CAYLEY COLOR GROUPS AND HAMILTON LINES

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IT IS a fairly old problem [5] to decide whether a graph has a Hamilton line: a closed connected path of edges without multiple points and containing each vertex of the graph. The question is related, for example, to the well-known chess problem of the knight and the four-color problem. Little headway has been made so far even with graphs in the plane.

For every group G, given together with a set of generators, Cayley defined a unique graph G^*, the "Cayley color group" [1, 2]. Thus, one can ask for "Hamilton lines of a group" when the generators are specified. As may be expected, the existence of such lines is reflected in the algebra of G; an interpretation is given below. The algebraic counterpart of a Hamilton line may well find group-theoretical application; for example, it may facilitate computation of the first cohomology group of G.

On the other hand, if one has a graph with an associated algebraic structure, the task of finding Hamilton lines is made easier. Using group properties, I find a set of them for one kind of presentation of the symmetric groups, S_n; using algebraic as well as geometric properties, I prove their existence for certain other presentations.

My interest in the matter was first aroused by problems of musical composition for chain ringing (the method of ringing church bells in England) [3, 4]. The composer must, in effect, find a Hamilton line of a group's graph; for mechanical reasons he is under stringent restrictions which, of course, do not bind the mathematically motivated treatment.

All undefined terms used and all theorems stated without proof may be found in [5] and [6].

I. Let G be a finite group of order m, $g_1, g_3, \ldots, g_{2r-1}$ a fixed set of generators not containing the identity element of G, and $g_2 = g_1^{-1}, \ldots, g_{2r} = g_{2r-1}^{-1}$. Let W designate a word in the symbols $g_1, \ldots, g_{2r}$ with the following properties: $W = g_{i_1} g_{i_2} \ldots$ has g-length m-1 and contains no segment that is a relation in G; that is

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$$W = g_{i_1} \cdot \cdots \cdots g_{i_{m-1}}$$

and distinct non-empty initial segments $g_{i_1} \ldots g_{i_k}$ are distinct elements of G. If for some symbol g_{i_m} in the set $(g_1, \ldots, g_{2r}) = (g)$ there exists a word W for which Wg_{i_m} is the identity element in G, I will say that G is unicursally generated by that subset of (g) actually occurring in Wg_{i_m}. (The term was used in yet another sense in the literature [7]. This will be touched upon later.)

The graph of G on the symbols $g_1, \ldots, g_{2r}$ is a set of m vertices (points) connected by directed edges (lines) as follows: from each vertex v issue $2r - k$ edges, one for each g_i not equal to its inverse in G and one for each pair g_i, g_{i+1} if $g_{i+1} = g_i^{-1} = g_{i+1}^{-1}$. If the edge $\overrightarrow{v_1v_2}$ corresponds to g_{2i-1} at v_1, then $\overrightarrow{v_2v_1}$ corresponding to $g_{2i} = g_{2i-1}^{-1}$ at v_2 is not drawn separately but $\overrightarrow{v_1v_2}$ bearing an arrow that points to v_2 and the name g_{2i-1} represents g_{2i} when traversed from v_2.

If any vertex v is designated as the identity element e of G, then a vertex v_1 represents g_i when $\overrightarrow{vv_1}$ represents g_i; similarly, the end point v_2 of $\overrightarrow{v_1v_2} = g_j$ will then represent g_ig_j. In this fashion, any directed connected sequence of edges (path) represents a word in the free group F_n on the generators $g_1, g_3, \ldots, g_{2r-1}$, of which G is a factor group. A closed path represents a word that is the identity in G. If an open path exists containing each vertex just once, it represents a word W as defined above; if this path is closed, it represents Wg_{i_m} and is a Hamilton line.

Thus, if G is unicursally generated by a given set of elements the corresponding graph has a Hamilton line given by Wg_{i_m} and every Hamilton line gives rise to a representative Wg_{i_m} of the identity element e of G.

II. There is another fairly obvious algebraic counterpart to the Hamilton line. Suppose that a finite group G is generated by $g_1, \ldots, g_k = (g)$ unicursally, and $Wg_{i_m} = g_{i_1} \cdots g_{i_{m-1}} g_{i_m}$. If (g) contains g_i^{-1} along with g_i, for some i, let $h_1, \ldots, h_n = (h)$ be a maximal subset of (g) in which this does not happen. Then the free group F_n on the generators (h) has a normal subgroup N for which $G = F_n/N$.

The $m - 1$ initial segments, $s_1 = g_{i_1}, s_2 = g_{i_1}g_{i_2}, \ldots, s_{m-1} = W$, of the word W are distinct elements of G and $s_i \neq e$ for any i. Setting $s_0 = e$, the m elements $s_0, \ldots, s_{m-1}$ clearly form a Schreier system of coset representatives of N in F_n. It is well known that the set of all distinct elements of F_n of the form

$$s_i h_k (\overline{s_i h_k})^{-1} \neq e, \; k: \; 1, \ldots, n, \; i: \; 0, \ldots, m-1$$

generates N freely. Here $\overline{(s_i h_k)}$ is the coset representative s_j of $s_i h_k$.

Hence the existence of a Hamilton line implies the existence of a set of free generators of N which are of the form $s_i h_k s_j^{-1}$ with s_i and s_j initial segments of the one element W of F_n.

The word W represents the element $g_{i_m}^{-1}$ in G since the Hamilton line is closed. If one dropped the one condition that the line be closed, that is if W were not to represent a generator or its inverse in G, its meaning for N would remain the same; only in this case the free generators of N would not contain an element of the form Wg_{i_m}.

III. The Hamilton lines (H) of a group's graph fall naturally into equivalence classes according to the word Wg_{i_m} that describes them. If H_1 and H_2 belong to the same word Wg_{i_m} they are algebraically identical; they need not consist of the same set of edges, as may be seen by shifting the vertex that serves as the identity element. Since direction of the path is not essential here, the lines described by Wg_{i_m}, respectively, by $(Wg_{i_m})^{-1} = g_{i_m}^{-1} \ldots g_{i_1}^{-1} = g_{j_1} \ldots g_{j_m}$ are identical. Furthermore, any cyclically reduced word which is cyclically equivalent to Wg_{i_m}—such as, say, $g_{i_2} g_{i_3} \ldots g_{i_m} g_{i_1}$—gives a Hamilton line on the same set of edges occurring when Wg_{i_m} is used.

These considerations yield equivalence classes of Hamilton lines with each class represented by a cyclic word Wg_{i_m} as well as by its inverse.

The simplest cases are: groups on one generator, g, where the graph is a circle, Wg_{i_m} is a power of g, and groups on two involutory generators, g_1 and g_2 (that is, $g_1^2 = g_2^2 = e$), where the graph is again a circle but $Wg_{i_m} = (g_1 g_2)^k$ or $(g_2 g_1)^k$. In general, however, one Wg_{i_m} defines many geometrically distinct lines.

IV. If the group G, given with the set of generators (g), is known to contain a subgroup H unicursally generated by (g), $H = (g_{i_1}, g_{i_1} g_{i_2}, \ldots, \prod_{j:1}^{h} g_{i_j})$, it can happen that coset representatives $(e, c_1, \ldots)$ can be found for H in G such that the coset diagram [2] has a Hamilton line. In this case the table of coset decomposition of G by H is of the form

$$
\begin{array}{llll}
e & Xg_{k_1} & Xg_{k_1}\cdot Xg_{k_2} \;\ldots & Xg_{k_1} \ldots\ldots Xg_{k_r} \\
 & & \vdots & \vdots \\
g_{t_1} & Xg_{k_1}\cdot g_{t_1} & \vdots & \vdots \\
 & & \vdots & \vdots \\
g_{t_1}\cdot g_{t_2} & Xg_{k_1}\cdot g_{t_1}\cdot g_{t_2} & \vdots & \vdots \\
\vdots & \vdots & & \\
\vdots & \vdots & & \\
X & Xg_{k_1}\cdot X & \cdot & g_i
\end{array}
$$

where $X = \prod_{j:1}^{h} g_{t_j}$, $c_1 = Xg_{k_1}$, $c_2 = Xg_{k_1}Xg_{k_2}$, etc., and the left-hand side of the indentity $Xg_{k_1}\cdot Xg_{k_2} \ldots Xg_{k_r}\cdot X = g_i$ is the word W. Thus one gets a Hamilton line by tracing on the graph a path given by the ordered set of elements of H (column 1) and following with the other cosets in order.

While this observation can be exploited—as in one of the results below—it is not always useful. For example, the cyclic group of order 6, on the generators g_1 of order 3 and g_2 of order 2 has the sole Hamilton line given by $g_1g_1g_2g_1^{-1}g_1^{-1}g_2$, which, even viewed as a cyclic word, does not correspond to a coset decomposition of the group.

V. In this connection the concept of Sainte-Laguë [7] referred to before is of some relevance. His definition, adapted to the graph of a group, is: a directed path P on a directed graph G^* is unicursal if each edge belonging to P that represents a given generator is described by P in one sense (that is, positively always, or negatively always, with respect to the orientation of G^*). To avoid confusion, I will say that such a path is coherently oriented (with respect to the given generator).

In the case of involutory generators the definition is vacuous. If at least one generator, g_1, has order higher than 2, to demand coherence of orientation of a Hamilton line is to bar either g_1 or g_1^{-1} from the word Wg_{i_m} that is to describe the line. This can be a serious limitation. For example, the symmetric group S_3 on the generators $S = (123)$ and $T = (12)$ has such a word, namely $SSTSST$, but S_4 on the generators $S = (1234)$ and $T = (12)$ does not. (It requires a somewhat lengthy argument to show that all its Hamilton lines contain both S and S^{-1}.) The same applies to the example above.

VI. Combining some of the notions above leads to the question of unicursal extensions $G_1 \subset G_2 \subset \ldots \subset G$: that of the existence of a Hamilton line H_{i+1} of G_{i+1} a segment of which "describes" G_i. For given $G_1 \subset G_2 \subset \ldots \subset G_i \subset \ldots$ the question is whether G_i has a Hamilton line H_i containing all but one edge of H_{i-1} (assuming of course that the graph of G_{i-1} is a connected subgraph of the graph of G_i). If so, then a recursive algebraic formula for the H_j may be found.

In view of the fact that every group has a graph with a Hamilton line—for choose the m distinct generators $g_1, \ldots, g_m$ comprising the whole group; then any two vertices are connected by an edge (cf. [5])—the question makes sense only for a specified generator system. Clearly, the number of elements in the system must grow with i; thus one can ask for minima. What follows is a recursive formula for the nested sequence of the symmetric groups S_n on a set of generators minimal in this sense.

If t_i is the transposition $(i, i+1)$, i: $1, \ldots, n-1 \geq 2$, $g_1 = t_1$, $g_2 = t_2$, $g_3 = t_1t_3$, $g_4 = t_2t_4$, $\ldots$, $g_{2k-1} = t_1t_3\ldots t_{2k-1}$, $g_{2k} = t_2t_4\ldots t_{2k}$, $\ldots$ then for the symmetric group S_n on the generators $g_1, \ldots, g_{n-1}$ the word W_n recursively defined by

$$H_n = (W_{n-1}g_{n-1})^n = W_ng_{n-1}$$

gives a Hamilton line of the group's graph.

Proof: For $n = 3$, $W_3g_2 = (g_1g_2)^3$ satisfies the definition of W (section I) and $H_3 = (g_1g_2)^2g_1g_2 = W_3g_2$ with $W_3 = g_2$ an identity in S_3. For $n = 4$, $W_3g_3 = g_2g_3$ is an identity in S_4 and so either side is of order 4; the powers $(W_3g_3)^k$, k: $1, \ldots, 4$, are a full set of coset representatives of S_3 in S_4. Assuming now H_{n-1} to give a Hamilton line for S_{n-1} with $W_{n-1} = g_{n-2}$ in S_{n-1}, one gets $W_{n-1} = g_{n-2}$ in S_n and $W_{n-1}g_{n-1} = g_{n-2}g_{n-1}$ there; the latter identity has the product of the $n-1$ distinct transpositions t_i of S_n on either side which has order n; its powers are a full set of representatives of the cosets of S_{n-1} in S_n. This verifies the formula.

Since S_3 is not cyclic and since a recursive formula requires that a full set of generators of S_n occur among those of S_{n+m}, $m \geq 0$, $n-1$ is the minimal number of generators to give this result. This concludes the proof.

However, $n-1$ is not the least number of generators on which every S_n has a Hamilton line. As a combination of geometric and algebraic arguments will show, every symmetric group has a minimal set of involutory generators on which its graph has a Hamilton line.

VII. In a graph, a cyclically ordered set $v_1, v_2, \ldots, v_k, v_1$ of distinct vertices defines a polygon if two consecutive vertices are always connected by an edge. Two polygons are neighbors if they share at least one edge. A graph is of degree n if each vertex is on just n edges.

A connected graph of degree 3 has a Hamilton line if there is a set P of (pairwise disjoint) polygons containing every vertex just once and a set Q of (pairwise disjoint) rectangles containing every vertex just once and such that no member of P contains every vertex of a member of Q.

Proof: The edges belonging to P form a set of circles; designate these by (P). Then (P) "spans" the vertices: contains each just once. Every vertex v is on just one polygon P_i of P and so no two members of P have an edge in common; similarly for Q. If Q_1 in Q has two distinct neighbors, P_1 and P_2, then Q_1 and P_1 have just one edge E_1, Q_1, and P_2 have just one edge E_2, in common. E_1 and E_2 being disjoint form a pair of opposite sides of $Q_1 = E_1E_3E_2E_4$. Then the edges of P_1, P_2, and Q_1—with E_1 and E_2 left off—form a circle C. Let (P') designate the set gotten from (P) when P_1 and P_2 are replaced by C; then (P') spans the vertices of the graph and consists of disjoint circles.

Suppose the same operation applied to (P'), thus reducing the number of circles, and that this process is repeated as long as possible, yielding a set of circles (P^*). The circles in (P^*) will be disjoint and span the vertices. If (P^*) contains more than one circle, then, since the graph is connected, there are two circles, C_1 and C_2, in it with some vertex v_1 of C_1 connected to a vertex v_2 of C_2 by an edge E. If now E is on a polygon Q_1 of Q, whose parallel sides not involving E may be denoted by v_1p_1 and v_2p_2, then v_1p_1 and v_2p_2 belong to C_1, respectively, C_2; replacing them by the edges v_1v_2 and p_1p_2 joins C_1 and C_2 to one circle—contrary to the assumption that the number of circles was minimal. If E is not on any Q_i of Q, let E_2 and E_3 be the other two edges issuing from v_1. Then, E_2 and E_3 are on C_1 and must be on some Q_1 of Q, so C_1 contains three vertices and two consecutive sides of Q_1. It is clear from the construction that there had to be a P_1 in P containing these vertices and sides of Q_1; however, the fourth vertex of Q_1 is not on P_1 by hypothesis and so can be on no polygon of P. This is impossible. Hence, (P^*) is just one circle spanning the vertices, as claimed.

For arbitrary $n \geq 3$, let t_i designate the transposition $(i, i+1)$, i: $1, \ldots, n-1$, $g_1 = t_1$, $g_2 =$ product of all the t_i with i odd, $g_3 =$ product of all the t_i with i even. Then the elements g_1, g_2, g_3 generate S_n and the corresponding graph has a Hamilton line.

Proof: The two elements g_1 and g_2g_3 are known to generate S_n and so g_1, g_2, g_3 also generate. Since each of these elements has order 2, the graph has degree 3.

On the graph of a group every relation $w(g) = g_{j_1} \ldots g_{j_k} = e$ between the generators gives rise to a closed path; this path can be chosen to contain any given vertex. When $w(g)$ contains no proper segment that is itself an identity of the group, this closed path is a polygon.

For $n > 4$, $R_1 = (g_1g_2)^2$ and $R_2 = (g_1g_3)^6$ are relations of S_n; it is easy to check that they give rise to polygons in the graph of S_n. Since $g_k = g_k^{-1}$ in S_n for each k, the edges g_k and g_k^{-1} coincide. Thus if Q_1 is a rectangle through v_1, v_2, v_3, and v_4 defined by R_1, it is the only such rectangle through any one of these vertices. There are therefore $n/4$ disjoint rectangles, the set Q, that span the graph. Similarly, R_2 gives rise to $n/6$ disjoint polygons, the set P, that span the graph. As the degree is 3, the fact that g_2 does not occur in R_2 insures that no member of P contains every vertex of a member of Q. By the previous result, the graph has a Hamilton line.

For $n = 3$, the relation $(g_1g_2)^3$, and for $n = 4$ the relations $(g_1g_2)^2$, $(g_1g_3)^3$ accomplish the same. This concludes the proof.

The minimum number of generators for a symmetric group is two. Clearly, neither of the methods just used avails here. Solutions are easy to find in the graph of S_n on the generators $S = (12\ldots n)$ and $T = (12)$ when n is small, but I have been unable to find one for every n.

Remark. As with other problems connected with Hamilton lines of a graph, the problem of finding such a line on the graph of a group can be posed as a variant of the chess-problem of the knight [5]. This is a question as to whether the knight can, in just 64 moves, reach every square on the chessboard and be back at the (arbitrary) start.

For a group on the distinct generators $g_1, g_2 = g_1^{-1}, \ldots, g_{2k} = g_{2k-1}^{-1}, \ldots, g_t$, with $g_i^2 = 1$ for $i > 2k$, let $M = (a_{ij})$ be a matrix of m rows and t columns. Let $w_1, w_2 \ldots, w_m$ be a fixed ordering of the elements of the group G in question, and $a_{ij} = w_ig_j$. Then the jth column of M is $Gg_j = G = (w_1g_j, w_2g_j, \ldots) = (a_{1j}, a_{2j}, \ldots) = (w_{h_1}, w_{h_2}, \ldots)$. Any set of m places of the matrix containing m distinct entries forms a "chessboard". The permissible "moves" from the place (ij) are to the places (kx), where $a_{ij} = w_k$, x: $1, \ldots, t$. Can the "board" be covered in m "moves" that return to the (arbitrary) start? This manner of putting the problem indicates an examination of the regular permutation group G_r of G.

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