A micro Lie theory for state estimation in robotics

本文是 Lecture-Lie theory for the roboticist 的后序文档，对应的paper为Joan Sola的著作A micro Lie theory for state estimation in robotics，这篇paper简明扼要地介绍了Lie theory以及在状态估计中的应用，非常值得精读，在 Lecture-Lie theory for the roboticist 中已经整理过Lie theory中的一些基本概念，本文不再赘述Lie theory的基本概念，关注的重点为**Derivatives on Lie groups**。

Pre-knowledge

The Exponential map

Intuitive notion

Exponential map **exactly** converts elements of the Lie algebra into elements of the group

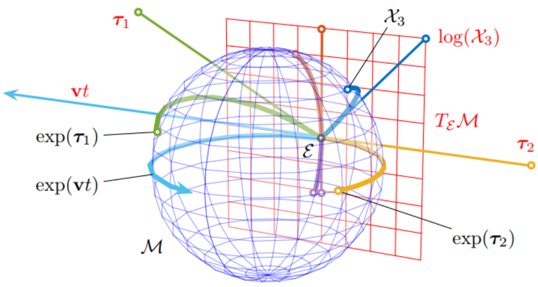


Figure 1. Representation of the relation between the Lie group and the Lie algebra. The Lie algebra $T_{\mathcal{E}}\mathcal{M}$ (red plane) is the tangent space to the Lie group's manifold $\mathcal{M}$ (here represented as a blue sphere) at the identity $\mathcal{E}$. Through the exponential map, each straight path $\mathbf{v}t$ through the origin on the Lie algebra produces a path $\exp(\mathbf{v}t)$ around the manifold which runs along the respective geodesic. Conversely, each element of the group has an equivalent in the Lie algebra. This relation is so profound that (nearly) all operations in the group, which is curved and nonlinear, have an exact equivalent in the Lie algebra, which is a linear vector space. Though the sphere in $\mathbb{R}^3$ is not a Lie group (we just use it as a representation that can be drawn on paper), that in $\mathbb{R}^4$ is, and describes the group of unit quaternions —see Fig. 4 and Ex. 5.

Definition

In order to provide a more generic definition of the exponential map, let us define the tangent increment $\boldsymbol{\tau} \triangleq \mathbf{v}t \in \mathbb{R}^m$ as velocity per time, so that we have $\boldsymbol{\tau}^\wedge = \mathbf{v}^\wedge t \in \mathfrak{m}$ a point in the Lie algebra. The exponential map, and its inverse the logarithmic map, can be now written as,

$$\exp: \quad \mathfrak{m} \to \mathcal{M} \quad ; \quad \boldsymbol{\tau}^\wedge \mapsto \mathcal{X} = \exp(\boldsymbol{\tau}^\wedge) \tag{14}$$

$$\log: \quad \mathcal{M} \to \mathfrak{m} \quad ; \quad \mathcal{X} \mapsto \boldsymbol{\tau}^\wedge = \log(\mathcal{X}) \ . \tag{15}$$

Closed forms of the exponential in multiplicative groups are obtained by writing the absolutely convergent Taylor series,

$$\exp(\boldsymbol{\tau}^\wedge) = \mathcal{E} + \boldsymbol{\tau}^\wedge + \tfrac{1}{2}\boldsymbol{\tau}^{\wedge 2} + \tfrac{1}{3!}\boldsymbol{\tau}^{\wedge 3} + \cdots \ , \tag{16}$$

Key Properties of the exponential map

$$\exp((t+s)\boldsymbol{\tau}^\wedge) = \exp(t\boldsymbol{\tau}^\wedge)\exp(s\boldsymbol{\tau}^\wedge) \tag{17}$$

$$\exp(t\boldsymbol{\tau}^\wedge) = \exp(\boldsymbol{\tau}^\wedge)^t \tag{18}$$

$$\exp(-\boldsymbol{\tau}^\wedge) = \exp(\boldsymbol{\tau}^\wedge)^{-1} \tag{19}$$

$$\exp(\mathcal{X}\boldsymbol{\tau}^\wedge\mathcal{X}^{-1}) = \mathcal{X}\exp(\boldsymbol{\tau}^\wedge)\mathcal{X}^{-1} \ , \tag{20}$$

- 式(17)证明

根据矩阵指数的性质：

如果 $\mathbf{XY} = \mathbf{YX}$，那么 $\exp(\mathbf{X})\exp(\mathbf{Y}) = \exp(\mathbf{X}+\mathbf{Y})$，由此可知

$$\exp(t\boldsymbol{\tau}^\wedge)\exp(s\boldsymbol{\tau}^\wedge) = \exp((t+s)\boldsymbol{\tau}^\wedge) \tag{A.1}$$

其中矩阵指数的性质可以由BCH近似得到

- 式(18)证明

根据式(17)

$$
\begin{aligned}
\exp(t\boldsymbol{\tau}^{\wedge}) = \exp(\boldsymbol{\tau}^{\wedge} + (t-1)\boldsymbol{\tau}^{\wedge}) &= \exp(\boldsymbol{\tau}^{\wedge})\exp((t-1)\boldsymbol{\tau}^{\wedge}) \\
&= \exp(\boldsymbol{\tau}^{\wedge})\exp(\boldsymbol{\tau}^{\wedge})\exp((t-2)\boldsymbol{\tau}^{\wedge}) \\
&= \ldots \\
&= \exp(\boldsymbol{\tau}^{\wedge})^{t}
\end{aligned} \tag{A.2}
$$

- 式(19)证明

令式(17)中$t=-1$，即可得式(19)

- 式(20)证明

这4个性质中**式(20)最为重要**，会在**adjoint**以及**derivative**中多次使用，推导过程如下

$$
\begin{aligned}
\exp(\mathcal{X}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1}) &= \mathbf{I} + \mathcal{X}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1} + \frac{1}{2!}(\mathcal{X}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1})^2 + \frac{1}{3!}(\mathcal{X}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1})^3 + \ldots \\
&= \mathbf{I} + \mathcal{X}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1} + \frac{1}{2!}\mathcal{X}(\boldsymbol{\tau}^{\wedge})^2\mathcal{X}^{-1} + \frac{1}{3!}\mathcal{X}(\boldsymbol{\tau}^{\wedge})^3\mathcal{X}^{-1} + \ldots \\
&= \mathcal{X}\exp(\boldsymbol{\tau}^{\wedge})\mathcal{X}^{-1}
\end{aligned} \tag{A.3}
$$

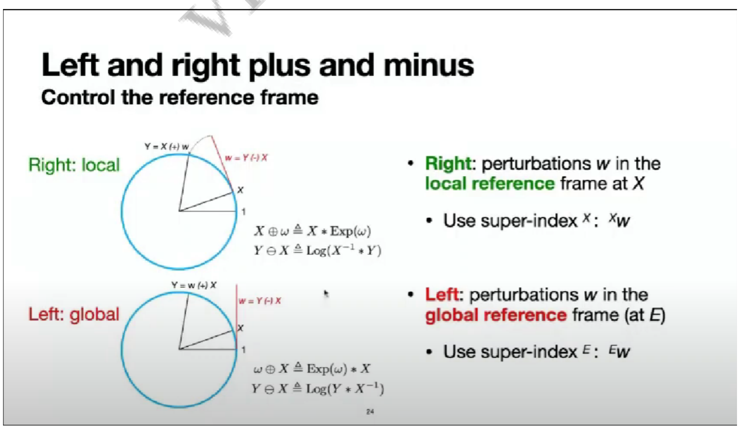
Plus and minus operators

Definition

$$
\begin{aligned}
\text{right-}\oplus: &\quad \mathcal{Y} = \mathcal{X} \oplus {}^{\mathcal{X}}\boldsymbol{\tau} \triangleq \mathcal{X} \circ \operatorname{Exp}({}^{\mathcal{X}}\boldsymbol{\tau}) \in \mathcal{M} && (25) \\
\text{right-}\ominus: &\quad {}^{\mathcal{X}}\boldsymbol{\tau} = \mathcal{Y} \ominus \mathcal{X} \triangleq \operatorname{Log}(\mathcal{X}^{-1} \circ \mathcal{Y}) \in T_{\mathcal{X}}\mathcal{M}\,. && (26)
\end{aligned}
$$

$$
\begin{aligned}
\text{left-}\oplus: &\quad \mathcal{Y} = {}^{\mathcal{E}}\boldsymbol{\tau} \oplus \mathcal{X} \triangleq \operatorname{Exp}({}^{\mathcal{E}}\boldsymbol{\tau}) \circ \mathcal{X} \in \mathcal{M} && (27) \\
\text{left-}\ominus: &\quad {}^{\mathcal{E}}\boldsymbol{\tau} = \mathcal{Y} \ominus \mathcal{X} \triangleq \operatorname{Log}(\mathcal{Y} \circ \mathcal{X}^{-1}) \in T_{\mathcal{E}}\mathcal{M}\,. && (28)
\end{aligned}
$$

这两种定义的主要区别在于**tangent vector**所对应的**reference frame**不同，**left operators**对应manifold的**identity**处的tangent vector，而**right operators**对应manifold的$\mathcal{X}$点处的tangent vector，后文若没有特别强调，都使用**right operators**。定义两种同一操作的不同operator的**根本原因是群上的运算在很多情况下不满足交换律**



Adjoint and adjoint matrix

Adjoint

为了探究manifold上的元素$\mathcal{X}$使用left operator和right operator结果相同时，$\mathcal{X}$点处的**tangent space与ε点处的tangent space**的对应关系，可以得到

$$
{}^{\varepsilon}\boldsymbol{\tau} \oplus \mathcal{X} = \mathcal{X} \oplus {}^{\mathcal{X}}\boldsymbol{\tau} \tag{A.4}
$$

将式(25)(27)代入式(A.4)可得

$$
\begin{aligned}
\exp({}^{\varepsilon}\boldsymbol{\tau}^{\wedge})\mathcal{X} &= \mathcal{X}\exp({}^{\mathcal{X}}\boldsymbol{\tau}^{\wedge}) \\
\exp({}^{\varepsilon}\boldsymbol{\tau}^{\wedge}) &= \mathcal{X}\exp({}^{\mathcal{X}}\boldsymbol{\tau}^{\wedge})\mathcal{X}^{-1}
\end{aligned} \tag{A.5}
$$

将指数映射的重要性质式(20)代入式(A.5)得

$$\exp({}^{\varepsilon}\boldsymbol{\tau}^{\wedge}) = \exp(\mathcal{X}\,{}^{\mathcal{X}}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1}) \tag{A.6}$$

即

$${}^{\varepsilon}\boldsymbol{\tau}^{\wedge} = \mathcal{X}\,{}^{\mathcal{X}}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1} \tag{A.7}$$

式(A.7)描述了从两个切空间的对应关系，而adjoint就是用来描述这种对应关系

1) The adjoint: We thus define the *adjoint* of $\mathcal{M}$ at $\mathcal{X}$, noted $\mathrm{Ad}_{\mathcal{X}}$, to be

$$\mathrm{Ad}_{\mathcal{X}} : \mathfrak{m} \to \mathfrak{m}; \quad \boldsymbol{\tau}^{\wedge} \mapsto \mathrm{Ad}_{\mathcal{X}}(\boldsymbol{\tau}^{\wedge}) \triangleq \mathcal{X}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1}, \tag{29}$$

so that ${}^{\mathcal{E}}\boldsymbol{\tau}^{\wedge} = \mathrm{Ad}_{\mathcal{X}}({}^{\mathcal{X}}\boldsymbol{\tau}^{\wedge})$. This defines the *adjoint action* of the group on its own Lie algebra. The adjoint has two interesting (and easy to prove) properties,

$$\begin{aligned}
\text{Linear}: &\quad \mathrm{Ad}_{\mathcal{X}}(a\boldsymbol{\tau}^{\wedge} + b\boldsymbol{\sigma}^{\wedge}) = a\mathrm{Ad}_{\mathcal{X}}(\boldsymbol{\tau}^{\wedge}) \\
&\qquad\qquad\qquad\qquad\qquad + b\mathrm{Ad}_{\mathcal{X}}(\boldsymbol{\sigma}^{\wedge}) \\
\text{Homomorphism}: &\quad \mathrm{Ad}_{\mathcal{X}}(\mathrm{Ad}_{\mathcal{Y}}(\boldsymbol{\tau}^{\wedge})) = \mathrm{Ad}_{\mathcal{X}\mathcal{Y}}(\boldsymbol{\tau}^{\wedge}) .
\end{aligned}$$

Adjoint matrix

上一小节定义的adjoint描述了**lie algebra**形式的tangent space的映射关系，在实际应用中，我们通常使用**Cartesian**形式的tangent space，因此需要定义一个变换来描述**Cartesian**形式的tangent space，或者称为**tangent vector之间的映射关系**

2) The adjoint matrix: Since $\mathrm{Ad}_{\mathcal{X}}()$ is linear, we can find an equivalent matrix operator $\mathbf{Ad}_{\mathcal{X}}$ that maps the Cartesian tangent vectors ${}^{\mathcal{E}}\boldsymbol{\tau} \cong {}^{\mathcal{E}}\boldsymbol{\tau}^{\wedge}$ and ${}^{\mathcal{X}}\boldsymbol{\tau} \cong {}^{\mathcal{X}}\boldsymbol{\tau}^{\wedge}$,

$$\mathbf{Ad}_{\mathcal{X}} : \mathbb{R}^m \to \mathbb{R}^m; \quad {}^{\mathcal{X}}\boldsymbol{\tau} \mapsto {}^{\mathcal{E}}\boldsymbol{\tau} = \mathbf{Ad}_{\mathcal{X}}\,{}^{\mathcal{X}}\boldsymbol{\tau}, \tag{30}$$

which we call the *adjoint matrix*. This can be computed by applying $^{\vee}$ to (29), thus writing

$$\mathbf{Ad}_{\mathcal{X}}\,\boldsymbol{\tau} = (\mathcal{X}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1})^{\vee}, \tag{31}$$

上图中式(30)为adjoint matrix的定义式，而式(31)则是通常计算adjoint matrix的公式，通过**将式(31)的右侧展开后构造成式(A.8)的形式**（$\mathbf{Z}$ 为推导结果），得到 $\mathbf{Ad}_{\mathcal{X}}$ 的最终形式

$$\mathbf{Ad}_{\mathcal{X}}\boldsymbol{\tau} = \mathbf{Z}\boldsymbol{\tau} \tag{A.8}$$

Examples

SE(3) adjoint

Example 6: The adjoint matrix of $SE(3)$

The $SE(3)$ group of rigid body motions (see App. D) has group, Lie algebra and vector elements,

$$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}, \quad \boldsymbol{\tau}^{\wedge} = \begin{bmatrix} [\boldsymbol{\theta}]_{\times} & \boldsymbol{\rho} \\ \mathbf{0} & 0 \end{bmatrix}, \quad \boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix}.$$

The adjoint matrix is identified by developing (31) as

$$\begin{aligned}
\mathbf{Ad}_{\mathbf{M}}\boldsymbol{\tau} &= (\mathbf{M}\boldsymbol{\tau}^{\wedge}\mathbf{M}^{-1})^{\vee} = \cdots = \\
&= \left(\begin{bmatrix} \mathbf{R}[\boldsymbol{\theta}]_{\times}\mathbf{R}^{\top} & -\mathbf{R}[\boldsymbol{\theta}]_{\times}\mathbf{R}^{\top}\mathbf{t} + \mathbf{R}\boldsymbol{\rho} \\ \mathbf{0} & 0 \end{bmatrix}\right)^{\vee} \\
&= \left(\begin{bmatrix} [\mathbf{R}\boldsymbol{\theta}]_{\times} & [\mathbf{t}]_{\times}\mathbf{R}\boldsymbol{\theta} + \mathbf{R}\boldsymbol{\rho} \\ \mathbf{0} & 0 \end{bmatrix}\right)^{\vee} \\
&= \begin{bmatrix} [\mathbf{t}]_{\times}\mathbf{R}\boldsymbol{\theta} + \mathbf{R}\boldsymbol{\rho} \\ \mathbf{R}\boldsymbol{\theta} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & [\mathbf{t}]_{\times}\mathbf{R} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix}
\end{aligned}$$

where we used $[\mathbf{R}\boldsymbol{\theta}]_{\times} = \mathbf{R}[\boldsymbol{\theta}]_{\times}\mathbf{R}^{\top}$ and $[\mathbf{a}]_{\times}\mathbf{b} = -[\mathbf{b}]_{\times}\mathbf{a}$. So the adjoint matrix is

$$\mathbf{Ad}_{\mathbf{M}} = \begin{bmatrix} \mathbf{R} & [\mathbf{t}]_{\times}\mathbf{R} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \in \mathbb{R}^{6\times 6}.$$

SO(3) adjoint

本小节有以下3个重要公式，其中式(A.9)是SO(3)上的重要性质，也是推导式(A.10)和式(A.11)的前提

✅

$$\mathbf{R}\mathbf{v}^{\wedge}\mathbf{R}^{\top}=(\mathbf{R}\mathbf{v})^{\wedge} \tag{A.9}$$

✅

$$\mathbf{Ad}_{\mathbf{R}}=\mathbf{R} \tag{A.10}$$

✅

$$\mathbf{R}\exp(\mathbf{v}^{\wedge})\mathbf{R}^{\top}=\exp((\mathbf{R}\mathbf{p})^{\wedge}) \tag{A.11}$$

- 式(A.9)证明

$$\begin{aligned}
& (\mathbf{R}\mathbf{v})^{\wedge}=\mathbf{R}\mathbf{v}^{\wedge}\mathbf{R}^{T} \\
\Leftrightarrow\quad & (\mathbf{R}\mathbf{v})^{\wedge}\mathbf{R}=\mathbf{R}\mathbf{v}^{\wedge} \\
\Leftrightarrow\quad & \forall\mathbf{u}\in\mathbb{R}^{3},(\mathbf{R}\mathbf{v})^{\wedge}\mathbf{R}\mathbf{u}=\mathbf{R}\mathbf{v}^{\wedge}\mathbf{u} \\
\Leftrightarrow\quad & \forall\mathbf{u}\in\mathbb{R}^{3},(\mathbf{R}\mathbf{v})\times(\mathbf{R}\mathbf{u})=\mathbf{R}(\mathbf{v}\times\mathbf{u})
\end{aligned}$$

最后一式利用向量叉乘的旋转变换不变性（参看 Wikipedia）可证，即对于任意 $\mathbf{v},\mathbf{u}\in\mathbb{R}^{3}$ ，总有

$$(\mathbf{R}\mathbf{v})\times(\mathbf{R}\mathbf{u})=\mathbf{R}(\mathbf{v}\times\mathbf{u})$$

这一点，可以调用空间想象力，从三维几何的角度来理解：**v, u** 是任意两个三维向量，(**v**×**u**) 是一个和 **v, u** 都垂直、大小为 **|v||u|**sin(**u**,**v**) 的三维向量；将 **v, u, v**×**u** 三个向量都经过同一个旋转，它们的相对位姿和模长都不会改变，所以 (**Rv**) 和 (**Ru**) 的叉乘仍是 **R(v**×**u)**。

- 式(A.10)证明

联合式(A.8)和式(A.9)可得

$$\begin{aligned}
\mathbf{Ad}_{\mathbf{R}}\boldsymbol{\tau} &= (\mathbf{R}\boldsymbol{\tau}^{\wedge}\mathbf{R}^{-1})^{\vee} \\
&= (\mathbf{R}\boldsymbol{\tau}^{\wedge}\mathbf{R}^{\mathbf{T}})^{\vee} \\
&= ((\mathbf{R}\boldsymbol{\tau})^{\wedge})^{\vee} \\
&= \mathbf{R}\boldsymbol{\tau}
\end{aligned} \tag{A.12}$$

即

$$\mathbf{Ad}_{\mathbf{R}}=\mathbf{R} \tag{A.13}$$

- 式(A.11)证明

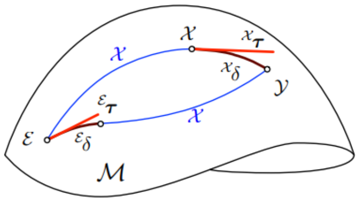
对式(A.9)两边同时取exp并代入式(20)得

$$\begin{aligned}
\exp((\mathbf{R}\mathbf{v})^{\wedge}) &= \exp(\mathbf{R}\mathbf{v}^{\wedge}\mathbf{R}^{\top}) \\
&= \mathbf{R}\exp(\mathbf{v}^{\wedge})\mathbf{R}^{\top}
\end{aligned} \tag{A.14}$$

Intuitive

一图胜千言，下图揭示了manifold上Adjoint的本质。后文中如果没有特殊强调，**adjoint matrix简称为adjoint**

> Vectors of the tangent space at $\mathcal{X}$ can be transformed to the tangent space at the **identity** $\boldsymbol{\varepsilon}$ through a linear transform. This transform is called the adjoint $\mathbf{Ad}_{\mathcal{X}}$, it follows: ${}^{\varepsilon}\boldsymbol{\tau}=\mathbf{Ad}_{\mathcal{X}}\,{}^{\mathcal{X}}\boldsymbol{\tau}$



$$\mathcal{Y} = {}^{\mathcal{E}}\delta \circ \mathcal{X} = \mathcal{X} \circ {}^{\mathcal{X}}\delta$$
$$\mathcal{Y} = {}^{\mathcal{E}}\boldsymbol{\tau} \oplus \mathcal{X} = \mathcal{X} \oplus {}^{\mathcal{X}}\boldsymbol{\tau}$$
$${}^{\mathcal{E}}\boldsymbol{\tau} = \mathrm{Ad}_{\mathcal{X}}\, {}^{\mathcal{X}}\boldsymbol{\tau}$$

Figure 7. Two paths, $\mathcal{X} \circ {}^{\mathcal{X}}\delta$ and ${}^{\mathcal{E}}\delta \circ \mathcal{X}$, join the origin $\mathcal{E}$ with the point $\mathcal{Y}$. They both compose the element $\mathcal{X}$ with increments or 'deltas' expressed either in the local frame, ${}^{\mathcal{X}}\delta$, or in the origin, ${}^{\mathcal{E}}\delta$. Due to non-commutativity, the elements ${}^{\mathcal{X}}\delta$ and ${}^{\mathcal{E}}\delta$ are not equal. Their associated tangent vectors ${}^{\mathcal{X}}\boldsymbol{\tau} = \mathrm{Log}({}^{\mathcal{X}}\delta)$ and ${}^{\mathcal{E}}\boldsymbol{\tau} = \mathrm{Log}({}^{\mathcal{E}}\delta)$ are therefore unequal too. They are related by the linear transform ${}^{\mathcal{E}}\boldsymbol{\tau} = \mathbf{Ad}_{\mathcal{X}}\, {}^{\mathcal{X}}\boldsymbol{\tau}$ where $\mathbf{Ad}_{\mathcal{X}}$ is the adjoint of $\mathcal{M}$ at $\mathcal{X}$.

Properties

式(32)根据式(A.4)和式(30)得证，式(33)和式(34)暂未能证明

$$\begin{aligned} \mathcal{X} \oplus \boldsymbol{\tau} &= (\mathbf{Ad}_{\mathcal{X}}\, \boldsymbol{\tau}) \oplus \mathcal{X} && (32)\\ \mathbf{Ad}_{\mathcal{X}^{-1}} &= \mathbf{Ad}_{\mathcal{X}}{}^{-1} && (33)\\ \mathbf{Ad}_{\mathcal{X}\mathcal{Y}} &= \mathbf{Ad}_{\mathcal{X}} \mathbf{Ad}_{\mathcal{Y}}\ . && (34) \end{aligned}$$

Derivatives on Lie groups

Jacobian

- 如果使用right operator，计算过程如下，结果称为 f 的right jacobian

$$\frac{{}^{\mathcal{X}}Df(\mathcal{X})}{D\mathcal{X}} \triangleq \lim_{\boldsymbol{\tau}\to 0} \frac{f(\mathcal{X} \oplus \boldsymbol{\tau}) \ominus f(\mathcal{X})}{\boldsymbol{\tau}} \qquad \in \mathbb{R}^{n\times m} \qquad (41a)$$

which develops as,

$$\begin{aligned} &= \lim_{\boldsymbol{\tau}\to 0} \frac{\mathrm{Log}\left(f(\mathcal{X})^{-1} \circ f(\mathcal{X} \circ \mathrm{Exp}(\boldsymbol{\tau}))\right)}{\boldsymbol{\tau}} && (41b)\\ &= \left.\frac{\partial \mathrm{Log}\left(f(\mathcal{X})^{-1} \circ f(\mathcal{X} \circ \mathrm{Exp}(\boldsymbol{\tau}))\right)}{\partial \boldsymbol{\tau}}\right|_{\boldsymbol{\tau}=0}. && (41c) \end{aligned}$$

当 $\boldsymbol{\tau}$ 是小量时有

$$f(\mathcal{X} \oplus {}^{\mathcal{X}}\boldsymbol{\tau}) \xrightarrow[{}^{\mathcal{X}}\boldsymbol{\tau}\to 0]{} f(\mathcal{X}) \oplus \frac{{}^{\mathcal{X}}Df(\mathcal{X})}{D\mathcal{X}}\, {}^{\mathcal{X}}\boldsymbol{\tau} \qquad \in \mathcal{N}\ . \qquad (43)$$

- 如果使用left operator，计算过程，结果称为 f 的left jacobian

$$\begin{aligned} \frac{{}^{\mathcal{E}}Df(\mathcal{X})}{D\mathcal{X}} &\triangleq \lim_{\boldsymbol{\tau}\to 0} \frac{f(\boldsymbol{\tau} \oplus \mathcal{X}) \ominus f(\mathcal{X})}{\boldsymbol{\tau}} \qquad \in \mathbb{R}^{n\times m} && (44)\\ &= \lim_{\boldsymbol{\tau}\to 0} \frac{\mathrm{Log}(f(\mathrm{Exp}(\boldsymbol{\tau}) \circ \mathcal{X}) \circ f(\mathcal{X})^{-1})}{\boldsymbol{\tau}}\\ &= \left.\frac{\partial \mathrm{Log}\left(f(\mathrm{Exp}(\boldsymbol{\tau}) \circ \mathcal{X}) \circ f(\mathcal{X})^{-1}\right)}{\partial \boldsymbol{\tau}}\right|_{\boldsymbol{\tau}=0}, \end{aligned}$$

当 $\boldsymbol{\tau}$ 是小量时有

$$f({}^{\mathcal{E}}\boldsymbol{\tau} \oplus \mathcal{X}) \xrightarrow[{}^{\mathcal{E}}\boldsymbol{\tau}\to 0]{} \frac{{}^{\mathcal{E}}Df(\mathcal{X})}{D\mathcal{X}}\, {}^{\mathcal{E}}\boldsymbol{\tau} \oplus f(\mathcal{X}) \qquad \in \mathcal{N}\ . \qquad (45)$$

- Jacobian的意义

Jacobian是 $f(\mathcal{X})$ **对** $\mathcal{X}$ **的导数**，通过 $\oplus$ 和 $\ominus$ 操作（本质是**exponential map**）,我们把极小量的变化表达在tangent space这个**vector space**中，jacobian描述的就是这两个tangent space之间的映射关系，right jacobian描述的是 $T_{\mathcal{X}}M$ **到** $T_{f(\mathcal{X})}N$ **的映射**，right jacobian第 i 列 j_i 就是 $T_{f(\mathcal{X})}N$ 对 $T_{\mathcal{X}}M$ 的第 i 维的导数，left jacobian描述的是 $T_{\varepsilon}M$ **到** $T_{\varepsilon}N$ **的映射**，left jacobian第 i 列 j_i 就是 $T_{\varepsilon}N$ 对 $T_{\varepsilon}M$ 的第 i 维的导数

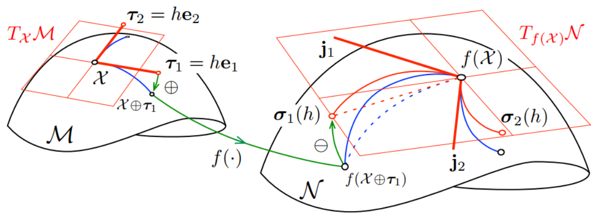


Figure 8. Right Jacobian of a function $f : \mathcal{M} \to \mathcal{N}$. The perturbation vectors in the canonical directions, $\boldsymbol{\tau}_i = h\mathbf{e}_i \in T_{\mathcal{X}}\mathcal{M}$, are propagated to perturbation vectors $\boldsymbol{\sigma}_i \in T_{f(\mathcal{X})}\mathcal{N}$ through the processes of plus, apply $f()$, and minus (green arrows), obtaining $\boldsymbol{\sigma}_i(h) = f(\mathcal{X}\oplus h\mathbf{e}_i)\ominus f(\mathcal{X})$. For varying values of h, notice that in $\mathcal{M}$ the perturbations $\boldsymbol{\tau}_i(h) = h\mathbf{e}_i$ (thick red) produce paths in $\mathcal{M}$ (blue) along the geodesic (recall Fig. 1). Notice also that in $\mathcal{N}$, due to the non-linearity of $f(\cdot)$, the image paths (solid blue) are generally not in the geodesic (dashed blue). These image paths are lifted onto the tangent space $T_{f(\mathcal{X})}\mathcal{N}$, producing smooth curved paths (thin solid red). The column vectors $\mathbf{j}_i$ of $\mathbf{J}$ (thick red) are the derivatives of the lifted paths evaluated at $f(\mathcal{X})$, *i.e.*, $\mathbf{j}_i = \lim_{h\to 0}\boldsymbol{\sigma}_i(h)/h$. Each $h\mathbf{e}_i \in T_{\mathcal{X}}\mathcal{M}$ gives place to a $\mathbf{j}_i \in T_{f(\mathcal{X})}\mathcal{N}$, and thus the resulting Jacobian matrix $\mathbf{J} = [\mathbf{j}_1 \cdots \mathbf{j}_m] \in \mathbb{R}^{n\times m}$ linearly maps vectors from $T_{\mathcal{X}}\mathcal{M} \cong \mathbb{R}^m$ to $T_{f(\mathcal{X})}\mathcal{N} \cong \mathbb{R}^n$.

- Left jacobian与right jacobian的转换关系

$$\frac{{}^{\mathcal{E}}Df(\mathcal{X})}{D\mathcal{X}}\mathbf{Ad}_{\mathcal{X}} = \mathbf{Ad}_{f(\mathcal{X})}\frac{{}^{\mathcal{X}}Df(\mathcal{X})}{D\mathcal{X}} \,. \tag{46}$$

式(46)也可写作

$$\frac{{}^{\mathcal{E}}Df(\mathcal{X})}{D\mathcal{X}} = \mathbf{Ad}_{f(\mathcal{X})}\frac{{}^{\mathcal{X}}Df(\mathcal{X})}{D\mathcal{X}}\mathbf{Ad}_{\mathcal{X}}{}^{-1} \,. \tag{57}$$

We use the notations $\mathbf{J}_{\mathcal{X}}^{f(\mathcal{X})} \triangleq \frac{Df(\mathcal{X})}{D\mathcal{X}}$ and $\mathbf{J}_{\mathcal{X}}^{\mathcal{Y}} \triangleq \frac{D\mathcal{Y}}{D\mathcal{X}}$. We notice also that $\mathbf{Ad}_{\mathcal{X}}{}^{-1}$ should rather be implemented by $\mathbf{Ad}_{\mathcal{X}^{-1}}$ —see (33, 34) and the comment below them.

这些映射关系的直观表示如下所示

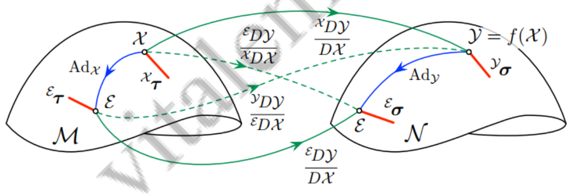


Figure 9. Linear maps between all tangent spaces involved in a function $\mathcal{Y} = f(\mathcal{X})$, from $\mathcal{M}$ to $\mathcal{N}$. The linear maps ${}^{\mathcal{E}}\boldsymbol{\tau} = \mathbf{Ad}_{\mathcal{X}}\,{}^{\mathcal{X}}\boldsymbol{\tau}$, ${}^{\mathcal{E}}\boldsymbol{\sigma} = \mathbf{Ad}_{\mathcal{Y}}\,{}^{\mathcal{Y}}\boldsymbol{\sigma}$, ${}^{\mathcal{E}}\boldsymbol{\sigma} = \frac{{}^{\mathcal{E}}D\mathcal{Y}}{D\mathcal{X}}\,{}^{\mathcal{E}}\boldsymbol{\tau}$, and ${}^{\mathcal{Y}}\boldsymbol{\sigma} = \frac{{}^{\mathcal{X}}D\mathcal{Y}}{D\mathcal{X}}\,{}^{\mathcal{X}}\boldsymbol{\tau}$, form a loop (solid) that leads to (46). The crossed Jacobians (dashed) form more mapping loops leading to (47,48).

Chain rule

For $\mathcal{Y} = f(\mathcal{X})$ and $\mathcal{Z} = g(\mathcal{Y})$ we have $\mathcal{Z} = g(f(\mathcal{X}))$. The chain rule simply states,

$$\frac{D\mathcal{Z}}{D\mathcal{X}} = \frac{D\mathcal{Z}}{D\mathcal{Y}}\frac{D\mathcal{Y}}{D\mathcal{X}} \quad \text{or} \quad \mathbf{J}_{\mathcal{X}}^{\mathcal{Z}} = \mathbf{J}_{\mathcal{Y}}^{\mathcal{Z}}\mathbf{J}_{\mathcal{X}}^{\mathcal{Y}} \,. \tag{58}$$

We prove it here for the right Jacobian using (43) thrice,

$$\begin{aligned} g(f(\mathcal{X}))\oplus\mathbf{J}_{\mathcal{X}}^{\mathcal{Z}}\boldsymbol{\tau} \leftarrow g(f(\mathcal{X}\oplus\boldsymbol{\tau})) &\to g(f(\mathcal{X})\ \oplus\mathbf{J}_{\mathcal{X}}^{\mathcal{Y}}\boldsymbol{\tau}) \\ &\to g(f(\mathcal{X}))\oplus\mathbf{J}_{\mathcal{Y}}^{\mathcal{Z}}\mathbf{J}_{\mathcal{X}}^{\mathcal{Y}}\boldsymbol{\tau} \end{aligned}$$

with the arrows indicating limit as $\boldsymbol{\tau} \to 0$, and so $\mathbf{J}_{\mathcal{X}}^{\mathcal{Z}} = \mathbf{J}_{\mathcal{Y}}^{\mathcal{Z}}\mathbf{J}_{\mathcal{X}}^{\mathcal{Y}}$. The proof for the left and crossed Jacobians is akin,

Elementary Jacobian

下文中 ${}^{R}\mathbf{J}$ 表示right jacobian， ${}^{L}\mathbf{J}$ 表示left jacobian，如无特别提示， $\mathbf{J}$ 表示 ${}^{R}\mathbf{J}$ ，inverse和composition的jacobian的一般形式如下

	$\mathbf{J}_{\mathcal{X}}^{\mathcal{X}^{-1}}$	$\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\mathcal{Y}}$	$\mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\mathcal{Y}}$
${}^{R}\mathbf{J}$	$-\mathbf{Ad}_{\mathcal{X}}$	$\mathbf{Ad}_{\mathcal{Y}}^{-1}$	$\mathbf{I}$
${}^{L}\mathbf{J}$	$-\mathbf{Ad}_{\mathcal{X}}^{-1}$	$\mathbf{I}$	$\mathbf{Ad}_{\mathcal{X}}$

Inverse

令 $f(\mathcal{X})=\mathcal{X}^{-1}$

Right Jacobian

$$
\begin{aligned}
{}^{R}\mathbf{J}_{\mathcal{X}}^{\mathcal{X}^{-1}} = \frac{Df(\mathcal{X})}{D\mathcal{X}} &= \lim_{\boldsymbol{\tau}\to 0}\frac{f(\mathcal{X}\oplus\boldsymbol{\tau})\ominus f(\mathcal{X})}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0}\frac{\mathrm{Log}((\mathcal{X}^{-1})^{-1}(\mathcal{X}\mathrm{Exp}(\boldsymbol{\tau}))^{-1})}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0}\frac{\mathrm{Log}(\mathcal{X}\mathrm{Exp}(\boldsymbol{\tau})^{-1}\mathcal{X}^{-1})}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0}\frac{\log(\exp(-\mathcal{X}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1}))^{\vee}}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0}\frac{(-\mathcal{X}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1})^{\vee}}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0}\frac{-\mathbf{Ad}_{\mathcal{X}}\boldsymbol{\tau}}{\boldsymbol{\tau}} \\
&= -\mathbf{Ad}_{\mathcal{X}}
\end{aligned} \tag{A.15}
$$

Left Jacobian

$$
\begin{aligned}
{}^{L}\mathbf{J}_{\mathcal{X}}^{\mathcal{X}^{-1}} = \frac{Df(\mathcal{X})}{D\mathcal{X}} &= \lim_{\boldsymbol{\tau}\to 0}\frac{f(\boldsymbol{\tau}\oplus\mathcal{X})\ominus f(\mathcal{X})}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0}\frac{\mathrm{Log}((\mathrm{Exp}(\boldsymbol{\tau})\mathcal{X})^{-1}(\mathcal{X}^{-1})^{-1}))}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0}\frac{\mathrm{Log}(\mathcal{X}^{-1}\mathrm{Exp}(\boldsymbol{\tau})^{-1}\mathcal{X})}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0}\frac{\log(\exp(-\mathcal{X}^{-1}\boldsymbol{\tau}^{\wedge}\mathcal{X}))^{\vee}}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0}\frac{(-\mathcal{X}^{-1}\boldsymbol{\tau}^{\wedge}\mathcal{X})^{\vee}}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0}\frac{-\mathbf{Ad}_{\mathcal{X}^{-1}}\boldsymbol{\tau}}{\boldsymbol{\tau}} \\
&= -\mathbf{Ad}_{\mathcal{X}}^{-1}
\end{aligned} \tag{A.16}
$$

将式(A.15)代入式(57)验证

$$
\begin{aligned}
{}^{L}\mathbf{J}_{\mathcal{X}}^{\mathcal{X}^{-1}} &= \mathbf{Ad}_{\mathcal{X}^{-1}}\ {}^{R}\mathbf{J}_{\mathcal{X}}^{\mathcal{X}^{-1}}\,\mathbf{Ad}_{\mathcal{X}}^{-1} \\
&= \mathbf{Ad}_{\mathcal{X}}^{-1}\ {}^{R}\mathbf{J}_{\mathcal{X}}^{\mathcal{X}^{-1}}\,\mathbf{Ad}_{\mathcal{X}}^{-1} \\
&= \mathbf{Ad}_{\mathcal{X}}^{-1}(-\mathbf{Ad}_{\mathcal{X}})\mathbf{Ad}_{\mathcal{X}}^{-1} \\
&= -\mathbf{Ad}_{\mathcal{X}}^{-1}
\end{aligned} \tag{A.17}
$$

Right Composition

令 $f(\mathcal{X})=\mathcal{X}\mathcal{Y}$

Right Jacobian

$$
\begin{aligned}
{}^{R}\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\mathcal{Y}} = \frac{Df(\mathcal{X})}{D\mathcal{X}} &= \lim_{\boldsymbol{\tau}\to 0}\frac{f(\mathcal{X}\oplus\boldsymbol{\tau})\ominus f(\mathcal{X})}{\tau} \\
&= \lim_{\boldsymbol{\tau}\to 0}\frac{\mathrm{Log}((\mathcal{X}\mathcal{Y})^{-1}(\mathcal{X}\mathrm{Exp}(\boldsymbol{\tau})\mathcal{Y}))}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0}\frac{\mathrm{Log}(\mathcal{Y}^{-1}\mathrm{Exp}(\boldsymbol{\tau})\mathcal{Y})}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0}\frac{\log(\exp(\mathcal{Y}^{-1}\boldsymbol{\tau}^{\wedge}\mathcal{X}))^{\vee}}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0}\frac{(\mathcal{Y}^{-1}\boldsymbol{\tau}^{\wedge}\mathcal{Y})^{\vee}}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0}\frac{\mathbf{Ad}_{\mathcal{Y}^{-1}}\boldsymbol{\tau}}{\boldsymbol{\tau}} \\
&= \mathbf{Ad}_{\mathcal{Y}^{-1}} \\
&= \mathbf{Ad}_{\mathcal{Y}}^{-1}
\end{aligned} \tag{A.18}
$$

Left Jacobian

$$
\begin{aligned}
{}^{L}\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\mathcal{Y}} = \frac{Df(\mathcal{X})}{D\mathcal{X}} &= \lim_{\tau \to 0} \frac{f(\boldsymbol{\tau} \oplus \mathcal{X}) \ominus f(\mathcal{X})}{\boldsymbol{\tau}} \\
&= \lim_{\tau \to 0} \frac{\mathrm{Log}((\mathrm{Exp}(\boldsymbol{\tau})\mathcal{X}\mathcal{Y})(\mathcal{X}\mathcal{Y})^{-1})}{\boldsymbol{\tau}} \\
&= \lim_{\tau \to 0} \frac{\mathrm{Log}((\mathrm{Exp}(\boldsymbol{\tau}))}{\boldsymbol{\tau}} \\
&= \mathbf{I}
\end{aligned} \tag{A.19}
$$

将式(A.18)代入式(57)验证

$$
\begin{aligned}
{}^{L}\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\mathcal{Y}} &= \mathbf{Ad}_{\mathcal{X}\mathcal{Y}}\, {}^{R}\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\mathcal{Y}}\, \mathbf{Ad}_{\mathcal{X}}^{-1} \\
&= \mathbf{Ad}_{\mathcal{X}} \mathbf{Ad}_{\mathcal{Y}} \mathbf{Ad}_{\mathcal{Y}}^{-1} \mathbf{Ad}_{\mathcal{X}}^{-1} \\
&= \mathbf{I}
\end{aligned} \tag{A.20}
$$

Left Composition

令 $f(\mathcal{Y}) = \mathcal{X}\mathcal{Y}$

Right Jacobian

$$
\begin{aligned}
{}^{R}\mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\mathcal{Y}} = \frac{Df(\mathcal{Y})}{D\mathcal{Y}} &= \lim_{\tau \to 0} \frac{f(\mathcal{Y} \oplus \boldsymbol{\tau}) \ominus f(\mathcal{Y})}{\boldsymbol{\tau}} \\
&= \lim_{\tau \to 0} \frac{\mathrm{Log}((\mathcal{X}\mathcal{Y})^{-1}(\mathcal{X}\mathcal{Y}\mathrm{Exp}(\boldsymbol{\tau})))}{\boldsymbol{\tau}} \\
&= \lim_{\tau \to 0} \frac{\mathrm{Log}(\mathrm{Exp}(\boldsymbol{\tau}))}{\boldsymbol{\tau}} \\
&= \mathbf{I}
\end{aligned} \tag{A.21}
$$

Left Jacobian

$$
\begin{aligned}
{}^{L}\mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\mathcal{Y}} = \frac{Df(\mathcal{Y})}{D\mathcal{Y}} &= \lim_{\tau \to 0} \frac{f(\boldsymbol{\tau} \oplus \mathcal{Y}) \ominus f(\mathcal{Y})}{\boldsymbol{\tau}} \\
&= \lim_{\tau \to 0} \frac{\mathrm{Log}((\mathcal{X}\mathrm{Exp}(\boldsymbol{\tau})\mathcal{Y})(\mathcal{X}\mathcal{Y})^{-1})}{\boldsymbol{\tau}} \\
&= \lim_{\tau \to 0} \frac{\mathrm{Log}(\mathcal{X}\mathrm{Exp}(\boldsymbol{\tau})\mathcal{X}^{-1})}{\boldsymbol{\tau}} \\
&= \mathbf{Ad}_{\mathcal{X}}
\end{aligned} \tag{A.22}
$$

将式(A.21)代入式(57)验证

$$
\begin{aligned}
{}^{L}\mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\mathcal{Y}} &= \mathbf{Ad}_{\mathcal{X}\mathcal{Y}}\, {}^{R}\mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\mathcal{Y}}\, \mathbf{Ad}_{\mathcal{Y}}^{-1} \\
&= \mathbf{Ad}_{\mathcal{X}} \mathbf{Ad}_{\mathcal{Y}}\, \mathbf{I}\, \mathbf{Ad}_{\mathcal{Y}}^{-1} \\
&= \mathbf{Ad}_{\mathcal{X}}
\end{aligned} \tag{A.23}
$$

Jacobians of Manifold

Manifold的jacobian是**group对vector的导数**，在计算过程中转换为Manifold的**tangent space对vector的导数**，根据选取的tangent space在 $\mathcal{X}$ 处还是在identity处，分为right jacobian和left jacobian

	定义	推导	BCH近似
Right Jacobian	$\mathbf{J}_r(\boldsymbol{\tau}) = \frac{{}^{\tau}D\mathrm{Exp}(\boldsymbol{\tau})}{D\boldsymbol{\tau}}$	$\begin{aligned} \mathrm{Exp}(\boldsymbol{\tau}+\delta\boldsymbol{\tau}) &= \mathrm{Exp}(\boldsymbol{\tau}) \oplus \frac{{}^{\tau}D\mathrm{Exp}(\boldsymbol{\tau})}{D\boldsymbol{\tau}}\delta\boldsymbol{\tau} \\ &= \mathrm{Exp}(\boldsymbol{\tau})\mathrm{Exp}(\frac{{}^{\tau}D\mathrm{Exp}(\boldsymbol{\tau})}{D\boldsymbol{\tau}}\delta\boldsymbol{\tau}) \\ &= \mathrm{Exp}(\boldsymbol{\tau})\mathrm{Exp}(\mathbf{J}_r\delta\boldsymbol{\tau}) \end{aligned} \quad (A.24)$	$\begin{aligned} \mathrm{Exp}(\boldsymbol{\tau}+\delta\boldsymbol{\tau}) &\approx \mathrm{Exp}(\boldsymbol{\tau})\,\mathrm{Exp}(\mathbf{J}_r(\boldsymbol{\tau})\delta\boldsymbol{\tau}) && (68) \\ \mathrm{Exp}(\boldsymbol{\tau})\,\mathrm{Exp}(\delta\boldsymbol{\tau}) &\approx \mathrm{Exp}(\boldsymbol{\tau}+\mathbf{J}_r^{-1}(\boldsymbol{\tau})\,\delta\boldsymbol{\tau}) && (69) \\ \mathrm{Log}(\mathrm{Exp}(\boldsymbol{\tau})\,\mathrm{Exp}(\delta\boldsymbol{\tau})) &\approx \boldsymbol{\tau}+\mathbf{J}_r^{-1}(\boldsymbol{\tau})\,\delta\boldsymbol{\tau}\ . && (70) \end{aligned}$
Left Jacobian	$\mathbf{J}_l(\boldsymbol{\tau}) = \frac{{}^{\varepsilon}D\mathrm{Exp}(\boldsymbol{\tau})}{D\boldsymbol{\tau}}$	$\begin{aligned} \mathrm{Exp}(\boldsymbol{\tau}+\delta\boldsymbol{\tau}) &= \frac{{}^{\varepsilon}D\mathrm{Exp}(\boldsymbol{\tau})}{D\boldsymbol{\tau}}\delta\boldsymbol{\tau} \oplus \mathrm{Exp}(\boldsymbol{\tau}) \\ &= \mathrm{Exp}(\frac{{}^{\varepsilon}D\mathrm{Exp}(\boldsymbol{\tau})}{D\boldsymbol{\tau}}\delta\boldsymbol{\tau})\mathrm{Exp}(\boldsymbol{\tau}) \\ &= \mathrm{Exp}(\mathbf{J}_l\delta\boldsymbol{\tau})\mathrm{Exp}(\boldsymbol{\tau}) \end{aligned} \quad (A.25)$	$\begin{aligned} \mathrm{Exp}(\boldsymbol{\tau}+\delta\boldsymbol{\tau}) &\approx \mathrm{Exp}(\mathbf{J}_l(\boldsymbol{\tau})\delta\boldsymbol{\tau})\,\mathrm{Exp}(\boldsymbol{\tau}) && (72) \\ \mathrm{Exp}(\delta\boldsymbol{\tau})\,\mathrm{Exp}(\boldsymbol{\tau}) &\approx \mathrm{Exp}(\boldsymbol{\tau}+\mathbf{J}_l^{-1}(\boldsymbol{\tau})\,\delta\boldsymbol{\tau}) && (73) \\ \mathrm{Log}(\mathrm{Exp}(\delta\boldsymbol{\tau})\,\mathrm{Exp}(\boldsymbol{\tau})) &\approx \boldsymbol{\tau}+\mathbf{J}_l^{-1}(\boldsymbol{\tau})\,\delta\boldsymbol{\tau}\ . && (74) \end{aligned}$

在Joan Sola的另一本著作Quaternion kinematics for the error-state Kalman filter的p42中，直观地描述了 $\mathbf{J}_r$ 的意义，即将参数空间 $\boldsymbol{\theta}$ 附近的扰动 $\delta\boldsymbol{\theta}$ 映射到 $\mathbf{R}$ 处的tangent space中 ，同理可知 $\mathbf{J}_l$ 是将参数空间中 $\boldsymbol{\theta}$ 附近的扰动 $\delta\boldsymbol{\theta}$ 映射到manifold的identity处的tangent space中

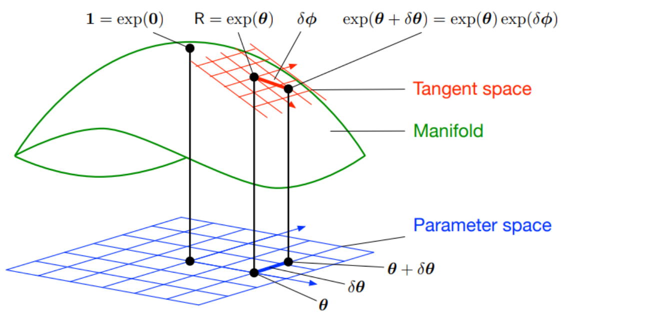


Figure 12: The right Jacobian $\mathbf{J}_r = \partial\delta\boldsymbol{\phi}/\partial\delta\boldsymbol{\theta}$ maps variations $\delta\boldsymbol{\theta}$ around the parameter $\boldsymbol{\theta}$ into variations $\delta\boldsymbol{\phi}$ on the vector space tangent to the manifold at the point $\operatorname{Exp}\boldsymbol{\theta}$.

$\mathbf{J}_r$ 和 $\mathbf{J}_l$ 的关系由两个等式来描述：

we can relate left- and right- Jacobians with the adjoint,

$$\mathbf{Ad}_{\operatorname{Exp}(\boldsymbol{\tau})} = \mathbf{J}_l(\boldsymbol{\tau})\,\mathbf{J}_r{}^{-1}(\boldsymbol{\tau})\ . \tag{75}$$

Also, the chain rule allows us to relate $\mathbf{J}_r$ and $\mathbf{J}_l$,

$$\begin{aligned}\mathbf{J}_r(-\boldsymbol{\tau}) &\triangleq \mathbf{J}_{-\boldsymbol{\tau}}^{\operatorname{Exp}(-\boldsymbol{\tau})} = \mathbf{J}_{\boldsymbol{\tau}}^{\operatorname{Exp}(-\boldsymbol{\tau})}\mathbf{J}_{-\boldsymbol{\tau}}^{\boldsymbol{\tau}} = \mathbf{J}_{\boldsymbol{\tau}}^{\operatorname{Exp}(\boldsymbol{\tau})^{-1}}(-\mathbf{I}) \\ &= -\mathbf{J}_{\operatorname{Exp}(\boldsymbol{\tau})}^{\operatorname{Exp}(\boldsymbol{\tau})^{-1}}\mathbf{J}_{\boldsymbol{\tau}}^{\operatorname{Exp}(\boldsymbol{\tau})} = \mathbf{Ad}_{\operatorname{Exp}(\boldsymbol{\tau})}\mathbf{J}_r(\boldsymbol{\tau}) \\ &= \mathbf{J}_l(\boldsymbol{\tau})\ .\end{aligned} \tag{76}$$

- 式(75)的证明如下

根据BCH近似有

$$\operatorname{Exp}(\boldsymbol{\tau}+\delta\boldsymbol{\tau}) = \operatorname{Exp}(\mathbf{J}_l\delta\boldsymbol{\tau})\operatorname{Exp}(\boldsymbol{\tau}) = \operatorname{Exp}(\boldsymbol{\tau})\operatorname{Exp}(\mathbf{J}_r\delta\boldsymbol{\tau}) \tag{A.26}$$

对于式(A.26)的后两项

$$\begin{aligned}\operatorname{Exp}(\mathbf{J}_l\delta\boldsymbol{\tau})\operatorname{Exp}(\boldsymbol{\tau}) &= \operatorname{Exp}(\boldsymbol{\tau})\operatorname{Exp}(\mathbf{J}_r\delta\boldsymbol{\tau}) \\ \operatorname{Exp}(\mathbf{J}_l\delta\boldsymbol{\tau}) &= \operatorname{Exp}(\boldsymbol{\tau})\operatorname{Exp}(\mathbf{J}_r\delta\boldsymbol{\tau})\operatorname{Exp}(\boldsymbol{\tau})^{-1} \\ \exp((\mathbf{J}_l\delta\boldsymbol{\tau})^\wedge) &= \exp(\operatorname{Exp}(\boldsymbol{\tau})(\mathbf{J}_r\delta\boldsymbol{\tau})^\wedge\operatorname{Exp}(\boldsymbol{\tau})^{-1}) \\ (\mathbf{J}_l\delta\boldsymbol{\tau})^\wedge &= \operatorname{Exp}(\boldsymbol{\tau})(\mathbf{J}_r\delta\boldsymbol{\tau})^\wedge\operatorname{Exp}(\boldsymbol{\tau})^{-1} \\ \mathbf{J}_l\delta\boldsymbol{\tau} &= (\operatorname{Exp}(\boldsymbol{\tau})(\mathbf{J}_r\delta\boldsymbol{\tau})^\wedge\operatorname{Exp}(\boldsymbol{\tau})^{-1})^\vee \\ \mathbf{J}_l\delta\boldsymbol{\tau} &= \mathbf{Ad}_{\operatorname{Exp}(\boldsymbol{\tau})}\mathbf{J}_r\delta\boldsymbol{\tau} \\ \mathbf{J}_l &= \mathbf{Ad}_{\operatorname{Exp}(\boldsymbol{\tau})}\mathbf{J}_r\end{aligned} \tag{A.27}$$

Group action

Group action的jacobian并没有统一的表达形式，下文以SO(3)和SE(3)为例进行推导

4) Group action: For $\mathcal{X} \in \mathcal{M}$ and $v \in \mathcal{V}$, we define with (41a)

$$\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\cdot v} \triangleq \frac{{}^{\mathcal{X}}D\mathcal{X}\cdot v}{D\mathcal{X}} \tag{77}$$

$$\mathbf{J}_{v}^{\mathcal{X}\cdot v} \triangleq \frac{{}^{v}D\mathcal{X}\cdot v}{Dv}\ . \tag{78}$$

Since group actions depend on the set $\mathcal{V}$, these expressions cannot be generalized. See the appendices for reference.

SO(3)

- Right jacobian

$$
\begin{aligned}
{}^R\mathbf{J}_{\mathcal{X}}^{\mathcal{X}v} &= \lim_{\tau\to 0}\frac{\mathcal{X}\mathrm{Exp}(\boldsymbol{\tau})\mathbf{v}-\mathcal{X}\mathbf{v}}{\tau} \\
&= \lim_{\tau\to 0}\frac{\mathcal{X}(\mathbf{I}+\boldsymbol{\tau}^{\wedge})\mathbf{v}-\mathcal{X}\mathbf{v}}{\boldsymbol{\tau}} \\
&= \lim_{\tau\to 0}\frac{\mathcal{X}\boldsymbol{\tau}^{\wedge}\mathbf{v}}{\boldsymbol{\tau}} \\
&= \lim_{\tau\to 0}\frac{-\mathcal{X}\mathbf{v}^{\wedge}\boldsymbol{\tau}}{\boldsymbol{\tau}} \\
&= -\mathcal{X}\mathbf{v}^{\wedge}
\end{aligned} \tag{A.28}
$$

- Left Jacobian

$$
\begin{aligned}
{}^L\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\mathbf{v}} &= \lim_{\tau\to 0}\frac{\mathrm{Exp}(\boldsymbol{\tau})\mathcal{X}\mathbf{v}-\mathcal{X}\mathbf{v}}{\boldsymbol{\tau}} \\
&= \lim_{\tau\to 0}\frac{(\mathbf{I}+\boldsymbol{\tau}^{\wedge})\mathcal{X}\mathbf{v}-\mathcal{X}\mathbf{v}}{\boldsymbol{\tau}} \\
&= \lim_{\tau\to 0}\frac{\boldsymbol{\tau}^{\wedge}\mathcal{X}\mathbf{v}}{\boldsymbol{\tau}} \\
&= \lim_{\tau\to 0}\frac{-(\mathcal{X}\mathbf{v})^{\wedge}\boldsymbol{\tau}}{\boldsymbol{\tau}} \\
&= -(\mathcal{X}\mathbf{v})^{\wedge}
\end{aligned} \tag{A.29}
$$

SE(3)

令

$$
\mathcal{X}=\begin{bmatrix}\mathbf{R} & \mathbf{t}\\ 0 & 1\end{bmatrix} \in SE(3)\subset\mathbb{R}^{4\times 4} \tag{A.30}
$$

对应的lie algebra

$$
\boldsymbol{\tau}^{\wedge}=\begin{bmatrix}\boldsymbol{\theta}^{\wedge} & \boldsymbol{\rho}\\ 0 & 1\end{bmatrix}\subset se(3), \boldsymbol{\tau}=\begin{bmatrix}\boldsymbol{\rho}\\ \boldsymbol{\theta}\end{bmatrix}\in\mathbb{R}^6 \tag{A.31}
$$

- Right jacobian

$$
\begin{aligned}
{}^R\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\mathbf{v}} &= \lim_{\tau\to 0}\frac{\mathcal{X}\mathrm{Exp}(\boldsymbol{\tau})\mathbf{v}-\mathcal{X}\mathbf{v}}{\boldsymbol{\tau}} \\
&= \lim_{\tau\to 0}\frac{\mathcal{X}(\mathbf{I}+\boldsymbol{\tau}^{\wedge})\mathbf{v}-\mathcal{X}\mathbf{v}}{\boldsymbol{\tau}} \\
&= \lim_{\tau\to 0}\frac{\mathcal{X}\boldsymbol{\tau}^{\wedge}\mathbf{v}}{\boldsymbol{\tau}} \\
&= \lim_{\tau\to 0}\frac{\begin{bmatrix}\mathbf{R} & \mathbf{t}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{\theta}^{\wedge} & \boldsymbol{\rho}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{v}\\ 1\end{bmatrix}}{\begin{bmatrix}\boldsymbol{\rho}\\ \boldsymbol{\theta}\end{bmatrix}} \\
&= \lim_{\tau\to 0}\frac{\begin{bmatrix}\mathbf{R}\boldsymbol{\theta}^{\wedge}\mathbf{v}+\mathbf{R}\boldsymbol{\rho}\\ 0\end{bmatrix}}{\begin{bmatrix}\boldsymbol{\rho}\\ \boldsymbol{\theta}\end{bmatrix}} \\
&= \begin{bmatrix}\mathbf{R} & -\mathbf{R}\mathbf{v}^{\wedge}\\ 0 & 0\end{bmatrix} \iff \begin{bmatrix}\mathbf{R} & -\mathbf{R}\mathbf{v}^{\wedge}\end{bmatrix}
\end{aligned} \tag{A.32}
$$

- Left Jacobian

$$
\begin{aligned}
{}^L\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\mathbf{v}} &= \lim_{\boldsymbol{\tau}\to 0} \frac{\mathrm{Exp}(\boldsymbol{\tau})\mathcal{X}\mathbf{v} - \mathcal{X}\mathbf{v}}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0} \frac{(\mathbf{I}+\boldsymbol{\tau}^\wedge)\mathcal{X}\mathbf{v} - \mathcal{X}\mathbf{v}}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0} \frac{\boldsymbol{\tau}^\wedge\mathcal{X}\mathbf{v}}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0} \frac{\begin{bmatrix}\boldsymbol{\theta}^\wedge & \boldsymbol{\rho} \\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{R} & \mathbf{t} \\ 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{v} \\ 1\end{bmatrix}}{\begin{bmatrix}\boldsymbol{\rho} \\ \boldsymbol{\theta}\end{bmatrix}} \\
&= \lim_{\boldsymbol{\tau}\to 0} \frac{\begin{bmatrix}\boldsymbol{\theta}^\wedge(\mathbf{R}\mathbf{v}+\mathbf{t})+\boldsymbol{\rho} \\ 0\end{bmatrix}}{\begin{bmatrix}\boldsymbol{\rho} \\ \boldsymbol{\theta}\end{bmatrix}} \\
&= \begin{bmatrix}\mathbf{I} & -(\mathbf{R}\mathbf{v}+\mathbf{t})^\wedge \\ 0 & 0\end{bmatrix} \iff \begin{bmatrix}\mathbf{I} & -(\mathbf{R}\mathbf{v}+\mathbf{t})^\wedge\end{bmatrix}
\end{aligned}
\tag{A.33}
$$

Useful, but deduced, Jacobian blocks

1) Log map: For $\boldsymbol{\tau} = \mathrm{Log}(\mathcal{X})$, and from (70),

$$
\mathbf{J}_{\mathcal{X}}^{\mathrm{Log}(\mathcal{X})} = \mathbf{J}_r^{-1}(\boldsymbol{\tau}) \ . \tag{79}
$$

2) Plus and minus: We have

$$
\begin{aligned}
\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\oplus\boldsymbol{\tau}} &= \mathbf{J}_{\mathcal{X}}^{\mathcal{X}\circ(\mathrm{Exp}(\boldsymbol{\tau}))} &&= \mathbf{Ad}_{\mathrm{Exp}(\boldsymbol{\tau})}{}^{-1} && (80)\\
\mathbf{J}_{\boldsymbol{\tau}}^{\mathcal{X}\oplus\boldsymbol{\tau}} &= \mathbf{J}_{\mathrm{Exp}(\boldsymbol{\tau})}^{\mathcal{X}\circ(\mathrm{Exp}(\boldsymbol{\tau}))}\mathbf{J}_{\boldsymbol{\tau}}^{\mathrm{Exp}(\boldsymbol{\tau})} &&= \mathbf{J}_r(\boldsymbol{\tau}) && (81)
\end{aligned}
$$

and given $\mathcal{Z} = \mathcal{X}^{-1}\circ\mathcal{Y}$ and $\boldsymbol{\tau} = \mathcal{Y}\ominus\mathcal{X} = \mathrm{Log}(\mathcal{Z})$,

$$
\begin{aligned}
\mathbf{J}_{\mathcal{X}}^{\mathcal{Y}\ominus\mathcal{X}} &= \mathbf{J}_{\mathcal{Z}}^{\mathrm{Log}(\mathcal{Z})}\mathbf{J}_{\mathcal{X}^{-1}}^{\mathcal{Z}}\mathbf{J}_{\mathcal{X}}^{\mathcal{X}^{-1}} &&= -\mathbf{J}_l^{-1}(\boldsymbol{\tau}) && (82)\\
\mathbf{J}_{\mathcal{Y}}^{\mathcal{Y}\ominus\mathcal{X}} &= \mathbf{J}_{\mathcal{Z}}^{\mathrm{Log}(\mathcal{Z})}\mathbf{J}_{\mathcal{Y}}^{\mathcal{Z}} &&= \mathbf{J}_r^{-1}(\boldsymbol{\tau}) \ . && (83)
\end{aligned}
$$

- 式(79)证明

令 $f(\mathcal{X}) = \mathrm{Log}\mathcal{X}$

$$
\begin{aligned}
\mathbf{J}_{\mathcal{X}}^{\mathrm{Log}\mathcal{X}} &= \lim_{\boldsymbol{\tau}\to\mathbf{0}} \frac{f(\mathcal{X}\oplus\boldsymbol{\tau})\ominus f(\mathcal{X})}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to\mathbf{0}} \frac{\mathrm{Log}(\mathcal{X}\mathrm{Exp}(\boldsymbol{\tau})) - \mathrm{Log}(\mathcal{X})}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to\mathbf{0}} \frac{\mathrm{Log}(\mathrm{Exp}(\mathrm{Log}(\mathcal{X})+\mathbf{J}_r^{-1}\boldsymbol{\tau})) - \mathrm{Log}(\mathcal{X})}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to\mathbf{0}} \frac{\mathrm{Log}(\mathcal{X})+\mathbf{J}_r^{-1}\boldsymbol{\tau} - \mathrm{Log}(\mathcal{X})}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to\mathbf{0}} \frac{\mathbf{J}_r^{-1}\boldsymbol{\tau}}{\boldsymbol{\tau}} \\
&= \mathbf{J}_r^{-1}
\end{aligned}
\tag{A.34}
$$

- 式(80)证明

令 $f(\mathcal{X}) = \mathcal{X}\oplus\boldsymbol{\tau} = \mathcal{X}\mathrm{Exp}(\boldsymbol{\tau})$

$$
\begin{aligned}
\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\oplus\boldsymbol{\tau}} &= \lim_{\boldsymbol{\theta}\to 0} \frac{f(\mathcal{X}\oplus\boldsymbol{\theta})\ominus f(\mathcal{X})}{\boldsymbol{\theta}} \\
&= \lim_{\boldsymbol{\theta}\to 0} \frac{\mathrm{Log}(\mathrm{Exp}(\boldsymbol{\tau})^{-1}\mathcal{X}^{-1}\mathcal{X}\mathrm{Exp}(\boldsymbol{\theta})\mathrm{Exp}(\boldsymbol{\tau}))}{\boldsymbol{\theta}} \\
&= \lim_{\boldsymbol{\theta}\to 0} \frac{\mathrm{Log}(\mathrm{Exp}(\boldsymbol{\tau})^{-1}\mathrm{Exp}(\boldsymbol{\theta})\mathrm{Exp}(\boldsymbol{\tau}))}{\boldsymbol{\theta}} \\
&= \lim_{\boldsymbol{\theta}\to 0} \frac{\log(\exp(\mathrm{Exp}(\boldsymbol{\tau})^{-1}(\boldsymbol{\theta})^\wedge\mathrm{Exp}(\boldsymbol{\tau})))^\vee}{\boldsymbol{\theta}} \\
&= \lim_{\boldsymbol{\theta}\to 0} \frac{(\mathrm{Exp}(\boldsymbol{\tau})^{-1}(\boldsymbol{\theta})^\wedge\mathrm{Exp}(\boldsymbol{\tau}))^\vee}{\boldsymbol{\theta}} \\
&= \lim_{\boldsymbol{\theta}\to 0} \frac{\mathbf{Ad}_{\mathrm{Exp}(\boldsymbol{\tau})^{-1}}\boldsymbol{\theta}}{\boldsymbol{\theta}} \\
&= \mathbf{Ad}_{\mathrm{Exp}(\boldsymbol{\tau})}^{-1}
\end{aligned}
\tag{A.35}
$$

- 式(81)证明

可以使用式(81)中的链式法则，也可以用定义来推导

令 $f(\boldsymbol{\tau})=\mathcal{X}\oplus\boldsymbol{\tau}=\mathcal{X}\mathrm{Exp}(\boldsymbol{\tau})$

$$
\begin{aligned}
\mathbf{J}_{\boldsymbol{\tau}}^{\mathcal{X}\oplus\boldsymbol{\tau}} &= \lim_{\delta\boldsymbol{\tau}\to 0}\frac{f(\boldsymbol{\tau}+\delta\boldsymbol{\tau})\ominus f(\boldsymbol{\tau})}{\delta\boldsymbol{\tau}} \\
&= \lim_{\delta\boldsymbol{\tau}\to 0}\frac{\mathrm{Log}(\mathrm{Exp}(\boldsymbol{\tau})^{-1}\mathcal{X}^{-1}\mathcal{X}\mathrm{Exp}(\boldsymbol{\tau}+\delta\boldsymbol{\tau}))}{\delta\boldsymbol{\tau}} \\
&= \lim_{\delta\boldsymbol{\tau}\to 0}\frac{\mathrm{Log}(\mathrm{Exp}(\boldsymbol{\tau})^{-1}\mathrm{Exp}(\boldsymbol{\tau})\mathrm{Exp}(\mathbf{J}_r\delta\boldsymbol{\tau}))}{\delta\boldsymbol{\tau}} \\
&= \lim_{\delta\boldsymbol{\tau}\to 0}\frac{\mathbf{J}_r\delta\boldsymbol{\tau}}{\delta\boldsymbol{\tau}} \\
&= \mathbf{J}_r
\end{aligned} \tag{A.36}
$$

- 式(82)推导

$$
\begin{aligned}
\mathbf{J}_{\mathcal{X}}^{\mathcal{Y}\ominus\mathcal{X}} &= \mathbf{J}_{(\mathcal{X}^{-1}\circ\mathcal{Y})}^{\mathrm{Log}(\mathcal{X}^{-1}\circ\mathcal{Y})}\ \mathbf{J}_{\mathcal{X}^{-1}}^{(\mathcal{X}^{-1}\circ\mathcal{Y})}\ \mathbf{J}_{\mathcal{X}}^{\mathcal{X}^{-1}} \\
(79,65,62) &= \quad \mathbf{J}_r^{-1}(\boldsymbol{\tau})\ \ \mathbf{Ad}_{\mathcal{Y}}^{-1}\ \ (-\mathbf{Ad}_{\mathcal{X}}) \\
(33,34) &= -\mathbf{J}_r^{-1}(\boldsymbol{\tau})\,\mathbf{Ad}_{\mathcal{Y}^{-1}\mathcal{X}} \\
&= -\mathbf{J}_r^{-1}(\boldsymbol{\tau})\,\mathbf{Ad}_{\mathrm{Exp}(\boldsymbol{\tau})}{}^{-1} \\
(75) &= -\mathbf{J}_l^{-1}(\boldsymbol{\tau})\ .
\end{aligned}
$$

- 式(83)推导见原式

Cheet Sheet

Joan Sola在github上开源了paper的相关代码和常用的Lie_theory_cheat_sheet，以下整理了SO(3)和SE(3)上常用的Jacobian形式，便于快速查询并使用

		$\mathbf{J}_{\mathcal{X}}^{\mathcal{X}^{-1}}$	$\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\mathcal{Y}}$	$\mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\mathcal{Y}}$	$\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\mathbf{p}}$	$\mathbf{J}_{\mathbf{p}}^{\mathcal{X}\mathbf{p}}$
$\mathcal{X}=\mathbf{R}_a\in\mathbf{SO}(3)$ $\mathcal{Y}=\mathbf{R}_b\in\mathbf{SO}(3)$	${}^R\mathbf{J}$	$-\mathbf{R}_a$	$\mathbf{R}_b^\top$	$\mathbf{I}_3$	$-\mathbf{R}_a[\mathbf{p}]_\times$	$\mathbf{R}_a$
	${}^L\mathbf{J}$	$-\mathbf{R}_a^\top$	$\mathbf{I}_3$	$\mathbf{R}_a$	$-[\mathbf{R}_a\mathbf{p}]_\times$	$\mathbf{R}_a$
$\mathcal{X}=\begin{bmatrix}\mathbf{R}_a & \mathbf{t}_a\\ 0 & 1\end{bmatrix}\in\mathbf{SE}(3)$ $\mathcal{Y}=\begin{bmatrix}\mathbf{R}_b & \mathbf{t}_b\\ 0 & 1\end{bmatrix}\in\mathbf{SE}(3)$	${}^R\mathbf{J}$	$-\begin{bmatrix}\mathbf{R}_a & [\mathbf{t}]_\times\mathbf{R}_a\\ 0 & \mathbf{R}_a\end{bmatrix}$	$\begin{bmatrix}\mathbf{R}_b^\top & -\mathbf{R}_b^\top[\mathbf{t_b}]_\times\\ 0 & \mathbf{R}_b^\top\end{bmatrix}$	$\mathbf{I}_6$	$\begin{bmatrix}\mathbf{R}_a & -\mathbf{R}_a[\mathbf{p}]_\times\end{bmatrix}_{3\times 6}$	$\mathbf{R}_a$
	${}^L\mathbf{J}$	$-\begin{bmatrix}\mathbf{R}_a^\top & -\mathbf{R}_a^\top[\mathbf{t}]_\times\\ 0 & \mathbf{R}_a^\top\end{bmatrix}$	$\mathbf{I}_6$	$\begin{bmatrix}\mathbf{R}_a^\top & -\mathbf{R}_a^\top[\mathbf{t_a}]_\times\\ 0 & \mathbf{R}_a^\top\end{bmatrix}$	$\begin{bmatrix}\mathbf{I}_3 & -[\mathbf{Rp}+\mathbf{t}]_\times\end{bmatrix}_{3\times 6}$	$\mathbf{R}_a$

REFERENCES

- Youtube-Lie theory for the roboticist Joan Solà
- A micro Lie theory for state estimation in robotics
- (Rv)^ = Rv^R' 的一种证明
- Quaternion kinematics for the error-state Kalman filter
- A Micro Lie Theory 论文理解
- Lie theory cheat sheet
- 李群和李代数（四）：姿态估计小结
- 流形,李群及刚体变换(上)
- SLAM中李群、李代数常用公式
- 李群(具有连续的群结构的实流形或者复流形) - 知乎
- 在位姿估计问题中应用李群/流形的发展脉络、动机与直觉(框架草稿)
- 位姿估计问题应用李群的思路与直觉