Chapter 3: Matrices: Exercises 3.3
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Exercises 3.3

In Exercises 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10, find the inverse of the given matrix (if it exists) using Theorem 3.8.

1. $\begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

5. $\begin{bmatrix} \frac{3}{4} & \frac{3}{5} \\ \frac{5}{6} & \frac{2}{3} \end{bmatrix}$

6. $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

7. $\begin{bmatrix} -1.5 & -4.2 \\ 0.5 & 2.4 \end{bmatrix}$

8. $\begin{bmatrix} 3.55 & 0.25 \\ 8.52 & 0.60 \end{bmatrix}$

9. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

10. $\begin{bmatrix} 1/a & 1/b \\ 1/c & 1/d \end{bmatrix}$, where neither a, b, c, nor d is 0

In Exercises 11 and 12, solve the given linear system using the method of Example 3.25.

11. $$\begin{aligned} 2x + y &= -1 \\ 5x + 3y &= 2 \end{aligned}$$

12. $$\begin{aligned} x_1 - x_2 &= 2 \\ x_1 + 2x_2 &= 5 \end{aligned}$$

13. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and $\mathbf{b}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

   a. Find A^{-1} and use it to solve the three systems $A\mathbf{x} = \mathbf{b}_1$, $A\mathbf{x} = \mathbf{b}_2$, and $A\mathbf{x} = \mathbf{b}_3$.

   b. Solve all three systems at the same time by row reducing the augmented matrix $[A \mid \mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3]$ using Gauss-Jordan elimination.

   c. Carefully count the total number of individual multiplications that you performed in (a) and in (b). You should discover that, even for this 2×2 example, one method uses fewer operations.

   For larger systems, the difference is even more pronounced, and this explains why computer systems do not use one of these methods to solve linear systems.

14. Prove Theorem 3.9(b).

15. Prove Theorem 3.9(d).

16. Prove that the $n \times n$ identity matrix I_n is invertible and that $I_n^{-1} = I_n$.

17. a. Give a counterexample to show that $(AB)^{-1} \neq A^{-1}B^{-1}$ in general.

    b. Under what conditions on A and B is $(AB)^{-1} = A^{-1}B^{-1}$? Prove your assertion.

18. By induction, prove that if $A_1, A_2, \ldots, A_n$ are invertible matrices of the same size, then the product $A_1A_2 \cdots A_n$ is invertible and $(A_1A_2 \cdots A_n)^{-1} = A_n^{-1} \cdots A_2^{-1}A_1^{-1}$.

19. Give a counterexample to show that $(A + B)^{-1} \neq A^{-1} + B^{-1}$ in general.

In Exercises 20, 21, 22, and 23, solve the given matrix equation for *X*. Simplify your answers as much as possible. (In the words of Albert Einstein, “Everything should be made as simple as possible, but not simpler.”) Assume that all matrices are invertible.

20. $XA^{-1} = A^3$

21. $AXB = (BA)^2$

22. $(A^{-1}X)^{-1} = (AB^{-1})^{-1}(AB)^2$

23. $ABXA^{-1}B^{-1} = I + A$

In Exercises 24, 25, 26, 27, 28, 29, and 30, let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -1 & 3 \\ 2 & 1 & -1 \end{bmatrix}$$

In each case, find an elementary matrix E that satisfies the given equation.

24. $EA = B$

25. $EB = A$

26. $EA = C$

27. $EC = A$

28. $EC = D$

29. $ED = C$

30. Is there an elementary matrix E such that $EA = D$? Why or why not?

In Exercises 31, 32, 33, 34, 35, 36, 37, and 38, find the inverse of the given elementary matrix.

31. $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

32. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

33. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

34. $\begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$

35. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

36. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

37. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}, c \neq 0$

38. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, c \neq 0$

In Exercises 39 and 40, find a sequence of elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \cdots E_2 E_1 A = I$. Use this sequence to write both *A* and A^{-1} as products of elementary matrices.

39. $A = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}$

40. $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$

41. Prove Theorem 3.13 for the case of $AB = I$.

42. a. Prove that if *A* is invertible and $AB = O$, then $B = O$.

    b. Give a counterexample to show that the result in part (a) may fail if *A* is not invertible.

43. a. Prove that if *A* is invertible and $BA = CA$, then $B = C$.

    b. Give a counterexample to show that the result in part (a) may fail if *A* is not invertible.

44. A square matrix *A* is called **idempotent** if $A^2 = A$. (The word ***idempotent*** comes from the Latin *idem*, meaning “same,” and *potere*, meaning “to have power.” Thus, something that is idempotent has the “same power” when squared.)

    a. Find three idempotent 2×2 matrices.

b. Prove that the only invertible idempotent $n \times n$ matrix is the identity matrix.

45. Show that if A is a square matrix that satisfies the equation $A^2 - 2A + I = O$, then $A^{-1} = 2I - A$.

46. Prove that if a symmetric matrix is invertible, then its inverse is symmetric also.

47. Prove that if A and B are square matrices and AB is invertible, then both A and B are invertible.

In Exercises 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, and 63, use the Gauss-Jordan method to find the inverse of the given matrix (if it exists).

48. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

49. $\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$

50. $\begin{bmatrix} 3 & -4 \\ -6 & 8 \end{bmatrix}$

51. $\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}$

52. $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 5 & 1 \\ 2 & 3 & 0 \end{bmatrix}$

53. $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 2 \\ 2 & 3 & -1 \end{bmatrix}$

54. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

55. $\begin{bmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 1 & a \end{bmatrix}$

56. $\begin{bmatrix} 0 & a & 0 \\ b & 0 & c \\ 0 & d & 0 \end{bmatrix}$

57. $\begin{bmatrix} 0 & -1 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 1 & -1 & 3 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

58. $\begin{bmatrix} \sqrt{2} & 0 & 2\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$

59. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{bmatrix}$

60. $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \text{over } \mathbb{Z}_2$

61. $\begin{bmatrix} 4 & 2 \\ 3 & 4 \end{bmatrix} \text{over } \mathbb{Z}_5$

62. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \text{over } \mathbb{Z}_3$

63. $\begin{bmatrix} 1 & 5 & 0 \\ 1 & 2 & 4 \\ 3 & 6 & 1 \end{bmatrix} \text{over } \mathbb{Z}_7$

Partitioning large square matrices can sometimes make their inverses easier to compute, particularly if the blocks have a nice form. In Exercises 64, 65, 66, 67, and 68, verify by block multiplication that the inverse of a matrix, if partitioned as shown, is as claimed. (Assume that all inverses exist as needed.)

64. $\begin{bmatrix} A & B \\ O & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ O & D^{-1} \end{bmatrix}$

65. $\begin{bmatrix} O & B \\ C & I \end{bmatrix}^{-1} = \begin{bmatrix} -(BC)^{-1} & (BC)^{-1}B \\ C(BC)^{-1} & I - C(BC)^{-1}B \end{bmatrix}$

66. $\begin{bmatrix} I & B \\ C & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - BC)^{-1} & -(I - BC)^{-1}B \\ -C(I - BC)^{-1} & I + C(I - BC)^{-1}B \end{bmatrix}$

67. $\begin{bmatrix} O & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} -\left(BD^{-1}C\right)^{-1} & \left(BD^{-1}C\right)^{-1}BD^{-1} \\ D^{-1}C\left(BD^{-1}C\right)^{-1} & D^{-1} - D^{-1}C\left(BD^{-1}C\right)^{-1}BD^{-1} \end{bmatrix}$

68. $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$, where $P = \left(A - BD^{-1}C\right)^{-1}$, $Q = -PBD^{-1}$, $R = -D^{-1}CP$, and $S = D^{-1} + D^{-1}CPBD^{-1}$

In Exercises 69, 70, 71, and 72, partition the given matrix so that you can apply one of the formulas from Exercises 64, 65, 66, 67, and 68, and then calculate the inverse using that formula.

69. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

70. The matrix in $\begin{bmatrix} \sqrt{2} & 0 & 2\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$

71. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

72. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & 1 \\ -1 & 5 & 2 \end{bmatrix}$