Strong formulation We will work on a one-dimensional Helmholtz problem. Let $\Omega \subset \mathbb{R}$ be an open bounded region whose border $\delta\Omega$ is sufficiently smooth. Our strong formulation is the following:

$$\text{Find } u \text{ such that: } \begin{cases} \Delta u + k^2 u = -f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$\Delta = \nabla^2$ is the Laplace operator, u could be the unknown acoustic pressure field, k is the wavenumber and f is a source term. In general, u and k can be complex-valued. However, this study will be conducted on real-valued cases, with $k > 0$.

Galerkin formulation The usual Galerkin formulation for our problem is the following:

$$\text{Find } u^h \in \mathcal{V}^h \text{ such that } \forall v^h \in \mathcal{V}^h,\ a_G(u^h, v^h) = \left\langle f, v^h \right\rangle_{L^2(\Omega)}$$

Where $\mathcal{V}^h \subset H_0^1(\Omega)$ is finite-dimensional, $\langle u, v \rangle_{L^2(\Omega)}$ is the $L^2(\Omega)$ scalar product, and a_G is the following sesquilinear form:

$$\begin{aligned} a_G : H_0^1(\Omega)^2 &\to \mathbb{C} \\ (u, v) &\mapsto \langle \nabla u, \nabla v \rangle_{L^2(\Omega)} - k^2 \langle u, v \rangle_{L^2(\Omega)} \end{aligned}$$

We will use a mesh method to build $\mathcal{V}^h$, partitioning Ω in mutually exclusive elements.

GLS The Galerkin/least-squares formulation uses the previous formulation and adds other terms whose purpose is to minimize the square of the residue over the element interiors $\tilde{\Omega}$. Their contribution is weighted by a parameter τ :

$$\text{Find } u^h \in \mathcal{V}^h,\ \forall v^h \in \mathcal{V}^h,\ a_{GLS}(u^h, v^h) = \left\langle f, v^h \right\rangle_{L^2(\Omega)} + \left\langle \tau f, \mathcal{L} v^h \right\rangle_{L^2(\tilde{\Omega})}$$

Where $\mathcal{L}. = \Delta. + k^2.$ is the Helmholtz operator, and a_{GLS} is the follwing sesquilinear form:

$$\begin{aligned} a_{GLS} : H_0^1(\Omega)^2 &\to \mathbb{C} \\ (u, v) &\mapsto a_G(u, v) + \langle \tau \mathcal{L} u, \mathcal{L} v \rangle_{L^2(\tilde{\Omega})} \end{aligned}$$

1 Dispersion analysis

The Helmholtz equation suffers from the pollution effect, causing numerical wavenumbers to lose accuracy with the physical wavenumber increasing. In this section,

Exact solutions When considering the sourceless ($f = 0$) and free-space version of our problem, the exact solutions are: $\forall x \in \mathbb{R},\ u(x) = Ce^{ikx}$

Numerical resolution Using linear interpolation with nodal shape functions N_i that respect the partition of unity and Kronecker delta properties yields the linear system $AU^h = 0$, where U^h is a the vector of nodal values and A is the impedence matrix.
When using our Galerkine formulation, the matrix is $A_{G,ij} = \sum_{e \in E} K^e_{ij} - k^2 M^e_{ij}$ where E is the set of elements whose boundary contain both nodes i and j, and K^e_{ij} and M_{ij^e} are respectively stiffness and masse terms assembled using shape functions i and j over the element e.

Dispersion relation We consider, without loss of generality, the stencil formed by an interior node 0 and its neighbours -1 et $+1$. The "dispersion relation" linear equation between U^h_0 and the neighboring nodal values is:

$$A_{0,-1}U^h_{-1} + A_{0,0}U^h_0 + A_{0,+1}U^h_{+1} = 0 \tag{1}$$

$$\begin{aligned}
A_{0,-1} &= \frac{h^-_x k^2}{6} + \frac{1}{h^-_x} \\
A_{0,0} &= \frac{h^-_x k^2}{3} + \frac{h^+_x k^2}{3} - \frac{1}{h^-_x} - \frac{1}{h^+_x} \\
A_{0,1} &= \frac{h^+_x k^2}{6} + \frac{1}{h^+_x}
\end{aligned}$$

We assume that the numerical solution can be defined by $u^h(x) = ce^{ik^h x}$ where k^h is the numerical wavenumber. $h^-_x = \alpha^- h_x$ and $h^+_x = \alpha^+ h_x$

Theorem 1.1 For the 1D Gakerine method stencil with linear finite elements of varying sizes, the numerical wave number k^h is linked to the exact wavenumber when $k^h h \to 0$ and $kh \to 0$:

$$k^h \approx k - \frac{1}{24}\left(\alpha^{-2} + \alpha^{+2} - \alpha^-\alpha^+\right) k^3 h_x^2 \tag{2}$$

Proof. Substituting the nodel numerical values in the dispertion relation (1) yields a relation between k and k^h that we can solve for k:

$$k^2 = \frac{6}{h^+_x h^-_x} \frac{h^-_x \left(1 - \cos\left(h^+_x k^h\right)\right) + h^+_x \left(1 - \cos\left(h^-_x k^h\right)\right)}{h^-_x \left(2 + \cos\left(h^-_x k^h\right)\right) + h^+_x \left(2 + \cos\left(h^+_x k^h\right)\right)} \tag{3}$$

However, it cannot be written as $k^h = f(k)$ because of the cosines of varying wavenumbers. We need to use a Taylor expansion on the cosines when $h_x k^h \to 0$:

$$\cos\left(\alpha^\pm h_x k^h\right) = 1 - \frac{\alpha^{\pm 2}}{2}(h_x k^k)^2 + \frac{\alpha^{\pm 4}}{24}(h_x k^h)^4 + O((h_x k^h)^6) \tag{4}$$

Substituting this expansion in (3) and simplifying by common factors yields:

$$k^2 h_x^2 = \frac{6\left(12\left(\alpha^- + \alpha^+\right)(h_x k^h)^2 - \left(\alpha^{-3} + \alpha^{+3}\right)(h_x k^h)^4\right) + O((h_x k^h)^6)}{72\left(\alpha^- + \alpha^+\right) - 12\left(\alpha^{-3} + \alpha^{+3}\right)(h_x k^h)^2 + \left(\alpha^{-5} + \alpha^{+5}\right)(h_x k^h)^4 + +O((h_x k^h)^6)}$$

This ratio of polynomials is inconvenient. Let's rewrite $(kh_x)^2$:

$$(kh_x)^2 = X\frac{1}{1-Y} \tag{5}$$
$$X = (h_x k^h)^2 + \chi(h_x k^h)^4 + O((h_x k^h)^6) \tag{6}$$
$$Y = \gamma_1(h_x k^h)^2 + \gamma_2(h_x k^h)^4 + O((h_x k^h)^6) \tag{7}$$
$$\chi = -\frac{1}{12}\left(\alpha^{-2} + \alpha^{+2} - \alpha^-\alpha^+\right) \tag{8}$$
$$\gamma_1 = \frac{1}{6}\left(\alpha^{-2} + \alpha^{+2} - \alpha^-\alpha^+\right) = -2\chi \tag{9}$$
$$\gamma_2 = -\frac{1}{72}\frac{\alpha^{-5} + \alpha^{+5}}{\alpha^- + \alpha^+} \tag{10}$$

$Y \to 0$ as $h_x k^h \to 0$, thus yielding the following Taylor expansion:

$$\frac{1}{1-Y} = 1 + \gamma_1(h_x k^h)^2 + (\gamma_1^2 + \gamma_2)(h_x k^h)^4 + O((h_x k^h)^6) \tag{11}$$

A polynomial expression of $(kh_x)^2$ is now available as a product of (7) and (11):

$$(kh_x)^2 = (h_x k^h)^2 - \chi(h_x k^h)^4 + O((h_x k^h)^6) \tag{12}$$

We need an expansion of $(h_x k^h)^2$ as powers of $(kh)^2$. This is possible thanks to a series reversion (cite). The first step is writing the Taylor expansion we are trying to achieve:

$$(k^h h_x)^2 \approx \alpha_1((kh_x)^2)^1 + \alpha_2((kh_x)^2)^2 \tag{13}$$

Then, substituting (13) into (12) allows us to determine the right coefficients for the expansion:

$$(k^h h_x)^2 \approx (kh_x)^2 + \chi(kh_x)^4 \tag{14}$$

The last step is extracting the Taylor series of (14), which is made under the hypothesis that $k^h h_x > 0$:

$$k^h h_x \approx kh_x + \frac{1}{2}\chi(kh_x)^3 \tag{15}$$

This concludes the proof. □

2 Optimal GLS parameter

$$\tau k^2 = 1 - \frac{6}{k^2 h_x^- h_x^+}\frac{\alpha^-\left(1-\cos\left(h_x^+ k\right)\right) + \alpha^+\left(1-\cos\left(h_x^- k\right)\right)}{\alpha^-\left(2+\cos\left(h_x^- k\right)\right) + \alpha^+\left(2+\cos\left(h_x^+ k\right)\right)}$$