Math 421, Sections 1&3
Homework 9
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Problem 1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function that satisfies $f(0) = 0$ and $f'(0) = 0$. Define the function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) \cdot \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that g is differentiable at 0 and $g'(0) = 0$.

Solution: Type your solution to problem 1 here.

We want to prove that g is differentiable at 0 by showing that $\lim_{g \to 0} \frac{g(a+h)-g(a)}{h}$ exists and equals to 0.

For $h \neq 0$, $\frac{g(0+h)-g(a)}{h} = \frac{f(h)\sin\left(\frac{1}{h}\right)-0}{h} = \frac{f(0+h)-f(0)}{h} \cdot \sin\left(\frac{1}{h}\right)$. Since $f'(0) = 0$ we know that $\lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = 0$. Therefore from lecture we know that $\lim_{h \to 0} \left(\frac{f(0+h)-f(0)}{h} \cdot \sin\left(\frac{1}{h}\right) \right) = 0$. Therefore, g is differentiable at 0 and $g'(0) = 0$. $\square$

Problem 2. Prove that the function $f : \mathbb{R} \to \mathbb{R}, f(x) = |x|^3$ is twice differentiable at any point $a \in \mathbb{R}$, but is not three-times differentiable at 0.

Solution: Type your solution to problem 2 here.

For this problem we have the function

$$f(x) = \begin{cases} x^3 & \text{for } x \geq 0 \\ -x^3 & \text{for } x < 0 \end{cases}.$$

Need to prove $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $f' : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'''(0) = \lim_{h\to 0} \frac{f''(0+h)-f''(0)}{h}$ does not exist.

1)
Case $a > 0$: Then $f(x) = x^3 \forall x \in (a-\delta, a+\delta)$ for $\delta = a$ So the value of f'(a) is the same as for x^3. So $f'(a) = 3a^2$
Case $a < 0$: Then $f(x) = -x^3 \forall x \in (a-\delta, a+\delta)$ for $\delta = |a|$ So the value of f'(a) is the same as for $-x^3$. So $f'(a) = -3a^2$
Case $a = 0$: For $h \neq 0$

$$\frac{f(0+h) - f(0)}{h} = \begin{cases} \frac{h^3-0}{h} = h^2 & \text{for } h > 0 \\ \frac{-h^3-0}{h} = -h^2 & \text{for } h < 0 \end{cases}.$$

So $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = 0$
Alltogether, f is differentiable at $a \forall a \in \mathbb{R}$ and

$$f'(a) = \begin{cases} 3a^2 & \text{for } x \geq 0 \\ -3a^2 & \text{for } x < 0 \end{cases}.$$

2)
Case $a > 0$: Then $f'(x) = 3x^2 \forall x \in (a-\delta, a+\delta)$ for $\delta = a$ So the value of f''(a) is the same as for $3x^2$. So $f''(a) = 6a$
Case $a < 0$: Then $f'(x) = -3x^2 \forall x \in (a-\delta, a+\delta)$ for $\delta = |a|$ So the value of f''(a) is the same as for $-3x^2$. So $f''(a) = -6a$
Case $a = 0$: For $h \neq 0$

$$\frac{f'(0+h) - f'(0)}{h} = \begin{cases} \frac{3h^2-0}{h} = 3h & \text{for } h > 0 \\ \frac{-3h^2-0}{h} = -3h & \text{for } h < 0 \end{cases}.$$

So $f''(0) = \lim_{h\to 0} \frac{f'(0+h)-f'(0)}{h} = 0$
Alltogether, f' is differentiable at $a \forall a \in \mathbb{R}$ and

$$f''(a) = \begin{cases} 6a & \text{for } x \geq 0 \\ -6a & \text{for } x < 0 \end{cases}.$$

3)
Claim: $f'''(0)$ Does not exist For $h \neq 0$

$$\frac{f''(0+h)-f''(0)}{h} = \begin{cases} \frac{6h-0}{h} = 6 & \text{for } h > 0 \\ \frac{-6h-0}{h} = -6 & \text{for } h < 0 \end{cases}.$$

So $\lim_{h\to 0^+} \frac{f''(0+h)-f''(0)}{h} = 6$ and $\lim_{h\to 0^-} \frac{f''(0+h)-f''(0)}{h} = -6$
Therefore $\lim_{h\to 0} \frac{f''(0+h)-f''(0)}{h}$ Does not exist $\square$

Problem 3. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are twice differentiable at any $a \in \mathbb{R}$. Prove that the product fg is also twice differentiable at any $a \in \mathbb{R}$. (Hint: There is no need for limits here. Just apply the product rule a few times.)

Solution: Type your solution to problem 3 here.

Since f and g are twice differentiable, we know that $f''(a)$ and $g''(a)$ exist for any $a \in \mathbb{R}$. Applying the product rule to find the first derivative $(fg)'$:

$$(fg)' = f'g + fg'.$$

using the product rule again to find the second derivative:

$$(fg)'' = (f'g + fg')' = (f''g + 2f'g' + fg'') = f''g + 2f'g' + fg''.$$

All the derivatives involved in $(fg)''$ are derivatives of f and g, and since f and g are twice differentiable, f'', g'', f', and g' exist.
Therefore, the product fg is twice differentiable at any $a \in \mathbb{R}$. $\square$

Problem 4. Prove that for any $n \in \mathbb{N}$, the function $f : (0, \infty) \to \mathbb{R}, f(x) = x^{\frac{n}{2}}$ is differentiable with derivative $f'(x) = \frac{n}{2}x^{\frac{n}{2}-1}$. (Hint: If n is even, then we already know this statement is true. For n odd, write $f(x) = x^{\frac{n-1}{2}} \cdot \sqrt{x}$ and use the product rule.)

Solution: Type your solution to problem 4 here.

When n is even, $\frac{n}{2} \in \mathbb{N}$. Applying the prop from class: $f'(x) = \frac{n}{2}x^{\frac{n}{2}-1}$.
When n is odd: $f(x) = x^{\frac{n-1}{2}} \cdot \sqrt{x}$. Since $\frac{n-1}{2} \in \mathbb{N}$ when n is odd, we know that $\frac{d}{dx}\left(x^{\frac{n-1}{2}}\right) = \frac{n-1}{2}x^{\frac{n}{2}-\frac{3}{2}}$.
Since $\mathbb{N} \subset (0, \infty)$, $\frac{d}{dx}(\sqrt{n}) = \frac{1}{2}x^{-\frac{1}{2}}$. Using the product rule, we get that $\frac{d}{dx}\left(x^{\frac{n-1}{2}} \cdot \sqrt{x}\right) = \left(\frac{n-1}{2}x^{\frac{n}{2}-\frac{3}{2}}\right)\left(x^{\frac{1}{2}}\right) + \left(\frac{1}{2}x^{-\frac{1}{2}}\right)\left(x^{\frac{n-1}{2}}\right) = \frac{n-1}{2}x^{\frac{n}{2}-1} + \frac{1}{2}x^{\frac{n}{2}-1} = \frac{n}{2}x^{\frac{n}{2}-1}$.

Therefore, $f(x) = x^{\frac{n}{2}}$ is differentiable with derivative $f'(x) = \frac{n}{2}x^{\frac{n}{2}-1}$ $\square$