1 Problem statement

1.1 Strong formulation

We will work on a two-dimensional Helmholtz problem. Let $\Omega \subset \mathbb{R}^2$ be an open bounded region whose border $\delta\Omega$ is sufficiently smooth. Our strong formulation is the following:

$$\text{Find } u \text{ such that: } \begin{cases} \Delta u + k^2 u = -f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$\Delta = \nabla^2$ is the Laplace operator, u could be the unknown acoustic pressure field, k is the wavenumber and f is a source term. In general, u and k can be complex-valued. However, the paramter studies will be conducted on real-valued cases, with $k > 0$. The extension to complex-valued cases should be straightforward.

1.2 Galerkin formulation

The usual Galerkin formulation for our problem is the following:

$$\text{Find } u^h \in \mathcal{V}^h \text{ such that } \forall v^h \in \mathcal{V}^h, \ a_G(u^h, v^h) = \left\langle f, v^h \right\rangle_{L^2(\Omega)}$$

Where $\mathcal{V}^h \subset H_0^1(\Omega)$ is finite-dimensional, $\langle u, v \rangle_{L^2(\Omega)}$ is the $L^2(\Omega)$ scalar product, and a_G is the following sesquilinear form:

$$a_G : H_0^1(\Omega)^2 \to \mathbb{C}$$
$$(u, v) \mapsto \langle \nabla u, \nabla v \rangle_{L^2(\Omega)} - k^2 \langle u, v \rangle_{L^2(\Omega)}$$

We will use a mesh method to build $\mathcal{V}^h$, partitioning Ω in mutually exclusive elements.

1.3 GLS

The Galerkin/least-squares formulation uses the previous formulation and adds other terms whose purpose is to minimize the square of the residue over the element interiors $\tilde{\Omega}$. Their contribution is weighted by a parameter τ :

$$\text{Find } u^h \in \mathcal{V}^h, \ \forall v^h \in \mathcal{V}^h, \ a_{GLS}(u^h, v^h) = \left\langle f, v^h \right\rangle_{L^2(\Omega)} + \left\langle \tau f, \mathcal{L}v^h \right\rangle_{L^2(\tilde{\Omega})}$$

Where $\mathcal{L}. = \Delta. + k^2.$ is the Helmholtz operator, and a_{GLS} is the follwing sesquilinear form:

$$a_{GLS} : H_0^1(\Omega)^2 \to \mathbb{C}$$
$$(u, v) \mapsto a_G(u, v) + \langle \tau \mathcal{L}u, \mathcal{L}v \rangle_{L^2(\tilde{\Omega})}$$

2 Dispersion analysis and optimal parameter

The Helmholtz equation suffers from the pollution effect, causing numerical wavenumbers to lose accuracy with the physical wavenumber increasing.

Exact solutions When considering the sourceless ($f = 0$) and free-space version of our problem, the exact solutions are $\forall x \in \mathbb{R}, \ u(x) = Ce^{i(k_x x + k_y y)}$. With $k^2 = k_x^2 + k_y^2$ Our studies will revolve around this case.

$$\sum_{i,j \in -1,0,1} A_{G,0,0}^{i,j} U_{i,j}^h = 0 \tag{1}$$

Setting $2h_\xi = h_\xi^- + h_\xi^+$ with ξ being x or y.
The values of the nonzero entries are the following:
Four corner nodes ($i, j \in \{-, +\}$):

$$A_{G,0,0}^{i,j} = \frac{k^2 h_x^i h_y^j}{36} + \frac{h_x^i}{6h_y^j} + \frac{h_y^j}{6h_x^i}$$

Four side nodes:

$$A_{G,0,0}^{\pm,0} = \frac{k^2 h_x^{\pm} h_y}{9} + \frac{2h_y}{3h_x^{\pm}} - \frac{h_x^{\pm}}{3} \frac{h_y}{h_y^- h_y^+}$$

$$A_{G,0,0}^{0,\pm} = \frac{k^2 h_x h_y^{\pm}}{9} + \frac{2h_x}{3h_y^{\pm}} - \frac{h_y^{\pm}}{3} \frac{h_x}{h_x^- h_x^+}$$

Center node:

$$A_{G,0,0}^{0,0} = \frac{4k^2 h_x h_y}{9} - \frac{4h_x}{3} \frac{h_y}{h_y^- h_y^+} - \frac{4h_y}{3} \frac{h_x}{h_x^- h_x^+}$$

Setting $\alpha_\xi^{\pm} h_\xi = h_\xi^{\pm}$ with ξ being x or y.
Four side nodes:

$$A_{G,0,0}^{\pm,0} = \frac{k^2 h_x^{\pm} h_y}{9} + \frac{2h_y}{3h_x^{\pm}} - \frac{h_x^{\pm}}{3\alpha_y^- \alpha_y^+ h_y}$$

$$A_{G,0,0}^{0,\pm} = \frac{k^2 h_x h_y^{\pm}}{9} + \frac{2h_x}{3h_y^{\pm}} - \frac{h_y^{\pm}}{3\alpha_x^- \alpha_x^+ h_x}$$

Center node:

$$A_{G,0,0}^{0,0} = \frac{4k^2 h_x h_y}{9} - \frac{4h_x}{3\alpha_y^- \alpha_y^+ h_y} - \frac{4h_y}{3\alpha_x^- \alpha_x^+ h_x}$$

$$k^2 = \frac{6\left(2f_x^+ h_x^- \left((h_x^+)^2 h_y - (h_y^+)^2 h_y^- - h_y^+ (h_y^-)^2\right) + 2f_x^- h_x^+ \left((h_x^-)^2 h_y - (h_y^+)^2 h_y^- - h_y^+ (h_y^-)^2\right) + 2f_y^+ h_y^- \left(h_x\right)}{}$$

$$\gamma_0 + \sum_{\xi \in \{x,y\}} \sum_{i \in \{-,+\}} \gamma_\xi^i f_\xi^i + \tag{2}$$

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Terme indépendant en haut

$$+ 12h_x^- h_y^+ \left(\left(h_x^+\right)^2 + \left(h_y^-\right)^2 \right)$$

$$+ 12h_x^- h_y^- \left(\left(h_x^+\right)^2 + \left(h_y^+\right)^2 \right)$$

$$+ 12h_x^+ h_y^+ \left(\left(h_x^-\right)^2 + \left(h_y^-\right)^2 \right)$$

$$+ 12h_x^+ h_y^- \left(\left(h_x^-\right)^2 + \left(h_y^+\right)^2 \right)$$

Termes en f en haut

$$12f_x^+ h_x^- \left(\left(h_x^+\right)^2 h_y - \left(h_y^+\right)^2 h_y^- - h_y^+ \left(h_y^-\right)^2 \right) + 12f_x^- h_x^+ \left(\left(h_x^-\right)^2 h_y - \left(h_y^+\right)^2 h_y^- - h_y^+ \left(h_y^-\right)^2 \right)$$

$$+12f_y^+ h_y^- \left(h_x \left(h_y^+\right)^2 - \left(h_x^+\right)^2 h_x^- - h_x^+ \left(h_x^-\right)^2 \right) + 12f_y^- h_y^+ \left(h_x \left(h_y^-\right)^2 - h_x^+ \left(h_x^-\right)^2 - \left(h_x^+\right)^2 h_x^- \right)$$

Terme indépendant en bas (en facteur des 4h)

$$16 h_x h_y$$

Termes en f en bas (en facteur des 4h)

$$4f_x^+ h_x^+ h_y \qquad + \qquad 4f_x^- h_x^- h_y \qquad + \qquad 4f_y^+ h_x h_y^+ \qquad + \qquad 4f_y^- h_x h_y^-$$

Termes en g en bas (en facteur des 4h)

$$+h_x^+ h_y^+ g^{++} + h_x^+ h_y^- g^{+-} + h_x^- h_y^+ g^{-+} + h_x^- h_y^- g^{--}$$

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