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(1) $f(n) = 4n^2 - 3n + 2 \quad g(n) = n^2 + 6n + 4$

To prove :- $f(n) = \theta(g(n))$

Proof :- To find c_1, we need to find c_1 such that

$$c_1 g(n) \le f(n) \quad \forall\, n \geq n_0 \ \& \ n_0 = 0$$

$$\Rightarrow c_1[n^2 + 6n + 4] \le 4n^2 - 3n + 2 \qquad 20c_1 \le 12$$

Pick $c_1 = \frac{1}{4}$, the above becomes $\qquad 5 \le 12$

$$\Rightarrow 15n^2 - 18n + 4 \geq 0$$

which holds true for all $n \geq n_0 = 0$

We also need to find c_2 such that

$$f(n) \le c_2 g(n) \quad \forall\, n \geq n_0 \ \& \ n_0 = 0$$

$$\Rightarrow 4n^2 - 3n + 2 \le c_2[n^2 + 6n + 4]$$

Pick $c_2 = 1$, the above becomes

$$\Rightarrow 3n^2 - 9n - 2 \le 0$$

which holds true for all $n \geq n_0 = 0$

So if $\boxed{c_1 = \frac{1}{4},\ c_2 = 1 \text{ and } n_0 = 0}$, all the required conditions hold.

(2) Rank of the functions by increasing growth rate :-

$n^{1/\ln(n)}$, $\ln^{1.5} n$, $\sqrt{2}^{\lg(n)}$, $n^{0.5}$, $2^{\lg(n)}$, n, $\ln(n)^{\ln(n)}$, $n^2 + n\ln(n)$
$g_1(n)$ $g_2(n)$ $g_3(n)$ $g_4(n)$ $g_5(n)$ $g_6(n)$ $g_7(n)$ $g_8(n)$

$n^{\ln(n)}$, 2.5^n, $n!$, $(n+1)!$
$g_9(n)$ $g_{10}(n)$ $g_{11}(n)$ $g_{12}(n)$

Comparing $g_1(n)$ and $g_2(n)$ using Limit's theorem : $g_3(n)$

$$\Rightarrow C = \lim_{n\to\infty} \frac{g_1(n)}{g_2(n)} = \lim_{n\to\infty} \frac{n^{1/\ln(n)}}{\ln^{1.5} n} = \lim_{n\to\infty} \frac{e}{\ln^{1.5} n} = 0 \Rightarrow C = 0$$

$\therefore C = 0 \Rightarrow g_1(n) = o(g_2(n)) \Rightarrow g_1(n) = O(g_2(n))$

Comparing $g_2(n)$ and $g_3(n)$ using Limit's ru :-

$$\Rightarrow C = \lim_{n\to\infty} \frac{g_3(n)}{g_4(n)} = \lim_{n\to\infty} \frac{\ln^{1.5} n}{\sqrt{n}} = \lim_{n\to\infty} \frac{\frac{1}{2}\ln^{0.5} n \times \frac{1}{n}}{-\frac{1}{2} \times n^{-1/2}} \quad \left[\text{L'Hospital's rule}\right]$$

$$= \lim_{n\to\infty} -\left[\frac{\ln^{0.5} n}{\sqrt{n}}\right]$$

$$= \lim_{n\to\infty} -\left[\frac{-\frac{1}{2}\ln^{-0.5} n \times \frac{1}{n}}{-\frac{1}{2} \times n^{-1/2}}\right]$$

$$= \lim_{n\to\infty} -\left[\frac{1}{\sqrt{n\ln(n)}}\right]$$

$$= 0$$

$\therefore C = 0 \Rightarrow g_2(n) = o(g_3(n)) \Rightarrow g_2(n) = O(g_3(n))$

Comparing $g_4(n)$ and $g_5(n)$ using Limit's test:-

$g_3(n) = (\sqrt{2})^{\log(n)} = \sqrt{n}$ and $g_4 = \sqrt{n}$, since the 2 functions are the same, from Limit's rule $\Rightarrow C = 1$

$\because C = 1 \Rightarrow g_3(n) = \Theta(g_4(n))$

$$\Rightarrow C = \lim_{n \to \infty} \frac{g_4(n)}{g_5(n)} = \lim_{n \to \infty} \frac{\sqrt{n}}{2^{\lg(n)}} = \lim_{n \to \infty} \frac{\sqrt{n}}{n}$$

$$= \lim_{n \to \infty} \frac{1}{\sqrt{n}}$$

$$= \frac{1}{\infty}$$

$$\Rightarrow C = 0$$

$\because C = 0 \Rightarrow g_4(n) = o(g_5(n)) \Rightarrow g_4(n) = O(g_5(n))$

As $g_5(n) = 2^{\lg(n)} = n$ which is equal to $g_6(n) = n$

$\therefore$ applying Limit's test $\Rightarrow C = 1$

$\because C = 1 \Rightarrow g_5(n) = \Theta(g_6(n))$

Comparing $g_6(n)$ and $g_7(n)$ using Limit's test

$$\Rightarrow C = \lim_{n \to \infty} \frac{g_6(n)}{g_7(n)} = \lim_{n \to \infty} \frac{n}{\ln(n)^{\ln(n)}}$$

For all $R(n)$, $R(n) = e^{\ln(R(n))}$

$$\Rightarrow C = \lim_{n\to\infty} e^{\ln\left[\frac{n}{\ln(n)^{\ln(n)}}\right]} = \lim_{n\to\infty} e^{\ln(n) - \ln\left[\ln(n)^{\ln(n)}\right]}$$

$$= e^{\lim_{n\to\infty}\left[\ln(n) - \ln(n)\ln(\ln(n))\right]}$$

$$= e^{\lim_{n\to\infty}\left[\ln(n)(1 - \ln(\ln(n)))\right]}$$

$$= e^{\lim_{n\to\infty}\left[\ln(n)\right] \times \lim_{n\to\infty} \ln(\ln(n))}$$

$$= e^{\infty \times -\infty}$$

$$= 0 \text{ [approximately]}$$

$$\therefore C = 0 \Rightarrow g_6(n) = o(g_7(n)) \Rightarrow g_6(n) = O(g_7(n))$$

Comparing $g_7(n)$ and $g_8(n)$ using limit's theorem

$$\Rightarrow C = \lim_{n\to\infty} \frac{g_7(n)}{g_8(n)} = \lim_{n\to\infty} \frac{\ln(n)^{\ln(n)}}{n^2 + n\ln(n)}$$

$$\forall R(n) \Rightarrow R(n) = e^{\ln(R(n))}$$

$$\Rightarrow C = \lim_{n\to\infty} e^{\ln\left[\frac{\ln(n)^{\ln(n)}}{n^2+n\ln(n)}\right]} =$$

$$= e^{\lim_{n\to\infty} \ln(n)\left[\ln(\ln(n))\right] - \left[\ln(n^2 + n\ln(n))\right]}$$

Solving exponential power seperately

$$\Rightarrow \lim_{n\to\infty} \left[\ln(n) \left[\ln(\ln(n)) - \frac{\ln(n^2+n\ln(n))}{\ln(n)} \right] \right]$$

$$\Rightarrow \lim_{n\to\infty} \left[\ln(n) \left[\ln(\ln(n)) - \ln(n+\ln(n)) \right] \right]$$

$$\Rightarrow \lim_{n\to\infty} [\ln(n)] \times \lim_{n\to\infty} [\underbrace{\ln(\ln(n))}_{p} - \underbrace{\ln(n+\ln(n))}_{q}]$$

As $n\to\infty$, $p \ll q$ and the second limit gets closer to $-\infty$, which simplies the above equation to

$$\Rightarrow C = e^{\infty \times -\infty} \approx e^{-\infty} = 0$$

$$\therefore C=0 \Rightarrow g_7(n) = o(g_8(n)) \Rightarrow g_7(n) = O(g_8(n))$$

Comparing $g_8(n)$ and $g_9(n)$ using Limit's theorem:-

$$\Rightarrow C = \lim_{n\to\infty} \frac{g_8(n)}{g_9(n)} = \lim_{n\to\infty} \frac{n^2+n\ln(n)}{n^{\ln(n)}}$$

$$\Rightarrow C = \lim_{n\to\infty} \frac{1}{n^{\ln(n)-2}} + \underbrace{\lim_{n\to\infty} \frac{n\ln(n)}{n^{\ln(n)}}}_{(i)}$$

The eq (i), as $n\to\infty$, the denominator becomes larger than the numerator.

For e.g., if $n=10 \Rightarrow n\ln(n) \approx 27$ and $n^{\ln(n)} \approx 200$.

Hence as $n \to \infty$, the eq(i) tends to zero.

$\therefore$ $c = 0 + 0 \Rightarrow c = 0$

Hence

$$c = 0 \Rightarrow g_8(n) = o(g_9(n))$$

$$\Rightarrow g_8(n) = O(g_9(n))$$

Comparing $g_9(n)$ and $g_{10}(n)$ using Limit's theorem:-

$$\Rightarrow c = \lim_{n\to\infty} \frac{n^{\ln(n)}}{2.5^n}$$

As $n \to \infty$, the denominato is very huge when compared to the numerator. Let's assume $n=10 \Rightarrow n^{\ln(n)} \approx 200.71$ and $2.5^n = 9536.74$. Hence as $n \to \infty$, the above value gets closer and closer to 0

$$\Rightarrow \lim_{n\to\infty} \frac{n^{\ln(n)}}{2.5^n} = 0$$

$\therefore$ $c = 1 \Rightarrow g_9(n) = o(g_{10}(n)) \Rightarrow g_9(n) = O(g_{10}(n))$

Comparing $g_{10}(n)$ and $g_{11}(n)$ using Limit's rules:-

$$\Rightarrow c = \lim_{n\to\infty} \frac{g_{10}(n)}{g_{11}(n)} = \lim_{n\to\infty} \frac{2.5^n}{n!} = \lim_{n\to\infty} \frac{2.5^n}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$$ [Applying String Formula]

$$= \lim_{n\to\infty} \frac{2.5^n e^n}{\sqrt{2\pi n}(n)^n}$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \left[\frac{2.5e}{n}\right]^n \times \frac{1}{\sqrt{n}}$$

As $n \to \infty$, the denominator becomes very large and the value of $\frac{2.5e}{n}$ tends to zero. Hence when $n \to \infty$, the value is closer to 0.

$$\therefore \quad c = \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \left[\frac{2.5e}{n}\right]^n \times \frac{1}{\sqrt{n}} = 0$$

$$\therefore \quad c = 0 \Rightarrow g_{10}(n) = o(g_{11}(n))$$

$$\Rightarrow g_{10}(n) = O(g(n))$$

Comparing $g_{11}(n)$ and $g_{12}(n)$ using Limit's test:-

$$\Rightarrow c = \lim_{n\to\infty} \frac{g_{11}(n)}{g_{12}(n)} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} \Rightarrow c = 0$$

$$\therefore \quad c = 0 \Rightarrow g_{11}(n) = o(g_{12}(n)) = g_{11}(n) = O(g_{12}(n))$$

$\therefore$ The order is:-

$n^{1/\ln(n)}$, $\ln^{1.5} n$, $\{\sqrt{2}^{\lg(n)}, n^{0.5}\}$, $\{2^{\lg n}, n\}$, $\ln(n)^{\ln(n)}$, $n^2 + n\ln(n)$, $n^{\ln(n)}$, 2.5^n, $n!$, $(n+1)!$

③ $f(n)$ and $g(n)$ are asymptotically positive functions

1. $f(n) = \Omega(g(n)) \Rightarrow g(n) = O(f(n))$

Suppose $f(n) = \Omega(g(n))$. By definition, it means that $\exists\, c_1 > 0,\ n_0 \geq 0$ so that $\forall\, n \geq n_0,\ 0 \leq c_1 g(n) \leq f(n)$

Since by definition of $f(n) = O(g(n)) \Rightarrow$ $\exists\, c_1 > 0,\ n_0 \geq 0$ so that $\forall\, n \geq n_0,\ 0 \leq f(n) \leq c_1 g(n)$

As for the same constants c_1 and n_0, the definitions are contrasting each other, we can say

if $\boxed{f(n) = \Omega((n)) \Rightarrow g(n) = O(f(n))}$

2. $f(n) = O(g(n)) \Rightarrow \ln(f(n)) = O(\ln(g(n)))$ where $\ln(g(n)) \geq 1$ and $\ln(f(n)) \geq 1\ \forall$ sufficiently large n.

Suppose $f(n) = O(g(n))$. By definition it means that $\exists\, c_1 > 0,\ n_0 \geq 0$ so that $\forall\, n \geq n_0,\ 0 \leq f(n) \leq c_1 g(n)$

To prove $\ln(f(n)) = O(\ln(g(n)))$ with the above conditions we need to prove $c_2 = \ln(c_1) \geq 0$, $n_0' = \ln(n_0) \geq 0$ so that $\forall\, n \geq n_0'$

$$0 \leq \ln(f(n)) \leq \ln[c_2 g(n)]$$

As we already know that $c_1 > 0$ and applying natural logarithm "ln" for c_1, we get the range of $c_2 [= \ln c_1]$.

Value of c_1 can also be 1, which implies that $c_2 = \ln(c_1) = \ln(1) = 0$ which means that c_2 can have values less than or equal to zero i.e.,

$$c \leq 0 \ \forall \ 0 < c_1 \leq 1 \quad \text{--- (i)}$$

Since the above equation (i) doesn't satisfy the conditions required for Big-O notation, the statement

$f(n) = O(g(n)) \Rightarrow \ln(f(n)) = O(\ln(g(n)))$ isn't true.

3. $f(n) = \Theta(h(n))$ and $g(n) = \Theta(h(n))$

$$\Rightarrow f(n) \times g(n) = \Theta(h(n)^2)$$

Using Limit's theorem :-

$$c_1 = \lim_{n \to \infty} \frac{f(n)}{h(n)} > 0 \quad \text{and} \quad c_2 > \lim_{n \to \infty} \frac{g(n)}{h(n)} > 0$$

Applying Limit's theorem on the 3^{rd} equation :-

$$c_3 = \lim_{n \to \infty} \frac{f(n) \times g(n)}{h(n)^2}$$

$$\Rightarrow C_3 = \lim_{n\to\infty} \frac{f(n)}{h(n)} \times \lim_{n\to\infty} \frac{g(n)}{h(n)}$$

$$\Rightarrow C_3 = C_1 \times C_2$$

$$\because C_1 > 0 \quad \text{and} \quad C_2 > 0$$

$$\Rightarrow C_3 > 0 \ [\text{a constant}]$$

From Limit's theorem

$$\because C_3 > 0 \Rightarrow f(n) \times g(n) = \theta(h(n))^2$$

4. $f(n) = \theta(g(n)) \Rightarrow 3^{f(n)} = \theta(3^{g(n)})$

From Limit's theorem

$$\Rightarrow C_1 = \lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$$

$$\Rightarrow C_1 > 0$$

We need to prove Limit's theorem for the second equation which is :- $3^{f(n)} = \theta(3^{g(n)})$

$$\therefore C_2 = \lim_{n\to\infty} \frac{f_1(n)}{f_2(n)}$$

where $f_1(n) = 3^{f(n)}$ & $3^{g(n)}$

$$\Rightarrow C_2 = \lim_{n\to\infty} \frac{3^{f(n)}}{3^{g(n)}} \Rightarrow C_2 = \lim_{n\to\infty} 3^{f(n)-g(n)}$$

$$\Rightarrow C_2 = \lim_{n\to\infty} 3^{g(n)\left(\frac{f(n)}{g(n)} - 1\right)}$$

$$\Rightarrow C_2 = 3^{\lim_{n\to\infty} g(n) \times \lim_{n\to\infty}\left(\frac{f(n)}{g(n)} - 1\right)}$$

$$\Rightarrow C_2 = 3^{\lim_{n\to\infty} g(n) \times c_2 - \lim_{n\to\infty} 1}$$

$\because$ the above value is not a constant and can be any value b/w $[-\infty, \infty]$, therefore it does not satisfy the required condition.

$$\therefore f_1(n) = o(g(n)) \not\Rightarrow 3^{f_1(n)} = \theta(3^{g(n)})$$

5. $g(n) = o(f(n)) \Rightarrow f(n) + g(n) = \theta(f(n))$

Let's assume that the above statement is true

$$\Rightarrow \lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$$

Let's check limit test for RHS

$$\Rightarrow \lim_{n\to\infty} \frac{f(n) + g(n)}{f(n)} = \lim_{n\to\infty}\left(1 + \frac{g(n)}{f(n)}\right)$$

$$= 1 + 0 = 1$$

$$\Rightarrow (f(n) + g(n)) = \theta(f(n))$$

Hence proved

(4) Number of distinct ways of choosing n objects from 2n objects is given by:-

$$\binom{2n}{n} = \frac{(2n)!}{n!\,n!} \quad \text{--- (i)}$$

To prove :- $\binom{2n}{n} = \theta\left(\frac{4^n}{n^{1/2}}\right)$

Stirling formula states that $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot \left[1+\theta\left(\frac{1}{n}\right)\right]$

Substituting above in (i)

$$\frac{(2n)!}{n!\,n!} = \frac{\sqrt{2\pi(2n)}\left(\frac{2n}{e}\right)^{2n}}{\left[\sqrt{2\pi(n)}\left(\frac{n}{e}\right)^n\right]^2} = \frac{2\sqrt{\pi n}\left(\frac{2n}{e}\right)^{2n}}{2\pi n\left(\frac{n}{e}\right)^{2n}}$$

$$\Rightarrow \frac{(2n)!}{n!\,n!} = \frac{4^n}{\sqrt{\pi n}} \quad \text{--- (i)}$$

Using Limit test, let $f(n) = \binom{2n}{n}$, $g(n) = \frac{4^n}{n^{1/2}}$

$$\Rightarrow C = \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty}\left[\binom{2n}{n} \Big/ \frac{4^n}{n^{1/2}}\right]$$

Substituting (i)

$$\Rightarrow C = \lim_{n\to\infty}\left[\frac{4^n}{[\pi n]^{1/2}} \Big/ \frac{4^n}{n^{1/2}}\right] \Rightarrow C = \lim_{n\to\infty} \frac{1}{\pi^{1/2}}$$

$$\Rightarrow C = \frac{1}{\sqrt{\pi}}$$

As from Limit's test, $C>0$, then

$\Rightarrow f(n) = \theta(g(n))$

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(5) Asymptotically tight bounds for the following:-

(a) $\sum_{k=1}^{n} k^2 + 4k^{1.5}$

Solution:-

As $a=1$ and $b=n$, which implies that

$$\int_0^n (x^2+4x^{1.5})\,dx \le \sum_{i=1}^{n} x^2+4x^{1.5} \le \int_1^{n+1} (x^2+4x^{1.5})\,dx$$

Solving the lower bound seperately:-

$$\int_0^n (x^2+4x^{1.5})\,dx = \left[\frac{1}{3}x^3 + \frac{8}{5}x^{2.5}\right]_0^n$$

$$= \left[\frac{n^3}{3}+\frac{8}{5}n^{2.5}\right] - [0+0]$$

$$= n^{2.5}\left[\frac{\sqrt{n}}{3}+\frac{8}{5}\right]$$

Solving the upper bound seperately

$$\int_1^{n+1} (x^2+4x^{1.5})\,dx = \left[\frac{x^3}{3}+\frac{8x^{2.5}}{5}\right]_1^{n+1}$$

$$= \left[\frac{(n+1)^3}{3}+\frac{8(n+1)^{2.5}}{5}\right] - \left[\frac{1}{3}+\frac{8}{5}\right]$$

$$= (n+1)^{2.5}\left[\frac{\sqrt{n+1}}{3}+\frac{8}{5}\right] - \left[\frac{29}{15}\right]$$

$\therefore$ the asymptotically tight bounds are :-

$$n^{2.5}\left[\frac{\sqrt{n}}{3}+\frac{8}{5}\right] \leq \sum_{i=1}^{n} k^2+4k \leq (n+1)^{2.5}\left[\frac{\sqrt{n+1}}{3}+\frac{8}{5}\right]-\frac{29}{15}$$

(b) $\sum_{k=1}^{n} 2k\ln k + 3k$

As $a=1$ and $b=n$, which implies that

$$\int_0^n (2k\ln k + 3k)\,dk \leq \sum_{k=1}^{n} 2k\ln k + 3k \leq \int_1^{n+1} (2k\ln k + 3k)\,dk$$

Solving the upper bound seperately

$$\int_0^n (2k\ln k + 3k)\,dk = \int_0^n (2k\ln k)\,dk + \int_0^n (3k)\,dk$$

Applying the formula $\int u.dv = uv - \int v\,du$

$$= 2\left[\left[\frac{k^2}{2}(\ln k)\right]_0^n - \int_0^n \frac{k^2}{2}\left(\frac{1}{k}\right)dk\right] + \left[\frac{3k^2}{2}\right]_0^n$$

$$= 2\left[\left(\frac{n^2\ln(n)}{2} - 0\right) - \frac{1}{2}\left(\frac{k^2}{2}\right)_0^n\right] + \frac{3n^2}{2}$$

$$= 2\left[\frac{n^2\ln(n)}{2} - \frac{n^2}{4}\right] + \frac{3n^2}{2}$$

$$= n^2\ln(n) - \frac{n^2}{2} + \frac{3n^2}{2}$$

$$= n^2[\ln(n) - 1]$$

Solving the lower bound seperately

$$\int_1^{n+1} (2k\ln k + 3k)\,dk = 2\left[\left[\frac{k^2}{2}(\ln k)\right]_1^{n+1} - \int_1^{n+1}\left(\frac{k}{2}\right)dk\right] + \left[\frac{3k^2}{2}\right]_1^{n+1}$$

$$\Rightarrow 2\left[\frac{(n+1)^2}{2}\ln(n+1) - \frac{1}{4}\left[(n+1)^2 - 1\right]\right] + \frac{3}{2}\left[(n+1)^2 - 1\right]$$

$$\Rightarrow (n+1)^2 \ln(n+1) - \frac{\left[(n+1)^2 - 1\right]}{2}$$

$$\Rightarrow (n+1)^2\left[\ln(n+1) - \frac{1}{2}\right] + \frac{1}{2}$$

$\therefore$ the asymptotically tight bounds are :-

$$n^2\left[\ln(n) - 1\right] \leq \sum_{i=1}^{n} (2k\ln(k))\,dk \leq (n+1)^2\left[\ln(n+1) - \frac{1}{2}\right] + \frac{1}{2}$$

⑥ 1. $T(n) = T(2n/3) + n^2$

Using Master's theorem :-

$a = 1 \quad b = \frac{3}{2} \quad f(n) = n^2 \quad k = 2$

$\log_b a = \log_{1.5} 1 = 0$

As $k > \log_b a \Rightarrow T(n) = \theta(n^2)$

$\therefore$ this satisfies case 3 we also need to check another condition which is

$a f(n/b) \leq c f(n) \;\forall\; c < 1$

and $c = \frac{a}{b^k} < 1$

$\therefore C = \frac{1}{\left(\frac{3}{2}\right)^2} = \frac{4}{9} < 1 \quad \therefore 1.f\left(\frac{2n}{3}\right) \leq \frac{4}{9} f\left(\frac{2n}{3}\right)$

$\Rightarrow f\left(\frac{2n}{3}\right) \leq \frac{4}{9} f\left(\frac{2n}{3}\right)$

$\therefore$ both the conditions of case 3 of Master's theorem are matching $\Rightarrow T(n) = \theta(n^2)$

2. $T(n) = 4T(n/2) + n^2$

$a = 4 \quad b = 2 \quad f(n) = n^2 \quad k = 2$

$\log_b a = \log_2 4 = 2$

$\therefore k = \log_b a, \; p = 0$

we can use Master's theorem case (ii) to say:

$$T(n) = \theta(n^2 \log n)$$

3. $T(n) = 9T(n/3) + n^{1.5}$

$a = 9 \quad b = 3 \quad f(n) = n^{1.5} \quad k = 1.5 \quad p = 0$

$$\log_b a = 2$$

$\therefore \; k < \log_b a$

according to the case (i) of Master's theorem:-

$$T(n) = \theta(n^{\log_3 9}) = \theta(n^2)$$

$$\Rightarrow T(n) = \theta(n^2)$$

4. $T(n) = 2T(n/4) + \sqrt{n}$

$a = 2 \quad b = 4 \quad f(n) = \sqrt{n} \quad k = 0.5$

$$\log_b a = \log_4 2 = \log_4 4^{0.5} = 0.5$$

$\therefore \; k = \log_b a$

this satisfies the case (ii) of master's theorem

and hence $\Rightarrow T(n) = \theta(n^{0.5} \log n)$

$$\Rightarrow T(n) = \theta(\sqrt{n} \log n)$$

5. $T(n) = T(n-2) + n - 1$

As Master's theorem cannot be used for the above equation, we can use recursive substitution.

$$\therefore T(n) = T(n-2) + n - 1$$
$$= T(n-4) + (n-3) + n - 1$$
$$= T(n-6) + (n-5) + (n-3) + (n-1)$$

$$= (n-1) + (n-3) + \dots\dots + 3 + 1$$

Since the above is sum of n odd numbers, we can say that the above value is n^2

$$\therefore T(n) = n^2$$

$$\Rightarrow T(n) = O(n^2)$$

$\therefore$ The asymptotically tight bound for the above equation is

$$T(n) = \Theta(n^2)$$

6.

$$T(n) = \begin{cases} \theta(1) & n \le 2 \\ T(n^{1/3}) + 3 & n > 2 \end{cases}$$

Let's simplify equation for $n > 2$

$\therefore \quad T(n) = T(n^{1/3}) + 3$

$= [T(n^{1/9}) + 3] + 3$

$= [T(n^{1/27}) + 3] + 3 + 3$

Repeat this k times

$\Rightarrow \quad T(n) = T(n^{1/3^k}) + 3k$

When $n^{\frac{1}{3^k}} = 2$, the recursion stops $\Rightarrow n^{\frac{1}{3^k}} = 2$

Applying $\log_2$ on both sides

$\Rightarrow \quad \frac{1}{3^k} \log_2 n = 1 \Rightarrow \log_2 n = 3^k \Rightarrow \log_3 \log_2 n = k$

$\therefore \; T(n) = T(2) + 3 \log_3 \log_2 n$

$\therefore$ the time complexity for the above equation is

$\Rightarrow T(n) = 3 \log_3 \log_2 n$

$\Rightarrow \quad T(n) = \theta(\log_3 \log_2 n)$

7 1. $a_0 = 1 \quad a_1 = 2$

$\forall\, n \geq 0 \quad a_{n+2} = 3a_{n+1} - 2a_n$

Characteristic equation :- $x^2 = 3x - 2 \Rightarrow x^2 - 3x + 2 = 0$

The roots for the above equation can be found out by $\Rightarrow x = \dfrac{3 \pm \sqrt{9-8}}{2}$

$= 2$ or 1

$\therefore \alpha_1 = 2$ and $\alpha_2 = 1$

The solution has the form:

$a_n = a_1 (2)^n + a_2 . (1)^n$

$\Rightarrow a_n = 2^n a_1 + a_2$

$\therefore$ For $n = 0 \Rightarrow a_0 = 1 = a_1 + a_2$ ... (i)

$n = 1 \Rightarrow a_1 = 2 = 2a_1 + a_2$ — (ii)

$\therefore \quad 2(i) - (ii) \Rightarrow (2a_1 + 2a_2) - (2a_1 + a_2) = 2 - 2$

$\Rightarrow \quad a_2 = 0$ ---- (iii)

Substituting (iii) in (i)

$\Rightarrow 1 = a_1$

$\therefore \quad a_n = 2^n$ is the characteristic equation

2. $b_0 = 1 \quad b_1 = 2 \quad b_2 = 4$

$\forall\ n \geq 0 \quad b_{n+3} = 3b_{n+2} - 4b_n$

$\therefore$ The characteristic equation

$\Rightarrow x^3 = 3x^2 - 4 \Rightarrow x^3 - 3x^2 + 4 = 0$

Substituting $x = -1 \Rightarrow -1 - 3 + 4 = 0$ -- (i)

Hence $x = -1$ is one of the roots

$\Rightarrow (x+1)(x^2 - 4x + 4) = 0$

$\Rightarrow (x+1)(x-2)(x-2) = 0$

$\therefore \alpha_1 = -1 \quad \alpha_2 = 2 \quad \alpha_3 = 2$

$\therefore$ the solution has the form:

$b_n = b_1(-1)^n + b_2(2^n) + b_3 n(2^n)$

$\therefore$ if $n = 0 \Rightarrow b_0 = 1 = b_1 + b_2$ ---- (i)

$n = 1 \Rightarrow b_1 = 2 = -b_1 + 2b_2 + 2b_3$ ---- (ii)

$n = 2 \Rightarrow b_2 = 4 = b_1 + 4b_2 + 8b_3$ --- (iii)

(iii) $- 4$(ii) $\Rightarrow b_1 + 4b_2 + 8\cancel{b_3} + 4b_1 - 8b_2 - 8\cancel{b_3} = 4 - 8$

$\Rightarrow 5b_1 - 4b_2 = -4$

$\Rightarrow 5b_1 - 4[1 - b_1] = -4$ [From (i)]

$\Rightarrow 5b_1 - 4 + 4b_1 = -4$

$\Rightarrow b_1 = 0 \Rightarrow b_2 = 1$

Sub in (ii) $\Rightarrow 2 = 0 + 2 + 2b_3$

$\Rightarrow b_3 = 0$

$\therefore$ Characteristic equation is

$\Rightarrow \quad b_n = 2^n$