Chapter 3: Matrices: Exercises 3.1
Book Title: Linear Algebra: A Modern Introduction
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Exercises 3.1

Let $A = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, $D = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}$,
$E = \begin{bmatrix} 4 & 2 \end{bmatrix}$, $F = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

In Exercises 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, and 16, compute the indicated matrices (if possible).

1. $A + 2D$
2. $2D - 5A$
3. $B - C$
4. $B - C^T$
5. AB
6. B^2
7. $D + BC$
8. BB^T
9. $E(AF)$
10. $F(AF)$
11. FE
12. EF
13. $B^T C^T - (CB)^T$
14. $DA - AD$
15. A^3

16. $(I_2 - A)^2$

17. Give an example of a nonzero 2×2 matrix A such that $A^2 = O$.

18. Let $A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$. Find 2×2 matrices B and C such that $AB = AC$ but $B \neq C$.

19. A factory manufactures three products (doohickies, gizmos, and widgets) and ships them to two warehouses for storage. The number of units of each product shipped to each warehouse is given by the matrix

$$A = \begin{bmatrix} 200 & 75 \\ 150 & 100 \\ 100 & 125 \end{bmatrix}$$

(where a_{ij} is the number of units of product *i* sent to warehouse *j* and the products are taken in alphabetical order). The cost of shipping one unit of each product by truck is \$1.50 per doohickey, \$1.00 per gizmo, and \$2.00 per widget. The corresponding unit costs to ship by train are \$1.75, \$1.50, and \$1.00, respectively. Organize these costs into a matrix *B* and then use matrix multiplication to show how the factory can compare the cost of shipping its products to each of the two warehouses by truck and by train.

20. Referring to Exercise 19, suppose that the unit cost of distributing the products to stores is the same for each product but varies by warehouse because of the distances involved. It costs \$0.75 to distribute one unit from warehouse 1 and \$1.00 to distribute one unit from warehouse 2. Organize these costs into a matrix *C* and then use matrix multiplication to compute the total cost of distributing each product.

In Exercises 21 and 22, write the given system of linear equations as a matrix equation of the form $A\mathbf{x} = \mathbf{b}$.

21. $$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 - 5x_3 &= 4 \end{aligned}$$

22. $$\begin{aligned} -x_1 \phantom{{}- x_2{}} + 2x_3 &= 1 \\ x_1 - x_2 \phantom{{}+ x_3{}} &= -2 \\ x_2 + x_3 &= -1 \end{aligned}$$

In Exercises 23, 24, 25, 26, 27, and 28, let

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 1 \\ -1 & 6 & 4 \end{bmatrix}$$

23. Use the matrix-column representation of the product to write each column of AB as a linear combination of the columns of A.

24. Use the row-matrix representation of the product to write each row of AB as a linear combination of the rows of B.

25. Compute the outer product expansion of AB.

26. Use the matrix-column representation of the product to write each column of BA as a linear combination of the columns of B.

27. Use the row-matrix representation of the product to write each row of BA as a linear combination of the rows of A.

28. Compute the outer product expansion of BA.

In Exercises 29 and 30, assume that the product AB makes sense.

29. Prove that if the columns of B are linearly dependent, then so are the columns of AB.

30. Prove that if the rows of A are linearly dependent, then so are the rows of AB.

In Exercises 31, 32, 33, and 34, compute AB by block multiplication, using the indicated partitioning.

31. A equals matrix (3, 4 [1, negative 1, 0, 0, 0, 1, 0, 0, 0, 0, 2, 3]). In the matrix, a horizontal dashed line is

32. A equals matrix (2, 4 [2, 3, 1, 0, 4, 5, 0, 1]). In the matrix, a vertical dashed line is drawn

33. A equals matrix (4, 4 [1, 2, 1, 0, 3, 4, 0, 1, 1, 0, negative 1, 1, 0, 1, 1, negative 1]). In the matrix, a horizontal dashed

34. A equals matrix (4, 4 [1, 0, 0, 1, 0, 1, 0, 2, 0, 0, 1, 3, 0, 0, 0, 4]). In the matrix, a horizontal dashed line is

35. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$.

 a. Compute $A^2, A^3, \ldots, A^7$.

 b. What is A^{2025}? Why?

36. Let $B = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. Find, with justification, B^{2025}.

37. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find a formula for $A^n (n \geq 1)$ and verify your formula using mathematical induction.

38. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

 a. Show that $A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$.

 b. Prove, by mathematical induction, that

$$A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix} \qquad \text{for } n \geq 1$$

39. In each of the following, find the 4×4 matrix $A = [a_{ij}]$ that satisfies the given condition:

 a. $a_{ij} = (-1)^{i+j}$

 b. $a_{ij} = j - i$

 c. $a_{ij} = (i-1)^j$

 d. $a_{ij} = \sin\left(\frac{(i+j-1)\pi}{4}\right)$

40. In each of the following, find the 6×6 matrix $A = [a_{ij}]$ that satisfies the given condition:

a. $a_{ij} = \begin{cases} i+j & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases}$

b. $a_{ij} = \begin{cases} 1 & \text{if } |i-j| \leq 1 \\ 0 & \text{if } |i-j| > 1 \end{cases}$

c. $a_{ij} = \begin{cases} 1 & \text{if } 6 \leq i+j \leq 8 \\ 0 & \text{otherwise} \end{cases}$

41. Prove Theorem 3.1(a).

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