Final Transition Scheme (Including Different c Scenarios):

Given a line through (a, b) described by

$$y - b = k(x - a),$$

with $k = \tan\theta$, the probability that an exponential vector (X, Y) (with density $\lambda_1\lambda_2 e^{-\lambda_1 x - \lambda_2 y}$) lies below the line in the first quadrant is

$$I(k) = \int_{x=x_0}^{\infty} \lambda_1 e^{-\lambda_1 x}\left[1 - e^{-\lambda_2\left(k(x-a)+b\right)}\right] dx, \quad \text{where } x_0 = \max\left\{0,\, a - \frac{b}{k}\right\}.$$

Replacing k with $\tan\theta$, we rewrite the line as

$$x\cos\theta + y\sin\theta = a\cos\theta + b\sin\theta,$$

and define the threshold

$$c = a\cos\theta + b\sin\theta.$$

Expressing the point in polar coordinates via $a = r\cos\phi$ and $b = r\sin\phi$ gives

$$c = r\cos(\theta - \phi).$$

For intermediate values of c (neither asymptotically small nor large), the optimal separation direction is obtained by solving the implicit equation

$$\frac{d}{d\theta} I(\theta) = 0,$$

where

$$I(\theta) = \int_{x=\max\{0, a - \frac{b}{\tan\theta}\}}^{\infty} \lambda_1 e^{-\lambda_1 x} \left[1 - e^{-\lambda_2(\tan\theta(x-a)+b)}\right] dx.$$

This integral formulation, with the lower limit depending on θ via the $\max$ operator, fully characterizes the behavior of the separation probability for intermediate c, linking the geometric parameter $c = r\cos(\theta - \phi)$ with the slope parameter $k = \tan\theta$; the resulting equation must be solved (typically numerically) to determine the optimal angle θ^* (and hence the optimal $k^* = \tan\theta^*$).

Different c Scenarios:

For **small** c (i.e., when the threshold is close to the origin), the approximation

$$\max\{0, a - b/\tan\theta\} \approx 0$$

simplifies the integral, leading to

$$k^* \approx \frac{\lambda_1}{\lambda_2} = \tan\theta^*.$$

For **large** c (i.e., when the threshold is far from the origin), the probability integral is dominated by the exponential decay, and we can apply the approximation

$$I(k) \approx \log\left(\frac{c}{a}\right),$$

resulting in the asymptotic behavior

$$k^* \approx \frac{a}{b} = \tan \phi.$$

This logarithmic correction comes from the fact that, for large values of c, the integrand tends to approach the boundary of the domain, and thus the result becomes proportional to $\log(c)$.

For **intermediate** c, the situation is more complex, and no direct analytical simplifications hold. In this case, the optimization must be conducted numerically using

$$\frac{d}{d\theta} \int_{x=\max\{0,a-b/\tan\theta\}}^{\infty} \lambda_1 e^{-\lambda_1 x} \left[1 - e^{-\lambda_2(\tan\theta(x-a)+b)}\right] dx = 0.$$