Finite Automata 07

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Example

Show that $C = \{w \mid w$ has an equal number of 0s and 1s$\}$ is not regular.

- Assume that C is regular
- Since C is regular, the Pumping lemma says that for any string $s \in C$ of length at least p, s can be divided into $s = xyz$ satisfying the following conditions:
  1. $xy^iz \in C$ for any $i \geq 0$
  2. $|y| > 0$
  3. $|xy| \leq p$
- Let $s = (01)^p$
  - $s \in C$ — good
  - $|s| = 2p \geq p$ — good

Example

Show that $C = \{w \mid w \text{ has an equal number of 0s and 1s}\}$ is not regular.

- $s = (01)^p$
- Let $x = \varepsilon$, $y = 01$, and $z = (01)^{p-1}$ and check all three conditions:
  1. $\varepsilon(01)^i(01)^{p-1} \in C$ for any $i \geq 0$
     - Every time you insert a y, you add equal number of 0 and 1
  2. $|01| = 2 > 0$
  3. $|\varepsilon 01| = 2 \leq p$

  All three condition can be true (no contradiction)
- **Important**: No contradiction means nothing
  - You also cannot conclude that C is regular
- But if we pick $s = 0^p1^p$, we will get a contradiction
  - Same kind of proof as in previous example but focus on the number of 0s and 1s (no pattern)

Show that $C = \{w \mid w$ has an equal number of 0s and 1s$\}$ is not regular

Assume that C is regular. Since C is regular, the Pumping lemma says that for any string $s \in C$ of length at least p, s can be divided into $s = xyz$ satisfying the following conditions:

1. $xy^iz \in C$ for any $i \geq 0$
2. $|y| > 0$
3. $|xy| \leq p$

Let $s = 0^p1^p$. Since s starts with p 0s, to satisfy the third condition, x and y are strings that contain nothing but 0s. In other words, $x = 0^j$ for any $j \geq 0$, and $y = 0^k$ for any $k > 0$. Note that k must be greater than 0 because $|y| = |0^k| = k$, and the condition 2 says that $|y| > 0$. Since $x = 0^j$ and $y = 0^k$, $z = 0^{p-(j+k)}1^p$. Let $i = 2$. We have

$$\begin{aligned}
xy^iz &= xy^2z \\
&= xyyz \\
&= 0^j0^k0^k0^{p-(j+k)}1^p \\
&= 0^{p+k}1^p
\end{aligned}$$

For the string $0^{p+k}1^p$ to be in C, the number of 0s must be equal to the number of 1s. In other words, $p + k$ must be equal to p. This requires k to be 0. But since k must be greater than 0, $xy^2z \notin B$ — contradiction. Therefore, C is not regular.

Example

Show that $D = \{w \mid w$ has more number of 0s than number of 1s$\}$ is not regular.

- As usual, assume that D is regular and followed by the statement from the Pumping lemma
- If you pick $s = 0^{2p}1^p$, you will not get a contradiction
  - $x = 0^j$ for any $j \geq 0$
  - $y = 0^k$ for any $k > 0$
  - $z = 0^{2p-(j+k)}1^p$
  - For $i \geq 2$ in xy^iz, you add more 0s which makes the result string still have more 0s than 1s
  - For $i = 0$
    - $xy^0z = xz = 0^j0^{2p-(j+k)}1^p = 0^{2p-k}1^p$
    - If k is 1, $2p - 1 > p$ (the number of 0s is greater than the number of 1s) — no contradiction
    - If k is $p - 1$, $2p - (p - 1) = p + 1 > p$ — no contradiction
    - Note that $0 < k \leq p$ from (2) and (3)
    - If there is a k that works, no contradiction

Example

- Let's try $s = 1^p 0^{2p}$. Note that $s \in D$ and $|s| \geq p$.
  - Again, to satisfy (2) and (3), we have
    - $x = 1^j$ for any $j \geq 0$
    - $y = 1^k$ for any $k > 0$
    - $z = 1^{p-(j+k)} 0^{2p}$
  - We have $xy^iz = 1^j(1^k)^i 1^{p-(j+k)} 0^{2p} = 1^{p+ki-k} 0^{2p}$
  - If we increase i, we increase the number of 1s
  - To get a contradiction, we need the number of 1s to be greater than or equal to the number of 0s
    - In other words, $p + ki - k \geq 2p$

$$
\begin{aligned}
p + ki - k &\geq 2p \\
ki - k &\geq p \\
k(i-1) &\geq p \\
i - 1 &\geq p/k \\
i &\geq (p/k) + 1
\end{aligned}
$$

- $xy^{(p/k)+1}z = 1^{2p}0^{2p} \notin D$ — contradiction

Show that $D = \{w \mid w$ has more number of 0s than number of 1s$\}$ is not regular

Assume that D is regular. Since D is regular, the Pumping lemma says that for any string $s \in D$ of length at least p, s can be divided into $s = xyz$ satisfying the following conditions:

1. $xy^iz \in D$ for any $i \geq 0$
2. $|y| > 0$
3. $|xy| \leq p$

Let $s = 1^p0^{p+1}$. Since s starts with p 1s, to satisfy the third condition, x and y are strings that contain nothing but 1s. In other words, $x = 1^j$ for any $j \geq 0$, and $y = 1^k$ for any $k > 0$. Note that k must be greater than 0 because $|y| = |1^k| = k$, and the condition 2 says that $|y| > 0$. Since $x = 1^j$ and $y = 1^k$, $z = 1^{p-(j+k)}0^{p+1}$. Let $i = 2$. We have

$$\begin{aligned} xy^iz = xy^2z &= xyyz \\ &= 1^j1^k1^k1^{p-(j+k)}0^{p+1} \\ &= 1^{p+k}0^{p+1} \end{aligned}$$

For the string $1^{p+k}0^{p+1}$ to be in D, the number of 0s must be greater than the number of 1s. In other words, $p+1$ must be greater than $p+k$. This requires k to be 0. But since k must be greater than 0, $xy^2z \notin D$ — contradiction. Therefore, D is not regular.

Rule of Thumb

- If a condition of the language is about inequality ($<$, $\leq$, $>$, $\geq$), pick a string that is right at the border line to break the condition
  - $D = \{w \mid w$ has more number of 0s than number of 1s$\}$
    - 1^p0^{p+1} needs xy^2z to add at least one 1
    - 0^p1^{p-1} needs xy^0z to take out at least one 0
    - No need to find a large value of i

Example

Show that $E = \{0^{(i^2)} \mid i \geq 0\}$ is not regular.

- Let's try to understand this language first
  - If $i = 0$, $0^{(0^2)} = 0^0 = \varepsilon$
  - If $i = 1$, $0^{(1^2)} = 0^1 = 0$
  - If $i = 2$, $0^{(2^2)} = 0^4 = 0000$
  - If $i = 3$, $0^{(3^2)} = 0^9 = 000000000$, and so on
- Thus, we have

$$\begin{aligned} E = \{w \mid\ & w \text{ contains nothing but 0s and} \\ & \text{the number of 0s is } i^2 \text{ for some } i \geq 0\} \end{aligned}$$

- **Important**: You cannot pick $s = 0^p$
  - There is nothing to guarantee that $p = i^2$ for some $i \geq 0$
  - We need to pick $s = 0^{(p^2)}$

Example

Show that $E = \{0^{i^2} \mid i \geq 0\}$ is not regular.

- As usual, assume that E is regular and followed by the statement from the Pumping lemma
- Let $s = 0^{(p^2)}$
  - Note that $s \in E$ and
  - $|s| = p^2 \geq p$.
- From the second and third conditions ($|y| > 0$ and $|xy| \leq p$), we have

$$0 < |y| \leq p$$

- Note that since $s = xyz$ and $|s| = p^2$, $|xyz| = p^2$.

Example

- Let's do some analysis about xy^2z
  - $p^2 = |xyz|$
  - $|xyz| < |xy^2z| = |xyyz|$ because $|y| > 0$
  - $|xyyz| = |xyz| + |y| = p^2 + |y|$
  - $p^2 + |y| \leq p^2 + p$ because $|y| \leq p$
  - $p^2 + p < p^2 + 2p + 1 = (p+1)^2$
- Note that the string xy^2z can in E if $|xy^2z| = q^2$ for some q
- Above analysis shows that $p^2 < |xy^2z| < (p+1)^2$
- But there is no q such that $p^2 < q^2 < (p+1)^2$
- Thus, $xy^2z \notin E$ — contradiction
- E is not regular