Differential Equations	Name
MAA121/MAG131 Sheet 2	Number
Hand in by 25/02/2010	Year
	Mark: /10
	date marked: / /2010

Please attach your working, with this sheet at the front.

These handouts and other support materials can be downloaded at:
`http://www-maths.swan.ac.uk/staff/vm/ODE/ODE.html`

The mark for this assignment does not count towards your final mark.

1. Classify the following equations as ordinary or partial and give their order. In each case identify the unknown function and independent variable(s).

(a) Bessel's equation (ν is a parameter):

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

(a) Burger's equation (ν is a parameter):

$$\frac{\partial u}{\partial t} - \nu\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0.$$

(c) van der Pol's equation (m, k, a and b are parameters):

$$m\ddot{x} + kx = a\dot{x} - b\dot{x}^3.$$

(d) equation

$$\frac{dy}{dt} = t - y^2.$$

2. Find the function $g(x)$ defined for $x > -1$ that has slope $\log(1 + x)$ and passes through the origin

Turn over.

3. Find the general solution of the following initial value problems:

(a)

$$\frac{dy}{dt} = \sin(t) + \cos(t), \qquad y(\pi) = 0,$$

(b) (use partial fractions)

$$\frac{dy}{dx} = \frac{1}{x^2 - 1}, \qquad y(0) = 0,$$

(c)

$$\frac{dU}{dt} = 4t \log(t), \qquad U(0) = 1.$$

4. A car of mass $m = 1000\ kg$ is travelling at a speed $v_0 = 70$ mph ($\sim 31\ m/s$) when it suddenly starts to brake. The brakes apply a constant force $k = 6500$ N (one newton is one $kg\ m/s^2$). Using Newton's second law of motion

$$m\dot{v} = -k,$$

where $v(t)$ is the speed of the car at time t, find out how long does it take the car to stop and how far does it travel before it comes to rest. What is the stopping distance of the car if $v_0 = 80$ mph ($\sim 36\ m/s$)?

Differential Equations
MAA121/MAG131 Sheet 2 (Solutions)

1. (a) 2nd order ODE; unknown function $y(x)$, independent variable x;
(b) 2nd order PDE; unknown function $u(x,t)$, independent variables x and t;
(c) 2nd order ODE; unknown function $x(t)$, independent variable t;
(d) 1st order ODE; unknown function $y(t)$, independent variable t.

2. The required function $g(x)$ satisfies the initial value problem

$$\frac{dg}{dx} = \log(1+x), \quad g(0) = 0.$$

Integrating both sides of the differential equation

$$\int g'(x)dx = \int \log(1+x)dx,$$

we obtain the general solution

$$g(x) = (1+x)\log(1+x) - x + C.$$

Substituting the initial data $g(0) = 0$ into the general solution, we obtain

$$\underbrace{g(0)}_{=0} = C,$$

so we conclude that $C = 0$ and the required solution of the initial value problem is

$$g(x) = (1+x)\log(1+x) - x.$$

3. (a) Integrating, we obtain

$$y(t) = \int \sin(t) + \cos(t)\, dt = -\cos(t) + \sin(t) + C,$$

which is the general solution of the equation.

To find the particular solution that passes through the origin we need $y(0) = 0$, i.e.

$$\underbrace{y(\pi)}_{=0} = -\cos(\pi) + \sin(\pi) + C = 1 + C,$$

so $C = -1$ and thus the required solution is

$$y(t) = -\cos(t) + \sin(t) - 1.$$

(b) By using partial fractions, we obtain

$$\frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right),$$

and so we need to solve

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right).$$

Integrating, we obtain

$$\begin{aligned} y(x) &= \int \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) \\ &= \frac{1}{2}\left(\log|x-1| - \log|x+1|\right) + C = \frac{1}{2}\log\frac{|x-1|}{|x+1|} + C, \end{aligned}$$

which is the general solution of the equation. To find the particular solution that passes through the origin we need $y(0) = 0$, i.e.

$$\underbrace{y(0)}_{=0} = \frac{1}{2}\log\frac{|-1|}{|+1|} + C,$$

so $C = 0$ and thus the required solution is

$$y(x) = \frac{1}{2}\log\frac{|x-1|}{|x+1|}.$$

(c) Integrating, we obtain

$$U(t) = \int 4t\log(t)\,dt = 2t^2\log(t) - t^2 + C.$$

To find the particular solution that passes through the origin we need $U(0) = 1$, i.e.

$$\underbrace{U(0)}_{=1} = C,$$

so $C = 1$ and thus the required solution is

$$U(t) = 2t^2\log(t) - t^2 + 1.$$

4. First of all, we need to solve the initial value problem

$$m\dot{v} = -k, \qquad v(0) = v_0.$$

The general solution of the differential equation is

$$v(t) = -\frac{k}{m}t + C.$$

To satisfy the initial condition $v(0) = v_0$ we need to set $C = v_0$, so the solution of the initial value problem is

$$v(t) = -\frac{k}{m}t + v_0.$$

The *stopping time* T of the car is the solution of the equation

$$v(t) = 0,$$

or, using the explicit form of the solution

$$-\frac{k}{m}t + v_0 = 0.$$

Thus we obtain

$$T = \frac{mv_0}{k}.$$

Since the velocity of the car is the time derivative of the postion x,

$$\dot{x} = v,$$

or, using again the explicit form of the solution we have

$$\dot{x} = -\frac{k}{m}t + v_0.$$

Integrating again, we obtain

$$x(t) = v_0 t - \frac{k}{2m}t^2 + C.$$

Assume that at time $t = 0$ the car is at the zero position, that is

$$x(0) = 0.$$

Then $C = 0$ and therefore the stopping distance is computed as

$$x(T) = v_0 T - \frac{k}{2m}T^2 = \frac{m}{2k}v_0^2.$$

Finally, using the values $m = 1000\ kg$ and $k = 6500$ N, we evaluate that if $v_0 = 31\ m/s$ then

$$x(T_*) = \frac{1000}{2 \times 6500}(31)^2 \simeq 74m,$$

and if $v_0 = 36\ m/s$ then

$$x(T_*) = \frac{1000}{2 \times 6500}(36)^2 \simeq 100m.$$