span - the set of all their **linear combinatons**.

$$a\overrightarrow{v} + b\overrightarrow{w}$$

linear dependent

$$\overrightarrow{u} = a\overrightarrow{v} + b\overrightarrow{w}$$

linear independent

$$\overrightarrow{u} \neq a\overrightarrow{v} + b\overrightarrow{w}$$

basis is a set of linearly independent vectors that span the full space.

linear transformation - Lines remain lines; Origin remains fixed - Grid lines remain parallel and evenly spaced.

Matrix

A matrix represents a specific linear transformation.

Multiplying a matrix by a vextor is to apply that transformation to that vector.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Composition applying one transformation then another.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$Shear \leftarrow Rotation = Composition$$

*Read from right to left: f(g(x))

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e \\ g \end{bmatrix} (\hat{i})$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} f \\ h \end{bmatrix} (\hat{j})$$

Oder does matter:

$$M1M2 \neq M2M1$$

Associality:

$$(AB)C = A(BC)$$

Determinant

The factor by which a linear reansformation changes any area.

$$det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc$$

3D - volume of the **parallelepiped**. When det = 0, colunms must be linearly dependent

$$det(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}) = a * det(\begin{bmatrix} e & f \\ h & i \end{bmatrix}) - b * det(\begin{bmatrix} d & f \\ g & i \end{bmatrix}) + c * det(\begin{bmatrix} d & e \\ g & h \end{bmatrix})$$

$$det(M1M2) = det(M1)det(M2)$$

Linear System of Equations

$$2x + 5y + 3z = -3$$

$$4x + 0y + 8z = 0$$

$$1x + 3y + 0z = 2$$

$$\Downarrow$$

$$\begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

$$A\overrightarrow{x} = \overrightarrow{v}$$

Inverse transformation:

$$A^{-1} \rightarrow \begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix}^{-1}$$

Identity transformation:

$$A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

When det(A) is not 0 (Space does not get squished into a zero area region): A inverse exists

When det(A) == 0, cannot inverse A => Cannot un-squish a line to turn it into a plane.

Rank:

The number of dimensions in the output of a transformation (colunm space).

The output of a transformation is a line, it's one-dimensional => The transformation has a **rank** of one.

full rank

Colunm Space of A:

Set of all possible outputs of Av => **span** of the colunm of the matrix

Zero point is always in the colunm space

Null space / Kernel:

The space of all vectors that become null (land on the zero vector)

When v happens to be zero pointer, the null space is all the possible solusions.

Nonsquare matrices

2D input => 3D output: Mapping two dimensions to three dimensions:

$$3 \times 2 Matrix$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}$$

All the vectors land is a 2D plance slicing through the origin of 3D space.

Full rank: the number of dimensions in this column space is the same as the number of dimensions of the input space.

Two colunms: input space has two basis vectors.

Three rows: the landing spot for each of those basis vectors is described with three seperate coordinates.

3D input => 2D output

$$2 \times 3 Matrix$$

$$\begin{bmatrix} 2 & 0 & 4 \\ -1 & 1 & 5 \end{bmatrix}$$

3 basis vectors, 2 coordinates for each landing spots.

2D input => 1D output

$$1 \times 2 Matrix$$

$$\begin{bmatrix} 2 & 4 \end{bmatrix}$$

Dot products and duality

Projection

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 1 \cdot 3 + 2 \cdot 4$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 1 \cdot 3 + 2 \cdot 4$$

$$\overrightarrow{v} \cdot \overrightarrow{w} = (Projected\ \ \overrightarrow{w})(Length\ \ \overrightarrow{v})$$

Any time you have a 2d-to-1d linear transformation, it's associated with some vector.

$$1 \times 2 matrices \longleftrightarrow 2d\ \ vectors$$

Cross products

Area of parallelogram

$$\overrightarrow{v} \times \overrightarrow{w} = -\overrightarrow{w} \times \overrightarrow{v}$$

Negative if v is in the left of w.

$$\overrightarrow{v} \times \overrightarrow{w} = det(\begin{bmatrix} -3 & 2 \\ 1 & 1 \end{bmatrix}) = -3 \cdot 1 - 2 \cdot 1 = -5$$

$$\overrightarrow{v} \times \overrightarrow{w} = \overrightarrow{p}$$

Perpendicular to the parallelogram, with length the area of it

$$\begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} \times \begin{bmatrix} w1 \\ w2 \\ w3 \end{bmatrix} = det(\begin{bmatrix} \hat{i} & v1 & w1 \\ \hat{j} & v2 & w2 \\ \hat{k} & v3 & w3 \end{bmatrix})$$

$$= \hat{i}(v2 \cdot w3 - w2 \cdot v3) + \hat{j}(v3 \cdot w1 - w3 \cdot v1) + \hat{k}(v1 \cdot w2 - w1 \cdot v2)$$

$$= \begin{bmatrix} v2 \cdot w3 - w2 \cdot v3 \\ v3 \cdot w1 - w3 \cdot v1 \\ v1 \cdot w2 - w1 \cdot v2 \end{bmatrix}$$

1. Define a 3d-to-1d linear transformation in terms of v and w
2. Find its dual vector
3. Show that the dual is v X w

$$\begin{bmatrix} p1 \\ p2 \\ p3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = det(\begin{bmatrix} x & v1 & w1 \\ y & v2 & w2 \\ z & v3 & w3 \end{bmatrix})$$

Cramer's Rule

if

$$T(\overrightarrow{v}) \cdot T(\overrightarrow{w}) = \overrightarrow{v} \cdot \overrightarrow{w}$$

for all v and w

then: **T is Orthonotmal**

$$\begin{bmatrix} cos(30^\circ) & -sin(30^\circ) \\ sin(30^\circ) & cos(30^\circ) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} cos(30^\circ) \\ sin(30^\circ) \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -sin(30^\circ) \\ cos(30^\circ) \end{bmatrix}$$

All areas get scaled by det(A)

$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$Area = y$$

$$Area = det(A)y$$

$$y = \frac{Area}{det(A)} = \frac{det(\begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix})}{det(\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix})}$$

$$x = \frac{Area}{det(A)} = \frac{det(\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix})}{det(\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix})}$$

in 3-D

$$z = det(\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z \end{bmatrix})$$

Change of basis

basis vectors: i j

A : Jennifer's basis vectors, written in out coordinates

xj, yj : vector in her coordinates

xo, yo : same vector in our coordinates

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$A \begin{bmatrix} xj \\ yj \end{bmatrix} = \begin{bmatrix} xo \\ yo \end{bmatrix}$$

translate a matrix:

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \vec{v}$$

Eigenvectors and Eigenvalues

Eigenvector: vector remains on its own span, on a line streached by a linear transformation (without getting rotated off the span)

Eigenvalue: the factor it stretched or squashed during the transformation

$$A\vec{v} = \lambda\vec{v} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \vec{v} = (\lambda I)\vec{v}$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

$$det(A - \lambda I) = 0$$

$$det(\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}) = 0$$

$$det(\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = (2-\lambda)(3-\lambda) - 2 \cdot 1 = 0$$

There could be no eigenvectors: rotation

Use eigenvectors as basis

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Abstract vector spaces

WHAT IS A VECTOR

$$(f+g)(x) = f(x) + g(x)$$

$$\begin{bmatrix} x1 \\ y1 \\ z1 \end{bmatrix} + \begin{bmatrix} x2 \\ y2 \\ z2 \end{bmatrix} = \begin{bmatrix} x1+x2 \\ y1+y2 \\ z1+z2 \end{bmatrix}$$

$$(2f)(x) = 2f(x)$$

$$2 \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

Linear transformation <-> derivative

definition of linearity: Additivity and Scaling.

Derivativ is linear

$$L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w})$$

$$\frac{d}{dx}(x^3 + x^2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2)$$

$$L(c\vec{v}) = cL(\vec{v})$$

$$\frac{d}{dx}(4x^3) = 4\frac{d}{dx}(x^3)$$

Our current space: All polynomials

$$\frac{d}{dx}(1x^3 + 5x^2 + 4x + 5) = 3x^2 + 10x + 4$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 3 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 5 \\ 1 \\ \dots \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 \\ 2 \cdot 5 \\ 3 \cdot 1 \\ 0 \\ \dots \end{bmatrix}$$

Axioms