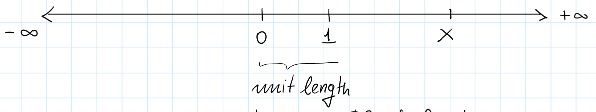
Points and Vectors in Space

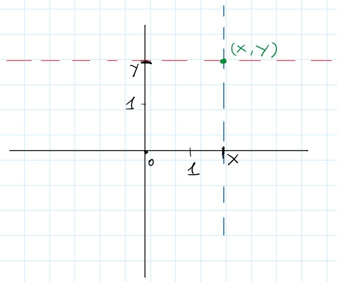
- Fixing a unit length, and a point 0 on a straight line, we can use a real number x to represent any point on the line and vice versa:



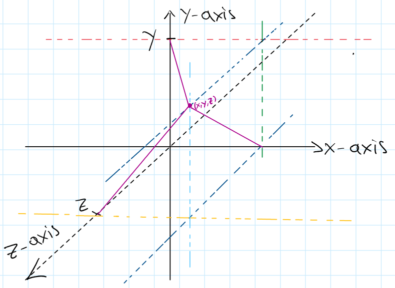
e.g. meter, inch, Planck length, ...

We denote from now on the set of real numbers by $\mathbb{R}$ and we write $x \in \mathbb{R}$ to mean that "x is an element in the set of real numbers", in other words x is a real number.

A pair of real numbers $(x, y) \in \mathbb{R}^2$, where $\mathbb{R}^2$ is the set of pairs of real numbers, represents a point on the plane:



A triple of numbers $(x, y, z) \in \mathbb{R}^3$ represents a point in 3-dimensional space ($\rightsquigarrow$ add another axis to $\mathbb{R}^2$).



$\rightsquigarrow$ Analogously we define a point in an n-dimensional space to be an n-tuple of numbers:

$$(x_1, \ldots, x_n) \in \mathbb{R}^n \quad \text{where } n \in \mathbb{Z}_+ \text{ (positive integer).}$$

Mental Yoga: Lets represent your lecturer by a point, e.g. his center of mass. How would you go about describing his position in the lecture room?

$\Rightarrow$ You could go a step further and think of time as an additional dimension, a fourth dimension in addition to the 3-spatial dimensions.

Example There is no reason to limit ourselves to spatial and time dimension. We can use points on $\mathbb{R}$ to represent any quantity taking values in the real numbers. One could for example assign a real-dimension to each one of the big sectors that contribute in the GDP of the US:

x_1- axis: Finance
x_2- axis: Agriculture
x_3- axis: Health care
x_4- axis: Construction
$\vdots$
x_n - axis: Education
(often put at the bottom of the list by politicians)

Say we denote the point associated to the 2023 contributions by $A \in \mathbb{R}^n$ and the point of 2024 contributions by $B \in \mathbb{R}^n$. Each point in this n-dimensional space encodes the information of all of these sectors individually. We will soon introduce the notion of a vector, which will allow us to compare the contributions between the two years.

Definition 1.1 "Addition of points in $\mathbb{R}^n$"

Let $A = (a_1, \dots, a_n) \in \mathbb{R}^n$, $B = (b_1, \dots, b_n) \in \mathbb{R}^n$ i.e. points in an n-dimensional space, then we define their sum $A+B$ to be the point whose coordinates are $A+B := (a_1+b_1, \dots, a_n+b_n) \in \mathbb{R}^n$ (another point in $\mathbb{R}^n$).

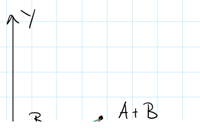
E.g. $A = (-1, \pi, 3) \in \mathbb{R}^3$
$B = (\sqrt{2}, 7, -2) \in \mathbb{R}^3$
then $A+B = (-1+\sqrt{2}, \pi+7, 1) \in \mathbb{R}^3$

One says that addition is defined in a "component-wise" manner. Note that:

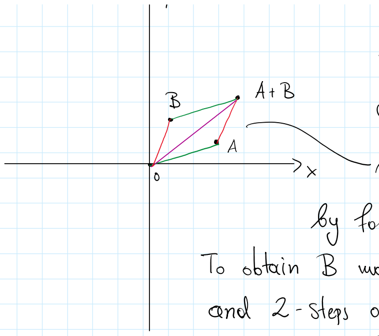
1. $(A+B)+C = A+(B+C)$
2. $A+B = B+A$
3. For $O = (0, \dots, 0)$ we have: $A+O = O+A = A$ for all $A \in \mathbb{R}^n$ (we also write $\forall A \in \mathbb{R}^n$).
4. For $A = (a_1, \dots, a_n)$ we write $-A$ for the point in $\mathbb{R}^n$ such that $A+(-A) = O$, that is $-A = (-a_1, \dots, -a_n)$.

The reason these properties hold, is that they hold for all components individually and since addition in higher-dimensions is defined componentwise, we follow that the properties of addition are true also in higher dimensions.

Now lets see what addition looks like geometrically:



$A = (3,1)$
$B = (1,2)$
$\Big\}$ $A+B = (4,3)$



$$\left.\begin{array}{l} A=(3,1) \\ B=(1,2) \end{array}\right\} A+B=(4,3)$$

$O=(0,0)$

We get a parallelogram by following the coordinates

To obtain B make 1 step on the x-axis and 2-steps on the y. Then to get A+B, start from A instead from O and do again 1 step on x-axis and 2 on y-axis.

Note that since $A+B=B+A$ we can reach the sum of the points in two different ways. Note also that the line segments between O and B, and between A and A+B are the hypotenuses of right triangles whose legs are of the same length and are paralles (red line-segments above)

Definition 1.2: Multiplication of points by scalars.

Let $c \in \mathbb{R}$, also called a scalar, and let $A=(a_1, \dots, a_n) \in \mathbb{R}^n$ then we define the multiplication:

$$c \cdot A := (c \cdot a_1, \dots, c \cdot a_n)$$

That is c multiplies each component of A.

↳ * $c(A+B) = c \cdot A + c \cdot B$
* $c \cdot A = A \cdot c$
* If $c_1, c_2 \in \mathbb{R}$ two scalars, then

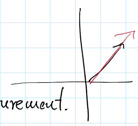
$$(c_1+c_2) \cdot A = c_1 A + c_2 A \qquad \text{(linearity).}$$

and $(c_1 \cdot c_2) \cdot A = c_1 (c_2 \cdot A) = c_2 \cdot (c_1 A)$

↳ Example: Let $A=(1,2)$, and $c=3$ then $cA = (3,6)$.

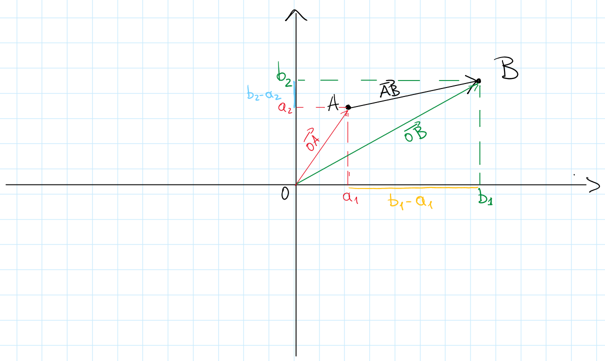
Two ways of thinking about multiplication by a scalar:

1) It streches the line segment
2) It "re-scales" the unit measurement. sending $1 \xrightarrow{\text{to}} c$.



Vectors (Located Vectors).

Definition 1.3 A (located) vector is an ordered pair of points, which we write $\overrightarrow{AB}$ (not a product!)



$A = (a_1, a_2)$

$B = (b_1, b_2)$

Definition: 1.4 : "Equivalent Vectors"

Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be two located vectors. We shall say that they are equivalent or equal if

$$\underline{B - A = D - C}.$$

"Same direction, same length"

Before seeing what this means geometrically, note that every located vector $\overrightarrow{AB}$ is equal to one whose starting point is at the origin, since

$$B - A = (B - A) - O$$

$$\Longleftrightarrow \quad \overrightarrow{AB} = \overrightarrow{O(B-A)}$$

This means that up to equivalence all loc. vectors start at the origin. This is a very important realisation. From now on, when we say the word vector we will implicitly think of a vector with starting point at the origin, unless stated otherwise, or unless is obvious otherwise. (so we drop the terminology "located" for brevity.

Note also that there is a unique vector whose start is at the origin and is equivalent / equal to $\overrightarrow{AB}$, namely the vector $\overrightarrow{O(B-A)}$.

In particular this means that we can describe uniquely any vector up to equivalence with just a single point in $\mathbb{R}^n$. The vector associated to a point $P = (x_1, \dots, x_n)$ is then $\overrightarrow{OP}$ where $O = (0, \dots, 0)$ the origin in the n-dimensional space.

All rules on addition of tupels in $\mathbb{R}^n$ transfer directly to rules on addition of vectors. Similarly with the rules on multiplication by scalars.

↳ Geometric intuition:

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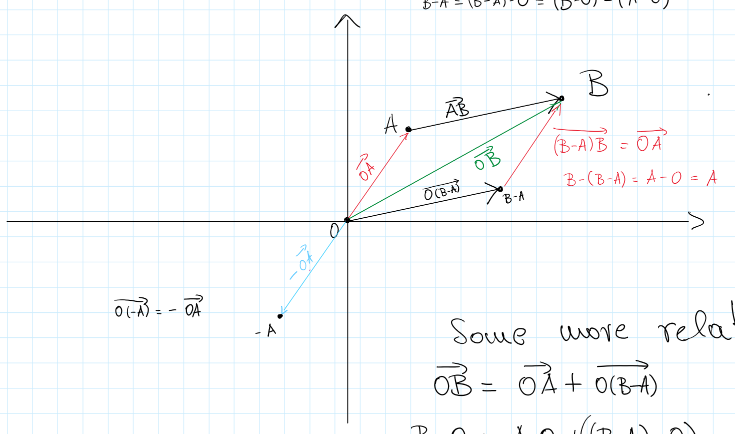
$\overrightarrow{AB} = \overrightarrow{O(B-A)} = \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OB} + (-\overrightarrow{OA})$.

$B-A = (B-A)-O = (B-O)-(A-O)$



Some more relations:

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{O(B-A)}$

$B-O = A-O + ((B-A)-O)$

Example: Please work out the following example:

Let $P = (1,-1,3)$, $Q = (2,4,1)$

$A = (4,-2,5)$, $B = (5,3,3)$

Question: Are the vectors $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ equivalent and if so, why?

Solution: $Q-P = (1,5,-2) = B-A \implies \overrightarrow{PQ} = \overrightarrow{AB}$.

Notation: "From points to vectors"

While for points we used the n-tupel notation $P = (p_1, \dots, p_n) \in \mathbb{R}^n$, when we talk about vectors with start point the origin $O = (0,\dots,0)$ and end point P, i.e. $\overrightarrow{OP}$ we will write:

$$\overrightarrow{OP} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

(or some times you will see also $\overrightarrow{OP} = [p_1, \dots, p_n]$

↳ The reason will become clear later, it's mostly better for computations.

Let me stress these crucial points in other words:

1. To each point $A \in \mathbb{R}^n$, we can associate a unique vector $\overrightarrow{OA}$ that starts from the origin and ends on A.

2. Any vector $\overrightarrow{AB}$ is equivalent to a unique vector $\overrightarrow{O(B-A)}$ that starts at the origin and for which $\overrightarrow{AB} = \overrightarrow{O(B-A)}$ holds. In particular there is a unique vector associated to $\overrightarrow{AB}$

there is a unique vector associated to $\overrightarrow{AB}$ that is "pointing in the same direction and has the same length as $\overrightarrow{AB}$.

3. From (1) and (2), by transitivity, we have a set-theoretic 1:1 - correspondence between:

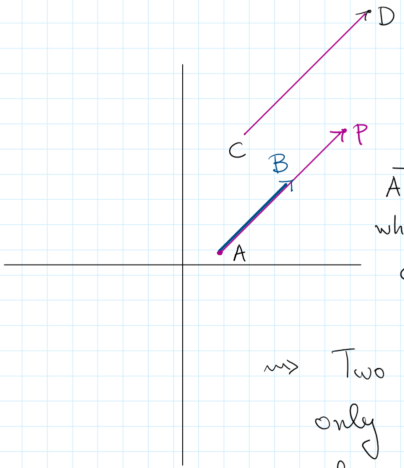
$$\{\text{points in } \mathbb{R}^n\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{Vectors in n-dim} \\ \text{space with starting} \\ \text{point at the origin}\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{vectors in an} \\ \text{n-dim. space} \\ \text{up to equivalence}\end{array}\right\}$$

$$A \in \mathbb{R}^n \longleftrightarrow \overrightarrow{OA}$$

Def. 1.5: "Parallel Vectors":

Two vectors $\overrightarrow{AB}$, $\overrightarrow{PQ}$ are said to be parallel if there is a scalar $c \in \mathbb{R}$ s.t. $B-A = c\cdot(Q-P)$.

Moreover, they are said to have the same direction if $c > 0$, and have the opposite direction if $c < 0$.



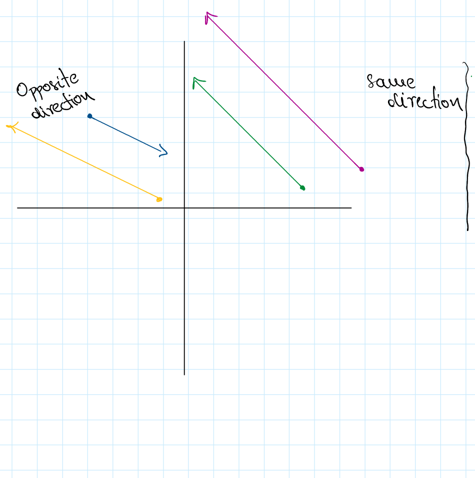
Parallel vectors.

$$\overrightarrow{AP} = c\cdot\overrightarrow{AB} = \overrightarrow{CD}$$

where

$$c = \frac{P-A}{B-A}.$$

⇝ Two vectors are parallel if and only if they are scalar multiples of each other (up to vector-equivalence).



Linear Combinations and Co-ordinates.

Definition 1.6: Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be vectors in an n-dimensional space and let $c_1, \ldots, c_n \in \mathbb{R}$ be scalars.

Then we call the sum:

$$\sum_{i=1}^{n} c_i \cdot v_i = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

a linear combination of the vectors $v_1, \ldots, v_n$.

We call the scalars c_i in a linear combination the coefficients of the linear combination.

We call the set of all linear combinations of the given vectors $v_1, \ldots, v_n$, the linear span of $\{v_1, \ldots, v_n\}$:

$$\mathrm{Span}(v_1, \ldots, v_n) = \{ \lambda_1 v_1 + \cdots + \lambda_n v_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{R} \}.$$

I like to write $\langle v_1, \ldots, v_n \rangle_{\mathbb{R}}$ instead of $\mathrm{Span}(v_1, \ldots, v_n)$

or just $\langle v_1, \ldots, v_n \rangle$.

Examples:

The vector $v = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ is a linear combination of the following vectors: $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix}$ since

$$v = 3 \cdot v_1 + 2 v_2 - 1 \cdot v_3.$$

We also say that the vector v is an element of the span of the set of vectors $\{v_1, v_2, v_3\}$ or "v lies in the span of $\{v_1, v_2, v_3\}$", and write:

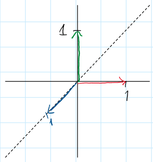
$$v \in \langle v_1, v_2, v_3 \rangle_{\mathbb{R}}$$

So the span $\langle v_1, \ldots, v_n \rangle$ of a set of vectors is the set of all vectors we can obtain by all possible rescalings of $\{v_1, \ldots, v_n\}$ and sums thereof $\rightsquigarrow$ i.e. all possible linear combinations.

Definition 1.7: Let $\{v_1, \ldots, v_n\}$ be a set of vectors in $\mathbb{R}^n$. We say that the set of vectors is linearly dependent if there exist scalars $c_1, \ldots, c_n$ not all zero such that $c_1 \cdot v_1 + \cdots + c_n v_n = 0 = (0, \ldots, 0)$. Otherwise we call $\{v_1, \ldots, v_n\}$ a set of linear independent such that $c_1 \cdot v_1 + \dots + c_n v_n = 0 = (0,\dots,0)$. Otherwise we call $\{v_1,\dots,v_n\}$ a set of linear independent vectors.

Expl. Take the elementary unit vectors in $\mathbb{R}^3$:

$$\left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$



This set is lin. independent, since there is no-non-zero solution to the equation:

$$c_1 \cdot e_1 + c_2 e_2 + c_3 v_3 = 0$$

$\iff c_1 \cdot e_1 + c_2 e_2 = -c_3 \cdot v_3$, solving this equation would mean that I can write $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as a linear combination of e_1, e_2; but this is clearly impossible:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\iff \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -c_3 \end{bmatrix} \iff \begin{bmatrix} c_1 + 0 \\ 0 + c_2 \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -c_3 \end{bmatrix}$$

$$\iff \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -c_3 \end{bmatrix} \iff \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases} \rightsquigarrow c_1, c_2, c_3 \text{ all zero so } c_1, c_2, c_3 \text{ lin Ind.}$$

↳ This is a first instance of a system of linear equations!

↳ We will study this concept in depth later

This following notion of an equivalence relation we didn't define in class and I don't expect you to know it, but If you really want to know what the term "equivalence" means, for example equivalence of vectors or "equality up to equivalence", well here is what the definition of an equivalence is... We might do it properly down the line if we decide to talk about congruence relations and modular arithmetic which is extremely important for computer scientists.

Def. Equivalence Relation:

Is a binary relation "~" on a set $\mathcal{U}$ characterised by the following properties For all $a, b, c \in \mathcal{U}$ we have:

i) $a \sim a$ "reflexivity)

ii) $a \sim b \iff b \sim a$ "symmetry"

iii) If $a \sim b$ and $b \sim c$ then $a \sim c$.

↳ we define the equivalence class of an element a in $\mathcal{U}$

↳ we define the equivalence class of an element a in $\mathcal{U}$ with respect to $\sim$ as:

$[a] := \{ x \in \mathcal{U} \mid x \sim a \}$.

i.e. (all elements in $\mathcal{U}$ that are equivalent to a.

⇝ Leads to the concept of an Affine Space.