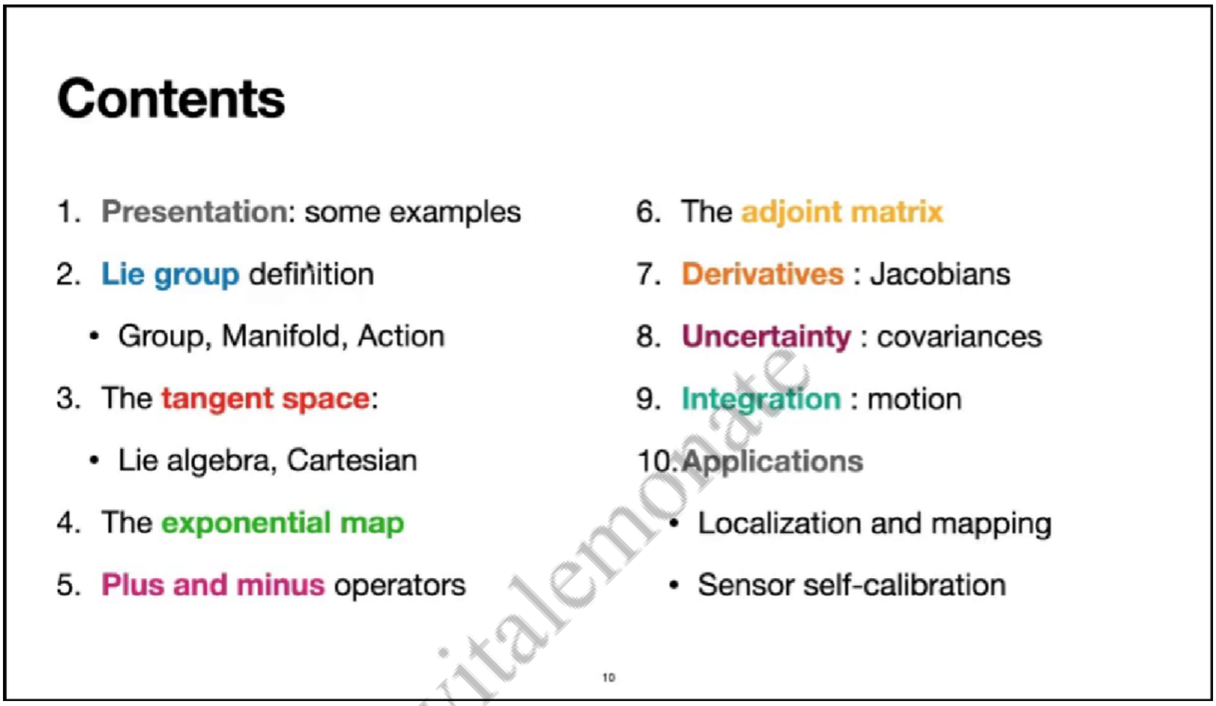
Lecture-Lie theory for the roboticist

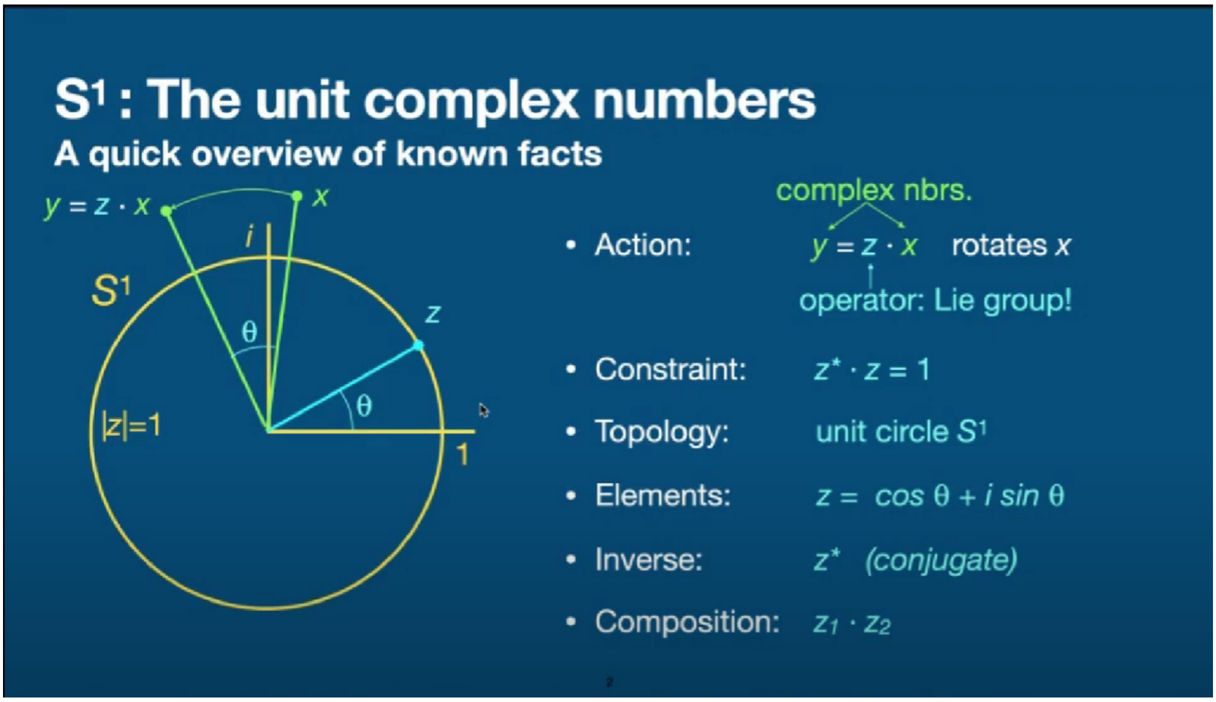
本文是学习Joan Sola在2020年和2022年的Robotics & AI Summer School的lecture的笔记，二者的内容基本相同，下文的截图主要基于2022年最新的lecture，也有少量截图来自2020年的lecture。此外，本lecture主要的参考文档为Joan Sola的著作A micro Lie theory for state estimation in robotics

Contents

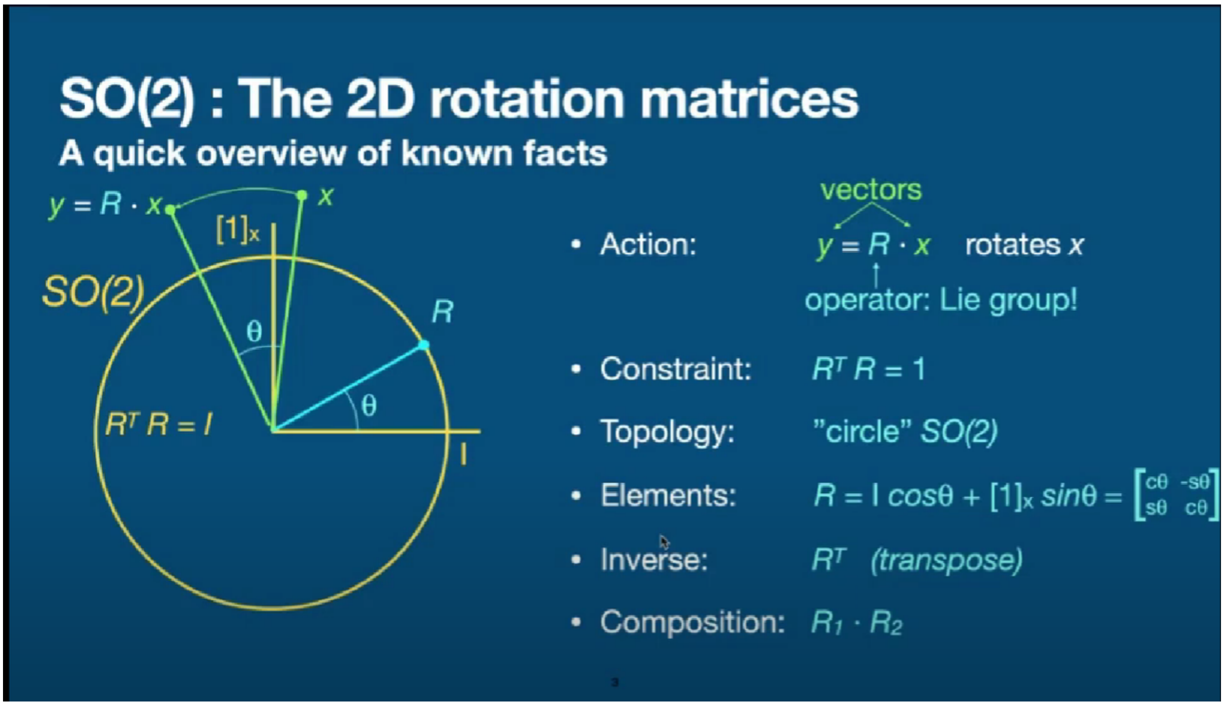


Examples

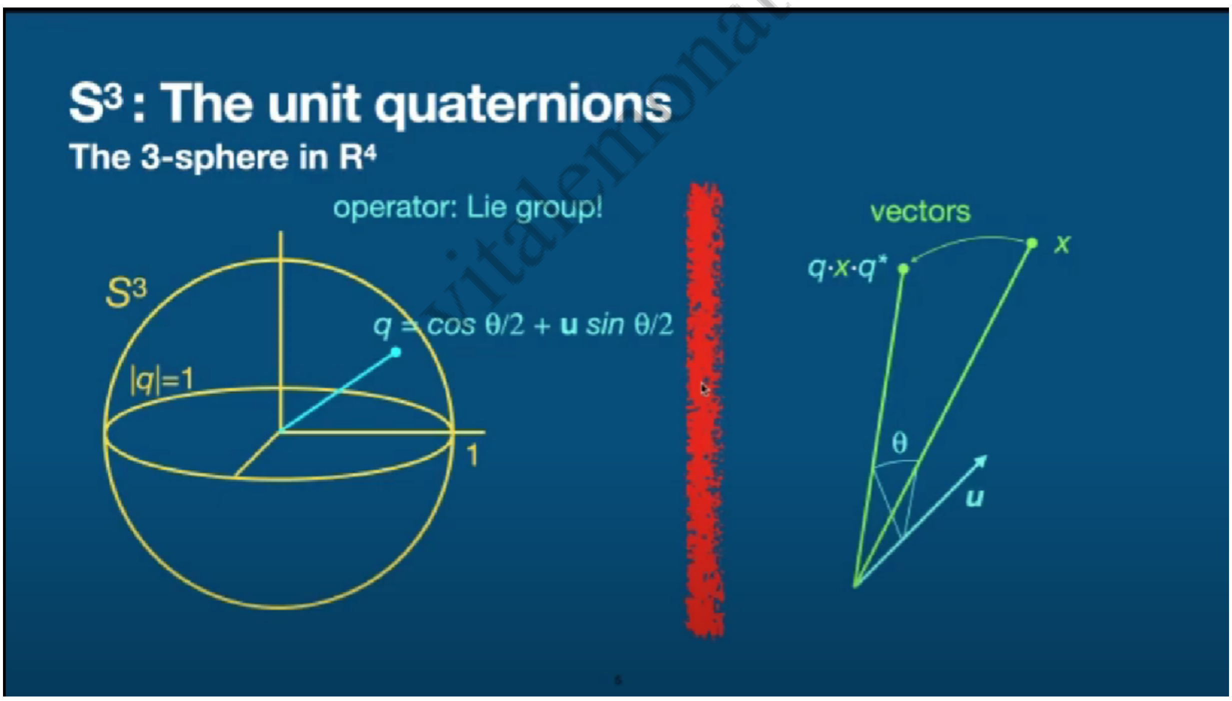
- S^1 ： The unit complex numbers(the sphere of the **one dimension)**



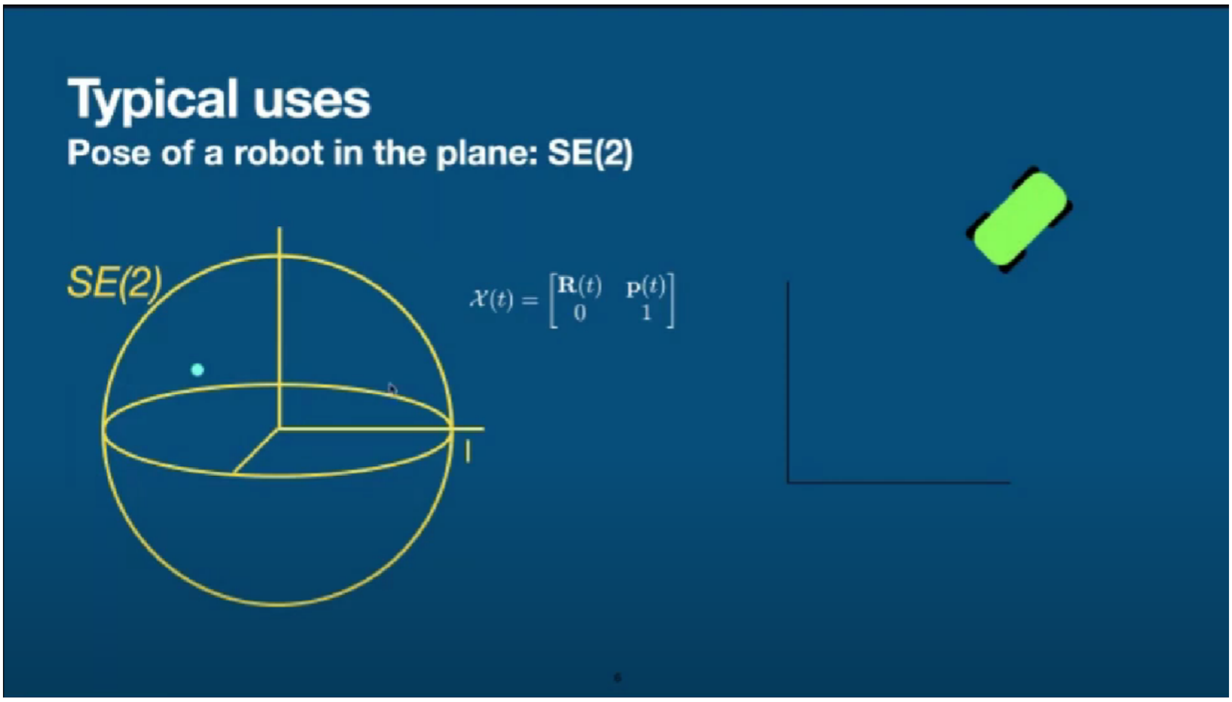
- $SO(2)$： 2D rotation group(二维特殊正交群)
  - x, y 和 $\mathbf{R}$ 属于不同的set， $\mathbf{R}$ 的基是 $(I, [1]_{\times})$ ，而 x, y 的基是 $(1,0),(0,1)$



- S^3： The unit quaternions(3-sphere in R^4)



- $SE(2)$： A pose(rigid transform) in 2-d Euclidean space(二维特殊欧式群)
  - 下图左侧的曲面图并不是精确的，只是一个抽象曲面



A pose (rigid transformation) in two-dimensional Euclidean space can be uniquely determined by means of a 3×3 homogeneous matrix with this structure:

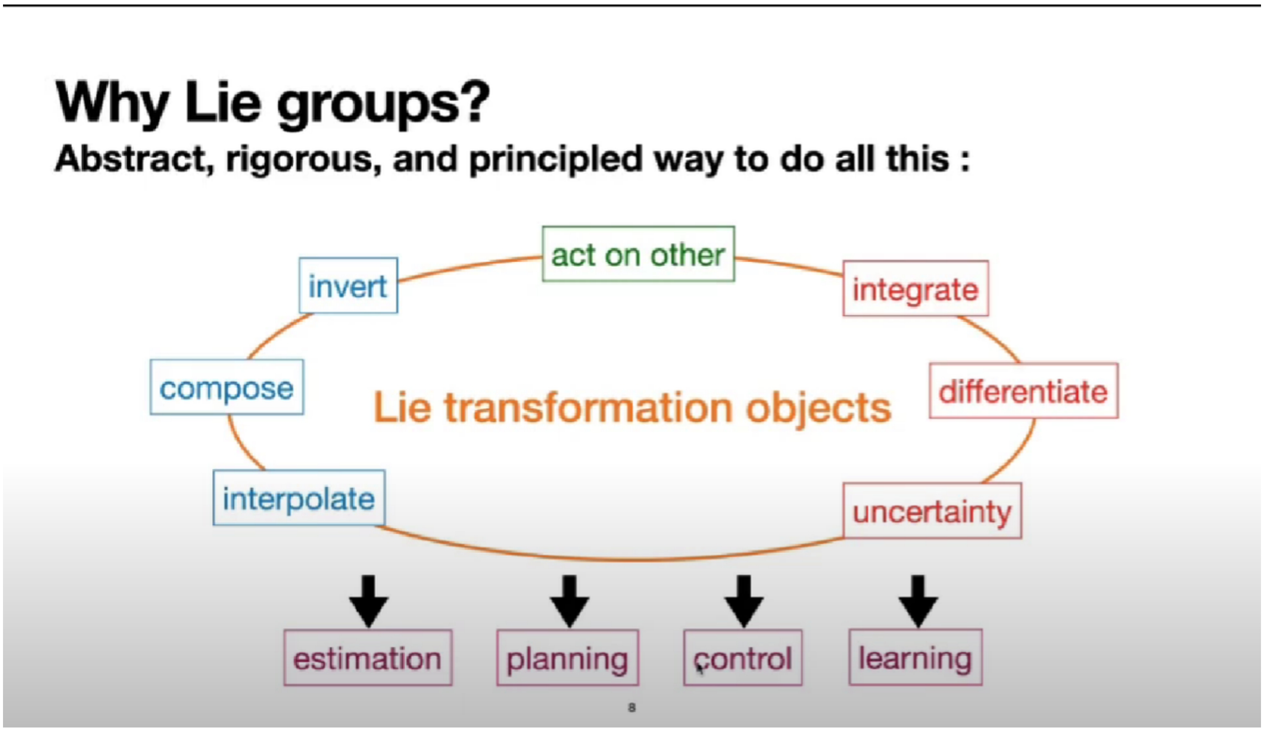
$$\mathbf{T}_2 = \left(\begin{array}{c|c} \mathbf{R}_2 & \mathbf{t} \\ \hline \mathbf{0}_{1\times 2} & 1 \end{array} \right) = \left(\begin{array}{cc|c} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ \hline 0 & 0 & 1 \end{array} \right) \tag{B.1}$$

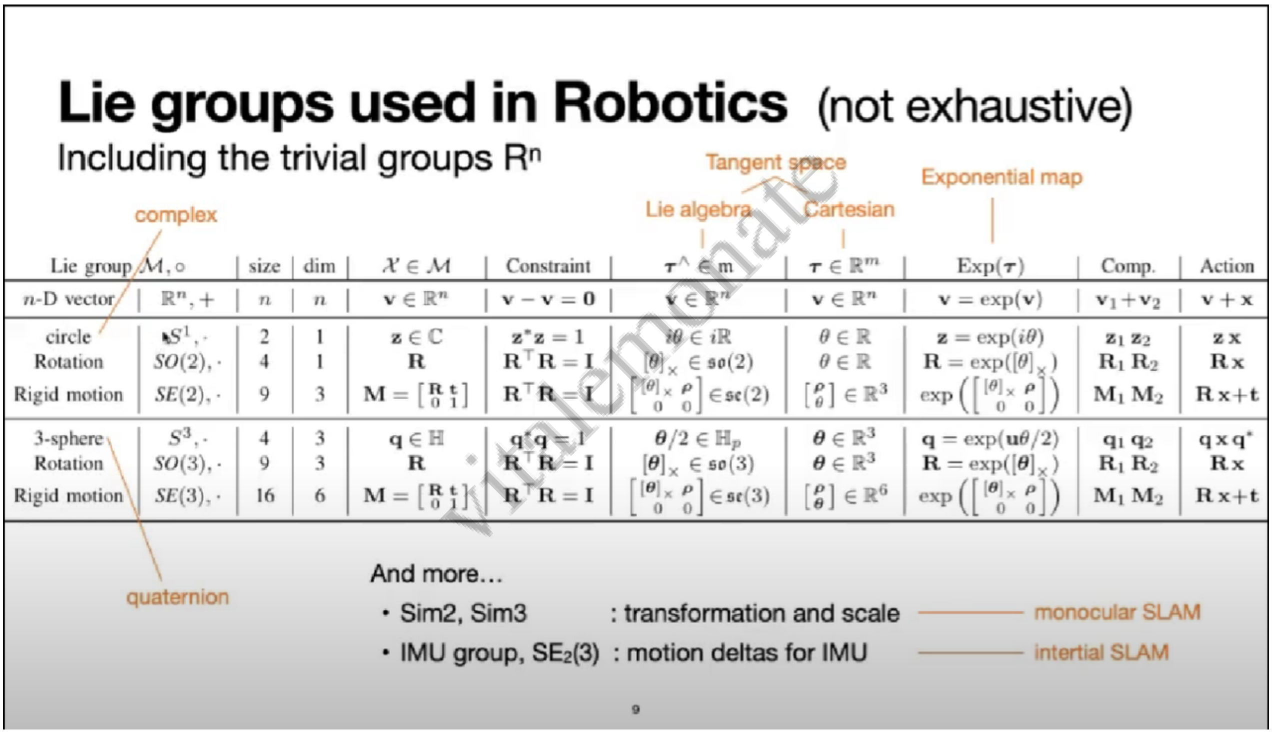
where the three degrees of freedom of the 2D transformation are the (x, y) translation and the rotation of ϕ radians.

The $\mathbf{R}_2$ belongs to the group $SO(2)$, and $\mathbf{T}_2$ to $SE(2)$.

- $SE(3)$: A pose(rigid transform) in 3-d Euclidean space(三维特殊欧式群)

Why Lie Groups



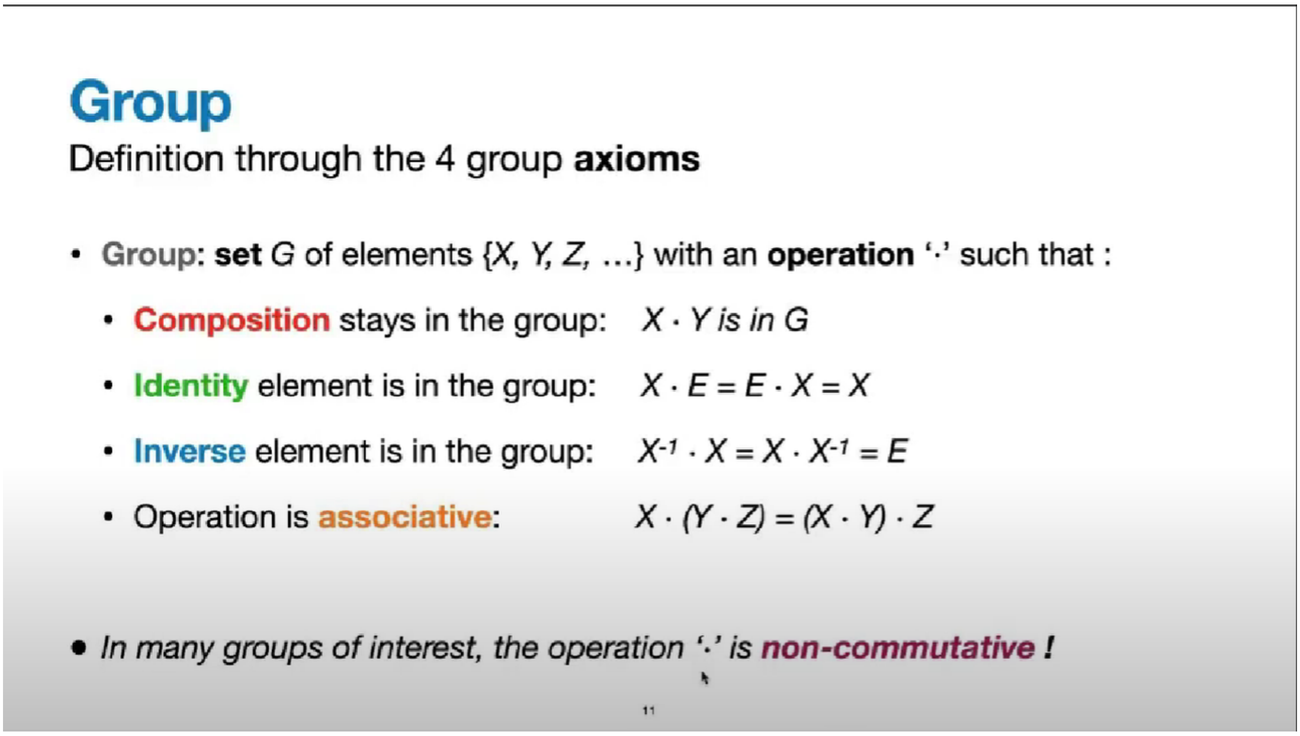


Lie group	$\mathcal{M}, \circ$	size	dim	$\mathcal{X} \in \mathcal{M}$	Constraint	$\boldsymbol{\tau}^\wedge \in \mathfrak{m}$	$\boldsymbol{\tau} \in \mathbb{R}^m$	$\mathrm{Exp}(\boldsymbol{\tau})$	Comp.	Action
n-D vector	$\mathbb{R}^n, +$	n	n	$\mathbf{v} \in \mathbb{R}^n$	$\mathbf{v} - \mathbf{v} = \mathbf{0}$	$\mathbf{v} \in \mathbb{R}^n$	$\mathbf{v} \in \mathbb{R}^n$	$\mathbf{v} = \exp(\mathbf{v})$	$\mathbf{v}_1 + \mathbf{v}_2$	$\mathbf{v} + \mathbf{x}$
circle	$S^1, \cdot$	2	1	$\mathbf{z} \in \mathbb{C}$	$\mathbf{z}^*\mathbf{z} = 1$	$i\theta \in i\mathbb{R}$	$\theta \in \mathbb{R}$	$\mathbf{z} = \exp(i\theta)$	$\mathbf{z}_1\,\mathbf{z}_2$	$\mathbf{z}\,\mathbf{x}$
Rotation	$SO(2), \cdot$	4	1	$\mathbf{R}$	$\mathbf{R}^\top\mathbf{R} = \mathbf{I}$	$[\theta]_\times \in \mathfrak{so}(2)$	$\theta \in \mathbb{R}$	$\mathbf{R} = \exp([\theta]_\times)$	$\mathbf{R}_1\,\mathbf{R}_2$	$\mathbf{R}\,\mathbf{x}$
Rigid motion	$SE(2), \cdot$	9	3	$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}$	$\mathbf{R}^\top\mathbf{R} = \mathbf{I}$	$\begin{bmatrix} [\theta]_\times & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(2)$	$\begin{bmatrix} \boldsymbol{\rho} \\ \theta \end{bmatrix} \in \mathbb{R}^3$	$\exp\left(\begin{bmatrix} [\theta]_\times & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix}\right)$	$\mathbf{M}_1\,\mathbf{M}_2$	$\mathbf{R}\,\mathbf{x} + \mathbf{t}$
3-sphere	$S^3, \cdot$	4	3	$\mathbf{q} \in \mathbb{H}$	$\mathbf{q}^*\mathbf{q} = 1$	$\boldsymbol{\theta}/2 \in \mathbb{H}_p$	$\boldsymbol{\theta} \in \mathbb{R}^3$	$\mathbf{q} = \exp(\mathbf{u}\theta/2)$	$\mathbf{q}_1\,\mathbf{q}_2$	$\mathbf{q}\,\mathbf{x}\,\mathbf{q}^*$
Rotation	$SO(3), \cdot$	9	3	$\mathbf{R}$	$\mathbf{R}^\top\mathbf{R} = \mathbf{I}$	$[\boldsymbol{\theta}]_\times \in \mathfrak{so}(3)$	$\boldsymbol{\theta} \in \mathbb{R}^3$	$\mathbf{R} = \exp([\boldsymbol{\theta}]_\times)$	$\mathbf{R}_1\,\mathbf{R}_2$	$\mathbf{R}\,\mathbf{x}$
Rigid motion	$SE(3), \cdot$	16	6	$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}$	$\mathbf{R}^\top\mathbf{R} = \mathbf{I}$	$\begin{bmatrix} [\boldsymbol{\theta}]_\times & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3)$	$\begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix} \in \mathbb{R}^6$	$\exp\left(\begin{bmatrix} [\boldsymbol{\theta}]_\times & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix}\right)$	$\mathbf{M}_1\,\mathbf{M}_2$	$\mathbf{R}\,\mathbf{x} + \mathbf{t}$

Lie Group definition

Group

- 满足4个公理的set和operation构成一个group
- 很多group是**不满足交换律**的(**non-commutative**)

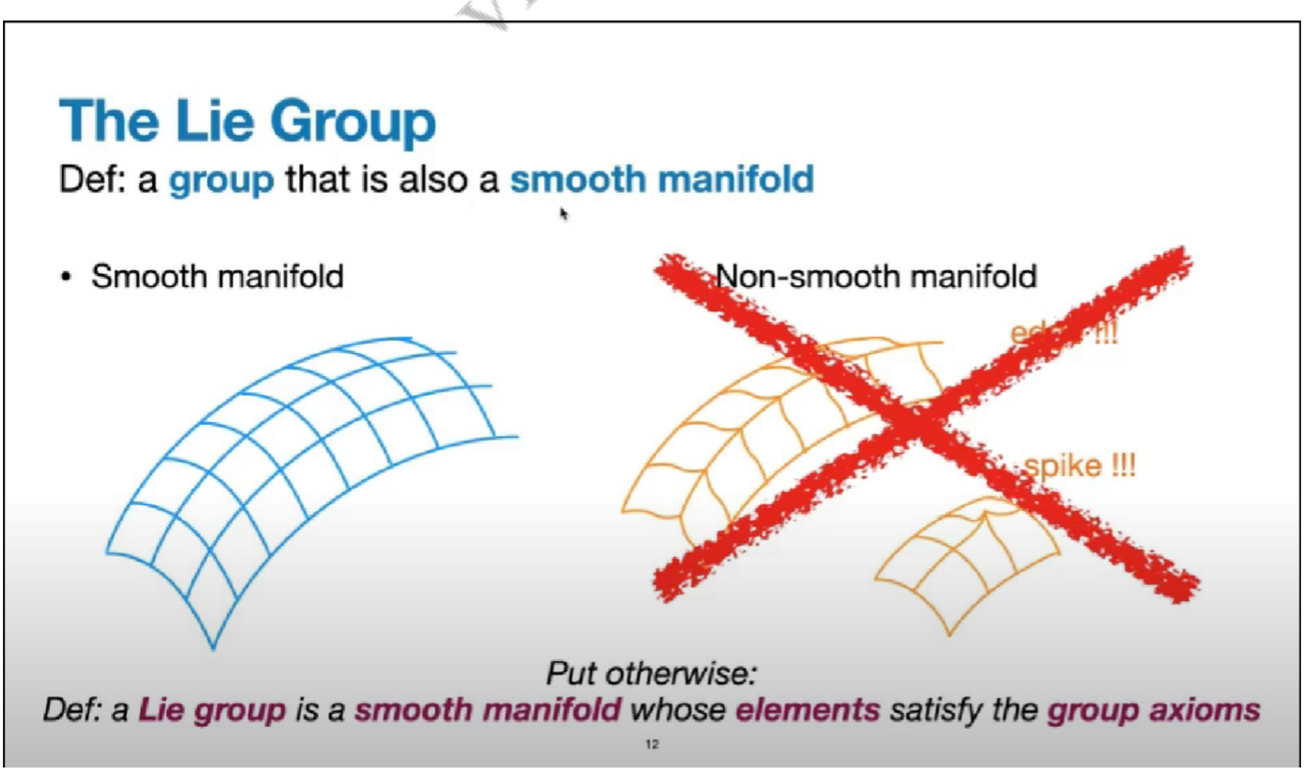


Manifold

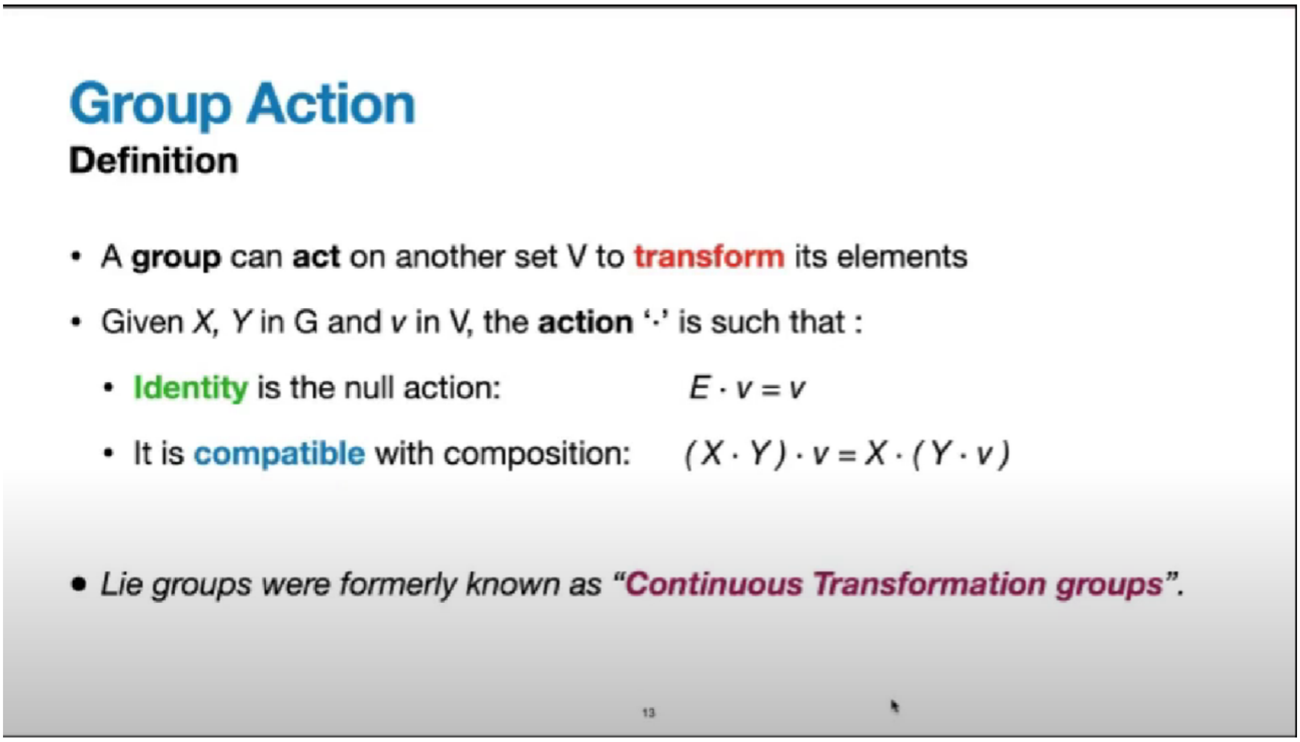
- **Wikipage: 流形**（英语：Manifolds）是**可以局部欧几里得空间化**的一个拓扑空间，是欧几里得空间中的曲线、曲面等概念的推广。欧几里得空间就是最简单的流形的实例。地球表面这样的球面则是一个稍微复杂的例子。一般的流形可以通过把许多平直的片折弯并粘连而成
- 粗略地说，流形是局部看近似线性空间的抽象**曲面（而不是曲面围成的实心部分）**

Lie Group

- 具有group性质的**smooth manifold**

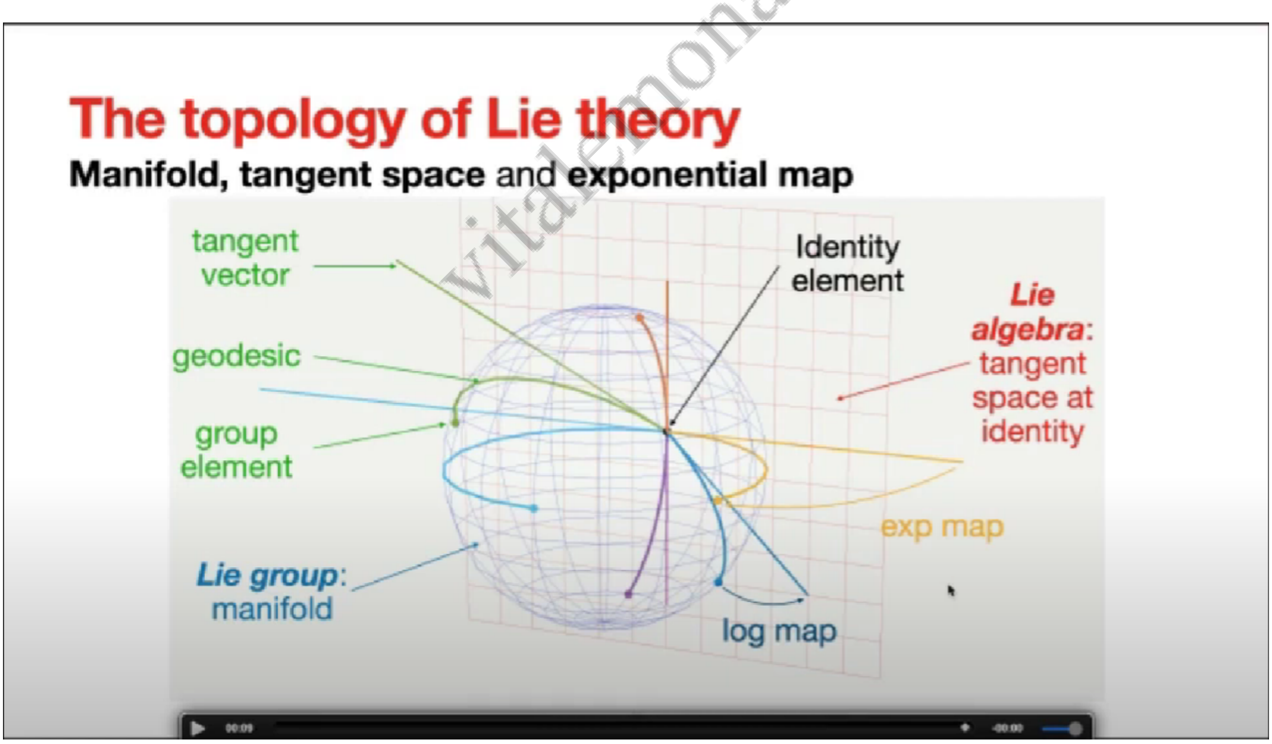


Group Action

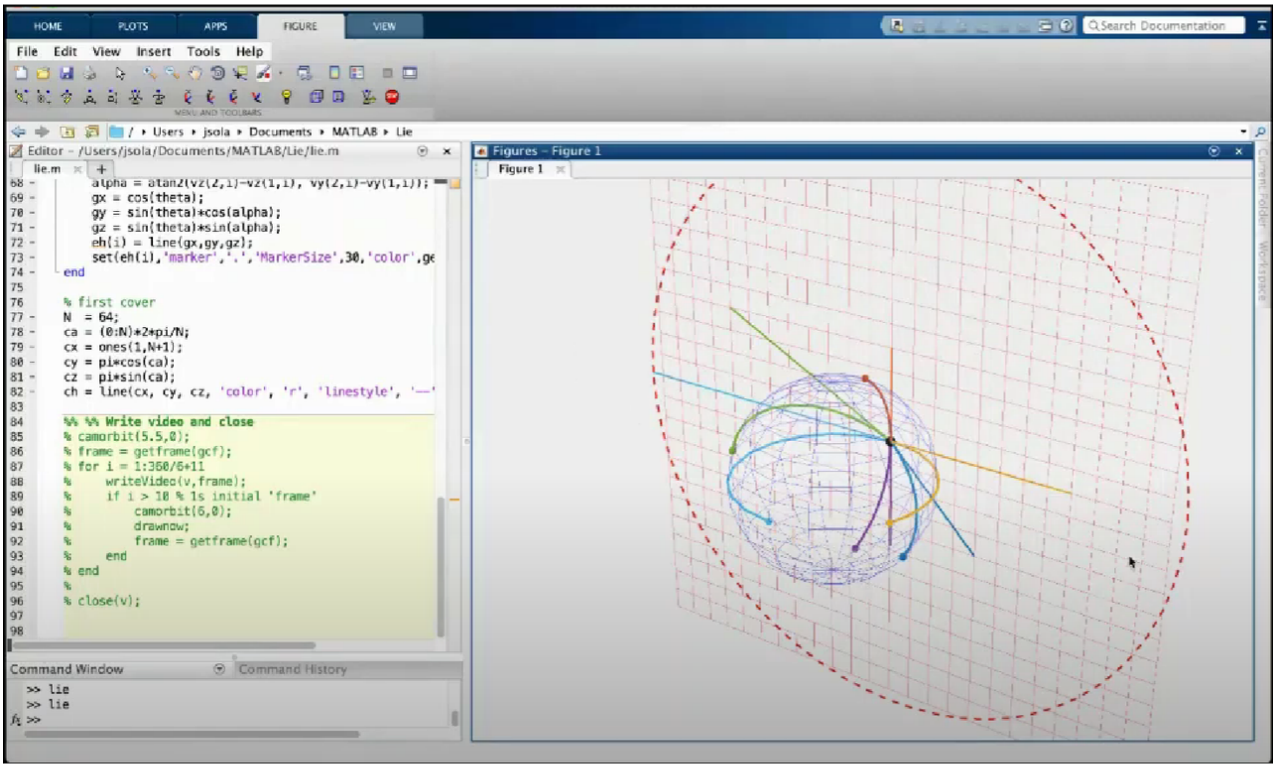


The topology of Lie Theory

- exactly converts elements of the Lie algebra into elements of the group.
  The log map is the inverse operation



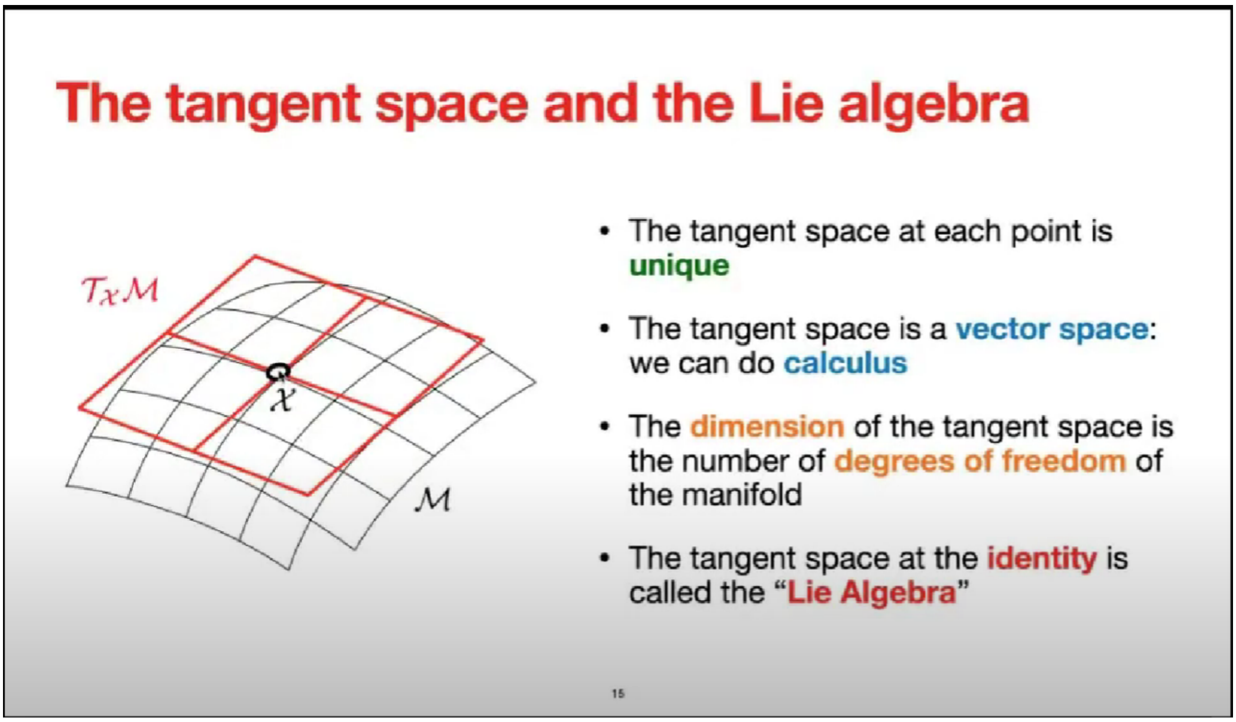
- Tangent sapce will cover manifold **multiple times** and the first it covers is the time where the red circle's radius is pi and it will end up at the **antipode point（对径点）**"



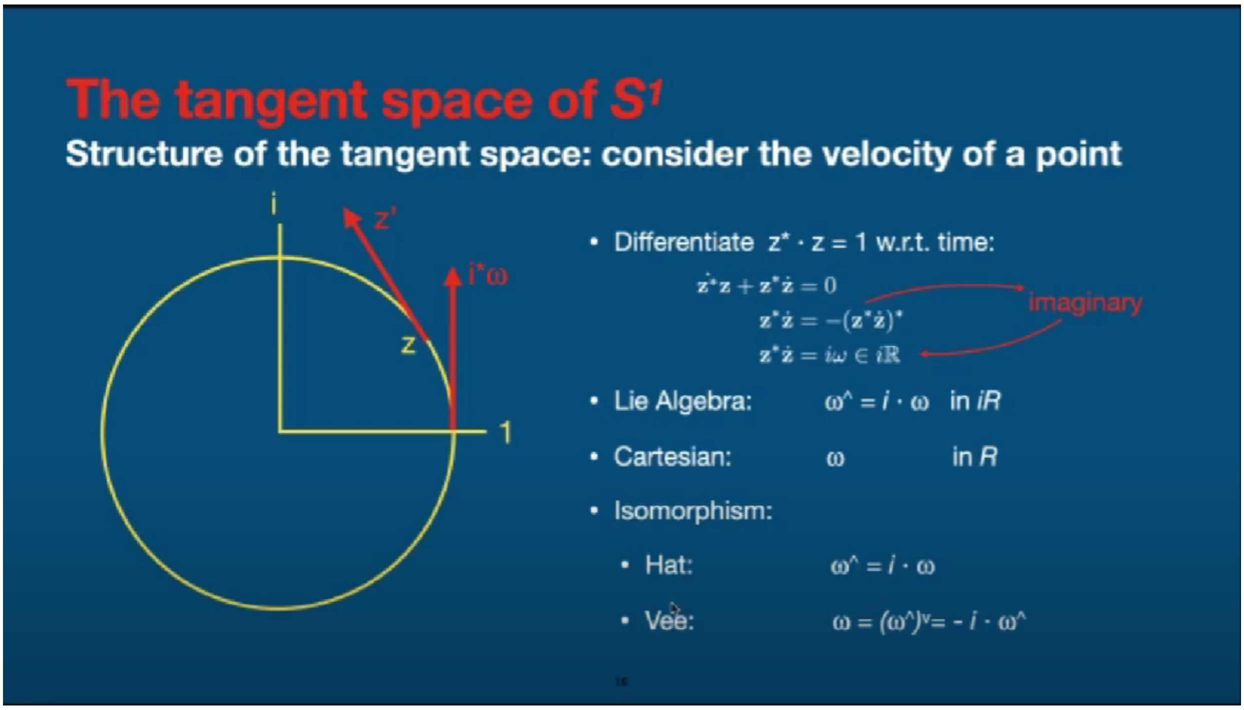
Tangent space and Lie Algebra

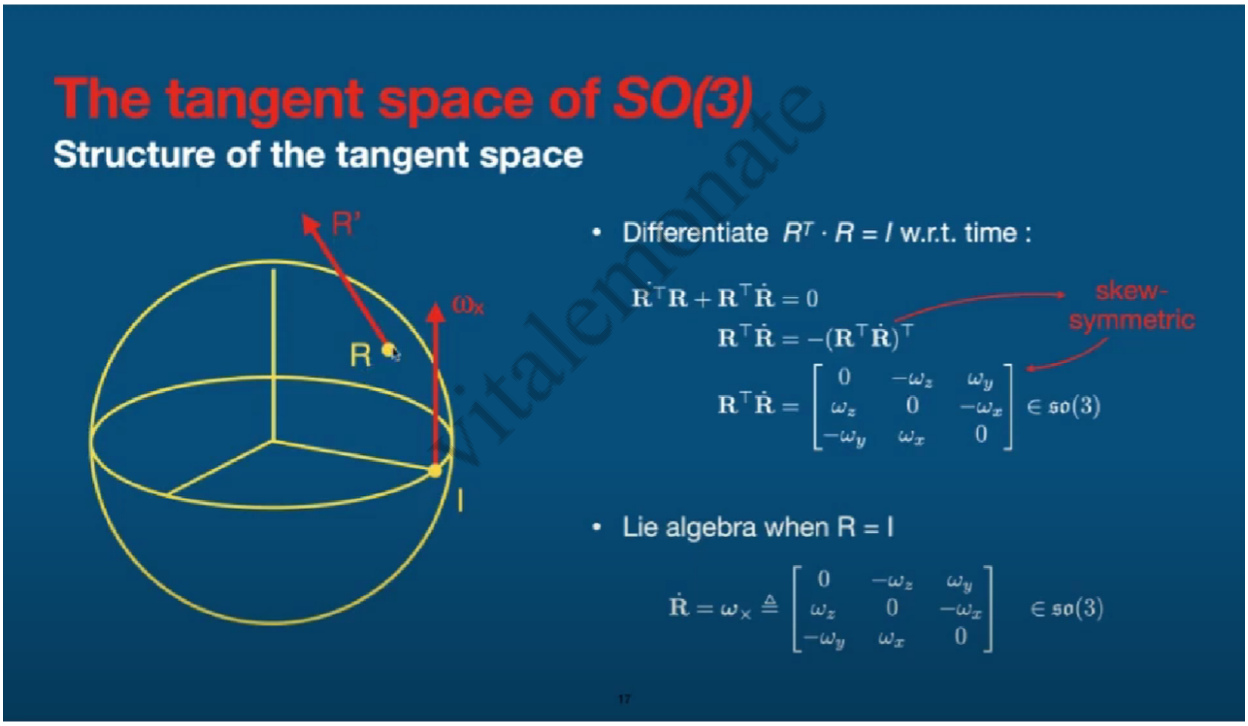
Definition

- Wiki page: **切空间**（Tangent space）是在某一点所有的切向量组成的线性空间
- 由于manifold是smooth的，因此manifold上每个点处的tangent space是**唯一的**
- tangent space是向量空间
- tangent space空间的维度是manifold的自由度
- 特殊的tangent space：group上所有点处的tangent space是的**结构是相同的("The manifold has the same structure everywhere")**，比如下图中所有点处的tangent space都是二维平面，但各点处的tangent space**并不是同一个平面(not the same)，称identity处的tangent space为lie algebra**



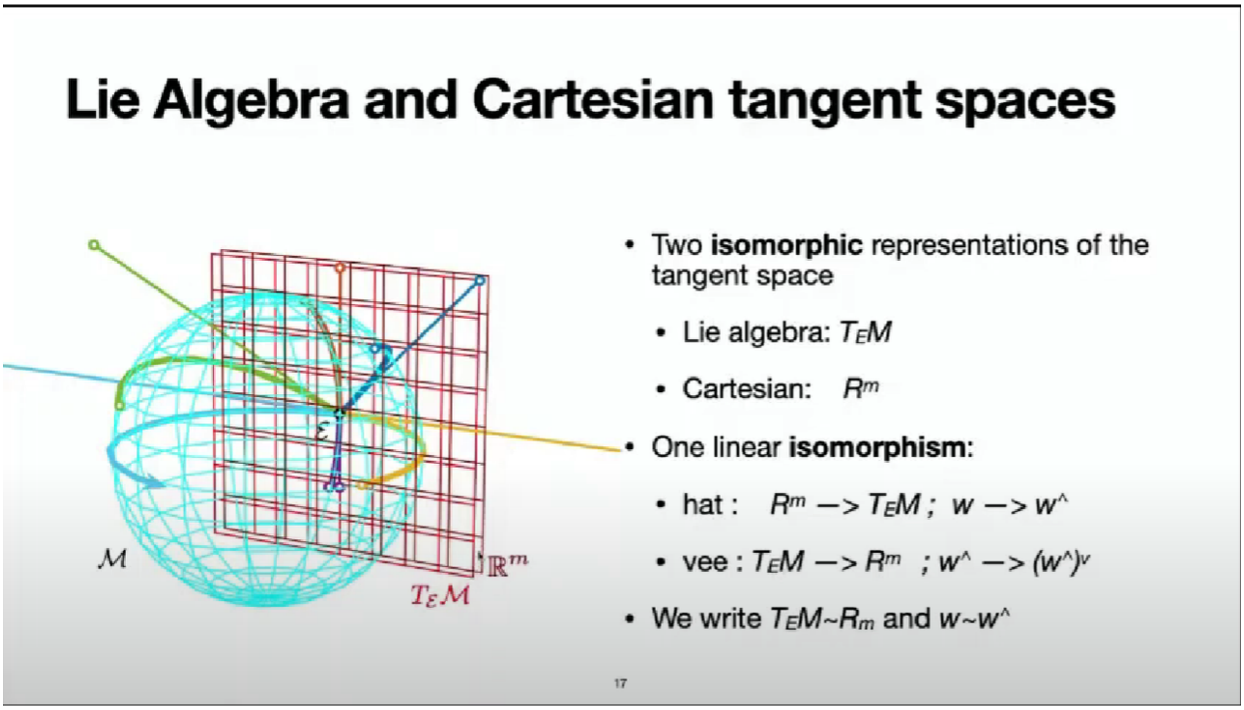
Examples

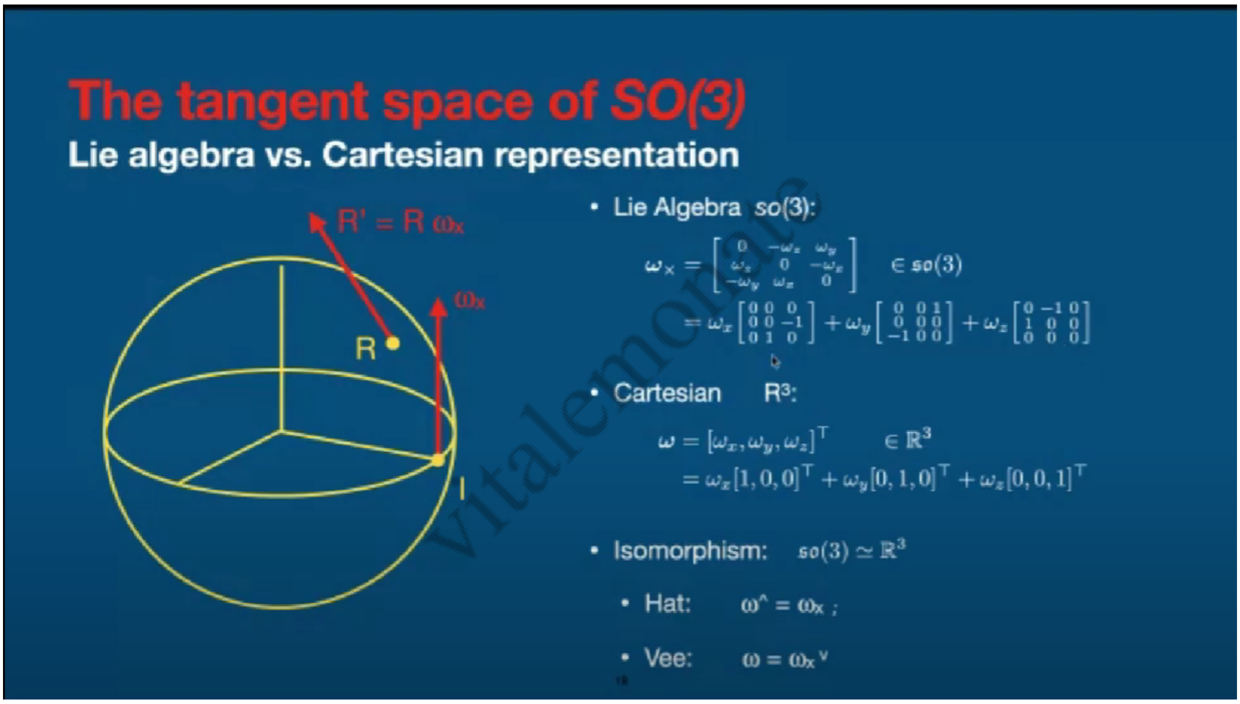




Lie algebra vs. Cartesian representation

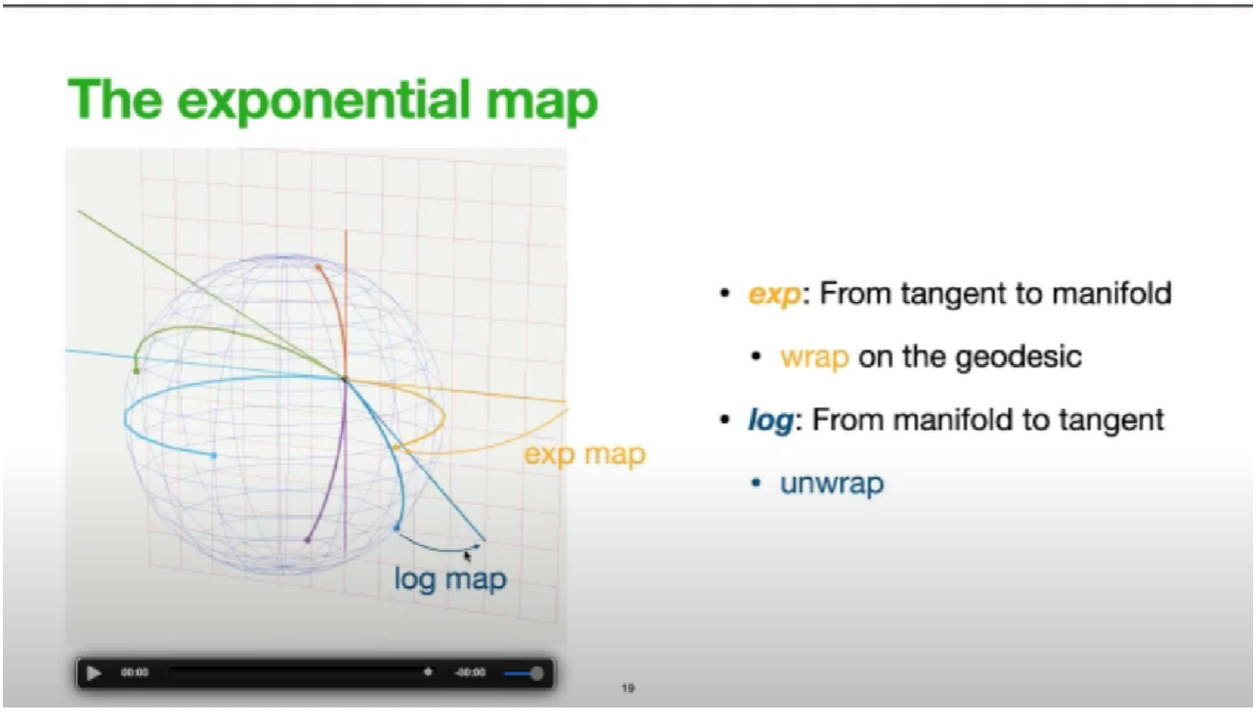
- Tangent space有两种同构(isomorphism)的表示形式：**Lie algebra和Cartesian**





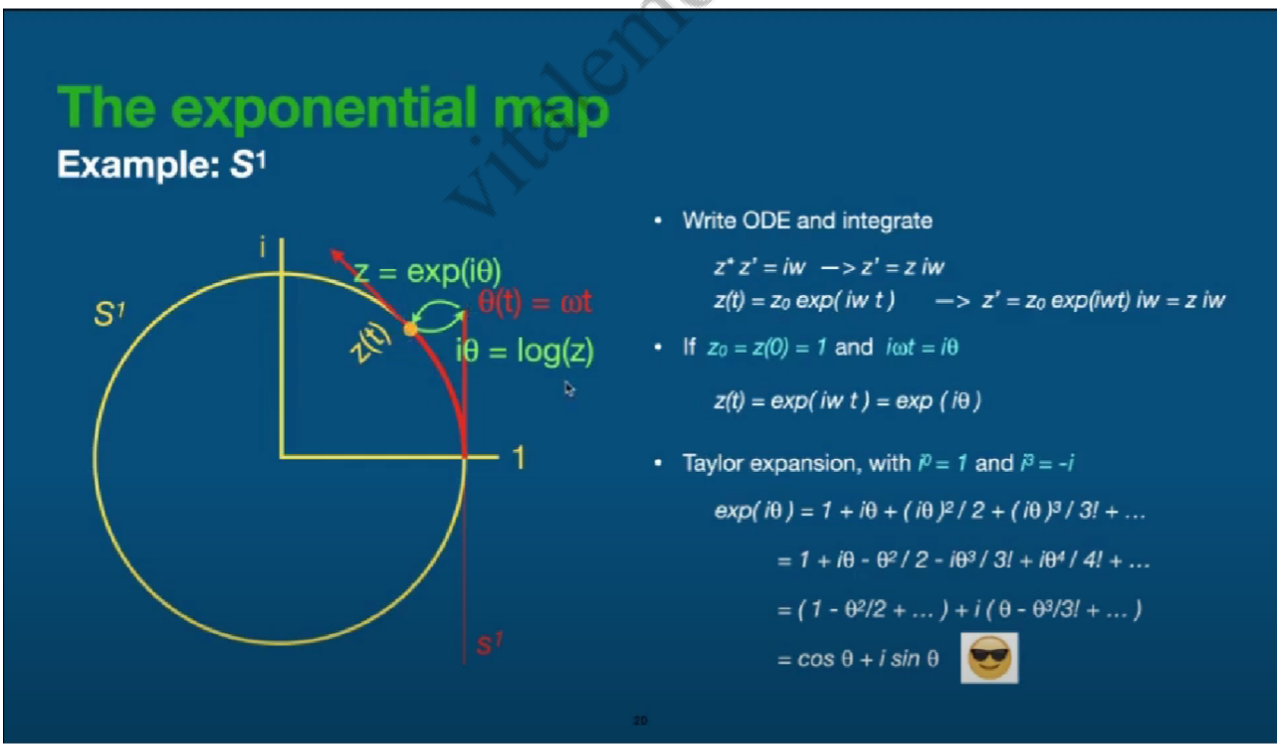
The Exponential map

Definition

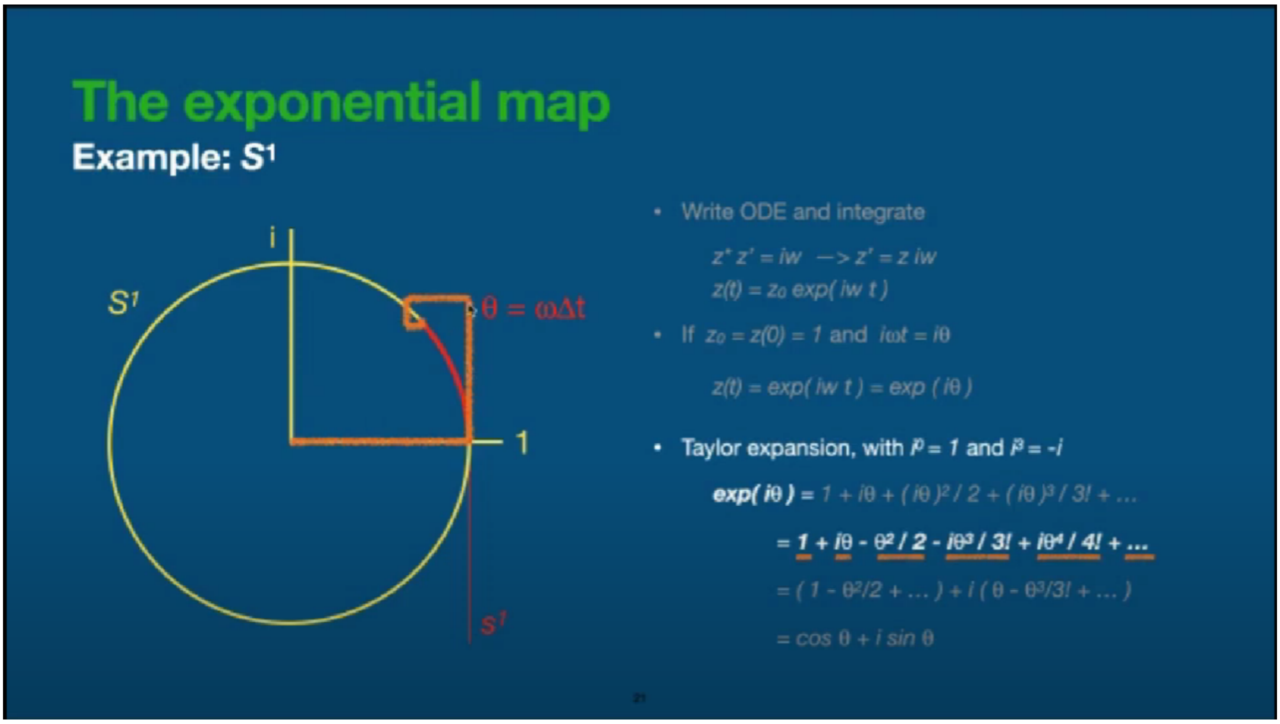


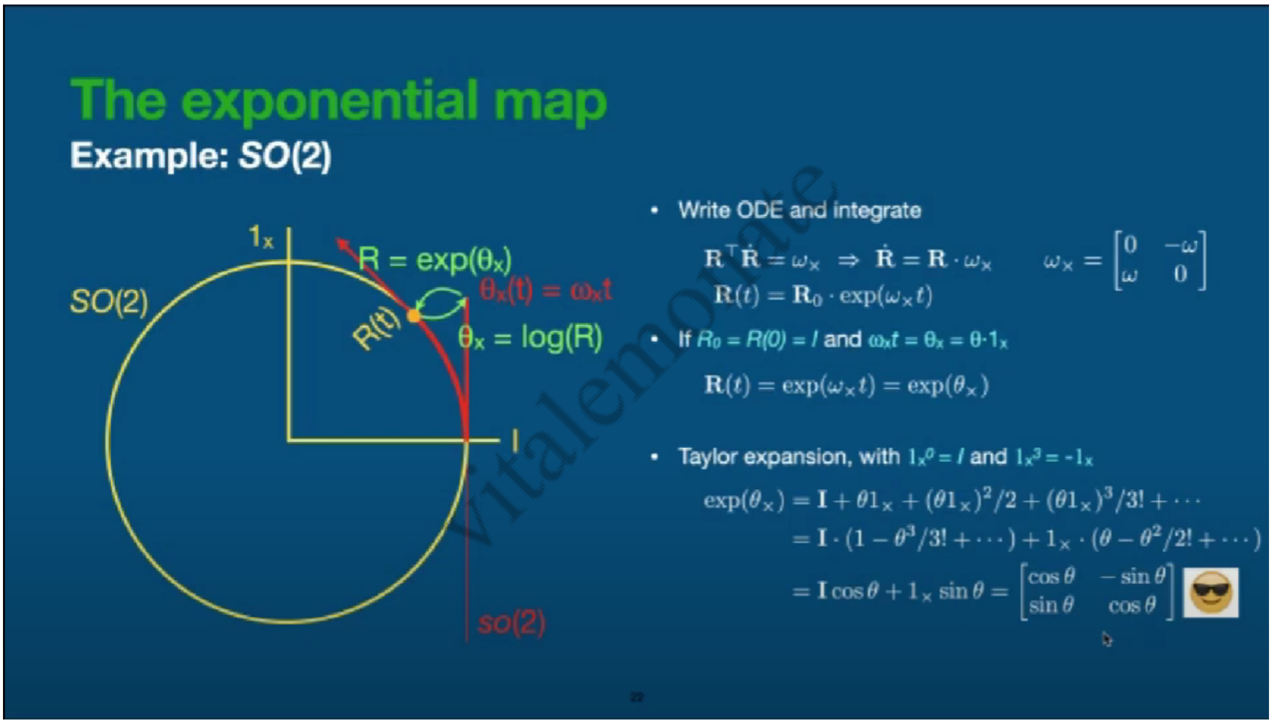
Examples

- 推导思路：根据manifold的约束方程，对时间 t 求导，构建 z 的ODE方程，最终得到lie algebra到lie group的exponential map
- 因为整个推导过程没有**近似(approximation)的步骤**，每一步都是精确(exact)的，因此“**exponential map is exact”**

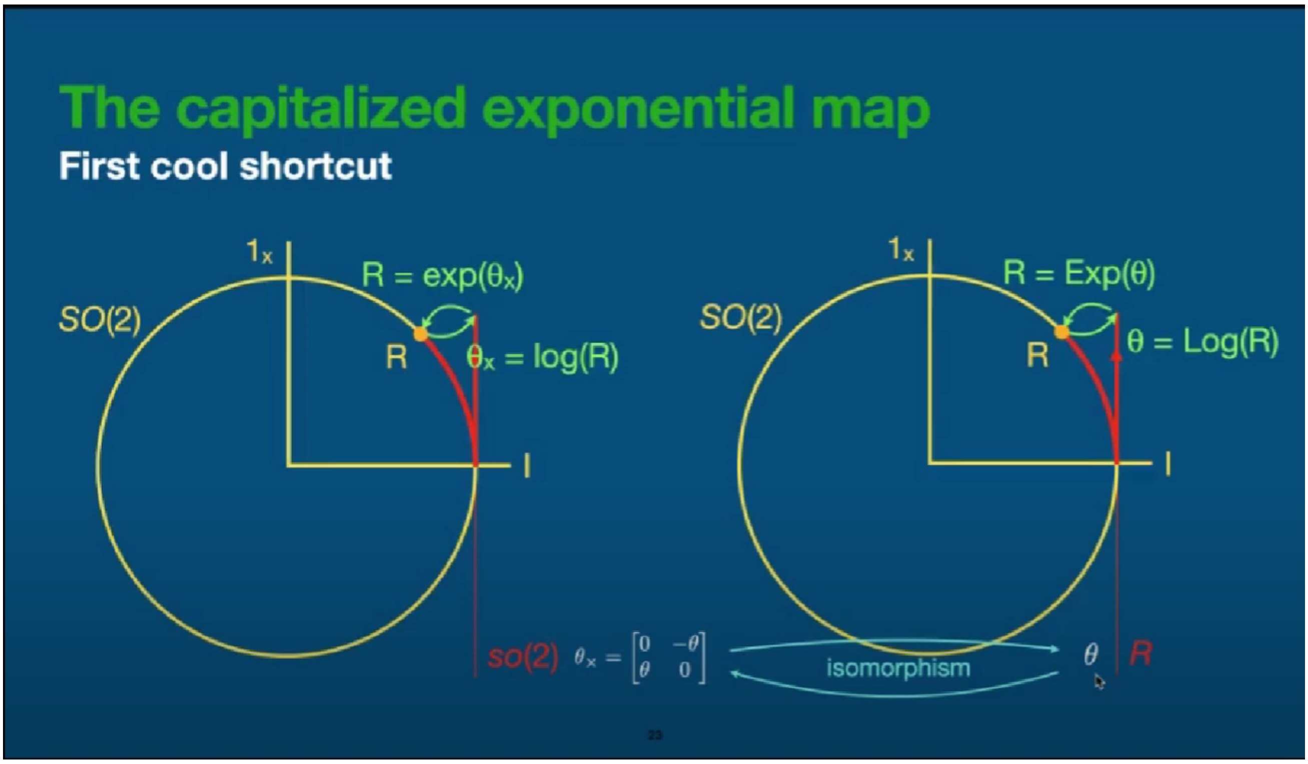


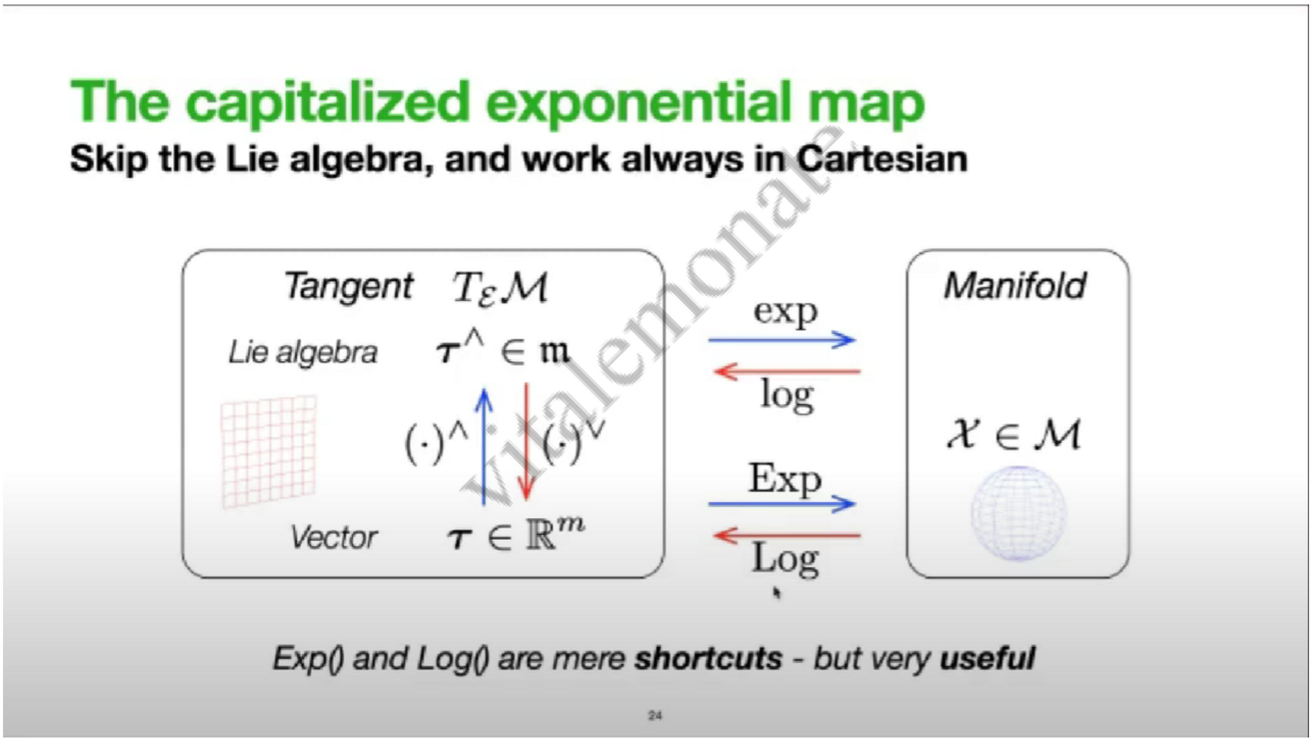
- 根据 exp 的泰勒展开式，可以直观地描述exponential map的过程





The capitalized expoential map

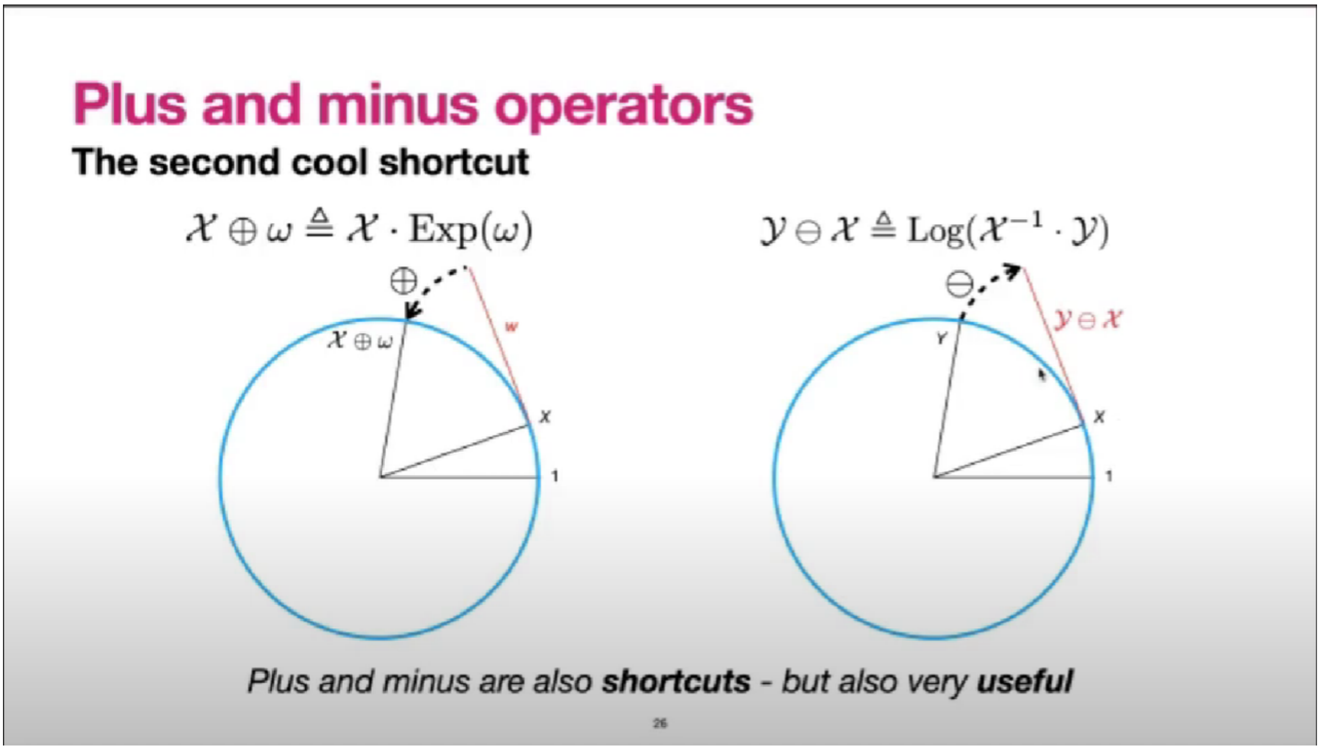




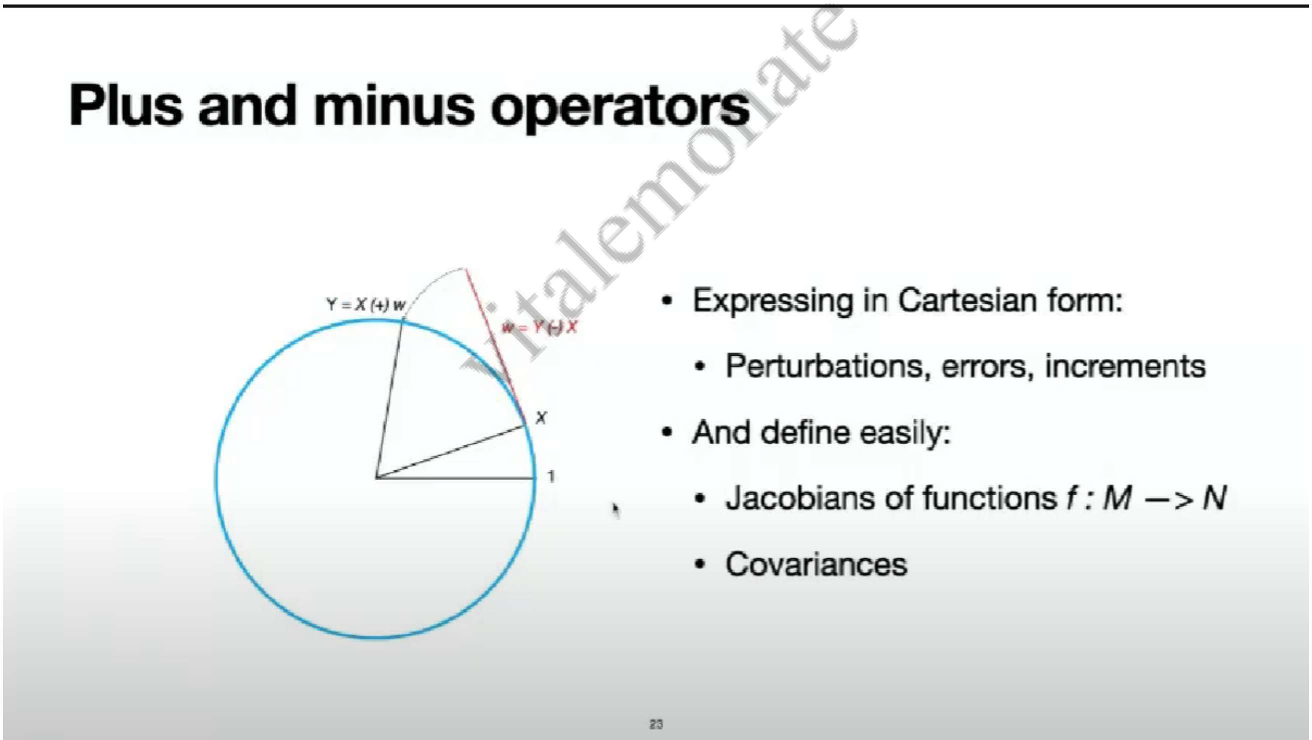
Plus and minus operators

Definition

- $\mathcal{X} \oplus \boldsymbol{\omega}$ 的结果是group上的元素，而 $\mathcal{Y} \ominus \mathcal{X}$ **是vector**

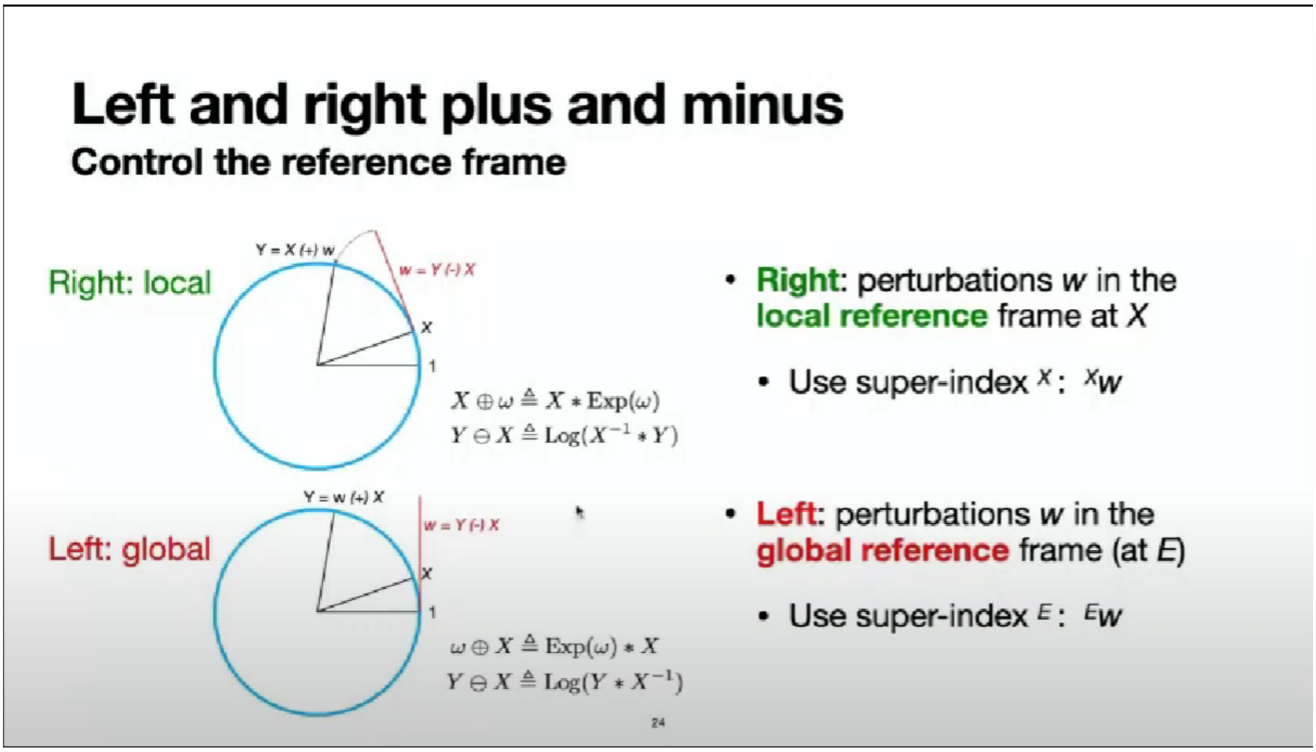


- 使用Cartesian形式的tangent space的优势是方便构造perturbations, errors, increments以及jacobians和covarians



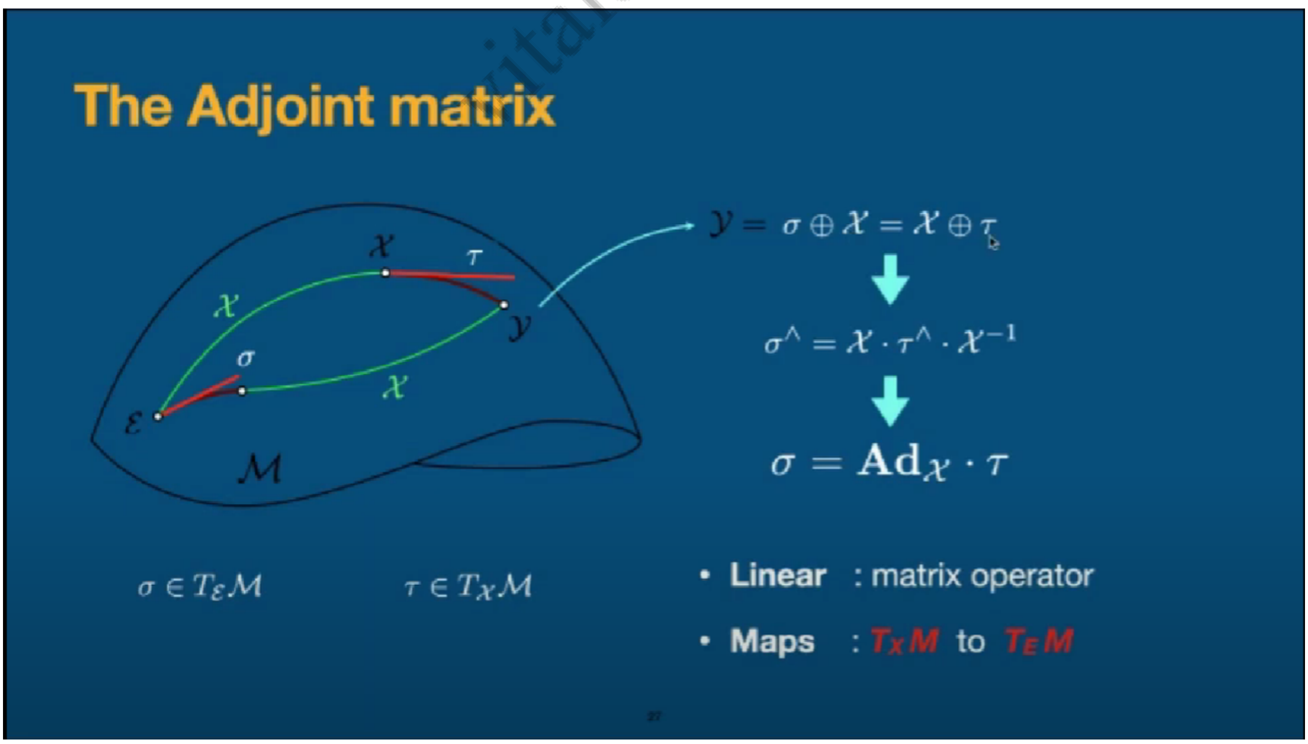
Left and right plus and minus

- 由于group通常**不满足交换律(non-commutative)**，因此left和right operator的结果往往是不同的
- 当使用right operator时，tangent vector是表示在 $\mathcal{X}$ 点所在的local reference frame下的
- 当使用left operator时，tangent vector是表示在identity点所在的global reference frame下的
- 当group满足**交换律时，**二者的结果相同，比如SO(2)

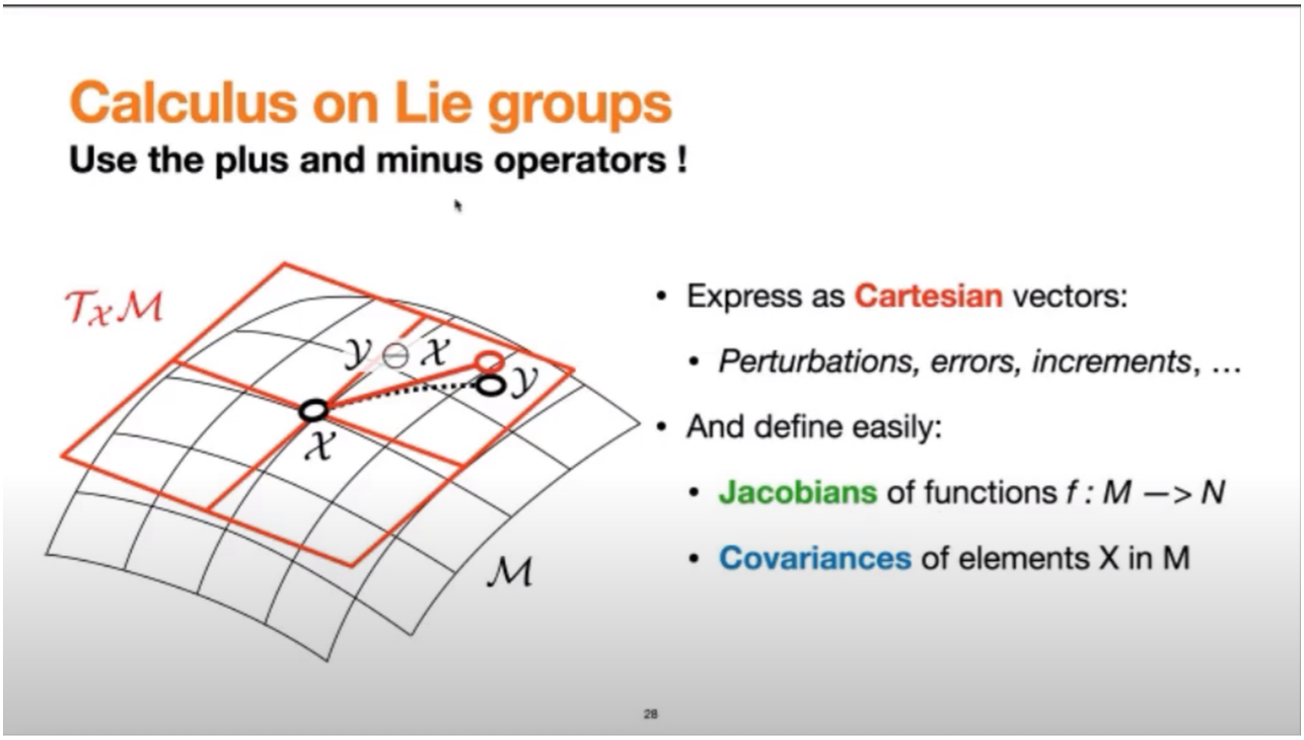


The Adjoint matrix

- The adjoint **linearly** and **exactly** transforms **tangent vectors at** χ to the tangent space **at the identity**
- 当group满足**交换律**时，$\boldsymbol{\sigma}$ **和** $\boldsymbol{\tau}$ **相同**，此时 $\mathbf{Ad}_{\mathcal{X}} = \mathbf{I}$，一个比较典型的例子就是SO(2)



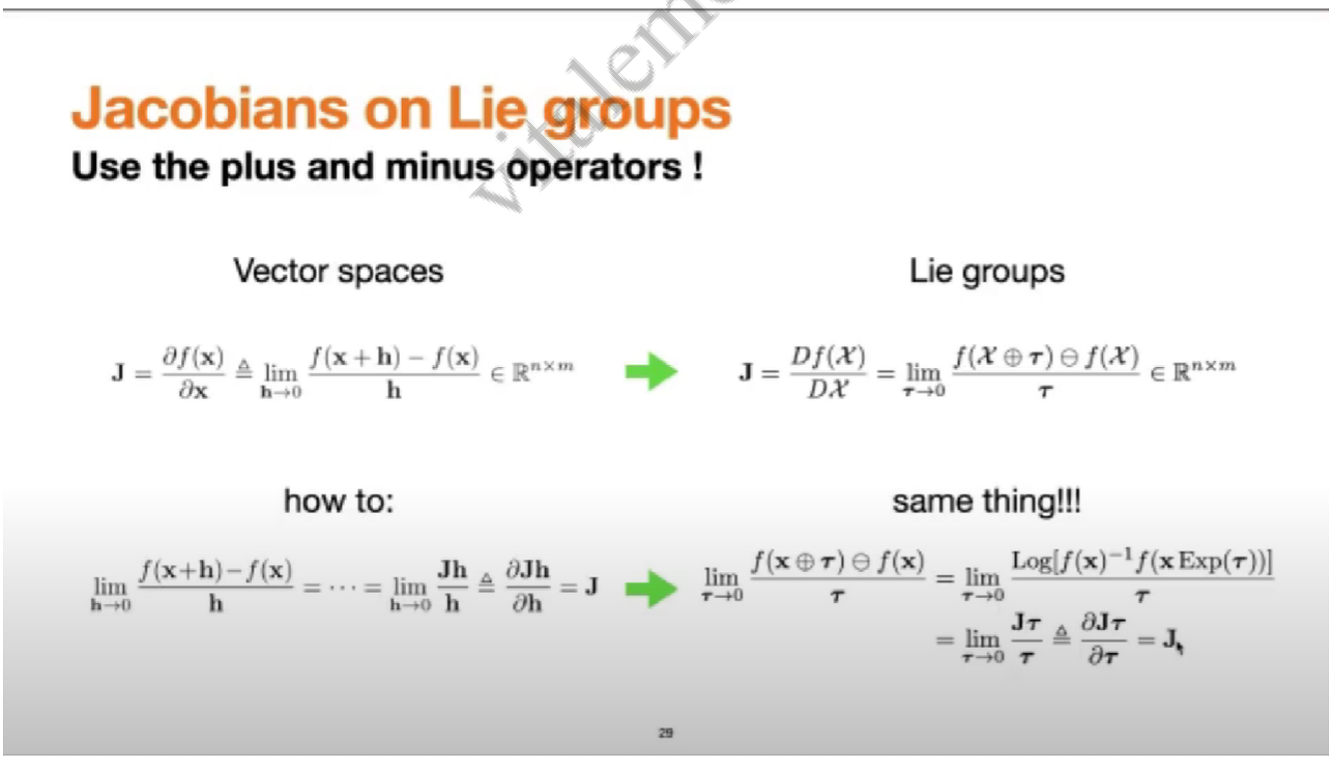
Calculus on Lie groups



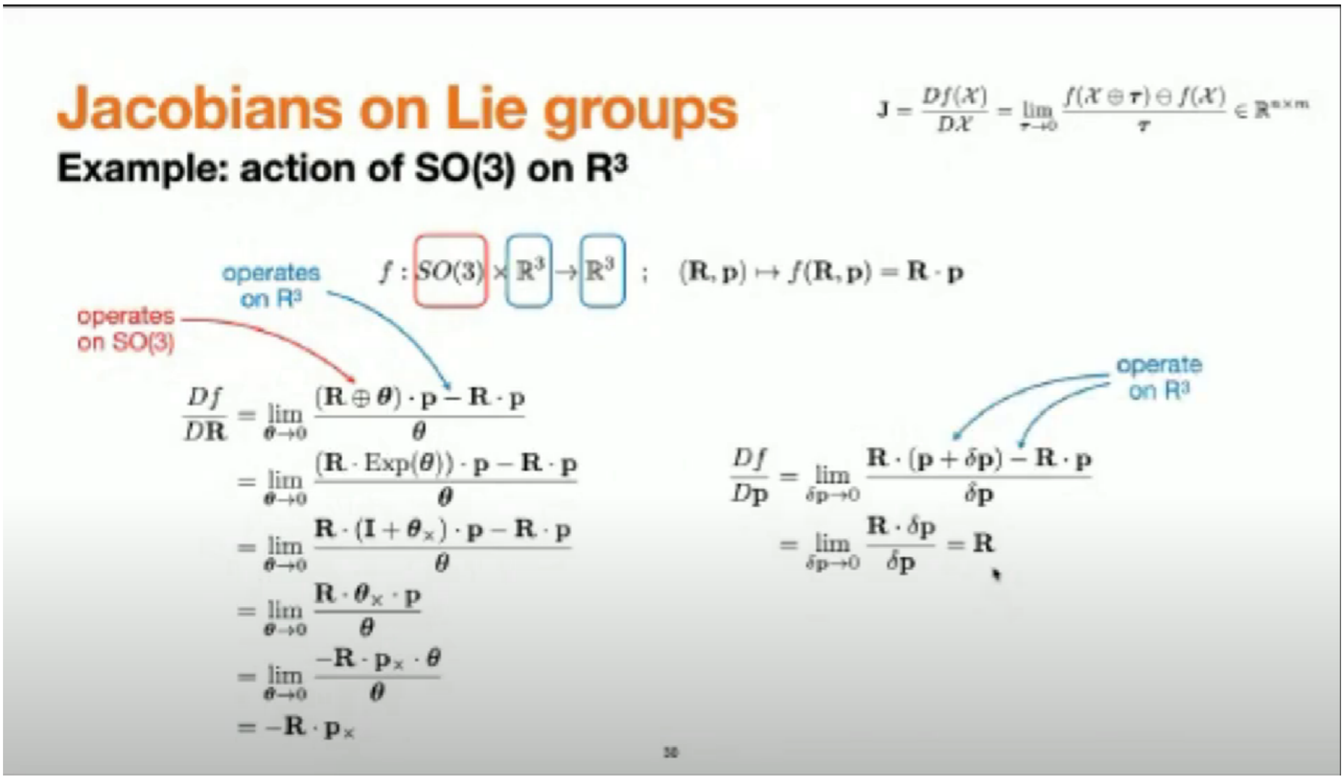
Jacobians

Definition

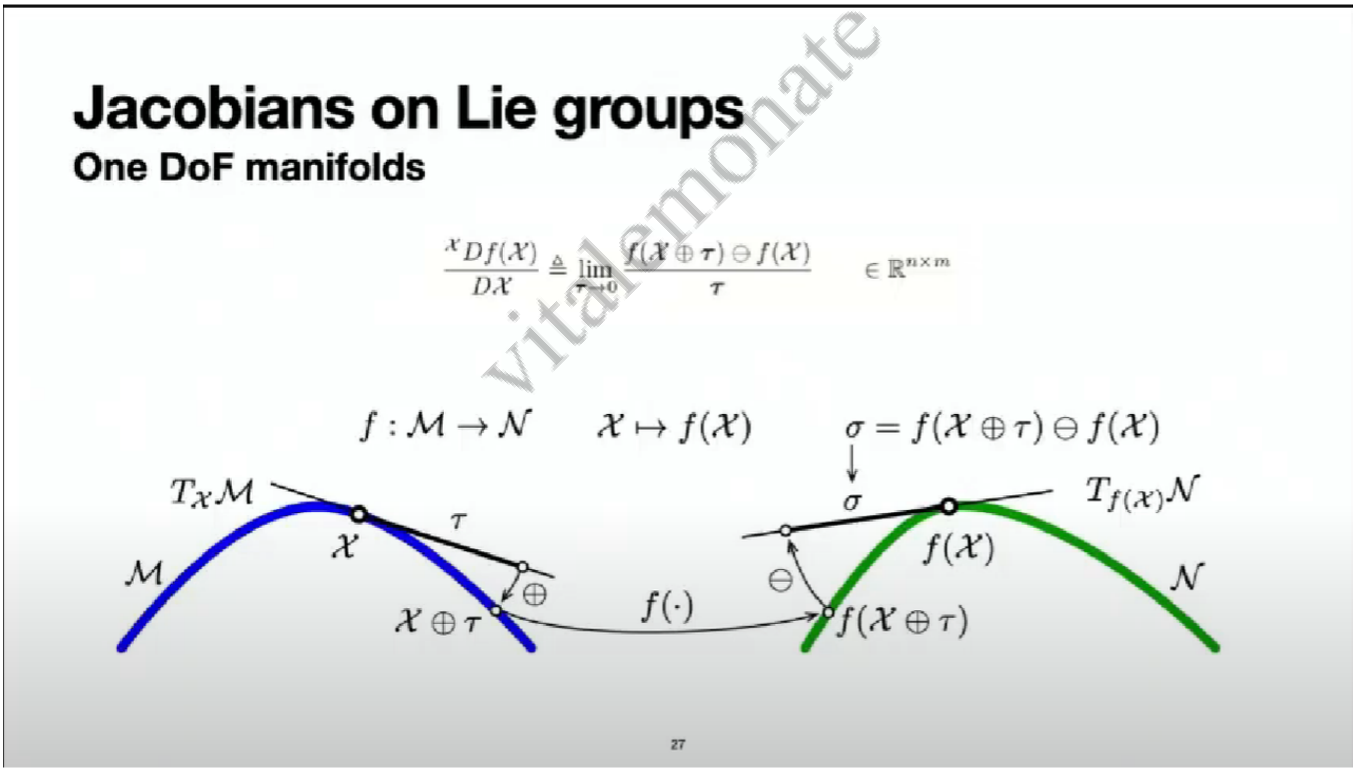
- 求导时要注意元素是在Vector space在Lie Group，**vector space**上的加减运算向量的**"+"，"-"**，**Lie group**上元素的加减运算分别为 $\oplus$ **和** $\ominus$

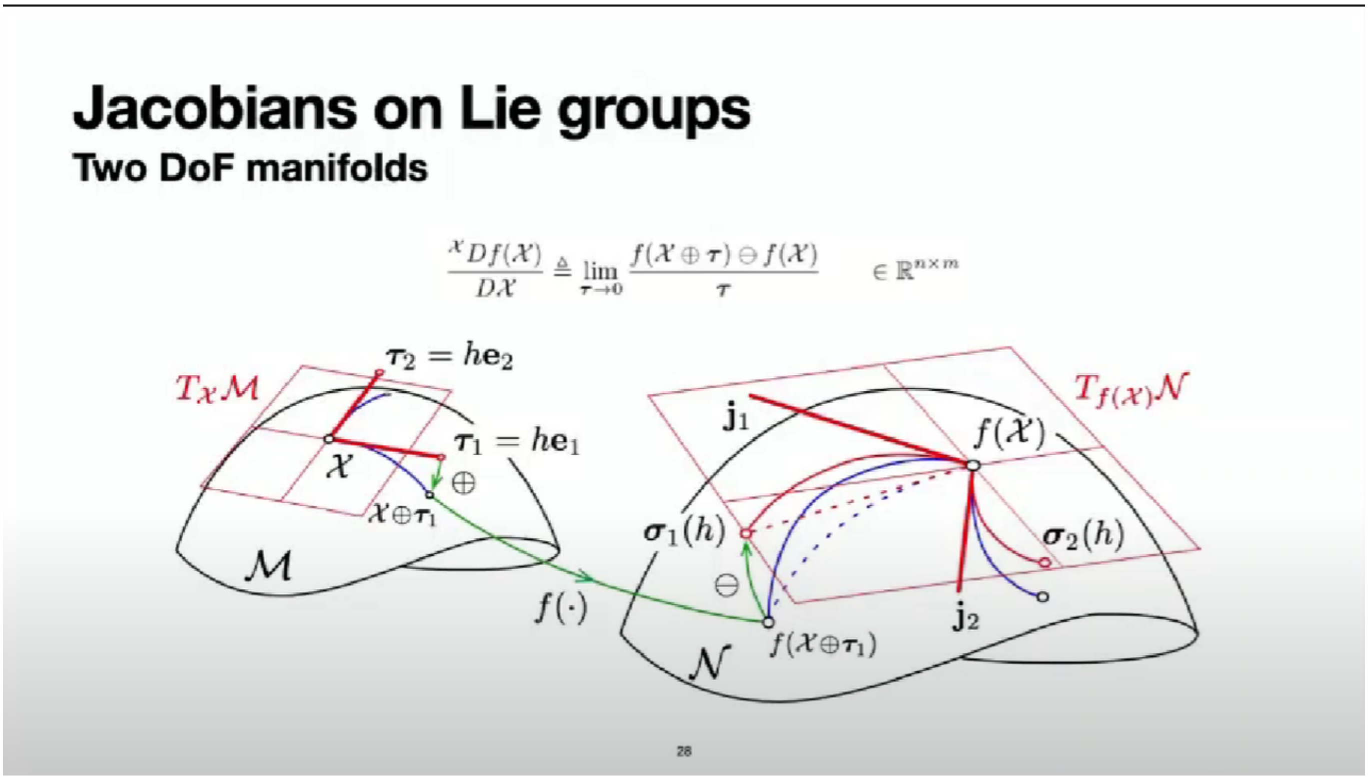


Example

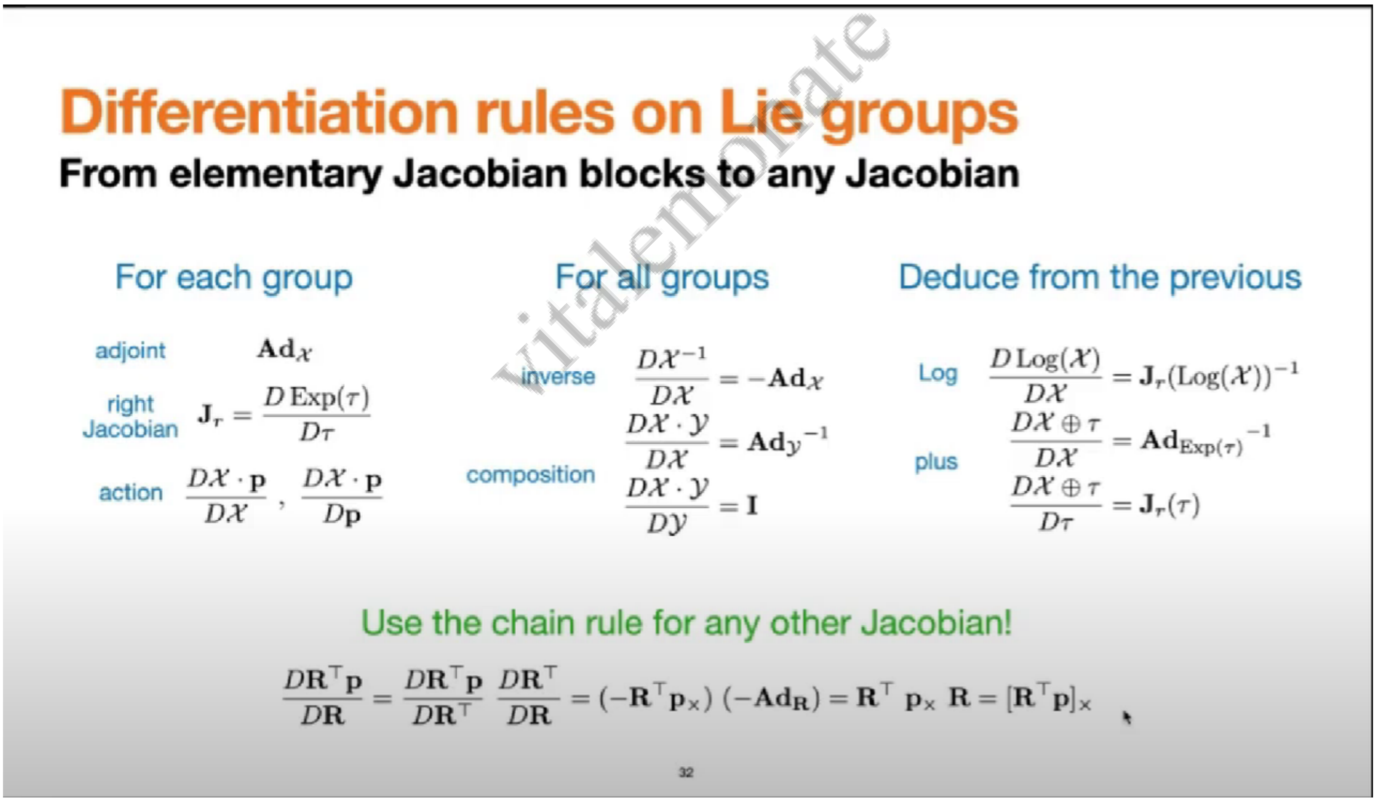


Graphical interpretation

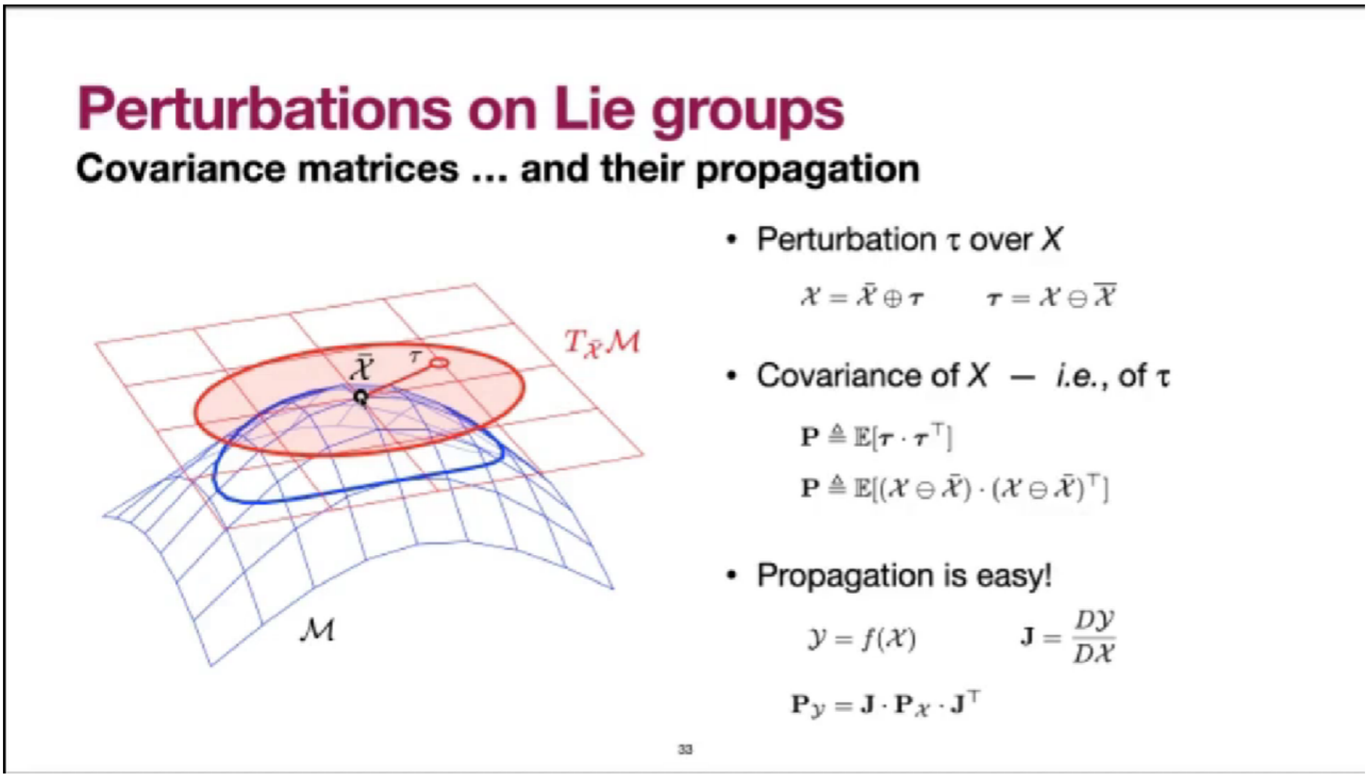




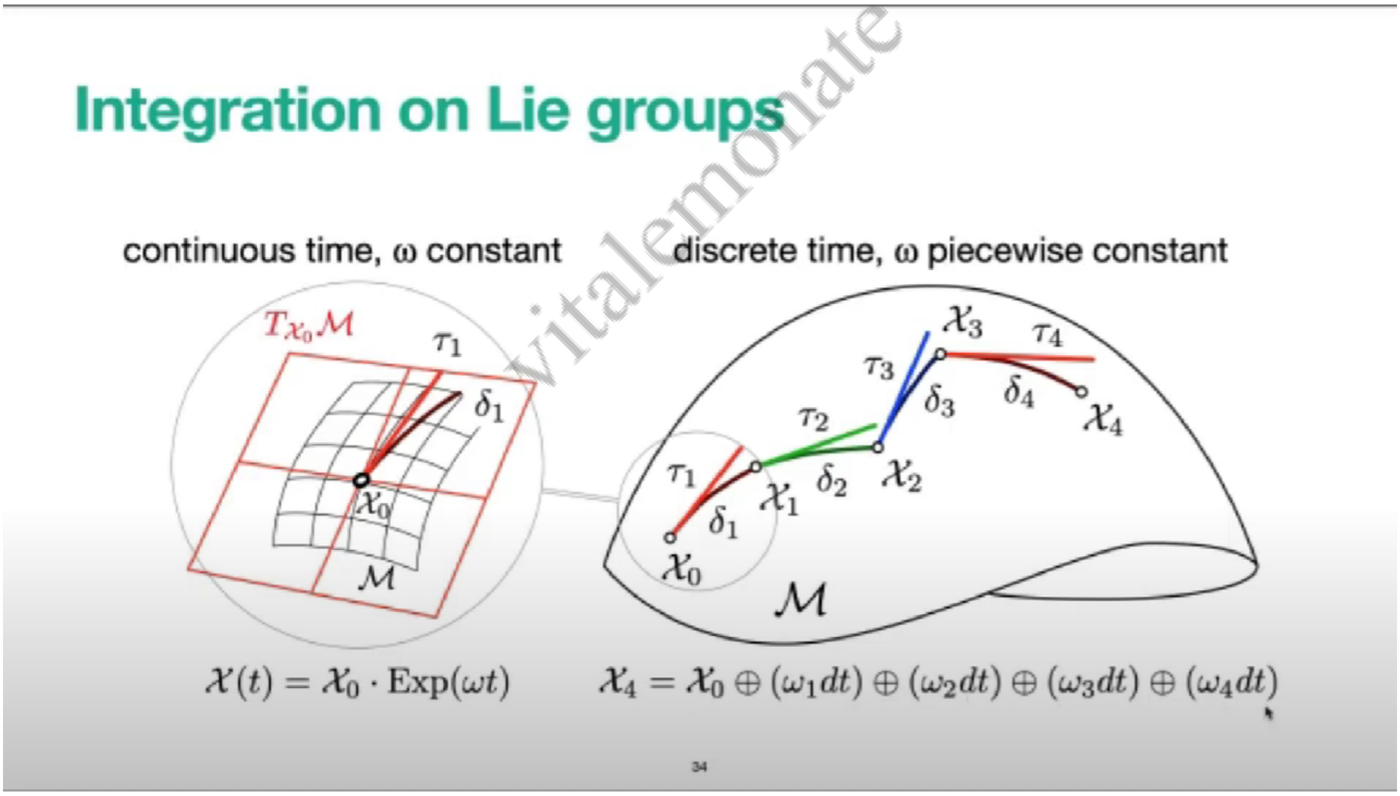
Differentiation rules on Lie groups



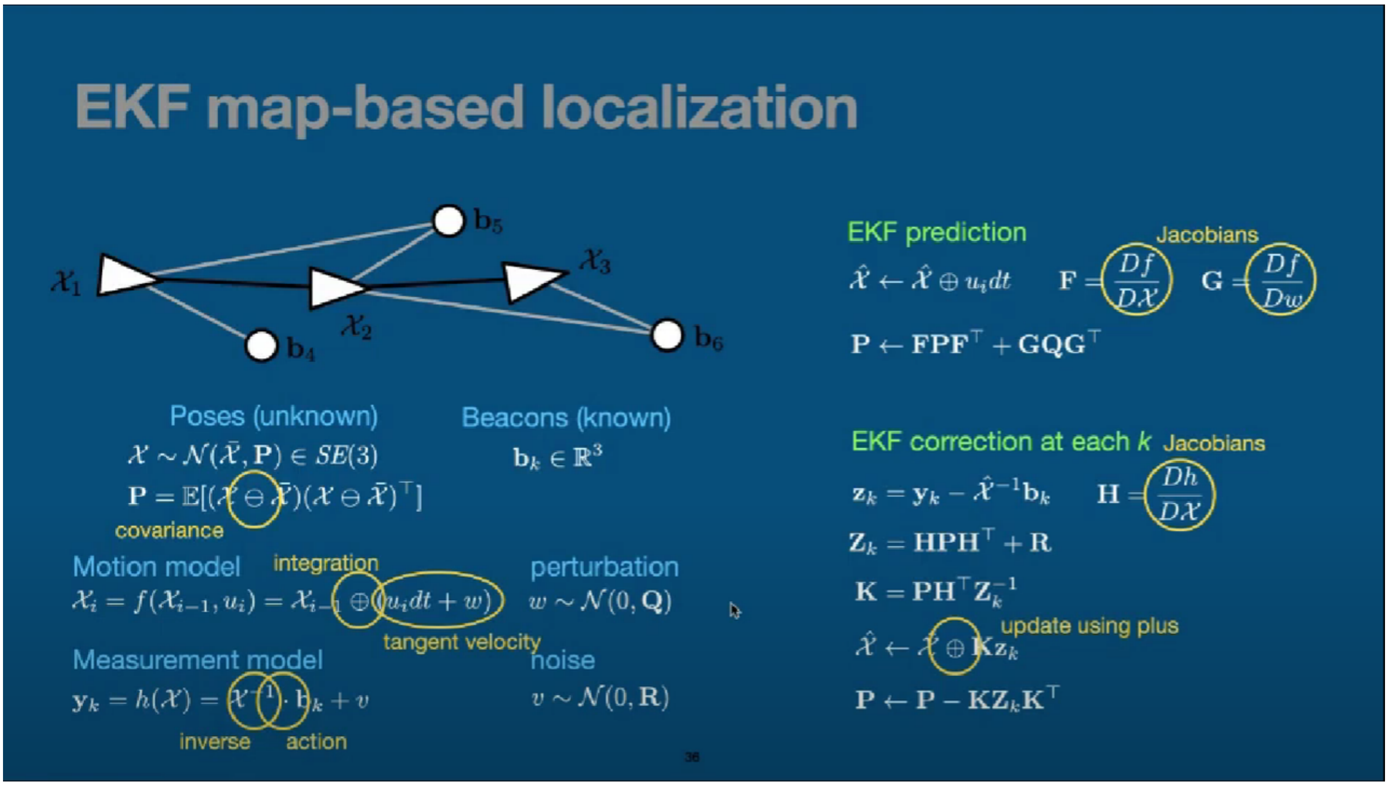
Perturbations on Lie groups



Intergration on Lie groups



EKF map-based localization


EKF map-based localization
$\mathcal{X}_1$ $\mathcal{X}_2$ $\mathcal{X}_3$ $\mathbf{b}_4$ $\mathbf{b}_5$ $\mathbf{b}_6$
Poses (unknown)
$\mathcal{X} \sim \mathcal{N}(\bar{\mathcal{X}}, \mathbf{P}) \in SE(3)$
$\mathbf{P} = \mathbb{E}[(\mathcal{X} \ominus \bar{\mathcal{X}})(\mathcal{X} \ominus \bar{\mathcal{X}})^\top]$
covariance
Beacons (known)
$\mathbf{b}_k \in \mathbb{R}^3$
Motion model
integration
$\mathcal{X}_i = f(\mathcal{X}_{i-1}, u_i) = \mathcal{X}_{i-1} \oplus (u_i dt + w)$
tangent velocity
perturbation
$w \sim \mathcal{N}(0, \mathbf{Q})$
Measurement model
$\mathbf{y}_k = h(\mathcal{X}) = \mathcal{X}^{-1} \cdot \mathbf{b}_k + v$
inverse
action
noise
$v \sim \mathcal{N}(0, \mathbf{R})$
EKF prediction
Jacobians
$\hat{\mathcal{X}} \leftarrow \hat{\mathcal{X}} \oplus u_i dt$
$\mathbf{F} = \frac{Df}{D\mathcal{X}}$ $\mathbf{G} = \frac{Df}{Dw}$
$\mathbf{P} \leftarrow \mathbf{F}\mathbf{P}\mathbf{F}^\top + \mathbf{G}\mathbf{Q}\mathbf{G}^\top$
EKF correction at each k Jacobians
$\mathbf{z}_k = \mathbf{y}_k - \hat{\mathcal{X}}^{-1}\mathbf{b}_k$
$\mathbf{H} = \frac{Dh}{D\mathcal{X}}$
$\mathbf{Z}_k = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}$
$\mathbf{K} = \mathbf{P}\mathbf{H}^\top \mathbf{Z}_k^{-1}$
update using plus
$\hat{\mathcal{X}} \leftarrow \hat{\mathcal{X}} \oplus \mathbf{K}\mathbf{z}_k$
$\mathbf{P} \leftarrow \mathbf{P} - \mathbf{K}\mathbf{Z}_k\mathbf{K}^\top$
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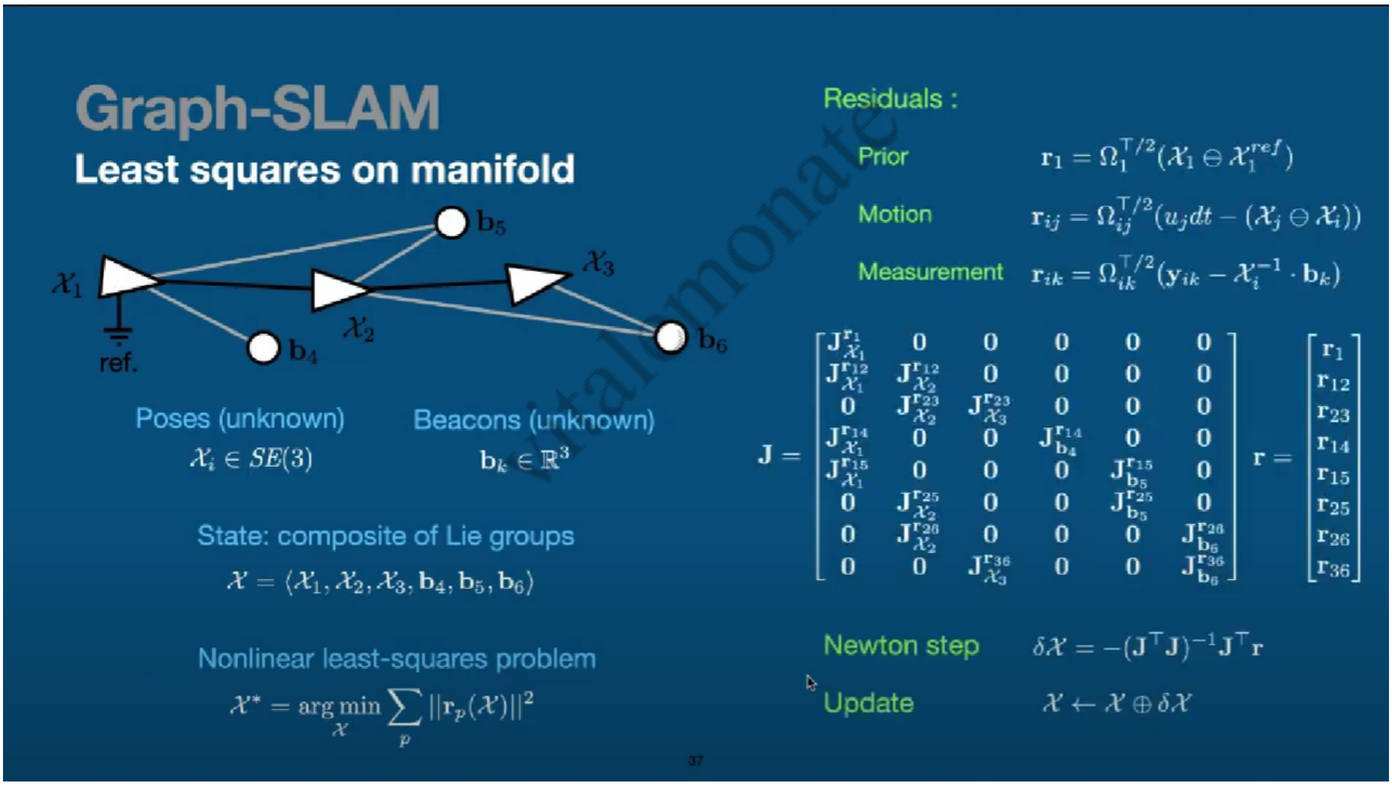
Graph-SLAM
Least squares on manifold
$\mathcal{X}_1$ $\mathcal{X}_2$ $\mathcal{X}_3$ $\mathbf{b}_4$ $\mathbf{b}_5$ $\mathbf{b}_6$
ref.
Poses (unknown)
$\mathcal{X}_i \in SE(3)$
Beacons (unknown)
$\mathbf{b}_k \in \mathbb{R}^3$
State: composite of Lie groups
$\mathcal{X} = \langle \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathbf{b}_4, \mathbf{b}_5, \mathbf{b}_6 \rangle$
Nonlinear least-squares problem
$\mathcal{X}^* = \arg\min_{\mathcal{X}} \sum_p ||\mathbf{r}_p(\mathcal{X})||^2$
Residuals :
Prior $\mathbf{r}_1 = \Omega_1^{\top/2}(\mathcal{X}_1 \ominus \mathcal{X}_1^{ref})$
Motion $\mathbf{r}_{ij} = \Omega_{ij}^{\top/2}(u_j dt - (\mathcal{X}_j \ominus \mathcal{X}_i))$
Measurement $\mathbf{r}_{ik} = \Omega_{ik}^{\top/2}(\mathbf{y}_{ik} - \mathcal{X}_i^{-1} \cdot \mathbf{b}_k)$
$\mathbf{J} = \begin{bmatrix} \mathbf{J}^{\mathbf{r}_1}_{\mathcal{X}_1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{J}^{\mathbf{r}_{12}}_{\mathcal{X}_1} & \mathbf{J}^{\mathbf{r}_{12}}_{\mathcal{X}_2} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{J}^{\mathbf{r}_{23}}_{\mathcal{X}_2} & \mathbf{J}^{\mathbf{r}_{23}}_{\mathcal{X}_3} & 0 & 0 & 0 \\ \mathbf{J}^{\mathbf{r}_{14}}_{\mathcal{X}_1} & 0 & 0 & \mathbf{J}^{\mathbf{r}_{14}}_{\mathbf{b}_4} & 0 & 0 \\ \mathbf{J}^{\mathbf{r}_{15}}_{\mathcal{X}_1} & 0 & 0 & 0 & \mathbf{J}^{\mathbf{r}_{15}}_{\mathbf{b}_5} & 0 \\ 0 & \mathbf{J}^{\mathbf{r}_{25}}_{\mathcal{X}_2} & 0 & 0 & \mathbf{J}^{\mathbf{r}_{25}}_{\mathbf{b}_5} & 0 \\ 0 & \mathbf{J}^{\mathbf{r}_{26}}_{\mathcal{X}_2} & 0 & 0 & 0 & \mathbf{J}^{\mathbf{r}_{26}}_{\mathbf{b}_6} \\ 0 & 0 & \mathbf{J}^{\mathbf{r}_{36}}_{\mathcal{X}_3} & 0 & 0 & \mathbf{J}^{\mathbf{r}_{36}}_{\mathbf{b}_6} \end{bmatrix}$ $\mathbf{r} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_{12} \\ \mathbf{r}_{23} \\ \mathbf{r}_{14} \\ \mathbf{r}_{15} \\ \mathbf{r}_{25} \\ \mathbf{r}_{26} \\ \mathbf{r}_{36} \end{bmatrix}$
Newton step $\delta\mathcal{X} = -(\mathbf{J}^\top\mathbf{J})^{-1}\mathbf{J}^\top\mathbf{r}$
Update $\mathcal{X} \leftarrow \mathcal{X} \oplus \delta\mathcal{X}$
vitalmonate
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