Image Processing

Mathematical preliminaries

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Notation and definitions

- 1-D continuous signal: $F(x)$, $u(x)$, $s(t)$
- 1-D sampled signal: $u(n)$, u_n
- Continuous image -> function of 2 independent variables: $u(x,y)$, $v(x,y)$, $f(x,y)$
- Sampled image -> 2 or higher D sequence of real numbers: $U_{m,n,,}$ $v(m,n)$, $u(i,j,k)$
- Complex conjugate of a complex variable z: z^*
- 2-D separable function: $f(x,y)=f(x)f(y)$
- 1-D Dirac Delta: $\delta(x)=0$, $x\neq 0$ and $\lim \int_{\varepsilon}^{\varepsilon} \delta(x)\,dx = 1$
- 2-D Dirac Delta: $\delta(x,y)= \delta(x)\,\delta(y)$
  - Kronecker Delta: $\delta(m,n)= \delta(m)\,\delta(n)$
  - Properties – p. 12 (Jain)

specify integer indices of arrays and vectors. The symbol roman j will represent $\sqrt{-1}$. The complex conjugate of a complex variable such as z, will be denoted by z^*. Certain symbols will be redefined at appropriate places in the text to keep the notation clear.

Table 2.1 lists several well-known one-dimensional functions that will be often encountered. Their two-dimensional versions are functions of the *separable form*

$$f(x, y) = f_1(x)f_2(y) \tag{2.1}$$

For example, the two-dimensional delta functions are defined as

$$\text{Dirac:} \quad \delta(x, y) = \delta(x)\delta(y) \tag{2.2a}$$

$$\text{Kronecker:} \quad \delta(m, n) = \delta(m)\delta(n) \tag{2.2b}$$

which satisfy the properties

$$\left.\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y')\delta(x - x', y - y')\,dx'\,dy' &= f(x, y) \\ \lim_{\epsilon \to 0}\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon} \delta(x, y)\,dx\,dy &= 1, \end{aligned}\right\} \tag{2.3}$$

$$\left.\begin{aligned} x(m, n) &= \sum\sum_{m', n' = -\infty}^{\infty} x(m', n')\delta(m - m', n - n') \\ \sum\sum_{m, n = -\infty}^{\infty} \delta(m, n) &= 1 \end{aligned}\right\} \tag{2.4}$$

The definitions and properties of the functions rect(x, y), sinc(x, y), and comb(x, y) can be defined in a similar manner.

TABLE 2.1 Some Special Functions

Function	Definition	Function	Definition
Dirac delta	$\delta(x) = 0,\ x \neq 0$	*Rectangle*	$\text{rect}(x) = \begin{cases} 1, & \lvert x\rvert \le \frac{1}{2} \\ 0, & \lvert x\rvert > \frac{1}{2} \end{cases}$
	$\lim_{\epsilon \to 0}\int_{-\epsilon}^{\epsilon} \delta(x)\,dx = 1$	*Signum*	$\text{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$
Sifting property	$\int_{-\infty}^{\infty} f(x')\delta(x - x')\,dx' = f(x)$	*Sinc*	$\text{sinc}(x) = \frac{\sin \pi x}{\pi x}$
Scaling property	$\delta(ax) = \frac{\delta(x)}{\lvert a\rvert}$	*Comb*	$\text{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n)$
Kronecker delta	$\delta(n) = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$	*Triangle*	$\text{tri}(x) = \begin{cases} 1 - \lvert x\rvert, & \lvert x\rvert \le 1 \\ 0, & \lvert x\rvert > 1 \end{cases}$
Sifting property	$\sum_{m=-\infty}^{\infty} f(m)\delta(n - m) = f(n)$		

Linear Systems

$$x(m,n) \xrightarrow{T} T(y(m,n))$$

Linear systems and shift invariance

- Imaging systems -> modeled as 2-D linear system
- For linear system let x(m,n) ->input and y(m,n)->output: $y(m,n)=T[x(m,n)]$
  - Linear – iff obeys linear superposition

$$T[a_1x_1(m,n)+a_2x_2(m,n)]=a_1T[x_1(m,n)]+ a_2T[x_2(m,n)]=a_1y_1(m,n)+a_2y_2(m,n)$$

- If i/p is 2-D koneker Delta at (m′,n′), the o/p at (m,n) is:

$$h(m,n;m',n')\cong T[\delta(m-m',n-n')]$$

  - This is **impulse response** of the system
  - For imaging system, this is the o/p image due to an ideal point source at (m′,n′) in i/p plane.

Linear systems and shift invariance

- **Point spread function** (PSF) – when i/p and o/p represents +ve quantity (such as intensity of light in imaging system)
- **Region of support** (ROS) of impulse response function is the smallest closed region in the (m,n) plane outside which impulse is zero
- **Finite impulse response** (FIR)/ **Infinite impulse response** (IIR) system – if impulse response has finite/infinite ROS
- Output of any linear system is obtained using its impulse response –
  - i.e. any linear linear system is completely defined by its impulse response
  - P. 13 (Jain)

2.3 LINEAR SYSTEMS AND SHIFT INVARIANCE

A large number of imaging systems can be modeled as two-dimensional linear systems. Let $x(m, n)$ and $y(m, n)$ represent the input and output sequences, respectively, of a two-dimensional system (Fig. 2.1), written as

$$y(m, n) = \mathcal{H}[x(m, n)] \tag{2.5}$$

This system is called *linear* if and only if any linear combination of two inputs $x_1(m, n)$ and $x_2(m, n)$ produces the same combination of their respective outputs $y_1(m, n)$ and $y_2(m, n)$, i.e., for arbitrary constants a_1 and a_2

linear: Scale invariant

$$\begin{aligned}\mathcal{H}[a_1 x_1(m, n) + a_2 x_2(m, n)] &= a_1 \mathcal{H}[x_1(m, n)] + a_2 \mathcal{H}[x_2(m, n)] \\ &= a_1 y_1(m, n) + a_2 y_2(m, n)\end{aligned} \tag{2.6}$$

This is called *linear superposition*. When the input is the two-dimensional Kronecker delta function at location (m', n'), the output at location (m, n) is defined as

$$h(m, n; m', n') \triangleq \mathcal{H}[\delta(m - m', n - n')] \tag{2.7}$$

and is called the *impulse response* of the system. For an imaging system, it is the image in the output plane due to an ideal point source at location (m', n') in the input plane. In our notation, the semicolon (;) is employed to distinguish the input and output pairs of coordinates.

PSF is +ve impulse response

The impulse response is called the *point spread function* (PSF) when the inputs and outputs represent a positive quantity such as the intensity of light in imaging systems. The term *impulse response* is more general and is allowed to take negative as well as complex values. The *region of support* of an impulse response is the smallest closed region in the m, n plane outside which the impulse response is zero. A system is said to be a *finite impulse response* (FIR) or an *infinite impulse response* (IIR) system if its impulse response has finite or infinite regions of support, respectively.

The output of any linear system can be obtained from its impulse response and the input by applying the superposition rule of (2.6) to the representation of (2.4) as follows:

$$\begin{aligned}y(m, n) &= \mathcal{H}[x(m, n)] \\ &= \mathcal{H}\left[\sum_{m'}\sum_{n'} x(m', n')\delta(m - m', n - n')\right] \rightarrow \text{Shift invariance} \\ &= \sum_{m'}\sum_{n'} x(m', n')\mathcal{H}[\delta(m - m', n - n')] \rightarrow \text{Scale invariance} \\ \Rightarrow \quad y(m, n) &= \sum_{m'}\sum_{n'} x(m', n')h(m, n; m', n') \rightarrow \text{Superposition}\end{aligned} \tag{2.8}$$

$$x(m, n) \longrightarrow \boxed{\mathcal{H}[\cdot]} \longrightarrow y(m, n)$$

Figure 2.1 A system.

A system is called *spatially invariant* or *shift invariant* if a translation of the input causes a translation of the output. Following the definition of (2.7), if the impulse occurs at the origin we will have

$$\mathcal{H}[\delta(m, n)] = h(m, n; 0, 0)$$

Hence, it must be true for shift invariant systems that

$$h(m, n; m', n') \triangleq \mathcal{H}[\delta(m - m', n - n')]$$
$$= h(m - m', n - n'; 0, 0)$$
$$\Rightarrow \quad h(m, n; m', n') = h(m - m', n - n') \tag{2.9}$$

i.e., the impulse response is a function of the two displacement variables only. This means the shape of the impulse response does not change as the impulse moves about the m, n plane. A system is called *spatially varying* when (2.9) does not hold. Figure 2.2 shows examples of PSFs of imaging systems with separable or circularly symmetric impulse responses.

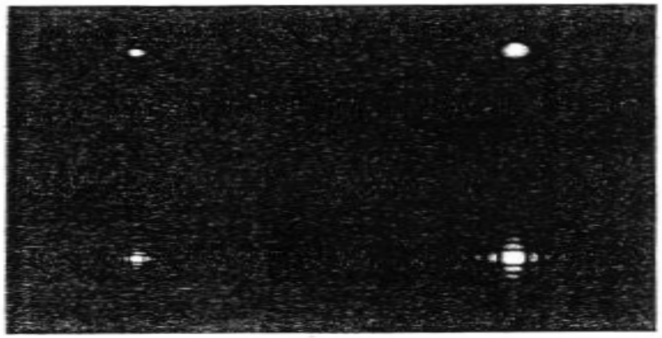
For shift invariant systems, the output becomes

Convolution: for LSI

$$y(m, n) = \sum_{m', n' = -\infty}^{\infty}\sum h(m - m', n - n')x(m', n') \tag{2.10}$$

which is called the *convolution* of the input with the *impulse response*. Figure 2.3 shows a graphical interpretation of this operation. The impulse response array is rotated about the origin by 180° and then shifted by (m, n) and overlayed on the array $x(m', n')$. The sum of the product of the arrays $\{x(\cdot,\cdot)\}$ and $\{h(\cdot,\cdot)\}$ in the overlapping regions gives the result at (m, n). We will use the symbol $\circledast$ to denote the convolution operation in both discrete and continuous cases, i.e.,

$$g(x, y) = h(x, y) \circledast f(x, y) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x - x', y - y')f(x', y')dx'\,dy'$$
$$y(m, n) = h(m, n) \circledast x(m, n) \triangleq \sum_{m', n' = -\infty}^{\infty}\sum h(m - m', n - n')x(m', n') \tag{2.11}$$



a	b
c	d

Figure 2.2 Examples of PSFs. (a) Circularly symmetric PSF of average atmospheric turbulence causing small blur; (b) atmospheric turbulence PSF causing large blur; (c) separable PSF of a diffraction limited system with square aperature; (d) same as (c) but with smaller aperture.

Linear systems and shift invariance

- Spatially/shift invariant – if a translation in input causes a translation in output
  - For impulse at the origin: $h(m,n;0,0) \cong T[\delta(m-0,n-0)]$
  - For shift invariant system: $h(m,n;m',n') \cong T[\delta(m-m',n-n')]$

$$= h(m-m',n-n';0,0)$$

Hence, $h(m,n;m',n') = h(m-m',n-n') \qquad (1)$

- function of displacement variables only;
- shape of impulse does not change as impulse moves about (m,n) plane
- If (1) is not true -> **spatially varying** system
- Fig. 2.2 (Jain p. 14) -> example of PSF with separable/circularly symmetric impulse response
- Convolution – see next

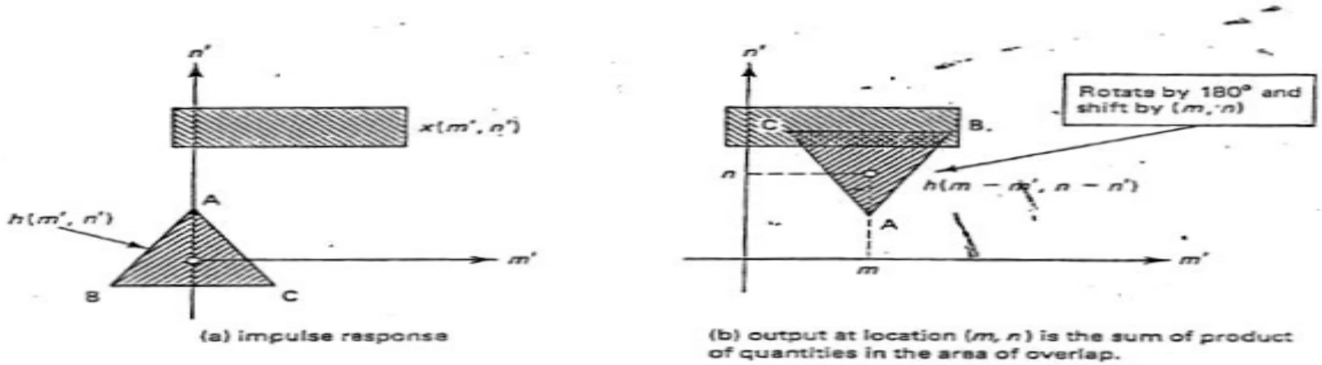
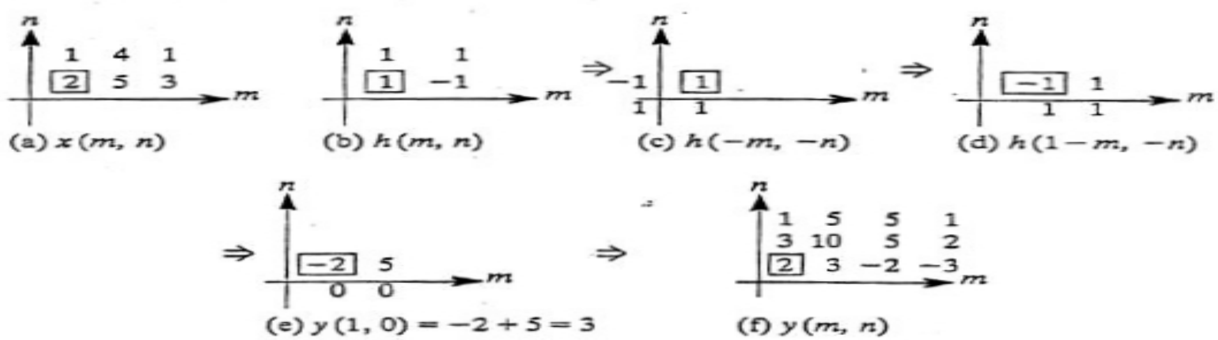


Figure 2.3 Discrete convolution in two dimensions

The convolution operation has several interesting properties, which are explored in Problems 2.2 and 2.3.

Example 2.1 (Discrete convolution)

Consider the 2×2 and 3×2 arrays $h(m, n)$ and $x(m, n)$ shown next, where the boxed element is at the origin. Also shown are the various steps for obtaining the convolution of these two arrays. The result $y(m, n)$ is a 4×3 array. In general, the convolution of two arrays of sizes $(M_1 \times N_1)$ and $(M_2 \times N_2)$ yields an array of size $[(M_1 + M_2 - 1) \times (N_1 + N_2 - 1)]$ (Problem 2.5).



2.4 THE FOURIER TRANSFORM

Two-dimensional transforms such as the Fourier transform and the Z-transform are of fundamental importance in digital image processing as will become evident in the subsequent chapters. In one dimension, the Fourier transform of a complex

Note that $X(\omega)$ is periodic with period 2π. Hence it is sufficient to specify it over one period.

The Fourier transform pair of a two-dimensional sequence $x(m, n)$ is defined as

$$X(\omega_1, \omega_2) \triangleq \sum\sum_{m,n=-\infty}^{\infty} x(m, n) \exp[-j(m\omega_1 + n\omega_2)], \qquad -\pi \le \omega_1, \omega_2 < \pi \quad (2.24)$$

$$x(m, n) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} X(\omega_1, \omega_2) \exp[j(m\omega_1 + n\omega_2)]\, d\omega_1\, d\omega_2 \quad (2.25)$$

Now $X(\omega_1, \omega_2)$ is periodic with period 2π in each argument, i.e.,

$$X(\omega_1 \pm 2\pi, \omega_2 \pm 2\pi) = X(\omega_1 \pm 2\pi, \omega_2) = X(\omega_1, \omega_2 \pm 2\pi) = X(\omega_1, \omega_2) \quad (2.25)$$

Often, the sequence $x(m, n)$ in the series in (2.24) is absolutely summable, i.e.,

$$\sum\sum_{m,n=-\infty}^{\infty} |x(m, n)| < \infty \quad (2.26)$$

Analogous to the continuous case, $H(\omega_1, \omega_2)$, the Fourier transform of the shift invariant impulse response is called *frequency response*. The Fourier transform of sequences has many properties similar to the Fourier transform of continuous functions. These are summarized in Table 2.4.

TABLE 2.4 Properties and Examples of Fourier Transform of Two-Dimensional Sequences

Property	Sequence	Transform
	$x(m, n), y(m, n), h(m, n), \cdots$	$X(\omega_1, \omega_2), Y(\omega_1, \omega_2), H(\omega_1, \omega_2), \cdots$
Linearity	$a_1 x_1(m, n) + a_2 x_2(m, n)$	$a_1 X_1(\omega_1, \omega_2) + a_2 X_2(\omega_1, \omega_2)$
Conjugation	$x^*(m, n)$	$X^*(-\omega_1, -\omega_2)$
Separability	$x_1(m) x_2(n)$	$X_1(\omega_1) X_2(\omega_2)$
Shifting	$x(m \pm m_0, n \pm n_0)$	$\exp[\pm j(m_0\omega_1 + n_0\omega_2)] X(\omega_1, \omega_2)$
Modulation	$\exp[\pm j(\omega_{01} m + \omega_{02} n)] x(m, n)$	$X(\omega_1 \mp \omega_{01}, \omega_2 \mp \omega_{02})$
Convolution	$y(m, n) = h(m, n) \circledast x(m, n)$	$Y(\omega_1, \omega_2) = H(\omega_1, \omega_2) X(\omega_1, \omega_2)$
Multiplication	$h(m, n) x(m, n)$	$\left(\frac{1}{4\pi^2}\right) H(\omega_1, \omega_2) \circledast X(\omega_1, \omega_2)$
Spatial correlation	$c(m, n) = h(m, n) \star x(m, n)$	$C(\omega_1, \omega_2) = H(-\omega_1, -\omega_2) X(\omega_1, \omega_2)$
Inner product	$I = \sum\sum_{m,n=-\infty}^{\infty} x(m, n) y^*(m, n)$	$I = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} X(\omega_1, \omega_2) Y^*(\omega_1, \omega_2)\, d\omega_1\, d\omega_2$
Energy conservation	$\mathscr{E} = \sum\sum_{m,n=-\infty}^{\infty} \|x(m, n)\|^2$	$\mathscr{E} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \|X(\omega_1, \omega_2)\|^2\, d\omega_1\, d\omega_2$
	$\sum\sum_{m,n=-\infty}^{\infty} \exp[j(m\omega_{01} + n\omega_{02})]$	$4\pi^2 \delta(\omega_1 - \omega_{01}, \omega_2 - \omega_{02})$
	$\delta(m, n)$	$\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \exp[-j(\omega_1 m + \omega_2 n)]\, d\omega_1\, d\omega_2$

Z-transform or Laurent series

- Useful generalization of Fourier series
- For 2-D complex sequence x(m,n)

$$X(z_1, z_2)=\Sigma\Sigma^{\infty}_{m,n=\infty} x(m,n)\, z_1^{-m}\, z_2^{-n} \qquad (1)$$

  Where z_1, z_2 are complex variables
- The set of values for which this set converges uniformly is called **region of convergence** (ROC) -> unit circle (poles and zeros)
- Z-transform of the impulse response of a linear shift invariant system is called **transfer function**
- Using convolution theorem for Z-transform

$$Y(z_1, z_2)= H(z_1, z_2)X(z_1, z_2)$$
$$=> H(z_1, z_2)= Y(z_1, z_2)/X(z_1, z_2)$$

- Similar relationship in frequency domain using Fourier transform

2.5 THE Z-TRANSFORM OR LAURENT SERIES

A useful generalization of the Fourier series is the Z-transform, which for a two-dimensional complex sequence $x(m, n)$ is defined as

$$X(z_1, z_2) = \sum\sum_{m,n=-\infty}^{\infty} x(m, n) z_1^{-m} z_2^{-n} \tag{2.27}$$

where z_1, z_2 are complex variables. The set of values of z_1, z_2 for which this series converges uniformly is called the *region of convergence*. The Z-transform of the impulse response of a linear shift invariant discrete system is called its *transfer function*. Applying the convolution theorem for Z-transforms (Table 2.5) we can transform (2.10) as

$$Y(z_1, z_2) = H(z_1, z_2) X(z_1, z_2)$$

$$\Rightarrow \qquad H(z_1, z_2) = \frac{Y(z_1, z_2)}{X(z_1, z_2)}$$

i.e., the transfer function is also the ratio of the Z-transforms of the output and the input sequences. The inverse Z-transform is given by the double contour integral

$$x(m, n) = \frac{1}{(j2\pi)^2} \oint\oint X(z_1, z_2) z_1^{m-1} z_2^{n-1} \, dz_1 \, dz_2 \tag{2.28}$$

where the contours of integration are counterclockwise and lie in the region of convergence. When the region of convergence includes the unit circles $|z_1| = 1, |z_2| = 1$, then evaluation of $X(z_1, z_2)$ at $z_1 = \exp(j\omega_1), z_2 = \exp(j\omega_2)$ yields the Fourier transform of $x(m, n)$. Sometimes $X(z_1, z_2)$ is available as a finite series (such as the transfer function of FIR filters). Then $x(m, n)$ can be obtained by inspection as the coefficient of the term $z_1^{-m} z_2^{-n}$.

TABLE 2.5 Properties of the Two-Dimensional Z-Transform

Property	Sequence	Z-Transform
	$x(m, n), y(m, n), h(m, n), \cdots$	$X(z_1, z_2), Y(z_1, z_2), H(z_1, z_2), \cdots$
Rotation	$x(-m, -n)$	$X(z_1^{-1}, z_2^{-1})$
Linearity	$a_1 x_1(m, n) + a_2 x_2(m, n)$	$a_1 X_1(z_1, z_2) + a_2 X_2(z_1, z_2)$
Conjugation	$X^*(m, n)$	$X^*(z_1^*, z_2^*)$
Separability	$x_1(m) x_2(n)$	$X_1(z_1) X_2(z_2)$
Shifting	$x(m \pm m_0, n \pm n_0)$	$z_1^{\pm m_0} z_2^{\pm n_0} X(z_1, z_2)$
Modulation	$a^m b^n x(m, n)$	$X\left(\frac{z_1}{a}, \frac{z_2}{b}\right)$
Convolution	$h(m, n) \circledast x(m, n)$	$H(z_1, z_2) X(z_1, z_2)$
Multiplication	$x(m, n) y(m, n)$	$\left(\frac{1}{2\pi j}\right)^2 \oint_{C_1}\oint_{C_2} X\left(\frac{z_1}{z_1'}, \frac{z_2}{z_2'}\right) Y(z_1', z_2') \frac{dz_1'}{z_1'} \frac{dz_2'}{z_2'}$

Causality and stability

- **Causal** if, 1-D shift invariant system-> at any time output is not affected by future input
  - Impulse response $h(n)=0$ for $n<0$
  - Transfer function has one-sided Laurent series

    $$H(z)=\Sigma^{\infty}_{n=0}h(n)\, z^{-n} \qquad (1)$$
  - Causal, if $x(n)=0,\ n<0$
  - Anti-causal, if $x(n)=0,\ n\geq 0$
  - Non-causal, if neither
- **Stable**, if output remains uniformly bounded for any bounded input
  - Impulse response is absolutely summable

    $$\Sigma^{\infty}_{n=-\infty}|h(n)|<\infty$$
  - Implications on poles and zeros in unit circle!

Causality and Stability

A one-dimensional shift invariant system is called causal if its output at any time is not affected by future inputs. This means its impulse response $h(n) = 0$ for $n < 0$ and its transfer function must have a one-sided Laurent series, i.e.,

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \tag{2.29}$$

Extending this definition, any sequence $x(n)$ is called *causal* if $x(n) = 0, n < 0$; *anticausal* if $x(n) = 0, n \geq 0$, and *noncausal* if it is neither causal nor anticausal.

A system is called *stable* if its output remains uniformly bounded for any bounded input. For linear shift invariant systems, this condition requires that the impulse response should be absolutely summable (prove it!), i.e.,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \tag{2.30}$$

This means $H(z)$ cannot have any poles on the unit circle $|z| = 1$. If this system is to be *causal and stable,* then the convergence of (2.29) at $|z| = 1$ implies the series must converge for all $|z| \geq 1$, i.e., the poles of $H(z)$ must lie *inside* the unit circle.

Causal & stable - poles inside unit circle

In two dimensions, a linear shift invariant system is stable when

$$\sum_{m} \sum_{n} |h(m, n)| < \infty \tag{2.31}$$

which implies the region of convergence of $H(z_1, z_2)$ must include the unit circles, i.e., $|z_1| = 1, |z_2| = 1$.

2.6 OPTICAL AND MODULATION TRANSFER FUNCTIONS

For a spatially invariant imaging system, its *optical transfer function* (OTF) is defined as its normalized frequency response, i.e.,

$$\text{OTF} = \frac{H(\xi_1, \xi_2)}{H(0, 0)} \tag{2.32}$$

The *modulation transfer function* (MTF) is defined as the magnitude of the OTF, i.e.,

$$\text{MTF} = |\text{OTF}| = \frac{|H(\xi_1, \xi_2)|}{|H(0, 0)|} \tag{2.33}$$

Similar relations are valid for discrete systems. Figure 2.5 shows the MTFs of systems whose PSFs are displayed in Fig. 2.2. In practice, it is often the MTF that is measurable. The phase of the frequency response is estimated from physical considerations. For many optical systems, the OTF itself is positive.

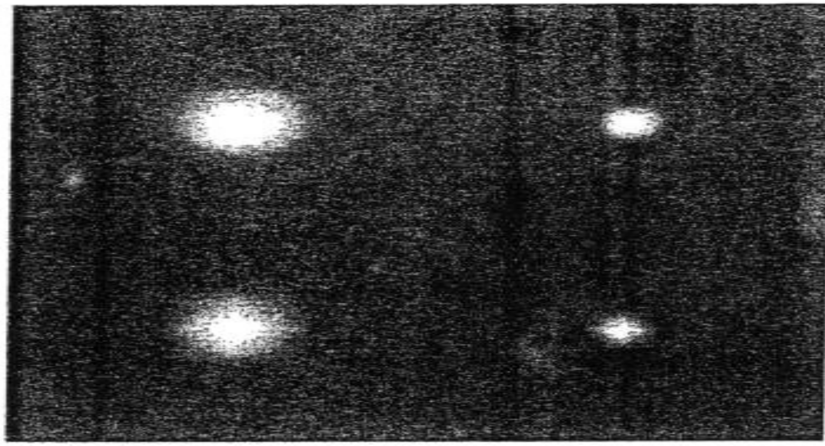


Figure 2.5 MTFs of systems whose PSFs are displayed in Figure 2.2.

Example 2.2

The impulse response of an imaging system is given as $h(x, y) = 2 \sin^2[\pi(x - x_0)]/[\pi(x - x_0)]^2 \sin^2[\pi(y - y_0)]/[\pi(y - y_0)]^2$. Then its frequency response is $H(\xi_1, \xi_2) = 2 \operatorname{tri}(\xi_1, \xi_2) \exp[-j2\pi(x_0 \xi_1 + y_0 \xi_2)]$, and OTF $= \operatorname{tri}(\xi_1, \xi_2) \exp[-j2\pi(x_0 \xi_1 + y_0 \xi_2)]$, MTF $= \operatorname{tri}(\xi_1, \xi_2)$.

2.7 MATRIX THEORY RESULTS

Vectors and Matrices

Often one- and two-dimensional sequences will be represented by vectors and matrices, respectively. A column vector $\mathbf{u}$ containing N elements is denoted as

$$\mathbf{u} \triangleq \{u(n)\} = \begin{bmatrix} u(1) \\ u(2) \\ \cdot \\ \cdot \\ \cdot \\ u(N) \end{bmatrix} \tag{2.34}$$

The nth element of the vector $\mathbf{u}$ is denoted by $u(n)$, u_n, or $[\mathbf{u}]_n$. Unless specified otherwise, all vectors will be column vectors. A column vector of size N is also called an $N \times 1$ vector. Likewise, a row vector of size N is called a $1 \times N$ vector.

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A matrix $\mathbf{A}$ of size $M \times N$ has M rows and N columns and is defined as

$$\mathbf{A} \triangleq \{a(m, n)\} = \begin{bmatrix} a(1,1) & a(1,2) & \cdots & a(1,N) \\ a(2,1) & & & \\ \vdots & \vdots & & \vdots \\ a(M,1) & a(M,2) & \cdots & a(M,N) \end{bmatrix} \tag{2.35}$$

The element in the mth row and nth column of matrix $\mathbf{A}$ is written as $[\mathbf{A}]_{m,n} \triangleq a(m, n) \triangleq a_{m,n}$. The nth column of $\mathbf{A}$ is denoted by $\mathbf{a}_n$, whose mth element is written as $a_n(m) = a(m, n)$. When the starting index of a matrix is not $(1, 1)$, it will be so indicated. For example,

$$\mathbf{A} = \{a(m, n) \qquad 0 \le m, n \le N - 1\}$$

represents an $N \times N$ matrix with starting index $(0, 0)$. Common definitions from matrix theory are summarized in Table 2.6.

In two dimensions it is often useful to visualize an image as a matrix. The matrix representation is simply a $90°$ clockwise rotation of the conventional two-dimensional Cartesian coordinate representation:

$$x(m, n) = \begin{array}{c|ccc} n \uparrow & 2 & -1 & -3 \\ & 4 & 0 & 5 \\ & 1 & 2 & 3 \\ \hline & & & \rightarrow m \end{array} \Rightarrow \mathbf{X} = \begin{array}{cc} & \rightarrow n \\ m \downarrow & \begin{bmatrix} 1 & 4 & 2 \\ 2 & 0 & -1 \\ 3 & 5 & -3 \end{bmatrix} \end{array}$$

Row and Column Ordering

Sometimes it is necessary to write a matrix in the form of a vector, for instance, when storing an image on a disk or a tape. Let

$$\boldsymbol{x} \triangleq \mathcal{O}\{x(m, n)\}$$

be a one-to-one ordering of the elements of the array $\{x(m, n)\}$ into the vector $\boldsymbol{x}$. For an $M \times N$ matrix, a mapping used often is called the *lexicographic or dictionary ordering.* This is a *row-ordered vector* and is defined as

$$\begin{aligned} \boldsymbol{x}^T &= [x(1,1)x(1,2)\ldots x(1,N)x(2,1)\ldots x(2,N)\ldots x(M,1)\ldots x(M,N)]^T \\ &\triangleq \mathcal{O}_r\{x(m, n)\} \end{aligned} \tag{2.36a}$$

Thus $\boldsymbol{x}^T$ is the row vector obtained by stacking each row to the right of the previous row of $\mathbf{X}$. Another useful mapping is the column by column stacking, which gives a *column-ordered vector* as

$$\boldsymbol{x}^T = [x(1,1)x(2,1)\ldots x(M,1)x(1,2)\ldots x(M,2)\ldots x(1,M)\ldots x(M,N)]^T$$

TABLE 2.6 Matrix Theory Definitions

Item	Definition	Comments
Matrix	$\mathbf{A} = \{a(m, n)\}$	m = row index, n = column index
Transpose	$\mathbf{A}^T = \{a(n, m)\}$	Rows and columns are interchanged.
Complex conjugate	$\mathbf{A}^* = \{a^*(m, n)\}$	
Conjugate transpose	$\mathbf{A}^{*T} = \{a^*(n, m)\}$	
Identity matrix	$\mathbf{I} = \{\delta(m - n)\}$	A square matrix with unity along its diagonal.
Null matrix	$\mathbf{O} = \{0\}$	All elements are zero.
Matrix addition	$\mathbf{A} + \mathbf{B} = \{a(m, n) + b(m, n)\}$	$\mathbf{A}, \mathbf{B}$ have same dimensions.
Scalar multiplication	$\alpha\mathbf{A} = \{\alpha a(m, n)\}$	
Matrix multiplication	$c(m, n) \triangleq \sum_{k=1}^{K} a(m, k)\, b(k, n)$	$\mathbf{C} \triangleq \mathbf{AB}$, $\mathbf{A}$ is $M \times K$, $\mathbf{B}$ is $K \times N$, $\mathbf{C}$ is $M \times N$. $\mathbf{AB} \neq \mathbf{BA}$.
Commuting matrices	$\mathbf{AB} = \mathbf{BA}$	Not true in general.
Vector inner product	$\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \mathbf{x}^{*T}\mathbf{y} = \sum_n x^*(n)\, y(n)$	Scalar quantity. If zero, $\mathbf{x}$ and $\mathbf{y}$ are called orthogonal.
Vector outer product	$\mathbf{xy}^T = \{x(m)\, y(n)\}$	$\mathbf{x}$ is $M \times 1$, $\mathbf{y}$ is $N \times 1$, outerproduct is $M \times N$; is a rank 1 matrix.
Symmetric	$\mathbf{A} = \mathbf{A}^T$	
Hermitian	$\mathbf{A} = \mathbf{A}^{*T}$	A real symmetric matrix is Hermitian. All eigenvalues are real.
Determinant	$\lvert\mathbf{A}\rvert$	For square matrices only.
Rank [A]	Number of linearly independent rows or columns.	
Inverse, $\mathbf{A}^{-1}$	$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$	For square matrices only.
Singular	$\mathbf{A}^{-1}$ does not exist	$\lvert\mathbf{A}\rvert = 0$
Trace	$\mathrm{Tr}[\mathbf{A}] = \sum_n a(n, n)$	Sum of the diagonal elements.
Eigenvalues, λ_k	All roots $\lvert\mathbf{A} - \lambda_k\mathbf{I}\rvert = 0$	
Eigenvectors, $\boldsymbol{\phi}_k$	All solutions $\mathbf{A}\boldsymbol{\phi}_k = \lambda_k\boldsymbol{\phi}_k$, $\boldsymbol{\phi}_k \neq \mathbf{0}$	
ABCD lemma	$(\mathbf{A} - \mathbf{BCD})^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{C}^{-1} - \mathbf{DA}^{-1}\mathbf{B})^{-1}\mathbf{DA}^{-1}$	$\mathbf{A}, \mathbf{C}$ are nonsingular.

$$= \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{x}_N \end{bmatrix} \triangleq \mathcal{O}_c\{x(m, n)\} \qquad (2.36b)$$

where $\mathbf{x}_n$ is the nth column of $\mathbf{X}$.

Transposition and Conjugation Rules

$$\begin{cases} 1.\ \mathbf{A}^{*T} = [\mathbf{A}^T]^* \\ 2.\ [\mathbf{AB}]^T = \mathbf{B}^T\mathbf{A}^T \\ 3.\ [\mathbf{A}^{-1}]^T = [\mathbf{A}^T]^{-1} \\ 4.\ [\mathbf{AB}]^* = \mathbf{A}^*\mathbf{B}^* \end{cases}$$

Note that the *conjugate transpose* is denoted by $\mathbf{A}^{*T}$. In matrix theory literature, a simplified notation $\mathbf{A}^*$ is often used to denote the conjugate transpose of $\mathbf{A}$. In the theory of image transforms (Chapter 5), we will have to distinguish between $\mathbf{A}$, $\mathbf{A}^*$, $\mathbf{A}^T$ and $\mathbf{A}^{*T}$ and hence the need for the notation.

Toeplitz and Circulant Matrices

A *Toeplitz* matrix $\mathbf{T}$ is a matrix that has constant elements along the main diagonal and the subdiagonals. This means the elements $t(m, n)$ depend only on the difference $m - n$, i.e., $t(m, n) = t_{m-n}$. Thus an $N \times N$ Toeplitz matrix is of the form

$$\mathbf{T} = \begin{bmatrix} t_0 & t_{-1} & & \cdots & t_{-N+1} \\ t_1 & t_0 & t_{-1} & & t_{-N+2} \\ t_2 & & \ddots & & \cdot \\ \cdot & & & \ddots & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & t_{-1} \\ t_{N-1} & \cdots & t_2 & t_1 & t_0 \end{bmatrix} \qquad (2.37)$$

and is completely defined by the $(2N - 1)$ elements $\{t_k, -N + 1 \le k \le N - 1\}$. Toeplitz matrices describe the input-output transformations of one-dimensional linear shift invariant systems (see Example 2.3) and correlation matrices of stationary sequences.

Normal convolution

A matrix $\mathbf{C}$ is called *circulant* if each of its rows (or columns) is a circular shift of the previous row (or column), i.e.,

$$\mathbf{C} = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{N-1} \\ c_{N-1} & c_0 & c_1 & & c_{N-2} \\ \cdot & & \ddots & \ddots & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ c_2 & & & & c_1 \\ c_1 & c_2 & \cdots & c_{N-1} & c_0 \end{bmatrix} \qquad (2.38)$$

Note that **C** is also Toeplitz and

$$c(m, n) = c((m - n) \quad \text{modulo } N) \tag{2.39}$$

Circular convolution

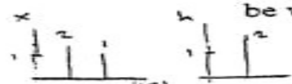
Circulant matrices describe the input-output behavior of one-dimensional linear periodic systems (see Example 2.4) and correlation matrices of periodic sequences.

Example 2.3 (Linear convolution as a Toeplitz matrix operation)

The output of a shift invariant system with impulse response $h(n) = n$, $-1 \le n \le 1$, and with input $x(n)$, which is zero outside $0 \le n \le 4$, is given by the convolution

$$y(n) = h(n) \circledast x(n) = \sum_{k=0}^{4} h(n-k)x(k)$$

Note that $y(n)$ will be zero outside the interval $-1 \le n \le 5$. In vector notation, this can be written as a 7×5 Toeplitz matrix operating on a 5×1 vector, namely,



$$\begin{bmatrix} y(-1) \\ y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \end{bmatrix}$$

$$\underset{4\times3}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}} = \underset{4\times1}{\begin{bmatrix} \\ \\ \end{bmatrix}}$$

• size: $M_1 + M_2 - 1$

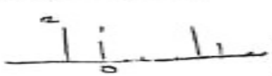
Example 2.4 (Circular convolution as a circulant matrix operation)

If two convolving sequences are periodic, then their convolution is also periodic and can be represented as

• wrap around

• size: $= M_1 = M_2$

$$y(n) = \sum_{k=0}^{N-1} h(n-k)x(k), \qquad 0 \le n \le N-1$$



where $h(-n) = h(N-n)$ and N is the period. For example, let $N = 4$ and $h(n) = n + 3$ (modulo 4). In vector notation this gives

$$\underset{3\times3}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}} = \underset{3\times1}{\begin{bmatrix} \\ \\ \end{bmatrix}}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 3 & 2 & 1 \\ 1 & 0 & 3 & 2 \\ 2 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

Thus the input-to-output transformation of a circular convolution is described by a circulant matrix.

Orthogonal and Unitary Matrices

An *orthogonal* matrix is such that its inverse is equal to its transpose, i.e., **A** is orthogonal if

$$\mathbf{A}^{-1} = \mathbf{A}^T$$

or

$$\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I} \tag{2.40}$$

Hermitian matrices, the eigenvectors corresponding to distinct eigenvalues are orthogonal. For repeated eigenvalues, their eigenvectors form a subspace that can be orthogonalized to yield a complete set of orthogonal eigenvectors. Normalization of these eigenvectors yields an orthonormal set, i.e., the unitary matrix Φ, whose columns are these eigenvectors. The matrix Φ is also called the eigenmatrix of R.

2.8 BLOCK MATRICES AND KRONECKER PRODUCTS

In image processing, the analysis of many problems can be simplified substantially by working with block matrices and the so-called Kronecker products. For example, the two-dimensional convolution can be expressed by simple block matrix operations.

Block Matrices

Any matrix $\mathcal{A}$ whose elements are matrices themselves is called a *block matrix*; for example,

$$\mathcal{A} = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \cdots & \mathbf{A}_{1,n} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} & \mathbf{A}_{2,n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \mathbf{A}_{m,1} & \mathbf{A}_{m,2} \cdots & \mathbf{A}_{m,n} \end{bmatrix} \tag{2.47}$$

is a block matrix where $\{\mathbf{A}_{i,j}\}$ are $p \times q$ matrices. The matrix $\mathcal{A}$ is called an $m \times n$ block matrix of basic dimension $p \times q$. If $\mathbf{A}_{i,j}$ are square matrices (say, $p \times p$), then we also call $\mathcal{A}$ to be an $m \times n$ block matrix of basic dimension p.

If the block structure is Toeplitz, ($\mathbf{A}_{i,j} = \mathbf{A}_{i-j}$) or circulant ($\mathbf{A}_{i,j} = \mathbf{A}_{((i-j) \text{ modulo } n)}$, $m = n$) then $\mathcal{A}$ is called *block Toeplitz* or *block circulant*, respectively. Additionally, if each block itself is Toeplitz (or circulant), then $\mathcal{A}$ is called *doubly block Toeplitz* (or doubly block circulant). Finally, if $\{\mathbf{A}_{i,j}\}$ are Toeplitz (or circulant) but ($\mathbf{A}_{i,j} \neq \mathbf{A}_{i-j}$) then $\mathcal{A}$ is called a *Toeplitz block* (or *circulant block*) matrix. Note that a doubly Toeplitz (or circulant) matrix need not be fully Toeplitz (or circulant), i.e., the scaler elements of $\mathcal{A}$ need not be constants along the subdiagonals.

Example 2.6

Consider the two-dimensional convolution

$$y(m, n) = \sum_{m'=0}^{2} \sum_{n'=0}^{1} h(m - m', n - n')x(m', n'), \qquad 0 \le m \le 3, \quad 0 \le n \le 2$$

where the $x(m, n)$ and $h(m, n)$ are defined in Example 2.1. We will examine the block structure of the matrices when the input and output arrays are mapped into column-ordered vectors. Let $\mathbf{x}_n$ and $\mathbf{y}_n$ be the column vectors. Then

$$\mathbf{y}_n = \sum_{n'=0}^{1} \mathbf{H}_{n-n'} \mathbf{x}_{n'}, \qquad \mathbf{H}_n = \{h(m - m', n), \quad 0 \le m \le 3, \quad 0 \le m' \le 2\},$$

where

$$\mathbf{x}_0 = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}, \qquad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \qquad \mathbf{H}_0 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\mathbf{H}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{H}_{-1} = \mathbf{0}, \qquad \mathbf{H}_2 = \mathbf{0}$$

Defining $\mathscr{y}$ and $\mathscr{x}$ as column-ordered vectors, we get

$$\mathscr{y} = \begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \overset{12\times 6}{\begin{bmatrix} \mathbf{H}_0 & \mathbf{0} \\ \mathbf{H}_1 & \mathbf{H}_0 \\ \mathbf{0} & \mathbf{H}_1 \end{bmatrix}} \overset{6\times 1}{\begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{bmatrix}} \triangleq \mathscr{H}\mathscr{x}$$

Spatially invariant: obeys scaling & superposition

where $\mathscr{H}$ is a doubly Toeplitz 3×2 block matrix of basic dimensions 4×3. However, the matrix $\mathscr{H}$ as a whole is not Toeplitz because $[\mathscr{H}]_{m,n} \neq [\mathscr{H}]_{m-n}$ (show it!). Hence the one-dimensional system $\mathscr{y} = \mathscr{H}\mathscr{x}$ is linear but not spatially invariant, even though the original two-dimensional system is. Alternatively, $\mathscr{y} = \mathscr{H}\mathscr{x}$ does not represent a one-dimensional convolution operation although it does represent a two-dimensional convolution.

Example 2.7

Block circulant matrices arise when the convolving arrays are periodic. For example, let

$$y(m,n) = \sum_{m'=0}^{2} \sum_{n'=0}^{3} h(m-m', n-n')x(m', n'), \qquad 0 \le m \le 2, \quad 0 \le n \le 3$$

where $h(m, n)$ is doubly periodic with periods $(3, 4)$, i.e., $h(m,n) = h(m+3, n+4)$, $\forall m, n$. The array $h(m, n)$ over one array period is shown next:

n	$m=0$	$m=1$	$m=2$
3	1	0	1
2	2	2	0
1	3	5	1
$n=0$	4	8	3

$h(m, n)$

In terms of column vectors of $x(m, n)$ and $y(m, n)$, we can write

$$\mathbf{y}_n = \sum_{n'=0}^{3} \mathbf{H}_{n-n'}\mathbf{x}_{n'}, \qquad 0 \le n \le 3$$

where $\mathbf{H}_n$ is a periodic sequence of 3×3 circulant matrices with period 4, given by

$$\mathbf{H}_0 = \begin{bmatrix} 4 & 3 & 8 \\ 8 & 4 & 3 \\ 3 & 8 & 4 \end{bmatrix}, \qquad \mathbf{H}_1 = \begin{bmatrix} 3 & 1 & 5 \\ 5 & 3 & 1 \\ 1 & 5 & 3 \end{bmatrix}, \qquad \mathbf{H}_2 = \begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix}, \qquad \mathbf{H}_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Written as a column-ordered vector equation, the output becomes

$$\mathscr{y} = \begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{H}_0 & \mathbf{H}_3 & \mathbf{H}_2 & \mathbf{H}_1 \\ \mathbf{H}_1 & \mathbf{H}_0 & \mathbf{H}_3 & \mathbf{H}_2 \\ \mathbf{H}_2 & \mathbf{H}_1 & \mathbf{H}_0 & \mathbf{H}_3 \\ \mathbf{H}_3 & \mathbf{H}_2 & \mathbf{H}_1 & \mathbf{H}_0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \triangleq \mathscr{H} x$$

where $\mathbf{H}_{-n} = \mathbf{H}_{4-n}$. Now $\mathscr{H}$ is a doubly circulant, 4 × 4 block matrix of basic dimension 3 × 3.

Kronecker Products

If $\mathbf{A}$ and $\mathbf{B}$ are $M_1 \times M_2$ and $N_1 \times N_2$ matrices, respectively, then their Kronecker product is defined as

$$\mathbf{A} \otimes \mathbf{B} \triangleq \{a(m, n)\mathbf{B}\} = \begin{bmatrix} a(1, 1)\mathbf{B} & \cdots & a(1, M_2)\mathbf{B} \\ \vdots & & \vdots \\ a(M_1, 1)\mathbf{B} & \cdots & a(M_1, M_2)\mathbf{B} \end{bmatrix} \tag{2.48}$$

This is an $M_1 \times M_2$ block matrix of basic dimension $N_1 \times N_2$. Note that $\mathbf{A} \otimes \mathbf{B} \neq \mathbf{B} \otimes \mathbf{A}$. Kronecker products are useful in generating high-order matrices from low-order matrices, for example, the fast Hadamard transforms that will be studied in Chapter 5. Several properties of Kronecker products are listed in Table 2.7. A particularly useful result is the identity

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}) \tag{2.49}$$

It expresses the matrix multiplication of two Kronecker products as a Kronecker product of two matrices. For $N \times N$ matrices, it will take $O(N^6) + O(N^4)$ oper-

TABLE 2.7 Properties of Kronecker Products

1. $(\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C}$
2. $(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C})$
3. $\alpha(\mathbf{A} \otimes \mathbf{B}) = (\alpha\mathbf{A}) \otimes \mathbf{B} = \mathbf{A} \otimes (\alpha\mathbf{B})$, where α is scalar.
4. $(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T$
5. $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$
6. $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD})$
7. $\mathbf{A} \otimes \mathbf{B} = (\mathbf{A} \otimes \mathbf{I})(\mathbf{I} \otimes \mathbf{B})$
8. $\prod_{k=1}^{l} (\mathbf{A}_k \otimes \mathbf{B}_k) = \left(\prod_{k=1}^{l} \mathbf{A}_k\right) \otimes \left(\prod_{k=1}^{l} \mathbf{B}_k\right)$, where $\mathbf{A}_k$ and $\mathbf{B}_k$ are square matrices
9. $\det(\mathbf{A} \otimes \mathbf{B}) = (\det \mathbf{A})^m (\det \mathbf{B})^n$, where $\mathbf{A}$ is $m \times m$ and $\mathbf{B}$ is $n \times n$
10. $\mathrm{Tr}(\mathbf{A} \otimes \mathbf{B}) = [\mathrm{Tr}(\mathbf{A})][\mathrm{Tr}(\mathbf{B})]$
11. If $r(\mathbf{A})$ denotes the rank of a matrix $\mathbf{A}$, then $r(\mathbf{A} \otimes \mathbf{B}) = r(\mathbf{A})\, r(\mathbf{B})$.
12. If $\mathbf{A}$ and $\mathbf{B}$ are unitary, then $\mathbf{A} \otimes \mathbf{B}$ is also unitary.
13. If $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$, $\mathbf{C}\xi_k = \gamma_k \xi_k$, $\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i$, $\mathbf{B}\mathbf{y}_j = \mu_j \mathbf{y}_j$, then $\xi_k = \mathbf{x}_i \otimes \mathbf{y}_j$, $\gamma_k = \lambda_i \mu_j$, $1 \le i \le m$, $1 \le j \le n$, $1 \le k \le mn$.

Random Signals

- Stationary process – a random sequence $u(n)$ is -
  - **Strict-sense stationary**, $\{u(l), 1 \le l \le k\}$ is the same as that of the shifted sequence $\{u(l+m), 1 \le l \le k\}$, for any integer and any length k.
  - **Wide-sense stationary** if,

    $E[u(n)] = \mu = \text{constant}$

    $E[u(n)u)(n')] = r(n-n')$ -> covariance

  i.e., $r(n,n') = r(n-n')$.
- We use wide sense-stationary for any random process
  - A Gaussian process is completely specified by its mean and covariance -> thus, strict-sense and wide sense stationarity are same.

Orthogonality and Independence

- 2 random variables x and y are **independent** iff their joint probability density function (pdf) is product of their marginal densities,

  $p_{x,y}(x,y)= p_x(x)p_y(y)$

  - 2 random sequences x(n) and y(n) are independent iff for every n and n', the random variable x(n) and x(n') are independent.
- Random variables x and y are **orthogonal** if

  $$E[xy)]=0$$

  - Uncorrelated if, $E[xy^*]=(E[x])(E[y^*])$

    $$\text{Or, } E[(x-\mu_x)(y-\mu_y)^*]=0$$

  - Thus, zero mean uncorrelated random variables are also orthogonal
  - Gaussian random variables which are uncorrelated are also independent

ations to compute the left side, whereas only $O(N^4)$ operations are required to compute the right side. This principle is useful in developing fast algorithms for multiplying matrices that can be expressed as Kronecker products.

Example 2.8

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Then

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ 3 & 4 & -3 & -4 \end{bmatrix}, \quad \mathbf{B} \otimes \mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & -1 & 2 & -2 \\ 3 & 3 & 4 & 4 \\ 3 & -3 & 4 & -4 \end{bmatrix}$$

Note the two products are not equal.

Separable Operations

Consider the transformation (on an $N \times M$ image $\mathbf{U}$)

$$\mathbf{V} \triangleq \mathbf{A}\mathbf{U}\mathbf{B}^T$$

or

$$v(k, l) = \sum_m \sum_n a(k, m) u(m, n) b(l, n) \tag{2.50}$$

This defines a class of separable operations, where $\mathbf{A}$ operates on the columns of $\mathbf{U}$ and $\mathbf{B}$ operates on the rows of the result. If $\mathbf{v}_k$ and $\mathbf{u}_m$ denote the kth and mth row vectors of $\mathbf{V}$ and $\mathbf{U}$, respectively, then the preceding series becomes

$$\mathbf{v}_k^T = \sum_m a(k, m)[\mathbf{B}\mathbf{u}_m^T] = \sum_m [\mathbf{A} \otimes \mathbf{B}]_{k,m} \mathbf{u}_m^T$$

where $[\mathbf{A} \otimes \mathbf{B}]_{k,m}$ is the (k, m)th block of $\mathbf{A} \otimes \mathbf{B}$. Thus if $\mathbf{U}$ and $\mathbf{V}$ are *row-ordered* into vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively, then

$$\mathbf{V} = \mathbf{A}\mathbf{U}\mathbf{B}^T \Rightarrow \boldsymbol{v} = (\mathbf{A} \otimes \mathbf{B})\boldsymbol{u}$$

i.e., the separable transformation of (2.50) maps into a Kronecker product operating on a vector.

2.9 RANDOM SIGNALS

Definitions

A complex discrete random signal or a discrete random process is a sequence of random variables $u(n)$. For complex random sequences, we define

$$\text{Mean} \triangleq \mu_u(n) \triangleq \mu(n) = E[u(n)] \tag{2.51}$$

$$\text{Variance} \triangleq \sigma_u^2(n) = \sigma^2(n) = E[|u(n) - \mu(n)|^2] \tag{2.52}$$

$$\begin{aligned}\text{Covariance} &= \text{Cov}[u(n), u(n')] \triangleq r_u(n, n') \triangleq r(n, n') \\ &= E\{[u(n) - \mu(n)][u^*(n') - \mu^*(n')]\}.\end{aligned} \tag{2.53}$$

$$\begin{aligned}\text{Cross covariance} &\triangleq \text{Cov}[u(n), v(n')] \triangleq r_{uv}(n, n') \\ &= E\{[u(n) - \mu_u(n)][v^*(n') - \mu_v^*(n')]\}\end{aligned} \tag{2.54}$$

$$\begin{aligned}\text{Autocorrelation} &\triangleq a_{uu}(n, n') \triangleq a(n, n') = E[u(n)u^*(n')] \\ &= r(n, n') + \mu(n)\mu^*(n')\end{aligned} \tag{2.55}$$

$$\text{Cross-correlation} = a_{uv}(n, n') = E[u(n)v^*(n')] = r_{uv}(n, n') + \mu_u(n)\mu_v^*(n') \tag{2.56}$$

The symbol E denotes the mathematical expectation operator. Whenever there is no confusion, we will drop the subscript u from the various functions. For an $N \times 1$ vector $\mathbf{u}$, its mean, covariance, and other properties are defined as

$$E[\mathbf{u}] = \boldsymbol{\mu} = \{\mu(n)\} \quad \text{is an } N \times 1 \text{ vector,} \tag{2.57}$$

$$\text{Cov}[\mathbf{u}] \triangleq E(\mathbf{u} - \boldsymbol{\mu})(\mathbf{u}^* - \boldsymbol{\mu}^*)^T \triangleq \mathbf{R}_u \triangleq \mathbf{R} = \{r(n, n')\} \quad \text{is an } N \times N \text{ matrix} \tag{2.58}$$

$$\text{Cov}[\mathbf{u}, \mathbf{v}] \triangleq E(\mathbf{u} - \boldsymbol{\mu}_u)(\mathbf{v}^* - \boldsymbol{\mu}_v^*)^T \triangleq \mathbf{R}_{uv} = \{r_{uv}(n, n')\} \quad \text{is an } N \times N \text{ matrix} \tag{2.59}$$

Now $\boldsymbol{\mu}$ and $\mathbf{R}$ represent the mean vector and the covariance matrix, respectively, of the vector $\mathbf{u}$.

Gaussian or Normal Distribution

The probability density function of a random variable u is denoted by $p_u(\mathscr{u})$. For a Gaussian random variable

$$p_u(\mathscr{u}) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{-|\mathscr{u} - \mu|^2}{2\sigma^2}\right\}, \tag{2.60}$$

where μ and σ^2 are its mean variance and $\mathscr{u}$ denotes the value the random variable takes. For $\mu = 0$ and $\sigma^2 = 1$, this is called the *standard normal distribution.*

Gaussian Random Processes

A sequence, possibly infinite, is called a Gaussian (or normal) random process if the joint probability density of any finite sub-sequence is a Gaussian distribution. For example, for a Gaussian sequence $\{u(n), 1 \le n \le N\}$ the joint density would be

$$p_{\mathbf{u}}(\mathscr{u}) = p_{\mathbf{u}}(\mathscr{u}_1, \mathscr{u}_2, \ldots, \mathscr{u}_N) = [(2\pi)^{N/2}|\mathbf{R}|^{1/2}]^{-1} \exp\{-\tfrac{1}{2}(\mathscr{u} - \boldsymbol{\mu})^{*T}\mathbf{R}^{-1}(\mathscr{u} - \boldsymbol{\mu})\} \tag{2.61}$$

where $\mathbf{R}$ is the covariance matrix of $\mathbf{u}$ and is assumed to be nonsingular.

Stationary Processes

A random sequence $u(n)$ is said to be *strict-sense stationary* if the joint density of any partial sequence $\{u(l), 1 \le l \le k\}$ is the same as that of the shifted sequence

$\{u(l+m), 1 \le l \le k\}$, for any integer m and any length k. The sequence $u(n)$ is called *wide-sense stationary* if

$$E[u(n)] = \mu = \text{constant}$$
$$E[u(n)u^*(n')] = r(n-n') \tag{2.62}$$

This implies $r(n, n') = r(n-n')$, i.e., the covariance matrix of $\{u(n)\}$ is Toeplitz.

Unless stated otherwise, we will imply wide-sense stationarity whenever we call a random process stationary. Since a Gaussian process is completely specified by the mean and covariance functions, for such a process wide-sense stationarity is the same as strict-sense stationarity. In general, although strict-sense stationarity implies stationarity in the wide sense, the converse is not true.

We will denote the covariance function of a stationary process $u(n)$ by $r(n)$, the implication being

$$r(n) = \text{Cov}[u(n), u(0)] = \text{Cov}[u(n'+n), u(n')], \qquad \forall n', \forall n \tag{2.63}$$

Using the definitions of covariance and autocorrelation functions, it can be shown that the arrays $r(n, n')$ and $a(n, n')$ are conjugate symmetric and nonnegative definite, i.e.,

$$\textit{Symmetry:} \quad r(n, n') = r^*(n', n), \qquad \forall n, n' \tag{2.64}$$

$$\textit{Nonnegativity:} \quad \sum_n \sum_{n'} x(n) r(n, n') x^*(n') \ge 0, \qquad x(n) \ne 0, \forall n \tag{2.65}$$

This means the covariance and autocorrelation matrices are Hermitian and nonnegative definite.

Markov Processes

A random sequence $u(n)$ is called Markov-p, or pth-order Markov, if the conditional probability of $u(n)$ given the entire past is equal to the conditional probability of $u(n)$ given only $u(n-1), \ldots, u(n-p)$, i.e.,

$$\text{Prob}[u(n)|u(n-1), u(n-2), \ldots] = \text{Prob}[u(n)|u(n-1), \ldots, u(n-p)], \qquad \forall n \tag{2.66a}$$

A Markov-1 sequence is simply called Markov. A Markov-p scalar sequence can also be expressed as a $(p \times 1)$ Markov-1 vector sequence. Another interpretation of a pth-order Markov sequence is that if the "present," $\{u(j), n-p \le j \le n-1\}$, is known, then the "past," $\{u(j), j < n-p\}$, and the "future," $\{u(j), j \ge n\}$, are independent. This definition is useful in defining Markov random fields in two dimensions (see Chapter 6). For Gaussian Markov-p sequences it is sufficient that the conditional expectations satisfy the relation

$$E[u(n)|u(n-1), u(n-2), \ldots] = E[u(n)|u(n-1), \ldots, u(n-p)], \qquad \forall n \tag{2.66b}$$

Example 2.9 (Covariance matrix of stationary sequences)

The covariance function of a first-order stationary Markov sequence $u(n)$ is given as

$$r(n) = \rho^{|n|}, \qquad |\rho| < 1, \forall n \tag{2.67}$$

Information and entropy

- Suppose a source (image), which generates a discrete set of independent messages (gray levels) r_k, with probabilities p_k, k=1,…,L.
  - Then **information** associated with r_k is defined as,

    $$I_k = -\log_2 p_k \text{ bits; since } \Sigma^{\infty}_{k=1} p_k = 1$$

  - Each $p_k \leq 1$ and I_k is nonnegative
- **Entropy** is defined as the average information generated by the source,

  $$\text{Entropy, } H = \Sigma^{\infty}_{k=1} p_k \log_2 p_k \text{ bits/message}$$

  - For a digital image, source is independent pixels; its entropy can be estimated from its histogram.

Information

Suppose there is a source (such as an image), which generates a discrete set of independent messages (such as gray levels) r_k, with probabilities p_k, $k = 1, \ldots, L$. Then the information associated with r_k is defined as

$$I_k = -\log_2 p_k \quad \text{bits} \tag{2.111}$$

Since

$$\sum_{k=1}^{L} p_k = 1 \tag{2.112}$$

each $p_k \leq 1$ and I_k is nonnegative. This definition implies that the information conveyed is large when an unlikely message is generated.

Entropy

The *entropy* of a source is defined as the average information generated by the source, i.e.,

$$\text{Entropy, } H = -\sum_{k=1}^{L} p_k \log_2 p_k \quad \text{bits/message} \tag{2.113}$$

For a digital image considered as a source of independent pixels, its entropy can be estimated from its histogram. For a given L, the entropy of a source is maximum for uniform distributions, i.e., $p_k = 1/L$, $k = 1, \ldots, L$. In that case

$$\max_{p_k} H = -\sum_{k=1}^{L} \frac{1}{L} \log_2 \frac{1}{L} = \log_2 L \quad \text{bits} \tag{2.114}$$

The entropy of a source gives the lower bound on the number of bits required to encode its output. In fact, according to Shannon's noiseless coding theorem [11, 12], it is possible to code without distortion a source of entropy H bits using an average of $H + \epsilon$ bits/message, where $\epsilon > 0$ is an arbitrarily small quantity. An alternate form of this theorem states that it is possible to code the source with H bits such that the distortion in the decoded message could be made arbitrarily small.

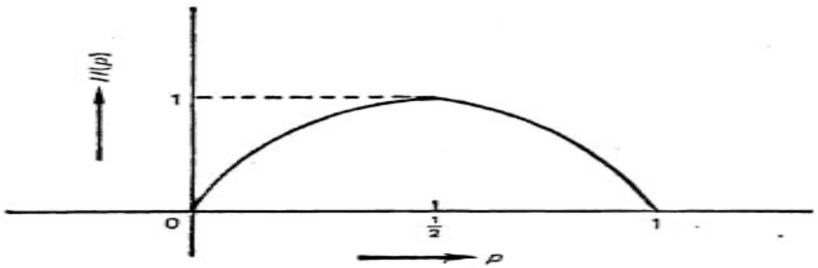


Figure 2.8 Entropy of a binary source.