

A Galerkin method for the 1D Helmholtz equation

An introduction to PDEs and Numerical Mathematics through the example
of wave propagation

V. de Nodrest¹

¹CentraleSupélec, Université Paris Saclay

October 6, 2025





1 First

First first

2 Bibliography



1 First

First first

2 Bibliography

Many physical problems involve the wave equation, which describes wave propagation in media that are homogeneous, isotropic, linear et non dispersive:

The wave equation

$$\frac{\partial^2 p}{\partial t^2} = c^2 \Delta p$$

$\Delta = \nabla^2$ is the Laplacian, a differential operator.

p is a physical phenomenon propagated through one of the aforementioned media at a speed c .

It depends on both space \mathbf{r} and time t .

Assuming the solution is separable in time and space ($p(\mathbf{r}, t) = u(\mathbf{r})T(t)$), the wave equation can be rewritten as such:

$$\frac{\Delta u}{u} = \frac{1}{c^2} \frac{d^2 T}{dt^2}$$

The left-hand side only depends on space and the right-hand side only depends on time. In order to be equal in any situation, both members need to be equal to the same constant. This constant is set to $-k^2$ for convenience:

$$\begin{aligned} \frac{\Delta u}{u} &= -k^2 \\ \frac{1}{c^2} \frac{d^2 T}{dt^2} &= -k^2 \end{aligned}$$

Rearranging the first equation (space-dependent) yields:

The homogeneous Helmholtz equation

$$\Delta u + k^2 u = 0$$

It is possible to account for sources using f , a function with compact support, thus yielding:

The inhomogeneous Helmholtz equation

$$\Delta u + k^2 u = f$$

We consider the following problem, a simple yet telling example of a situation involving the Helmholtz equation:

Our 1D Helmholtz problem

$$u'' + k^2 u = 0 \text{ in }]0, 1[\quad (1)$$

$$u'(0) = ik \quad (2)$$

$$u'(1) = iku(1) \quad (3)$$

Our 1D Helmholtz problem

$$u'' + k^2 u = 0 \text{ in }]0, 1[\quad (1)$$

$$u'(0) = ik \quad (2)$$

$$u'(1) = iku(1) \quad (3)$$

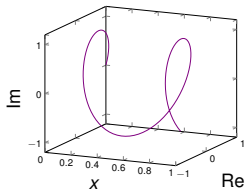
- u is a 1D complex-valued function, at least two times derivable
- $f = 0$, so this problem "sourceless"/homogeneous in the domain
- (2), assigning a value to the derivative, is called a Neumann "flux" boundary condition.
- (3), establishing a linear relationship between the value and the derivative, is called a Robin "impedance" boundary condition.

The problem can be solved with linear PDE tools, yielding an unique solution:

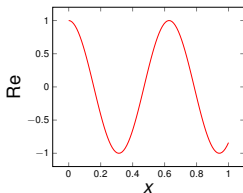
The "Euler wave" exact solution

$$\forall x \in [0, 1], u(x) = e^{ikx}$$

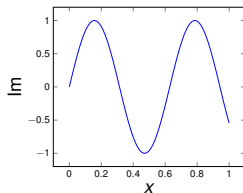
Our problem is a simple case of a well-posed problem.



(a) 3D complex plot



(b) Real part



(c) Imaginary part

Figure: Exact solution with $k = 10$

Let's forget about our exact solution and apply conventional PDE tools, which are mandatory for harder problems and for the Galerkin methods.

We will assume that u exists and is in $V = H^2(0, 1)$ (u is two times derivable and u , u' and u'' can all be squared than integrated over $]0, 1[$).

Then we must have, for every v in V :

$$\int_{]0,1[} u'' \bar{v} + k^2 \int_{]0,1[} u \bar{v} = 0$$

Integrating by parts and noting $\langle u, v \rangle_{L^2} = \int_{]0,1[} u \bar{v}$, we get:

Our weak formulation

$$\forall v \in V, \quad k^2 \langle u, v \rangle_{L^2} - \langle u', v' \rangle_{L^2} + ik \left[u(1) \overline{v(1)} - \overline{v(0)} \right] = 0$$

This is only one of the many possible weak formulations we could have obtained. Other choices could have been made regarding:

- The space to which u belongs (test functions)
- The space to which v belongs (weighting functions)
- The norm and the scalar product

We need to rewrite our weak formulation in the standardized form:

Variational formulation

Find $u \in V_1$ such that

$$\forall v \in V_2, a(u, v) = l(v)$$

Where:

- V_1 and V_2 are Hilbert spaces (with their respective scalar products)
- a is sesquilinear (bilinear in \mathbb{R})
- l is antilinear (linear in \mathbb{R})

First, note that $(V, \langle \cdot, \cdot \rangle_{L^2})$ is a Hilbert space.

Based on the weak formulation, we deduce our forms:

Our sesquilinear form

$$a: V \times V \longrightarrow \mathbb{C}$$

$$(u, v) \longmapsto k^2 \langle u, v \rangle_{L^2} - \langle u', v' \rangle_{L^2} + ik \left[u(1) \overline{v(1)} \right]$$

Our antilinear form

$$l: V \longrightarrow \mathbb{C}$$

$$v \longmapsto ik \left[\overline{v(0)} \right]$$



Continuity of the sesquilinear form





Straightforward numerical solving of the variational formulation is not possible, as there are infinitely many possible trial functions and weighting functions. We discretize those spaces, thus yielding:

Galerkin equation

Find $u \in V_1^h$ such that

$$\forall v \in V_2^h, a(u, v) = l(v)$$

Where $V_1^h \subset V_1$ and $V_2^h \subset V_2$ are finite.

Be reminded that in our case, $V_1 = V_2 = H^2(0, 1)$.

One might notice that the error is orthogonal to the subspaces:

$$a(u - u^h, v^h) = a(u, v^h) - a(u^h, v^h) = f(v^h) - f(v^h) = 0$$



Matrix form of the Galerkin equation



1 First

First first

2 Bibliography

