Contents

Chapter 1

Generalizing Tukey Depth in a Uniformly Distributed Unit Ball in R^p

1.1 Definition and Notation

Let $B^p = \{ x \in R^p : \|x\| \leq 1 \}$ be the closed unit ball in R^p. We assume a **uniform** (isotropic) probability measure on B^p. Hence, for any measurable $A \subset B^p$,

$$F(A) \;=\; \frac{\mathrm{Vol}(A)}{\mathrm{Vol}(B^p)},$$

where $\mathrm{Vol}(\cdot)$ denotes the usual Lebesgue volume in p dimensions.

Closed Halfspace and Tukey (Halfspace) Depth. For a direction $u \in S^{p-1}$ (the unit sphere) and a point $\theta \in R^p$, define the closed halfspace

$$H(u, \theta) \;=\; \{ x \in R^p : \; \langle x - \theta,\, u \rangle \;\geq\; 0 \}.$$

Its boundary hyperplane $\langle x - \theta,\, u \rangle = 0$ passes through θ. Given a probability measure F on R^p, the *Tukey depth* of θ is

$$\mathrm{depth}(\theta; F) \;=\; \inf_{u \in S^{p-1}} \; F\big(H(u, \theta)\big).$$

In the following, we focus on computing $\mathrm{depth}(\theta; F)$ explicitly when F is uniform on B^p.

1.2 Geometry of Halfspaces in the Ball

1.2.1 Key Observation: Height from the Center

To illustrate the concept, consider the unit disk B^2 with center O and a point θ in its interior (with $\|\theta\| < 1$). The measure of a closed halfspace $H(u, \theta) \cap B^2$ depends on the perpendicular distance from O to the hyperplane $\langle x - \theta,\, u \rangle = 0$. For a given direction $u \in S^1$ (the unit circle), this distance is

$$d \;=\; |\langle \theta,\, u \rangle|.$$

Among all directions u that ensure $\theta \in H(u, \theta)$, the minimal volume of $H(u, \theta) \cap B^2$ is achieved when d is maximized. Geometrically, this optimal situation occurs when u is chosen parallel to the vector from the origin to θ, i.e.,

$$u \;=\; \frac{\theta}{\|\theta\|}\,.$$

In that case, the boundary hyperplane becomes

$$\langle x, \theta \rangle \;=\; \|\theta\|^2,$$

and the associated halfspace is

$$H\Big(\frac{\theta}{\|\theta\|}, \theta\Big) \;=\; \Big\{x \in R^2 : \langle x, \theta \rangle \geq \|\theta\|^2\Big\}.$$

The Tukey depth of θ is given by the ratio of the area of the cap $H\Big(\frac{\theta}{\|\theta\|}, \theta\Big) \cap B^2$ to the total area of B^2:

$$\operatorname{depth}(\theta; F) \;=\; \frac{\operatorname{Vol}\Big(H\Big(\frac{\theta}{\|\theta\|}, \theta\Big) \cap B^2\Big)}{\operatorname{Vol}(B^2)}.$$

Figure 1.1 (Figure 1) illustrates this geometry. In the figure, the unit disk is shown with center O and a point θ. A family of lines through θ is depicted; among these, the dashed line is orthogonal to the vector from O to θ and attains the greatest distance from O. This optimal line determines the smallest cap (in terms of area), and its relative area (with respect to the entire disk) defines the Tukey depth of θ.

Figure 1.1: In R^2, for a point θ in the unit disk, the optimal halfspace is determined by the line orthogonal to the vector from the origin O to θ. This line maximizes the distance from O and minimizes the area of the cap cut off from the disk, thereby defining the Tukey depth.

1.3 Volume of the Corresponding Spherical Cap

1.3.1 Integral Form for the Volume

Without loss of generality, by rotational symmetry we can align θ with the first coordinate axis. Then the condition $\langle x, \theta \rangle \geq \|\theta\|^2$ becomes simply

$$x_1 \;\geq\; \|\theta\|,$$

where x_1 denotes the first coordinate of x. Consequently,

$$\big\{\, x \in B^p : \langle x, \theta \rangle \geq \|\theta\|^2 \big\} \;=\; \{\, x : \|x\| \leq 1,\; x_1 \geq \|\theta\|\}.$$

To find its volume, we slice the unit ball $\|x\| \leq 1$ at each fixed $r \in [\|\theta\|, 1]$. Each slice is a $(p-1)$-dimensional ball of radius $\sqrt{1-r^2}$. Hence

$$\operatorname{Vol}\big(H(\theta) \cap B^p\big) \;=\; \int_{\|\theta\|}^{1} \operatorname{vol}\big(B^{p-1}\big)\,\big(1-r^2\big)^{\frac{p-1}{2}}\, dr.$$

1.3.2 Final Depth Formula

We start from the expression

$$\operatorname{depth}(\theta; F) \;=\; \frac{1}{\operatorname{Vol}(B^p)} \int_{\|\theta\|}^{1} \operatorname{vol}\big(B^{p-1}\big)\,\big(1-r^2\big)^{\frac{p-1}{2}}\, dr,$$

where the total volume of the p-ball and the $(p-1)$-ball are given by

$$\mathrm{Vol}(B^p) \;=\; \frac{\pi^{\frac{p}{2}}}{\Gamma\!\left(\frac{p}{2}+1\right)} \quad \text{and} \quad \mathrm{vol}\!\left(B^{p-1}\right) \;=\; \frac{\pi^{\frac{p-1}{2}}}{\Gamma\!\left(\frac{p-1}{2}+1\right)}.$$

Substituting these expressions into the depth formula, we obtain

$$\mathrm{depth}(\theta;F) \;=\; \frac{\Gamma\!\left(\frac{p}{2}+1\right)}{\pi^{\frac{p}{2}}}\,\frac{\pi^{\frac{p-1}{2}}}{\Gamma\!\left(\frac{p-1}{2}+1\right)}\int_{\|\theta\|}^{1}\left(1-r^2\right)^{\frac{p-1}{2}}dr.$$

This prefactor simplifies to

$$\frac{\Gamma\!\left(\frac{p}{2}+1\right)}{\Gamma\!\left(\frac{p-1}{2}+1\right)}\,\frac{1}{\pi^{1/2}}.$$

To evaluate the integral, we substitute $u=r^2$ so that $du=2r\,dr$ and hence

$$dr = \frac{du}{2\sqrt{u}},$$

with the limits of integration changing from $r=\|\theta\|$ to $u=\|\theta\|^2$ and from $r=1$ to $u=1$. Thus, the integral becomes

$$\int_{\|\theta\|}^{1}\left(1-r^2\right)^{\frac{p-1}{2}}dr \;=\; \frac{1}{2}\int_{\|\theta\|^2}^{1}u^{-\frac{1}{2}}\left(1-u\right)^{\frac{p-1}{2}}du.$$

By the definition of the incomplete Beta function,

$$B_z(a,b) \;=\; \int_0^z t^{\,a-1}\,(1-t)^{\,b-1}\,dt,$$

with $a=\frac{1}{2}$ and $b=\frac{p+1}{2}$, the above integral can be written as

$$\int_{\|\theta\|^2}^{1}u^{-\frac{1}{2}}\left(1-u\right)^{\frac{p-1}{2}}du \;=\; B\!\left(\tfrac{1}{2},\tfrac{p+1}{2}\right) - B_{\|\theta\|^2}\!\left(\tfrac{1}{2},\tfrac{p+1}{2}\right).$$

Hence,

$$\int_{\|\theta\|}^{1}\left(1-r^2\right)^{\frac{p-1}{2}}dr \;=\; \frac{1}{2}\left[B\!\left(\tfrac{1}{2},\tfrac{p+1}{2}\right) - B_{\|\theta\|^2}\!\left(\tfrac{1}{2},\tfrac{p+1}{2}\right)\right].$$

Next, recall the identity that connects the Beta function with Gamma functions:

$$B(a,b) \;=\; \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}.$$

Thus,

$$B\!\left(\tfrac{1}{2},\tfrac{p+1}{2}\right) \;=\; \frac{\Gamma\!\left(\frac{1}{2}\right)\Gamma\!\left(\frac{p+1}{2}\right)}{\Gamma\!\left(\frac{p}{2}+1\right)}.$$

The regularized incomplete Beta function is defined as

$$I_z(a,b) \;=\; \frac{B_z(a,b)}{B(a,b)}.$$

Combining these relations, we express the integral as

$$\int_{\|\theta\|}^{1} \left(1-r^2\right)^{\frac{p-1}{2}}\, dr \;=\; \frac{1}{2}\, B\!\left(\tfrac{1}{2}, \tfrac{p+1}{2}\right) \left[1 - I_{\|\theta\|^2}\!\left(\tfrac{1}{2}, \tfrac{p+1}{2}\right)\right].$$

Substituting back into the expression for $\operatorname{depth}(\theta; F)$, we have

$$\operatorname{depth}(\theta;F) \;=\; \frac{\Gamma\!\left(\frac{p}{2}+1\right)}{\Gamma\!\left(\frac{p-1}{2}+1\right)}\, \frac{1}{\pi^{1/2}} \cdot \frac{1}{2}\, B\!\left(\tfrac{1}{2}, \tfrac{p+1}{2}\right) \left[1 - I_{\|\theta\|^2}\!\left(\tfrac{1}{2}, \tfrac{p+1}{2}\right)\right].$$

Substitute the Beta function in terms of Gamma functions:

$$\frac{1}{2}\, B\!\left(\tfrac{1}{2}, \tfrac{p+1}{2}\right) \;=\; \frac{1}{2}\, \frac{\Gamma\!\left(\frac{1}{2}\right)\Gamma\!\left(\frac{p+1}{2}\right)}{\Gamma\!\left(\frac{p}{2}+1\right)}.$$

This cancels with the $\Gamma\!\left(\frac{p}{2}+1\right)$ in the numerator:

$$\operatorname{depth}(\theta;F) \;=\; \frac{1}{2}\, \frac{\Gamma\!\left(\frac{1}{2}\right)\Gamma\!\left(\frac{p+1}{2}\right)}{\Gamma\!\left(\frac{p-1}{2}+1\right)\pi^{1/2}} \left[1 - I_{\|\theta\|^2}\!\left(\tfrac{1}{2}, \tfrac{p+1}{2}\right)\right].$$

Since $\Gamma\!\left(\frac{1}{2}\right) = \sqrt{\pi}$, we obtain the final closed-form expression:

$$\boxed{\operatorname{depth}(\theta;F) \;=\; \frac{1}{2}\left[1 - I_{\|\theta\|^2}\!\left(\tfrac{1}{2}, \tfrac{p+1}{2}\right)\right], \quad \text{for } \|\theta\| < 1.}$$

This derivation explicitly shows how the volumes of the $(p-1)$- and p-dimensional balls (involving π and Γ functions) combine with the integral expression to yield the regularized incomplete Beta function in the final formula.

1.3.3 Examples

One Dimension ($p = 1$): In one dimension, the unit ball is the interval $B^1 = [-1, 1]$. For a point $\theta \in [-1, 1]$ (with $\|\theta\| = |\theta|$), a halfspace is simply a closed ray. For instance, when $\theta > 0$ the natural choice is the ray

$$H(1,\theta) = \{x \in R : x \geq \theta\}.$$

Its length is $1 - \theta$ while the total length of $[-1, 1]$ is 2. Thus the Tukey depth is

$$\operatorname{depth}(\theta;F) = \frac{1-\theta}{2},$$

or, more symmetrically,

$$\operatorname{depth}(\theta;F) = \frac{1-|\theta|}{2}.$$

It is easy to verify that substituting $p = 1$ into the Beta-function expression yields the same result.

Chapter 2

Projection Method for Bivariate Student-t Depth

2.1 Introduction and Background

This chapter derives Tukey's (halfspace) depth for a bivariate Student-t distribution using a projection method that exploits the elliptical symmetry of the distribution. In such distributions, the depth of a point reduces to a univariate tail probability of a t-distribution and can be written in closed form via the regularized incomplete Beta function.

A random vector $\mathbf{X} \in R^2$ follows a bivariate Student-t distribution with location $\mu \in R^2$, scale matrix $\Sigma \in R^{2\times 2}$, and $\nu > 0$ degrees of freedom,

$$\mathbf{X} \sim t_\nu(\mu, \Sigma),$$

if its density is

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\pi\,\nu)\sqrt{\det(\Sigma)}} \left[1 + \frac{(\mathbf{x}-\mu)^\top \Sigma^{-1}(\mathbf{x}-\mu)}{\nu}\right]^{-\frac{\nu+2}{2}}.$$

A key property is that for any unit vector $\mathbf{u} \in S^1$ the projection $\mathbf{u}^\top \mathbf{X}$ is univariate t_ν with location $\mathbf{u}^\top \mu$ and scale $\sqrt{\mathbf{u}^\top \Sigma \mathbf{u}}$.

2.2 Tukey's Depth and Its Geometric Interpretation

Tukey's (halfspace) depth for a point $\mathbf{x} \in R^2$ with respect to a distribution F is defined as

$$\mathrm{Depth}(\mathbf{x}; F) \;=\; \inf_{\mathbf{u}\in S^1} F\Big(\{\mathbf{z} \in R^2 : \mathbf{u}^\top \mathbf{z} \geq \mathbf{u}^\top \mathbf{x}\}\Big).$$

Equivalently, it is the infimum over directions of the univariate tail probabilities,

$$\mathrm{Depth}(\mathbf{x}; F) \;=\; \inf_{\mathbf{u}\in S^1} \Big[1 - F_{\mathbf{u}^\top \mathbf{X}}(\mathbf{u}^\top \mathbf{x})\Big].$$

Geometrically, one considers all closed halfspaces whose boundaries pass through $\mathbf{x}$ and selects the one with the smallest probability mass. For elliptical distributions such as the Student-t, symmetry implies that the minimal halfspace is obtained when the direction $\mathbf{u}$ is chosen along the line joining μ and $\mathbf{x}$. That is, the optimal direction is

$$\mathbf{u}_* = \frac{\mathbf{x} - \mu}{\|\mathbf{x} - \mu\|}.$$

In this case the halfspace becomes

$$H\Big(\mathbf{u}_*, \mathbf{x}\Big) \;=\; \Big\{\mathbf{z} \in R^2 : \mathbf{u}_*^\top \mathbf{z} \geq \mathbf{u}_*^\top \mathbf{x}\Big\},$$

and the depth reduces to

$$\text{Depth}\big(\mathbf{x}; t_\nu(\mu, \Sigma)\big) \;=\; 1 - F_{T_\nu}\Big(\frac{\mathbf{u}_*^\top \mathbf{x} - \mu_Y}{s}\Big),$$

where

$$\mu_Y = \mathbf{u}_*^\top \mu, \quad s = \sqrt{\mathbf{u}_*^\top \Sigma\, \mathbf{u}_*},$$

and T_ν denotes a standard t_ν variable.

2.3 Projection Method and Closed-Form Expression

Once the optimal direction is fixed, the projected variable

$$Y = \mathbf{u}_*^\top \mathbf{X} \sim t_\nu\Big(\mu_Y, s^2\Big)$$

and its tail probability is given by

$$P(Y \geq \mathbf{u}_*^\top \mathbf{x}) \;=\; 1 - F_{T_\nu}\Big(\frac{\mathbf{u}_*^\top \mathbf{x} - \mu_Y}{s}\Big).$$

The standard t-CDF $F_{T_\nu}(z)$ can be written in terms of the regularized incomplete Beta function $I_x(a, b)$. In particular, one has

$$F_{T_\nu}(z) \;=\; \frac{1}{2} + \frac{z}{|z|}\,\frac{1}{2}\, I_{\frac{\nu}{\nu + z^2}}\Big(\frac{\nu}{2}, \frac{1}{2}\Big), \quad z \neq 0.$$

Thus, the explicit closed-form expression for Tukey's depth becomes

$$\text{Depth}(\mathbf{x}; t_\nu(\mu, \Sigma)) \;=\; 1 - \left\{\frac{1}{2} + \frac{\text{sgn}\Big(\frac{\mathbf{u}_*^\top \mathbf{x} - \mu_Y}{s}\Big)}{2}\, I_{\frac{\nu}{\nu + \Big(\frac{\mathbf{u}_*^\top \mathbf{x} - \mu_Y}{s}\Big)^2}}\Big(\frac{\nu}{2}, \frac{1}{2}\Big)\right\}.$$

This formula expresses the depth in terms of standard functions and highlights that, due to elliptical symmetry, the depth depends solely on the one-dimensional tail probability in the direction of $\mathbf{x} - \mu$.

Remark. When $\mathbf{x} = \mu$, symmetry ensures that the depth achieves its maximum value, typically 1/2 for such symmetric distributions.