EMBEDDING THEOREMS FOR DISCRETE DYNAMICAL SYSTEMS AND TOPOLOGICAL FLOWS

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ABSTRACT. In this paper, we investigate the embeddings for topological flows. We prove an embedding theorem for discrete topological system. Our results apply to suspension flows via constant function, and for this case we show an embedding theorem for suspension flows and give a new proof of Gutman-Jin embedding theorem.

1. Introduction

A pair (X, \mathbb{R}) is called a \mathbb{R} -flow (real flow or topological flow) if X is a topological compact space and the Ableian group \mathbb{R} acts on X continuously, i.e., 0.x = x and r.(s.x) = (r+s).x for all $r, s \in \mathbb{R}$ and $x \in X$. In this paper, we are interested in the embedding property of the topological flows. Firstly, we consider a special topological flow called solenoid [NS60, V.8.15]. Let $Y_n = [0, n!]$ such that 0 is identified with n!. The solenoid is defined as

$$Y = \{(x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} Y_n : x_n = x_{n+1} \mod n! \},$$

with \mathbb{R} -action $\Psi_r: (x_n)_{n\in\mathbb{N}} \mapsto (x_n+r \mod n!)_{n\in\mathbb{N}}$ for all $r\in\mathbb{R}$. Clearly, the solenoid (Y,Ψ) is a minimal topological flow. Our first main result are stated as follows.

Proposition 1.1. Let (Z, Φ) be an extension of the solenoid (Y, Ψ) . Then (Z, Φ) is topologically conjugate to the suspension flow under the constant function.

A pair (X,T) is called a discrete topological (dynamical) system if X is a topological compact space and T is a homeomorphism on X. In other words, T induces a continuous \mathbb{Z} -action on X by $n.x = T^n(x)$ for $n \in \mathbb{Z}$ and $x \in X$.

Mean dimension was introduced by Gromov [Gro99], and was further investigated by Lindenstrauss and Weiss [LW00] as an invariant of topological dynamical systems. Several applications and interesting relations with other subjects have been studied. In recent years the relation with the so-called embedding problem has attracted considerable attention. Roughly speaking, the problem is which system (X, T) can be embedded in the shifts on the Hilbert cubes $(([0,1]^N)^{\mathbb{Z}}, \sigma)$, where N is a natural number and σ is the (left) shift on $([0,1]^N)^{\mathbb{Z}}$. Gutman and

Tsukamoto [GT20], as well as Gutman, Qiao and Tsukamoto [GQT19], had found a way that the discrete system can be firstly embedded in the space of bounded band-limited real functions and then the Hilbert cubes.

For a discrete system (X,T), it is said to satisfy the marker property if for each positive integer N there exists an open set $U \subset X$ satisfying that

$$U \cap T^{-n}U = \emptyset$$
 (for $0 < n < N$) and $X = \bigcup_{n \in \mathbb{Z}} T^n U$.

For example, an extension of an aperiodic minimal system has the marker property [Lin99, Lemma 3.3]. Obviously, the marker property implies the aperiodicity. See [Gut15, Gut17] where the marker property was developed. Gutman, Qiao and Tsukamoto [GQT19] proved the embedding theorem of discrete topological system, which asserts that if a topological dynamical system (X,T) satisfies the marker property and the mean dimension $\operatorname{mdim}(X,T) < \frac{a}{2}$ then we can embed it in the shift on $\mathcal{B}(a)$, where $\mathcal{B}(a)$ is the space of bounded band-limited real functions (see Section 4.3). In fact, they proved the result for \mathbb{Z}^k -action. In this paper, we show that not only there is an embedding from (X,T) to $\mathcal{B}(a)$, but also it satisfies that different \mathbb{Z} -orbits are embedded in different \mathbb{R} -orbits.

Theorem 1.2. If a topological dynamical system (X,T) satisfies the marker property and $\operatorname{mdim}(X,T) < \frac{a}{2}$ then we can embed it in the shift on $\mathcal{B}(a)$ via h and h satisfies that if there exists $x, x' \in X$ and $r \in \mathbb{R}$ such that $\Phi_r(h(x)) = h(x')$ then $r \in \mathbb{Z}$ and $x' = T^r x$.

Actually, using the same proof, we can prove the above result for \mathbb{Z}^k -action. Since we focus on \mathbb{R} -action (as well as \mathbb{Z} -action) in the current paper, we only give the proof for \mathbb{Z} -action.

Using Theorem 1.2 and Proposition 1.1, we give a new proof of Gutman-Jin embedding theorem [GJ20, Theorem 5.1] as follows.

Corollary 1.3. Let (X, Φ) be an extension of the solenoid. Suppose that $\operatorname{mdim}(X, \Phi) < a/2$. Then (X, Φ) can be embedded in B(a).

We organize the paper as follows. Firstly, in Section 2, we recall basic notions of discrete topological systems and topological flows. In Section 3, we investigate the properties of the solenoid and its extensions. In Section 4, we recall basic properties of mean dimension of topological flows. In Section 5, we prove Theorem 1.2. In Section 6, we show an embedding theorem for suspension flows and consequently prove Corollary 1.3. Finally, in Section 7, we discuss several open problems.

Notations.

- $\mathbb{R}_{>0} = (0, \infty), \, \mathbb{R}_{>0} = [0, \infty) \text{ and } \mathbb{R}_{<0} = (-\infty, 0].$
- Denote a topological flow by $(X, \Phi) = (X, (\Phi_t)_{t \in \mathbb{R}})$ or (X, \mathbb{R}) whenever we do not emphasis the precise \mathbb{R} -action on X.

2. Preliminary

In this section, we recall several notions of discrete topological systems and topological flows.

2.1. Suspension flow and extension. Let (Z, ρ) be a compact metric space and $T: Z \to Z$ a homeomorphism. Let $f: Z \to \mathbb{R}_{>0}$ be a continuous map. The suspension flow of T under f, written by (Z_f, T_f) , is the flow $(T_t)_{t\in\mathbb{R}}$ on the space

$$Z_f := \{(x,t) : 0 \le t \le f(x), x \in Z\}/(x,f(x)) \sim (Tx,0)$$

induced by the time translation T_t on $Z \times \mathbb{R}$ defined by $T_t(x, s) = (x, t + s)$.

2.2. **Cross-section.** A cross-section of time $\eta > 0$ is a subset $S \subset X$ such that the restriction of Φ on $S \times [-\eta, \eta]$ is one-to-one and onto its image. Moreover, it is said to be global if there is $\xi > 0$ such that $\Phi(S \times [-\xi, \xi]) = X$. The flow interior of the cross-section S is defined by

$$\operatorname{Int}^{\Phi}(S) := \operatorname{Int}(\Phi_{(-\gamma,\gamma)}(S)) \cap S,$$

for any $0 < \gamma < \eta$. The flow boundary of S is defined as

$$\partial^{\Phi} S := \overline{S} \setminus \operatorname{Int}^{\Phi}(S).$$

We remark that the definitions of $\operatorname{Int}^{\Phi}(S)$ and $\partial^{\Phi}S$ do not depend on the choice of $0 < \gamma < \eta$.

Now we consider a global closed cross-section S of time $\eta > 0$. Let $\xi > 0$ with $\Phi(S \times [-\xi, \xi]) = X$. Let $t_S : X \to \mathbb{R}$ be the first time in S. Obviously, t_S is bounded from above by 2ξ (and from below by 2η when restricted on S). However, the first return time t_S is not continuous on S. In general, t_S is a lower semicontinuous positive function. Denote by $\mathcal{C}_S \subset S$ the set of continuity points of t_S . The first return map $T_S : S \to S$ is defined by $x \mapsto \Phi(x, t_S(x))$.

The first return map T_S is continuous on \mathcal{C}_S but not on S in general.

Lemma 2.1. Let
$$x \in S \setminus C_S$$
. Then $\Phi_{t_S(x)}(x) \in \partial^{\Phi} S$.

Proof. Since $x \in S \setminus C_S$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of points in S converging to x with

$$(2\cdot 1) t_S(x) < \lim_{n \to \infty} t_S(x_n) \le 2\xi.$$

Fix a small $0 < \gamma < \eta$. By $(2\cdot 1)$, the point $\Phi_{t_S(x)}(x_n)$ lies out $\operatorname{Int}(\Phi_{(-\gamma,\gamma)}(S))$ when n is sufficiently large. So does $\Phi_{t_S(x)}(x)$.

A cross-section S of (X, Φ) is called a *Poincaré cross-section* if the map Φ is a surjective local homeomorphism from $S \times \mathbb{R}$ to X.

Lemma 2.2. Let S be a closed cross-section of topological flow (X, Φ) . Then the following are equivalent.

(1) S is a Poincaré cross-section.

- (2) S is a global closed cross-section with empty flow boundary.
- (3) (X, Φ) is topologically conjugate to the suspension flow $(S_{t_S}, (T_S)_{t_S})$ where $t_S : S \to \mathbb{R}_{>0}$ is the first return time restricted on S and $T_S : S \to S$ the first return map.

Proof. The equivalence between (1) and (2) follows by [Bur19, Lemma 2.9].

- $(2) \Rightarrow (3)$: Since S has the empty flow boundary, by Lemma 2.1, we see that t_S and T_S are continuous. It follows that the map $(x,t) \mapsto \Phi(x,t)$ is an isomorphism from $(S_{t_S},(T_S)_{t_S})$ to (X,Φ) , where the inverse map is $x \mapsto (T_S^{-1}(x), t_S^{-1}(x))$.
- $(3) \Rightarrow (1)$: It is not hard to check that $(T_S)_{t_S}$ is a surjective local homeomorphism from $S \times \mathbb{R}$ to S_{t_S} .
- 2.3. Inverse limit of dynamical systems. A pair (X, G) is called a G-system if X is compact space, G is a topological group and G acts on X continuously. For example, $G = \mathbb{R}$ or \mathbb{Z} .

Let $\{(X_n, G_n)\}_{n\in\mathbb{N}}$ be a sequence of systems. Suppose that for every pair m > n there exists a factor map $\sigma_{m,n} : X_m \to X_n$ such that for any triple m > n > l the diagram

$$(X_m, G_m)$$

$$\downarrow^{\sigma_{m,l}} \qquad \downarrow^{\sigma_{m,n}}$$

$$(X_l, G_l) \leftarrow^{\sigma_{n,l}} (X_n, G_n)$$

commutes. Set

$$X = \{ x \in \prod_{n \in \mathbb{N}} X_n : \sigma_{m,n}(x_m) = x_n, \forall m > n \}.$$

Clearly, the space X is closed in $\prod_{n\in\mathbb{N}}X_n$ and thus compact. Let $\sigma_n:X\to X_n$ be the projection map for each $n\in\mathbb{N}$. Define the action $G:X\to X$ by

$$G:(x_n)_{n\in\mathbb{N}}\to (G_nx_n)_{n\in\mathbb{N}}.$$

We call the G-system (X, G) the *inverse limit* of the family $\{(X_n, G_n)\}_{n \in \mathbb{N}}$ via $\sigma = (\sigma_{m,n})_{m,n \in \mathbb{N}, m > n}$ and write

$$(X,G) = \underline{\lim}(X_n,G_n).$$

It is clear that if G_n are the same for every $n \in \mathbb{N}$ then $G = G_n$.

3. Solenoid and its extension

Let $Y_n = [0, n!]$ such that 0 is identified with n!. Let $\Psi_r^{(n)}$ be the flow on Y_n defined by $x \mapsto x + r \mod n!$ for $r \in \mathbb{R}$ and $x \in Y_n$. The solenoid is defined by $Y = \varprojlim (Y_n, \Psi^{(n)})$. Clearly, the solenoid is a minimal flow. For $x \in Y$, we write $x = (x_n)_{n \geq 1}$ where x_n is the projection of x on Y_n . Let $S_n := \{x \in Y : x_1 = x_2 = \cdots = x_n = 0\}$. Clearly, the set S_n is closed. Now we state our main result of this section.

Proposition 3.1 (=Proposition 1.1). Let (Z, Φ) be an extension of the solenoid (Y, Ψ) via π . Then (Z, Φ) is topologically conjugate to the suspension flow $((\pi^{-1}(S_n))_{f_n}, (T_{\pi^{-1}(S_n)})_{f_n})$ for each $n \geq 1$ where f_n is the constant function n!.

We study firstly the solenoid and then its extensions.

Lemma 3.2. For every $n \ge 1$, S_n is a Poincaré cross-section of (Y, Ψ) .

Proof. Fix $n \geq 1$. By Lemma 2.2, it is sufficient to show that S_n is a global cross-section with empty flow boundary. It is clear that

$$\Psi_r(S_n) = \{ x \in Y : x_k \equiv r \mod k! \text{ for } 1 \le k \le n \}$$
$$= \{ x \in Y : x_n \equiv r \mod n! \},$$

for $r \in \mathbb{R}$. It follows that Ψ is injective on $S_n \times [0, n!)$ and $\Psi_{[0,n!]}(S_n) = Y$. That is, the set S_n is a global cross-section. Note that $\Psi_{(-\gamma,\gamma)}(S_n) = \{x \in Y : x_n \in [0,\gamma) \cup (n!-\gamma,n!]\}$ which is open (because $[0,\gamma) \cup (n!-\gamma,n!]$ is open in Y_n) for $0 < \gamma < n!/2$. Thus we have that $\operatorname{Int}(\Psi_{(-\gamma,\gamma)}(S_n)) = \Psi_{(-\gamma,\gamma)}(S_n)$ for $0 < \gamma < n!/2$ and consequently that $\operatorname{Int}^{\Psi}(S_n) = S_n$. This implies that $\partial^{\Psi}(S_n) = \emptyset$.

Lemma 3.3. The solenoid (Y, Ψ) is topological conjugate to the suspension flow $((S_n)_{f_n}, (T_{S_n})_{f_n})$ for each $n \ge 1$ where f_n is the constant function n!.

Proof. Let $n \geq 1$. It is clear that the first return time $t_{S_n} \equiv n!$. Combining Lemma 3.2 and Lemma 2.2, we complete the proof.

Now we consider the extensions of solenoid.

Lemma 3.4. For every $n \ge 1$, $\pi^{-1}(S_n)$ is a Poincaré cross-section of (Z, Φ) .

Proof. Fix $n \geq 1$. Since S_n is closed and π is continuous, we have that $\pi^{-1}(S_n)$ is closed. We claim that Φ is injective on $\pi^{-1}(S_n) \times [0, n!)$. In fact, if not, then there are $x, x' \in \pi^{-1}(S_n)$ and $t, t' \in [0, n!)$ such that $(x, t) \neq (x', t')$ and $\Phi_t(x) = \Phi_{t'}(x')$. It follows that

$$\Phi_{t-t'}(x) = x'.$$

Then we have $\pi(\Phi_{t-t'}(x)) = \pi(x')$, implying that $\Psi_{t-t'}(\pi(x)) = \pi(x')$. Since $\pi(x), \pi(x') \in S_n$ and |t-t'| < n!, by Lemma 3.2, we obtain that t = t' and $\pi(x) = \pi(x')$. Combining this with (3·1), we see that x = x', which is a contradiction. Thus Φ is injective on $\pi^{-1}(S_n) \times [0, n!)$. It follows that $\pi^{-1}(S_n)$ is a closed cross-section.

Since $\Psi_{[0,n!]}(S_n) = Y$, we obtain that

$$\Phi_{[0,n!]}(\pi^{-1}(S_n)) = \pi^{-1}(\Psi_{[0,n!]}(S_n)) = \pi^{-1}(Y) = Z.$$

Since $\Psi_{(-\gamma,\gamma)}(S_n)$ is open and π is continuous, we obtain that

$$Int(\Phi_{(-\gamma,\gamma)}(\pi^{-1}(S_n))) = Int(\pi^{-1}(\Psi_{(-\gamma,\gamma)}(S_n))) = \pi^{-1}(\Psi_{(-\gamma,\gamma)}(S_n)),$$

for $\gamma \in (0, n!/2)$. Thus we have that

$$\operatorname{Int}(\Phi_{(-\gamma,\gamma)}(\pi^{-1}(S_n))) \cap \pi^{-1}(S_n) = \pi^{-1}(\Psi_{(-\gamma,\gamma)}(S_n)) \cap \pi^{-1}(S_n)$$
$$= \pi^{-1}(\Psi_{(-\gamma,\gamma)}(S_n) \cap S_n)$$
$$= \pi^{-1}(S_n).$$

This means that $\operatorname{Int}^{\Phi}(\pi^{-1}(S_n)) = \pi^{-1}(S_n)$ and consequently that $\partial^{\Phi}(\pi^{-1}(S_n)) = \emptyset$. We conclude that $\pi^{-1}(S_n)$ is a Poincaré cross-section of (Z, Φ) by Lemma 2.2.

Now we give the proof of Proposition 3.1.

Proof of Proposition 3.1. Let $n \geq 1$. It is clear that the first return time $t_{\pi^{-1}(S_n)} \equiv n!$. Combining Lemma 3.4 and Lemma 2.2, we complete the proof.

We remark that the discrete topological system $(\pi^{-1}(S_n), T_{\pi^{-1}(S_n)})$ is the extension of the minimal system (S_n, T_{S_n}) via $\pi_{|\pi^{-1}(S_n)}$.

4. Mean dimension of \mathbb{R} -flow

In this section, we recall several notions related to mean dimension of \mathbb{R} -flow.

4.1. **Mean dimension of** \mathbb{R} **-flow.** Let (X,d) be a compact metric space. For $\epsilon > 0$ and Y a topological space, a continuous map $f: X \to Y$ is called a (d,ϵ) -embedding if for any $y \in Y$ we have diam $(f^{-1}(y)) < \epsilon$. Note that the identity map from X to itself is a (d,ϵ) -embedding for every $\epsilon > 0$. Denote by dim the Lebesgue covering dimension. Define

$$\operatorname{Widim}_{\epsilon}(X, d) = \min_{K} \dim(K),$$

where K runs over all compact metrizable space such that there is a (d, ϵ) -embedding $f: X \to K$.

Let (X, \mathbb{R}) be a topological flow. For R > 0, we define the metric d_R by

$$d_R(x,y) = \sup_{0 \le r \le R} d(r.x, r.y), \forall x, y \in X.$$

The topology on X is compatible with the metric d_R for R > 0. The (topological) mean dimension of (X, \mathbb{R}) is defined by

$$\operatorname{mdim}(X, \mathbb{R}) = \lim_{\epsilon \to 0} \lim_{R \to \infty} \frac{\operatorname{Widim}_{\epsilon}(X, d_R)}{R}.$$

The limit above exists due to Ornstein-Weiss' Lemma (see [LW00, Theorem 6.1]).

In what follows, we summarize several properties of mean dimension of topological flows which are obtained in [GJ20].

Proposition 4.1 ([GJ20], Proposition 2.5). Let (X, Φ) be a topological flow. Then $\operatorname{mdim}(X, \Phi) = \operatorname{mdim}(X, \Phi_1)$.

Proposition 4.2 ([GJ20], Proposition 2.6). Let (X, Φ) be a topological flow. If the topological entropy $h(X, \Phi)$ is finite, then $mdim(X, \Phi) = 0$.

Combining [Lin99, Theorem 4.3] and [GJ20, Proposition 3.3], we have the following proposition.

Proposition 4.3. Let (Z,T) be an extension of nontrivial minimal system. Let $f: X \to \{1\}$ be the constant function. Then $mdim(Z_f, T_f) = mdim(Z,T)$.

4.2. \mathbb{R} -shift on $B_1(V[a,b])$. A function $f:\mathbb{R}\to\mathbb{C}$ is called a *Schwartz function* (or *rapidly decreasing function*) if it is infinitely differentiable and satisfies

$$||f||_{\alpha,\beta} := \sup_{x \in \mathbb{R}} |x^{\alpha} f^{(\beta)}(x)| < \infty,$$

for all $\alpha, \beta \in \mathbb{N}$. The *Schwartz space* $\mathcal{S}(\mathbb{R})$ is defined as the space of all Schwartz functions. Recall that a *tempered distribution* on \mathbb{R} is a continuous linear functional on $\mathcal{S}(\mathbb{R})$. We remarks that bounded continuous functions are tempered distributions.

For L^1 -function $f: \mathbb{R} \to \mathbb{C}$, the Fourier transformations of f are defined by

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i \xi x} dx, \ \overline{\mathcal{F}}(f)(\xi) = \int_{\mathbb{R}} f(x)e^{2\pi i \xi x} dx.$$

Moreover, if additionally $\mathcal{F}(f) \in L^1(\mathbb{R})$ (resp. $\overline{\mathcal{F}}(f) \in L^1(\mathbb{R})$), then $\overline{\mathcal{F}}(\mathcal{F}(f)) = f$ (resp. $\mathcal{F}(\overline{\mathcal{F}}(f)) = f$).

The Fourier transformations \mathcal{F} and $\overline{\mathcal{F}}$ of tempered distribution ϕ are defined by the tempered distributions satisfying that

$$\langle \mathcal{F}(\phi), \Psi \rangle = \langle \phi, \overline{\mathcal{F}}(\Psi) \rangle, \langle \overline{\mathcal{F}}(\phi), \Psi \rangle = \langle \phi, \mathcal{F}(\Psi) \rangle, \forall \Psi \in \mathcal{S}(\mathbb{R}).$$

For a < b and tempered distribution ϕ , we say that the *support* of ϕ is contained in [a,b] and write $\operatorname{supp}(\phi) \subset [a,b]$ if $\langle \phi, \Psi \rangle = 0$ for any $\Psi \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp}(\Psi) \cap [a,b] = \emptyset$.

Let a < b be real numbers. We define $B_1(V[a, b])$ as the space of all continuous functions $f : \mathbb{R} \to \mathbb{C}$ satisfying the following conditions:

- supp $\mathcal{F}(f) \subset [a,b]$.
- $||f||_{\infty} \le 1$.

The space $B_1(V[a,b])$ is metric and compact under the distance d induced by the distance on $C(\mathbb{R},\mathbb{D})$, that is,

$$d(f,g) = \sum_{n=1}^{\infty} \frac{\|f \cdot 1_{[-n,n]} - g \cdot 1_{[-n,n]}\|_{\infty}}{2^n}, \text{ for } f, g \in B_1(V[a,b]),$$

where \mathbb{D} is the unit disk in \mathbb{C} . The topology induced by the distance d is compatible with the standard topology of tempered distribution.

For $r \in \mathbb{R}$, we denote by τ_r the translation by r, that is, $\tau_r(f)(x) = f(x+r)$ for function $f : \mathbb{R} \to \mathbb{C}$ and $x \in \mathbb{R}$. Since $\mathcal{F}(\tau_r f)(\xi) =$

 $e^{2\pi i r \xi} \mathcal{F}(f)(\xi)$, one sees that $\operatorname{supp}(\mathcal{F}(\tau_r f)) = \operatorname{supp}(\mathcal{F}(f))$. It follows that $(\tau_r)_{r \in \mathbb{R}}$ induces the \mathbb{R} -shift on $B_1(V(a,b))$, denoted by $(B_1(V[a,b]), \mathbb{R})$.

Lemma 4.4 ([GQT19], Footnote 4). $\operatorname{mdim}(B_1(V[a,b]), \mathbb{R}) = b - a$.

- 4.3. \mathbb{R} -shift on $\mathcal{B}(a)$. Let a > 0. We define $\mathcal{B}(a)$ as the space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:
 - supp $\mathcal{F}(f) \subset [-\frac{a}{2}, \frac{a}{2}].$
 - $||f||_{\infty} \leq 1$.

Similarly to the space $B_1(V[b_1, b_2])$ for $b_1 < b_2$, the space $\mathcal{B}(a)$ is metric and compact under the distance d induced by the distance on $C(\mathbb{R}, [-1, 1])$. The translation $(\tau_r)_{r \in \mathbb{R}}$ induces the \mathbb{R} -shift on $\mathcal{B}(a)$, denoted by $(\mathcal{B}(a), \mathbb{R})$.

Lemma 4.5 ([GQT19], Footnote 4). $\operatorname{mdim}(\mathcal{B}(a), \mathbb{R}) = 2a$.

The relation between $B_1(V[a,b])$ and $\mathcal{B}(2b)$ is shown as follows.

Lemma 4.6. Let b > a > 0. Then there is an embedding from $B_1(V[a,b])$ to $\mathcal{B}(2b)$.

Proof. It is easy to check that $\phi \mapsto \frac{1}{2}(\phi + \overline{\phi})$ is the embedding.

5. Strongly embedding Theorem for discrete systems

Gutman, Qiao and Tsukamoto [GQT19] proved the embedding theorem as follows. In fact, they proved the result for \mathbb{Z}^k -action and we state it for \mathbb{Z} -action in current paper for sake of simplicity.

Theorem 5.1 ([GQT19], Main Theorem 2). If a topological dynamical system (X,T) satisfies the marker property and $mdim(X,T) < \frac{a}{2}$ then we can embed it in the shift on $\mathcal{B}(a)$.

For a discrete system (X,T) and a topological flow (Y,Φ) , we say that (X,T) is strongly embedded in (Y,Φ) if (X,T) is embedded in (Y,Φ_1) via h and h satisfies that if there exists $x,x' \in X$ and $r \in \mathbb{R}$ such that $\Phi_r(h(x)) = h(x')$ then $r \in \mathbb{Z}$ and $x' = T^r x$. Actually, it means that different \mathbb{Z} -orbits are embedded in different \mathbb{R} -orbits.

Building on Theorem 5.1, we show the following theorem.

Theorem 5.2 (=Theorem 1.2). If a topological dynamical system (X, T) satisfies the marker property and $\operatorname{mdim}(X, T) < \frac{a}{2}$ then we can strongly embed it in the shift on $\mathcal{B}(a)$.

Actually, the above result also holds for \mathbb{Z}^k -action by the same proof. Since we consider \mathbb{R} -action (as well as \mathbb{Z} -action) in our paper, we only give the proof for \mathbb{Z} -action in what follows.

The proof of Theorem 5.2 follows by the main strategy of Theorem 5.1. The key point of our proof is to make the parameter r_1 vary with respect to the parameter L (see Section 5.2). The proof itself of Theorem 5.1 is highly sophisticated and deeply technical. We will

not repeat several proofs of technical lemmas and refer to [GQT19] for detail.

5.1. **Tilling.** A collection $W = \{W_n\}_{n \in \mathbb{Z}^k}$ of sets is said to be a *tiling* of \mathbb{R}^k if $\bigcup_{n \in \mathbb{Z}^k} W_n = \mathbb{R}^k$ and the Lebesgue measure of $W_n \cap W_m$ vanishes for distinct $n, m \in \mathbb{Z}^k$. Moreover, it called a *convex tiling* of \mathbb{R}^k if W_n are convex for all $n \in \mathbb{Z}^k$.

Let (X,T) be a discrete topological system with marker property. Let M>0 be an integer. Then there exists an open set $U\subset X$ such that

$$U \cap T^{-n}U = \emptyset$$
 (for $0 < |n| < N$) and $X = \bigcup_{n \in \mathbb{Z}} T^n U$.

Then there exists an integer $M_1 > M$ and a compact set $F \subset U$ such that $X = \bigcup_{|n| < M_1} T^n F$. Choose a continuous map $\phi : X \to [0, 1]$ satisfying that $\operatorname{supp}(\phi) \subset U$ and h = 1 on F. Let

$$S(x) = \left\{ \left(n, \frac{1}{\phi(T^n x)} \right) : n \in \mathbb{Z}, \phi(T^n x) > 0 \right\}$$

be the discrete set in \mathbb{R}^2 . The Voronoi tiling $\mathcal{V}(x) = \{V_0(x,n)\}_{n \in \mathbb{Z}}$ is defined as follows: if $\phi(T^n x) = 0$ then $V_0(x,n) = \emptyset$; if $\phi(T^n x) > 0$ then

$$V_0(x,n) = \left\{ u \in \mathbb{R}^2 : \left| u - \left(n, \frac{1}{\phi(T^n x)} \right) \right| \le |u - p|, \forall p \in \mathcal{S}(x) \right\}.$$

Clearly, the sets $V_0(x,n)$ form a tiling of \mathbb{R}^2 . Let $\pi: \mathbb{R}^2 \to \mathbb{R}$ be the projection on the first coordinate. Let $H = (M_1 + 1)^2$ and

$$W_0(x,n) = \pi \left(V_0(x,n) \cap (\mathbb{R} \times \{-H\}) \right).$$

Then the sets $W_0(x, n)$ form a tiling of \mathbb{R} . Moreover, the tiling $\{W_0(x, n)\}_{n \in \mathbb{Z}}$ is \mathbb{Z} -equivariant, i.e. $W_0(T^n x, m) = -n + W_0(x, n + m)$.

Lemma 5.3 ([GQT19], Claim 5.11). Let $x \in X$ and $n \in \mathbb{Z}$ with $h(T^n x) > 0$. Then the following hold.

- (1) $V_0(x,n)$ contains the ball $B_{M/2}(n,1/h(T^nx))$.
- (2) $W_0(x,n)$ is contained in $B_{M_1+1}(n)$.
- (3) If $W_0(x, n) \neq \emptyset$ then $h(T^n x) > 1/2$.

Let $\epsilon > 0$. Pick $1 < c < \frac{1}{1-\epsilon}$. Let $M_2 = \frac{(c-1)H}{H+2}M$. Notice that $M_2 \approx (1-c^{-1})M$ as M_1 is sufficient large. Using the same method of [GQT19, Claim 5.12], we obtain the following lemma.

Lemma 5.4. Let $x \in X$ and $n \in \mathbb{Z}$ with $h(T^n x) > 0$ and $W_0(x, n) \neq \emptyset$. Then $W_0(x, n)$ contains the ball of radius M_2 .

5.2. Tiling-Like Band-Limited Map. Let a > 0 and $b = a + \delta/2$. Let L > 0. For r > 0 and $u \in \mathbb{C}$, we define $D_r(u)$ as the closed disk centered at u of radius r in \mathbb{C} . Define

$$\Theta_L: \mathbb{C} \to \mathbb{C}, z \mapsto e^{\pi i b z} \sin(\frac{\pi z}{L}).$$

Pick $r_1 = r_1(L)$ such that

- $0 < r_1 < \min\{\frac{1}{16}, \frac{1}{L}\}.$ For $|z| < r_1$, we have that

$$\pi \left| b \sin(\frac{\pi z}{L}) + \frac{1}{L} \cos(\frac{\pi z}{L}) \right| > \frac{3}{L}.$$

We remark here that the choice of r_1 is the key point in our proof. In [GQT19, Notation 5.3], they choose r_1 independent of L > 1. Here, we choose $r_1 \to 0$ as $L \to \infty$.

Let

$$\Omega = \{ z \in \mathbb{C} : |\mathrm{Im}(z)| < 1 \}.$$

Define

$$\theta_L = \min \left\{ \frac{9}{16L}, \inf\{|\Theta_L(z)| : z \in \Omega \setminus \bigcup_{n \in \mathbb{Z}} D_{r_1}(Ln)\} \right\}.$$

We choose a Schwartz function $\chi_1 : \mathbb{R} \to \mathbb{R}$ satisfying that

- supp $\mathcal{F}(\chi_1) \subset B_{\delta/8}(0)$.
- $\int_{\mathbb{D}} \chi_1(x) dx = 1$.

Let $E = E(L, \theta_L) > 0$ such that for all $z \in \Omega$, we have

$$\|(\Theta_L)_{|\Omega}\|_{\infty} \int_{\mathbb{R}\setminus B_E(\operatorname{Re}(z))} |\chi_1(z-t)| dt < \frac{\theta_L}{2}.$$

Recall that the tilling $\{W_0(x,n)\}_{n\in\mathbb{Z}}$ was defined in Section 5.1. For $x \in X$, define $\Phi(x) : \mathbb{C} \to \mathbb{C}$ as

$$\Phi(x)(z) = \sum_{n \in \mathbb{Z}} \Theta_L(z - n) \int_{W_0(x,n)} \chi_1(z - t) dt.$$

Lemma 5.5 ([GQT19], Lemma 5.9, Lemma 5.14). For $x \in X$, the function $\Phi(x)$ satisfies the following properties.

- (1) $\|\Phi(x)_{\mathbb{R}}\|_{\infty} \leq K_1$, where $K_1 = \int_{\mathbb{R}} |\chi_1(t)| dt$.
- (2) If $L > 4/\delta$ then the support of $\mathcal{F}(\Phi(x)_{\mathbb{R}})$ is contained in $(a/2, a/2 + \delta/2).$
- (3) $\Phi(x)$ is \mathbb{Z} -equivariant, i.e. $\Phi(T^n x)(z) = \Phi(x)(z+n)$.
- (4) If x_n converges to x in X then $\Phi(x_n)$ converges to $\Phi(x)$ uniformly over every compact subset of \mathbb{C} .
- (5) For $z \in \Omega$, if $Re(z) \in Int_E W_0(x, n)$ and $\Phi(x)(z) = 0$ then there exists $m \in \mathbb{Z}$ satisfying $z \in D_{r_1}(n+Lm)$, where $Int_EW_0(x,n) =$ $\{y \in W_0(x,n) : B_E(y) \in W_0(x,n)\}.$
- (6) If $n, m \in \mathbb{Z}$ with $n + Lm \in Int_{E+1}W_0(x, n)$, then there exists $z \in D_{r_1}(n + LM)$ such that $\Phi(x)(z) = 0$.
- 5.3. Main proposition. As in [GQT19], we pick L sufficient large (i.e. it satisfies [GQT19, Condition 5.13]) and M large (i.e. it satisfies [GQT19, Equation (5.9)]). Moreover, we assume $M_2 = M_2(M)$ satisfies the following condition.

Condition 5.6. $M_2 > 4L + E + 1$.

Combing this condition with Lemma 5.4 and Lemma 5.5 (5) & (6), we obtain the following lemma.

Lemma 5.7. If $W(x,n) \neq \emptyset$ then

- there exists $m \in \mathbb{Z}$ such that $n+Lm, n+L(m+1) \in Int_EW_0(x,n)$;
- there exists $z_1 \in D_{r_1}(n + Lm)$ and $z_2 \in D_{r_1}(n + L(m+1))$ such that $\Phi(x)(z_1) = \Phi(x)(z_2) = 0$;
- $\Phi(x)(z) \neq 0$ for all z with $Re(z) \in (n + Lm + r_1, n + L(m + 1) r_1)$.

The main proposition is shown as follows. We remark that comparing to [GQT19, Proposition 3.1], the statement (3) in the following proposition is crucial to prove Theorem 5.2.

Proposition 5.8. Let (X,T) be a dynamical system satisfying the marker property and $mdim(X,T) < \frac{a}{2}$. Let $f: X \to \mathcal{B}(a)$ be a \mathbb{Z} -equivalent continuous map. There exists a \mathbb{Z} -equivalent continuous map $g: X \to \mathcal{B}(a+\delta)$ such that

- (1) $||g(x) f(x)|| < \delta \text{ for all } x \in X.$
- (2) g is a (d, δ) -embedding.
- (3) If there exists $x, y \in X$ and $r \in [-1/2, 1/2]$ such that $\tau_r(g(y)) = g(x)$ then $|r| \leq 2r_1$.

Proof. (1) and (2) follows by [GQT19, Proposition 3.1] where

$$g(x) = g_1(x) + g_2(x)$$

satisfying that $g_2(x) = \frac{\delta}{2K_1} \operatorname{Re}(\Phi(x)_{|\mathbb{R}})$ and $\operatorname{supp}(\mathcal{F}(g_1(x))) \cap \operatorname{supp}(\mathcal{F}(g_2(x))) = \emptyset$. It remains to prove (3). Notice that if g(x) = g(y) then $g_2(x) = g_2(y)$ implying that $\Phi(x) = \Phi(y)^1$. Thus (3) follows by Lemma 5.9.

Lemma 5.9. If there exists $x, y \in X$ and $r \in [-1/2, 1/2]$ such that $\tau_r(\Phi(y)) = \Phi(x)$ then $|r| \leq 2r_1$.

Proof. Without loss of generality, we assume $r \geq 0$. Due to Lemma 5.4 and Condition 5.6, we see that if $W_0(x, n_x) \cap W_0(y, m) \neq \emptyset$ and $W_0(y, m_1)$, $W_0(y, m)$ and $W_0(y, m_2)$ are successive nonempty intervals then there exists $m_y \in \{m_1, m, m_2\}$ satisfying that

$$(5.1) |W_0(x, n_x) \cap W_0(y, m_y)| \ge \frac{2M_2 - 2E}{2} > 4L.$$

Let $n \in \mathbb{Z}$ with $W_0(x,n) \neq \emptyset$. By Condition 5.6 and Equation (5·1), there exists $n_y \in \mathbb{Z}$ such that $|W_0(x,n) \cap W_0(y,m_y)| > 4L$. Let m, z_1 be as Lemma 5.7, that is, $\Phi(x)(z_1) = 0$ and $z_1 \in D_{r_1}(n + Lm)$, and satisfy that $[n + Lm - r_1, n + L(m+1) + r_1] \subset W_0(y, n_y)$. If $r \in (2r_1, 1)$ then

$$\Phi(y)(z_1 + r) = \Phi(x)(z_1) = 0.$$

¹The supports of $\mathcal{F}(\Phi(x))$ and $\mathcal{F}(\overline{\Phi(x)})$ are distinct.

However $D_{r_1}(z_1+r) \subset D_{2r_1}(n+Lm+r)$ which doesn't contain the integer points on real axis. This is a contradiction to Lemma 5.5 (5) (with respect to the tiling $\{W_0(y,n)\}_{n\in\mathbb{Z}}$). Therefore we conclude that $r \leq 2r_1$.

5.4. **Proof of Theorem 5.2.** We first show the lemma which roughly says that the collection of functions satisfying Proposition 5.8 (3) is open, and then use this lemma to prove Theorem 5.2.

Lemma 5.10. Let a > 1 and (X,T) be a discrete dynamical system. Let $r_1, \eta > 0$. Let $g: X \to \mathcal{B}(a)$ be a \mathbb{Z} -equivariant continuous map satisfying that if there exists $x, y \in X$ and $r \in [-1/2, 1/2]$ such that $\tau_r(g(y)) = g(x)$ then $|r| \leq 2r_1$. Then there exists $\epsilon > 0$ such that for any \mathbb{Z} -equivariant continuous map $h: X \to \mathcal{B}(a + \eta)$ with $\sup_{x \in X} \|h(x) - g(x)\|_{\mathbb{R},\infty} < \epsilon$ if there exists $x, y \in X$ and $r \in [-1/2, 1/2]$ such that $\tau_r(h(y)) = h(x)$ then $|r| < 3r_1$.

Proof. We prove it by contradiction. Assume there exist $r_n \in [-1/2, -3r_1] \cup [3r_1, 1/2]$, \mathbb{Z} -equivariant continuous maps $h_n : X \to \mathcal{B}(a+\eta)$ and points $x_n, y_n \in X$ such that $x_n \to x, y_n \to y, r_n \to r \in [-1/2, -3r_1] \cup [3r_1, 1/2]$ as $n \to \infty$,

$$\tau_{r_n}(h_n(y_n)) = h_{r_n}(x_n), \forall n \in \mathbb{N},$$

and

$$\sup_{x \in X} \|h_n(x) - g(x)\|_{\mathbb{R}, \infty} < \frac{1}{n}, \forall n \in \mathbb{N}.$$

Taking n tend to ∞ , we obtain that

$$\tau_r(g(y)) = g(x),$$

which is a contradiction.

Now we prove the main result in this section.

Proof of Theorem 5.2. Without loss of generality, we assume diam(X, d) < 1. Pick a strictly increasing sequence $\{a_i\}_{i\in\mathbb{N}}$ such that

$$0 < a_i < a \text{ and } \operatorname{mdim}(X) < \frac{a_i}{2}, \forall i \in \mathbb{N}.$$

We will inductively define positive numbers ϵ_n , r_n and \mathbb{Z} -equivariant continuous (d, 1/n)-embedding map $h_n : X \to \mathcal{B}(a_n)$ such that if there exists $x, y \in X$ and $r \in [-1/2, 1/2]$ such that $\tau_r(h_n(y)) = h_n(x)$ then $|r| \leq 2r_n$. For n = 1, we define

$$\epsilon_1 = 1, h_1(x) = 0 (\forall x \in X) \text{ and } r_1 = \frac{1}{4}.$$

Clearly, h_1 is a (d, 1)-embedding map and satisfies that if there exists $x, y \in X$ and $r \in [-1/2, 1/2]$ such that $\tau_r(h_1(y)) = h_1(x)$ then $|r| \le 1/2$. Suppose ϵ_n, r_n and h_n are well defined. Since h_n is a (d, 1/n)-embedding map, there exists $0 < \epsilon'_n < \epsilon_n/2$ such that if a \mathbb{Z} -equivariant continuous map $h: X \to \mathcal{B}(a)$ satisfies $\sup_{x \in X} \|h_n(x) - h(x)\|_{\infty} < \epsilon'_n$,

then h is also a (d, 1/n)-embedding map. Moreover, by Lemma 5.10, there exists $\epsilon_n'' > 0$ such that for any \mathbb{Z} -equivariant continuous map $h: X \to \mathcal{B}(a)$ with $\sup_{x \in X} \|h_n(x) - h(x)\|_{\mathbb{R},\infty} < \epsilon_n''$ if there exists $x, y \in X$ and $r \in [-1/2, 1/2]$ such that $\tau_r(h(y)) = h(x)$ then $|r| < 3r_n$. Let $\epsilon_{n+1} = \min\{\epsilon_n', \epsilon_n''\}$. Let

$$\delta = \min \left\{ \frac{1}{n+1}, \frac{\epsilon_{n+1}}{2}, a_{n+1} - a_n \right\}.$$

We choose a large number L satisfying Condition 5.6 and $r_{n+1} = r_{n+1}(L)$ as in Section 5.2 with $0 < r_{n+1} < r_n/2$. By Proposition 5.8, we can find a \mathbb{Z} -equivariant continuous map $h_{n+1}: X \to \mathcal{B}(a_n)$ such that

- (1) $||g(x) f(x)||_{\infty} < \delta$ for all $x \in X$.
- (2) g is a (d, δ) -embedding.
- (3) If there exists $x, y \in X$ and $r \in [-1/2, 1/2]$ such that $\tau_r(g(y)) = g(x)$ then $|r| \leq 2r_{n+1}$.

Therefore, ϵ_{n+1}, r_{n+1} and h_{n+1} are constructed. We will show that the limit of h_n exists and is what we desire. A simple computation shows that for $n > m \ge 1$ and $x \in X$,

$$||h_n(x) - h_m(x)||_{\infty} \le \sum_{\ell=m}^{n-1} ||h_{\ell}(x) - h_{\ell+1}(x)||_{\infty}$$
$$< \sum_{\ell=m}^{n-1} \frac{\epsilon_{\ell+1}}{2} < \sum_{\ell=1}^{\infty} 2^{\ell} \epsilon_{m+1} = \epsilon_{m+1},$$

which tends to 0 as $m \to \infty$. It follows that $\{h_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence and thus the limit exists, say $h: X \to \mathcal{B}(a)$. Furthermore, it is \mathbb{Z} -equivariant continuous and satisfies that

$$\sup_{x \in X} \|h_n(x) - h(x)\|_{\infty} < \epsilon_n, \forall n \in \mathbb{N}.$$

Therefore, by the definition of ϵ_n , we obtain that h is a (d, 1/n)-embedding and satisfies that if there exists $x, y \in X$ and $r \in [-1/2, 1/2]$ such that $\tau_r(h(y)) = h(x)$ then $|r| < 3r_n$ for all $n \in \mathbb{N}$. It follows that h is an embedding and if there exists $x, y \in X$ and $r \in [-1/2, 1/2]$ such that $\tau_r(h(y)) = h(x)$ then r = 0. It remains to show that h is a strongly embedding. Suppose there exists $x, y \in X$ and $r \in \mathbb{R}$ such that $\Phi_r(h(y)) = h(x)$. Let

$$m \in \{k \in \mathbb{Z} : |r - k| \le |r - n|, \forall n \in \mathbb{Z}\}.$$

Then $|r-m| \leq [-1/2, 1/2]$. It follows that

$$h(x) = \tau_r(h(y)) = \tau_{r-m}(h(T^m y)),$$

implying that r = m and consequently $x = T^m y$. This completes the proof.

6. Embedding Theorem for Suspension flow

In this section, we study the embedding of suspension flows. The main result is as follows.

Theorem 6.1. Let (Z,T) be a discrete topological system. Let $f: Z \to \{1\}$ be the constant function. Let a > 0. Then the discrete topological system (Z,T) can be strongly embedded in B(a) if and only if the suspension flow (Z_f,T_f) can be embedded in B(a).

Proof. Suppose $H: Z_f \to \mathcal{B}(a)$ is an embedding. Then it is easy to check that (Z,T) can be strongly embedded in $\mathcal{B}(V[a,b])$ via $H_{|Z}: Z \to \mathcal{B}(a)$.

Now suppose $h: Z \to \mathcal{B}(a)$ is a strong embedding. Define $h_f: Z_f \to \mathcal{B}(a)$ by $h_f(x,t) = \tau_t(h(x))$. We claim that h_f is an embedding. In fact, if there exists two distinct points $(x,t), (y,s) \in Z_f$ such that $h_f(x,t) = h_f(y,s)$, then we have

$$\tau_t(h(x)) = \tau_s(h(y)).$$

It follows that $t - s \in \mathbb{Z}$ and $T^{t-s}x = y$, implying that t = s and x = y. This is a contradiction. we complete the proof.

We remark that Theorem 6.1 may fail when f is not the constant function 1. This is mainly because Proposition 4.3 may not hold for such case, that is, the mean dimension of suspension flow under non-constant function may no longer equal the one of discrete topological flow.

Applying Theorem 1.2 to Theorem 6.1, we have the following corollary.

Corollary 6.2. Let (Z,T) be a discrete topological system. Suppose that (Z,T) satisfies the marker property and has $mdim(Z,T) < \frac{a}{2}$. Let $f: Z \to \{1\}$ be the constant function. Then the suspension flow (Z_f, T_f) can be embedded in $\mathcal{B}(a)$.

Now we have a new proof of [GJ20, Theorem 5.1] as follows.

Corollary 6.3 (=Corollary 1.3). Let (X, Φ) be an extension of the solenoid. Suppose that $\operatorname{mdim}(X, \Phi) < a/2$. Then (X, Φ) can be embedded in $\mathcal{B}(a)$.

Proof. By Proposition 3.1, (X, Φ) is topologically conjugate to a suspension flow (Z_f, T_f) for some discrete topological system (Z, T) where f is the constant function 1. By Proposition 4.3, we see that $\operatorname{mdim}(Z, T) = \operatorname{mdim}(X, \Phi) < b - a$. By remark at the end of Section 3, (Z, T) is an extension of nontrivial minimal system and thus satisfies marker property. By Theorem 1.2, (Z, T) can be strongly embedded in $\mathcal{B}(a)$. It follows from Theorem 6.1 that (X, Φ) can be embedded in $\mathcal{B}(a)$.

7. Further discussion

In this section, we discuss several open problems. Lindenstrauss and Tsukamoto [LT14] conjectured that

Conjecture 7.1. Let (X,T) be a topological dynamical system. Suppose that $\operatorname{mdim}(X,T) < \frac{d}{2}$ and

$$\frac{\dim(\{x: T^n x = x\})}{n} < \frac{d}{2} \text{ for all } n \ge 1.$$

Then there is an embedding from (X,T) into $(([0,1]^d)^{\mathbb{Z}},\sigma)$.

This conjecture holds generically [GQS18, Appendix A], but it is still widely open in general. Since there is the embedding from $(\mathcal{B}(a), \mathbb{Z})$ to $(([0,1]^d)^{\mathbb{Z}}, \sigma)$ for a < d by sampling theory (see also [GT20, Lemma 1.5]), Gutman, Qiao and Tsukamoto [GQT19] conjectured that

Conjecture 7.2. Let (X,T) be a topological dynamical system. Suppose that $\operatorname{mdim}(X,T) < \frac{a}{2}$ and

$$\frac{\dim(\{x: T^n x = x\})}{n} < \frac{a}{2} \text{ for all } n \ge 1.$$

Then there is an embedding from (X,T) into $\mathcal{B}(a)$.

We have introduced the strong embedding in the current paper. We then conjecture that

Conjecture 7.3. Let (X,T) be a topological dynamical system. Suppose that $\operatorname{mdim}(X,T) < \frac{a}{2}$ and

$$\frac{\dim(\{\mathbf{x}: \mathbf{T}^n\mathbf{x} = \mathbf{x}\})}{n} < \frac{a}{2} \text{ for all } n \ge 1.$$

Then there is a strong embedding from (X,T) into $\mathcal{B}(a)$.

The relation of above conjectures is as follows:

Conjecture
$$7.3 \Rightarrow$$
 Conjecture $7.2 \Rightarrow$ Conjecture 7.1 .

Finally, using the same idea in [GQS18, Appendix A], we can show that the conjecture 7.3 holds generically (but it is still open in general).

Theorem 7.4. The conjecture 7.3 holds generically.

Proof. Due to [Hoc08, Corollary 3.6] and [GQS18, Appendix A], the zero-dimensional aperiodic system is generic. Since the zero-dimensional aperiodic system has mean dimension zero and satisfies marker property (see [Dow06]), by Theorem 1.2, we obtain that Conjecture 7.3 holds generically.

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