Chapter 3: Matrices: Exercises 3.4
Book Title: Linear Algebra: A Modern Introduction
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Exercises 3.4

In Exercises 1, 2, 3, 4, 5, and 6, solve the system $A\mathbf{x} = \mathbf{b}$ using the given LU factorization of A.

1. $A = \begin{bmatrix} -2 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 0 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 1 & -2 \\ -2 & 3 & -4 \\ 4 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -\frac{5}{4} & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & -2 \\ 0 & 4 & -6 \\ 0 & 0 & -\frac{7}{2} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & -4 & 0 \\ 3 & -1 & 4 \\ -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$

5. $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 6 & -4 & 5 & -3 \\ 8 & -4 & 1 & 0 \\ 4 & -1 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 2 & -1 & 5 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & -1 & 5 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 4 & 3 & 0 \\ -2 & -5 & -1 & 2 \\ 3 & 6 & -3 & -4 \\ -5 & -8 & 9 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$

$\mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ -1 \\ 0 \end{bmatrix}$

In Exercises 7, 8, 9, 10, 11, and 12, find an LU factorization of the given matrix.

7. $\begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix}$

8. $\begin{bmatrix} 2 & -4 \\ 3 & 1 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 7 & 9 \end{bmatrix}$

10. $\begin{bmatrix} 2 & 2 & -1 \\ 4 & 0 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 6 & 3 & 0 \\ 0 & 6 & -6 & 7 \\ -1 & -2 & -9 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 2 & 2 & 2 & 1 \\ -2 & 4 & -1 & 2 \\ 4 & 4 & 7 & 3 \\ 6 & 9 & 5 & 8 \end{bmatrix}$

Generalize the definition of LU factorization to nonsquare matrices by simply requiring U to be a matrix in row echelon form. With this modification, find an LU factorization of the matrices in Exercises 13 and 14.

13. $\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 3 & 3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ -2 & -7 & 3 & 8 & -2 \\ 1 & 1 & 3 & 5 & 2 \\ 0 & 3 & -3 & -6 & 0 \end{bmatrix}$

For an invertible matrix with an LU factorization $A = LU$, both L and U will be invertible and $A^{-1} = U^{-1}L^{-1}$. In Exercises 15 and 16, find L^{-1}, U^{-1}, and

A^{-1} for the given matrix.

15. A in $A = \begin{bmatrix} -2 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & 6 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

16. A in $A = \begin{bmatrix} 2 & -4 & 0 \\ 3 & -1 & 4 \\ -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$

The inverse of a matrix can also be computed by solving several systems of equations using the method of Example 3.34. For an $n \times n$ matrix *A*, to find its inverse we need to solve $AX = I_n$ for the $n \times n$ matrix *X*. Writing this equation as $A[\mathbf{x}_1 \quad \mathbf{x}_2 \cdots \mathbf{x}_n] = [\mathbf{e}_1 \quad \mathbf{e}_2 \cdots \mathbf{e}_n]$, using the matrix-column form of AX, we find that we need to solve *n* systems of linear equations: $A\mathbf{x}_1 = \mathbf{e}_1, A\mathbf{x}_2 = \mathbf{e}_2, \ldots, A\mathbf{x}_n - \mathbf{e}_n$. Moreover, we can use the factorization $A = LU$ to solve each one of these systems.

In Exercises 17 and 18, use the approach just outlined to find A^{-1} for the given matrix. Compare with the method of Exercises 15 and 16.

17. A in $A = \begin{bmatrix} -2 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & 6 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

18. A in $A = \begin{bmatrix} 2 & -4 & 0 \\ 3 & -1 & 4 \\ -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$

In Exercises 19, 20, 21, and 22, write the given permutation matrix as a product of elementary (row interchange) matrices.

19. $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

20. $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

21. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

22. $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 23, 24, and 25, find a $P^T LU$ factorization of the given matrix A.

23. $A = \begin{bmatrix} 0 & 1 & 4 \\ -1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix}$

24. $A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ -1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$

25. $A = \begin{bmatrix} 0 & -1 & 1 & 3 \\ -1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

26. Prove that there are exactly $n!$ $n \times n$ permutation matrices.

In Exercises 27 and 28, solve the system $A\mathbf{x} = \mathbf{b}$ using the given factorization $A = P^T LU$. Because $PP^T = I$, $P^T LU\mathbf{x} = \mathbf{b}$ can be rewritten as $LU\mathbf{x} = P\mathbf{b}$. This system can then be solved using the method of Example 3.34.

27. $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -\frac{5}{2} \end{bmatrix} = P^T LU,$

$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$

28. $A = \begin{bmatrix} 8 & 3 & 5 \\ 4 & 1 & 2 \\ 4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = P^T LU,$

$\mathbf{b} = \begin{bmatrix} 16 \\ -4 \\ 4 \end{bmatrix}$

29. Prove that a product of unit lower triangular matrices is unit lower triangular.

30. Prove that every unit lower triangular matrix is invertible and that its inverse is also unit lower triangular.

An ***LDU* factorization** of a square matrix A is a factorization $A = LDU$, where L is a unit lower triangular matrix, D is a diagonal matrix, and U is a unit upper triangular matrix (upper triangular with 1s on its diagonal). In Exercises 31 and 32, find an LDU factorization of A.

31. A in $A = \begin{bmatrix} -2 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

32. A in $A = \begin{bmatrix} 2 & -4 & 0 \\ 3 & -1 & 4 \\ -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$

33. If A is symmetric and invertible and has an LDU factorization, show that $U = L^T$.

34. If A is symmetric and invertible and $A = LDL^T$ (with L unit lower triangular and D diagonal), prove that this factorization is unique. That is, prove that if we also have $A = L_1 D_1 L_1^T$ (with L_1 unit lower triangular and D_1 diagonal), then $L = L_1$ and $D = D_1$.

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