Chapter 3: Matrices: Exercises 3.5
Book Title: Linear Algebra: A Modern Introduction
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Exercises 3.5

In Exercises 1, 2, 3, and 4, let S be the collection of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ that satisfy the given property. In each case, either prove that S forms a subspace of $\mathbb{R}^2$ or give a counterexample to show that it does not.

1. $x = 0$
2. $x \geq 0, y \geq 0$
3. $y = 2x$
4. $xy \geq 0$

In Exercises 5, 6, 7, and 8, let S be the collection of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ that satisfy the given property. In each case, either prove that S forms a subspace of $\mathbb{R}^3$ or give a counterexample to show that it does not.

5. $x = y = z$
6. $z = 2x, y = 0$
7. $x - y + z = 1$
8. $|x - y| = |y - z|$
9. Prove that every line through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.
10. Suppose S consists of all points in $\mathbb{R}^2$ that are on the x-axis or the y-axis (or both). (S is called the *union* of the two axes.) Is S a subspace of $\mathbb{R}^2$? Why or why not?

In Exercises 11 and 12, determine whether **b** is in col(A) and whether **w** is in row(A), as in Example 3.41.

11. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 0 \\ 3 & -1 & -5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{w} = [1 \quad -3 \quad -3]$

13. In Exercise 11, determine whether $\mathbf{w}$ is in row(A), using the method described in the Remark following Example 3.41.

14. In Exercise 12, determine whether $\mathbf{w}$ is in row(A), using the method described in the Remark following Example 3.41.

15. If A is the matrix in Exercise 11, is $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$ in null(A)?

16. If A is the matrix in Exercise 12, is $\mathbf{v} = \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix}$ in null(A)?

In Exercises 17, 18, 19, and 20, give bases for row(A), col(A), and null(A).

17. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

18. $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 5 & 1 \\ 1 & -1 & -2 \end{bmatrix}$

19. $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$

20. $A = \begin{bmatrix} 3 & -6 & -1 & 0 & -2 \\ 2 & -4 & 0 & 2 & 1 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}$

In Exercises 21, 22, 23, and 24, find bases for row(A) and col(A) in the given exercises using A^T.

21. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

22. $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 5 & 1 \\ 1 & -1 & -2 \end{bmatrix}$

23. $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$

24. $$A = \begin{bmatrix} 3 & -6 & -1 & 0 & -2 \\ 2 & -4 & 0 & 2 & 1 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}$$

25. Explain carefully why your answers to Exercises 17 and 21 are both correct even though there appear to be differences.

26. Explain carefully why your answers to Exercises 18 and 22 are both correct even though there appear to be differences.

In Exercises 27, 28, 29, and 30, find a basis for the span of the given vectors.

27. $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

28. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

29. $[2 \quad -3 \quad 1], [1 \quad -1 \quad 0], [4 \quad -4 \quad 1]$

30. $[3 \quad 1 \quad -1 \quad 0], [0 \quad -1 \quad 2 \quad -1], [4 \quad 3 \quad 8 \quad 3]$

For Exercises 31 and 32, find bases for the spans of the vectors in the given exercises from among the vectors themselves.

31. $[2 \quad -3 \quad 1], [1 \quad -1 \quad 0], [4 \quad -4 \quad 1]$

32. $[3 \quad 1 \quad -1 \quad 0], [0 \quad -1 \quad 2 \quad -1], [4 \quad 3 \quad 8 \quad 3]$

33. Prove that if R is a matrix in echelon form, then a basis for row(R) consists of the nonzero rows of R.

34. Prove that if the columns of A are linearly independent, then they must form a basis for col(A).

For Exercises 35, 36, 37, and 38, give the rank and the nullity of the matrices in the given exercises.

35. $$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

36. $$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 5 & 1 \\ 1 & -1 & -2 \end{bmatrix}$$

37. $$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

38. $$A = \begin{bmatrix} 3 & -6 & -1 & 0 & -2 \\ 2 & -4 & 0 & 2 & 1 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}$$

39. If A is a 3×5 matrix, explain why the columns of A must be linearly dependent.

40. If A is a 4×2 matrix, explain why the rows of A must be linearly dependent.

41. If A is a 3×5 matrix, what are the possible values of nullity(A)?

42. If A is a 4×2 matrix, what are the possible values of nullity(A)?

In Exercises 43 and 44, find all possible values of rank(A) as a varies.

43. $$A = \begin{bmatrix} 1 & 2 & a \\ -2 & 4a & 2 \\ a & -2 & 1 \end{bmatrix}$$

44. $$A = \begin{bmatrix} a & 2 & -1 \\ 3 & 3 & -2 \\ -2 & -1 & a \end{bmatrix}$$

Answer Exercises 45, 46, 47, and 48 by considering the matrix with the given vectors as its columns.

45. Do $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{R}^3$?

46. Do $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ form a basis for $\mathbb{R}^3$?

47. Do $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{R}^4$?

48. Do $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ form a basis for $\mathbb{R}^4$?

49. Do $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{Z}_2^3$?

50. Do $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{Z}_3^3$?

In Exercises 51 and 52, show that **w** is in $\mathrm{span}(\mathcal{B})$ and find the coordinate vector $[\mathbf{w}]_{\mathcal{B}}$.

51. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}, \mathbf{w} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$

52. $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} \right\}, \mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

In Exercises 53, 54, 55, and 56, compute the rank and nullity of the given matrices over the indicated $\mathbb{Z}_p$.

53. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ over $\mathbb{Z}_2$

54. $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 0 & 0 \end{bmatrix}$ over $\mathbb{Z}_3$

55. $\begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 4 & 0 \end{bmatrix}$ over $\mathbb{Z}_5$

56. $\begin{bmatrix} 2 & 4 & 0 & 0 & 1 \\ 6 & 3 & 5 & 1 & 0 \\ 1 & 0 & 2 & 2 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ over $\mathbb{Z}_7$

57. If A is $m \times n$, prove that every vector in null(A) is orthogonal to every vector in row(A).

58. If A and B are $n \times n$ matrices of rank n, prove that AB has rank n.

59.

(a) Prove that $\mathrm{rank}(AB) \leq \mathrm{rank}(B)$. [**Hint:** Review Exercise 29 in Section 3.1.]

(b) Give an example in which $\mathrm{rank}(AB) < \mathrm{rank}(B)$.

60.

(a) Prove that $\mathbf{rank}(AB) \leq \mathbf{rank}(A)$. [**Hint:** Review Exercise 30 in Section 3.1 or use transposes and Exercise 59(a).]

(b) Give an example in which $\mathbf{rank}(AB) < \mathbf{rank}(A)$.

61.

(a) Prove that if *U* is invertible, then $\mathbf{rank}(UA) = \mathbf{rank}(A)$. [**Hint:** $A = U^{-1}(UA)$.]

(b) Prove that if *V* is invertible, then $\mathbf{rank}(AV) = \mathbf{rank}(A)$.

62. Prove that a nonzero $m \times n$ matrix *A* has rank 1 if and only if *A* can be written as the outer product $\mathbf{uv}^T$ of a vector $\mathbf{u} \neq \mathbf{0}$ in $\mathbb{R}^m$ and $\mathbf{v} \neq \mathbf{0}$ in $\mathbb{R}^n$.

63. If an $m \times n$ matrix *A* has rank *r*, prove that *A* can be written as the sum of *r* matrices, each of which has rank 1. [**Hint:** Find a way to use Exercise 62.]

64. Prove that, for $m \times n$ matrices *A* and *B*, rank $(A + B) \leq \mathbf{rank}(A) + \mathbf{rank}(B)$.

65. Let *A* be an $n \times n$ matrix such that $A^2 = O$. Prove that $\mathbf{rank}(A) \leq n/2$. [**Hint:** Show that $\mathbf{col}(A) \subseteq \mathbf{null}(A)$ and use the Rank Theorem.]

66. Let *A* be a skew-symmetric $n \times n$ matrix. (Refer to page 145.)

(a) Prove that $\mathbf{x}^T A\mathbf{x} = 0$ for all **x** in $\mathbb{R}^n$.

(b) Prove that $I + A$ is invertible. [**Hint:** Show that $\mathbf{null}(I + A) = \{\mathbf{0}\}$.]

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