Lab 5: Power Spectral Density, Noise, and Symbol Timing Information

1 Introduction

The two concepts that are most fundamental to the realistic modeling of communication systems are the randomness of the source signal or message to be transmitted and the constraints imposed by the communication channel. A channel constraint that may be either random or deterministic or part of both is the impulse response and the bandwidth of the channel. Average or peak power limitation is regarded as another channel constraint, even though it may actually be imposed by transmitter design considerations and/or FCC (Federal Communications Commission) rules. A third component, that almost always needs to be modeled as a random process, is noise that is present in all communication systems. Although some of the noise in a received signal may in reality come from the front end of the receiver, noise is generally attributed to the channel. Together with the power limitation of the transmitted signal, it forms a constraint in terms of the signal-to-noise ratio (SNR) that the receiver has to work with.

1.1 Energy and Power

The **energy** of a (possibly complex-valued) CT signal $x(t)$ in the time interval $t_1 \le t < t_2$ is given by

$$E_x(t_1, t_2) = \int_{t_1}^{t_2} x(t)\, x^*(t)\, dt = \int_{t_1}^{t_2} |x(t)|^2\, dt\,,$$

where $x^*(t)$ is the complex conjugate of $x(t)$. The quantity $|x(t)|^2$ is called the **instantaneous power** of $x(t)$ at time t. The **average power** of $x(t)$ in $t_1 \le t < t_2$ is given by

$$P_x(t_1, t_2) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2\, dt\,.$$

For theoretical computations, the total energy and average power of $x(t)$ for all times $-\infty < t < \infty$ are important quantities and defined as (if they exist)

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt\,, \qquad \text{and} \qquad P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt\,.$$

Note that if $x(t)$ is periodic with period T_1, then the computation of the average power simplifies to

$$P_x = \frac{1}{T_1} \int_{T_1} |x(t)|^2\, dt\,,$$

where the integration is taken over one full period of $x(t)$, starting at any convenient point in time.

For a DT sequence x_n, energy and power in the time (or index) interval $n_1 \le n < n_2$ are defined as

$$E_x[n_1, n_2] = \sum_{n=n_1}^{n_2-1} |x_n|^2 \,, \qquad \text{and} \qquad P_x[n_1, n_2] = \frac{1}{n_2 - n_1} \sum_{n=n_1}^{n_2-1} |x_n|^2 \,,$$

and, for $-\infty < n < \infty$, as (if they exist)

$$E_x = \sum_{n=-\infty}^{\infty} |x_n|^2 \,, \qquad \text{and} \qquad P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_n|^2 \,.$$

The quantity $|x_n|^2$ is the instantaneous power of x_n at index n.

If the DT sequence x_n is regarded as an approximation for the CT waveform $x(t)$, sampled with rate F_s, then

$$E_x(t_1, t_2) = \int_{t_1}^{t_2} |x(t)|^2 \, dt \approx \frac{1}{F_s} \sum_{n=n_1}^{n_2-1} |x_n|^2 = \frac{E_x[n_1, n_2]}{F_s} \,, \qquad \text{and}$$

$$P_x(t_1, t_2) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 \, dt \approx \frac{1}{(t_2 - t_1)\, F_s} \sum_{n=n_1}^{n_2-1} |x_n|^2 \approx P_x[n_1, n_2] \,,$$

if $n_1 \approx t_1 \, F_s$ and $n_2 \approx t_2 \, F_s$, and F_s is chosen large enough for the given $x(t)$.

1.2 Parseval's Identity

For deterministic signals energy or power in the time domain is related to energy or power in the frequency domain through **Parseval's identity** in its various forms:

FS:
$$\frac{1}{T_1} \int_{T_1} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |X_k|^2 \,, \qquad X_k = \frac{1}{T_1} \int_{T_1} x(t) \, e^{-j2\pi kt/T_1} \, dt$$

FT:
$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df \,, \qquad X(f) = \int_{-\infty}^{\infty} x(t) \, e^{-j2\pi ft} \, dt$$

DTFT:
$$\sum_{n=-\infty}^{\infty} |x_n|^2 = \int_1 |X(\phi)|^2 \, d\phi \,, \qquad X(\phi) = \sum_{n=-\infty}^{\infty} x_n \, e^{-j2\pi \phi n}$$

DFT:
$$\sum_{n=0}^{N-1} |x_n|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X_k|^2 \,, \qquad X_k = \sum_{n=0}^{N-1} x_n \, e^{-j2\pi kn/N}$$

1.3 ESD for Deterministic Signals

For energy signals (i.e., signals with finite energy), the distribution of energy in the frequency domain is of interest. The following definitions apply to deterministic signals.

Definition: The (normalized) **energy spectral density (ESD)** of a CT signal $x(t)$ is defined as (if it exists)

$$|X(f)|^2 \,.$$

The units of $|X(f)|^2$ are volts^2-sec/Hz if $x(t)$ is a voltage signal.

Definition: The **correlation function** between two CT signals $v(t)$ and $w(t)$ is defined as (if it exists)

$$\psi_{vw}(\tau) = \int_{-\infty}^{\infty} v(t+\tau)\, w^*(t)\, dt \,.$$

If $w(t) = v(t)$, then $\psi_{vv}(\tau)$ is called **autocorrelation function**, otherwise it is called **cross-correlation function**.

Theorem: Energy spectral density and autocorrelation for CT signals are related through the FT (if it exists)

$$|X(f)|^2 = \Psi_{xx}(f) = \int_{-\infty}^{\infty} \psi_{xx}(\tau)\, e^{-j2\pi f\tau}\, d\tau \,.$$

Definition: The **energy spectral density (ESD)** of a DT signal x_n is defined as (if it exists)

$$|X(\phi)|^2 \,.$$

Note that this is periodic in the normalized frequency ϕ with period 1.

Definition: The **correlation function** between two DT signals v_n and w_n is defined as (if it exists)

$$\psi_{vw}[m] = \sum_{n=-\infty}^{\infty} v_{n+m}\, w_n^* \,.$$

If $w_n = v_n$, then $\psi_{vv}[m]$ is called **autocorrelation function**, otherwise it is called **cross-correlation function**.

Theorem: Energy spectral density and autocorrelation for DT signals are related through the DTFT (if it exists)

$$|X(\phi)|^2 = \Psi_{xx}(\phi) = \sum_{m=-\infty}^{\infty} \psi_{xx}[m]\, e^{-j2\pi\phi m} \,.$$

1.4 PSD for Deterministic Signals

For power signals (i.e., signals with finite power), the distribution of power in the frequency domain is of interest. The following definitions are for deterministic signals.

Definition: The (normalized) **power spectral density (PSD)** of a CT signal $x(t)$ is defined as (if it exists)

$$S_x(f) = \lim_{T\to\infty} \frac{1}{2T}\, |X_T(f)|^2 , \qquad \text{where} \quad X_T(f) = \int_{-T}^{T} x(t)\, e^{-j2\pi f t}\, dt \,.$$

For a voltage signal $x(t)$ the units of $S_x(f)$ are volts^2/Hz. If $x(t)$ is only available in the interval $t_1 \le t < t_2$, then the PSD is defined as

$$S_{x(t_1,t_2)}(f) = \frac{1}{t_2 - t_1}\, |X_{(t_1,t_2)}(f)|^2 , \qquad \text{where} \quad X_{(t_1,t_2)}(f) = \int_{t_1}^{t_2} x(t)\, e^{-j2\pi f t}\, dt \,.$$

To keep the notation simple, $S_x(f)$ will be used instead of $S_{x(t_1,t_2)}(f)$ if it is clear from the context that $x(t)$ is only available for $t_1 \le t < t_2$.

Definition: If $x(t)$ is a periodic CT signal with period T_1, then its FT consists of impulses and its PSD is defined as

$$S_x(f) = \sum_{k=-\infty}^{\infty} |X_k|^2\, \delta\big(f - \frac{k}{T_1}\big) , \qquad \text{where} \quad X_k = \frac{1}{T_1} \int_{T_1} x(t)\, e^{-j2\pi k t/T_1}\, dt \,.$$

To check that this is indeed the right definition, integrate $S_x(f)$ over all f to obtain

$$\begin{aligned}\int_{-\infty}^{\infty} S_x(f)\, df &= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |X_k|^2\, \delta\big(f - \frac{k}{T_1}\big)\, df = \sum_{k=-\infty}^{\infty} |X_k|^2 \underbrace{\int_{-\infty}^{\infty} \delta\big(f - \frac{k}{T_1}\big)\, df}_{=1} \\ &= \sum_{k=-\infty}^{\infty} |X_k|^2 = \frac{1}{T_1} \int_{T_1} |x(t)|^2\, dt = P_x \,,\end{aligned}$$

where the last equality follows from Parseval's identity for the FS.

Definition: The **power spectral density (PSD)** of a DT signal x_n is defined as (if it exists)

$$S_x(\phi) = \lim_{N\to\infty} \frac{|X_N(\phi)|^2}{2N+1} , \qquad \text{where} \quad X_N(\phi) = \sum_{n=-N}^{N} x_n\, e^{-j2\pi\phi n} \,.$$

Note that $S_x(\phi)$ is periodic in the normalized frequency ϕ with period 1. If x_n is only available in the interval $n_1 \le n < n_2$, then the PSD is defined as

$$S_{x[n_1,n_2]}(\phi) = \frac{1}{n_2 - n_1}\, |X_{[n_1,n_2]}(\phi)|^2 , \qquad \text{where} \quad X_{[n_1,n_2]}(\phi) = \sum_{n=n_1}^{n_2-1} x_n\, e^{-j2\pi\phi n} \,.$$

To keep the notation simple, $S_x(\phi)$ will be used instead of $S_{x[n_1,n_2]}(\phi)$ if it is clear from the context that x_n is only available for $n_1 \le n < n_2$.

Approximation for $S_x(f)$ Using DFT/FFT. Let $x_n = x(nT_s)$, $0 \le n < N$ be the DT sequence that is obtained from sampling the CT waveform $x(t)$ at times $t = nT_s = n/F_s$. Assume that the sampling rate F_s was chosen large enough and the frequency resolution F_s/N was chosen small enough so that aliasing effects due to sampling are negligible. Then

$$X\Big(\frac{kF_s}{N}\Big) \approx \frac{X_k}{F_s} \,,$$

where $X(kF_s/N)$ is the (approximate) FT of $x(t)$ and X_k are the DFT coefficients of x_n, $0 \le n < N$. Thus, the DFT (or FFT) can be used to approximate the PSD $S_x(f)$ of $x(t)$ at $f = kF_s/N$ as

$$S_x\big(\frac{kF_s}{N}\big) = \frac{|X(kF_s/N)|^2}{N/F_s} \approx \frac{|X_k/F_s|^2}{N/F_s} = \frac{|X_k|^2}{NF_s} \,.$$

Quite often x_n, $0 \le n < N$, is split up into $m = \lfloor N/N' \rfloor > 1$ blocks of length $N' < N$. The PSDs of these blocks are then computed individually and the average over the m PSDs is displayed as the final result. Assuming that x_n and $x_{n-iN'}$, $i = 1, 2, \ldots$, have similar statistical properties and are approximately independent of each other, this yields results similar to the ones obtained from averaging over several independent but statistically identical systems.

1.5 Introduction to Random Processes

Power spectral density (PSD) for deterministic signals is of interest mainly for the analysis and the display of an observed signal with finite duration. For almost all theoretical considerations and analytical computations, communications signals have to be modeled as random processes. The main reason for this is that, by definition, deterministic signals can be predicted and thus there is no need for communication.

A CT random process signal is usually a **power signal**, i.e., it has finite power, but infinite energy. But since it is not periodic, its FT is not of the form of a train of impulses multiplied by FS coefficients. In fact, the FT of a random power signal may not even exist, and thus another approach has to be taken to determine the PSD of random processes.

Let $x(t)$ be a **CT random process** and define the random variables $x_i = x(t_i)$ for $i = 1, 2, \ldots, k$. Then the joint **cdf (cumulative distribution function)** $F_{\mathbf{x}}(\alpha)$ and joint **pdf (probability density function)** $f_{\mathbf{x}}(\alpha)$ of $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ are defined as (if they exist)

$$F_{x_1,x_2,\ldots,x_k}(\alpha_1, \alpha_2, \ldots, \alpha_k) = Pr\{x_1 \le \alpha_1, x_2 \le \alpha_2, \ldots, x_k \le \alpha_k\} \,,$$

$$f_{x_1,x_2,\ldots,x_k}(\alpha_1, \alpha_2, \ldots, \alpha_k) = \frac{\partial^k F_{x_1,x_2,\ldots,x_k}(\alpha_1, \alpha_2, \ldots, \alpha_k)}{\partial\alpha_1\, \partial\alpha_2\, \cdots\, \partial\alpha_k} \,.$$

A random process is considered to be completely specified if $F_{x_1,\ldots,x_k}(\alpha_1, \ldots, \alpha_k)$ can (in principle at least) be constructed for any finite set of time instants $t_1, \ldots, t_k$.

Random processes are often characterized in terms of averages of one or more random variables. Of particular interest are the first and second moments of a single random variable, which are defined as follows.

Definition: The **first** and **second moments** (or the mean and the average power) of a CT random process $x(t)$ at (fixed) time t_1 are the quantities

$$m_x(t_1) = E[x(t_1)] = \int_{-\infty}^{\infty} \alpha \, f_{x(t_1)}(\alpha) \, d\alpha \,,$$

$$P_x(t_1) = E[|x(t_1)|^2] = \int_{-\infty}^{\infty} |\alpha|^2 \, f_{x(t_1)}(\alpha) \, d\alpha \,.$$

A related quantity is the **variance** or the second central moment of $x(t)$ at time t_1, defined as

$$\sigma_x^2(t_1) = E[|x(t_1) - m_x(t_1)|^2] \,.$$

Among the averages over pairs of random variables the correlation functions are important, with particular emphasis on the autocorrelation function as defined next.

Definition: The **(auto)correlation function** of a possibly complex-valued CT random process $x(t)$ at (fixed) times t_1, t_2 is defined as

$$R_x(t_1, t_2) = E[x(t_1)x^*(t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha_1 \alpha_2^* \, f_{x(t_1),x(t_2)}(\alpha_1, \alpha_2) \, d\alpha_1 d\alpha_2 \,.$$

The second moment or power at time t_1 is related to the autocorrelation function by

$$P_x(t_1) = R_x(t_1, t_1) = E[|x(t_1)|^2] \,.$$

In general the statistical properties of a random process depend on the absolute values of the time instants $t_1, t_2, \ldots, t_k$. A process for which

$$F_{x(t_1),x(t_2),\ldots,x(t_k)}(\alpha_1, \alpha_2, \ldots \alpha_k) = F_{x(t_1+T),x(t_2+T),\ldots,x(t_k+T)}(\alpha_1, \alpha_2, \ldots \alpha_k) \,, \quad \text{all } T \,,$$

is called **strict-sense stationary**. To obtain results that are practical on one hand and mathematically tractable on the other hand, the following much less restrictive form of stationarity is commonly used in communications.

Definition: A CT random process $x(t)$ for which

$$m_x(t_1) = m_x(t_1 + T), \quad \text{and} \quad R_x(t_1, t_2) = R_x(t_1 + T, t_2 + T) \,,$$

for all T is said to be a **wide-sense stationary (WSS)** process. In this case

$$m_x(t_1) = m_x \,, \quad \text{and} \quad R_x(t_1, t_2) = R_x(t_1 - t_2, 0) = R_x(t_1 - t_2) \,.$$

The definitions for DT random processes are very similar. The main difference is that random variables can only be defined for discrete time instants, and not everywhere along the time axis as for CT processes. Let $x[n]$ be a **DT random process** and define the random variables $x_i = x[n_i]$ for $i = 1, 2, \ldots, k$. Then the joint cdf and pdf of $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ are defined as (if they exist)

$$F_{x_1,x_2,\ldots,x_k}(\alpha_1, \alpha_2, \ldots, \alpha_k) = Pr\{x_1 \leq \alpha_1, x_2 \leq \alpha_2, \ldots, x_k \leq \alpha_k\} ,$$

$$f_{x_1,x_2,\ldots,x_k}(\alpha_1, \alpha_2, \ldots, \alpha_k) = \frac{\partial^k F_{x_1,x_2,\ldots,x_k}(\alpha_1, \alpha_2, \ldots, \alpha_k)}{\partial\alpha_1\, \partial\alpha_2\, \cdots\, \partial\alpha_k} .$$

The first and second moments and the autocorrelation function are given in the next two definitions.

Definition: The **first** and **second moments** (or the mean and the average power) of a DT random process $x[n]$ at (fixed) time index n_1 are the quantities

$$m_x[n_1] = E\big[x[n_1]\big] = \int_{-\infty}^{\infty} \alpha\, f_{x[n_1]}(\alpha)\, d\alpha ,$$

$$P_x[n_1] = E\big[|x[n_1]|^2\big] = \int_{-\infty}^{\infty} |\alpha|^2\, f_{x[n_1]}(\alpha)\, d\alpha .$$

Definition: The **(auto)correlation function** of a possibly complex-valued DT random process $x[n]$ at (fixed) time indexes n_1, n_2 is defined as

$$R_x[n_1, n_2] = E\big[x[n_1]x^*[n_2]\big] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha_1\alpha_2^*\, f_{x[n_1],x[n_2]}(\alpha_1, \alpha_2)\, d\alpha_1 d\alpha_2 .$$

Related quantities are

$$\text{Variance at time index } n_1: \qquad \sigma_x^2[n_1] = E\big[|x[n_1] - m_x[n_1]|^2\big]$$

$$\text{Power at time index } n_1: \qquad R_x[n_1, n_1] = P_x[n_1] = E\big[|x[n_1]|^2\big] .$$

The definition of a WSS DT process is

Definition: A DT random process $x[n]$ for which

$$m_x[n_1] = m_x[n_1 + N], \quad \text{and} \quad R_x[n_1, n_2] = R_x[n_1 + N, n_2 + N] ,$$

for all N is said to be a **wide-sense stationary (WSS)** process. In this case

$$m_x[n_1] = m_x , \quad \text{and} \quad R_x[n_1, n_2] = R_x[n_1 - n_2, 0] = R_x[n_1 - n_2] .$$

1.6 PSD for Random Processes

The reason why WSS processes play an eminent role in communications and signal processing is because the autocorrelation function of such processes is related to the PSD of the process through the following

Theorem: Wiener-Khintchine. The autocorrelation function R_x and the PSD S_x for CT and DT random processes $x(t)$ and $x[n]$, respectively, are related by the following Fourier transform pairs

$$\textbf{CT:} \qquad R_x(\tau) = E[x(t)\, x^*(t-\tau)] \iff S_x(f) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j2\pi f \tau}\, d\tau \ ,$$

$$\textbf{DT:} \qquad R_x[m] = E[x_n\, x^*_{n-m}] \iff S_x(\phi) = \sum_{m=-\infty}^{\infty} R_x[m]\, e^{-j2\pi\phi m} \ .$$

The next statement defines a subset of WSS processes with a special property in the frequency domain.

Definition: A WSS process is said to be **white** if its mean is zero and its PSD is constant for all frequencies, i.e., if $m_x = 0$ and (for CT and DT processes, respectively)

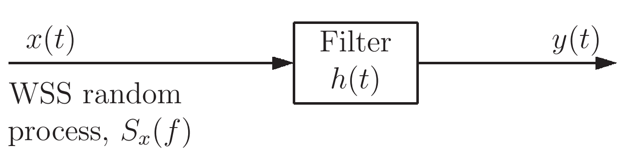
$$S_x(f) = \frac{\mathcal{N}_0}{2} \Leftrightarrow R_x(\tau) = \frac{\mathcal{N}_0}{2}\, \delta(\tau) \ , \quad \text{or} \quad S_x(\phi) = \frac{\mathcal{N}_0}{2T_s} \Leftrightarrow R_x[m] = \frac{\mathcal{N}_0}{2T_s}\, \delta_m \ .$$

Note that a white CT process $x(t)$ is a theoretical concept that cannot exist in practice since it would require infinite power. However, at the output of a filter with finite passband, e.g., an ideal LPF with cutoff frequency f_L, the power contribution from a white process becomes finite, e.g., $2f_L\, \mathcal{N}_0/2 = f_L\, \mathcal{N}_0$, and thus practical.

If $x(t)$ or $x[n]$ is a Gaussian process (i.e., the joint pdf of any k observation instants is jointly Gaussian), with zero mean and a constant non-zero PSD for all frequencies, then $x(t)$ or $x[n]$ is called **white Gaussian noise**. A common noise model used for communication systems is **additive white Gaussian noise (AWGN)**. For practical modeling the implicit assumption is that a communication receiver always has a filter at the front end, and the noise only has to be white in the passband of this filter. This ensures that the effect of white noise can be modeled without the need for infinite power.

1.7 PSD of Filtered WSS Process

Let $x(t)$ be a WSS CT random process which is passed through a linear and time-invariant (LTI) system with response $h(t) \Leftrightarrow H(f)$ as shown below.



The output

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\mu)\, x(t-\mu)\, d\mu \;,$$

is a new random process with

$$\begin{aligned} R_y(t_1,t_2) = E[y(t_1)\, y^*(t_2)] &= E\Big[\int_{-\infty}^{\infty} h(\mu)\, x(t_1-\mu)\, d\mu \int_{-\infty}^{\infty} h^*(\nu)\, x^*(t_2-\nu)\, d\nu \Big] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\mu) h^*(\nu) \underbrace{E[x(t_1-\mu)\, x^*(t_2-\nu)]}_{=R_x(t_1-t_2+\nu-\mu)} d\mu d\nu \;. \end{aligned}$$

Thus, setting $t_1 - t_2 \to \tau$ and $\nu - \mu \to \lambda$ yields

$$\begin{aligned} R_y(\tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\mu) \quad \underbrace{h^*(\lambda+\mu)}_{=\int H^*(f) e^{-j2\pi f(\lambda+\mu)} df} \quad R_x(\tau+\lambda)\, d\mu\, d\lambda \\ &= \int_{-\infty}^{\infty} H^*(f) \underbrace{\int_{-\infty}^{\infty} h(\mu)\, e^{-j2\pi f\mu}\, d\mu}_{=H(f)} \underbrace{\int_{-\infty}^{\infty} R_x(\tau+\lambda)\, e^{-j2\pi f\lambda}\, d\lambda}_{=S_x(f)\, e^{j2\pi f\tau}}\, df \\ &= \int_{-\infty}^{\infty} |H(f)|^2\, S_x(f)\, e^{j2\pi f\tau}\, df \;. \end{aligned}$$

Noting that the last expression is just an inverse Fourier transform, the PSDs of $x(t)$ and $y(t)$ are thus related by

$$S_y(f) = |H(f)|^2\, S_x(f) \;.$$

In the special case when $x(t)$ is a white process (e.g., white Gaussian noise) with $S_x(f) = \mathcal{N}_0/2$, the PSD of $y(t)$ is

$$S_y(f) = \frac{\mathcal{N}_0}{2}\, |H(f)|^2 \;.$$

1.8 PSD of WSS Process Multiplied by Sinusoid

Let $s(t)$ be a WSS random process (e.g., a baseband message signal) with PSD $S_s(f)$ and consider the random process $x(t)$ obtained by

$$x(t) = s(t)\, \cos(2\pi f_c t + \theta_c) \;, \qquad \theta_c \in [0, 2\pi) \;,$$

i.e., $s(t)$ is multiplied by a sinusoidal carrier with deterministic frequency f_c and random phase θ_c. Assuming that $s(t)$ and θ_c are statistically independent, the autocorrelation function of $x(t)$ can be computed as

$$\begin{aligned}
R_x(t_1,t_2) &= E[x(t_1)\,x^*(t_2)] = E\big[s(t_1)\,\cos(2\pi f_c t_1+\theta_c)\,s^*(t_2)\,\cos(2\pi f_c t_2+\theta_c)\big] \\
&= \underbrace{E[s(t_1)\,s^*(t_2)]}_{=R_s(t_1-t_2)}\;\; E\big[\underbrace{\cos(2\pi f_c t_1+\theta_c)\,\cos(2\pi f_c t_2+\theta_c)}_{=[\cos(2\pi f_c(t_1-t_2))+\cos(2\pi f_c(t_1+t_2)+2\theta_c)]/2}\big] \\
&= \frac{1}{2}\,R_s(t_1-t_2)\,\big[\cos(2\pi f_c(t_1-t_2)) + \underbrace{E[\cos(2\pi f_c(t_1+t_2)+2\theta_c)]}_{=0 \text{ if } \theta_c \text{ uniform}}\big]\;.
\end{aligned}$$

Thus, assuming that the random phase θ_c is uniformly distributed in $[0,2\pi)$ (reflecting the fact that if many identical but independent communication systems were run, their phases would be uncorrelated),

$$R_x(\tau) = \frac{1}{2}\,R_s(\tau)\,\cos(2\pi f_c\tau) \quad\Longleftrightarrow\quad S_x(f) = \frac{1}{4}\,\big[S_s(f-f_c)+S_s(f+f_c)\big]\;.$$

That is, $S_x(f)$ is obtained by shifting $S_s(f)$ to the right and to the left by f_c and dividing it by 4.

1.9 PSD of PAM Signals

Let a_n be a DT WSS random process with baud rate $F_B = 1/T_B$ and consider the general PAM signal

$$s(t) = \sum_{n=-\infty}^{\infty} a_n\,p(t-nT_B)\;,$$

where $p(t)$ is a deterministic pulse. It is desired to determine the PSD $S_s(f)$ of the CT random process $s(t)$ in terms of the PDF $S_a(\phi)$ of a_n and the FT $P(f)$ of $p(t)$. The autocorrelation function of $a(t)$ is

$$\begin{aligned}
R_s(t_1,t_2) &= E[s(t_1)\,s^*(t_2)] = E\big[\sum_{k=-\infty}^{\infty} a_k\,p(t_1-kT_B)\,\sum_{\ell=-\infty}^{\infty} a_\ell^*\,p^*(t_2-\ell T_B)\big] \\
&= \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} \underbrace{E[a_k\,a_\ell^*]}_{=R_a[k-\ell]}\;\; p(t_1-kT_B)\,p^*(t_2-\ell T_B) \\
&= \sum_{m=-\infty}^{\infty} R_a[m]\,\sum_{k=-\infty}^{\infty} p(t_1-kT_B)\,p^*(t_2-(k-m)T_B)\;,
\end{aligned}$$

where the substitution $k-\ell \to m$ was used for the last equality. Since $R_s(t_1,t_2)$ depends on the absolute values of t_1 and t_2 (and not just on t_1-t_2), $s(t)$ is not a WSS process. However, $s(t)$ is **cyclostationary** with period T_B, i.e., for any integer k it satisfies

$$E[s(t+kT_B)] = E[s(t)]\;,\quad \text{and} \quad R_s(t_1+kT_B,t_2+kT_B) = R_s(t_1,t_2)\;.$$

Define the **time-averaged autocorrelation function** $\bar{R}_s(\tau)$, averaged over one period of $R_s(t_1, t_2)$ as

$$\begin{aligned}
\bar{R}_s(\tau) &= \frac{1}{T_B}\int_0^{T_B} R_s(t+\tau, t)\, dt \\
&= \frac{1}{T_B}\sum_{m=-\infty}^{\infty} R_a[m] \sum_{k=-\infty}^{\infty}\int_0^{T_B} p(t-kT_B+\tau)\, p^*(t-kT_B+mT_B)\, dt \\
&= \frac{1}{T_B}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty} R_a[m]\, p(t+\tau) \qquad \underbrace{p^*(t+mT_B)}_{=\int P^*(f)e^{-j2\pi f(t+mT_B)}df} \qquad dt \\
&= \frac{1}{T_B}\int_{-\infty}^{\infty} P^*(f) \underbrace{\sum_{m=-\infty}^{\infty} R_a[m]\, e^{-j2\pi m f T_B}}_{=S_a(fT_B)} \underbrace{\int_{-\infty}^{\infty} p(t+\tau)\, e^{-j2\pi f t}\, dt}_{=P(f)e^{j2\pi f\tau}}\, df \\
&= \frac{1}{T_B}\int_{-\infty}^{\infty} S_a(fT_B)\, |P(f)|^2\, e^{j2\pi f\tau}\, df\ .
\end{aligned}$$

Since the last expression is just an inverse FT, the desired expression for $S_s(f)$ is

$$S_s(f) = F_B\, S_a(fT_B)\, |P(f)|^2\ .$$

Example: PSD of Unipolar Binary PAM. In this case $a_n \in \{0, 1\}$. Assuming that the a_n are iid (independent, identically distributed) with $Pr\{a_n = 0\} = Pr\{a_n = 1\}$, the autocorrelation function of the DT random process a_n is

$$R_a[m] = E[a_{n+m}\, a_n^*] = \left\{ \begin{array}{ll} 1/2\,, & m = 0\,, \\ 1/4\,, & m \neq 0\,. \end{array} \right. \qquad \Longrightarrow \qquad R_a[m] = \frac{1}{4}\,(\delta_m + 1)\ .$$

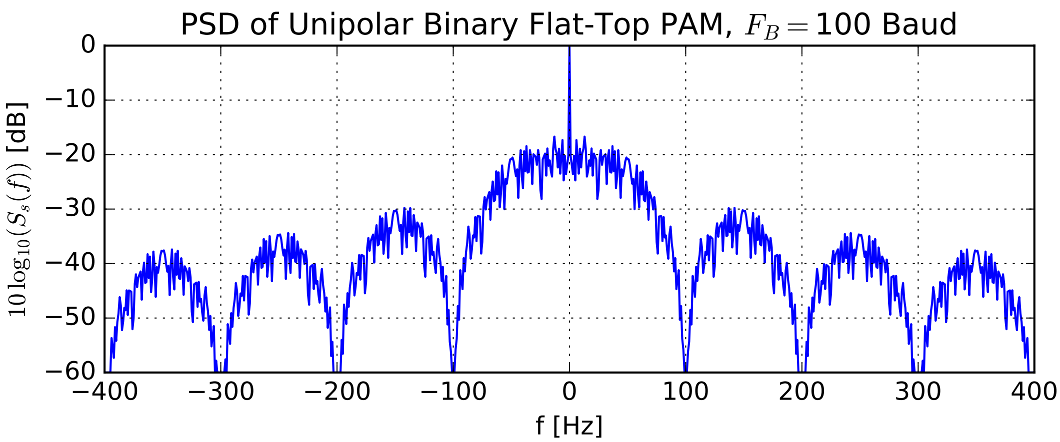
Taking the DTFT of $R_a[m]$ yields

$$S_a(fT_B) = \sum_{m=-\infty}^{\infty} R_a[m]\, e^{-j2\pi m f T_B} = \frac{1}{4} + \frac{1}{4}\sum_{k=-\infty}^{\infty}\delta(fT_B - k) = \frac{1}{4} + \frac{F_B}{4}\sum_{k=-\infty}^{\infty}\delta(f - kF_B)\ .$$

Therefore, unipolar binary PAM with pulse $p(t)$ has PSD

$$S_s(f) = \frac{F_B\, |P(f)|^2}{4} + \frac{F_B^2}{4}\sum_{k=-\infty}^{\infty}\delta(f - kF_B)|P(kF_B)|^2\ .$$

An example of $S_s(f)$ for a random unipolar binary flat-top (i.e., rectangular $p(t)$) PAM signal with $F_B = 100$ bits/sec is shown in the graph below.



Example: PSD of Polar Binary PAM. In this case $a_n \in \{-1, +1\}$. Assuming that the a_n are iid (independent, identically distributed) with $Pr\{a_n = -1\} = Pr\{a_n = +1\}$, the autocorrelation function of the DT random process a_n is

$$R_a[m] = E[a_{n+m}\, a_n^*] = \begin{cases} 1\,, & m = 0\,, \\ 0\,, & m \neq 0\,. \end{cases} \qquad \Longrightarrow \qquad R_a[m] = \delta_m\,.$$

Taking the DTFT of $R_a[m]$ yields

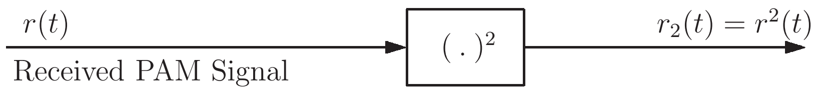
$$S_a(fT_B) = \sum_{m=-\infty}^{\infty} R_a[m]\, e^{-j2\pi m f T_B} = 1 \qquad \Longrightarrow \qquad S_s(f) = F_B\, |P(f)|^2\,.$$

1.10 Symbol Timing Information

One way to extract symbol timing information from a received PAM signal

$$r(t) = \sum_{m=-\infty}^{\infty} a_m\, q(t - mT_B)\,,$$

with baud rate $F_B = 1/T_B$ is to pass $r(t)$ through a squaring device as shown below.



The output $r_2(t) = t^2(t)$ of the squaring device is

$$r_2(t) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_k\, a_\ell\, q(t - kT_B)\, q(t - \ell T_B)\,,$$

and thus the autocorrelation function of $r^2(t)$ is

$$
\begin{aligned}
R_{r2}(t_1,t_2) &= E[r_2(t_1)\, r_2^*(t_2)] \\
&= E\Big[\sum_k \sum_\ell \sum_m \sum_n a_k\, a_\ell\, a_m^*\, a_n^*\, q(t_1{-}kT_B)\, q(t_1{-}\ell T_B)\, q^*(t_2{-}mT_B)\, q^*(t_2{-}nT_B)\Big] \\
&= \sum_k \sum_\ell \sum_m \sum_n E[a_k\, a_\ell\, a_m^*\, a_n^*] q(t_1{-}kT_B)\, q(t_1{-}\ell\, T_B) q^*(t_2{-}mT_B)\, q^*(t_2{-}nT_B)\,.
\end{aligned}
$$

To simplify this expression, assume that a_n is a stationary DT random process and assume that the a_n are iid (independent and identically distributed) random variables with zero mean. This implies that

$$
E[a_k\, a_\ell\, a_m^*\, a_n^*] = E[a_k]\, E[a_\ell]\, E[a_m^*]\, E[a_n^*] = 0 \qquad \text{if } k,\ell,m,n \text{ are all different}\,.
$$

More specifically, under the above assumptions $E[a_k a_\ell a_m^* a_n^*]$ can only be nonzero if either $k=\ell$ and $m=n$, or $k=m$ and $\ell=n$, or $k=n$ and $\ell=m$. Therefore

$$
\begin{aligned}
R_{r2}(t_1,t_2) = {} & \sum_k \sum_m \underbrace{E[a_k^2\,(a_m^*)^2]}_{=\,R_{a2}[k-m]}\; q^2(t_1{-}kT_B)\,(q^*(t_2{-}mT_B))^2 + \\
& + 2\sum_k \sum_\ell \underbrace{E[|a_k|^2\,|a_\ell|^2]}_{=\,R_{m2}[k-\ell]} q(t_1{-}kT_B)\, q(t_1{-}\ell T_B)\, q^*(t_2{-}kT_B)\, q^*(t_2{-}\ell T_B) + \\
& - 2\sum_k \underbrace{E[|a_k|^4]}_{=\,E[|a|^4]}\; q^2(t_1{-}kT_B)\,(q^*(t_2{-}kT_B))^2\,.
\end{aligned}
$$

Setting $n \leftarrow k{-}m$, $k{-}n \leftarrow m$, and $n \leftarrow k{-}\ell$, $k{-}n \leftarrow \ell$, respectively, yields

$$
\begin{aligned}
R_{r2}(t_1,t_2) = {} & \sum_n R_{a2}[n] \sum_k q^2(t_1{-}kT_B)\,(q^*(t_2{-}(k{-}n)T_B))^2 + \\
& + 2\sum_n R_{m2}[n] \sum_k q(t_1{-}kT_B)\, q(t_1{-}(k{-}n)T_B)\, q^*(t_2{-}kT_B)\, q^*(t_2{-}(k{-}n)T_B) + \\
& - 2\,E[|a|^4] \sum_k q^2(t_1{-}kT_B)\,(q^*(t_2{-}kT_B))^2\,.
\end{aligned}
$$

This is the autocorrelation function of a cyclostationary random process with period T_B. The time-averaged (over one period) autocorrelation function of $r^2(t)$ is computed as

$$
\begin{aligned}
\bar{R}_{r2}(\tau) &= \frac{1}{T_B}\int_{T_B} R_{r2}(t+\tau,t)\, dt \\
&= \frac{1}{T_B}\sum_n R_{a2}[n] \sum_k \int_{T_B} q^2(t{+}\tau{-}kT_B)\,(q^*(t{-}(k{-}n)T_B))^2\, dt + \\
&+ \frac{2}{T_B}\sum_n R_{m2}[n] \sum_k \int_{T_B} q(t{+}\tau{-}kT_B)\, q(t{+}\tau{-}(k{-}n)T_B)\, q^*(t{-}kT_B)\, q^*(t{-}(k{-}n)T_B)\, dt + \\
&- \frac{2}{T_B}\, E[|a|^4] \sum_k \int_{T_B} q^2(t{+}\tau{-}kT_B)\,(q^*(t{-}kT_B))^2\, dt
\end{aligned}
$$

The combinations of an infinite sum over k and an integration over a finite time interval of length T_B can be combined into integrals for $-\infty < t < \infty$ so that

$$\begin{aligned}\bar{R}_{r2}(\tau) &= \frac{1}{T_B}\sum_n R_{a2}[n]\int_t q^2(t+\tau)\,(q^*(t+nT_B))^2\,dt+\\ &+\frac{2}{T_B}\sum_n R_{m2}[n]\int_t q(t+\tau)\,q^*(t)\,q(t+\tau+nT_B)\,q^*(t+nT_B)\,dt+\\ &-\frac{2}{T_B}\,E[|a|^4]\int_t q^2(t+\tau)\,(q^*(t))^2\,dt\,.\end{aligned}$$

Next, use inverse FTs of the form

$$q(t+t_0) = \int_\alpha Q(\alpha)\,e^{j2\pi\alpha(t+t_0)}\,d\alpha\,, \qquad \text{and} \qquad q^*(t+t_0) = \int_\mu Q^*(\mu)\,e^{-j2\pi\mu(t+t_0)}\,d\mu\,,$$

to obtain

$$\begin{aligned}\bar{R}_{r2}(\tau) = \frac{1}{T_B}\sum_n R_{a2}[n]\int_t\int_\alpha Q(\alpha)e^{j2\pi\alpha(t+\tau)}d\alpha\int_\beta Q(\beta)e^{j2\pi\beta(t+\tau)}d\beta\times&\\ \times\int_\mu Q^*(\mu)e^{-j2\pi\mu(t+nT_B)}d\mu\int_\nu Q^*(\nu)e^{-j2\pi\nu(t+nT_B)}d\nu dt+&\\ +\frac{2}{T_B}\sum_n R_{m2}[n]\int_t\int_\alpha Q(\alpha)e^{j2\pi\alpha(t+\tau)}d\alpha\int_\beta Q(\beta)e^{j2\pi\beta(t+\tau+nT_B)}d\beta\times&\\ \times\int_\mu Q^*(\mu)e^{-j2\pi\mu t}d\mu\int_\nu Q^*(\nu)e^{-j2\pi\nu(t+nT_B)}d\nu dt+&\\ -\frac{2}{T_B}E[|a|^4]\int_t\int_\alpha Q(\alpha)e^{j2\pi\alpha(t+\tau)}d\alpha\int_\beta Q(\beta)e^{j2\pi\beta(t+\tau)}d\beta\times&\\ \times\int_\mu Q^*(\mu)e^{-j2\pi\mu t}d\mu\int_\nu Q^*(\nu)e^{-j2\pi\nu t}d\nu dt\,.&\end{aligned}$$

After changing the order of integration and regrouping terms this becomes

$$\bar{R}_{r2}(\tau) = \frac{1}{T_B}\sum_n R_{a2}[n]\int_\alpha\int_\beta Q(\alpha)Q(\beta)e^{j2\pi(\alpha+\beta)\tau}\int_\mu\int_\nu Q^*(\mu)Q^*(\nu)e^{-j2\pi(\mu+\nu)T_B n}\times$$
$$\times\underbrace{\int_t e^{-j2\pi(\mu+\nu-\alpha-\beta)t}dt}_{=\delta(\mu+\nu-\alpha-\beta)}\,d\alpha\,d\beta\,d\mu\,d\nu+$$
$$+\frac{2}{T_B}\sum_n R_{ma}[n]\int_\alpha\int_\beta Q(\alpha)Q(\beta)e^{j2\pi(\alpha+\beta)\tau}\int_\mu\int_\nu Q^*(\mu)Q^*(\nu)e^{-j2\pi(\nu-\beta)T_B n}\times$$
$$\times\underbrace{\int_t e^{-j2\pi(\mu+\nu-\alpha-\beta)t}dt}_{=\delta(\mu+\nu-\alpha-\beta)}\,d\alpha\,d\beta\,d\mu\,d\nu+$$
$$-\frac{2}{T_B}E[|a|^4]\int_\alpha\int_\beta Q(\alpha)Q(\beta)e^{j2\pi(\alpha+\beta)\tau}\int_\mu\int_\nu Q^*(\mu)Q^*(\nu)\times$$
$$\times\underbrace{\int_t e^{-j2\pi(\mu+\nu-\alpha-\beta)t}dt}_{=\delta(\mu+\nu-\alpha-\beta)}\,d\alpha\,d\beta\,d\mu\,d\nu\,.$$

Now, use the change of variables $f \leftarrow \alpha+\beta$, $f-\alpha \leftarrow \beta$, and $g \leftarrow \mu+\nu$, $g-\mu \leftarrow \nu$:

$$\bar{R}_{r2}(\tau) = \frac{1}{T_B}\int_f\int_\alpha Q(\alpha)Q(f-\alpha)e^{j2\pi f\tau}\int_g\int_\mu Q^*(\mu)Q^*(g-\mu)\underbrace{\sum_n R_{a2}[n]e^{-j2\pi g T_B n}}_{=S_{a2}(gT_B)}\times$$
$$\times\,\delta(g-f)\,df\,d\alpha\,dg\,d\mu+$$
$$+\frac{2}{T_B}\int_f\int_\alpha Q(\alpha)Q(f-\alpha)e^{j2\pi f\tau}\int_g\int_\mu Q^*(\mu)Q^*(g-\mu)\underbrace{\sum_n R_{m2}[n]e^{-j2\pi(g-\mu-f+\alpha)T_B n}}_{=S_{m2}((g-\mu-f+\alpha)T_B)}\times$$
$$\times\,\delta(g-f)\,df\,d\alpha\,dg\,d\mu+$$
$$-\frac{2}{T_B}E[|a|^4]\int_f\int_\alpha Q(\alpha)Q(f-\alpha)e^{j2\pi f\tau}\int_g\int_\mu Q^*(\mu)Q^*(g-\mu)\,\delta(g-f)\,df\,d\alpha\,dg\,d\mu\,.$$

Since $\delta(g-f)$ is zero unless $g=f$, this simplifies to

$$\bar{R}_{r2}(\tau) = \frac{1}{T_B}\int_f S_{a2}(fT_B)\left|\int_\alpha Q(\alpha)\,Q(f-\alpha)\,d\alpha\right|^2 e^{j2\pi f\tau}\,df+$$
$$+\frac{2}{T_B}\int_f\int_\alpha\int_\mu S_{m2}((\alpha-\mu)T_B)\,Q(\alpha)\,Q(f-\alpha)\,Q^*(\mu)\,Q^*(f-\mu)\,d\alpha\,d\mu\,e^{j2\pi f\tau}\,df+$$
$$-\frac{2}{T_B}\int_f E[|a|^4]\left|\int_\alpha Q(\alpha)\,Q(f-\alpha)\,d\alpha\right|^2 e^{j2\pi f\tau}\,df\,.$$

But, since this is in the form of an inverse FT, the PSD of the squared PAM signal $r^2(t)$ (in which a_n is assumed to be a stationary DT random process consisting of iid random variables with zero mean) is finally obtained as

$$\begin{aligned} S_{r2}(f) = F_B \Big(S_{a2}(fT_B) \Big| \int_\alpha Q(\alpha)\, Q(f-\alpha)\, d\alpha \Big|^2 + \\ + 2 \int_\alpha \int_\mu S_{m2}((\alpha-\mu)T_B)\, Q(\alpha)\, Q(f-\alpha)\, Q^*(\mu)\, Q^*(f-\mu)\, d\alpha\, d\mu + \\ - 2\, E[|a|^4] \Big| \int_\alpha Q(\alpha)\, Q(f-\alpha)\, d\alpha \Big|^2 \Big) . \end{aligned}$$

The PSDs $S_{a2}(\phi)$ and $S_{m2}(\phi)$ are obtained from

$$R_{a2}[n] = E[a_{k+n}^2\, (a_k^*)^2] \qquad \Longleftrightarrow \qquad S_{a2}(\phi) = \sum_n R_{a2}[n]\, e^{-j2\pi\phi n} ,$$

$$R_{m2}[n] = E[|a_{k+n}|^2\, |a_k|^2] \qquad \Longleftrightarrow \qquad S_{m2}(\phi) = \sum_n R_{m2}[n]\, e^{-j2\pi\phi n} .$$

Both $R_{a2}[n]$ and $R_{m2}[n]$ contain a strong dc component and thus both $S_{a2}(\phi)$ and $S_{m2}(\phi)$ have a component of the form

$$1 \qquad \Longleftrightarrow \qquad \sum_k \delta(\phi - k) ,$$

i.e., a component with impulses at normalized frequency $0, \pm 1, \pm 2, \ldots$. For the extraction of symbol timing information the first term in the formula for $S_{r2}(f)$ is used. Assuming that

$$S_{a2}(fT_B) = \sum_k \delta(fT_B - k) = \sum_k \delta(f/F_B - k) ,$$

this term is

$$F_B \sum_{k=-\infty}^{\infty} \delta(f/F_B - k) \Big| \int_{-\infty}^{\infty} Q(\nu)\, Q(f-\nu)\, d\nu \Big|^2 .$$

Thus, $r^2(t)$ has a spectral component at F_B if the convolution $Q(f) * Q(f)$ is nonzero at $f = F_B$.

2 Lab Experiments

E1. `showpsd` Python Function. (a) Add the `showpsd` function given below to the `showfun` module. It is used to plot (an approximation to) the PSD of a CT signal $x(t)$ from its samples $x(nT_s)$, taken with sampling rate $F_s = 1/T_s$.

```
def showpsd0(xt, Fs, ff_lim, N):
    """
    Plot (DFT/FFT approximation to) power spectral density (PSD) of x(t).
    Displays S_x(f) either linear and absolute or normalized in dB.
    >>>>> showpsd(xt, Fs, ff_lim, N) <<<<<
    where  xt:    sampled CT signal x(t)
           Fs:    sampling rate of x(t)
           ff_lim = [f1,f2,llim]
           f1:    lower frequency limit for display
           f2:    upper frequency limit for display
           llim = 0:  display S_x(f) linear and absolute
           llim < 0:  display 10*log_{10}(S_x(f))/max(S_x(f))
                      in dB with lower display limit llim dB
           N:     blocklength
    """

    # ***** Determine number of blocks, prepare x(t) *****
    N = int(min(N, len(xt)))   # N <= length(xt) needed
    NN = int(floor(len(xt)/float(N)))
                               # Number of blocks of length N
    xt = xt[0:N*NN]            # Truncate x(t) to NN blocks
    xNN = reshape(xt,(NN,N))   # NN row vectors of length N
    # ***** Compute DFTs/FFTs, average over NN blocks *****
    Sxf = np.power(abs(fft(xNN)),2.0)  # NN FFTs, mag squared
    if NN > 1:
        Sxf = sum(Sxf, axis=0)/float(NN)
    Sxf = Sxf/float(N*Fs)      # Correction factor DFT -> PSD
    Sxf = reshape(Sxf,size(Sxf))
    ff = Fs*array(arange(N),int64)/float(N)  # Frequency axis
    if ff_lim[0] < 0:          # Negative f1 case
        ixp = where(ff<0.5*Fs)[0]   # Indexes of pos frequencies
        ixn = where(ff>=0.5*Fs)[0]  # Indexes of neg frequencies
        ff = hstack((ff[ixn]-Fs,ff[ixp]))  # New freq axis
        Sxf = hstack((Sxf[ixn],Sxf[ixp]))  # Corresponding S_x(f)
    # ***** Determine maximum, trim to ff_lim *****
    maxSxf = max(Sxf)          # Maximum of S_x(f)
    ixf = where(logical_and(ff>=ff_lim[0], ff<ff_lim[1]))[0]
    ff = ff[ixf]               # Trim to ff_lim specs
    Sxf = Sxf[ixf]
    # ***** Plot PSD *****
    strgt = 'PSD Approximation, $F_s=${:d} Hz'.format(Fs)
    strgt = strgt + ', $\\Delta_f=${:.3g} Hz'.format(df)
    strgt = strgt + ', $NN=${:d}, $N=${:d}'.format(NN, N)
    f1 = figure()
    af1 = f1.add_subplot(111)
    af1.plot(ff, Sxf, '-b')
    af1.grid()
    af1.set_xlabel('f [Hz]')
    af1.set_ylabel(strgy)
    af1.set_title(strgt)
    show()
```

The function as given above will plot $S_x(f)$ linear and absolute over the frequency range specified in `ff_lim`. Try it using the following commands:

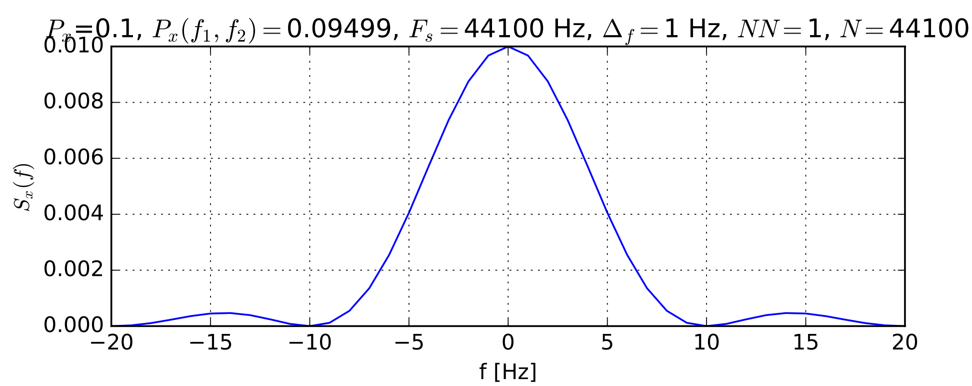
```
Fs = 44100                          # Sampling rate
f1 = 700                            # Test frequency 1
f2 = 720                            # Test frequency 1
tlen = 2                            # Duration in seconds
tt = arange(round(tlen*Fs))/float(Fs)  # Time axis
x1t = sin(2*pi*f1*tt)               # Sine with freq f1
x2t = 0.01*cos(2*pi*f2*tt)          # Attenuated cosine with freq f2
xt = x1t+x2t                        # Combined sinusoidal signal
showpsd(xt,Fs,[-1000 1000 0],Fs)   #Plot S_x(f)
```

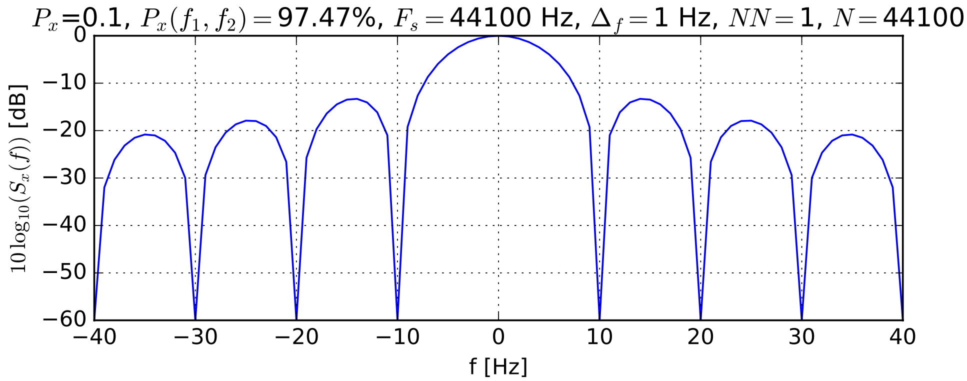
Obviously, it would be nice to be able to display $S_x(f)$ either absolute and linear or normalized in dB (so that the spectral lines of both sinusoids are visible). Add this enhancement to `showpsd` (as described in the function header), in a similar fashion as you did for the `showft` function.

(b) One useful feature of the PSD is that it can be used to compute the power over a range of frequencies, e.g., from f_1 to f_2, using the integral

$$P_x(f_1, f_2) = \int_{f_1}^{f_2} S_x(f)\, df \; .$$

Add a feature to `showpsd` so that it displays the total power P_x, as well as $P_x(f_1, f_2)$ for $f_1 =$ `ff_lim[0]` and $f_2 =$ `ff_lim[1]`, in the title bar. The following two graphs show this for a rectangular pulse of width 100 ms and amplitude 1. The total length of the signal is 1 sec and the sampling rate is $F_s = 44100$ Hz.

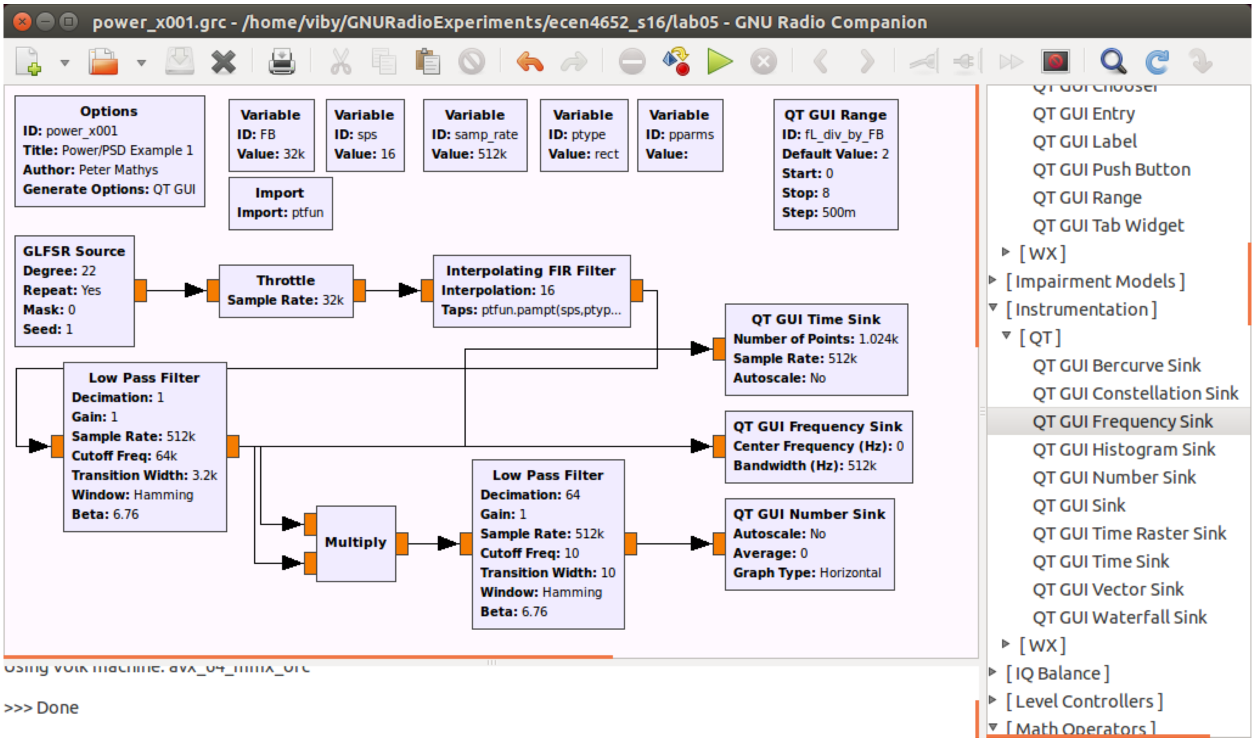




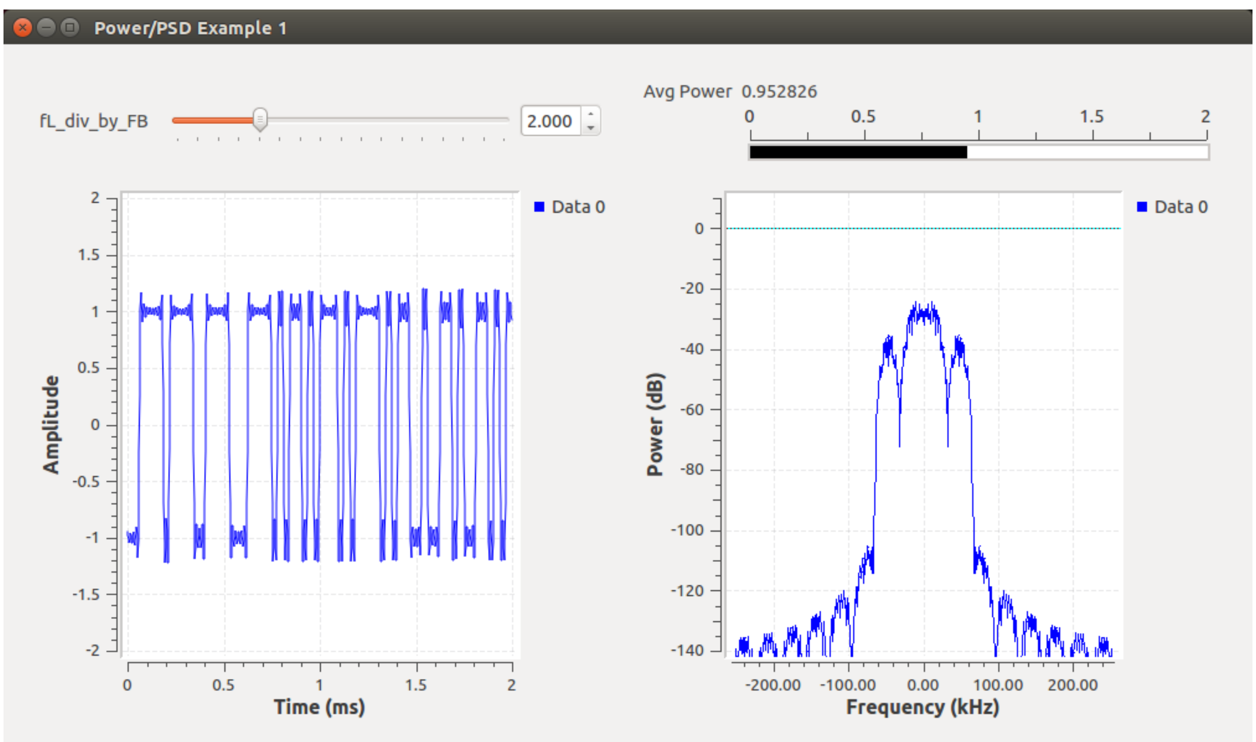
The first graph uses a linear and absolute vertical axis and displays $P_x(f_1, f_2)$ as an absolute value. The second graph shows $S_x(f)$ normalized (i.e., maximum at 0 dB) and in dB and displays $P_x(f_1, f_2)$ as a percentage of the total power P_x.

(c) Generate about 5 sec of random binary polar PAM signals with rectangular, triangular, and sinc (tail length ≈ 20, no windowing) pulses. Use $F_B = 100$ and $F_s = 44100$ Hz and display the PSDs of the three signals in dB, with lower limit -60 dB, over the frequency range $-500 \ldots 500$ Hz. Set the blocklength to F_s. Verify that the PSDs look right. A reasonable rule of thumb for the bandwidth of PAM signals is $2F_B$ when $p(t)$ is rectangular, F_B when $p(t)$ is triangular and $F_B/2$ when $p(t)$ is a sinc pulse. Determine how much of the total signal power is contained within these bandwidths for the rectangular, triangular and sinc pulse cases.

(d) PSD and Measuring Power in GNU Radio. The GNU Radio flowgraph shown below is used to generate a random polar binary PAM signal with baud rate F_B from different types of pulses $p(t)$ (`ptype = 'man', 'RCf', 'rect', 'sinc', 'tri'`) at the output of the Interpolating FIR Filter. The GLFSR (Galois Linear Feedback Shift Register) Source block generates a polar binary sequence (output symbols take on values of -1.0 or +1.0) with symbol rate F_B (=32 kbaud) at the output of the Throttle block. The Interpolating FIR Filter generates the PAM signal using the desired pulse shape $p(t)$ with `sps` samples per symbol from the `pampt` function in the `ptfun` module that you generated previously (you may still have to add the `'man'` and the `'RCf'` pulses). To look at the effect of bandlimiting the PAM signal, a Low Pass Filter with variable cutoff frequency $f_L = k/2 * F_B$, $k = 1, 2, 3, \ldots,$ `sps`, is used. The output from this filter is displayed in the time and frequency domains using a QT GUI Time Sink and a QT GUI Frequency Sink. To measure the average signal power, the output is also squared, further lowpass filtered, and displayed using a QT GUI Number Sink.



The output generated by the flowgraph is shown in the next screen snapshot for a PAM signal based on a rectangular pulse $p(t)$ after lowpass filtering at $2\,F_B$.



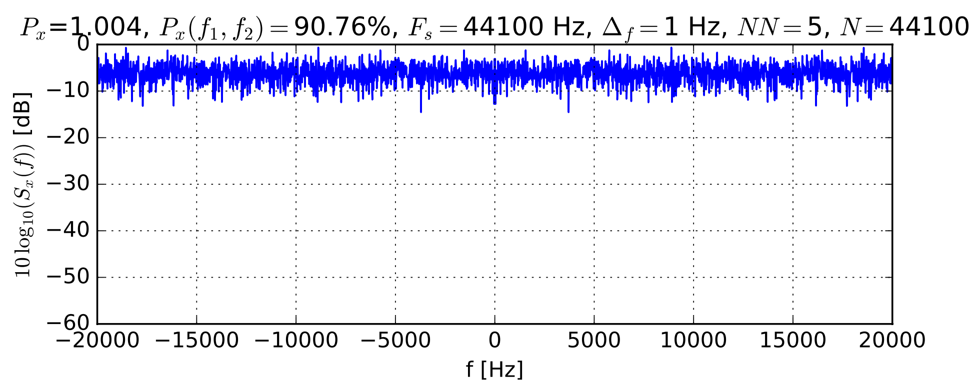
Your task: For the rule of thumb that rectangular type signals have a bandwidth of $2\,F_B$, rectangular type signals have a bandwidth of F_B, and 'sinc' type signals have a bandwidth of

$F_B/2$, use the above GNU Radio flowgraph to determine how much of the total signal power is contained in these bandwidths for PAM signals using `'man'`, `'RCf'`, `'rect'`, `'sinc'`, `'tri'` pulses $p(t)$. For the `'RCf'`, `'sinc'` pulses use a tail length of about $10\,T_B$. For `'RCf'` use $\alpha = 0.2$ and for `'sinc'` use $\beta = 0$.

E2. Generating and Estimating Noise. (a) If you need to generate white Gaussian noise in Python, you can use the following commands

```
Fs = 44100                          # Sampling rate
tlen = 5                            # Duration in seconds
tt = arange(round(tlen*Fs))/float(Fs)    # Time axis
nt = randn(len(tt))                 # Gaussian noise n(t)
```

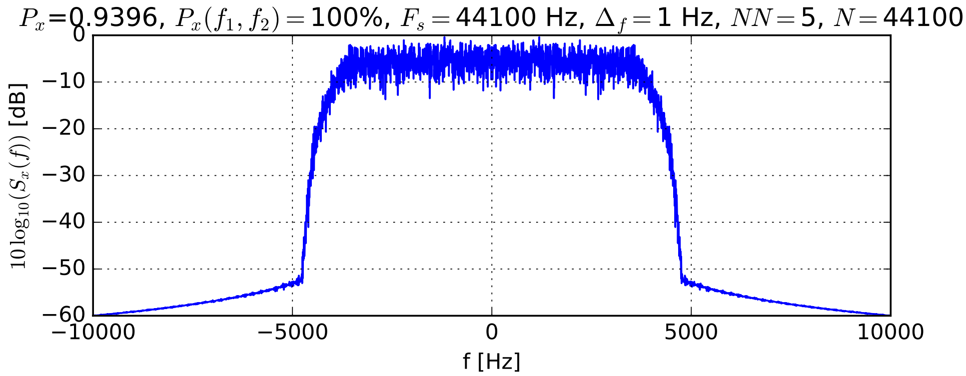
The PSD $S_n(f)$ of this $n(t)$ looks as follows:



It is often desirable to have bandlimited noise that has a constant PSD for all frequencies below some cutoff frequency f_L. This can either be obtained by filtering white noise with a lowpass filter, or by using a white noise sequence n_n with baud rate $2 * f_L$ as the input to a PAM signal generator with a 'sinc' or 'RCf' pulse. Here is some Python code to implement this:

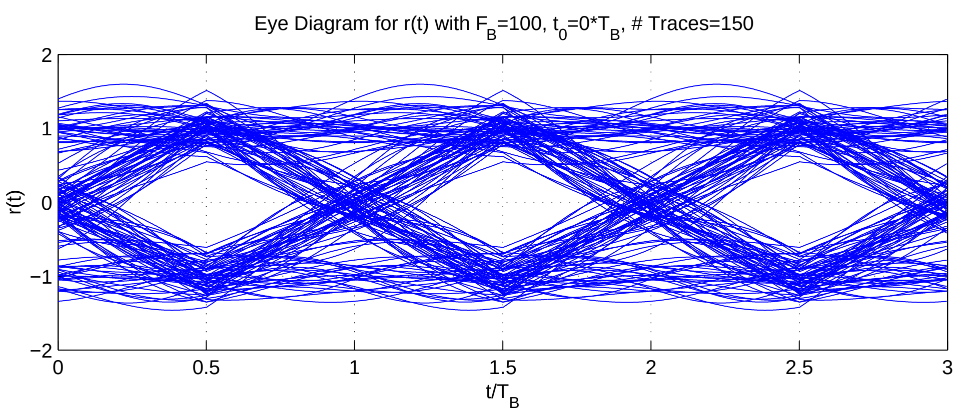
```
Fs = 44100                          # Sampling rate
nfL = *** enter fL value ***        # Noise cutoff frequency
tlen = 5                            # Duration in seconds
nn = randn(round(tlen*2*nfL))       # Gaussian noise, rate 2*nfL
nt = pam11(nn,2*nfL,Fs,'rcf',[20 0.2])
                                    #Bandlimited noise n(t), rate Fs
```

The PSD of this bandlimited noise is shown in the next graph for $f_L = 4000$ Hz.

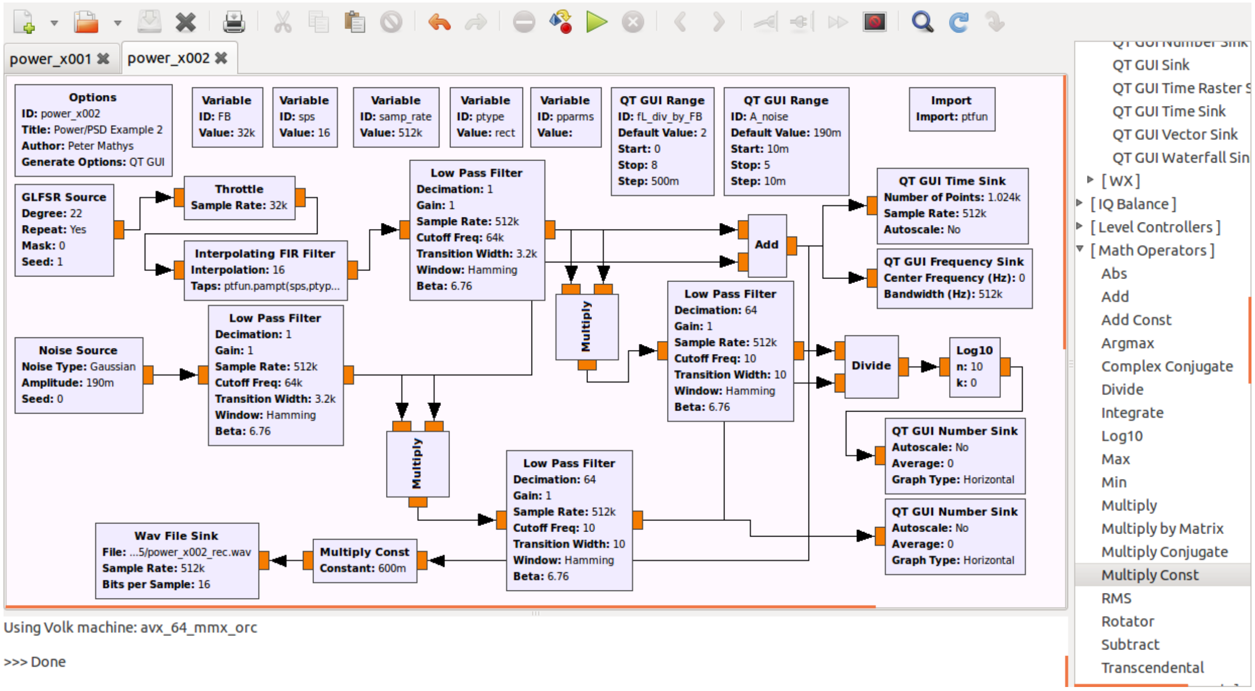


Using a sampling rate of $F_s = 44100$, generate 5 sec of white Gaussian noise (full bandwidth of $F_s/2$), of bandlimited Gaussian noise with $f_L = 4000$ Hz, and of bandlimited Gaussian noise with $f_L = 1000$ Hz. Plot the PSD's of all three signals and listen to each using the `sound` command in Matlab. Describe what you hear and how the different noises compare.

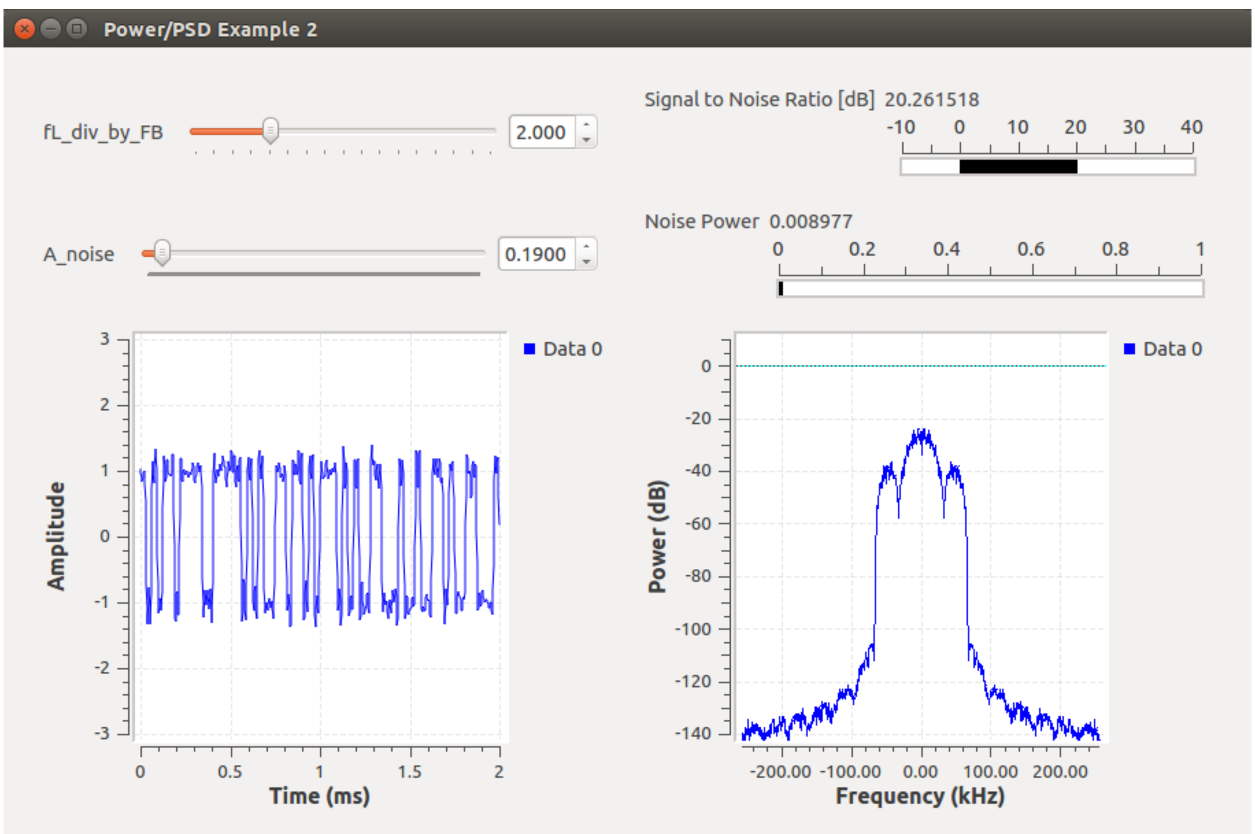
(b) Generate a random polar binary PAM signal $s(t)$ of length 2 sec with triangular $p(t)$. Use $F_B = 100$ and $F_s = 44100$. Then generate bandlimited Gaussian noise $n(t)$ with cutoff frequency $f_L = F_B$ (the rule of thumb bandwidth for PAM with triangular $p(t)$). Let $r(t) = s(t) + A\,n(t)$ be the received PAM signal. Use the `showeye` function to plot the eye diagram of $r(t)$. Adjust A such that the eye is approximately half closed as shown in the graph below. Determine at which signal-to-noise ratio (SNR) in dB this happens.



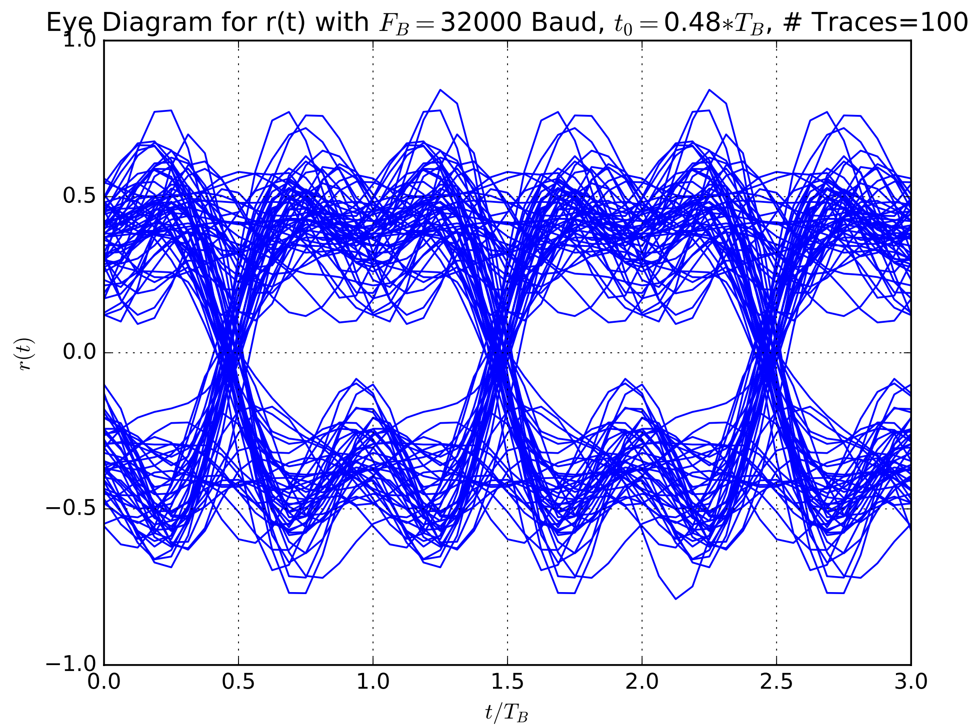
(c) PAM signal SNR estimation in GNU Radio. The GNU Radio flowgraph shown below generates PAM signals in the same way as the flowgraph described for experiment E1d. In addition, a Noise Source block, followed by a Low Pass Filter, is used to create bandlimited Gaussian noise that is added to the PAM signal. The PAM signal and the Gaussian noise are filtered by separate lowpass filters with the same characteristic so the the signal power and the noise power can be measured separately. The signal to noise ratio (SNR) is computed and displayed in decibels using a Divide block, a Log10 block, and a QT GUI Number Sink block. There is also a provision to save the bandlimited noisy PAM signal in 16-bit wav-file for later processing in Python.



A typical display that is obtained when running the flowgraph for a rectagular type PAM signal with 20 dB SNR is shown below.



Your task: Plot eye diagrams for rectangular PAM with bandwidth (both noise and PAM signal) $2\,F_B$ and SNRs of 20, 10, and 5 dB. Generate the bandlimited noisy PAM signals in GNU Radio and record the results in wav-files. Then use the `showeye` function in Python to plot the corresponding eye diagrams. Compare the results to the eyediagram of a rectangular PAM signal that is bandlimited to $F_B/2$ and has a SNR of 20 dB. An example of an eye diagram obtained in the way described above for a rectangular PAM signal with bandwidth of $2\,F_B$ and an SNR of 10 dB is shown next.



Notes: Remember that 16-bit wav-files can only record signals with amplitudes less than 1. Thus, the Constant in the Multiply Constant block before the Wav File Sink needs to be set accordingly, especially for small SNRs (when the noise is relatively large compared to the signal). Also, the filters in the GNU Radio flowgraph need some time to get into steady-state. Thus, the first few seconds of the wav-file recordings should be discarded to obtain characteristic eye diagrams.

E3. Timing Information from PAM Signals. (Experiment for ECEN 5002, optional for ECEN 4652) **(a)** Generate the same PAM signals as in E1(c). In addition, also generate about 5 sec of a random binary polar PAM signal using an RCf pulse with $\alpha = 0.5$ and tail length $k \approx 5$. Look at the PSDs of

$$s^2(t)\,, \qquad \text{and} \qquad |ds(t)/dt|\,,$$

where $s(t)$ is the PAM signal. The goal is to determine whether at least one of the two quantities has a spectral component at F_B that could be used to synchronize the receiver. What conclusions can you draw from the results?

(b) Look at the signals in `pamsig401.wav`, `pamsig402.wav`, and in `pr1sig401.wav` again. In particular, look at the PSD of the signal squared and/or raised to the 4-th power. Does it have a useful spectral component at F_B?

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