Differential Equations	Name
MAA121/MAG131 Sheet 4	Number
Hand in by 22/04/10	Year
	Mark: /10
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Please attach your working, with this sheet at the front.

These handouts can be downloaded on the Internet at:
`http://www-maths.swan.ac.uk/staff/vm/ODE/ODE.html`

1. Find solution for the following ODEs and initial value problems using the method of integrating factors:

   (a) $\dot{x} + tx = 4t, \quad x(0) = 2;$

   (b) $\dot{x} + 5x = t$ (find the general solution);

   (c) $\frac{dz}{dy} = z\tan(y) + \sin(y)$ (find the general solution).

2. *Newton's law of cooling* states that the temperature $T(t)$ of an object in surroundings of temperature $A(t)$ is governed by the ODE

$$\dot{T} = -k(T - A(t)),$$

where $k > 0$ measures the rate that heat is absorbed (or emitted) by the object.

Use Newton's law of cooling to solve the following problem. At 7 a.m. I made a cup of tea; after adding some milk it is about 90 °C. When I left at 7.30 a.m. the tea is still drinkable at about 45 °C. When I get back home at 8 a.m. the tea has cooled to 30 °C. What is the temperature of my house ?

3. Use an appropriate substitution to find general solutions of the ODEs:

   (a) $xy + y^2 + x^2 - x^2\frac{dy}{dx} = 0;$

   (b) $\frac{dy}{dx} = ky - y^2.$

Hint: (a) is a homogenous equation. Use substitution $u = \frac{y}{x}$. (b) is a Bernoulli equation. Use substitution $u = \frac{1}{y}$.

4. Find the general solution and the solution satisfying the specified initial condition of the following 2nd order linear ODEs:

   (a) $\ddot{x} - 3\dot{x} + 2x = 0$ with $x(0) = 2$ and $\dot{x}(0) = 6;$

   (b) $y'' - 4y' + 4y = 0$ with $y(0) = 0$ and $y'(0) = 3;$

   (c) $z'' - 4z' + 13z = 0$ with $z(0) = 7$ and $z'(0) = 42;$

Differential Equations
MAA121/MAG131 Sheet 4 (Solutions)

1. (*a*) The integrating factor for

$$\dot{x} + tx = 4t$$

is

$$I(t) = e^{\int t\,dt} = e^{t^2/2}.$$

Multiplying the equation by $I(t)$ we obtain

$$e^{t^2/2}\frac{dx}{dt} + te^{t^2/2}x = 4t,$$

or equivalently,

$$\frac{d}{dt}\left(e^{t^2/2}x\right) = 4te^{t^2/2}.$$

Integrating both sides

$$\int \frac{d}{dt}\left(e^{t^2/2}x\right)dt = \int 4te^{t^2/2}dt$$

we obtain

$$e^{t^2/2}x(t) = 4e^{t^2/2} + C.$$

Since $x(0) = 2$, we conclude that

$$C = x(0) - 4 = 2 - 4 = -2.$$

Therefore

$$x(t) = 4 - 2e^{-t^2/2}$$

is the solution of the initial value problem.

(*b*) The integrating factor for

$$\dot{x} + 5x = t$$

is

$$I(t) = e^{\int 5\,dt} = e^{5t}.$$

Multiplying the equation by $I(t)$ we obtain

$$e^{5t}\frac{dx}{dt} + te^{5t}x = e^{5t}t,$$

or equivalently,

$$\frac{d}{dt}\left(e^{5t}x\right) = e^{5t}t.$$

Integrating both sides

$$\int \frac{d}{dt}\left(e^{5t}x\right)dt = \int e^{5t}tdt$$

we obtain

$$e^{5t}x(t) = \frac{te^{5t}}{5} - \frac{e^{5t}}{25} + C,$$

so

$$x(t) = \frac{t}{5} - \frac{1}{25} + Ce^{-5t}$$

is the required general solution of the ODE.

(c) The integrating factor for

$$\frac{dz}{dy} - \tan(y)z = \sin(y)$$

is

$$I(t) = e^{\int -\tan(y)\,dt} = e^{\log(\cos(y))} = \cos(y).$$

Multiplying the equation by $I(t)$ we obtain

$$\cos(y)\frac{dz}{dy} - \cos(y)\tan(y)z = \cos(y)\sin(y),$$

or equivalently,

$$\frac{d}{dy}(\cos(y)z) = \cos(y)\sin(y) = \frac{1}{2}\sin(2y).$$

Integrating both sides

$$\int \frac{d}{dy}(\cos(y)z(y))\,dy = \int \frac{1}{2}\sin(2y)dy$$

we obtain

$$\cos(y)z(y) = -\frac{1}{4}\cos(2y) + C,$$

so

$$z(y) = -\frac{\cos(2y)}{4\cos(y)} + \frac{C}{\cos(y)}$$

is the required general solution of the ODE.

2. The differential equation for $T(t)$, the temperature of tea at time t, is

$$\dot{T} = -k(T - A),$$

where $k > 0$ is a cooling rate coefficient and A is the temperature of the house.

To solve the above differential equation, we can multiply the by the integrating factor e^{kt}. Then we have

$$\frac{d}{dt}\left(e^{kt}T\right) = kAe^{kt}.$$

Integrating on both sides, we obtain

$$(*) \qquad e^{kt}T(t) = Ae^{kt} + C.$$

Assuming that the temperature at an initial time t_0 is $T(t_0)$, and substituting into $(*)$ we conclude that

$$C = -Ae^{kt_0} + e^{kt_0}T(t_0) = e^{kt_0}(T(t_0) - A).$$

Substituting the above value of the constant C into $(*)$ we obtain

$$(**) \qquad T(t) = A + (T(t_0) - A)e^{k(t_0 - t)},$$

which give us the solution of the initial value problem.

We know that $T(7) = 90$, $T(7.5) = 45$, and $T(8) = 30$. Substituting this information into $(**)$ with $t_0 = 7$ we obtain

$$45 = A + (90 - A)e^{-k/2},$$

$$30 = A + (90 - A)e^{-k}.$$

From the 1st equation we conclude that

$$e^{-k/2} = \frac{45 - A}{90 - A}.$$

Noticing that $e^{-k} = \left(e^{-k/2}\right)^2$, we then obtain from the 2nd equation

$$30 = A + (90 - A)\left(\frac{45 - A}{90 - A}\right)^2,$$

that is

$$(30 - A)(90 - A) = (45 - A)^2,$$

which finally gives us the temperature of the house

$$A = 22.5\,.$$

3. (a) Divide through by x^2. Then the equation becomes

$$\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2} + 1.$$

This is homogeneous equation (see Lecture Notes 7.1.1). If we use substitution $u = \frac{y}{x}$ then

$$y = ux, \qquad \frac{dy}{dx} = x\frac{du}{dx} + u.$$

Substituting into the equation we obtain

$$x\frac{du}{dx} + u = u + u^2 + 1,$$

or equivalently,

$$x\frac{du}{dx} = u^2 + 1,$$

which is a separable equation. Separating the variables, we obtain

$$\frac{du}{u^2+1} = \frac{dx}{x}.$$

Integrating left and right hand side we get

$$\int \frac{du}{u^2+1} = \int \frac{1}{x},$$

and so

$$\arctan(u) = \log|x| + C,$$

or

$$u = \tan(\log|x| + C).$$

Since $y = xu$, we finally get

$$y(x) = x\tan(\log|x| + C),$$

which is the required general solution.

(b) This is a Logistic Equation, which could be solved by separating variables (see Lecture Notes 4.3). Instead, we observe that this is also a Bernoulli equation of order 2 hence we use a substitution $u = \frac{1}{y}$ (see lecture Notes 7.1.2). Then

$$y = \frac{1}{u}, \qquad \frac{dy}{dx} = -\frac{1}{u^2}\frac{du}{dx}.$$

Substituting into the equation we obtain

$$-\frac{1}{u^2}\frac{du}{dx} = \frac{k}{u} - \frac{1}{u^2},$$

or equivalently,

$$\frac{du}{dx} + ku = 1.$$

This is a linear equation with constant coefficients, which could be solved, e.g. by using integrating factors method, see (6.3) in the Lecture Notes. The general solution is

$$u = \frac{1}{k} + Ce^{-kx}.$$

Since $y = 1/u$, we finally get

$$y(x) = \frac{1}{\frac{1}{k} + Ce^{-kx}} = \frac{k}{1 + Cke^{-kx}},$$

which is the required general solution (compare this answer with the solution obtained in Section 4.3 of the Lecture Notes).

4. (a) $\ddot{x} - 3\dot{x} + 2x = 0$ with $x(0) = 2$ and $\dot{x}(0) = 6$.

The characteristic equation is

$$k^2 - 3k + 2 = 0.$$

The discriminant $D = 1 > 0$ and the roots of the characteristic equation are $k_1 = 1$ and $k_2 = 2$. Then the general solution is

$$x(t) = Ae^t + Be^{2t}.$$

To solve the initial value problem we first compute the derivative of the general solution:

$$\dot{x}(t) = Ae^t + 2Be^{2t}.$$

Substituting $t = 0$ into the general solution $x(t)$ and its derivative $\dot{x}(t)$ we obtain

$$\underbrace{x(0)}_{=2} = Ae^0 + Be^0 = A + B,$$

$$\underbrace{\dot{x}(0)}_{=6} = Ae^0 + 2Be^0 = A + 2B.$$

Therefore $A = -2$, $B = 4$ and the required solution of the initial value problem is

$$\boxed{x(t) = -2e^t + 4e^{2t}.}$$

(b) $\quad y'' - 4y' + 4y = 0$ with $y(0) = 0$ and $y'(0) = 3$.

The characteristic equation is

$$k^2 - 4k + 4 = 0.$$

The discriminant $D = 0$ and the repeated root of the characteristic equation is $k = 2$. Then the general solution is

$$y(t) = Ae^{2t} + Bte^{2t}.$$

To solve the initial value problem we first compute the derivative of the general solution:

$$\dot{y}(t) = (2A + B)e^{2t} + Bte^{2t}.$$

Substituting $t = 0$ into the general solution $y(t)$ and its derivative $\dot{y}(t)$ we obtain

$$\underbrace{y(0)}_{=0} = Ae^0 = A,$$

$$\underbrace{\dot{y}(0)}_{=3} = (2A + B)e^0 = 2A + B.$$

Therefore $A = 0$, $B = 3$ and the required solution of the initial value problem is

$$\boxed{y(t) = 3te^{2t}}.$$

(c) $\quad z'' - 4z' + 13z = 0$ with $z(0) = 7$ and $z'(0) = 42$.

The characteristic equation is

$$k^2 - 4k + 13 = 0.$$

The discriminant $D = -36 < 0$ and the complex roots of the characteristic equation are $k_{1,2} = 2 \pm 3i$. Then the general solution is

$$z(t) = e^{2t} \left(A\cos(3t) + B\sin(3t) \right).$$

To solve the initial value problem we first compute the derivative of the general solution:

$$\dot{z}(t) = e^{2t} \left((3B + 2A)\cos(3t) + (2B - 3A)\sin(3t) \right).$$

Substituting $t = 0$ into the general solution $z(t)$ and its derivative $\dot{z}(t)$ we obtain

$$\underbrace{z(0)}_{=7} = e^0 A = A,$$

$$\underbrace{\dot{z}(0)}_{=42} = e^0(3B + 2A) = 3B + 2A.$$

Therefore $A = 7$, $B = 28/3$ and the required solution of the initial value problem is

$$\boxed{z(t) = e^{2t} \left(7\sin(3t) + \frac{28}{3}\cos(3t) \right)}.$$