1 Problem statement

1.1 Strong formulation

We will work on a one-dimensional Helmholtz problem. Let $\Omega \subset \mathbb{R}$ be an open bounded region whose border $\delta\Omega$ is sufficiently smooth. Our strong formulation is the following:

$$\text{Find } u \text{ such that: } \begin{cases} \Delta u + k^2 u = -f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$\Delta = \nabla^2$ is the Laplace operator, u could be the unknown acoustic pressure field, k is the wavenumber and f is a source term. In general, u and k can be complex-valued. However, the parameter studies will be conducted on real-valued cases, with $k > 0$. The extension to complex-valued cases should be straightforward.

1.2 Galerkin formulation

The usual Galerkin formulation for our problem is the following:

$$\text{Find } u^h \in \mathcal{V}^h \text{ such that } \forall v^h \in \mathcal{V}^h,\ a_G(u^h, v^h) = \left\langle f, v^h \right\rangle_{L^2(\Omega)}$$

Where $\mathcal{V}^h \subset H_0^1(\Omega)$ is finite-dimensional, $\langle u, v \rangle_{L^2(\Omega)}$ is the $L^2(\Omega)$ scalar product, and a_G is the following sesquilinear form:

$$\begin{aligned} a_G : H_0^1(\Omega)^2 &\to \mathbb{C} \\ (u, v) &\mapsto \langle \nabla u, \nabla v \rangle_{L^2(\Omega)} - k^2 \langle u, v \rangle_{L^2(\Omega)} \end{aligned}$$

We will use a mesh method to build $\mathcal{V}^h$, partitioning Ω in mutually exclusive elements.

1.3 GLS formulation

The Galerkin/least-squares formulation uses the previous formulation and adds other terms whose purpose is to minimize the square of the residue over the element interiors $\tilde{\Omega}$. Their contribution is weighted by a parameter τ :

$$\text{Find } u^h \in \mathcal{V}^h, \forall v^h \in \mathcal{V}^h,\ a_{GLS}(u^h, v^h) = \left\langle f, v^h \right\rangle_{L^2(\Omega)} + \left\langle \tau f, \mathcal{L} v^h \right\rangle_{L^2(\tilde{\Omega})}$$

Where $\mathcal{L}. = \Delta. + k^2.$ is the Helmholtz operator, and a_{GLS} is the follwing sesquilinear form:

$$\begin{aligned} a_{GLS} : H_0^1(\Omega)^2 &\to \mathbb{C} \\ (u, v) &\mapsto a_G(u, v) + \langle \tau \mathcal{L} u, \mathcal{L} v \rangle_{L^2(\tilde{\Omega})} \end{aligned}$$

2 Dispersion analysis and optimal parameter

The Helmholtz equation suffers from the pollution effect, causing numerical wavenumbers to lose accuracy with the physical wavenumber increasing.

Exact solutions When considering the sourceless ($f = 0$) and free-space version of our problem, the exact solutions are $\forall x \in \mathbb{R},\ u(x) = Ce^{ikx}$. Our studies will revolve around this case.

2.1 Dispersion analysis

2.1.1 Numerical resolution

Using linear interpolation with nodal shape functions N_i that respect the partition of unity and Kronecker delta properties yields the linear system $AU^h = 0$, where U^h is a the vector of nodal values and A is the impedence matrix.
When using our Galerkine formulation, the matrix is $A_{G,ij} = \sum_{e \in E} K^e_{ij} - k^2 M^e_{ij}$ where E is the set of elements whose boundary contain both nodes i and j, and K^e_{ij} and M^e_{ij} are respectively stiffness and masse terms assembled using shape functions i and j over the element e.

2.1.2 Dispersion relation

We study, without loss of generality, the stencil formed by an interior node 0 and its neighbours -1 et $+1$. The distances between those nodes are h_x^- and h_x^+.
For convenience, we might use $h_x^- = \alpha_x^- h_x$, $h_x^+ = \alpha_x^+ h_x$ and $2h_x = h_x^- + h_x^+$.
The "dispersion relation" is the line 0 of the system:

$$A_{0,-1}U^h_{-1} + A_{0,0}U^h_0 + A_{0,+1}U^h_{+1} = 0 \tag{1}$$

The values of the coefficients are:

$$A_{0,-1} = \frac{h_x^- k^2}{6} + \frac{1}{h_x^-} \tag{2}$$

$$A_{0,0} = \frac{2h_x k^2}{3} - \frac{2}{\alpha_x^- \alpha_x^- h_x} \tag{3}$$

$$A_{0,+1} = \frac{h_x^+ k^2}{6} + \frac{1}{h_x^+} \tag{4}$$

2.1.3 Numerical solution

We assume that the numerical solution can be defined by $u^h(x) = Ce^{ik^h x}$ where k^h is the numerical wavenumber.

2.1.4 Dispersion relation

The numerical wave number k^h is asymptotically linked to the exact wavenumber when $k^h h_x \to 0$ and $kh_x \to 0$:

$$k^h \approx k - \frac{1}{24}\left({\alpha^-}^2 + {\alpha^+}^2 - \alpha^- \alpha^+\right) k^3 h_x^2 \tag{5}$$

Proof. Substituting the nodal numerical values in the dispertion relation Eq. (1) yields a relation between k and k^h that we can solve for k^2:

$$k^2 = \frac{6}{h_x^+ h_x^-} \frac{\alpha_x^- \left(1 - \cos h_x^+ k^h\right) + \alpha_x^+ \left(1 - \cos h_x^- k^h\right)}{\alpha_x^- \left(2 + \cos h_x^- k^h\right) + \alpha_x^+ \left(2 + \cos h_x^+ k^h\right)} \tag{6}$$

However, it cannot be exressed as $k^h = f(k)$ because of the cosines of varying wavenumbers. We need to use a Taylor expansion on the cosines when $h_x k^h \to 0$:

$$\cos\left(\alpha^{\pm} h_x k^h\right) = 1 - \frac{{\alpha^{\pm}}^2}{2}(h_x k^k)^2 + \frac{{\alpha^{\pm}}^4}{24}(h_x k^h)^4 + O((h_x k^h)^6) \tag{7}$$

Substituting this expansion in Eq. (6) and simplifying by common factors yields an expression for $(kh_x)^2$ as a ratio of polynomials. It can be rewritten as:

$$\begin{aligned}
&(kh_x)^2 = X\frac{1}{1-Y} \\
&X = (h_x k^h)^2 + \chi(h_x k^h)^4 + O((h_x k^h)^6) \\
&Y = \gamma_1(h_x k^h)^2 + \gamma_2(h_x k^h)^4 + O((h_x k^h)^6) \\
&\chi = -\frac{1}{12}\left({\alpha^-}^2 + {\alpha^+}^2 - \alpha^-\alpha^+\right) \\
&\gamma_1 = \frac{1}{6}\left({\alpha^-}^2 + {\alpha^+}^2 - \alpha^-\alpha^+\right) = -2\chi \\
&\gamma_2 = -\frac{1}{72}\frac{{\alpha^-}^5 + {\alpha^+}^5}{\alpha^- + \alpha^+}
\end{aligned}$$

$Y \to 0$ as $h_x k^h \to 0$, thus allowing the following Taylor expansion:

$$\frac{1}{1-Y} = 1 + \gamma_1(h_x k^h)^2 + (\gamma_1^2 + \gamma_2)(h_x k^h)^4 + O((h_x k^h)^6) \tag{8}$$

A polynomial expression of $(kh_x)^2$ is now available as a product of X and $\frac{1}{1-Y}$:

$$(kh_x)^2 = (h_x k^h)^2 - \chi(h_x k^h)^4 + O((h_x k^h)^6) \tag{9}$$

We need an expansion of $(h_x k^h)^2$ as powers of $(kh)^2$. This is possible thanks to a series reversion (cite). The first step is writing the Taylor expansion we are aiming for:

$$(k^h h_x)^2 \approx \alpha_1((kh_x)^2)^1 + \alpha_2((kh_x)^2)^2 \tag{10}$$

Then, substituting Eq. (10) into Eq. (9) allows us to determine the right coefficients for the expansion:

$$(k^h h_x)^2 \approx (kh_x)^2 + \chi(kh_x)^4 \tag{11}$$

The last step is extracting the Taylor series of Eq. (11), with $k^h h_x > 0$:

$$k^h h_x \approx kh_x + \frac{1}{2}\chi(kh_x)^3 \tag{12}$$

This concludes the proof. $\square$

2.2 Optimal GLS parameter

Our goal is to find a value for the parameter τ for each line of the linear system $AU^h = 0$. This parameter is designed with the goal of obtaining $k = k^h$ with the GLS method, thus enforcing wavenumber and nodal exactness of the numerical solution.

2.2.1 GLS numerical resolution

The procedure is similar to the one described in 2.1.1. However, this time, we have to account for GLS terms during the assembly procedure. As we are using linear shape functions, the Laplacian part of the residue vanishes. The k^2 part remains and the GLS impendence matrix is the following:

$$A_{GLS,ij} = \sum_{e \in E} K^e_{ij} - k^2(1 - \tau k^2) M^e_{ij}$$

2.2.2 Optimal GLS parameter

The optimal parameter τ_0 associated with the center node is:

$$\tau_0 k^2 = 1 - \frac{6}{k^2 h_x^- h_x^+} \frac{\alpha^- \left(1 - \cos h_x^+ k\right) + \alpha^+ \left(1 - \cos h_x^- k\right)}{\alpha^- \left(2 + \cos h_x^- k\right) + \alpha^+ \left(2 + \cos h_x^+ k\right)}$$

Proof. The main route when computing this parameter is writing the GLS dispersion relation as in 2.1.2 by replacing k^2 by $k^2(1 - \tau k^2)$, as demonstrated in 2.2.1. Then, we set all wavenumbers to k, thus enforcing numerical solution exactness. Finally, solving for τ yields the parameter making this possible. One might notice that it is possible to find a shortcut using Eq. (6).