1.2. Length and Angle. The Dot Product. Projections.

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In this lecture we will introduce the operation on pairs of vectors in $\mathbb{R}^n$, that will allow us to define lengths of vectors, but more importantly it will allow us to check whether two vectors are "orthogonal" to each other.

Recall Parallelism:

Transpose $A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $A^T = [a_1, \dots a_n]$.

Def. 1.? Let $A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ be two vectors in $\mathbb{R}^n$,

then we define the Scalar Product of A with B by (dot product).

$$A \cdot B = a_1 b_1 + a_2 \cdot b_2 + \dots + a_n \cdot b_n \in \mathbb{R}$$

$$= A^T \cdot B = [a_1, a_2, \dots, a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}.$$

↳ Thats the sum of the products of the components of u and v.

! The scalar product of two vectors is, as the name might suggest a scalar, i.e. a real number.

Example: $u = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$, $v = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$

Compute $u \cdot v = 1 \cdot (-3) + 2 \cdot 5 + (-3) \cdot 2 = 1$

$= v \cdot u$

Lets go back to our favourite $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$e_1^T \cdot e_1 = [1\ 0\ 0] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1$$

$$e_1^T \cdot e_2 = [1\ 0\ 0] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$e_i \cdot e_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Properties of the scalar product:

Let $v, u, w \in \mathbb{R}^n$ vectors and $c \in \mathbb{R}$. Then:

* $u \cdot v = v \cdot u$
* $u(v+w) = uv + uw$
* $(c\, u) \cdot v = c(u \cdot v)$
* $u \cdot u \geq 0$
* $u \cdot u = 0 \iff u = 0.$

Now using this gadjet we define the notion of length of a vector:

Def: The length of a vector $A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$ is defined by:

$$\|A\| = \sqrt{A \cdot A} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \in \mathbb{R}_{\geq 0}$$

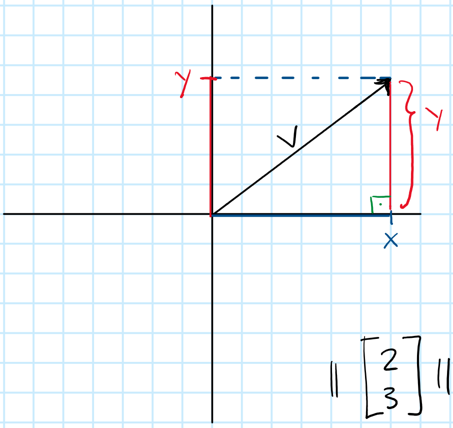
since $A \cdot A \in \mathbb{R}_{\geq 0}$,

[illegible]

since $A \cdot A \in \mathbb{R}_{\geq 0}$, in particular the square root of $A \cdot A$ is real (not complex).

⇝ How did we come up w/ this definition?

Let see this in 2-dimension: $v = \begin{bmatrix} x \\ y \end{bmatrix}$



$\|v\| = \sqrt{v \cdot v} = \sqrt{x^2 + y^2}$

"Pythagorean Thm."

$\left\| \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\| = \sqrt{2^2 + 3^2} = \sqrt{13}$.

Q: How about in 1-dim: $c \in \mathbb{R}$, $\|c\| = \sqrt{c^2} = |c|$.

Exercise:

Prove that 1. $\|v\| = 0 \iff v = 0$

2. $\|c \cdot v\| = |c| \cdot \|v\|$

Def: * A vector $v \in \mathbb{R}^n$ is called a unit vector iff $\|v\| = 1$.

* We define the normalisation of a vector as:

$\frac{v}{\|v\|}$, this is a unit vector pointing in the direction of v.

↳ The vectors $e_1, \ldots, e_n$; $e_i = \begin{array}{l} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \leftarrow$ i-th component

are a special case of unit vectors. We call them standard (orthogonal) unit vectors.

Example: Normalise $v = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, $\|v\| = \sqrt{14}$

$$\frac{v}{\|v\|} = \frac{1}{\sqrt{14}} \cdot \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Ex. (Proposition): Cauchy-Schwarz inequality.

Show: $|u \cdot v| \leq \|u\| \cdot \|v\|$ for $u, v \in \mathbb{R}^n$

In 2-dimensions:

$$|u \cdot v|^2 = (u_1 v_1 + u_2 v_2)^2 \leq (u_1^2 + u_2^2)(v_1^2 + v_2^2) = |u|^2 \cdot |v|^2$$

In n-dim:

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 - \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{(u_i v_j - u_j v_i)^2}$$

$$\geq 0.$$

clearly

since we are adding squares.

Proposition: The triangle inequality:

$$\|u + v\| \leq \|u\| + \|v\|$$

Pf:
$$\begin{aligned} \|u+v\|^2 &= (u+v) \cdot (u+v) \\ &= u \cdot u + 2(u \cdot v) + v \cdot v \\ &\leq \|u\|^2 + 2|u \cdot v| + \|v\|^2 \\ &\leq \|u\|^2 + 2\|u\| \cdot \|v\| + \|v\|^2 \\ &= (\|u\| + \|v\|)^2 \end{aligned}$$

since $\underbrace{u \cdot v}_{\text{can be negative}} \leq |u \cdot v|$

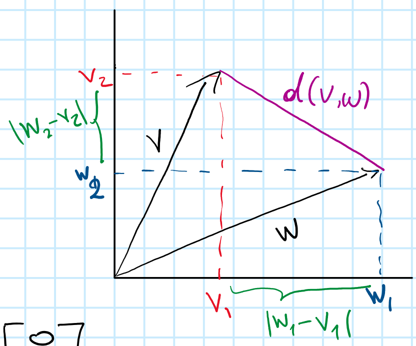
Cauchy-Schwarz.

-Schwarz ✓

$$= (\|u\| + \|v\|)^2.$$

Distance between two vectors.

Geometric intuition:



Def:

$$d(v,w) := \|v-w\|$$
$$= \sqrt{(w_1-v_1)^2 + (w_2-v_2)^2}.$$

Example: $u = \begin{bmatrix} \sqrt{2} \\ 1 \\ -1 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$

↳ asked to be done in class.

$$u - v = \begin{bmatrix} \sqrt{2} \\ -1 \\ 1 \end{bmatrix}$$

$$d(u,v) = \|u-v\| = \sqrt{(\sqrt{2})^2 + (-1)^2 + 1} = \sqrt{4} = 2.$$