Dynamic Programming | Set 24 (Optimal Binary Search Tree)

Given a sorted array *keys[0.. n-1]* of search keys and an array *freq[0.. n-1]* of frequency counts, where *freq[i]* is the number of searches to *keys[i]*. Construct a binary search tree of all keys such that the total cost of all the searches is as small as possible.

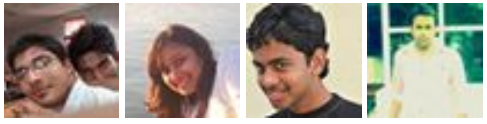
Let us first define the cost of a BST. The cost of a BST node is level of that node multiplied by its frequency. Level of root is 1.

Example 1

```
Input:  keys[] = {10, 12}, freq[] = {34, 50}
There can be following two possible BSTs
        10                       12
          \                     /
           12                 10
          I                     II
Frequency of searches of 10 and 12 are 34 and 50 respectively.
The cost of tree I is 34*1 + 50*2 = 134
The cost of tree II is 50*1 + 34*2 = 118
```

Example 2

```
Input:  keys[] = {10, 12, 20}, freq[] = {34, 8, 50}
There can be following possible BSTs
    10               12                 20         10               20
      \             /    \              /             \             /
      12          10     20           12                20         10
```

```
      \                        /               /          \
       20                     10              12           12
    I             II            III           IV           V
Among all possible BSTs, cost of the fifth BST is minimum.
Cost of the fifth BST is 1*50 + 2*34 + 3*8 = 142
```

1) Optimal Substructure:
The optimal cost for freq[i..j] can be recursively calculated using following formula.

$$optCost(i,j) = \sum_{k=i}^{j} freq[k] + \min_{r=i}^{j}[optCost(i, r-1) + optCost(r+1, j)]$$

We need to calculate ***optCost(0, n-1)*** to find the result.

The idea of above formula is simple, we one by one try all nodes as root (r varies from i to j in second term). When we make *rth* node as root, we recursively calculate optimal cost from i to r-1 and r+1 to j.
We add sum of frequencies from i to j (see first term in the above formula), this is added because every search will go through root and one comparison will be done for every search.

2) Overlapping Subproblems
Following is recursive implementation that simply follows the recursive structure mentioned above.

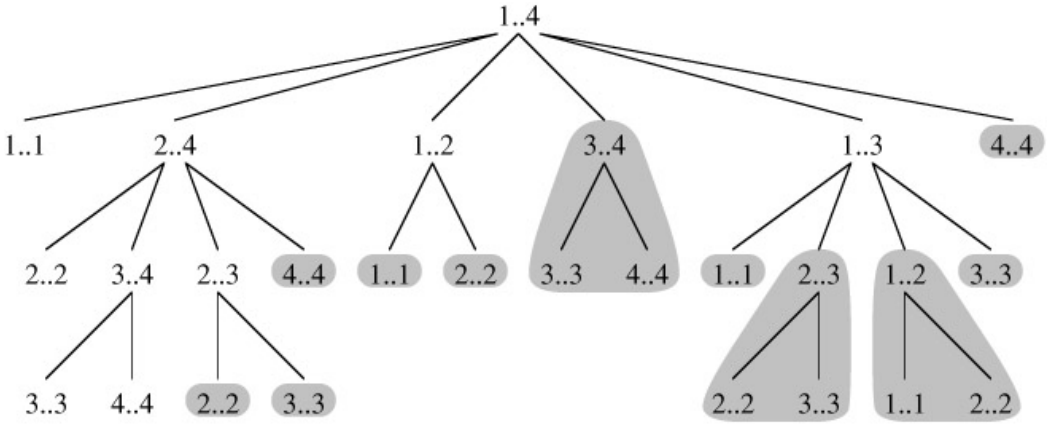
```
// A naive recursive implementation of optimal binary search tree prob
#include <stdio.h>
#include <limits.h>

// A utility function to get sum of array elements freq[i] to freq[j]
int sum(int freq[], int i, int j);

// A recursive function to calculate cost of optimal binary search tre
int optCost(int freq[], int i, int j)
{
   // Base cases
   if (j < i)      // If there are no elements in this subarray
     return 0;
   if (j == i)     // If there is one element in this subarray
     return freq[i];

   // Get sum of freq[i], freq[i+1], ... freq[j]
   int fsum = sum(freq, i, j);
```

```
    // Initialize minimum value
    int min = INT_MAX;

    // One by one consider all elements as root and recursively find co
    // of the BST, compare the cost with min and update min if needed
    for (int r = i; r <= j; ++r)
    {
        int cost = optCost(freq, i, r-1) + optCost(freq, r+1, j);
        if (cost < min)
            min = cost;
    }

    // Return minimum value
    return min + fsum;
}

// The main function that calculates minimum cost of a Binary Search T
// It mainly uses optCost() to find the optimal cost.
int optimalSearchTree(int keys[], int freq[], int n)
{
     // Here array keys[] is assumed to be sorted in increasing order.
     // If keys[] is not sorted, then add code to sort keys, and rearr
     // freq[] accordingly.
     return optCost(freq, 0, n-1);
}

// A utility function to get sum of array elements freq[i] to freq[j]
int sum(int freq[], int i, int j)
{
    int s = 0;
    for (int k = i; k <=j; k++)
        s += freq[k];
    return s;
}

// Driver program to test above functions
int main()
{
    int keys[] = {10, 12, 20};
    int freq[] = {34, 8, 50};
    int n = sizeof(keys)/sizeof(keys[0]);
    printf("Cost of Optimal BST is %d ", optimalSearchTree(keys, freq,
    return 0;
}
```

Output:

```
Cost of Optimal BST is 142
```

Time complexity of the above naive recursive approach is exponential. It should be noted that the above function computes the same subproblems again and again. We can see many subproblems being repeated in the following recursion tree for freq[1..4].



Since same suproblems are called again, this problem has Overlapping Subprolems property. So optimal BST problem has both properties (see this and this) of a dynamic programming problem. Like other typical Dynamic Programming(DP) problems, recomputations of same subproblems can be avoided by constructing a temporary array cost[][] in bottom up manner.

Dynamic Programming Solution

Following is C/C++ implementation for optimal BST problem using Dynamic Programming. We use an auxiliary array cost[n][n] to store the solutions of subproblems. cost[0][n-1] will hold the final result. The challenge in implementation is, all diagonal values must be filled first, then the values which lie on the line just above the diagonal. In other words, we must first fill all cost[i][i] values, then all cost[i][i+1] values, then all cost[i][i+2] values. So how to fill the 2D array in such manner> The idea used in the implementation is same as Matrix Chain Multiplication problem, we use a variable 'L' for chain length and increment 'L', one by one. We calculate column number 'j' using the values of 'i' and 'L'.

```
// Dynamic Programming code for Optimal Binary Search Tree Problem
#include <stdio.h>
#include <limits.h>
```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Aman Hi, Why arent we checking for conditions...

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newCoder3006 If the array contains negative numbers also. We...

Find subarray with given sum · 1 hour ago

newCoder3006 Code without using while loop. We can do it...

Find subarray with given sum · 1 hour ago

```
// A utility function to get sum of array elements freq[i] to freq[j]
int sum(int freq[], int i, int j);

/* A Dynamic Programming based function that calculates minimum cost o
   a Binary Search Tree. */
int optimalSearchTree(int keys[], int freq[], int n)
{
    /* Create an auxiliary 2D matrix to store results of subproblems *
    int cost[n][n];

    /* cost[i][j] = Optimal cost of binary search tree that can be
       formed from keys[i] to keys[j].
       cost[0][n-1] will store the resultant cost */

    // For a single key, cost is equal to frequency of the key
    for (int i = 0; i < n; i++)
        cost[i][i] = freq[i];

    // Now we need to consider chains of length 2, 3, ... .
    // L is chain length.
    for (int L=2; L<=n; L++)
    {
        // i is row number in cost[][]
        for (int i=0; i<=n-L+1; i++)
        {
            // Get column number j from row number i and chain length
            int j = i+L-1;
            cost[i][j] = INT_MAX;

            // Try making all keys in interval keys[i..j] as root
            for (int r=i; r<=j; r++)
            {
               // c = cost when keys[r] becomes root of this subtree
               int c = ((r > i)? cost[i][r-1]:0) +
                       ((r < j)? cost[r+1][j]:0) +
                       sum(freq, i, j);
               if (c < cost[i][j])
                  cost[i][j] = c;
            }
        }
    }
    return cost[0][n-1];
}

// A utility function to get sum of array elements freq[i] to freq[j]
int sum(int freq[], int i, int j)
{
```

```
    int s = 0;
    for (int k = i; k <=j; k++)
       s += freq[k];
    return s;
}

// Driver program to test above functions
int main()
{
    int keys[] = {10, 12, 20};
    int freq[] = {34, 8, 50};
    int n = sizeof(keys)/sizeof(keys[0]);
    printf("Cost of Optimal BST is %d ", optimalSearchTree(keys, freq,
    return 0;
}
```

Output:

```
Cost of Optimal BST is 142
```

Notes

1) The time complexity of the above solution is O(n^4). The time complexity can be easily reduced to O(n^3) by pre-calculating sum of frequencies instead of calling sum() again and again.

2) In the above solutions, we have computed optimal cost only. The solutions can be easily modified to store the structure of BSTs also. We can create another auxiliary array of size n to store the structure of tree. All we need to do is, store the chosen 'r' in the innermost loop.

Please write comments if you find anything incorrect, or you want to share more information about the topic discussed above.

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34 Comments GeeksforGeeks

Sort by Newest

Lohith Ravi • 4 days ago

we can also put the weights in a Max Heap with key->weight & Value->index .a
break the array at the index recursively. to form right and left subtree.

Forming the heap and taking = nlogn and that will be the complexity of algo as

NB • 25 days ago

Food for thought - This problem is similar to Huffman coding concept. (where
prefix code to symbols that occur more frequently). Huffman coding can be s
weights are given. This cant be directly be applied here because the the binary
binary SEARCH tree.

Guest • a month ago

Recursion tree for above example
opt(0,2)

root0 root1 root2
/ / \ \
opt(1,2) opt(0,0) opt(2,2) opt(0,1)
root1 root2 root0 root1
/ \ / \
/ \ / \
opt(2,2) opt(1,1) opt(1,1) opt(0,0)

Junyi Hu • 3 months ago

Very clear explanation!

Samar • 3 months ago

Following code is simple and easy to understand.

```
#include <iostream>
#include <cstdlib>

using namespace std;

struct node {
int data;
node *left;
node *right;
};

node * setroot(int n) {
node *p;
p = new node;
p->data = n;
p->left = NULL;
p->right = NULL;
```

see more

gautam • 4 months ago

Example that shows greedy doesn't work
keys[x,y,z,w] frequency[2, 8, 1, 9] and given that x<y<z<w so="" if="" you="" b
w(maximum="" frequency)="" you="" will="" get="" total="" cost="" 34="" ,="" w
bst="" with="" y(frequency="" 8)="" as="" a="" root="" you="" will="" get="" the=
shows="" that="" greedy="" won't="" work="" here.="">

gautam gautam • 4 months ago

Example that shows greedy doesn't work
keys[x,y,z,w] frequency[2,
8, 1, 9] and given that x<y<z<w. so="" choosing="" greedy="" w="" as=
34="" but="" choosing="" the="" y="" as="" root="" have="" cost="" 33.=

prashant jha • 5 months ago

```
#include<iostream>
#define infinity 999999
using namespace std;
struct s
{
int key;
int freq;
};
int fun(s *arr,int n)
{
if(n==0)
return 0;
if(n==1)
return (arr[0].freq);
s *left,*right;
int min=infinity;int i,j,m;
for(i=0;i<n;i++) {="" int="" r_cost="0,s1=0,s2=0,p1=0,p2=0;" for(j="0;j&amp;lt;n;j++)
r_cost="r_cost+arr[j].freq;" if(arr[j].key<arr[i].key)="" s1++;="" else="" s2++;="
```

see more

1 

jv • 8 months ago

regarding

2) In the above solutions, we have computed optimal cost only. The solutions
structure of BSTs also. We can create another auxiliary array of size n to stor
do is, store the chosen ‘r’ in the innermost loop.

i think this needs any array of NxN as we need to store r at every level and bac

Can you please explain how this can be done with array of size N only.

4

shiwakant.bharti • 9 months ago

Awesome post with amazing code and comments. Thank you admin!

Meanwhile there is a minor bug which leads to Exception in Java java.lang.Arra

The issue is in this code we have not created and array of size n+1. Also this i
no bound checking support there. Fixed code below.

```
            // i is row number in cost[][]
            for (int i = 0; i < n - L + 1; i++) {
```

2

Born Actor • 10 months ago

```
//to print the minimum cost and inorder traversal of teh tree formed



#include <iostream>
#include<string>
#include<sstream>
#include<iomanip>
#include <stdio.h>
#include <math.h>
#include <vector>
```

```
#include <stdlib.h>
using namespace std;
int keys[50];
int n;
int frequencies[50];
class node

{
```

see more

1

atul • a year ago

In the given example the output should be :134

(50*1)+(34*2)+(8*2)=134

i.e following tree :_

```
      20
     /  \
    10  12
```

but it seems to me the following details about the question is missing which w
correct:-

if ith value is considered as the root then 0 to i-1 elements lies on the left side
lies on the right side of ith node.

considering the fact that this tree is also valid

see more

Dheeraj atul • a year ago

Did u read the question? You don't even need to read the complete que
Tree. The example that you have given above is not a binary search tre

1

rajat rastogi • a year ago

Next problem will be print structure of optimal binary search tree in O(nlogn) ti

```
/* Paste your code here (You may delete these lines if not writing c
```

rajat rastogi rajat rastogi • a year ago

Correction, problem statment should be...Write algorithm to find Optim
time.

yashraj • a year ago

"We add sum of frequencies from i to j (see first term in the above formula), th
go through root and one comparison will be done for every search."

what does this mean.. i could not get. can you please explain in detail

1

jv yashraj • 8 months ago

lets say
keys[]={1,2,3}
frequency[]={3,10,5}
when you are considering 2 as root and finding the min cost

mincost(1, with size 1)+mincost(3, with size 1) + cost 2(=10) + s
moved to second level you need to add there cost also(=3+5)

so if see the total will become

cost of all keys + minat 1 + mniat 3

hope it clears this now

tejas • a year ago

Please change the condition to (i<=n-L).

1

rocker • a year ago

Why is the recurrence relation just considers (i,r-1) and (r+1, j) to be the subp
apart from root can be part of any of the subtrees.
The one defined here signifies that, elements from (i to r-1) or (r+1 to j) constit
case.

Unknown rocker • a year ago

Because the tree here is a BST. So it is obvious that we consider the s
root. Same statement follows for the right.

sreeram Unknown • a year ago

But that requires frequencies to be sorted right ?

or am i missing something ,,,

sreeram sreeram • a year ago

Oh no not frequencies ...keys ...i think here the assump

ammy • a year ago

Finding optimal BST can be done in o(n^2) using knuth's algorithm to find roots 1)<=root(i,j)<=root(i+1,j)...isn&#039t it??

Piyush • a year ago

In example 2, tree structure II has the least code, not V

Kartik ➚ Piyush • a year ago

Cost of IInd tree = 1*8 + 34*2 + 50*2 = 176

Which is more than cost of Vth tree.

Arvind B R • a year ago

The description is misleading "Construct a binary search tree of all keys such as small as possible." Here you have not constructed any binary search tree ,

1

Kartik ➚ Arvind B R • a year ago

The main algorithm for the given problem lies in finding out the total cos augmented to construct the tree as well. We will soon add code to con

BSTlover ➚ Kartik • 7 months ago

Can you share that solutions pls?

4

op • a year ago

good explanation

...Greedy algorithm....
make the most frequent element root, do the same for left and right subtrees

kapser op • a year ago

As OP said,

Why haven't you used Greedy algorithm ? (sort the frequencies by des
the keys).

Mathan Kumar kapser • a year ago

I think the question is to find optimal binary "search" tree... Not a

```
/* Paste your code here (You may delete these lines if
```

Kartik kapser • a year ago

Consider the following example
keys[] = {10, 12, 20};
freq[] = {100, 99, 98};
Among the 5 possible BSTs. following BST has the minimum c

```
   12
  /  \
10    20
Cost = 99*1 + 100*2 + 98*2 = 495
```

But according to Greedy, we should get following BST

```
     10
      \
       12
         \
          20

Cost = 100*1 + 99*2 + 98*3 =  595
```

The cost from Greedy approach is much more than the optima

Guru ➚ Kartik • a year ago

Greeedy works too. The above example has distributed always try to build a complete tree.

```
/* Paste your code here (You may delete these li
```

Kartik ➚ kapser • a year ago

Geedy algorithm doesn't always give the optimal solution. We v

2

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