

Lecture-Lie theory for the roboticist

本文是学习Joan Sola在2020年和2022年的Robotics & AI Summer School的lecture的笔记，二者的内容基本相同，下文的截图主要基于2022年最新的lecture，也有少量截图来自2020年的lecture。此外，本lecture主要的参考文档为Joan Sola的著作A micro Lie theory for state estimation in robotics

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 - Sensor self-calibration

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Examples

- S^1 : The unit complex numbers(the sphere of the one dimension)

S^1 : The unit complex numbers

A quick overview of known facts

$y = z \cdot x$

S^1

$|z|=1$

x

y

θ

i

- Action: $y = z \cdot x$ rotates x
operator: Lie group!
- Constraint: $z^* \cdot z = 1$
- Topology: unit circle S^1
- Elements: $z = \cos \theta + i \sin \theta$
- Inverse: z^* (conjugate)
- Composition: $z_1 \cdot z_2$

complex nbrs.

$y = z \cdot x$

\downarrow

operator: Lie group!

• Action: $y = z \cdot x$ rotates x

• Constraint: $z^* \cdot z = 1$

• Topology: unit circle S^1

• Elements: $z = \cos \theta + i \sin \theta$

• Inverse: z^* (conjugate)

• Composition: $z_1 \cdot z_2$

- $SO(2)$: 2D rotation group(二维特殊正交群)
 - x, y 和 \mathbf{R} 属于不同的set, \mathbf{R} 的基是 $(I, [1]_x)$, 而 x, y 的基是 $(1, 0), (0, 1)$

SO(2) : The 2D rotation matrices

A quick overview of known facts

- Action: $y = R \cdot x$ rotates x
- Constraint: $R^T R = I$
- Topology: "circle" $SO(2)$
- Elements: $R = I \cos \theta + [1]_x \sin \theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$
- Inverse: R^T (transpose)
- Composition: $R_1 \cdot R_2$

- S^3 : The unit quaternions(3-sphere in R^4)

S^3 : The unit quaternions

The 3-sphere in R^4

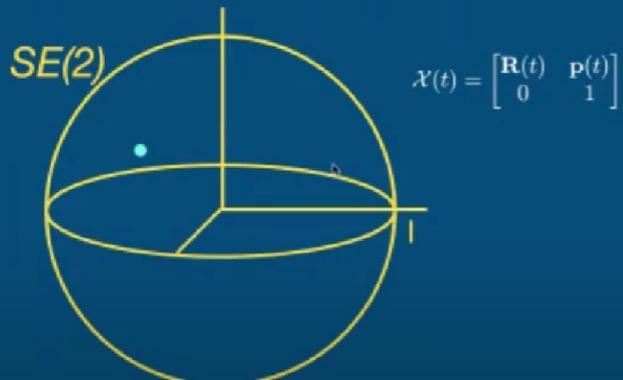
operator: Lie group!

$$q = \cos \theta/2 + \mathbf{u} \sin \theta/2$$

- $SE(2)$: A pose(rigid transform) in 2-d Euclidean space(二维特殊欧式群)
 - 下图左侧的曲面图并不是精确的，只是一个抽象曲面

Typical uses

Pose of a robot in the plane: $SE(2)$



A pose (rigid transformation) in two-dimensional Euclidean space can be uniquely determined by means of a 3×3 homogeneous matrix with this structure:

$$\mathbf{T}_2 = \left(\begin{array}{c|c} \mathbf{R}_2 & \mathbf{t} \\ \hline \mathbf{0}_{1 \times 2} & 1 \end{array} \right) = \left(\begin{array}{cc|c} \cos \phi & -\sin \phi & x \\ \sin \phi & \cos \phi & y \\ 0 & 0 & 1 \end{array} \right) \quad (\text{B.1})$$

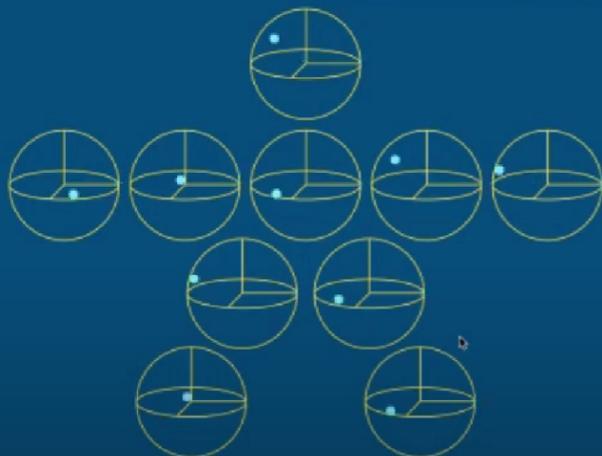
where the three degrees of freedom of the 2D transformation are the (x, y) translation and the rotation of ϕ radians.

The \mathbf{R}_2 belongs to the group $SO(2)$, and \mathbf{T}_2 to $SE(2)$.

- $SE(3)$: A pose(rigid transform) in 3-d Euclidean space(三维特殊欧式群)

Key interpretation

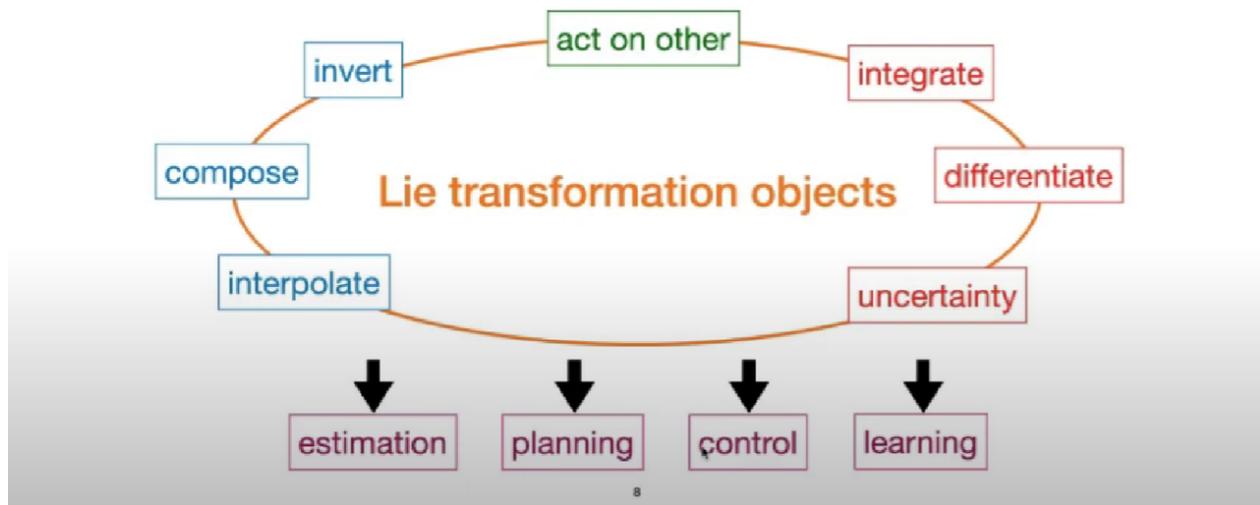
Pose of each limb in your humanoid : $SE(3)^n$



Why Lie Groups

Why Lie groups?

Abstract, rigorous, and principled way to do all this :



Lie groups used in Robotics (not exhaustive)

Including the trivial groups \mathbb{R}^n

Lie group \mathcal{M}, \circ	complex		Tangent space		Lie algebra		Cartesian		Exponential map		Comp.	Action
	size	dim	$\mathcal{X} \in \mathcal{M}$	Constraint	$\tau^\wedge \in \mathfrak{m}$	$\tau \in \mathbb{R}^m$	$\text{Exp}(\tau)$					
n -D vector	$\mathbb{R}^n, +$	n	$\mathbf{v} \in \mathbb{R}^n$	$\mathbf{v} - \mathbf{v} = \mathbf{0}$	$\mathbf{v} \in \mathbb{R}^n$	$\mathbf{v} \in \mathbb{R}^n$	$\mathbf{v} = \exp(\mathbf{v})$			$\mathbf{v}_1 + \mathbf{v}_2$		$\mathbf{v} + \mathbf{x}$
circle Rotation	\mathbb{S}^1, \cdot $SO(2), \cdot$	2 4	$\mathbf{z} \in \mathbb{C}$ \mathbf{R}	$\mathbf{z}^* \mathbf{z} = 1$ $\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$i\theta \in i\mathbb{R}$ $[\theta]_\times \in \mathfrak{so}(2)$	$\theta \in \mathbb{R}$	$\mathbf{z} = \exp(i\theta)$ $\mathbf{R} = \exp([\theta]_\times)$			$\mathbf{z}_1 \mathbf{z}_2$ $\mathbf{R}_1 \mathbf{R}_2$		$\mathbf{z} \mathbf{x}$ $\mathbf{R} \mathbf{x}$
Rigid motion	$SE(2), \cdot$	9	$\mathbf{M} = [\mathbf{R} \ \mathbf{t}]$	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$[\theta]_\times \rho \in \mathfrak{se}(2)$	$[\rho]_\theta \in \mathbb{R}^3$	$\exp([\theta]_\times \rho)$			$\mathbf{M}_1 \mathbf{M}_2$		$\mathbf{R} \mathbf{x} + \mathbf{t}$
3-sphere Rotation	S^3, \cdot $SO(3), \cdot$	4 9	$\mathbf{q} \in \mathbb{H}$ \mathbf{R}	$\mathbf{q}^* \mathbf{q} = 1$ $\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$\theta/2 \in \mathbb{H}_p$ $[\theta]_\times \in \mathfrak{so}(3)$	$\theta \in \mathbb{R}^3$	$\mathbf{q} = \exp(\mathbf{u}\theta/2)$ $\mathbf{R} = \exp([\theta]_\times)$			$\mathbf{q}_1 \mathbf{q}_2$ $\mathbf{R}_1 \mathbf{R}_2$		$\mathbf{q} \mathbf{x} \mathbf{q}^*$ $\mathbf{R} \mathbf{x}$
Rigid motion	$SE(3), \cdot$	16	$\mathbf{M} = [\mathbf{R} \ \mathbf{t}]$	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$[\theta]_\times \rho \in \mathfrak{se}(3)$	$[\rho]_\theta \in \mathbb{R}^6$	$\exp([\theta]_\times \rho)$			$\mathbf{M}_1 \mathbf{M}_2$		$\mathbf{R} \mathbf{x} + \mathbf{t}$

quaternion

And more...

- Sim2, Sim3 : transformation and scale
- IMU group, $SE_2(3)$: motion deltas for IMU

monocular SLAM
intertial SLAM

Lie Group definition

Group

- 满足4个公理的set和operation构成一个group
- 很多group是不满足交换律的(non-commutative)

Group

Definition through the 4 group **axioms**

- Group: **set** G of elements $\{X, Y, Z, \dots\}$ with an **operation** ‘ \cdot ’ such that :
 - **Composition** stays in the group: $X \cdot Y \text{ is in } G$
 - **Identity** element is in the group: $X \cdot E = E \cdot X = X$
 - **Inverse** element is in the group: $X^{-1} \cdot X = X \cdot X^{-1} = E$
 - Operation is **associative**: $X \cdot (Y \cdot Z) = (X \cdot Y) \cdot Z$
- In many groups of interest, the operation ‘ \cdot ’ is **non-commutative** !

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Manifold

- **Wikipage:** 流形 (英语: Manifolds) 是可以局部欧几里得空间化的一个拓扑空间，是欧几里得空间中的曲线、曲面等概念的推广。欧几里得空间就是最简单的流形的实例。地球表面这样的球面则是一个稍微复杂的例子。一般的流形可以通过把许多平直的片折弯并粘连而成
- 粗略地说，流形是局部看近似线性空间的抽象曲面 (而不是曲面围成的实心部分)

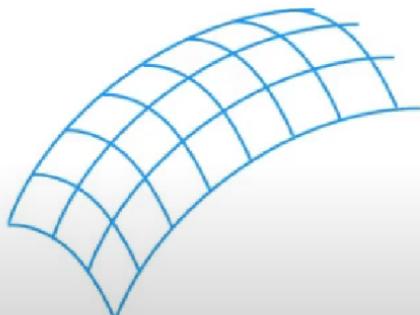
Lie Group

- 具有group性质的**smooth manifold**

The Lie Group

Def: a **group** that is also a **smooth manifold**

- Smooth manifold



Put otherwise:

Def: a **Lie group** is a **smooth manifold** whose **elements** satisfy the **group axioms**

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Group Action

Group Action

Definition

- A **group** can **act** on another set **V** to **transform** its elements
- Given X, Y in G and v in V , the **action** ‘ \cdot ’ is such that :
 - **Identity** is the null action: $E \cdot v = v$
 - It is **compatible** with composition: $(X \cdot Y) \cdot v = X \cdot (Y \cdot v)$
- *Lie groups were formerly known as “Continuous Transformation groups”.*

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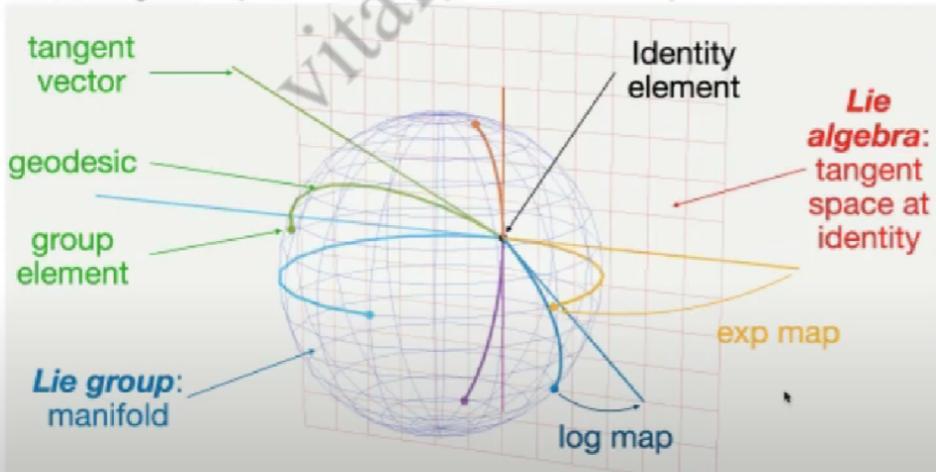
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The topology of Lie Theory

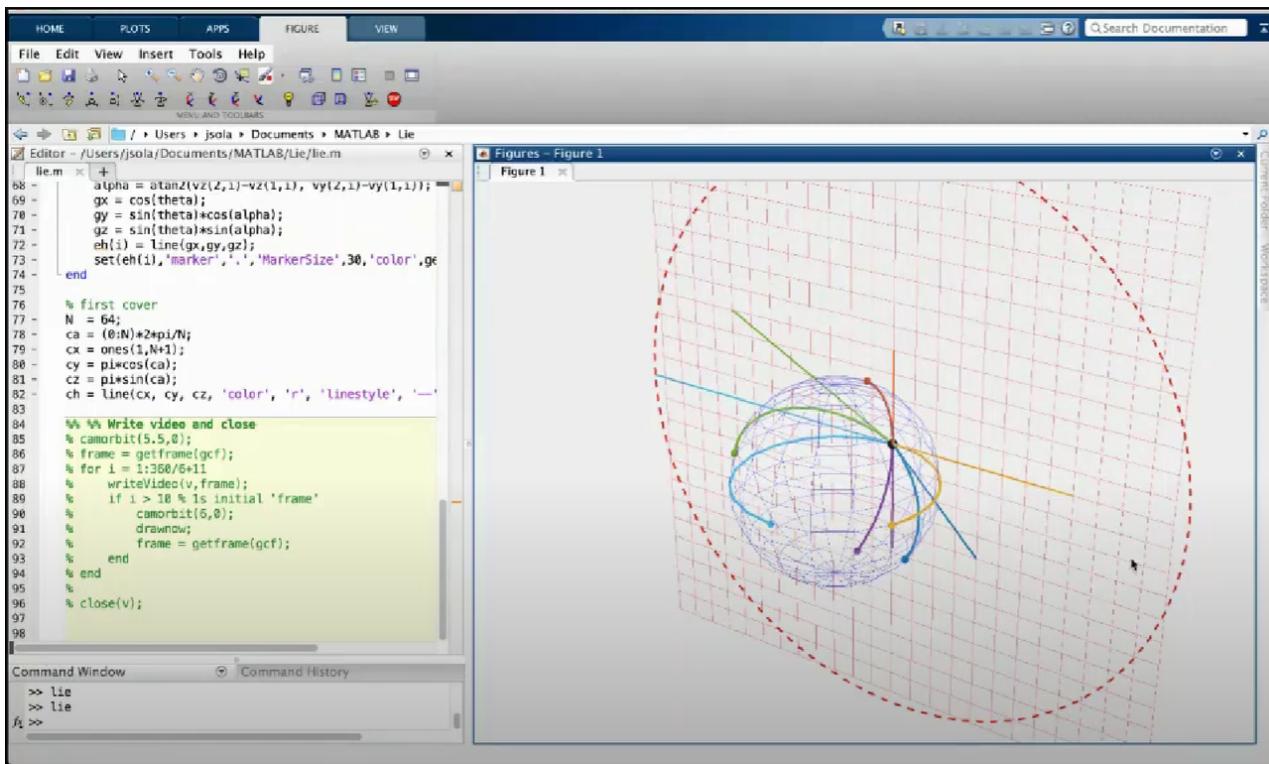
- exactly converts elements of the Lie algebra into elements of the group.
The log map is the inverse operation

The topology of Lie theory

Manifold, tangent space and exponential map



- Tangent space will cover manifold **multiple times** and the first it covers is the time where the red circle's radius is π and it will end up at the **antipode point** (对径点) "



Tangent space and Lie Algebra

Definition

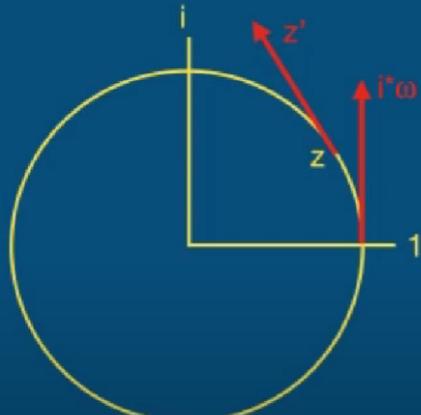
- [Wiki page](#): 切空间 (Tangent space) 是在某一点所有的切向量组成的线性空间
- 由于manifold是smooth的，因此manifold上每个点处的tangent space是唯一的
- tangent space是向量空间
- tangent space空间的维度是manifold的自由度
- 特殊的tangent space: group上所有点处的tangent space是的结构是相同的("The manifold has the same structure everywhere")，比如下图中所有点处的tangent space都是二维平面，但各点处的tangent space并不是同一个平面(not the same)，称identity处的tangent space为lie algebra

The tangent space and the Lie algebra

- The tangent space at each point is **unique**
- The tangent space is a **vector space**: we can do **calculus**
- The **dimension** of the tangent space is the number of **degrees of freedom** of the manifold
- The tangent space at the **identity** is called the "**Lie Algebra**"

The tangent space of S^1

Structure of the tangent space: consider the velocity of a point



- Differentiate $z^* \cdot z = 1$ w.r.t. time:

$$\begin{aligned} \dot{z}^* z + z^* \dot{z} &= 0 \\ z^* \dot{z} &= -(z^* \dot{z})^* \\ z^* \dot{z} &= i\omega \in i\mathbb{R} \end{aligned}$$

imaginary

- Lie Algebra: $\omega^\wedge = i \cdot \omega$ in $i\mathbb{R}$

- Cartesian: ω in \mathbb{R}

- Isomorphism:

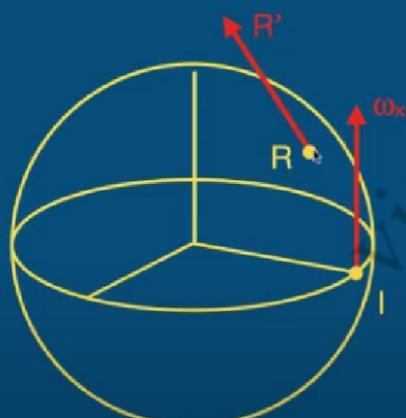
- Hat: $\omega^\wedge = i \cdot \omega$

- Vee: $\omega = (\omega^\wedge)^\vee = -i \cdot \omega^\wedge$

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The tangent space of $SO(3)$

Structure of the tangent space



- Differentiate $R^T \cdot R = I$ w.r.t. time:

$$\begin{aligned} R^T \dot{R} + R^T \dot{R}^T &= 0 \\ R^T \dot{R} &= -(R^T \dot{R})^T \\ R^T \dot{R} &= \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \in \mathfrak{so}(3) \end{aligned}$$

skew-symmetric

- Lie algebra when $R = I$

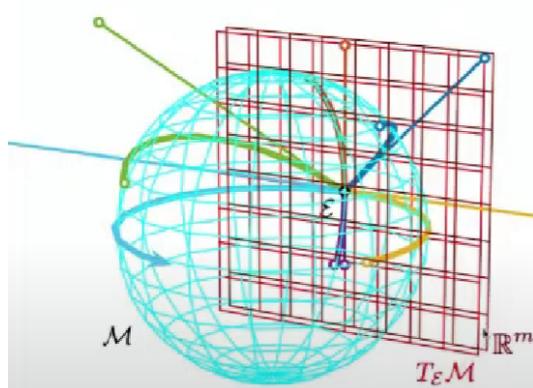
$$\dot{R} = \omega_x \triangleq \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \in \mathfrak{so}(3)$$

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Lie algebra vs. Cartesian representation

- Tangent space 有两种同构(isomorphism)的表示形式: Lie algebra 和 Cartesian

Lie Algebra and Cartesian tangent spaces

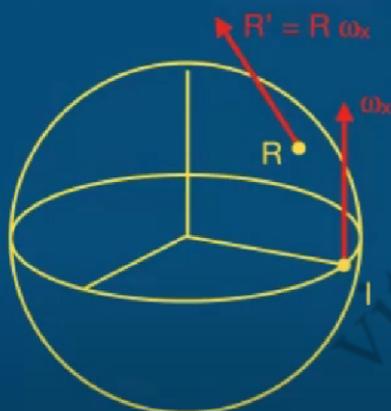


- Two **isomorphic** representations of the tangent space
 - Lie algebra: $T_E M$
 - Cartesian: \mathbb{R}^m
- One linear **isomorphism**:
 - hat : $\mathbb{R}^m \rightarrow T_E M ; w \rightarrow w^*$
 - vee : $T_E M \rightarrow \mathbb{R}^m ; w^* \rightarrow (w^*)^\vee$
- We write $T_E M \sim \mathbb{R}^m$ and $w \sim w^*$

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The tangent space of $SO(3)$

Lie algebra vs. Cartesian representation



- Lie Algebra $\mathfrak{so}(3)$:

$$\omega_x = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \in \mathfrak{so}(3)$$

$$= \omega_x \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + \omega_y \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \omega_z \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
- Cartesian \mathbb{R}^3 :

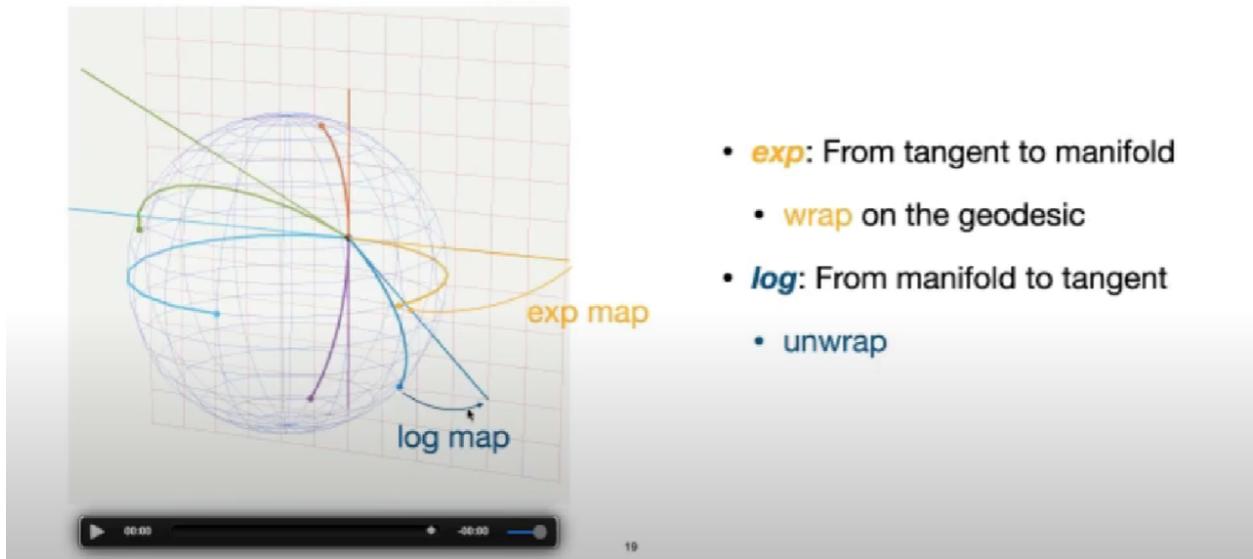
$$\omega = [\omega_x, \omega_y, \omega_z]^T \in \mathbb{R}^3$$

$$= \omega_x [1, 0, 0]^T + \omega_y [0, 1, 0]^T + \omega_z [0, 0, 1]^T$$
- Isomorphism: $\mathfrak{so}(3) \cong \mathbb{R}^3$
- Hat: $\omega^* = \omega_x$;
- Vee: $\omega = \omega_x^\vee$

The Exponential map

Definition

The exponential map



Examples

- 推导思路：根据manifold的约束方程，对时间 t 求导，构建 z 的ODE方程，最终得到lie algebra到lie group的exponential map
- 因为整个推导过程没有近似(approximation)的步骤，每一步都是精确(exact)的，因此“**exponential map is exact**”

The exponential map

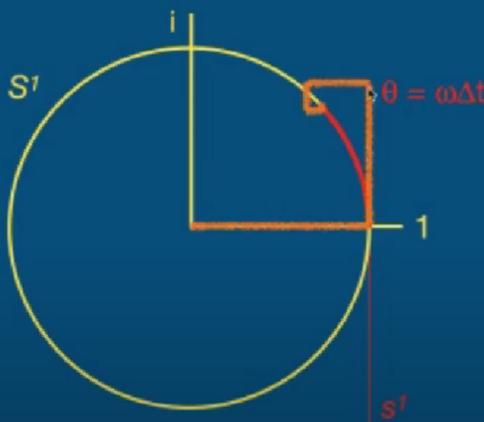
Example: S^1

- Write ODE and integrate
$$z^* z' = iw \rightarrow z' = z iw$$
$$z(t) = z_0 \exp(iw t) \rightarrow z' = z_0 \exp(iwt) iw = z iw$$
- If $z_0 = z(0) = 1$ and $i\omega t = i\theta$
$$z(t) = \exp(iw t) = \exp(i\theta)$$
- Taylor expansion, with $p = 1$ and $\bar{p} = -i$
$$\exp(i\theta) = 1 + i\theta + (i\theta)^2/2! + (i\theta)^3/3! + \dots$$
$$= 1 + i\theta - \theta^2/2 - i\theta^3/3! + i\theta^4/4! + \dots$$
$$= (1 - \theta^2/2 + \dots) + i(\theta - \theta^3/3! + \dots)$$
$$= \cos \theta + i \sin \theta$$

- 根据 \exp 的泰勒展开式，可以直观地描述exponential map的过程

The exponential map

Example: S^1



- Write ODE and integrate

$$z^+ z^- = iw \rightarrow z' = z iw$$

$$z(t) = z_0 \exp(iw t)$$

- If $z_0 = z(0) = 1$ and $i\omega t = i\theta$

$$z(t) = \exp(iw t) = \exp(i\theta)$$

- Taylor expansion, with $\vec{p} = 1$ and $\beta = -i$

$$\exp(i\theta) = 1 + i\theta + (i\theta)^2/2 + (i\theta)^3/3! + \dots$$

$$= 1 + i\theta - \theta^2/2 - i\theta^3/3! + i\theta^4/4! + \dots$$

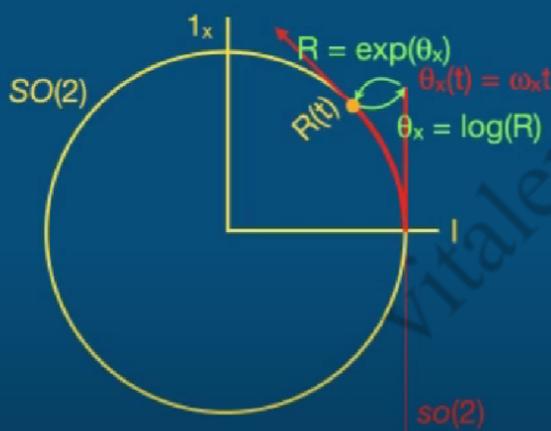
$$= (1 - \theta^2/2 + \dots) + i(\theta - \theta^3/3! + \dots)$$

$$= \cos \theta + i \sin \theta$$

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The exponential map

Example: $SO(2)$



- Write ODE and integrate

$$\mathbf{R}^\top \dot{\mathbf{R}} = \omega_x \Rightarrow \dot{\mathbf{R}} = \mathbf{R} \cdot \omega_x \quad \omega_x = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

- If $R_0 = R(0) = I$ and $\omega_x t = \theta_x = \theta \cdot 1_x$

$$\mathbf{R}(t) = \exp(\omega_x t) = \exp(\theta_x)$$

- Taylor expansion, with $1_x^0 = I$ and $1_x^3 = -1_x$

$$\exp(\theta_x) = I + \theta 1_x + (\theta 1_x)^2/2 + (\theta 1_x)^3/3! + \dots$$

$$= I \cdot (1 - \theta^3/3! + \dots) + 1_x \cdot (\theta - \theta^2/2! + \dots)$$

$$= I \cos \theta + 1_x \sin \theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

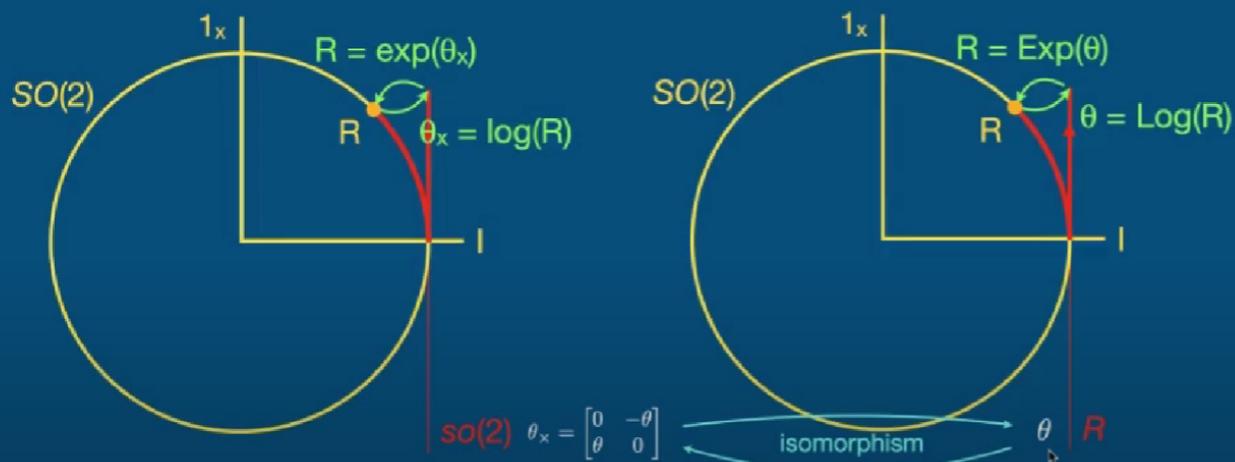


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The capitalized exponential map

The capitalized exponential map

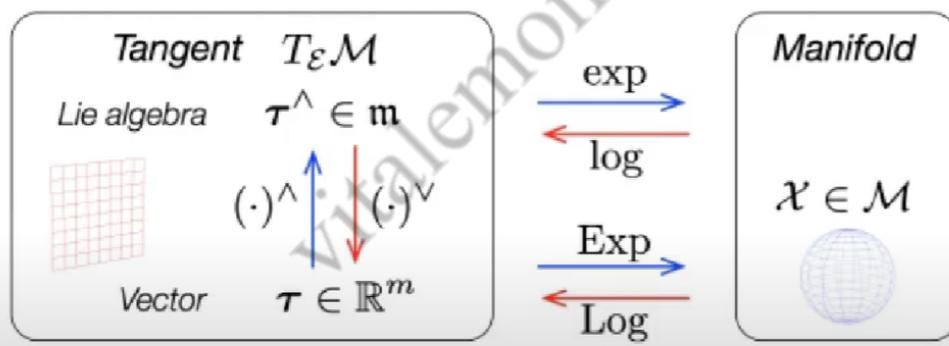
First cool shortcut



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The capitalized exponential map

Skip the Lie algebra, and work always in Cartesian



*Exp() and Log() are mere **shortcuts** - but very **useful***

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Plus and minus operators

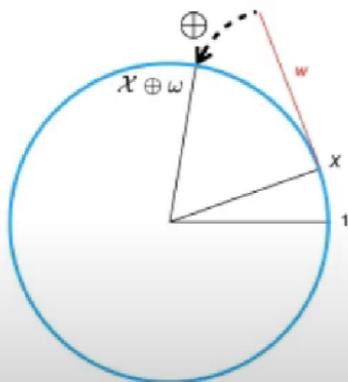
Definition

- $\mathcal{X} \oplus \omega$ 的结果是group上的元素，而 $\mathcal{Y} \ominus \mathcal{X}$ 是vector

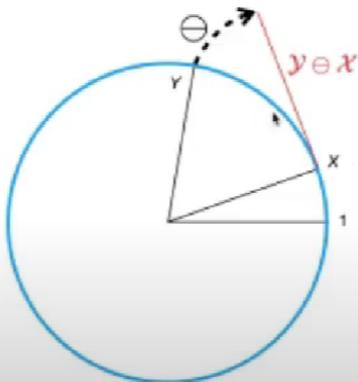
Plus and minus operators

The second cool shortcut

$$\mathcal{X} \oplus \omega \triangleq \mathcal{X} \cdot \text{Exp}(\omega)$$



$$\mathcal{Y} \ominus \mathcal{X} \triangleq \text{Log}(\mathcal{X}^{-1} \cdot \mathcal{Y})$$

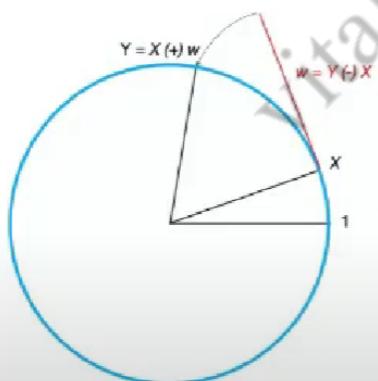


Plus and minus are also **shortcuts** - but also very **useful**

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- 使用Cartesian形式的tangent space的优势是方便构造perturbations, errors, increments以及Jacobians和covarians

Plus and minus operators



- Expressing in Cartesian form:
 - Perturbations, errors, increments
- And define easily:
 - Jacobians of functions $f: M \rightarrow N$
 - Covariances

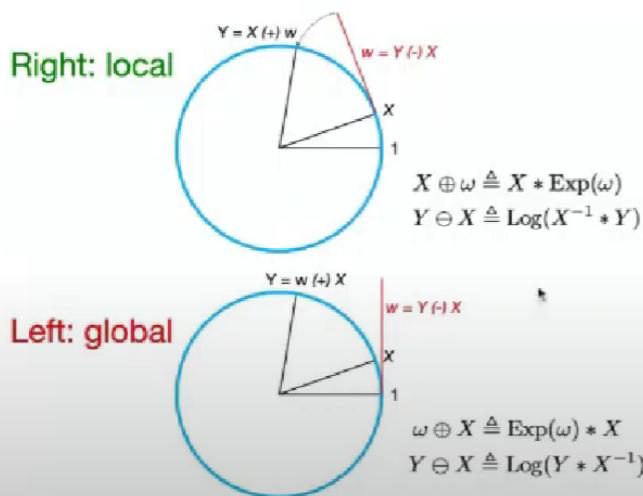
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Left and right plus and minus

- 由于group通常**不满足交换律**(non-commutative), 因此left和right operator的结果往往是不同的
- 当使用**right operator**时, tangent vector是表示在 \mathcal{X} 点所在的**local reference frame**下的
- 当使用**left operator**时, tangent vector是表示在identity点所在的**global reference frame**下的
- 当group**满足交换律时**, 二者的结果相同, 比如SO(2)

Left and right plus and minus

Control the reference frame



- **Right:** perturbations w in the **local reference** frame at X

- Use super-index X : x_w

- **Left:** perturbations w in the **global reference** frame (at E)

- Use super-index E : e_w

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The Adjoint matrix

- The adjoint **linearly** and **exactly** transforms **tangent vectors** at χ to the tangent space at **the identity**
- 当group满足交换律时， σ 和 τ 相同，此时 $\text{Ad}_\chi = \mathbf{I}$ ，一个比较典型的例子就是SO(2)

The Adjoint matrix

The diagram shows a manifold \mathcal{M} represented as a curved surface. A point ε is on the surface. Two tangent vectors $\sigma \in T_\varepsilon \mathcal{M}$ and $\tau \in T_\chi \mathcal{M}$ are shown originating from ε and χ respectively. A third vector y is also shown. The relationship between these vectors is given by the chain rule: $y = \sigma \oplus \chi = \chi \oplus \tau$. This leads to the formula for the Adjoint matrix: $\sigma^\wedge = \chi \cdot \tau^\wedge \cdot \chi^{-1}$, and finally $\sigma = \text{Ad}_\chi \cdot \tau$.

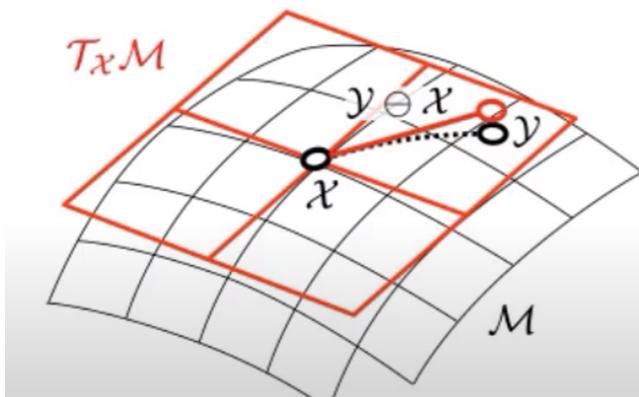
$\sigma \in T_\varepsilon \mathcal{M}$ $\tau \in T_\chi \mathcal{M}$

- **Linear** : matrix operator
- **Maps** : $T_\chi \mathcal{M}$ to $T_\varepsilon \mathcal{M}$

Calculus on Lie groups

Calculus on Lie groups

Use the plus and minus operators !



- Express as **Cartesian** vectors:
 - Perturbations, errors, increments, ...
- And define easily:
 - **Jacobians** of functions $f: M \rightarrow N$
 - **Covariances** of elements X in M

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Jacobians

Definition

- 求导时要注意元素是在Vector space在Lie Group, **vector space**上的加减运算向量的"+", "-", Lie group上元素的加减运算分别为 \oplus 和 \ominus

Jacobians on Lie groups

Use the plus and minus operators !

Vector spaces

$$J = \frac{\partial f(x)}{\partial x} \triangleq \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \in \mathbb{R}^{n \times m}$$



Lie groups

$$J = \frac{Df(x)}{Dx} = \lim_{\tau \rightarrow 0} \frac{f(x \oplus \tau) \ominus f(x)}{\tau} \in \mathbb{R}^{n \times m}$$

how to:

$$\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = \dots = \lim_{h \rightarrow 0} \frac{Jh}{h} \triangleq \frac{\partial Jh}{\partial h} = J$$



same thing!!!

$$\begin{aligned} \lim_{\tau \rightarrow 0} \frac{f(x \oplus \tau) \ominus f(x)}{\tau} &= \lim_{\tau \rightarrow 0} \frac{\text{Log}[f(x)^{-1} f(x \text{Exp}(\tau))]}{\tau} \\ &= \lim_{\tau \rightarrow 0} \frac{J\tau}{\tau} \triangleq \frac{\partial J\tau}{\partial \tau} = J, \end{aligned}$$

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Example

Jacobians on Lie groups

$$J = \frac{Df(\mathcal{X})}{D\mathcal{X}} = \lim_{\tau \rightarrow 0} \frac{f(\mathcal{X} \oplus \tau) \ominus f(\mathcal{X})}{\tau} \in \mathbb{R}^{n \times m}$$

Example: action of SO(3) on R³

$f : SO(3) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 ; \quad (\mathbf{R}, \mathbf{p}) \mapsto f(\mathbf{R}, \mathbf{p}) = \mathbf{R} \cdot \mathbf{p}$

operates on R³ *operates on SO(3)*

$$\begin{aligned} \frac{Df}{D\mathbf{R}} &= \lim_{\theta \rightarrow 0} \frac{(\mathbf{R} \oplus \theta) \cdot \mathbf{p} - \mathbf{R} \cdot \mathbf{p}}{\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{(\mathbf{R} \cdot \text{Exp}(\theta)) \cdot \mathbf{p} - \mathbf{R} \cdot \mathbf{p}}{\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{\mathbf{R} \cdot (\mathbf{I} + \theta \mathbf{\hat{x}}) \cdot \mathbf{p} - \mathbf{R} \cdot \mathbf{p}}{\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{\mathbf{R} \cdot \theta \mathbf{\hat{x}} \cdot \mathbf{p}}{\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{-\mathbf{R} \cdot \mathbf{p}_x \cdot \theta}{\theta} \\ &= -\mathbf{R} \cdot \mathbf{p}_x \end{aligned}$$

$$\begin{aligned} \frac{Df}{D\mathbf{p}} &= \lim_{\delta\mathbf{p} \rightarrow 0} \frac{\mathbf{R} \cdot (\mathbf{p} + \delta\mathbf{p}) - \mathbf{R} \cdot \mathbf{p}}{\delta\mathbf{p}} \\ &= \lim_{\delta\mathbf{p} \rightarrow 0} \frac{\mathbf{R} \cdot \delta\mathbf{p}}{\delta\mathbf{p}} = \mathbf{R} \end{aligned}$$

operate on R³

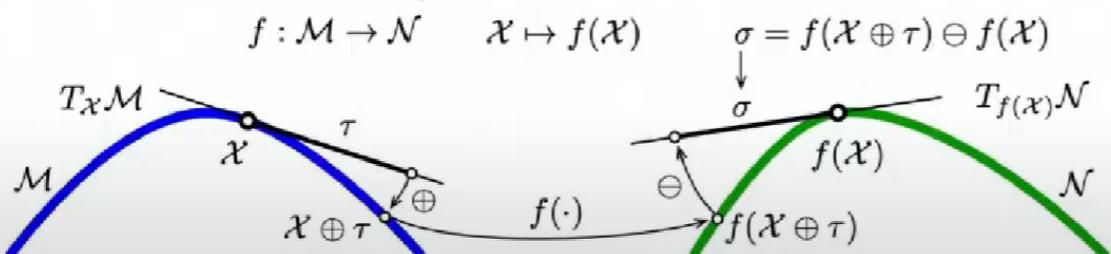
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Graphical interpretation

Jacobians on Lie groups

One DoF manifolds

$$\frac{}{}^{\mathcal{X}} Df(\mathcal{X}) \triangleq \lim_{\tau \rightarrow 0} \frac{f(\mathcal{X} \oplus \tau) \ominus f(\mathcal{X})}{\tau} \in \mathbb{R}^{n \times m}$$

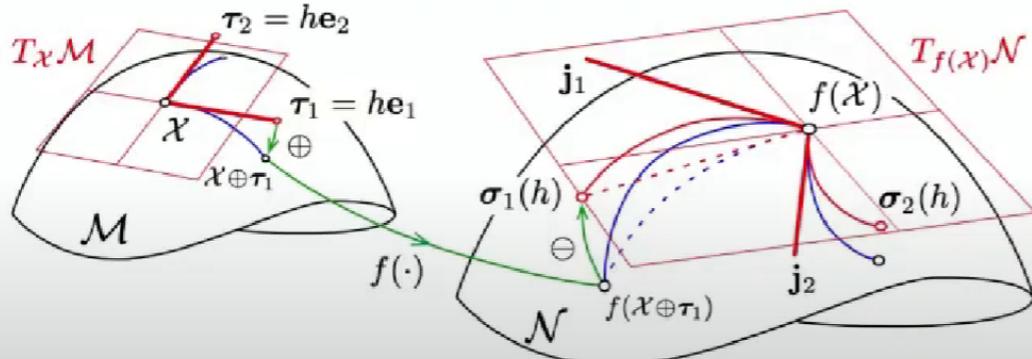


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Jacobians on Lie groups

Two DoF manifolds

$$\frac{^{\mathcal{X}}Df(\mathcal{X})}{D\mathcal{X}} \triangleq \lim_{\tau \rightarrow 0} \frac{f(\mathcal{X} \oplus \tau) \ominus f(\mathcal{X})}{\tau} \in \mathbb{R}^{n \times m}$$



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Differentiation rules on Lie groups

Differentiation rules on Lie groups

From elementary Jacobian blocks to any Jacobian

For each group	For all groups	Deduce from the previous
adjoint $\text{Ad}_{\mathcal{X}}$		
right Jacobian $\mathbf{J}_r = \frac{D \text{Exp}(\tau)}{D\tau}$	inverse $\frac{D\mathcal{X}^{-1}}{D\mathcal{X}} = -\text{Ad}_{\mathcal{X}}$	Log $\frac{D \text{Log}(\mathcal{X})}{D\mathcal{X}} = \mathbf{J}_r(\text{Log}(\mathcal{X}))^{-1}$
action $\frac{D\mathcal{X} \cdot \mathbf{p}}{D\mathcal{X}}, \frac{D\mathcal{X} \cdot \mathbf{p}}{D\mathbf{p}}$	composition $\frac{D\mathcal{X} \cdot \mathcal{Y}}{D\mathcal{X}} = \text{Ad}_{\mathcal{Y}}^{-1}$ $\frac{D\mathcal{X} \cdot \mathcal{Y}}{D\mathcal{Y}} = \mathbf{I}$	plus $\frac{D\mathcal{X} \oplus \tau}{D\mathcal{X}} = \text{Ad}_{\text{Exp}(\tau)}^{-1}$ $\frac{D\mathcal{X} \oplus \tau}{D\tau} = \mathbf{J}_r(\tau)$

Use the chain rule for any other Jacobian!

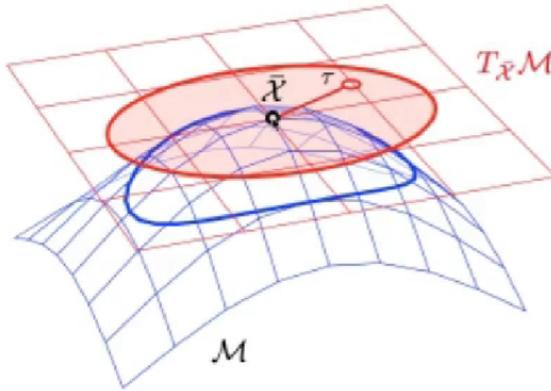
$$\frac{D\mathbf{R}^T \mathbf{p}}{D\mathbf{R}} = \frac{D\mathbf{R}^T \mathbf{p}}{D\mathbf{R}^T} \frac{D\mathbf{R}^T}{D\mathbf{R}} = (-\mathbf{R}^T \mathbf{p}_x) (-\text{Ad}_{\mathbf{R}}) = \mathbf{R}^T \mathbf{p}_x \mathbf{R} = [\mathbf{R}^T \mathbf{p}]_x$$

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Perturbations on Lie groups

Perturbations on Lie groups

Covariance matrices ... and their propagation



- Perturbation τ over X

$$X = \bar{X} \oplus \tau \quad \tau = X \ominus \bar{X}$$

- Covariance of X — i.e., of τ

$$\mathbf{P} \triangleq \mathbb{E}[\tau \cdot \tau^\top]$$

$$\mathbf{P} \triangleq \mathbb{E}[(X \ominus \bar{X}) \cdot (X \ominus \bar{X})^\top]$$

- Propagation is easy!

$$y = f(X) \quad \mathbf{J} = \frac{Dy}{DX}$$

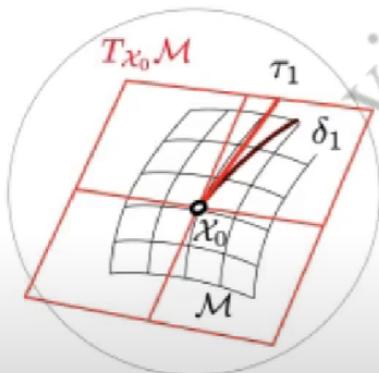
$$\mathbf{P}_y = \mathbf{J} \cdot \mathbf{P}_X \cdot \mathbf{J}^\top$$

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Integration on Lie groups

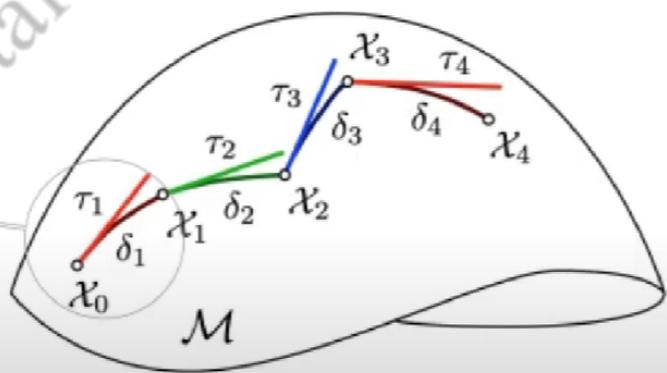
Integration on Lie groups

continuous time, ω constant



$$X(t) = X_0 \cdot \text{Exp}(\omega t)$$

discrete time, ω piecewise constant

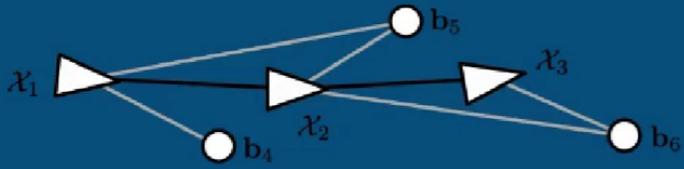


$$X_4 = X_0 \oplus (\omega_1 dt) \oplus (\omega_2 dt) \oplus (\omega_3 dt) \oplus (\omega_4 dt)$$

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EKF map-based localization

EKF map-based localization



Poses (unknown)

$$\mathcal{X} \sim \mathcal{N}(\bar{\mathcal{X}}, \mathbf{P}) \in SE(3)$$

$$\mathbf{P} = \mathbb{E}[(\mathcal{X} \ominus \bar{\mathcal{X}})(\mathcal{X} \ominus \bar{\mathcal{X}})^\top]$$

covariance

Motion model

$$\mathcal{X}_i = f(\mathcal{X}_{i-1}, u_i) = \mathcal{X}_{i-1} \oplus (u_i dt + w)$$

integration

$$w \sim \mathcal{N}(0, \mathbf{Q})$$

perturbation

Measurement model

$$\mathbf{y}_k = h(\mathcal{X}) = (\mathcal{X}^{-1} \cdot \mathbf{b}_k + v)$$

inverse action

Beacons (known)

$$\mathbf{b}_k \in \mathbb{R}^3$$

tangent velocity

$$v \sim \mathcal{N}(0, \mathbf{R})$$

noise

EKF prediction

$$\hat{\mathcal{X}} \leftarrow \hat{\mathcal{X}} \oplus u_i dt \quad \mathbf{F} = \frac{\partial f}{\partial \mathcal{X}} \quad \mathbf{G} = \frac{\partial f}{\partial w}$$

$$\mathbf{P} \leftarrow \mathbf{F}\mathbf{P}\mathbf{F}^\top + \mathbf{G}\mathbf{Q}\mathbf{G}^\top$$

EKF correction at each k

$$\mathbf{z}_k = \mathbf{y}_k - \hat{\mathcal{X}}^{-1} \mathbf{b}_k \quad \mathbf{H} = \frac{\partial h}{\partial \mathcal{X}}$$

$$\mathbf{Z}_k = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}$$

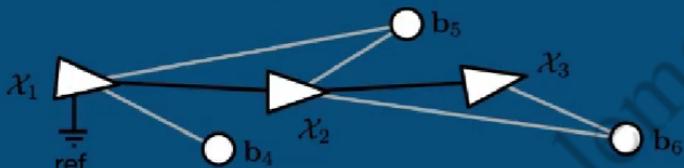
$$\mathbf{K} = \mathbf{P}\mathbf{H}^\top \mathbf{Z}_k^{-1}$$

$$\hat{\mathcal{X}} \leftarrow \hat{\mathcal{X}} \oplus \mathbf{K}\mathbf{z}_k$$

update using plus

$$\mathbf{P} \leftarrow \mathbf{P} - \mathbf{K}\mathbf{Z}_k\mathbf{K}^\top$$

Graph-SLAM Least squares on manifold



Poses (unknown)

$$\mathcal{X}_i \in SE(3)$$

Beacons (unknown)

$$\mathbf{b}_k \in \mathbb{R}^3$$

State: composite of Lie groups

$$\mathcal{X} = \langle \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathbf{b}_4, \mathbf{b}_5, \mathbf{b}_6 \rangle$$

Nonlinear least-squares problem

$$\mathcal{X}^* = \arg \min_{\mathcal{X}} \sum_p \|\mathbf{r}_p(\mathcal{X})\|^2$$

Residuals :

Prior $\mathbf{r}_1 = \Omega_1^{\top/2}(\mathcal{X}_1 \ominus \mathcal{X}_1^{ref})$

Motion $\mathbf{r}_{ij} = \Omega_{ij}^{\top/2}(u_j dt - (\mathcal{X}_j \ominus \mathcal{X}_i))$

Measurement $\mathbf{r}_{ik} = \Omega_{ik}^{\top/2}(\mathbf{y}_{ik} - \mathcal{X}_i^{-1} \cdot \mathbf{b}_k)$

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{\mathcal{X}_1}^{\mathbf{r}_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{J}_{\mathcal{X}_1}^{\mathbf{r}_{12}} & \mathbf{J}_{\mathcal{X}_2}^{\mathbf{r}_{12}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{\mathcal{X}_2}^{\mathbf{r}_{23}} & \mathbf{J}_{\mathcal{X}_3}^{\mathbf{r}_{23}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{J}_{\mathcal{X}_1}^{\mathbf{r}_{14}} & \mathbf{0} & \mathbf{0} & \mathbf{J}_{\mathbf{b}_4}^{\mathbf{r}_{14}} & \mathbf{0} & \mathbf{0} \\ \mathbf{J}_{\mathcal{X}_1}^{\mathbf{r}_{15}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{J}_{\mathbf{b}_5}^{\mathbf{r}_{15}} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{\mathcal{X}_2}^{\mathbf{r}_{25}} & \mathbf{0} & \mathbf{0} & \mathbf{J}_{\mathbf{b}_5}^{\mathbf{r}_{25}} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{\mathcal{X}_2}^{\mathbf{r}_{26}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{J}_{\mathbf{b}_6}^{\mathbf{r}_{26}} \\ \mathbf{0} & \mathbf{0} & \mathbf{J}_{\mathcal{X}_3}^{\mathbf{r}_{36}} & \mathbf{0} & \mathbf{0} & \mathbf{J}_{\mathbf{b}_6}^{\mathbf{r}_{36}} \end{bmatrix} \quad \mathbf{r} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_{12} \\ \mathbf{r}_{23} \\ \mathbf{r}_{14} \\ \mathbf{r}_{15} \\ \mathbf{r}_{25} \\ \mathbf{r}_{26} \\ \mathbf{r}_{36} \end{bmatrix}$$

Newton step $\delta \mathcal{X} = -(\mathbf{J}^\top \mathbf{J})^{-1} \mathbf{J}^\top \mathbf{r}$

Update $\mathcal{X} \leftarrow \mathcal{X} \oplus \delta \mathcal{X}$