On Euler Equations in higher order Sobolev spaces

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Abstract

We consider the integral functional of calculus of variations involving higher order derivatives. It is shown that any local minimum of the functional solves the associated strongly nonlinear elliptic problem in a certain weak sense in spite of the fact that no growth conditions are imposed on the zero order term.

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be an arbitrary open set, $m \in \mathbb{N}$ and $p \in (1, \infty)$. Consider the functional

$$I(u) = \int_\Omega A(x, u, \ldots, D^m u)\, dx,$$

where $A(x, \zeta) : \Omega \times \mathbb{R} \times \ldots \times \mathbb{R}^{mN} \to \mathbb{R}$ is a Carathéodory function of class $\mathcal{C}^1$ in the ζ–variable. Under appropriate ellipticity and polynomial growth conditions on A and its partial derivatives A_α the functional I is well–defined, coercive, weakly semicontinuous from below and differentiable on the Sobolev space $W_0^{m,p}(\Omega)$. The derivative of I is given by

$$I'(u)(w) = \langle \mathcal{A}(u), w \rangle \quad \text{for all} \quad w \in W_0^{m,p}(\Omega), \tag{1}$$

where

$$\mathcal{A}(u) = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \ldots, D^\alpha u)$$

is the corresponding to A quasilinear elliptic operator of order $2m$ and $\langle \cdot, \cdot \rangle$ is the duality between $W_0^{m,p}(\Omega)$ and its conjugate space $W^{-m,p'}(\Omega)$.

Now let $f(x, \xi) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$\sup_{|\xi| < \tau} |f(x, \xi)| \in L^1_{loc}(\Omega) \quad \text{for all} \ \ \tau \ge 0. \tag{2}$$

Let

$$F(x, \xi) = \int_0^\xi f(x, \tau)\, d\tau$$

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be the primitive of f. Let us consider the functional

$$J(u) = I(u) + \int_\Omega F(x,u)\,dx - \langle \mu, u \rangle,$$

with $\mu \in W^{-m,p'}(\Omega)$. Condition (2) implies that

$$\mathcal{C}_0^\infty(\Omega) \subseteq Dom(J) = \left\{ u \in W_0^{m,p}(\Omega) : F(x,u) \in L^1(\Omega) \right\}.$$

Let $mp \leq N$. Then $W_0^{m,p}(\Omega) \not\subset L^\infty(\Omega)$. If the *critical growth condition* of the type

$$|f(x,\xi)| \leq a|\xi|^{p^*-1} + b(x), \tag{3}$$

holds, when p^* denotes the critical Sobolev exponent, then $Dom(J) = W_0^{m,p}(\Omega)$ and J is differentiable on $W_0^{m,p}(\Omega)$. On the other hand if, for instance, Ω is bounded and the primitive F is bounded from below, then J admits a minimum point $u \in Dom(J)$ without any restriction on the growth of $|f(x,\xi)|$. It is thus natural to ask whether the Euler equation

$$\langle \mathcal{A}(u), w \rangle + \int_\Omega f(x,u) w\,dx = \langle \mu, w \rangle \tag{4}$$

corresponding to the functional J holds at u for functions w from a certain "test" subspace $\mathcal{W}$ of $W_0^{m,p}(\Omega)$. Or one can ask whether u solves, in a suitable weak sense, the quasilinear elliptic problem

$$\mathcal{A}(u) + f(x,u) = \mu, \tag{5}$$

associated to J although *no growth conditions* are imposed on $|f(x,\xi)|$.

The study of such "strongly nonlinear" problems goes back to earlier works of F. Browder, cf. [6]. The situation when $f(x,\xi)$ satisfy no a priori growth conditions but only the *sign condition*

$$f(x,\xi)\,\xi \geq 0 \tag{6}$$

has been considered in [16, 5, 7]. See also [10, 3] for a treatment in the frame of Orlicz–Sobolev spaces and further references therein. The existence of a solution for (5) has been established by using methods of pseudo–monotone operators or degree theory. Such a solution u satisfies the Euler equation (4) with respect to the test space $\mathcal{C}_0^\infty(\Omega)$ and satisfies the property

$$f(x,u) \in W^{-m,p'}(\Omega) \cap L^1_{loc}(\Omega) \quad \text{and} \quad f(x,u)\,u \in L^1(\Omega). \tag{7}$$

The treatment of the case $m > 1$ is much more involved, due to the lack of truncation operations in the higher order Sobolev spaces. As it has been pointed out by J. R. L. Webb [16], in this case must rely on the delicate approximation procedure for $W_0^{m,p}(\Omega)$ spaces introduced by L. I. Hedberg [12].

It seems that a variational approach in the study of strongly nonlinear elliptic problems has been initiated by J.–P. Gossez. In [2] it has been shown that in the case $m = 1$ the functional J is differentiable at $u \in Dom(J)$ with respect to the test space $\mathcal{C}_0^\infty(\Omega)$ if the *one sided growth condition* of the type

$$f(x,\xi)\,\xi \geq -a|\xi|^{p^*} - b(x) \tag{8}$$

holds. In [11] this result in a combination with a certain truncation technique has been applied to the nonlinearities $f(x,\xi) = f(\xi)$ and $\mu \in L^\infty(\Omega)$. For this case it has been proved that the solution for (5) exists *without any growth restriction* on $f(\xi)$. Such a solution satisfies (7) but in general is not a minimum of J.

In the recent paper [8], M. Degiovanni and S. Zani have considered a strongly nonlinear elliptic problem for $m = 1$, $p = 2$ and *without any growth restriction* on $|f(x,\xi)|$ by exploiting the crucial new idea of a variation of the test space for J. More precisely, it was shown that, for a given point $u \in Dom(J)$, one can construct a test space $\mathcal{W}_u$, dense in $W_0^{1,2}(\Omega)$, such that J is differentiable with respect to $\mathcal{W}_u$. This allowed to prove that any minimum u of J solves the equation (5) in a suitable weak sense. The same approach has been used in [9] in the framework of nonsmooth critical point theory.

The aim of the present paper is to generalize the results of [8] to the higher order situation. The technique we have to use differs considerably from the technique used in [8]. The reason is that there are no obvious truncation operations within the space $W_0^{m,p}(\Omega)$ with $m > 1$. Instead of it, use is made of an approximation technique related to the Hedberg's spectral synthesis theorem, cf. [1, Chapter 9]. Our basic result, Theorem 3, asserts that a function in the Sobolev space $W_0^{m,p}(\Omega)$ can be approximated by bounded functions from $W_0^{m,p}(\Omega)$ with compact support located away from the singularities of the given function. This argument allows to construct, for a point $u \in Dom(J)$, a subspace of test functions $\mathcal{W}_u$, dense in $W_0^{m,p}(\Omega)$, such that J is differentiable with respect to $\mathcal{W}_u$. For a given minimum u of J this implies that the Euler equation (4) holds at u with respect to $\mathcal{W}_u$. In general, such a test space $\mathcal{W}_u$ does not contain $\mathcal{C}_0^\infty(\Omega)$ and a minimum u does not satisfy (7). However it is still possible to identify $f(x,u)$ with a distribution from $W^{-m,p'}(\Omega)$ and to show that u solves (5) in $W^{-m,p'}(\Omega)$. In the following we consider the case $mp \leq N$, otherwise the results are trivial. For the potential theoretic notions mentioned in the paper we refer the reader to the monograph of D. R. Adams and L. I. Hedberg [1].

Acknowledgements. The author is grateful to Marco Degiovanni for many helpful discussions and to Walter Farkas, Niels Jacob and Massimo Lanza for their remarks and comments. The support of the DAAD – German Academic Exchange Service is greatfully acknowledged.

2 An approximation theorem

Let $\Omega \subseteq \mathbb{R}^N$ be an arbitrary open set, $m \in \mathbb{N}$ and $p \in (1, N/m]$. By $W^{m,p}(\Omega)$ we denote the Sobolev space of (equivalence classes of) real valued functions on Ω whose distributional derivatives of order up to m belong to the Lebesgue space $L^p(\Omega)$. The norm on $W^{m,p}(\Omega)$ is given by the formula

$$\|u\|_{m,p}^p = \sum_{|\alpha|\le m} \|D^\alpha u\|_{L^p}^p,$$

with the usual multi–index notation. By $W_0^{m,p}(\Omega)$ we denote the closure in $W^{m,p}(\Omega)$ of the space $\mathcal{C}_0^\infty(\Omega)$ of all smooth functions with compact support contained in Ω and by $W^{-m,p'}(\Omega)$, $p' = \frac{p}{p-1}$, the dual space to $W_0^{m,p}(\Omega)$. By $\langle\cdot,\cdot\rangle$ we denote the pairing in the duality of $W^{-m,p'}(\Omega)$ with $W_0^{m,p}(\Omega)$. For an open set $\tilde\Omega \subset \Omega$ we denote by $L_c^\infty(\tilde\Omega)$ the space of essentially bounded functions with compact support contained in $\tilde\Omega$.

Let $u \in W_0^{m,p}(\Omega)$. We can extend $u \in W_0^{m,p}(\Omega)$ by zero on all of $\mathbb{R}^N$ and obtain $u \in W^{m,p}(\mathbb{R}^N)$. By a theorem of A. P. Calderon $u \in W^{m,p}(\mathbb{R}^N)$ if and only if u can be represented as the Bessel potential

$$u = G_m * S, \qquad S \in L^p(\mathbb{R}^N),$$

where $''*''$ is the usual convolution and G_m is the Bessel kernel which can be defined as the inverse Fourier transform of $\hat G_m(\xi) = (1+|\xi|^2)^{-m/2}$. Moreover, there is an equivalence of norms, i.e.

$$a^{-1}\|S\|_{L_p} \le \|u\|_{m,p} \le a\|S\|_{L_p} \tag{9}$$

for some constant $a > 0$. Note, that the Bessel potential $u = G_m * S$ is everywhere defined on $\mathbb{R}^N$.

To the space $W^{m,p}(\mathbb{R}^N)$ one can associate a set function, called (m,p)–capacity, in the following way. Let $E \subset \mathbb{R}^N$ be an arbitrary set and

$$W_E = \{u = G_m * S \in W^{m,p}(\mathbb{R}^N) \;:\; u \ge 1 \text{ for all } x \in E\}.$$

Then the (m,p)–capacity can be defined by

$$C_{m,p}(E) = \inf\nolimits_{W_E} \|u\|_{m,p}^p \,.$$

If $W_E = \emptyset$ then $C_{m,p}(E) = \infty$.

Let $u : \Omega \to \mathbb{R}$ be a measurable function. We say that $x \in \Omega$ is a *singular point* of u if u is essentially unbounded in any neighborhood U_x of x in Ω. The set of all singular points of u is called the *singular set* of u and is denoted by $Sing(u)$. From the definition it is clear that $Sing(u)$ is a closed subset of Ω and $u \in L_{loc}^\infty(\Omega \setminus Sing(u))$. It is easy to construct the function $u \in L^p(\Omega)$ such that $Sing(u) = \Omega$. However for the functions from the Sobolev space $W_0^{m,p}(\Omega)$ we have the following result.

Theorem 1 *Let $u \in W_0^{m,p}(\Omega)$. Then $C_{m,p}(Sing(u)) = 0$.*

Proof. Let $u \in W_0^{m,p}(\Omega)$. Then, by [1, Proposition 6.1.2], the Bessel potential $\tilde u = G_m * S$ is an (m,p)–quasicontinuous representative of u. This means that

$$\tilde u = u \quad \text{a.e. in } \Omega$$

and that for any $\varepsilon > 0$ there is an open set $G_\varepsilon \subset \Omega$ such that $C_{m.p}(G_\varepsilon) < \varepsilon$ and the restriction of u to the complement of G_ε is a continuous function in the induced topology.

Clearly, $Sing(u) \subset G_\varepsilon$ for each $\varepsilon > 0$. Then, by the outer capacity property [1, Proposition 2.3.5],

$$C_{m,p}(Sing(u)) \leq \inf_{\varepsilon>0} C_{m,p}(G_\varepsilon) = 0,$$

which proves the theorem. □

Remark 1 For a set $E \subset \mathbb{R}^N$ and $mp \leq N$ it is easily seen that $mes(E) = 0$ implies $C_{m,p}(E) = 0$. More precisely, for a compact set $K \subset \mathbb{R}^N$ and $mp < N$ it is known that $C_{m,p}(K) = 0$ if and only if $dim(K) \leq N - mp$ where dim stands for a Hausdorff dimension of a set K. For $mp = N$ a description of sets of zero (m,p)–capacity can be given in terms of the so–called logarithmic Hausdorff measure, cf. [1, p.134–139] for details.

In the next theorem, for a relatively closed subset $E \subset \Omega$, by $W_0^{m,p}(\Omega) = W_0^{m,p}(\Omega \setminus E)$ we mean that each function $u \in W_0^{m,p}(\Omega)$ can be approximated in $\|\cdot\|_{m,p}$ norm by functions from $\mathcal{C}_0^\infty(\Omega \setminus E)$.

Theorem 2 *Let $E \subset \Omega$ be a relatively closed subset. Then the following statements are equivalent:*

(a) $C_{m,p}(E) = 0$*;*

(b) $W_0^{m,p}(\Omega) = W_0^{m,p}(\Omega \setminus E)$;

(c) *for all $v \in W_0^{m,p}(\Omega)$ there exists a sequence $(w_n) \subset W_0^{m,p}(\Omega) \cap L_c^\infty(\Omega \setminus E)$ such that*

i) $|w_n(x)| \leq |v(x)|$ *and* $w_n(x)v(x) \geq 0$ *a.e. in* Ω*;*

ii) $\lim_{n\to\infty} \|v - w_n\|_{m,p} = 0$.

In particular, $W_0^{m,p}(\Omega) \cap L_c^\infty(\Omega \setminus E)$ *is a dense linear subspace of* $W_0^{m,p}(\Omega)$.

Proof. We first prove the implication $(a) \Rightarrow (b)$. Let $v \in W_0^{m,p}(\Omega)$ and $(v_n) \subset \mathcal{C}_0^\infty(\Omega)$ be an approximating sequence converging to v in the $\|\cdot\|_{m,p}$ norm. Set

$$E_n = E \cap Supp(v_n).$$

Clearly $E_n \subset \Omega$ is a compact set. By the monotonicity of the capacity [1, Proposition 2.3.4] we have $C_{m,p}(E_n) = 0$.

By [1, Theorem 9.9.1], there exists a neighborhood $V_{E_n} \subset \Omega$ of E_n and a function $\omega_n \in C_0^\infty(\Omega)$ such that

$$0 \leq \omega_n \leq 1, \quad \omega_n = 1 \text{ on } V_{E_n} \quad \|\omega_n v_n\|_{m,p} \leq 1/n.$$

We set

$$w_n = (1 - \omega_n)v_n.$$

Then $(w_n) \subset \mathcal{C}_0^\infty(\Omega \setminus E)$ and

$$\lim_{n\to\infty} \|v - w_n\|_{m,p} \leq \lim_{n\to\infty} (\|v - v_n\|_{m,p} + \|\omega_n v_n\|_{m,p}) = 0,$$

which proves the statement.

We now prove the implication $(a) \Rightarrow (c)$. Let $v \in W_0^{m,p}(\Omega)$. By the spectral synthesis theorem of L. Hedberg [1, Theorem 9.1.3], for each $n \in \mathbb{N}$ there exists a function $\eta_n \in \mathcal{C}_0^\infty(\Omega)$ such that

$$0 \leq \eta_n \leq 1, \qquad \|v - \eta_n v\|_{m,p} \leq 1/n.$$

We set

$$E_n = Sing(\eta_n v) \cup (E \cap Supp(\eta_n v)).$$

Clearly $E_n \subset \Omega$ is a compact set. By Theorem 1 we have

$$C_{m,p}(Sing(\eta_n v)) = 0.$$

Then $C_{m,p}(E_n) = 0$ by the subadditivity [1, Proposition 2.3.6] and the monotonicity of the capacity.

Now, by [1, Theorem 9.9.1], there exists a neighborhood $V_{E_n} \subset \Omega$ of E_n and a function $\omega_n \in C_0^\infty(\Omega)$ such that

$$0 \leq \omega_n \leq 1, \quad \omega_n = 1 \; on \; V_{E_n} \quad \|\omega_n(\eta_n v)\|_{m,p} \leq 1/n.$$

We set

$$w_n = (1 - \omega_n)(\eta_n v).$$

Then

$$Supp(w_n) \subseteq Supp(\eta_n v) \setminus E_n.$$

By the compactness of $Supp(w_n)$ and by the construction of the set E_n, it follows that the function w_n is essentially bounded and that $w_n \in W_0^{m,p}(\Omega) \cap L_c^\infty(\Omega \setminus E)$. Furthermore

$$0 \leq (1 - \omega_n)\eta_n \leq 1.$$

Hence the statement (i) holds. Finally,

$$\lim_{n\to\infty} \|v - w_n\|_{m,p} \leq \lim_{n\to\infty} (\|v - \eta_n v\|_{m,p} + \|\omega_n(\eta_n v)\|_{m,p}) = 0,$$

which proves (ii) and the implication $(a) \Rightarrow (c)$.

We now prove the implication $(c) \Rightarrow (a)$. Assume that $C_{m,p}(E) > 0$. Then by the property of capacitable sets, see [1, p.28], there exists a compact set $K \subset E \subseteq \Omega$ such that

$$0 < C_{m,p}(K) \leq C_{m,p}(E).$$

We set

$$W_{K,\Omega} = W_K \cap W_0^{m,p}(\Omega),$$

and we note that $W_{K,\Omega} \neq \emptyset$ if $W_K \neq \emptyset$. Indeed, let $u \in W_K$ and $\eta \in C_0^\infty(\Omega)$ be a cut–off function such that $\eta = 1$ on K and $0 \leq \eta \leq 1$. Then $\eta u \in W_{K,\Omega}$. Moreover, it is clear that $W_{K,\Omega} \subseteq W_K$ and hence

$$\inf_{W_{K,\Omega}} \|u\|_{m,p}^p \geq C_{m,p}(K) > 0.$$

Now let $w \in W_0^{m,p}(\Omega) \cap L_c^\infty(\Omega \setminus E)$ and $v \in W_{K,\Omega}$ be represented as Bessel potentials

$$w = G_m * T, \quad v = G_m * S.$$

Then

$$w - v \in W_{K,\Omega}$$

and

$$\|v - w\|_{m,p}^p \geq C_{m,p}(K) > 0. \tag{10}$$

Since w is an arbitrary element in $W_0^{m,p}(\Omega) \cap L_c^\infty(\Omega \setminus E)$, inequality (10) proves the statement.

The implication $(b) \Rightarrow (a)$ can be treated in a similar way using the "smooth" version of (m,p)–capacity, cf. [1, Proposition 2.3.13]. □

Remark 2 The equivalence between (a) and (b) is known at least in part (cf. [14, Theorem 2.43] for the case $m = 1$). The direct proof of the implication $(a) \Rightarrow (c)$ (without using the spectral synthesis theorem arguments) for the case $\Omega = \mathbb{R}^N$ and empty "exceptional" set E has been given by L. Hedberg in [12, Lemma 5.2]. See also Theorem 3.4.1 of [1] and comments therein. The proof for an arbitrary open set $\Omega \subset \mathbb{R}^N$ and $E = \emptyset$ must rely on the spectral synthesis theorem, proved by L. Hedberg in 1981 (see a survey of L. Hedberg [13] and [1, Chapter 9] for a discussion). The usefulness of such results in the study of higher order strongly nonlinear elliptic problems has been pointed out by J. R. L. Webb in [16]. See also [5, 7, 10, 3, 2] for further related results and applications.

Our basic result from the point of view of applications is the following version of Theorem 2.

Theorem 3 *Let $u \in W_0^{m,p}(\Omega)$. Then for every $v \in W_0^{m,p}(\Omega)$ there exists a sequence $(w_n) \subset W_0^{m,p}(\Omega) \cap L_c^\infty(\Omega \setminus Sing(u))$ such that:*
i) $|w_n(x)| \leq |v(x)|$ and $w_n(x)v(x) \geq 0$ a.e. in Ω;
ii) $\lim_{n\to\infty} \|v - w_n\|_{m,p} = 0$.
In particular $W_0^{m,p}(\Omega) \cap L_c^\infty(\Omega \setminus Sing(u))$ is a dense linear subspace of $W_0^{m,p}(\Omega)$.

Proof. By Theorem 1 we already know that $C_{m,p}(Sing(u)) = 0$. Since $Sing(u)$ is a relatively closed subset of Ω the result follows from Theorem 2 by setting $E = Sing(u)$. □

Remark 3 For $m = 1$ and $p = 2$ Theorem 3 (in slightly different terms) has been proved by M. Degiovanni and S. Zani in [8, Theorem 2.3]. Their proof is based on the truncation properties of functions from $W_0^{1,2}(\Omega)$, an argument which provides a simple direct construction of an approximating sequence (w_n).

3 Brezis–Browder type result

Let $T : \Omega \to \mathbb{R}$ be a measurable function. Let E be a relatively closed subset of Ω of (m, p)–capacity zero. Let $T \in L^1_{loc}(\Omega \setminus E)$. Consider a linear functional

$$l_T(w) = \int_\Omega T(x)w(x)\, dx.$$

Clearly l_T is defined on $L^\infty_c(\Omega \setminus E)$. Suppose

$$\sup \{|l_T(w)| : w \in W_0^{m,p}(\Omega) \cap L^\infty_c(\Omega \setminus E),\ \|w\|_{m,p} \le 1\} < +\infty. \tag{11}$$

Then l_T is a bounded linear functional on $W_0^{m,p}(\Omega) \cap L^\infty_c(\Omega \setminus E)$. By Theorem 2 we know that $W_0^{m,p}(\Omega) \cap L^\infty_c(\Omega \setminus E)$ is a dense linear subspace of $W_0^{m,p}(\Omega)$. Hence there exists a unique continuous extension of l_T to $W_0^{m,p}(\Omega)$, which does not depend on the choice of the "exceptional" set E and on the corresponding initial domain $W_0^{m,p}(\Omega) \cap L^\infty_c(\Omega \setminus E)$. In this way one can identify T with l_T and to consider T as an element of $W^{-m,p'}(\Omega)$. We shall write

$$T \in W^{-m,p'}(\Omega) \cap L^1_{loc}(\Omega \setminus E)$$

if (11) holds. The following result gives an extension of the celebrated theorem of Brezis and Browder.

Theorem 4 *Let E be a relatively closed subset of Ω of (m, p)–capacity zero and $T \in W^{-m,p'}(\Omega) \cap L^1_{loc}(\Omega \setminus E)$. Let $v \in W_0^{m,p}(\Omega)$ and $\phi \in L^1(\Omega)$ be such that*

$$T(x)v(x) \ge \phi(x) \quad \textit{a.e. in } \Omega. \tag{12}$$

Then $Tv \in L^1(\Omega)$ and

$$\langle T, v\rangle = \int_\Omega Tv\, dx.$$

Proof. Let $(w_n) \in W_0^{m,p}(\Omega) \cap L^\infty_c(\Omega \setminus E)$ be the approximating sequence for v constructed in Theorem 2, c). Up to a subsequence, we can assume that

$$w_n(x) \to v(x) \quad \text{a.e. in } \Omega.$$

Since $T \in L^1_{loc}(\Omega \setminus E)$, it follows that $Tw_n \in L^1(\Omega)$ and that

$$\langle T, w_n\rangle = \int_\Omega Tw_n\, dx.$$

By Theorem 2, ii) (w_n) is converging to v in $W_0^{m,p}(\Omega)$. Hence

$$\lim_{n\to\infty} \langle T, w_n\rangle = \langle T, v\rangle.$$

On the other hand, by (12) and Fatou's lemma it follows that $Tv \in L^1(\Omega)$ and then

$$-\infty < \int_\Omega Tv\, dx \le \liminf_{n\to\infty} \int_\Omega Tw_n\, dx \le \langle T, v\rangle.$$

By Theorem 2, b–i) and by the dominated convergence theorem we conclude that

$$\lim_{n\to\infty}\int_\Omega Tw_n\,dx = \int_\Omega Tv,$$

which proves the theorem. $\square$

Remark 4 For the case $E = \emptyset$ the theorem has been proved by H. Brezis and F. Browder in [4] for the case $m = 1$ and in [5] for $m > 1$. For further related results and applications to nonlinear differential equations we refer to [7, 10, 3, 2]. A version of the theorem with a nonempty "exceptional" set E and with $m = 1$, $p = 2$ has been proved by M. Degiovanni and S. Zani [8, Theorem 2.8].

4 The Euler equation for J

Throughout this section we assume that a Carathéodory function $f(x, \xi)$ satisfies the condition

$$\sup_{|\xi|<\tau} |f(x,\xi)| \in L^1_{loc}(\Omega) \quad \text{for all } \tau \geq 0. \tag{13}$$

Then we have the following theorem.

Theorem 5 *Let $u : \Omega \to \mathbb{R}$ be a measurable function. Then*

$$f(x, u) \in L^1_{loc}(\Omega \setminus Sing(u)).$$

Proof. Assume that $\Omega \setminus Sing(u) \neq \emptyset$. Let $K \subset (\Omega \setminus Sing(u))$ be a compact set. From the definition of singular set $Sing(u)$, it follows that $u_{|K} \in L^\infty(K)$. Let $a = \|u_{|K}\|_{L_\infty}$. Then (13) implies that

$$|f(x,u)|_{|K} \leq \sup_{|\xi|\leq a} |f(x,\xi)| \in L^1(K).$$

Hence $f(x, u) \in L^1_{loc}(\Omega \setminus Sing(u))$ by the arbitrariness of K. $\square$

We now turn back to the functional

$$J(u) = I(u) + \int_\Omega F(x,u)\,dx - \langle \mu, u\rangle,$$

where

$$I(u) = \int_\Omega A(x, u, \ldots, D^m u)\,dx.$$

We assume that $\mu \in W^{-m,p'}(\Omega)$ and that the function $A(x, \zeta)$ satisfies the standard (m, p)–ellipticity and polynomial growth conditions, cf. [15]. Then the functional I is differentiable on the Sobolev space $W^{m,p}_0(\Omega)$ with the derivative given by (1) and

$$Dom(J) = \{u \in W^{m,p}_0(\Omega) : F(x,u) \in L^1(\Omega)\}.$$

Theorem 6 *Let $u \in Dom(J) \subseteq W_0^{m,p}(\Omega)$. Then J has a directional derivative given by the formula*

$$J'(u)(w) = \lim_{\tau \to 0} \frac{J(u + \tau w) - J(u)}{\tau} = \langle \mathcal{A}(u) - \mu, w \rangle + \int_\Omega f(x,u) w \, dx.$$

along any direction $w \in W_0^{m,p}(\Omega) \cap L_c^\infty(\Omega \setminus Sing(u))$.

Proof. It suffices to check only the differentiability of the zero–order term

$$J_F(u) = \int_\Omega F(x,u) \, dx.$$

Let $u \in Dom(J)$, $w \in W_0^{m,p}(\Omega) \cap L_c^\infty(\Omega \setminus Sing(u))$ and $\tau \in \mathbb{R}$, $\tau \neq 0$. By the mean value theorem there exists a measurable function $\theta_\tau(x)$ such that $0 \le \theta_\tau(x) \le 1$ and such that

$$\frac{1}{\tau} F(x, u + \tau w) - F(x,u) = f(x, u + \tau \theta_\tau w) w.$$

First, we observe that

$$Sing(u + \tau \theta_\tau w) = Sing(u),$$

because $\tau \theta_\tau w \in L_c^\infty(\Omega \setminus Sing(u))$. Then Theorem 5 implies that

$$f(x, u + \tau \theta_\tau w) \in L^1_{loc}(\Omega \setminus Sing(u)).$$

Hence

$$f(x, u + \tau \theta_\tau w) w \in L^1(\Omega).$$

By the dominated convergence theorem we obtain

$$J_F'(u)(w) = \lim_{\tau \to 0} \int_\Omega \frac{1}{\tau} (F(x, u + \tau w) - F(x,u)) \, dx =$$

$$\lim_{\tau \to 0} \int_\Omega f(x, u + \tau \theta_\tau w) w \, dx = \int_\Omega f(x,u) w \, dx$$

with $f(x,u)w \in L^1(\Omega)$. $\square$

Remark 5 By arguing as in the proof above one can show that for any $u \in Dom(J)$ the following holds:

$$u + \{W_0^{m,p}(\Omega) \cap L_c^\infty(\Omega \setminus Sing(u))\} \subseteq Dom(J).$$

In particular, $W_0^{m,p}(\Omega) \cap L_c^\infty(\Omega)$ is contained in $Dom(J)$.

Our main result is an immediate consequence of Theorem 6, generalizes to the higher order situation the result of M. Degiovanni and S. Zani [8, Theorem 3.4], and motivates considerations above.

Theorem 7 *Assume that $u \in Dom(J) \subseteq W_0^{m,p}(\Omega)$ is the local minimum of J. Then the Euler equation*

$$\langle \mathcal{A}(u), w\rangle + \int_\Omega f(x,u)w\, dx = \langle \mu, w\rangle \tag{14}$$

for J holds at u for all $w \in W_0^{m,p}(\Omega) \cap L_c^\infty(\Omega \setminus Sing(u))$ and u solves the equation

$$\mathcal{A}(u) + f(x,u) = \mu \quad in \;\; W^{-m,p'}(\Omega), \tag{15}$$

with

$$f(x,u) \in W^{-m,p'}(\Omega) \cap L^1_{loc}(\Omega \setminus Sing(u)). \tag{16}$$

Proof. Since $u \in Dom(J)$ is a local minimum for J, Theorem 6 and the classical Fermat principle immediately implies that the Euler equation (14) holds for all $w \in W_0^{m,p}(\Omega) \cap L_c^\infty(\Omega \setminus Sing(u))$.

Furthermore, by Theorem 5 we know that $f(x,u) \in L^1_{loc}(\Omega \setminus Sing(u))$ and by (14)

$$\int_\Omega f(x,u)w\, dx = \langle \mu - \mathcal{A}(u), w\rangle \le \|\mu - \mathcal{A}(u)\|_{-m,p'}$$

for $w \in W_0^{m,p}(\Omega) \cap L_c^\infty(\Omega \setminus Sing(u))$ and $\|w\|_{m,p} \le 1$. Then condition (11) holds and

$$f(x,u) \in W^{-m,p'}(\Omega) \cap L^1_{loc}(\Omega \setminus Sing(u)).$$

Since $W_0^{m,p}(\Omega) \cap L_c^\infty(\Omega \setminus Sing(u))$ is a dense linear subspace of $W_0^{m,p}(\Omega)$, Theorem 3 implies that u solves (15) in $W^{-m,p'}(\Omega)$. □

Remark 6 Condition (13) may be weakened. Instead of (13) one may assume that for a certain fixed relatively closed set $E \subset \Omega$ of (m,p)–capacity zero

$$\sup_{|\xi|<\tau} |f(x,\xi)| \in L^1_{loc}(\Omega \setminus E) \quad \text{for all } \; \tau \ge 0. \tag{17}$$

Then Theorem 7 remains true with the test space $W_0^{m,p}(\Omega) \cap L_c^\infty(\Omega \setminus (E \cup Sing(u)))$ and with

$$f(x,u) \in W^{-m,p'}(\Omega) \cap L^1_{loc}(\Omega \setminus (E \cup Sing(u)))$$

instead of (16).

Remark 7 From Theorem 4 it follows that if $u \in Dom(J)$ is a local minimum for J and if $w \in W_0^{m,p}(\Omega)$ is such that

$$f(x,u)w \ge \phi(x) \quad \text{or} \quad f(x,u)w \le \phi(x)$$

for some $\phi \in L_1(\Omega)$, then $f(x,u)w \in L_1(\Omega)$ and the Euler equation (14) holds for $w \in W_0^{m,p}(\Omega)$. In particular, the integral form of the Euler equation for J is satisfied "whenever it makes sense".

Using the Brezis–Browder type Theorem 4, one can show that under the one–sided growth conditions of the type (8) the Euler equation (14) is satisfied with respect to the functions from the test space $L_c^\infty(\Omega)$.

Theorem 8 *Assume that $u \in Dom(J) \subseteq W_0^{m,p}(\Omega)$ is a local minimum for J and that*

$$f(x,\xi)\,\xi \geq -a|\xi|^{p^*} - b(x), \tag{18}$$

where $p^* = \frac{pN}{N-mp}$ if $mp < N$ or p^* is any positive number if $mp = N$ and $a > 0$, $b \in L^1(\Omega)$. Then the Euler equation (14) holds at u for all $w \in L_c^\infty(\Omega)$ and u solves (15) in $W^{-m,p'}(\Omega)$ with

$$f(x,u) \in W^{-m,p'}(\Omega) \cap L^1_{loc}(\Omega) \quad \textit{and} \quad f(x,u)\,u \in L^1(\Omega). \tag{19}$$

Proof. By Theorem 7, we already know that

$$f(x,u) \in W^{-m,p'}(\Omega) \cap L^1_{loc}(\Omega \setminus Sing(u)).$$

Also $C_{m,p}(Sing(u)) = 0$ by Theorem 1. From condition (18), we deduce that

$$f(x,u)u \geq -a|u|^{p^*} - b(x) \in L^1(\Omega).$$

Then all of the assumptions of Theorem 4 hold and hence $f(x,u)\,u \in L^1(\Omega)$. Accordingly $f(x,u) \in W^{-m,p'}(\Omega) \cap L^1_{loc}(\Omega)$. Since u is a local minimum for J, then (14) holds for all $w \in L_c^\infty(\Omega)$. $\square$

ERRATUM

Theorem 1 is false. As a consequence, Theorem 3, Theorem 7 and Theorem 8 remain true only if one assumes in addition that $C_{m,p}(Sing(u)) = 0$.

The problem with our argument is in the erroneous choice of defining the singular set $Sing(u)$ on p. 96. To overcome the difficulty one can modify the definition as follows. Let $u \in W^{m,p}(\mathbb{R}^N)$ be represented as the Bessel potential $u = G_m * S$, $S \in L^p(\mathbb{R}^N)$. Then one can define the singular set of the function u as $Sing(u) = \{x \in \mathbb{R}^N : G_m * |S| = +\infty\} \cap Supp(u)$. With this definition all of the results of the paper remain true. However some statements and proofs require modification, the only essential one is in the proof of Theorem 2 on p. 97. Here, instead of spectral synthesis theorems, one can rely on smooth truncation techniques similar to those on p. 62–68 in [1]. The details will appear elsewhere.

We wish to thank Lars Hedberg for pointing out the error and further discussions.

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