mathbf{x} + \mathbf{y}$$

$$= \{(\lambda_a^2, -\lambda_a^2, 0)\} + \{(\lambda_b^2, -\lambda_b^2, 0)\}$$

$$= \{\lambda_a^2 + \lambda_b^2, -\lambda_a^2 - \lambda_b^2, 0\}$$

$$\notin B$$

Since, $\lambda_a^2 + \lambda_b^2 \neq \lambda^2$ (generally).
Thus, B is not a subspace of $\mathbb{R}^3$

9.3 Option (c)

$$C = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 | \xi_1 - 2\xi_2 + 3\xi_3 = \gamma, \gamma \in \mathbb{R}\}$$

Let $\alpha \in \mathbb{R}$ and $\mathbf{x} \in C$, then:

$$\alpha\mathbf{x}$$

$$= \{(\alpha\xi_1, \alpha\xi_2, \alpha\xi_3)\}$$

Since, $\xi_1 - 2\xi_2 + 3\xi_3 = \gamma$, then, $\alpha\xi_1 - 2\alpha\xi_2 + 3\alpha\xi_3 = \alpha\gamma \in \mathbb{R}$
Thus, $\alpha\mathbf{x} \in C$
Consider $\mathbf{x}, \mathbf{y} \in C$, then:

$$\mathbf{x}+\mathbf{y}$$

$$= \{(\xi_{1,a}, \xi_{2,a}, \xi_{3,a})\} + \{(\xi_{1,b}, \xi_{2,b}, \xi_{3,b})\}$$

$$= \{(\xi_{1,a}+\xi_{1,b}, \xi_{2,a}+\xi_{2,b}, \xi_{3,a}+\xi_{3,b})\}$$

Now, $\xi_{1,a} - 2\xi_{2,a} + 3\xi_{3,a} = \gamma_a \in \mathbb{R}$ and $\xi_{1,b} - 2\xi_{2,b} + 3\xi_{3,b} = \gamma_b \in \mathbb{R}$
Thus:

$$(\xi_{1,a}+\xi_{1,b}) - 2(\xi_{2,a}+\xi_{2,b}) + 3(\xi_{3,a}+\xi_{3,b}) = \gamma_a + \gamma_b \in \mathbb{R}$$

Thus, $\mathbf{x}+\mathbf{y} \in C$
Thus, C is a subspace of $\mathbb{R}^3$

9.4 Option (d)

$$D = \{(\xi_1, \xi_2, \xi_3)\,|\xi_2 \in \mathbb{Z}\}$$

Consider, $\alpha \in \mathbb{R}$ and $\mathbf{x} \in D$, then:

$$\alpha\mathbf{x}$$

$$= \{(\alpha\xi_1, \alpha\xi_2, \alpha\xi_3)\}$$

Since $\alpha \in \mathbb{R}$, it can't be said that (always) $\alpha\xi_2 \in \mathbb{Z}$, thus, $\alpha\mathbf{x} \notin D$
Thus, D is not a subspace of $\mathbb{R}^3$

10 Question 2.10

10.1 Part (a)

Using $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ as columns of a matrix we get:

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{bmatrix}$$

$R_3 \rightsquigarrow R_3 + R_2 - R_1$

$$\rightsquigarrow \begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 0 & -2 & 2 \end{bmatrix}$$

$R_1 \rightsquigarrow R_1 + 2R_2$, $R_3 \rightsquigarrow -\frac{1}{2}R_3$

$$\rightsquigarrow \begin{bmatrix} 0 & 3 & -3 \\ -1 & 1 & -3 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_1 \rightsquigarrow R_1 - 3R_3$, $R_2 \rightsquigarrow -R_2$

$$\rightsquigarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\rightsquigarrow \begin{bmatrix} 1 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\rightsquigarrow \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \rightsquigarrow R_1 + R_2$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

As the third column is not a pivot column, the vectors are NOT linearly independent.

10.2 Part (b)

Using $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ as columns of a matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_5 \rightsquigarrow R_5 - R_4$, $R_1 \rightsquigarrow R_1 - R_4 - R_3$

$$\rightsquigarrow \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \rightsquigarrow R_2 - 2R_3$

$$\rightsquigarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_4 \rightsquigarrow R_4 - R_2$

$$\rightsquigarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

All the three columns are pivot columns and that's why, the given vectors are linearly independent.

11 Question 2.11

Assume that:

$$\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \lambda_3 \mathbf{x}_3 = \mathbf{y}$$

This can be written in the following matrix form.

$$\mathbf{A}\breve{\ } = \mathbf{y}$$

Where,

$$\mathbf{A} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix}$$

$$\breve{\ } = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

The matrix equation can be represented using the following augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array}\right]$$

$R_3 \rightsquigarrow R_3 - R_2$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 0 & 1 & 2 & 7 \end{array}\right]$$

$R_2 \rightsquigarrow R_2 - R_1$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 1 & 2 & 7 \end{array}\right]$$

$R_3 \rightsquigarrow R_3 - R_2$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{array}\right]$$

$R_3 \rightsquigarrow \frac{1}{5} R_3$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$R_1 \rightsquigarrow R_1 - 2R_3$, $R_2 \rightsquigarrow R_2 + 3R_3$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$R_1 \rightsquigarrow R_1 - R_2$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Thus, $\lambda_1 = -6, \lambda_2 = 3, \lambda_3 = 2$
Therefore, $\mathbf{y} = -6\mathbf{x}_1 + 3\mathbf{x}_2 + 2\mathbf{x}_3$

12 Question 2.12

Reference: https://www.stat.uchicago.edu/~lekheng/courses/110s08/math110s-hw4sol.pdf

Let the three vectors spanning U_1 be $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$ and the three vectors spanning U_2 be $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$. If, $\mathbf{w} \in U_1 \cap U_2$, then there exist $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$ such that:

$$\alpha_1\mathbf{u_1} + \alpha_2\mathbf{u_2} + \alpha_3\mathbf{u_3} = \mathbf{w} = \beta_1\mathbf{v_1} + \beta_2\mathbf{v_2} + \beta_3\mathbf{v_3}$$

Using the above information, we can create the following augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -\beta_1 + 2\beta_2 - 3\beta_3 \\ 1 & -1 & 1 & -2\beta_1 - 2\beta_2 + 6\beta_3 \\ -3 & 0 & -1 & 2\beta_1 - 2\beta_3 \\ 1 & -1 & 1 & \beta_1 - \beta_3 \end{array}\right]$$

$R_2 \rightsquigarrow R_2 - R_1$, $R_3 \rightsquigarrow R_3 + 3R_1$, $R_4 \rightsquigarrow R_4 - R_1$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & -\beta_1 + 2\beta_2 - 3\beta_3 \\ 0 & -3 & 2 & -\beta_1 - 4\beta_2 + 9\beta_3 \\ 0 & 6 & -4 & -\beta_1 + 6\beta_2 - 11\beta_3 \\ 0 & -3 & 2 & 2\beta_1 - 2\beta_2 + 2\beta_3 \end{array}\right]$$

$R_4 \rightsquigarrow R_4 - R_2$, $R_3 \rightsquigarrow R_3 + 2R_2$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & -\beta_1 + 2\beta_2 - 3\beta_3 \\ 0 & -3 & 2 & -\beta_1 - 4\beta_2 + 9\beta_3 \\ 0 & 0 & 0 & -3\beta_1 - 2\beta_2 + 7\beta_3 \\ 0 & 0 & 0 & 3\beta_1 + 2\beta_2 - 7\beta_3 \end{array}\right]$$

$R_4 \rightsquigarrow R_4 + R_3$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & -\beta_1 + 2\beta_2 - 3\beta_3 \\ 0 & -3 & 2 & -\beta_1 - 4\beta_2 + 9\beta_3 \\ 0 & 0 & 0 & -3\beta_1 - 2\beta_2 + 7\beta_3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

For this system to be consistent, we must have $3\beta_1 + 2\beta_2 - 7\beta_3 = 0$
Hence,

$$\mathbf{w} = \beta_1 \begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix} + \beta_2 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix} + \beta_3 \begin{bmatrix} -3 \\ 6 \\ -2 \\ -1 \end{bmatrix}$$

$$\mathbf{w} = (7\beta_3 - 2\beta_2) \begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix} + \beta_2 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix} + \beta_3 \begin{bmatrix} -3 \\ 6 \\ -2 \\ -1 \end{bmatrix}$$

$$\mathbf{w} = \beta_2 \begin{bmatrix} 4 \\ 2 \\ -4 \\ -2 \end{bmatrix} + \beta_3 \begin{bmatrix} -10 \\ -8 \\ 12 \\ 6 \end{bmatrix}$$

for some $\beta_2, \beta_3 \in \mathbb{R}$. In other words,

$$U_1 \cap U_2 = \operatorname{span} \left\{ \begin{bmatrix} 4 \\ 2 \\ -4 \\ -2 \end{bmatrix}, \begin{bmatrix} -10 \\ -8 \\ 12 \\ 6 \end{bmatrix} \right\}$$

13 Question 2.13

13.1 Part (a) and (b)

The augmented matrix for U_1 is as follows:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & -2 & -1 & 0 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$R_4 \rightsquigarrow R_4 - R_1$, $R_2 \rightsquigarrow R_2 - R_1$, $R_3 \rightsquigarrow R_3 - 2R_1$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \rightsquigarrow -\frac{1}{2}R_2$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 \rightsquigarrow R_3 - R_2$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The above can be written as:

$$x_1 + x_3 = 0$$
$$\implies x_1 = -x_3$$
$$x_2 + x_3 = 0$$
$$\implies x_2 = -x_3$$

Thus, solution space is

$$U_1 = \text{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Since U_1 has only one independent vector, $\dim(U_1) = 1$
The augmented matrix for U_2 is as follows:

$$\left[\begin{array}{ccc|c} 3 & -3 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 7 & -5 & 2 & 0 \\ 3 & -1 & 2 & 0 \end{array}\right]$$

$R_1 \rightsquigarrow \frac{1}{3}R_1$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 7 & -5 & 2 & 0 \\ 3 & -1 & 2 & 0 \end{array}\right]$$

$R_2 \rightsquigarrow R_2 - R_1$, $R_2 \rightsquigarrow R_2 - 7R_1$, $R_3 \rightsquigarrow R_3 - 3R_1$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right]$$

$R_2 \rightsquigarrow \frac{1}{3}R_2$, $R_4 \rightsquigarrow R_4 - R_3$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 \rightsquigarrow R_3 - 2R_2$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \rightsquigarrow R_1 + R_2$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is the same as what we got in the case of U_1, thus:
Thus, solution space is

$$U_2 = \text{span}\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$$

Since U_2 has only one independent vector, $\dim(U_2) = 1$

13.2 Part (c)

Since bases of U_1 and U_2 are same, the basis of $U_1 \cap U_2$ is:

$$U_1 \cap U_2 = \text{span}\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$$

14 Question 2.14

14.1 Part (a) and (b)

Consider U_1 spanned by columns of $\mathbf{A_1}$, then:

$$\mathbf{A_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

$R_2 \rightsquigarrow R_1 - R_2$, $R_3 \rightsquigarrow R_3 - 2R_1$, $R_4 \rightsquigarrow R_4 - R_1$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \rightsquigarrow \frac{1}{2}R_2$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 \rightsquigarrow R_3 - R_2$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Only the first two columns are pivot columns and thus, basis of U_1 is:

$$U_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Thus, $\dim(U_1) = 2$
Consider U_2 spanned by columns of $\mathbf{A_2}$, then:

$$\mathbf{A_2} = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix}$$

$R_1 \rightsquigarrow \frac{1}{3}R_1$

$$\rightsquigarrow \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix}$$

$R_2 \rightsquigarrow R_2 - R_1$, $R_2 \rightsquigarrow R_2 - 7R_1$, $R_3 \rightsquigarrow R_3 - 3R_1$

$$\rightsquigarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 3 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$R_2 \rightsquigarrow \frac{1}{3}R_2$, $R_4 \rightsquigarrow R_4 - R_3$

$$\rightsquigarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 \rightsquigarrow R_3 - 2R_2$

$$\rightsquigarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \rightsquigarrow R_1 + R_2$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Only the first two columns are pivot columns and thus, basis of U_2 is:

$$U_2 = \text{span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -5 \\ -1 \end{bmatrix} \right\}$$

Thus, $\dim(U_2) = 2$

14.2 Part (c)

Let the two vectors spanning U_1 be $\mathbf{u_1}, \mathbf{u_2}$ and the two vectors spanning U_2 be $\mathbf{v_1}, \mathbf{v_2}$, and for some $\mathbf{w} \in U_1 \cap U_2$, we can write:

$$\alpha_1 \mathbf{u_1} + \alpha_2 \mathbf{u_2} = \mathbf{w} = \beta_1 \mathbf{v_1} + \beta_2 \mathbf{v_2}$$

The augmented matrix can be written as:

$$\left[\begin{array}{cc|c} 1 & 0 & 3\beta_1 - 3\beta_2 \\ 1 & -2 & \beta_1 + 2\beta_2 \\ 2 & 1 & 7\beta_1 - 5\beta_2 \\ 1 & 0 & 3\beta_1 - \beta_2 \end{array}\right]$$

$R_4 \rightsquigarrow R_4 - R_1$, $R_2 \rightsquigarrow R_1 - R_2$, $R_3 \rightsquigarrow R_3 - 2R_1$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 3\beta_1 - 3\beta_2 \\ 0 & 2 & 2\beta_1 - 5\beta_2 \\ 0 & 1 & \beta_1 + \beta_2 \\ 0 & 0 & 2\beta_2 \end{array}\right]$$

$R_2 \rightsquigarrow R_2 - 2R_3$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 3\beta_1 - 3\beta_2 \\ 0 & 0 & -7\beta_2 \\ 0 & 1 & \beta_1 + \beta_2 \\ 0 & 0 & 2\beta_2 \end{array}\right]$$

For this system to be consistent, $\beta_2 = 0$
Thus,

$$\mathbf{w} = \beta_1 \begin{bmatrix} 3 \\ 1 \\ 7 \\ 3 \end{bmatrix}$$

Hence, basis of $U_1 \cap U_2$ is:

$$U_1 \cap U_2 = \text{span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 7 \\ 3 \end{bmatrix} \right\}$$

15 Question 2.15

15.1 Part (a)

We just need to show the following are true for both F and G in order to show that they are subspaces of $\mathbb{R}^3$

1. $F \neq \phi$, in particular: $\in F$ (Similarly for G)
2. Closure of F (and G):

   a. With respect to the outer operation: $\forall \lambda \in \mathbb{R} \forall \mathbf{x} \in F : \lambda \mathbf{x} \in F$
   b. With respect to the inner operation: $\forall \mathbf{x}, \mathbf{y} \in F : \mathbf{x} + \mathbf{y} \in F$

[ ]: