1 Problem statement

1.1 Strong formulation

We will work on a two-dimensional Helmholtz problem. Let $\Omega \subset \mathbb{R}^2$ be an open bounded region whose border $\delta\Omega$ is sufficiently smooth. Our strong formulation is the following:

$$\text{Find } u \text{ such that: } \begin{cases} \Delta u + k^2 u = -f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$\Delta = \nabla^2$ is the Laplace operator, u could be the unknown acoustic pressure field, k is the wavenumber and f is a source term. In general, u and k can be complex-valued. However, the paramter studies will be conducted on real-valued cases, with $k > 0$. The extension to complex-valued cases should be straightforward.

1.2 Galerkin formulation

The usual Galerkin formulation for our problem is the following:

$$\text{Find } u^h \in \mathcal{V}^h \text{ such that } \forall v^h \in \mathcal{V}^h,\ a_G(u^h, v^h) = \left\langle f, v^h \right\rangle_{L^2(\Omega)}$$

Where $\mathcal{V}^h \subset H_0^1(\Omega)$ is finite-dimensional, $\langle u, v\rangle_{L^2(\Omega)}$ is the $L^2(\Omega)$ scalar product, and a_G is the following sesquilinear form:

$$\begin{aligned} a_G : H_0^1(\Omega)^2 &\to \mathbb{C} \\ (u, v) &\mapsto \langle \nabla u, \nabla v\rangle_{L^2(\Omega)} - k^2 \langle u, v\rangle_{L^2(\Omega)} \end{aligned}$$

We will use a mesh method to build $\mathcal{V}^h$, partitioning Ω in mutually exclusive elements.

1.3 GLS

The Galerkin/least-squares formulation uses the previous formulation and adds other terms whose purpose is to minimize the square of the residue over the element interiors $\tilde{\Omega}$. Their contribution is weighted by a parameter τ :

$$\text{Find } u^h \in \mathcal{V}^h, \forall v^h \in \mathcal{V}^h,\ a_{GLS}(u^h, v^h) = \left\langle f, v^h \right\rangle_{L^2(\Omega)} + \left\langle \tau f, \mathcal{L} v^h \right\rangle_{L^2(\tilde{\Omega})}$$

Where $\mathcal{L}. = \Delta. + k^2.$ is the Helmholtz operator, and a_{GLS} is the follwing sesquilinear form:

$$\begin{aligned} a_{GLS} : H_0^1(\Omega)^2 &\to \mathbb{C} \\ (u, v) &\mapsto a_G(u, v) + \langle \tau \mathcal{L} u, \mathcal{L} v\rangle_{L^2(\tilde{\Omega})} \end{aligned}$$

2 Dispersion analysis and optimal parameter

The Helmholtz equation suffers from the pollution effect, causing numerical wavenumbers to lose accuracy with the physical wavenumber increasing.

Exact solutions When considering the sourceless ($f = 0$) and free-space version of our problem, the exact solutions are $\forall x \in \mathbb{R},\ u(x) = Ce^{i(k_x x + k_y y)}$. With $k^2 = k_x^2 + k_y^2$ Our studies will revolve around this case.

2.1 Dispersion analysis

2.1.1 Numerical resolution

Using bilinear interpolation with nodal shape functions $N_{(i,j)}$ that respect the partition of unity and Kronecker delta properties yields the linear system $AU^h = 0$, where U^h is a the vector of nodal values and A is the impedence matrix.
When using our Galerkine formulation, the matrix is

$$A_{G,(i,j)(l,m)} = \sum_{e \in E} K^e_{(i,j)(l,m)} - k^2 M^e_{(i,j)(l,m)}$$

where E is the set of elements whose boundary contain both nodes of nodal coordinates (i,j) and (l,m), and $K^e_{(i,j)(l,m)}$ and $M^e_{(i,j)(l,m)}$ are respectively stiffness and masse terms assembled using shape functions (i,j) and (l,m) over the element e.

2.1.2 Dispersion relation

We study, without loss of generality, the stencil formed by an interior node $(0,0)$ and its eight neighbours. The four distances from the center node are denoted $h_\xi^\pm$, with ξ the dimension (x or y) and $\pm$ the direction. The elements are rectangular.
For convenience, we might use $h_\xi^\pm = \alpha_\xi^\pm h_\xi^\pm$ and $2h_\xi = h_\xi^- + h_\xi^+$.
The "dispersion relation" is the line 0 of the system:

$$\sum_{i,j \in \{-1,0,+1\}} A_{(0,0)(i,j)} U^h_{(i,j)}$$

In the following equations for convenience, $\{-,+\}$ could also be used to index neighboring nodes using directions.
The values of the four corner coefficients are, with $i,j \in \{-,+\}$:

$$A_{(0,0)(i,j)} = \frac{k^2 h_x^i h_y^j}{36} + \frac{h_x^i}{6h_y^j} + \frac{h_y^j}{6h_x^i} \tag{1}$$

The values of the four side nodes coefficients are:

$$A_{(0,0)(\pm,0)} = \frac{k^2 h_x^\pm h_y}{9} + \frac{2h_y}{3h_x^\pm} - \frac{h_x^\pm}{3\alpha_y^- \alpha_y^+ h_y} \tag{2}$$

$$A_{(0,0)(0,\pm)} = \frac{k^2 h_x h_y^\pm}{9} + \frac{2h_x}{3h_y^\pm} - \frac{h_y^\pm}{3\alpha_x^- \alpha_x^+ h_x} \tag{3}$$

The value of the center node coefficient is:

$$A_{(0,0)(0,0)} = \frac{4k^2 h_x h_y}{9} - \frac{4h_x}{3\alpha_y^- \alpha_y^+ h_y} - \frac{4h_y}{3\alpha_x^- \alpha_x^+ h_x}$$

2.1.3 Numerical solution

We assume that the numerical solution can be defined by $u^h(x) = Ce^{i(k_x^h x + k_y^h y)}$ with two numerical wavenumbers $k_x^h = k^h cos\theta$ and $k_y^h = k^h sin\theta$.

2.1.4 Dispersion relation

The numerical wave number k^h is asymptotically linked to the exact wavenumber when !!!:

$$k^h \approx !!! \tag{4}$$

Proof. Substituting the nodal numerical values in the dispertion relation section 2.1.2 yields a relation between that we can solve for k^2. Due to this expression being long, we will use the following notations:

$$\begin{aligned}
f_\xi^{\pm} &= cos\Big(k_\xi^h h_\xi^{\pm}\Big) \text{ with } \xi \in \{x, y\} \\
g^{--} &= cos\big(h_x^- k_x^h + h_y^- k_y^h\big) \\
g^{++} &= cos\big(h_x^+ k_x^h + h_y^+ k_y^h\big) \\
g^{-+} &= cos\big(h_x^- k_x^h - h_y^+ k_y^h\big) \\
g^{+-} &= cos\big(h_x^+ k_x^h - h_y^- k_y^h\big)
\end{aligned}$$

$$k^2 = \frac{\kappa_0 + \kappa_x^- f_x^- + \kappa_x^+ f_x^+ + \kappa_y^- f_y^- + \kappa_y^+ f_y^+ + \kappa^{--} g^{--} + \kappa^{++} g^{++} + \kappa^{-+} g^{-+} + \kappa^{+-} g^{+-}}{h_x^- h_x^+ h_y^- h_y^+ \left(\lambda_0 + \lambda_x^- f_x^- + \lambda_x^+ f_x^+ + \lambda_y^- f_y^- + \lambda_y^+ f_y^+ + \lambda^{--} g^{--} + \lambda^{++} g^{++} + \lambda^{-+} g^{-+} + \lambda^{+-} g^{+-}\right)} \tag{5}$$

$$k^2 = \frac{6\left(2f_x^+ h_x^- \left(\left(h_x^+\right)^2 h_y - \left(h_y^+\right)^2 h_y^- - h_y^+ \left(h_y^-\right)^2\right) + 2f_x^- h_x^+ \left(\left(h_x^-\right)^2 h_y - \left(h_y^+\right)^2 h_y^- - h_y^+ \left(h_y^-\right)^2\right) + 2f_y^+ h_y^- \left(h_x\right.\right.}{}$$

$$\gamma_0 + \sum_{\xi \in \{x,y\}} \sum_{i \in \{-,+\}} \gamma_\xi^i f_\xi^i + \tag{6}$$

Terme indépendant en haut

$$\begin{aligned}
&+ 12h_x^- h_y^+ \left(\left(h_x^+\right)^2 + \left(h_y^-\right)^2\right) \\
&\qquad + 12h_x^- h_y^- \left(\left(h_x^+\right)^2 + \left(h_y^+\right)^2\right) \\
&\qquad + 12h_x^+ h_y^+ \left(\left(h_x^-\right)^2 + \left(h_y^-\right)^2\right) \\
&\qquad + 12h_x^+ h_y^- \left(\left(h_x^-\right)^2 + \left(h_y^+\right)^2\right)
\end{aligned}$$

Termes en f en haut

$$\begin{aligned}
&12f_x^+ h_x^- \left(\left(h_x^+\right)^2 h_y - \left(h_y^+\right)^2 h_y^- - h_y^+ \left(h_y^-\right)^2\right) + 12f_x^- h_x^+ \left(\left(h_x^-\right)^2 h_y - \left(h_y^+\right)^2 h_y^- - h_y^+ \left(h_y^-\right)^2\right) \\
&+ 12f_y^+ h_y^- \left(h_x \left(h_y^+\right)^2 - \left(h_x^+\right)^2 h_x^- - h_x^+ \left(h_x^-\right)^2\right) + 12f_y^- h_y^+ \left(h_x \left(h_y^-\right)^2 - h_x^+ \left(h_x^-\right)^2 - \left(h_x^+\right)^2 h_x^-\right)
\end{aligned}$$

Terme indépendant en bas (en facteur des 4h)

$$16h_x h_y$$

Termes en f en bas (en facteur des 4h)

$$4f_x^+ h_x^+ h_y \quad + \quad 4f_x^- h_x^- h_y \quad + \quad 4f_y^+ h_x h_y^+ \quad + \quad 4f_y^- h_x h_y^-$$

Termes en g en bas (en facteur des 4h)

$$+h_x^+ h_y^+ g^{++} + h_x^+ h_y^- g^{+-} + h_x^- h_y^+ g^{-+} + h_x^- h_y^- g^{--}$$