

## 1 Problem statement

### 1.1 Strong formulation

We will work on a one-dimensional Helmholtz problem. Let  $\Omega \subset \mathbb{R}$  be an open bounded region whose border  $\delta\Omega$  is sufficiently smooth. Our strong formulation is the following:

$$\text{Find } u \text{ such that: } \begin{cases} \Delta u + k^2 u = -f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$\Delta = \nabla^2$  is the Laplace operator,  $u$  could be the unknown acoustic pressure field,  $k$  is the wavenumber and  $f$  is a source term. In general,  $u$  and  $k$  can be complex-valued. However, the parameter studies will be conducted on real-valued cases, with  $k > 0$ . The extension to complex-valued cases should be straightforward.

### 1.2 Galerkin formulation

The usual Galerkin formulation for our problem is the following:

$$\text{Find } u^h \in \mathcal{V}^h \text{ such that } \forall v^h \in \mathcal{V}^h, a_G(u^h, v^h) = \langle f, v^h \rangle_{L^2(\Omega)}$$

Where  $\mathcal{V}^h \subset H_0^1(\Omega)$  is finite-dimensional,  $\langle u, v \rangle_{L^2(\Omega)}$  is the  $L^2(\Omega)$  scalar product, and  $a_G$  is the following sesquilinear form:

$$\begin{aligned} a_G : H_0^1(\Omega)^2 &\rightarrow \mathbb{C} \\ (u, v) &\mapsto \langle \nabla u, \nabla v \rangle_{L^2(\Omega)} - k^2 \langle u, v \rangle_{L^2(\Omega)} \end{aligned}$$

We will use a mesh method to build  $\mathcal{V}^h$ , partitioning  $\Omega$  in mutually exclusive elements.

### 1.3 GLS formulation

The Galerkin/least-squares formulation uses the previous formulation and adds other terms whose purpose is to minimize the square of the residue over the element interiors  $\tilde{\Omega}$ . Their contribution is weighted by a parameter  $\tau$ :

$$\text{Find } u^h \in \mathcal{V}^h, \forall v^h \in \mathcal{V}^h, a_{GLS}(u^h, v^h) = \langle f, v^h \rangle_{L^2(\Omega)} + \langle \tau f, \mathcal{L}v^h \rangle_{L^2(\tilde{\Omega})}$$

Where  $\mathcal{L} = \Delta + k^2$  is the Helmholtz operator, and  $a_{GLS}$  is the following sesquilinear form:

$$\begin{aligned} a_{GLS} : H_0^1(\Omega)^2 &\rightarrow \mathbb{C} \\ (u, v) &\mapsto a_G(u, v) + \langle \tau \mathcal{L}u, \mathcal{L}v \rangle_{L^2(\tilde{\Omega})} \end{aligned}$$

## 2 Dispersion analysis and optimal parameter

The Helmholtz equation suffers from the pollution effect, causing numerical wavenumbers to lose accuracy with the physical wavenumber increasing.

**Exact solutions** When considering the sourceless ( $f = 0$ ) and free-space version of our problem, the exact solutions are  $\forall x \in \mathbb{R}$ ,  $u(x) = Ce^{ikx}$ . Our studies will revolve around this case.

## 2.1 Dispersion analysis

**Numerical resolution** Using linear interpolation with nodal shape functions  $N_i$  that respect the partition of unity and Kronecker delta properties yields the linear system  $AU^h = 0$ , where  $U^h$  is a the vector of nodal values and  $A$  is the impedance matrix.

When using our Galerkin formulation, the matrix is  $A_{G,ij} = \sum_{e \in E} K_{ij}^e - k^2 M_{ij}^e$  where  $E$  is the set of elements whose boundary contain both nodes  $i$  and  $j$ , and  $K_{ij}^e$  and  $M_{ij}^e$  are respectively stiffness and masse terms assembled using shape functions  $i$  and  $j$  over the element  $e$ .

**Dispersion relation** We study, without loss of generality, the stencil formed by an interior node 0 and its neighbours of indexes  $-1$  et  $+1$ . The distances between those nodes are  $h_x^-$  and  $h_x^+$ , as shown in the figure below:

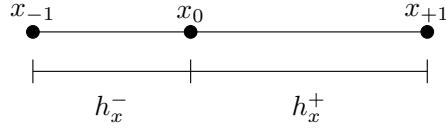


Figure 1: Node 0 and its neighbors

For convenience, we might use  $h_x^- = \alpha_x^- h$ ,  $h_x^+ = \alpha_x^+ h$  and  $2h = h_x^- + h_x^+$ . The "dispersion relation" is the line 0 of the system:

$$A_{0,-1}U_{-1}^h + A_{0,0}U_0^h + A_{0,+1}U_{+1}^h = 0 \quad (1)$$

The values of the coefficients are:

$$A_{0,-1} = \frac{h_x^- k^2}{6} + \frac{1}{h_x^-} \quad (2)$$

$$A_{0,0} = \frac{2hk^2}{3} - \frac{2}{\alpha_x^- \alpha_x^+ h} \quad (3)$$

$$A_{0,+1} = \frac{h_x^+ k^2}{6} + \frac{1}{h_x^+} \quad (4)$$

**Numerical solution** We assume that the numerical solution can be defined by  $u^h(x) = Ce^{ik^h x}$  where  $k^h$  is the numerical wavenumber.

**Theorem 2.1.1 (Dispersion relation)** *The numerical wave number  $k^h$  is asymptotically linked to the exact wavenumber  $k$  when  $k^h h \rightarrow 0$  and  $kh \rightarrow 0$ :*

$$k^h \approx k - \frac{1}{24} \left( \alpha_x^{-2} + \alpha_x^{+2} - \alpha_x^- \alpha_x^+ \right) k^3 h^2 \quad (5)$$

*Proof.* Substituting the nodal numerical values in the dispersion relation Eq. (1) yields a relation between  $k$  and  $k^h$  that we can solve for  $k^2$ :

$$k^2 = \frac{6}{h_x^+ h_x^-} \frac{\alpha_x^- (1 - \cos h_x^+ k^h) + \alpha_x^+ (1 - \cos h_x^- k^h)}{\alpha_x^- (2 + \cos h_x^- k^h) + \alpha_x^+ (2 + \cos h_x^+ k^h)} \quad (6)$$

However, it cannot be expressed as  $k^h = f(k)$  because of the cosines of varying wavenumbers. We need to use a Taylor expansion on the cosines when  $hk^h \rightarrow 0$ :

$$\cos(\alpha_x^\pm hk^h) = 1 - \frac{\alpha_x^{\pm 2}}{2}(hk^h)^2 + \frac{\alpha_x^{\pm 4}}{24}(hk^h)^4 + O((hk^h)^6) \quad (7)$$

Substituting this expansion in Eq. (6) and simplifying by common factors yields an expression for  $(kh)^2$  as a ratio of polynomials. It can be rewritten as:

$$\begin{aligned} (kh)^2 &= X \frac{1}{1 - Y} \\ X &= (hk^h)^2 + \chi(hk^h)^4 + O((hk^h)^6) \\ Y &= \gamma_1(hk^h)^2 + \gamma_2(hk^h)^4 + O((hk^h)^6) \\ \chi &= -\frac{1}{12} \frac{\alpha_x^{-3} + \alpha_x^{+3}}{\alpha_x^- + \alpha_x^+} = -\frac{1}{12} (\alpha_x^{-2} + \alpha_x^{+2} - \alpha_x^- \alpha_x^+) \\ \gamma_1 &= \frac{1}{6} \frac{\alpha_x^{-3} + \alpha_x^{+3}}{\alpha_x^- + \alpha_x^+} = -2\chi \\ \gamma_2 &= -\frac{1}{72} \frac{\alpha_x^{-5} + \alpha_x^{+5}}{\alpha_x^- + \alpha_x^+} \end{aligned}$$

$Y \rightarrow 0$  as  $hk^h \rightarrow 0$ , thus allowing the following Taylor expansion:

$$\frac{1}{1 - Y} = 1 + \gamma_1(hk^h)^2 + (\gamma_1^2 + \gamma_2)(hk^h)^4 + O((hk^h)^6) \quad (8)$$

A polynomial expression of  $(kh)^2$  is now available as a product of  $X$  and  $\frac{1}{1 - Y}$ :

$$(kh)^2 = (hk^h)^2 - \chi(hk^h)^4 + O((hk^h)^6) \quad (9)$$

We need an expansion of  $(hk^h)^2$  as powers of  $(kh)^2$ . This is possible thanks to a series reversion (cite). The first step is writing the Taylor expansion we are aiming for:

$$(k^h h)^2 \approx \alpha_1((kh)^2)^1 + \alpha_2((kh)^2)^2 \quad (10)$$

Then, substituting Eq. (10) into Eq. (9) allows us to determine the right coefficients for the expansion:

$$(k^h h)^2 \approx (kh)^2 + \chi(kh)^4 \quad (11)$$

The last step is extracting the Taylor series of Eq. (11), with  $k^h h > 0$ :

$$k^h h \approx kh + \frac{1}{2}\chi(kh)^3 \quad (12)$$

This concludes the proof.  $\square$

## 2.2 Optimal GLS parameter

Our goal is to find a value for the parameter  $\tau$  for each line of the linear system  $AU^h = 0$ . This parameter is designed with the goal of obtaining  $k = k^h$  with the GLS method, thus enforcing wavenumber and nodal exactness of the numerical solution.

**GLS numerical resolution** The procedure is similar to the one described in 2.1. However, this time, we have to account for GLS terms during the assembly procedure. As we are using linear shape functions, the Laplacian part of the residue vanishes. The  $k^2$  part remains and the GLS impedance matrix is the following:

$$A_{GLS,ij} = \sum_{e \in E} K_{ij}^e - k^2(1 - \tau k^2)M_{ij}^e$$

**Theorem 2.2.1 (Optimal GLS parameter)** *The optimal GLS parameter  $\tau_0$  associated with the center node is:*

$$\tau_0 k^2 = 1 - \frac{6}{k^2 h_x^- h_x^+} \frac{\alpha_x^- (1 - \cos h_x^+ k) + \alpha_x^+ (1 - \cos h_x^- k)}{\alpha_x^- (2 + \cos h_x^- k) + \alpha_x^+ (2 + \cos h_x^+ k)}$$

*Proof.* The most straightforward approach to computing this parameter is writing the GLS dispersion relation as in 2.1 by replacing  $k^2$  with  $k^2(1 - \tau k^2)$ , as demonstrated in 2.2. Then, setting all wavenumbers to  $k$ , enforces numerical solution exactness. Finally, solving for  $\tau$  yields the parameter making this possible.

One might notice that it is possible to find a shortcut using Eq. (6).