Exact solution. We consider the sourceless Helmholtz equation in homogeneous media $\Delta u + k^2 u = 0$ in $\mathbb{R}$. The solutions are defined by $u(x) = ce^{ikx}$ with any constant c.

Numerical solution We discretize the 1D domain and consider, without loss of generality, the stencil formed by an interior node 0 and its neighbours -1 et $+1$. The "dispersion relation" linear equation between U_0^h and the neighboring nodal values is:

$$A_{0,-1}U_{-1} + A_{0,0}U_0 + A_{0,+1}U_{+1} = 0 \tag{1}$$

$$\begin{aligned} A_{0,-1} &= \frac{h_x^- k^2}{6} + \frac{1}{h_x^-} \\ A_{0,0} &= \frac{h_x^- k^2}{3} + \frac{h_x^+ k^2}{3} - \frac{1}{h_x^-} - \frac{1}{h_x^+} \\ A_{0,1} &= \frac{h_x^+ k^2}{6} + \frac{1}{h_x^+} \end{aligned}$$

1 Dispersion analysis

We assume that the numerical solution can be defined by $u^h(x) = ce^{ik^h x}$ where k^h is the numerical wavenumber. $h_x^- = \alpha^- h_x$ and $h_x^+ = \alpha^+ h_x$

Theorem 1.1 For the 1D Gakerine method stencil with linear finite elements of varying sizes, the numerical wave number k^h is linked to the exact wavenumber when $k^h h \to 0$ and $kh \to 0$:

$$k^h \approx k - \frac{1}{24}\left({\alpha^-}^2 + {\alpha^+}^2 - \alpha^-\alpha^+\right) k^3 h_x^2 \tag{2}$$

Proof. Substituting the nodel numerical values in the dispertion relation (1) yields a relation between k and k^h that we can solve for k:

$$k^2 = \frac{6}{h_x^+ h_x^-} \frac{h_x^-\left(1-\cos\left(h_x^+ k^h\right)\right) + h_x^+\left(1-\cos\left(h_x^- k^h\right)\right)}{h_x^-\left(2+\cos\left(h_x^- k^h\right)\right) + h_x^+\left(2+\cos\left(h_x^+ k^h\right)\right)} \tag{3}$$

However, it cannot be written as $k^h = f(k)$ because of the cosines of varying wavenumbers. We need to use a Taylor expansion on the cosines when $h_x k^h \to 0$:

$$\cos\left(\alpha^\pm h_x k^h\right) = 1 - \frac{{\alpha^\pm}^2}{2}(h_x k^k)^2 + \frac{{\alpha^\pm}^4}{24}(h_x k^h)^4 + O((h_x k^h)^6) \tag{4}$$

Substituting this expansion in (3) and simplifying by common factors yields:

$$k^2 h_x^2 = \frac{6\left(12\left(\alpha^- + \alpha^+\right)(h_x k^h)^2 - \left({\alpha^-}^3 + {\alpha^+}^3\right)(h_x k^h)^4\right) + O((h_x k^h)^6)}{72\left(\alpha^- + \alpha^+\right) - 12\left({\alpha^-}^3 + {\alpha^+}^3\right)(h_x k^h)^2 + \left({\alpha^-}^5 + {\alpha^+}^5\right)(h_x k^h)^4 + +O((h_x k^h)^6)}$$

This ratio of polynomials is inconvenient. Let's rewrite $(kh_x)^2$:

$$(kh_x)^2 = X\frac{1}{1-Y} \tag{5}$$

$$X = (h_x k^h)^2 + \chi(h_x k^h)^4 + O((h_x k^h)^6) \tag{6}$$

$$Y = \gamma_1(h_x k^h)^2 + \gamma_2(h_x k^h)^4 + O((h_x k^h)^6) \tag{7}$$

$$\chi = -\frac{1}{12}\left(\alpha^{-2} + \alpha^{+2} - \alpha^-\alpha^+\right) \tag{8}$$

$$\gamma_1 = \frac{1}{6}\left(\alpha^{-2} + \alpha^{+2} - \alpha^-\alpha^+\right) = -2\chi \tag{9}$$

$$\gamma_2 = -\frac{1}{72}\frac{\alpha^{-5} + \alpha^{+5}}{\alpha^- + \alpha^+} \tag{10}$$

$Y \to 0$ as $h_x k^h \to 0$, thus yielding the following Taylor expansion:

$$\frac{1}{1-Y} = 1 + \gamma_1(h_x k^h)^2 + (\gamma_1^2 + \gamma_2)(h_x k^h)^4 + O((h_x k^h)^6) \tag{11}$$

A polynomial expression of $(kh_x)^2$ is now available as a product of (7) and (11):

$$(kh_x)^2 = (h_x k^h)^2 - \chi(h_x k^h)^4 + O((h_x k^h)^6) \tag{12}$$

We need an expansion of $(h_x k^h)^2$ as powers of $(kh)^2$. This is possible thanks to a series reversion (cite). The first step is writing the Taylor expansion we are trying to achieve:

$$(k^h h_x)^2 \approx \alpha_1((kh_x)^2)^1 + \alpha_2((kh_x)^2)^2 \tag{13}$$

Then, substituting (13) into (12) allows us to determine the right coefficients for the expansion:

$$(k^h h_x)^2 \approx (kh_x)^2 + \chi(kh_x)^4 \tag{14}$$

The last step is extracting the Taylor series of (14), which is made under the hypothesis that $k^h h_x > 0$:

$$k^h h_x \approx kh_x + \frac{1}{2}\chi(kh_x)^3 \tag{15}$$

This concludes the proof. $\square$

2 Optimal GLS parameter

$$\tau k^2 = 1 - \frac{6}{k^2 h_x^- h_x^+}\frac{\alpha^-\left(1-\cos\left(h_x^+ k\right)\right) + \alpha^+\left(1-\cos\left(h_x^- k\right)\right)}{\alpha^-\left(2+\cos\left(h_x^- k\right)\right) + \alpha^+\left(2+\cos\left(h_x^+ k\right)\right)}$$