# Blow-up of Solutions of Semilinear Parabolic Equations

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## Contents

# 1 Sufficient conditions for blow-up

In this section, we consider blow-up of solutions of the problem

$$(F) \begin{cases} u_t = \Delta u + f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \quad (\text{if } \partial\Omega \neq \emptyset) \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

where $\Omega$ is a bounded or unbounded domain in $\mathbb{R}^N$, $N \geq 1$, $u_0$ is continuous and $u_0 \in L^\infty(\Omega)$. We begin with the fundamental result concerning the existence of solutions.

**Theorem 1.** *If $f \in C^1$, then (F) has a unique (classical) solution $u$ defined for $t \in [0, T)$, $0 < T \leq \infty$. If $T < \infty$, then $\|u(\cdot, t)\|_\infty \to \infty$ as $t \to T$ and $u$ is said to blow up at time $T$.*

*Idea of proof.* Let $e^{t\Delta}$ be the heat semigroup in $\Omega$ with the Dirichlet boundary condition, that is, $u = e^{t\Delta}u_0$ is the solution of

$$\begin{aligned} u_t &= \Delta u & in & \quad \Omega \times (0, \infty), \\ u &= 0 & on & \quad \partial\Omega \times (0, \infty), \\ u(x,0) &= u_0(x) & in & \quad \Omega. \end{aligned}$$

In particular, if $\Omega = \mathbb{R}^N$, then

$$(e^{t\Delta}u_0)(x) = (4\pi t)^{-\frac{N}{2}} \int_{R^N} u_0(y) \exp\left(-\frac{|x-y|^2}{4t}\right) dy.$$

By using $e^{t\Delta}$, the problem (F) is converted into the integral equation

$$u(\cdot, t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f(u(\cdot, s)) ds.$$

With the Banach fixed point theorem, a unique solution is obtained, and the solution can be continued as long is it is bounded.                    q.e.d.

Next, we give a sufficient condition for the blowup of solutions.

**Theorem 2** (Kaplan, 1963 [24])**.** *Let $\Omega$ be bounded. Let $\lambda_1, \varphi_1$ denote the first eigenvalue and eigenfunction of $-\Delta$ with the Dirichlet boundary condition, that is,*

$$\begin{aligned} \Delta\varphi_1 + \lambda_1\varphi_1 &= 0 & in & \quad \Omega, \\ \varphi_1 &= 0 & on & \quad \partial\Omega, \\ \varphi_1 &> 0 \quad in \quad \Omega, & &\int_\Omega \varphi_1(x)dx = 1. \end{aligned}$$

*Assume that $f : [0, \infty) \to [0, \infty)$ is convex, $f(u) > 0$ for $u > 0$, and $\int_1^\infty \frac{du}{f(u)} < \infty$. Define $g(u) = f(u) - \lambda_1 u$ and let $A$ be the l arger root of $g(u) = 0$ if it has a root. Suppose that $u_0 \geq 0$ and that $\int_\Omega \varphi(x) u_0(x) dx > A$ if $g(u) = 0$ has a root, otherwise $u_0 \not\equiv 0$. Then $u$ blows up in finite time.*

*Proof.* Multiplying $u_t = \Delta u + f(u)$ by $\varphi_1$ and integrating over $\Omega$, we have

$$\frac{d}{dt} \int_\Omega u\varphi dx = -\lambda_1 \int_\Omega u\varphi dx + \int_\Omega f(u)\varphi dx$$

$$\geq -\lambda_1 \int_\Omega u\varphi dx + f\left(\int_\Omega u\varphi dx\right)$$

by the Jensen inequality. Set

$$y(t) := \int_\Omega u\varphi_1 dx.$$

Then $y(t)$ satisfies

$$y' \geq -\lambda_1 y + f(y) = g(y).$$

By assumption that $f$ is convex and $\int_1^\infty \frac{dy}{f(y)} < \infty$, we obtain

$$\lim_{u \to \infty} \frac{f(u)}{u} = \infty.$$

Then there exists $U > A$ such that for $u \geq U$

$$f(u) > 2\lambda_1 u.$$

It follows that

$$t \leq \int_{y(0)}^{y(t)} \frac{dy}{g(y)}$$

$$\leq \int_{y(0)}^\infty \frac{dy}{g(y)}$$

$$\leq \int_{y(0)}^U \frac{dy}{g(y)} + \int_U^\infty \frac{2}{f(y)} dy$$

$$< \infty.$$

This completes the proof.                                                    q.e.d.

**Remark 1.** *The assumption $\int_1^\infty \frac{du}{f(u)} < \infty$ is necessary. Indeed, if $\int_1^\infty \frac{du}{f(u)} = \infty$, then the solution of $v_t = f(v)$, $v(0) = \|u_0\|_\infty$ is a global supersolution for $u$.*

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**Remark 2.** *The assumption $\int_1^\infty \frac{du}{f(u)} < \infty$ is not sufficient for blow-up, if $f$ is not convex.* (See **Fila, Ninomiya and Vázquez [8]**).

**Remark 3.** *The proof does not necessarily imply that $y(t) \to \infty$ as $t \to T$.*

When $f(u) = u^p$, there appears a critical exponent for the blowup and global existence of positive solutions.

**Theorem 3** (Fujita, 1966 [12])**.** *Consider the equation*

$$u_t = \Delta u + u^p, \ x \in \mathbb{R}^N, \ p > 1.$$

(i) *If $p < 1 + \frac{2}{N}$, all positive solutions blow up.*

(ii) *If $p > 1 + \frac{2}{N}$, there exist both blowing up and global positive solutions.*

**Hayakawa, 1973 [21], Kobayashi, Sirao, Tanaka, 1977 [25]:** The critical case $p = 1 + \frac{2}{N}$ is as (i).

Why is $p = 1 + \frac{2}{N}$ critical? The following is an intuitive explanation. The fundamental solution of $u_t = \Delta u$ decays like $t^{-\frac{N}{2}}$, while the solution of $u_t = u^p$ is given by $C(T - t)^{-\frac{1}{p-1}}$. They are balanced if

$$\frac{N}{2} = \frac{1}{p-1} \ \Leftrightarrow \ p = 1 + \frac{2}{N}.$$

Better explanation is as follows. Define $v(y, s) = e^{\frac{s}{p-1}} u(e^{\frac{s}{2}} y, e^s - 1)$. Then $v$ solves

$$v_s = \Delta v + \frac{y}{2} \cdot \nabla v + \frac{1}{p-1} v + v^p, \ y \in \mathbb{R}^N, \ s > 0.$$

Here the principal eigenvalue of $Lv := \Delta v + \frac{y}{2} \cdot \nabla v$ in $L_\rho^2$ with $\rho(y) = e^{\frac{|y|^2}{4}}$ is given by $\lambda_1 = -\frac{N}{2}$. This implies that the trivial solution $v \equiv 0$ is stable if $\frac{N}{2} > \frac{1}{p-1}$, and is unstable if $\frac{N}{2} < \frac{1}{p-1}$. As a consequence of instability, the solution blows up.

Next, consider

$$\begin{aligned}
u_t &= \Delta u + f(u), &&x \in \Omega, \\
u &= 0, &&x \in \partial\Omega, \\
u(x, 0) &= u_0(x),
\end{aligned}$$

where $\Omega$ is a bounded domain. For $v \in C^1(\overline{\Omega})$, define

$$E(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \int_\Omega F(v) dx, \ F(v) = \int_0^v f(u) du.$$

**Lemma 1.** *If $u$ is a solution for $t \in [0, T)$, then*

$$\frac{d}{dt} E(u(\cdot, t)) = -\int_\Omega u_t^2 dx \leq 0.$$

*Proof.* We calculate

$$\frac{d}{dt} E(u(\cdot, t)) = \int_\Omega \nabla u \cdot \nabla u_t dx - \int_\Omega f(u) u_t dx$$

$$= -\int_\Omega (\Delta u + f(u)) u_t dx = -\int_\Omega u_t^2 dx \leq 0.$$

<div align="right">q.e.d.</div>

**Theorem 4** (Levine, 1973 [28]). *Let $\Omega$ be bounded and $f(u) = |u|^{p-1} u$, $p > 1$. If $u_0 \in C^1(\overline{\Omega})$ satisfies $E(u_0) < 0$, then $u$ blows up in finite time .*

*Proof.* Multiplying $u_t = \Delta u + |u|^{p-1} u$ by $u$ and integrating over $\Omega$, we have

$$\int_\Omega u u_t dx = \int_\Omega u \Delta u dx + \int_\Omega |u|^{p+1} dx,$$

and hence

$$\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 dx = -\int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^{p+1} dx$$

$$= -2E(u) + \frac{p-1}{p+1} \int_\Omega |u|^{p+1} dx.$$

From the Hölder inequality, it follows that

$$\int_\Omega |u|^{p+1} dx = \int_\Omega (u^2)^{\frac{p+1}{2}} dx \geq C(\Omega) \left( \int_\Omega u^2 dx \right)^{\frac{p+1}{2}}.$$

Define $y(t) := \int_\Omega u^2 dx$. Then the function $y(t)$ satisfies

$$\frac{1}{2} y' > C(\Omega)^{-1} \frac{p-1}{p+1} y^{\frac{p+1}{2}}.$$

Since $y$ cannot exist globally, the solution $u$ cannot exists globally either.

<div align="right">q.e.d.</div>

Consider the stationary problem

$$\text{(S)} \quad \begin{cases} v_{xx} + v^p = 0, \ x \in (-L, L), \ L > 0, \ p > 1, \\ v(\pm L) = 0. \end{cases}$$

**Theorem 5.** *For every $L > 0$ and $p > 1$, there is a unique positive solution of* (S).

*Proof.* At first, we prove that $x = 0$ is the only maximum point of a positive solution of (S). Assume $x = x_0$ is a maximum point of a positive solution of (S), $v(x_0) = M > 0$. Since a positive solution of (S) is concave by $v_{xx} < 0$ on $(-1, 1)$, this maximum point is unique. Multiplying $v_{xx} + v^p = 0$ by $v_x$, we have

$$v_x v_{xx} + v_x v^p = 0,$$

$$(\frac{1}{2} v_x^2 + \frac{1}{p+1} v^{p+1})_x = 0.$$

Integrating this over $[x_0, x]$, $x > x_0$, by $v_x(x_0) = 0$, we obtain

$$\frac{1}{2} v_x^2 + \frac{1}{p+1} v^{p+1} = \frac{1}{p+1} M^{p+1}.$$

By $v_x \leq 0$ on $[x_0, L]$, we have

$$v_x^2 = \frac{2}{p+1}(M^{p+1} - v^{p+1}),$$

$$v_x = -\sqrt{\frac{2}{p+1}} \sqrt{M^{p+1} - v^{p+1}}.$$

By separation of variables, it follows that for $x_0 \leq x \leq L$

$$\frac{dv}{\sqrt{M^{p+1} - v^{p+1}}} = -\sqrt{\frac{2}{p+1}} dx,$$

$$\int_{v(x)}^{M} \frac{dw}{\sqrt{M^{p+1} - w^{p+1}}} = \sqrt{\frac{2}{p+1}}(x - x_0).$$

By $v(L) = 0$, we obtain

$$\int_{0}^{M} \frac{dw}{\sqrt{M^{p+1} - w^{p+1}}} = \sqrt{\frac{2}{p+1}}(L - x_0).$$

Similarly, by $v_x \geq 0$ on $[L, x_0]$, we obtain

$$\int_{0}^{M} \frac{dw}{\sqrt{M^{p+1} - w^{p+1}}} = \sqrt{\frac{2}{p+1}}(L + x_0).$$

It follows that $x_0 = 0$.

Next, we prove that a positive solution of $(S)$ is even. As above, we have

$$\int_{v(x)}^{M} \frac{dw}{\sqrt{M^{p+1} - w^{p+1}}} = \sqrt{\frac{2}{p+1}}x \ \ for \ 0 \le x \le L,$$

$$\int_{v(x)}^{M} \frac{dw}{\sqrt{M^{p+1} - w^{p+1}}} = -\sqrt{\frac{2}{p+1}}x \ \ for \ 0 \le x \le L.$$

It follows that

$$\int_{v(x)}^{M} \frac{dw}{\sqrt{M^{p+1} - w^{p+1}}} = \int_{v(-x)}^{M} \frac{dw}{\sqrt{M^{p+1} - w^{p+1}}}.$$

Since

$$\int_{\alpha}^{M} \frac{dw}{\sqrt{M^{p+1} - w^{p+1}}}$$

is monotone decreasing on $[0, M]$ with $\alpha$, it follows that $v(x) = v(-x)$, and so a positive solution of (S) is even.

Finally, as a solution of (S) is even, it is sufficient to prove that there is a unique positive solution of

$$(\text{SR}) \quad \begin{cases} v_{xx} + v^p = 0, & x \in (0, L), \\ v_x(0) = v(L) = 0. \end{cases}$$

A positive solution of (SR) can be obtained as a solution

$$(\text{SR})' \quad \begin{cases} v_{xx} + v^p = 0, & x > 0, \\ v_x(0) = 0, & v(0) = M > 0. \end{cases}$$

Define

$$\varphi(M) := \int_{0}^{M} \frac{dw}{\sqrt{M^{p+1} - w^{p+1}}}.$$

Then it is sufficient to show for every $L > 0$ that $\varphi(M) = \sqrt{\frac{2}{p+1}}L$ has unique positive solution. We calculate $\varphi(M)$ as

$$\varphi(M) = \int_{0}^{M} \frac{dw}{\sqrt{M^{p+1} - w^{p+1}}} = M^{-\frac{p+1}{2}} \int_{0}^{M} \frac{dw}{\sqrt{1 - (\frac{w}{M})^{p+1}}},$$

and we change the variable $\frac{w}{M} = y$, $dw = Mdy$ to obtain

$$\varphi(M) = M^{-\frac{p-1}{2}} \int_{0}^{1} \frac{dy}{\sqrt{1 - y^{p+1}}},$$

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where $\int_0^1 \frac{dy}{\sqrt{1-y^{p+1}}}$ is a constant dependent on $p$. Define $C_p := \int_0^1 \frac{dy}{\sqrt{1-y^{p+1}}}$, then it follows that

$$\varphi(M) = C_p M^{-\frac{p-1}{2}}.$$

Since $\varphi$ is monotone decreasing and the range of $\varphi$ is $(0, \infty)$, $\varphi(M) = \sqrt{\frac{2}{p+1}} L$ has a unique positive solution for every $L > 0$.

<div align="right">q.e.d.</div>

**Theorem 6.** *Consider the problem*

$$\text{(P)} \quad \begin{cases} u_t = u_{xx} + u^p, & |x| < 1, \ p > 1, \\ u = 0, & |x| = 1, \\ u(x, 0) = u_0(x), & |x| < 1. \end{cases}$$

*If $u_0(x) \geq v(x)$, $u_0 \not\equiv v(x)$, $v$ is the solution of (S), then $u$ blows up in finite time.*

*Proof.* By the maximum principle, if $\tau > 0$, then $u(x, \tau) > v(x)$ for $|x| < 1$. Moreover $u_x(1, \tau) < v_x(1)$ and $u_x(-1, \tau) > v_x(-1)$. Hence there is $k > 1$ such that $u(x, \tau) > kv(x)$.

Thus, it is suffcient to consider the case where $u_0(x) = kv(x)$, $k > 1$. In this case, we have

$$(u_t)_t = (u_t)_{xx} + p u^{p-2} u_t, \qquad |x| < 1,$$
$$u_t = 0, \qquad |x| = 1,$$
$$u_t(x, 0) = k v_{xx} + k^p v^p > k(v_{xx} + v^p) = 0.$$

By the maximum principle, we obtain $u_t \geq 0$ as long as $u$ exists.

Suppose $u$ is global. If $u$ is bounded, there exists a solution $w(x)$ of (S) such that

$$u(x, t) \to w(x) \ as \ t \to \infty, \ w(x) > v(x), \ |x| < 1,$$

this is a contradiction with Theorem 5.

If $u$ is unbounded, then

$$u(x, t) \equiv u(-x, t), \ u_x(x, t) < 0 \ for \ x \in [0, 1),$$

and either

$$\text{(i)} \ \lim_{t \to \infty} u(x, t) < \infty, \ x \in (a, 1], \ a \in (0, 1],$$

or

$$\text{(ii)} \ \lim_{t \to \infty} u(x, t) = \infty, \ |x| < 1,$$

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occurs. In case (i), there exists $w$ such that

$$\lim_{t \to \infty} u(x,t) = w(x), \ x \in (a,1],$$

$$(S^*) \begin{cases} w_{xx} + w^p = 0, \\ w(a) = \infty, \ w(1) = 0, \end{cases}$$

but $(S^*)$ does not have a solution, because $w$ is concave. In case (ii), there is $t_0 > 0$ such that

$$\int_{-1}^{1} u(x,t_0)\varphi(x)dx > \lambda_1^{\frac{1}{p-1}} \quad (= A \text{ in Theorem 2}),$$

this is a contradiction with Theorem 2. $\hspace{4cm}$ q.e.d.

# 2 Blow-up set

**Definition 1.** *Let $u$ blow up in finite time $T > 0$. Then $x_0 \in \Omega$ is **a blow-up point** if $u(x_n, t_n) \to \infty$ for some $\{(x_n, t_n)\}_{n=1}^{\infty} \subset \Omega \times (0,T)$ such that $(x_n, t_n) \to (x_0, T)$ as $n \to \infty$. **The blow-up set** $B$ is the set of all blow-up points.*

**Weissler, 1984 [37]**: If $u$ is the solution of (P) with $u_0(x) = kv(x)$, where $v(x)$ is the unique stationary solution and $k > 1$, then $B = \{0\}$.

**Theorem 7** (Friedman and McLeod, 1985 [11]). *Consider the problem*

$$\begin{aligned} u_t &= u_{rr} + \tfrac{N-1}{r}u_r + u^p, \quad (r,t) \in (0,R) \times (0,\infty), \ p > 1, \\ u(R,t) &= u_r(0,t) = 0, \qquad t \in (0,\infty), \\ u(r,0) &= u_0(r), \qquad\qquad r \in [0,R]. \end{aligned}$$

*Let $u_0 \in C^2([0,R])$, $(u_0)_r < 0$ for $r \in (0,R]$, $(u_0)_{rr} < 0$. Then for any $q \in (1,p)$, there is $K = K(u_0, p, q, R)$ such that*

$$u(r,t) \leq Kr^{-\frac{2}{q-1}} \ \text{for} \ t \in (0,T).$$

*(If $T < \infty$, then $B = \{0\}$.)*

*Proof.* Define

$$J(r,t) = r^{N-1}u_r + c(r)F(u).$$

Assuming

$$\sup_{r>0} \frac{c(r)}{r^N} < \infty,$$

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we claim that

$$J_t + \frac{N-1}{r}J_r - J_{rr} + b(r,t)J = -c(r)A(r,t),$$

$$b(r,t) - bounded\ for\ 0 \le r \le R,$$

$$A(r,t) = pu^{p-1}F - u^pF' + \frac{c^2}{r^{2N-2}}F'F^2 - \frac{2c'}{r^{N-1}}F'F + \frac{2(N-1)}{r^N}cF'F$$

$$+ \left(c'' - \frac{N-1}{r}c'\right)\frac{F}{c},$$

and if we choose $c(r) = \epsilon r^{N+\delta}$, $\epsilon$, $\delta > 0$, $F(u) = u^\gamma$, $1 < \gamma < p$, then for any $\gamma \in (1,p)$, $\delta > 0$, there is $\epsilon > 0$ such that $A > 0$.

*Proof of claim.* We compute

$$J_t + \frac{N-1}{r}J_r - J_{rr} + c(r)A(r,t)$$
$$= \left\{pu^{p-1} + 2(N-1)cF'r^{-N} - \frac{c^3}{r^{2N-2}}(r^{N-1}u_r + cF)\right\}J.$$

The function

$$b(r,t) = pu^{p-1} + 2(N-1)cF'r^{-N} - (r^{N-1}u_r + cF)c^3/r^{2N-2}$$

is bounded for $0 < r < R$ by the assumption that $\sup_{r>0} c(r)/r^N < \infty$. We choose $c(r) = \epsilon r^{N+\delta}$, $0 < \epsilon < 1$, $\delta > 0$, $F(u) = u^\gamma$, $1 < \gamma < p$. Then we have

$$A(r,t) = u^\gamma\{(p-\gamma)u^{p-1} - 2\epsilon(\delta+1)\gamma r^\delta u^{\gamma-1}$$
$$+ \epsilon^2\gamma(\gamma-1)r^{2\delta+2}u^{3\gamma-2} + (N+\delta)\delta r^{-2}\}.$$

By $p > \gamma$, there exists $M > 0$ independent of $\epsilon$ such that if $u > M$, then $A > 0$. With $u(R,t) = 0$, there is $R_0 > 0$ independent of $\epsilon$ such that for $R_0 < r < R$, we have

$$(p-\gamma)u^{p-1} - 2\epsilon(\delta+1)\gamma r^\delta u^{\gamma-1} + \epsilon^2\gamma(\gamma-1)r^{2\delta+2}u^{3\gamma-2} + (N+\delta)\delta r^{-2} > 0,$$

and $A(r,t) > 0$ *for* $R_0 < r < R$. There is $m > 0$ such that $u(r,t) > m$ *for* $0 < r < R_0$. If $u \le M$, $0 < r < R_0$, then we obtain

$$A(r,t) \ge u^\gamma\{(p-\gamma)m^{p-1} - 2\epsilon(\delta+1)\gamma r^\delta M^{\gamma-1}$$
$$+ \epsilon^2\gamma(\gamma-1)r^{2\delta+2}m^{3\gamma-2} + (N+\delta)\delta r^{-2}\}.$$

If $\epsilon$ is small, then it follows that $A(r,t) > 0$. Consequently, the proof is complete. q.e.d.

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Moreover, we have

$$J(0, t) = 0, \quad J(R, t) \leq 0.$$

If $\epsilon$ is small, we obtain $J(r, 0) \leq 0$. By the maximum principle, it follows that $J \leq 0$ for $(r, t) \in [0, R] \times [0, T)$. Then we have

$$u_r \leq -\epsilon r^{1+\delta} u^{\gamma}, \quad \frac{1}{1-\gamma}(u^{1-\gamma})_r \leq -\epsilon r^{1+\delta}.$$

Integrating this over $[0, r]$, we obtain

$$u^{1-\gamma}(r, t) \geq u^{1-\gamma}(r, t) - u^{1-\gamma}(0, t) \geq \epsilon(\gamma - 1)\frac{1}{2+\gamma}r^{2+\gamma},$$

$$u(r, t) \leq Kr^{-\frac{2+\delta}{\gamma-1}},$$

and so we choose $\gamma$, $\delta$ such that $\frac{2+\delta}{\gamma-1} = \frac{2}{q-1}$. 

<div align="right">q.e.d.</div>

**Remark 4.** *Theorem 7 holds for $f(u)$ more general than $u^p$. It is essential that $f(u) \geq Cu \log^q u$, $u \geq U > 1$, $q > 2$.*

**Remark 5.** *If $f(u) = (u + a) \log^q(u + a)$, $a > 1$, then blow-up occurs if and only if $q > 1$. This follows from Theorem 2 and Remark 1.*

**Lacey, 1986 [25]**: Consider the problem

$$u_t = \Delta u + f(u), \quad x \in \Omega - bounded,$$
$$u = 0, \qquad\qquad x \in \partial\Omega.$$

(i) If $\Omega$ is large and $f(u) = (u + 2) \log^2(u + 2)$, then there is $\Omega_0 \subset \Omega$, $|\Omega_0| > 0$, $\overline{\Omega_0} \subset \Omega$, such that $B = \{\overline{\Omega_0}\}$ (regional blow-up), if $\Omega$ is small, then $B = \overline{\Omega}$.

(ii) If $f(u) = (u + 2) \log^q(u + 2)$, $1 < q < 2$, then $B = \overline{\Omega}$.

**Chen and Matano, 1989 [5]**: Consider the problem

$$u_t = u_{xx} + f(u), \quad |x| < 1,$$
$$u = 0, \qquad\qquad |x| = 1,$$

under some assumtion on $f$. Assume $T < \infty$, then $B$ consists of finite number of points. This number is smaller than or equal to the number of local maxima of $u_0$.

**Merle, 1992 [30]**: Consider the problem

$$
\begin{aligned}
u_t &= u_{xx} + u^p, & |x| &< 1, \ p > 1, \\
u &= 0, & |x| &= 1, \\
u(x,0) &= u_0(x), & |x| &\leq 1.
\end{aligned}
$$

Given any positive integer $k$ and $-1 < x_1 < \cdots < x_k < 1$, there is $u_0$ such that $u$ blows up at $t = T < \infty$ and $B = \{x_1, \ldots, x_k\}$.

**Giga and Kohn, 1989 [18]**: Consider the problem

$$
\begin{aligned}
u_t &= \Delta u + u^p, & x &\in \Omega, \\
u &= 0, & x &\in \partial\Omega, \\
u(x,0) &= u_0(|x|) \geq 0, & x &\in \Omega,
\end{aligned}
$$

where $\Omega = \{|x| < R\} \subset \mathbb{R}^N$, $p > 1$ and $(N-2)p < N+2$. Then there is $u_0$ such that $u$ blows up on a sphere $B = \{|x| = R_0\}$, $0 < R_0 < R$.

**Velázquez, 1993 [36]**: Consider the equation

$$
u_t = \Delta u + u^p, \ x \in \mathbb{R}^N, \ p > 1, \ (N-2)p < N+2.
$$

Since $(N-1)$-dimensional Hausdorff measure is finite in every compact sets, the dimension of $B$ is at most $(N-1)$ in every compact sets if $u \not\equiv C(T-t)^{-\frac{1}{p-1}}$.

# 3 Blow-up rate

In this section, we consider the blow-up rate of solutions for

$$
u_t = \Delta u + |u|^{p-1}u.
$$

Define $M(t) := \max_{\overline{\Omega}} u(\cdot, t)$, then

$$
M'(t) \leq M^p(t), \quad M(t) \geq ((p-1)(T-t))^{-\frac{1}{p-1}}.
$$

**Definition 2.** *Blow-up is of **type** I if $(T-t)^{-\frac{1}{p-1}} \|u(\cdot, t)\|_\infty \leq C$, otherwise blow-up is of **type** II.*

**Weissler, 1985 [38]**: Assume the condition

$$
(N-2)p < N+2, \ \Omega = \{|x| < R\} \subset \mathbb{R}^N, \ u - radial, \ u_r, \ u_{rr} \leq 0, \ u_t \geq 0.
$$

Then blow-up is type I.

**Friedman and McLeod, 1985 [11]**: Blow-up is of type I, if $\Omega$ is bounded, $u$, $u_t \geq 0$ ($u_0 \geq 0$, $\Delta u_0 + u_0^p \geq 0$ *in* $\Omega$).

*Proof.* Define $J(r,t) = u_t - \epsilon u^p$ for $\epsilon > 0$. Then we have

$$J_t - \Delta J - p u^{p-1} J = \epsilon p(p-1) u^{p-2} |\nabla u|^2 \geq 0,$$
$$J = 0 \text{ on } \partial\Omega, \quad J \geq 0 \text{ if } t = 0.$$

By the maximum principle, we obtain $J \geq 0$ *in* $\Omega \times (0, T)$. It follows that

$$u_t \geq \epsilon u^p,$$

and so we obtain

$$u(x,t) \leq ((p-1)\epsilon(T-t))^{-\frac{1}{p-1}}.$$

<div align="right">q.e.d.</div>

The results on type I blow-up are listed as follows:

- **Galaktionov and Posashkov, 1986 [13]**: N=1.

- **Giga and Kohn, 1985 [16], 1989, [18]**: $\Omega = \mathbb{R}^N$ or $\Omega$ is bounded, convex and either (i) $u \geq 0$, $(N-2)p < N+2$ or (ii) $(3N-4)p < 3N+8$.

- **Giga, Matsui and Sasayama, 2004 [19], [20]**: $\Omega = \mathbb{R}^N$ or $\Omega$ is bounded, convex, $(N-2)p < N+2$.

- **Filippas, Herrrero and Velázquez, 2000 [10]**: $\Omega = \{|x| < R\}$, $u$ is radial, $u_r \geq 0$, $p = \frac{N+2}{N-2}$.

- **Matano and Merle, 2004 [29]**: $\Omega = \{|x| < R\}$, $u$ is radial, $p > \frac{N+2}{N-2}$, $N > 2$, $p < p_{JL} = 1 + \frac{4}{N-4-2\sqrt{N-1}}$ if $N > 10$.

The results on type II blowup are listed as follows:

- **Herrero and Velázquez, 1993 [22]**: There are solutions with type II blow-up if $p > p_{JL}$.

- **Mizogichi, 2004 [32]**: The same result as [22] with a simlar but shorter proof.

In order to explain why type II blow-up occurs, we define

$$w(y, s) = (T - t)^{\frac{1}{p-1}} u(x, t), \quad y = \frac{x}{\sqrt{T - t}}, \quad s = -\log(T - t).$$

Then $w$ satisfies

$$\text{(W)} \quad w_s = \Delta w - \frac{y}{2} \nabla w + w^p - \frac{1}{p-1} w, \ y \in \mathbb{R}^N, \quad s > -\log T.$$

There is a solution $w$ such that

$$w(y, s) \to w^*(y) \ as \ s \to \infty,$$

where $w^*(y)$ is the singular stationary solution $C|y|^{-\frac{2}{p-1}}$.

# 4 Blow-up profile

In this section, we consider behavior at the blow-up time of solution for

$$u_t = \Delta u + u^p, \ p > 1.$$

We note that (W) has a stationary solution $k = (p - 1)^{-\frac{1}{p-1}}$.

**Giga and Kohn, 1987 [17]**: The constant $k$ is the only positive bounded stationary solution of (W). Moreover $w(y, s) \to k$ as $s \to \infty$ uniformly for $|y| \leq C, \ C > 0$.

**Herrero and Velázquez, 1992 [23]**: Consider the problem

$$u_t = u_{xx} + u^p, \quad x \in \mathbb{R},$$
$$u(x, 0) = u_0(x) \geq 0,$$

where $u_0$ is continuous, bounded, $u_0 \not\equiv C$ and such that $u$ blows up at $T < \infty$ with the blowup set $B = \{0\}$. Then, one of the following alternatives holds:

(a) $\lim\limits_{x \to 0} \left( \frac{|x|^2}{|\log |x||} \right)^{\frac{1}{p-1}} u(x, T) = \left( \frac{8p}{(p-1)^2} \right)^{\frac{1}{p-1}},$

(b) There exist a constant $C > 0$ and even integer $m \geq 4$ such that

$$\lim\limits_{x \to 0} |x|^{\frac{m}{p-1}} u(x, T) = C.$$

The case (a) occurs if $u_0$ has a single maximum. The case (b) with $m = 4$ occurs if $u_0$ has two local maxima for $t < T$ which merge at $T = t$.

**Bricmont and Kupiainen, 1994 [3]**: All profiles in (b) do occur.

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# 5   Complete blow-up

In this section, we consider

$$
(\mathrm{P})' \quad
\begin{cases}
u_t = \Delta u + u^p, & x \in \Omega : \text{bounded, } p > 1, \\
u = 0, & x \in \partial\Omega, \\
u(\cdot, 0) = u_0 \geq 0, \ u_0 \in L^\infty(\Omega).
\end{cases}
$$

**Definition 3** (Baras and Cohen, 1987 [2])**.** *Let $f_n(u) = \min\{u^p, n\}$, let $u_n$ be the solution of $(\mathrm{P})'$ with $f_n(u)$ instead of $u^p$, let $u$ be a solution of $(\mathrm{P})'$ which blows up at $t = T < \infty$. Then blow-up is* **complete** *if*

$$
\lim_{n \to \infty} u_n(x, t) = \infty \ \text{for} \ (x, t) \in \Omega \times (T, \infty).
$$

**Baras and Cohen, 1987 [2]:** Let $u_0$ be such that $u$ blows up. If $(N-2)p < N + 2$ or if $u_0 \in C^2$, $\Delta u_0 + u_0^p \geq 0$, then blow-up is complete.

# 6   Continuation after blow-up

**Definition 4.** *The function $u$ is* **a weak solution** *of $(P)'$ on $[0, T]$ if*

(a) $u \in C([0, T]; L^1(\Omega))$,

(b) $u^p \in L^1(\Omega \times (0, T))$,

(c) $\displaystyle \int_\Omega u(x, t_2)\psi(x, t_2)dx - \int_\Omega u(x, t_1)\psi(x, t_1)dx - \int_{t_1}^{t_2} \int_\Omega u\psi_t dxds$

$$
= \int_{t_1}^{t_2} \int_\Omega (u\Delta\psi + u^p\psi)dxds
$$

*for every $\psi \in C^2(\overline{\Omega} \times [0, T])$ with $\psi = 0$ on $\partial\Omega$, $0 \leq t_1 < t_2 \leq T$.*

*The function $u$ is* **a global weak solution** *if it is a weak solution on $[0, T]$ for every $T > 0$. If $\Omega = \mathbb{R}^N$, then we require that (c) holds for every $\psi \in C^2(\mathbb{R}^N \times [0, T])$ with compact support and (a), (b) hold for $L^1_{loc}$.*

**Theorem 8** (Ni, Sacks and Tavantzis, 1984 [34])**.** *Let $p \geq \frac{N+2}{N-2}$, $N > 2$, and $\Omega$ be convex. Then for every $f \in L^\infty(\Omega)$ with $f \geq 0$, $f \not\equiv 0$, there is $\lambda^* > 0$ such that the problem $(P)'$ with $u_0 = \lambda^* f$ has an unbounded global weak solution.*

*Idea of Proof.* Define

$$
\lambda^* = \sup\{\lambda > 0; \text{ the solution } u \text{ of } (P)' \text{ with } u_0 = \lambda f
$$
$$
\text{is a global classical solution, } u(\cdot, t) \to 0 \ as \ t \to \infty \}.
$$

Then $0 < \lambda^* < \infty$. The function $u(x,t;\lambda^* f)$ is not a global classical solution converging to zero. If $\Omega$ is convex, there is no positive stationary solution, and hence $u(x,t;\lambda^* f)$ cannot be uniformly bounded. Because it is a monotone limit of global classical solutions converging to zero, $u(x,t;\lambda^* f)$ must be a global weak solution.                                      q.e.d.

**Galaktionov and Vázquez, 1997 [15]:** Let $p \geq \frac{N+2}{N-2}$, $N > 2$, and let $p < 1 + \frac{6}{N-10}$ if $N > 10$. Suppose that $\Omega = \{|x| < R\}$ and $u_0$ is radial. Then the solution $u$ from Theorem 8 blows up in finite time.

**Mizoguchi, 2005 [31]:** The restriction $p < 1 + \frac{6}{N-10}$ if $N > 10$ can be removed.

Consider the problem

$$u_t = u_{xx} + a(x,t)u_x + b(x,t)u,$$
$$u(x_0,t_0) = 0, \ u_x(x_0,t_0) = 0, \ u_{xx}(x_0,t_0) = 0,$$

where $a(x,t)$ and $b(x,t)$ are bounded. Define

$$z(u(\cdot,t)) = \sharp\{\, r; \ u(r,t) = 0\,\}.$$

With the fact that $z(\cdot,t)$ is nonincreasing in time (Sturm, 1836) and $z(\cdot,t)$ decreases when a multiple zero occurs (Angenent, 1986 [1]), we obtain the following results.

**Fila, Matano and Poláčik, 2005 [7]:** Let $p \geq \frac{N+2}{N-2}$, $N > 2$, and let

$$p < p_{JL} = 1 + \frac{4}{N - 4 - 2\sqrt{N-1}}$$

if $N > 10$, $\Omega = \{|x| < R\}$ Suppose that $u_0$ is radial and such that $u$ blows up but it is a minimal global weak solution. Then there are $t_1 = T < t_2 < t_3 < \cdots < t_k < \infty$ such that $\|u(\cdot,t_j)\| = \infty$ and $\|u(\cdot,t_j)\| < \infty$ if $t \neq t_j$, $j = 1,\ldots,k$.

**Mizoguchi, 2005 [31]:** If $N > 10$, $p > p_{JL}$, $\Omega = \mathbb{R}^N$, then there is $u_0$ such that $k = 2$.

**Example 1.** *The function $u(r,t) = \{\varphi(t) + r^2\}^{-1}$ satisfies*

$$u_t = u_{rr} + \frac{N-1}{r}u_r + g(r,t)u^2,$$

*with*

$$g(r,t) = 2N - \varphi'(t) - \frac{8r^2}{\varphi(t) + r^2}.$$

*Assume $\varphi(t) \geq 0$. If $\varphi(T) = 0$, then $u$ blows up. If $\varphi(t) = 1 - e^{-(t-1)^2}$, then $u$ blows up at $t = 1$. If $N > 4$, then there are $0 < c_1 < c_2 < \infty$ such that $c_1 \leq g \leq c_2$. The function $u$ is a global weak solution if and only if $N > 4$.*

*Proof.* At first, we prove that there are $0 < c_1 < c_2 < \infty$ such that $c_1 \leq g \leq c_2$, if $N > 4$. To consider maximum and minimum of $g(\cdot, t)$, we calculate

$$g(r,t) = 2N - (t-1)e^{-(t-1)^2} + \frac{8r^2}{1 - e^{-(t-1)^2} + r^2},$$

$$2N - 8 - 2(t-1)e^{-(t-1)^2} \leq \min_r g \leq \max_r g \leq 2N - 2(t-1)e^{-(t-1)^2}.$$

We consider $f(t) = (t-1)e^{-(t-1)^2}$, we calculate

$$f'(t) = \{1 - 2(t-1)^2\}e^{-(t-1)^2}$$

and

$$f(0) = -e^{-1}, \ f(1 - \frac{1}{\sqrt{2}}) = -\frac{1}{\sqrt{2}}e^{-\frac{1}{2}}, \ f(1 + \frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}e^{-\frac{1}{2}}, \ \lim_{t \to \infty} f(t) = 0.$$

By this calculation, we have

$$-\frac{1}{\sqrt{2}}e^{-\frac{1}{2}} \leq f(t) \leq \frac{1}{\sqrt{2}}e^{-\frac{1}{2}}.$$

Consequently, we obtain

$$0 < 2N - 8 - \sqrt{2}e^{-\frac{1}{2}} \leq g \leq 2N + \sqrt{2}e^{-\frac{1}{2}} < \infty.$$

Next, we find a necessary and sufficient condition on $N$ such that $u$ is s global weak solution. Assume $N > 4$. Fix $T > 0$. By $0 \leq u(r,t) \leq 1/r^2$, the Lebesgue convergence theorem and $c_1 \leq g \leq c_2$, we have

$$u \in C([0,T]; L_{loc}^1(\mathbb{R}^N)), \quad g(r,t)u \in L_{loc}^1(\mathbb{R}^N \times (0,T)).$$

We take $\psi(x,t) \in C_0^2(\mathbb{R}^N \times [0,T])$, $0 \leq t_1 < t_2 \leq T$ and $R > 0$. Then we have

$$\int_{\mathbb{R}^N} u(x,t_2)\psi(x,t_2)dx - \int_{\mathbb{R}^N} u(x,t_1)\psi(x,t_1)dx - \int_{t_1}^{t_2} \int_{\mathbb{R}^N} u\psi_t dxds$$

$$= \int_{|x| \leq R} u(x,t_2)\psi(x,t_2)dx - \int_{|x| \leq R} u(x,t_1)\psi(x,t_1)dx + \int_{t_1}^{t_2} \int_{|x| > R} u_t\psi dxds$$

$$- \int_{t_1}^{t_2} \int_{|x| \leq R} u\psi_t dxds.$$

Moreover, we have

$$\int_{t_1}^{t_2}\int_{|x|>R}u_t\psi dxds = \int_{t_1}^{t_2}\int_{|x|>R}(\Delta u\psi + g(r,t)u\psi)dxds$$

$$= \int_{t_1}^{t_2}\int_{|x|>R}(u\Delta\psi + g(r,t)u\psi)dxds - \int_{t_1}^{t_2}\int_{|x|=R}\frac{\partial u}{\partial r}\psi dSdt$$

$$+ \int_{t_1}^{t_2}\int_{|x|=R}u\frac{\partial\psi}{\partial r}dSdt.$$

With $0 \le u \le 1/r^2$, we obtain

$$\left|\int_{|x|\le R}u(x,t_2)\psi(x,t_2)dx\right|, \left|\int_{|x|\le R}u(x,t_1)\psi(x,t_1)dx\right|$$

$$\le \int_{|x|\le R}\frac{1}{r^2}\|\psi\|_\infty dx = C\|\psi\|_\infty R^{N-2} \to 0 \ as \ R\to 0 \ ,$$

$$\left|\int_{t_1}^{t_2}\int_{|x|\le R}u\psi_t dxds\right| \le \int_{t_1}^{t_2}\int_{|x|\le R}\frac{1}{r^2}\|\psi_t\|_\infty dxds$$

$$= C(t_2-t_1)\|\psi_t\|_\infty R^{N-2} \to 0 \ as \ R\to 0,$$

and by $0 \ge \frac{\partial u}{\partial r} \ge -\frac{2}{r^3}$, we obtain

$$\left|\int_{t_1}^{t_2}\int_{|x|=R}\frac{\partial u}{\partial r}\psi dSdt\right| \le \int_{t_1}^{t_2}\int_{|x|=R}\frac{2}{r^3}\|\psi\|_\infty dSdt$$

$$= C(t_2-t_1)\|\psi\|_\infty R^{N-4} \to 0 \ as \ R\to 0 \ ,$$

$$\left|\int_{t_1}^{t_2}\int_{|x|=R}u\frac{\partial\psi}{\partial r}dSdt\right| \le \int_{t_1}^{t_2}\int_{|x|=R}\frac{1}{r^2}\|\nabla\psi\|_\infty dxds$$

$$= C(t_2-t_1)\|\nabla\psi\|_\infty R^{N-3} \to 0 \ as \ R\to 0.$$

By this calculation, it follows that as $R\to\infty$

$$\int_{\mathbb{R}^N}u(x,t_2)\psi(x,t_2)dx - \int_{\mathbb{R}^N}u(x,t_1)\psi(x,t_1)dx - \int_{t_1}^{t_2}\int_{\mathbb{R}^N}u\psi_t dxds$$

$$= \int_{t_1}^{t_2}\int_{R^N}(u\Delta\psi + g(r,t)u\psi).$$

This means that $u$ is a weak solution on $[0,T]$. Because $0 < T < \infty$ is arbitrary, $u$ is a global weak solution.

Finally, assume $u$ is a global weak solution. This needs $u \in C([0,T]; L^1_{loc}(\mathbb{R}^N))$ and $g(r,t)u \in L^1_{loc}(\mathbb{R}^N \times (0,T))$ for $0 < T < \infty$. By $u \in C([0,T]; L^1_{loc}(\mathbb{R}^N))$

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for $0 < T < \infty$, we obtain $u(x, 1) = \frac{1}{r^2} \in L^1_{loc}(\mathbb{R}^N)$. Then we need $N > 2$.
We consider $g(r, t)u \in L^1_{loc}(\mathbb{R}^N \times (0, T))$ for $0 < T < \infty$. With $1 - e^{-t} \leq t$,
we have

$$g(r, t) \geq 2N - 2(t - 1)^2 - \frac{8r^2}{(t - 1)^2 + r^2}.$$

If $r \leq C|t - 1|$, $|t - 1| \leq C$, we have

$$g(r, t) \geq 2N - 2C^2 - \frac{8C^2}{1 + C^2}.$$

If C is sufficiently small, obtain $g(r, t) \geq N$. By this, we obtain

$$\int_{1-C}^{1+C} \int_{r \leq C|t-1|} g(r,t)u(r,t)^2 dxdt \geq \int_{1-C}^{1+C} \int_{r \leq C|t-1|} N \frac{1}{\{(t-1)^2 + r^2\}^2} \omega_N r^{N-1} drdt$$

$$\geq A_N \int_{1-C}^{1+C} |t-1|^{N-5} \int_{r \leq C} \frac{r^{N-1}}{(1 + r^2)^2} drdt$$

$$= \infty, \ if \ N \leq 4.$$

This is a contradiction. It follows that $N > 4$.

Consequently, a necessary and sufficient condition on $N$ such that $u$ is a global weak solution is $N > 4$. q.e.d.

# 7  Boundedness of global solutions

In this section, we consider the boundedness of global solutions for

$$\begin{cases} u_t = \Delta u + f(u), & x \in \Omega \\ u = 0, & x \in \partial\Omega, \\ u(x, 0) = u_0(x), \ u_0 \in L^\infty. \end{cases}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$. We address the following

Question: Does blow-up in infinite time occur?

It is easy to see that the answer is "Yes" if $f(u) = \lambda u$, $\lambda > 1$.

A function $f$ is said to be superlinear, if there exists $\epsilon$, $A > 0$ such that

$$uf(u) \geq (2 + \epsilon) \int_0^u f(v)dv > 0 \ for \ \forall u \geq A.$$

What if $f$ is superlinear? The answer is "No" if

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(i) $N = 1$ $f$ is locally Lipschitz (Fila, 1992 [6]),

(ii) $N = 2$,$|f'(u)| \leq K e^{|u|^q}$, $0 < q < 2$ (Fila, 1992 [6]),

(iii) $N > 3$, $f$ is locally Lipschitz, $|f(u)| \leq C(1 + |u|^p)$, $(N-2)p < N + 2$ (Cazenave and Lions, 1984 [4]),

(iv) $N > 2$, $f(u) = u^p$, $u \geq 0$, $p > \frac{N+2}{N-2}$, $p < 1 + \frac{6}{N-10}$ if $N > 10$, $\Omega = B_R = \{|x| < R\}$, $u$ is radial (Galaktionov and Vázquez, 1994 [14]),

(v) $3 \leq N \leq 9$, $f(u) = e^u$, $\Omega = B_R$, $u$ is radial (Fila and Poláčik, 1999 [9]),

(vi) $N > 2$, $f \in C^1$, $f \geq 0$, $\lim_{u \to \infty} \frac{f(u)}{u^p} = 1$, $p > \frac{N+2}{N-2}$, $\Omega = B_R$, $u$ is radial, $p < p_{JL}$ if $N > 10$ (Chen, Fila and Guo).

The answer is "Yes", if

(vii) $f(u) = u^p$, $N > 2$, $p = \frac{N+2}{N-2}$, $\Omega = B_R$, $u$ is radial, (Ni, Sacks and Tavantzis, 1984 [34], Galaktionov and Vázquez, 1994 [14]),

(viii) $f(u) = 2(N-2)e^u$, $N > 9$, $\Omega = B_1$ (Lacey and Tzanetis, 1987 [27]).


**Explanations**
(vii) Define
$$\lambda^* := \sup\{\lambda > 0; \ u(\cdot, t; \lambda\varphi) \to 0 \ as \ t \to \infty\}$$

for $\varphi \in L^\infty(\Omega)$, $\varphi \geq 0$, $\varphi \not\equiv 0$. Then $u(\cdot, t; \lambda^*\varphi)$ is a global weak solution. Galaktionov and Vázquez, 1997 [15], showed that blow-up in finite time is complete. This implies that $u(\cdot, t; \lambda^*\varphi)$ is a global classical solution. If $u(\cdot, t; \lambda^*\varphi)$ is bounded, $u(\cdot, t; \lambda^*\varphi) \to 0$ $as$ $t \to \infty$ by Corollary 1, but 0 is stable, which contradicts the definition of $\lambda^*$. It follows that $u(\cdot, t; \lambda^*\varphi)$ is a classical unbounded solution.

**Pohožaev identity, 1965 [35]** (for a ball)
    If $u$ is a nontrivial solution of

$$\text{(S)} \quad \begin{cases} u_{rr} + \frac{N-1}{r}u_r + |u|^{p-1}u = 0, & 0 < r < 1, \\ u'(0) = u(1) = 0, \end{cases}$$

then

$$\left(N - 2 - \frac{2N}{p+1}\right) \int_0^1 r^{N-1}|u|^{p+1} dr = -u_r^2(1).$$

**Corollary 1.** *If $p \geq \frac{N+2}{N-2}$, $N \geq 2$, then there is no nontrivial solution of* (S).

*Proof for Pohožaev identity.* This equation is converted into

(1) $$(r^{N-1}u_r)_r + r^{N-1}|u|^{p-1}u = 0.$$

Multiplying (1) by $u$ and integrating it over $(0,1)$ by parts, we obtain

(2) $$\int_0^1 r^{N-1}u_r^2 dr = \int_0^1 r^{N-1}|u|^{p+1}dr.$$

Multiplying (1) by $ru_r$ and integrating it over $(0,1)$ by parts, we have

$$u_r^2(1) - \int_0^1 r^{N-1}u_r^2 dr - \int_0^1 r^N u_r u_{rr} dr + \int_0^1 r^N |u|^{p-1}uu_r dr = 0.$$

In view of $u_{rr} + \frac{N-1}{r}u_r + |u|^{p-1}u = 0$, this yields

(3) $$u_r^2(1) - (N-2)\int_0^1 r^{N-1}u_r^2 dr + 2\int_0^1 r^N |u|^{p-1}uu_r dr = 0.$$

Here we have

(4) $$\int_0^1 r^N |u|^{p-1}uu_r dr = -\frac{N}{p+1}\int_0^1 r^{N-1}|u|^{p+1}dr.$$

By combining (2),(3),(4), we obtain

$$\left(N - 2 - \frac{2N}{p+1}\right)\int_0^1 r^{N-1}|u|^{p+1}dr = -u_r^2(1).$$

<div align="right">q.e.d.</div>

(b) The function $\varphi^*(r) = -2\log r$ satisfies

$$\varphi_{rr} + \frac{N-1}{r}\varphi_r + 2(N-2)e^\varphi = 0, \ r > 0,$$
$$\varphi(1) = 0.$$

If $u_0 < \varphi^*$, then a solution $u$ is global and

$$u \to \varphi \ as \ t \to \infty.$$

Define
$$u_\lambda(x,t) = \begin{cases} \lambda^{p-1}u(\lambda x, \lambda^2 t) & \text{if} \ \ f(u) = u^p, \\ 2\lambda + u(e^\lambda x, e^{2\lambda}) & \text{if} \ \ f(u) = e^u. \end{cases}$$

The function $u_\lambda$ is a solution if $u$ is a solution. In (iv) and (v), the scaling invariance is used.

**Lemma 2.** *There is $v^*$ satisfying*

$$v_{rr} + \frac{N-1}{r}v_r + f(v) = 0, \quad v_r < 0 < v, \quad r \in (0, \epsilon), \quad \epsilon > 0,$$

$\lim_{r\to 0} r^{\frac{2}{p-1}} v^*(r) = K = K(p, N)$ *is the constant for which* $Kr^{-\frac{2}{p-1}}$ *satisfies*

$$\varphi_{rr} + \frac{N-1}{r}\varphi_r + \varphi^p = 0, \quad p > \frac{N}{N-2}.$$

**Ni and Sacks, 1985 [33]**: Every global classical solution is radially decreasing after some $t_0 > 0$.

*Proof of* (6). Suppose $u$ is a global classical solution, $u_r \leq 0$, $\|u(\cdot, t_i)\|_\infty \to \infty$ as $t_i \to \infty$. Set

$$M_i = u(0, t_i), \quad R_i = M_i^{-\frac{p-1}{2}}, \quad w_i(\rho, \tau) = R_i^m u(R_i\rho, t_i + R_i^2\tau), \quad m = \frac{2}{p-1}.$$

Then $w$ satisfies

$$w_{i,\tau} - \Delta w_i = w_i^p g_i, \quad g_i = \frac{f(M_i w_i)}{(M_i w_i)^p} \to 1 \text{ as } i \to \infty, \quad w_i(\cdot, 0) \to \varphi_1,$$

where $\varphi_1$ satisfies

$$\varphi_{rr} + \frac{N-1}{r}\varphi_r + \varphi^p = 0, \ r > 0,$$
$$\varphi_r(0) = 0, \ \varphi(0) = 1.$$

If $p < p_{JL}$, then

$$z(\varphi^* - \varphi_1) = \infty,$$

and so $z(\varphi_* - w_i(\cdot, 0))$ can be made arbitrarily large on $[0, \rho^*]$, if $\rho^*$, $i$ is large. Then $z(w_i(\cdot, 0) - v_i^*)$ is large on $[0, \rho^*]$, where $v_i^*(\rho) = R_i^m v^*(R_i\rho)$, $z(u(\cdot, t_i) - v^*)$ is large on $[0, \rho^* R_i]$. But $z(u(\cdot, t) - v^*)$ is nonincreasing in $t$, it is bounded by $z(u(\cdot, 0) - v^*)$, this is a contradiction. q.e.d.

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