Further Topics in Analysis

Lecture Notes 2006/07

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Contents

Notations

- $\mathbb{N}$ *set of natural numbers* $\{0, 1, 2, 3, \dots\}$
- $\mathbb{N}^+$ *set of positive natural numbers* $\{1, 2, 3, \dots\}$
- $\mathbb{Z}$ *set of integer numbers* $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- $\mathbb{Q}$ *set of rational numbers* $\{r = \frac{p}{q} \mid p \in \mathbb{Z},\ q \in \mathbb{N}^+,\ \mathrm{hcf}(p, q) = 1\}$
- $\mathbb{R}$ *set of real numbers*

Part I
Elements of Set Theory

RECOMMENDED TEXTS:

1. I. STEWART, D. TALL "The Foundations of Mathematics", Oxford University Press, 1977.
2. S. KRANTZ "Real Analysis and Foundations", Second Edition. Chapman and Hall/CRC Press, 2005.

This Lecture Notes, Current Exercises and Solutions to Exercises are available on the Web:
`http://www.maths.bris.ac.uk/~mavbm/fta.html`

In this part of the course we study the notion of *cardinality* of a set. An idea behind this notion is to assign a "size" to an infinite set, or "to count" "number of elements" in an infinite set. One can ask questions like "how many rational numbers are there ?". Or "is the set of real numbers bigger then the set of natural numbers ?" However rather then ask "how many elements" there are in a given set, one can compare two sets and ask if one has the same "number" of elements as another. This can be described by saying that sets X and Y are equivalent ("have the same number of elements") if there is a bijection $f : X \to Y$. That was an idea behind the (*Naive*)[1] *Set theory* theory, built by a German mathematician Georg Cantor (1845 – 1918) in the end of XIX's century.

1 Finite and infinite sets. Countable sets.

Recall that the notion of *set* is one of the basic notions. We can not, therefore, give its precise mathematical definition. Intuitively, *a set is a collection of elements.* [2]

Let X, Y be sets and $f : X \to Y$ is a function (a map) from X to Y. The function f is an *injection* if

$$(\forall x_1 \in X)(\forall x_2 \in X)[(f(x_1) = f(x_2) \Rightarrow (x_1 = x_2)].$$

This means that f is one–to–one correspondence between X and the range $Ran(f)$. The function f is a *surjection* if

$$(\forall y \in Y)(\exists x \in X)[f(x) = y].$$

This simply means that $Ran(f) = Y$. Finally, f is a *bijection* if f is an injection and surjection. In this case f establishes one-to-one correspondence between X and Y.

[1] We will see at the end of the Chapter that (Naive) Set Theory, although called Cantor's Set Theory leads to various *paradoxes.* Such paradoxes could be resolved in the framework of the modern *Axiomatic Set Theory.*

[2] This lack of a definition of set leads to paradoxes of Naive Set Theory, mentioned above

How do we decide how many elements there are in a set ? A possible answer is that we would count them. But what does it mean "to count" ? One can describe this as follows. We look at one element of the set and think "one". Then we look at a different element and think "two" and so on. We must be careful to make sure that all the elements have been counted and none of them has been counted twice. If our process of counting has stopped at a certain moment, we say that the set is finite.

Definition 1.1 A set X is called *finite* if for some $n \in \mathbb{N}^+$ there exists a bijection

$$f : \{1, 2, \dots, n\} \to X.$$

In this case we say also that X has n elements. We say that a set X is *infinite* if it is not finite.

Remark 1.2 We agree that the empty set $\emptyset$ is finite and has 0 elements.

A natural way to extend the process of "counting" to infinite sets is the following.

Definition 1.3 A set X is called *countable* if there exists a bijection

$$f : \mathbb{N}^+ \to X.$$

Remark 1.4 By Theorem **A1.7.14**[3] a function $f : \mathbb{N}^+ \to X$ is a bijection if and only if the inverse function $f^{-1} : X \to \mathbb{N}^+$ exists. In this case the inverse f^{-1} is a bijection too. In particular, X is countable if and only if there exists a bijection $f : X \to \mathbb{N}^+$.

Example 1.5 $\mathbb{N} = \{0, 1, 2, \dots\}$ is countable.

Proof. Define $f : \mathbb{N}^+ \to \mathbb{N}$ by $f(n) = n - 1$.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & \dots & n & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\ 0 & 1 & 2 & 3 & 4 & \dots & n-1 & \dots \end{array}$$

Then f is a bijection. $\square$

In this example we observe the first remarkable property of infinite sets. $\mathbb{N}^+$ is a subset of $\mathbb{N}$ so intuitively it should have fewer elements. Yet in the sense of a bijection between the sets, both $\mathbb{N}^+$ and $\mathbb{N}$ are countable (have the same "size"). The next example, constructed by Galileo in 1638, was considered as a paradox for over two centuries.

Example 1.6 The set of perfect squares $\mathcal{S} := \{1^2, 2^2, 3^2, \dots, n^2, \dots\}$ is countable.

Proof. Define $f : \mathbb{N}^+ \to \mathcal{S}$ by $f(n) = n^2$.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & \dots & n & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\ 1 & 4 & 9 & 16 & 25 & \dots & n^2 & \dots \end{array}$$

[3] In the following **Ax.y.z** refers to *Lecture Notes in Analysis* by Vitali Liskevich. You can download them on the Internet: `http://www.maths.bris.ac.uk/~maval/an1.html`

It is clear that f is a bijection. $\square$

The reason why Galileo's example was considered as a paradox is that intuitively the set $\mathcal{S}$ of perfect squares seems to be much "smaller" then $\mathbb{N}^+$. Yet, $\mathcal{S}$ is countable. Consider more examples.

Example 1.7 *The set of integers $\mathbb{Z}$ is countable.*

Proof. Define $f : \mathbb{N}^+ \to \mathbb{Z}$ by

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{1-n}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

The following diagram illustrates f:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & \dots & 2n & 2n+1 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \\ 0 & 1 & -1 & 2 & -2 & \dots & n & -n & \dots \end{array}$$

The inverse to f is a function $f^{-1} : \mathbb{Z} \to \mathbb{N}^+$ given by

$$f^{-1}(m) = \begin{cases} 2m, & \text{if } m \geq 1, \\ 1-2m, & \text{if } m \leq 0. \end{cases}$$

By Theorem **A1.7.14** we conclude that f is a bijection. $\square$

Remark 1.8 Notice that although f in the last example is a bijection, it does not preserve the order, in the sense that $m < n$ does not imply $f(m) < f(n)$.

Example 1.9 The set $\mathbb{N}^+ \times \mathbb{N}^+$ is countable.

Proof. We can write elements of $\mathbb{N}^+ \times \mathbb{N}^+$ as a rectangular array (figure on the left) and then read them off along the cross diagonal (figure on the right), first $(1,1)$, next $(1,2), (2,1)$, then $(1,3), (2,2), (3,1)$ and so on:

$$\begin{array}{c|ccccc} (n,m) & 1 & 2 & 3 & 4 & \dots \\ \hline 1 & (1,1) & (1,2) & (1,3) & (1,4) & \dots \\ 2 & (2,1) & (2,2) & (2,3) & \dots & \\ 3 & (3,1) & (3,2) & \dots & & \\ 4 & (4,1) & \dots & & & \\ \dots & \dots & & & & \end{array} \quad \xrightarrow{f(n,m)} \quad \begin{array}{c|ccccc} & & & & & \\ \hline & 1 & 2 & 4 & 7 & \dots \\ & 3 & 5 & 8 & \dots & \\ & 6 & 9 & \dots & & \\ & 10 & \dots & & & \\ & \dots & & & & \end{array}$$

This strings them out as a sequence, in which every element of $\mathbb{N}^+ \times \mathbb{N}^+$ is listed and none of them is counted twice:

$$\begin{array}{ccccccc} (1,1) & (1,2) & (2,1) & (1,3) & (2,2) & (3,1) & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 1 & 2 & 3 & 4 & 5 & 6 & \dots \end{array}$$

In this way we established a bijection between $\mathbb{N}^+ \times \mathbb{N}^+$ and $\mathbb{N}^+$. Such bijection $f : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ can be represented analytically by the formula

$$f(n,m) = \frac{(n+m-2)(n+m-1)}{2} + n.$$

Therefore $\mathbb{N}^+ \times \mathbb{N}^+$ is countable. $\square$

Remark 1.10 Let X be a countable set and $f : \mathbb{N}^+ \to X$ is a bijection. Define $x_1 := f(1)$, $x_2 := f(2)$, $x_3 := f(3)$, $\dots$, $x_n := f(n)$, $\dots$.

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & \dots & n & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\ x_1 & x_2 & x_3 & x_4 & x_5 & \dots & x_n & \dots \end{array}$$

Thus all elements of X can be "listed", or arranged in a (generated by the function f) *sequence*:

$$X = \{x_1, x_2, x_3, x_4, x_5, \dots, x_n, \dots\}.$$

In the following, if needed, we always assume that the elements of a countable set X can be written as a sequence.

We are going to prove several important properties of countable sets.

Lemma 1.11 *If a set X is countable then any subset $A \subset X$ is finite or countable.*

Proof. Assume A is infinite. We need to show that A is countable. Since X is countable there exists a bijection $f : \mathbb{N}^+ \to X$. Define $g : \mathbb{N}^+ \to A$ inductively by:

$g(1) := f(k)$ with the least k such that $f(k) \in A$;

having found $g(1), g(2), \dots, g(n)$, then

$g(n+1) := f(k)$ with the least k such that $f(k) \in A \setminus \{g(1), g(2), \dots, g(n)\}$.

It is clear that g is a bijection. □

Remark 1.12 Sometimes we say that a set X is *at most countable* if X is finite or countable.

Remark 1.13 Lemma 1.11 is very useful for proving that a certain set is countable. For example, let $\mathcal{P}$ be the set of all prime numbers. Obviously $\mathcal{P} \subset \mathbb{N}^+$. It is known that the set of primes is infinite. We conclude that $\mathcal{P}$ is countable.

Lemma 1.14 *Any infinite set has a countable subset $A \subset X$.*

Proof. We shall select a countably infinite subset A of X. Since no bijection exists between $\emptyset$ and X, X is nonempty and there exists some element in X which we call x_1. Define a function $g : \mathbb{N}^+ \to X$ inductively by $g(1) := x_1$, and if distinct elements $\{x_1, x_2, \dots, x_n\}$ have been found, then since g cannot give a bijection $g : \{1, 2, \dots, n\} \to X$, then must be another element, which we name $x_{n+1} \in X$, which is distinct from $x_1, x_2, \dots, x_n$. Define $g(n+1) := x_{n+1}$. Set

$$A := \{x_n \in X | n \in \mathbb{N}^+\}.$$

Thus A is a countable subset of X. □

Definition 1.15 A set Y is said to be *a proper subset* of a set X if $Y \subset X$ and $Y \neq X$.

Proposition 1.16 *If a set X is infinite then there exists a proper subset $Y \subset X$ and a bijection $f : X \to Y$.*

Proof. Let $A := \{x_1, x_2, \dots, x_n, \dots\}$ be a countable subset of X, constructed in Lemma 1.14. Let $B := \{x_1\} \subset A$. Set $Y := X \setminus B$. Define $f : X \to Y$ by

$$f(x_n) = x_{n+1} \quad \text{for } x_n \in A$$

and

$$f(x) = x \quad \text{for } x \in X \setminus A.$$

Then f is a bijection. □

Remark 1.17 Actually we proved that if X is infinite and $B = \{x_1\} \subset X$ then $Y := X \setminus B$ is infinite and there is a bijection $f : X \to Y$. Thus removing one element from an infinite set does not change the "size" of the set !

In a similar manner in the proof above one can select a countable subset $B \subset A$, set $Y := X \setminus B$ and yet there is a bijection $f : X \to Y$. Just select a countable subset $A := \{x_1, x_2, \dots, x_n, \dots\}$ of X, then let $B := \{x_{2k-1} | k \in \mathbb{N}^+\} \subset A$, $Y : X \setminus B$ and define $f : X \to Y$ by

$$f(x_n) = x_{2n} \quad \text{for } x_n \in A$$

and

$$f(x) = x \quad \text{for } x \in X \setminus A.$$

Then f is a bijection. So, starting from an infinite set X, we removed a countable subset B from X and still $Y := X \setminus B$ has the same "size" as X !

Lemma 1.18 *Let X and Y be countable sets. Then the cartesian product $X \times Y$ is countable.*

Proof. Let $f : \mathbb{N}^+ \to X$ and $g : \mathbb{N}^+ \to Y$ are bijections. Define the function $h : X \times Y \to \mathbb{N}^+ \times \mathbb{N}^+$ by the formula

$$h(x, y) := (f(x), g(y)).$$

To see that h is an injection suppose that $h(x_1, y_1) = h(x_2, y_2)$. This means that

$$(f(x_1), g(y_1)) = (f(x_2), g(y_2)).$$

So $f(x_1) = f(x_2)$ and $g(y_1) = g(y_2)$. Since f and g are injections it follows that $x_1 = x_2$ and $y_1 = y_2$, that is $(x_1, y_1) = (x_2, y_2)$. To check that h is a surjection, suppose $(n, m) \in \mathbb{N}^+ \times \mathbb{N}^+$. Then since f and g are both surjections, we can choose $x \in X$ and $y \in Y$ such that $f(x) = n$ and $g(y) = m$. Therefore, $h(x, y) = (n, m)$, as required.

Hence $h : X \times Y \to \mathbb{N}^+ \times \mathbb{N}^+$ is bijection. By Example 1.9, there is a bijection $\varphi : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$. It is easy to see that the composition $\varphi \circ h : X \times Y \to \mathbb{N}^+$ is also a bijection. Therefore $X \times Y$ is countable. □

Lemma 1.19 *Let $\{X_k\}_{k \in \mathbb{N}^+}$ be a countable collection of pairwise disjoint countable sets. Then the union $\cup_{k=1}^{\infty} X_k$ is countable.*

Proof. Since every set X_k is countable we can write elements of X_k in a list

$$X_k = \{x_{k1}, x_{k2}, x_{k3}, \dots, x_{kn}, \dots\}.$$

Therefore

$$
\begin{aligned}
X_1 &= \{x_{11}, x_{12}, x_{13}, \dots, x_{1n}, \dots\},\\
X_2 &= \{x_{21}, x_{22}, x_{23}, \dots, x_{2n}, \dots\},\\
X_3 &= \{x_{31}, x_{32}, x_{33}, \dots, x_{3n}, \dots\},\\
&\dots\dots\\
X_k &= \{x_{k1}, x_{k2}, x_{k3}, \dots, x_{kn}, \dots\},\\
&\dots\dots
\end{aligned}
$$

By the definition of the union

$$\cup_{k=1}^{\infty} X_k = \{x_{kn} | k \in \mathbb{N}^+, n \in \mathbb{N}^+\}.$$

Define a function $h : \cup_{k=1}^{\infty} X_k \to \mathbb{N}^+ \times \mathbb{N}^+$ by

$$h(x_{kn}) := (k, n).$$

Clearly h is a bijection. Since $\mathbb{N}^+ \times \mathbb{N}^+$ is countable we conclude that $\cup_{k=1}^{\infty} X_k$ is countable too. $\square$

Example 1.20 The set of rational numbers $\mathbb{Q}$ is countable.

Proof. For $n \in \mathbb{N}^+$ let

$$X_k := \left\{ \frac{i}{k} \mid i \in \mathbb{N}^+ \right\}$$

be the set of all positive fractions with denominator k. It is clear that for each $k \in \mathbb{N}^+$ the set X_k is countable. Let $X := \cup_{k=1}^{\infty} X_k$. By Lemma 1.19 we conclude that the set X is countable.

Let $\mathbb{Q}_+$ be the set of positive rationals. Obviously $\mathbb{Q}_+ \subset X$ and $\mathbb{Q}_+$ is infinite. Hence Lemma 1.11 implies that $\mathbb{Q}_+$ is countable.

In the same way we can prove that the set $\mathbb{Q}_-$ of negative rationals is countable. Then $\mathbb{Q} = \mathbb{Q}_- \cup \{0\} \cup \mathbb{Q}_+$ is countable by Lemma 1.19. $\square$

2 Uncountable sets. Continuum.

So far we proved that the sets $\mathbb{N}^+ \times \mathbb{N}^+$, $\mathbb{Q}$, etc., although they seem "much larger" than $\mathbb{N}^+$ are still countable. So, may be all infinite sets are countable ? The answer is no !

Definition 2.1 A set X is called *uncountable* if X is infinite not countable.

Roughly speaking, an uncountable set has "so many" elements that they can not be listed, or they can not be arranged in a sequence.

Theorem 2.2 *The open interval* $(0,1)$ *is uncountable.*

Proof. We prove this by contradiction. Assume that the interval $(0,1)$ is countable and $f : \mathbb{N}^+ \to (0,1)$ is a bijection. Then we can arrange all elements of $(0,1)$ in a sequence

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \dots, \alpha_k, \dots\}.$$

Represent each $\alpha_k \in (0,1)$ as a *decimal expansion*

$$\alpha_k = 0.\, a_{k1}\, a_{k2}\, a_{k3}\, a_{k4}\, a_{k5}\, \dots,$$

where we agree that if decimal expansion *terminates* [4] we will write it ending with a sequence of zeros, not a sequence of nines. Then all elements of the interval $(0,1)$ can be written as a sequence:

$$\begin{array}{cccccccccc}
\alpha_1 & = & 0. & \boxed{a_{11}} & a_{12} & a_{13} & a_{14} & a_{15} & \dots & a_{1k} & \dots \\
\alpha_2 & = & 0. & a_{21} & \boxed{a_{22}} & a_{23} & a_{24} & a_{25} & \dots & a_{2k} & \dots \\
\alpha_3 & = & 0. & a_{31} & a_{32} & \boxed{a_{33}} & a_{34} & a_{35} & \dots & a_{3k} & \dots \\
\alpha_4 & = & 0. & a_{41} & a_{42} & a_{43} & \boxed{a_{44}} & a_{45} & \dots & a_{4k} & \dots \\
\alpha_5 & = & 0. & a_{51} & a_{52} & a_{53} & a_{54} & \boxed{a_{55}} & \dots & a_{5k} & \dots \\
\dots & = & 0. & \dots & & & & & \ddots & & \\
\alpha_k & = & 0. & a_{k1} & a_{k2} & a_{k3} & a_{k4} & a_{k5} & \dots & \boxed{a_{kk}} & \dots \\
\dots & = & 0. & \dots & & & & & & & \ddots
\end{array}$$

Let $\beta \in (0,1)$ be the real number with the decimal expansion

$$\beta := 0.\, b_1\, b_2\, b_3\, b_4\, b_5\, \dots b_k\, \dots$$

where

$$b_k := \begin{cases} 1, & \text{if } a_{kk} \neq 1, \\ 2, & \text{if } a_{kk} = 1. \end{cases}$$

Then β is different from α_k for any $k \in \mathbb{N}^+$. The reason is that, for each $k \in \mathbb{N}^+$ the decimal expansion of β differs from that of α_k in its kth place. We arrived to a contradiction. Hence $(0,1)$ is uncountable. □

Remark 2.3 The method used in the proof of Theorem 2.2 is called *diagonalization* or *Cantor's diagonalization* because of the "diagonal" construction of the number β. Diagonalization is a powerful technique that can be used in many other proofs.

[4] See Proposition 2.11 below and discussion after it.

Definition 2.4 A set X is called a *continuum* (or *has a cardinality continuum*) if there exists a bijection

$$f : (0,1) \to X.$$

Example 2.5 Let $a, b \in \mathbb{R}$ and $a < b$. Then the open interval (a,b) is a continuum.

Proof. The function $f : (0,1) \to (a,b)$ defined by the formula

$$f(x) := (b-a)x + a$$

is a bijection with inverse $f^{-1} : (a,b) \to (0,1)$ given by

$$f^{-1}(y) = \frac{y-a}{b-a}.$$

Example 2.6 The intervals $[0,1)$, $(1,0]$, $[0,1]$ have a cardinality continuum.

Proof. (`Compare this proof with the proof of Proposition 1.16!`)
The function $f : [0,1) \to (0,1)$ defined by the formula

$$f(x) := \begin{cases} 1 - \frac{1}{n+1}, & \text{if } x = 1 - \frac{1}{n},\ n \in \mathbb{N}^+, \\ x, & \text{if } x \neq 1 - \frac{1}{n},\ n \in \mathbb{N}^+. \end{cases}$$

is a bijection with inverse

$$f^{-1}(y) := \begin{cases} 1 - \frac{1}{n}, & \text{if } y = 1 - \frac{1}{n+1},\ n \in \mathbb{N}^+, \\ y, & \text{if } y \neq 1 - \frac{1}{n+1},\ n \in \mathbb{N}^+. \end{cases}$$

The intervals $(1,0]$ and $[0,1]$ can be considered in a similar way. □

Example 2.7 The real line $\mathbb{R}$ is a continuum.

Proof. The function $g : (-1,1) \to \mathbb{R}$ defined by

$$g(x) := \frac{x}{1-|x|}$$

is a bijection with inverse $g^{-1} : \mathbb{R} \to (-1,1)$ given by

$$g^{-1}(y) = \frac{y}{1+|y|}.$$

By Example 2.5 there is a bijection $f : (0,1) \to (-1,1)$. Then the composition $g \circ f : (0,1) \to \mathbb{R}$ is a bijection. □

Example 2.8 The open square $(0,1) \times (0,1)$ is a continuum.

An idea of the proof. Let $(x,y) \in (0,1) \times (0,1)$. Represent (x,y) as a decimal expansion

$$(x,y) = (0.\,x_1\,x_2\,x_3\,x_{x4}\ \ldots,\ 0.\,y_1\,y_2\,y_3\,y_4\ \ldots).$$

The idea of the proof is to define the map $f : (0,1) \times (0,1) \to (0,1)$ by the formula

$$f((x,y)) := 0.\,x_1\,y_1\,x_2\,y_2\,x_3\,y_3\,x_4\,y_4\ \ldots$$

However the map f is not a bijection [try to understand why!].

To construct a bijection we shall divide the sequence of digits in the decimal expansions of x and y into 'blocks' in the following manner. Each digit different from nine which is not preceded by nine is to form a block by itself, but any run of consecutive nines is to form a single block with the 'non-nine's' digit which comes immediately after it. Now we form a real number $f(x,y)$ by stringing together the successive *blocks* of x and y, taken alternately. For instance, if

$$x = 0.\underline{4}\,\underline{7}\,\underline{91}\,\underline{0}\,\underline{992}... \quad \text{and} \quad y = 0.\underline{93}\,\underline{2}\,\underline{3}\,\underline{8}\,\underline{91}...$$

then

$$f(x,y) = 0.4\,93\,7\,2\,91\,3\,0\,8\,992\,91....$$

One can show that such defined map between $f : (0,1) \times (0,1) \to (0,1)$ is a bijection.

The interval $(0,1)$ is uncountable – another proof. [5] Our proof of Theorem 2.2 relies on the decimal expansion of the real numbers. The original Cantors's proof of Theorem 2.2, published in 1874, was different. It was based on the following "nested intervals" theorem, which is similar to Theorem A 3.4.5 from Analysis.

Theorem 2.9 *Let $([b_k, c_k])_{k\in\mathbb{N}^+}$ be a sequence of closed nonempty bounded intervals such that*

$$(\forall k \in \mathbb{N}^+)([b_{k+1}, c_{k+1}] \subset [b_k, c_k]).$$

Then $\cap_{k\in\mathbb{N}^+}[b_k, c_k] \neq \emptyset$.

Exercise 2.10 Prove Theorem 2.9.

Hint. Modify proof of Theorem A 3.4.5 from Analysis.

Original Cantor's proof of Theorem 2.2. Let $E \subset (0,1)$ be a countable subset of $(0,1)$, so E can be represented as

$$E = \{x_1, x_2, x_3, \dots\}.$$

Observe, that given a point $x \in (0,1)$ there exists a nonempty interval $[b,c] \subset (0,1)$ such that $x \notin [b,c]$.

Hence, we can choose a nonempty interval $[b_1, c_1] \in (0,1)$ such that $x_1 \notin [b_1, c_1]$. Now we proceed inductively. Let $n \in \mathbb{N}^+$. Having chosen a nonempty interval $([b_k, c_k])_{k=1}^n$ such that $x_k \notin [b_k, c_k]$ for $k = 1, \dots, n$, and such that $[b_{k+1}, c_{k+1}] \subset (b_k, c_k)$ for $k = 1, \dots, n-1$, we can choose nonempty interval $[b_{n+1}, c_{n+1}]$ such that $x_{n+1} \notin [b_{n+1}, c_{n+1}]$ and such that $[b_{n+1}, c_{n+1}] \subset (b_n, c_n)$.

In such a way we inductively defined a "nested" sequence of closed nonempty bounded intervals $([b_k, c_k])_{k\in\mathbb{N}^+}$ so that

$$(\forall k \in \mathbb{N}^+)[([b_{k+1}, c_{k+1}] \subset [b_k, c_k]) \wedge (x_k \notin [b_k, c_k])].$$

According to Theorem 2.9, there exists a point $x_* \in \cap_{k\in\mathbb{N}^+}[b_k, c_k]$. But since $x_n \notin \cap_{n\in\mathbb{N}^+}[b_k, c_k]$ for every $n \in \mathbb{N}^+$, we conclude that $x_* \notin E$. Therefore $E \neq (0,1)$, which means that the interval $(0,1)$ is uncountable. □

[5] The content of this paragraph will not be included into examination paper.

Decimal expansions. [6] Let $(a_k)_{k\in\mathbb{N}^+}$ be a sequence of numbers chosen from the set $\{0,1,2,\dots,9\}$. Then the series

$$\sum_{k=1}^{\infty}\frac{a_k}{10^k}$$

converges by comparison with a geometric progression to a sum $\alpha\in[0,1]$. We write

$$\alpha = 0.\,a_1\,a_2\,a_3\,a_4\,a_5\,\dots,$$

and say that the right–hand side is a *decimal expansion* of the real number α. For example,

$$\frac{1}{3}=0.333333\dots\,,\qquad \frac{1}{7}=\underbrace{142857}\underbrace{142857}\underbrace{142857}\dots\,,\qquad \frac{\pi}{4}=0.785398163397448\dots\dots\,.$$

Proposition 2.11 *Every real number $\alpha\in[0,1]$ has a unique decimal expansion unless it is a rational number of the form $\alpha=\frac{m}{10^n}$ for some $m\in\mathbb{N}^+$ and $n\in\mathbb{N}^+$ in which case α has precisely two decimal expansions.*

Sketch of the proof. Let $\alpha\in[0,1]$. Suppose that $\{a_1,a_2,\dots,a_k\}$ have been chosen from $\{0,1,2,\dots,9\}$ in such a way that

$$\epsilon_k := \alpha-\left(\frac{a_1}{10}+\frac{a_2}{10^2}+\cdots+\frac{a_k}{10^k}\right)$$

satisfies

$$0\le\epsilon_k<\frac{1}{10^k}.$$

Let a_{k+1} be chosen from $\{0,1,2,\dots,9\}$ so that a_{k+1} is maximized subject to the constraint $\epsilon_{k+1}\ge 0$. If $a_{k+1}<9$ then

$$\epsilon_{k+1}<\frac{1}{10^{k+1}},$$

because otherwise we could replace a_{k+1} by $a_{k+1}+1$. If $a_{k+1}=9$ then

$$\epsilon_{k+1}=\epsilon_k-\frac{9}{10^{k+1}}<\frac{1}{10^k}-\frac{9}{10^{k+1}}=\frac{1}{10^{k+1}}.$$

By the "sandwich rule" (**A3.3.7**) we conclude that $\lim_{k\to\infty}\epsilon_k=0$. Therefore

$$\alpha=\sum_{k=1}^{\infty}\frac{a_k}{10^k},$$

which gives the required decimal expansion of α.

We do not consider the issue of uniqueness of the decimal expansion except for the observation that, where two distinct expansions of α exist then $\alpha=\frac{m}{10^n}$ for some $m\in\mathbb{N}^+$ and $n\in\mathbb{N}^+$ and one of the expansions consists of zeros from some point and the other consists of nines from some point on. In this case we say that a decimal expansion of α *terminates*. For example,

$$\frac{1}{2}=\frac{5}{10}=0.5000\cdots=0.4999\dots\,.$$

If we agree that when decimal expansion terminates we will write it ending with a sequence of zeros, not a sequence of nines, then every real number $\alpha\in[0,1]$ has a unique decimal expansion.

[6]The content of this paragraph will not be included into examination paper.

3 Cardinality

In this paragraph we discuss a method to compare "the size" of arbitrary sets.

Definition 3.1 Let X, Y be sets. We say that X is *equivalent* to Y, or X has *the same cardinality* as Y, if there exists a bijection

$$f : X \to Y.$$

In this case we write $X \sim Y$ or $card(X) = card(Y)$.

Remark 3.2 Intuitively, $X \sim Y$ or $card(X) = card(Y)$ means that the sets X and Y have the same "size", or the same "number of elements".

Example 3.3 We already know that $\mathbb{N}^+ \sim \mathbb{Z} \sim \mathbb{Q} \sim \mathbb{N}^+ \times \mathbb{N}^+$ and $(0,1) \sim [0,1] \sim \mathbb{R} \sim (0,1) \times (0,1)$.

Proposition 3.4 *For any sets X, Y, Z:*
(a) *$X \sim X$ (reflexivity);*
(b) *if $X \sim Y$ then $Y \sim X$ (symmetry);*
(c) *if $X \sim Y$ and $Y \sim Z$ then $X \sim Z$ (transitivity).*

Remark 3.5 Properties (a), (b), (c) mean that "$\sim$" is an equivalence relation on the "collection of all sets". We will see however that the "collection of all sets" can not be a set! So "$\sim$" is not a relation in the sense of Definition **A1.6.1**, because the domain of "$\sim$" is not a set. However we still can consider the equivalence class $[X]$ of a given set X which is a *collection* of sets which are equivalent X. For example, the equivalence class $[\mathbb{N}^+]$ of the set of natural numbers $\mathbb{N}^+$ is a *collection* of all countable sets.

Proof. (a) The identity function $i_X : X \to X$ is a bijection.
(b) Suppose $X \sim Y$. Let $f : X \to Y$ be a bijection. Hence by Theorem **A1.7.14** there exists the inverse function $f^{-1} : Y \to X$. But now note that $(f^{-1})^{-1} = f : X \to Y$ is an inverse to f^{-1}. So by Theorem **A1.7.14** again, $f^{-1} : Y \to X$ is a bijection. Therefore $Y \sim X$.
(c) Suppose $X \sim Y$ and $Y \sim Z$. Let $f : X \to Y$ and $g : Y \to Z$ be bijections. Hence by Theorem **A1.7.14** there exist inverse functions $f^{-1} : Y \to X$ and $g^{-1} : Z \to Y$. Consider the composition $f \circ g : X \to Z$. It is easy to check that $(f \circ g)^{-1} = f^{-1} \circ g^{-1} : Z \to X$ is an inverse function to $f \circ g : X \to Z$. Hence $f \circ g : X \to Z$ is a bijection. Therefore $X \sim Z$. □

Exercise 3.6 Suppose $A \sim B$ and $C \sim D$. Prove that:
i) $(A \times C) \sim (B \times D)$;
ii) If A and C are disjoint and B and D are disjoint, then $(A \cup C) \sim (B \cup D)$.

Proof. See Home Exercises 2 Q 5.

Definition 3.7 Let X, Y be sets. We say that Y *dominates* X, or the cardinality of Y is *greater or equal* then the cardinality of X, if there exists an injection

$$f : X \to Y.$$

In this case we write $X \preccurlyeq Y$ or $card(X) \leq card(Y)$.

We say that the cardinality of Y is *strictly greater* then the cardinality of X, if there exists an injection
$$f : X \to Y$$
but there is no injection (and hence no bijection) from X to Y. In this case we write $X \prec Y$ or $card(X) < card(Y)$.

Remark 3.8 (a) Intuitively, $X \preccurlyeq Y$ or $card(X) \leq card(Y)$ means that the set X is "less or equal" then the set Y (or Y is "greater or equal" then X).

(b) If $X \sim Y$ then $X \preccurlyeq Y$ and $Y \preccurlyeq X$. To see this, consider a bijection $f : X \to Y$. Then $f : X \to Y$ is an injection and $f^{-1} : Y \to X$ is an injection too.

(c) If $A \subset X$ then the identity map $i_A : A \to X$ is an injection. Hence if $A \subset X$ then $A \preccurlyeq X$.

(d) If $X \prec Y$ then $X \preccurlyeq Y$ and $X \not\sim Y$. Intuitively this means that X is "less" then Y (or Y is "greater" then X).

Example 3.9 $\emptyset \prec \{1\} \prec \{1, 2\} \preccurlyeq \{3, 4\} \prec \{1, 2, \ldots, n\} \prec \mathbb{N}^+ \preccurlyeq \mathbb{Q} \prec \mathbb{R} \preccurlyeq (0, 1)$. However $\{1, 2\} \sim \{3, 4\}$, $\mathbb{N}^+ \sim \mathbb{Q}$ and $\mathbb{R} \sim (0, 1)$.

Exercise 3.10 Prove that for any sets X, Y, Z:
(a) $X \preccurlyeq X$ (reflexivity);
(b) if $X \preccurlyeq Y$ and $Y \preccurlyeq Z$ then $X \preccurlyeq Z$ (transitivity).

Proof. Similar to the proof of Proposition 3.4.

Exercise 3.11 Suppose $A \preccurlyeq B$ and $C \preccurlyeq D$. Prove that:
i) $(A \times C) \preccurlyeq (B \times D)$;
ii) If A and C are disjoint and B and D are disjoint, then $(A \cup C) \preccurlyeq (B \cup D)$.

Proof. Similar to Home Exercises 2 Q 5.

Remark 3.12 Properties (a) and (b) of Exercise 3.10 mean that "$\preccurlyeq$" is a symmetric and transitive relation on the "collection of all sets" (see Remark 3.5!). However "$\preccurlyeq$" is not an order relation, because it is not antisymmetric. This means that $X \preccurlyeq Y$ and $Y \preccurlyeq X$ does not imply that $X = Y$. For example, $\mathbb{N}^+ \sim \mathbb{Q}$, so $\mathbb{N}^+ \preccurlyeq \mathbb{Q}$ and $\mathbb{Q} \preccurlyeq \mathbb{N}^+$, but $\mathbb{N}^+ \neq \mathbb{Q}$.

Theorem 3.13 (Cantor–Schröder–Bernstein) *Let X and Y be sets. If $X \preccurlyeq Y$ and $Y \preccurlyeq X$ then $X \sim Y$.*

Proof. See Stewart and Tall, Theorem 6 on p.238.

The Cantor–Schröder–Bernstein is extremely useful when it is difficult to find a bijection between two sets. Instead, it is enough to construct two injections !

Example 3.14 $[0, 1] \sim (0, 1)$.

A simple proof via Cantor–Schröder–Bernstein Theorem. Define $f : (0, 1) \to [0, 1]$ and $g : [0, 1] \to (0, 1)$ by
$$f(x) := x, \qquad g(x) := \frac{1}{2}x + \frac{1}{4}.$$
It is clear that both f and g are injections. □

4 Power set. Hierarchy of cardinalities

Definition 4.1 Let X be a set. The *power set* of X, denoted $\mathcal{P}(X)$ or 2^X, is the set whose *elements* are all the subsets of X.

Example 4.2 Let $X = \{0, 1\}$. Then $2^X = \{\ \emptyset, \{0\}, \{1\}, \{0, 1\}\ \}$. Notice that $2^\emptyset = \{\emptyset\}$, so $2^\emptyset$ has one element.

Exercise 4.3 Let $X = \{1, 2, \ldots, N\}$. Then the set 2^X has 2^N elements.

Theorem 4.4 *Let X be a set. Then $X \prec 2^X$.*

Proof. It is clear that the function $f : X \to 2^X$ given by $f(x) := \{x\}$ is an injection, so $X \preccurlyeq 2^X$. We need to show $X \not\sim 2^X$, that is there is no surjection from X into 2^X.

Let $f : X \to 2^X$. Then $f(x) \in 2^X$, so $f(x)$ is a *subset* of X. We ask the question "does x belong to the *subset* $f(x)$?". The answer is always "yes" or "no". Select those elements for which the answer is "no" to get the subset

$$Z := \{x \in X | x \notin f(x)\}.$$

Notice that $Z \in 2^X$. We now assert that $Z \notin Im(f)$. Indeed, assume there exists $z \in X$ such that $f(z) = Z$. We ask the question "does z belong to $f(z) = Z$?". This leads to a contradiction, because

$$z \in Z \ \Rightarrow z \notin f(z) = Z,$$
$$z \notin Z \ \Rightarrow z \in f(z) = Z.$$

We conclude that g can not be a surjection. □

When applied to the set $\mathbb{N}^+$, the theorem says that $2^{\mathbb{N}^+}$ is uncountable. Actually one can prove that $2^{\mathbb{N}^+}$ is a continuum.

Proposition 4.5 $2^{\mathbb{N}^+} \sim \mathbb{R}$.

Proof. See Stewart and Tall, pp.240-241.

Exercise 4.6 Let $\mathcal{F} = \{f : [0, 1] \to \mathbb{R}\}$ be the set of *all* functions from the interval $[0, 1]$ to $\mathbb{R}$. Prove that $[0, 1] \prec \mathcal{F}$.

Hint. Consider the set $\mathcal{F}_0 = \{f_A : A \subset [0, 1]\}$ of all *characteristic functions* $f_A : [0, 1] \to \{0, 1\}$ of the form

$$f_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A, \end{cases}$$

here $A \in 2^{[0,1]}$ is a subset of $[0, 1]$. Establish a bijection between the set $\mathcal{F}_0$ and the set $2^{[0,1]}$ of all subsets of the interval $[0, 1]$.

Hierarchy of cardinalities. Theorem 4.4 leads us to a hierarchy of *different* cardinalities. We begin with $card(\mathbb{N}^+)$. Then $card(2^{\mathbb{N}^+}) = card(\mathbb{R})$ is strictly bigger. Then $card(2^{\mathbb{R}})$, and so on:

$$\emptyset \prec \{1\} \prec \{1, 2\} \prec \cdots \prec \{1, 2, \ldots, N\} \prec \cdots \prec \mathbb{N}^+ \prec 2^{\mathbb{N}^+} \sim \mathbb{R} \prec 2^{\mathbb{R}} \prec 2^{2^{\mathbb{R}}} \prec \ldots.$$

Continuum Hypothesis. In 1878, Cantor asked *is there a set X such that $\mathbb{N}^+ \prec X \prec \mathbb{R}$?* He conjectured that the answer was no, but he was not able to prove it. His conjecture is known as the continuum hypothesis.

Continuum Hypothesis. Let X be a set. If $\mathbb{N}^+ \prec X$ and $X \preccurlyeq \mathbb{R}$ then $X \sim \mathbb{R}$.

The Continuum Hypothesis was "resolved" by Gödel and Cohen in the middle of XX's century. They proved that in the framework of Naive Set Theory it is impossible to prove the continuum hypothesis and it is also impossible to disprove it. However in the framework of modern Axiomatic Set Theory the continuum hypothesis is *independent* of the axioms. This means that it is possible to construct axiomatic models of set theory in which this hypothesis is true and models in which it is false. Its status therefore is similar to that of the parallel postulate of Euclidean geometry.

Paradoxes of Naive Set Theory. In 1899 Cantor discovered a paradox which arises of one considers the set of all sets. What is the cardinal number of the set of all sets? Clearly it must be the greatest possible cardinal yet the cardinal of the set of all subsets of a set always has a greater cardinal than the set itself !

In 1902 Russel discover another paradox which is known as *Russel paradox.* Suppose that we assume the existence of the set y consisting of all sets x which do not belong to themselves, i.e.

$$y = \{x : x \notin x\}.$$

If $y \in y$ then y must satisfy the defining property of y, i.e. $y \notin y$. On the other hand, $y \notin y$, then y the defining property of y and then $y \in y$. In either case a contradiction is obtained.

The popular version of this paradox concerns a certain barber who shaves everyone in his town who does not shave himself. The question is the who shaves the barber ? All possible answers lead to contradictions and one conclude that there is no such barber !

Similarly, one concludes that it is meaningless to speak of "the set all sets which do not belong to themselves". It is also meaningless to speak of "the set all sets". There is no such set !

These paradoxes could be resolved in the framework of modern Axiomatic Set Theory. We do not study Axiomatic Set Theory in this course.

Part II
Further Topics in Analysis

RECOMMENDED TEXTS:

1. J. HOWIE "Real Analysis", Springer–Verlag, 2003.

2. S. KRANTZ "Real Analysis and Foundations", Second Edition. Chapman and Hall/CRC Press, 2005.

3. G. WANNER "Analysis by its History", Springer–Verlag, 1996.

This Lecture Notes, Current Exercises and Solutions to Exercises are available on the Web:
`http://www.maths.bris.ac.uk/~mavbm/fta.html`

5 Subsequences. Accumulation Points

If, for every positive integer $n \in \mathbb{N}^+$ we are given a number a_n then we speak of a sequence $(a_n)_{n\in\mathbb{N}^+}$ and write

$$(a_n)_{n\in\mathbb{N}^+} = (a_1, a_2, a_3, \ldots, a_n, \ldots).$$

Rigorously (see **A 3.1.1**), *a sequence* $(a_n)_{n\in\mathbb{N}^+}$ is defined as a function $f : \mathbb{N}^+ \to \mathbb{R}$. *A sequence* $(a_n)_{n\in\mathbb{N}^+}$ *converges* if there exists a number $a \in \mathbb{R}$ such that

$$(\forall\varepsilon > 0)\ (\exists N = N(\varepsilon) \in \mathbb{N}^+)(\forall n \in \mathbb{N}^+)[(n \geq N) \Rightarrow (|a_n - a| < \varepsilon)].$$

We then write

$$a = \lim_{n\to\infty} a_n \qquad \text{or} \qquad a_n \to a$$

and say that a is the limit of $(a_n)_{n\in\mathbb{N}^+}$. Intuitively, this means that the members a_n of a sequence $(a_n)_{n\in\mathbb{N}^+}$ approach arbitrary closely a number a for all n large enough.

A sequence $(a_n)_{n\in\mathbb{N}^+}$ *diverges* if it has no limit, i.e.

$$(\forall a \in \mathbb{R})\ (\exists\varepsilon > 0)\ (\forall N = N(\varepsilon) \in \mathbb{N}^+)(\exists n \in \mathbb{N}^+)[(n \geq N) \wedge (|a_n - a| \geq \varepsilon)].$$

We say that a sequence $(a_n)_{n\in\mathbb{N}^+}$ is *divergent to* $+\infty$ if

$$(\forall M > 0)\ (\exists N = N(M) \in \mathbb{N}^+)(\forall n \in \mathbb{N}^+)[(n \geq N) \Rightarrow (a_n > M)].$$

Similarly, a sequence $(a_n)_{n\in\mathbb{N}^+}$ is *divergent to* $-\infty$ if

$$(\forall M > 0)\ (\exists N = N(M) \in \mathbb{N}^+)(\forall n \in \mathbb{N}^+)[(n \geq N) \Rightarrow (a_n < -M)].$$

Finally, recall that a sequence $(a_n)_{n\in\mathbb{N}^+}$ is *bounded* if

$$(\exists M > 0)(\forall n \in \mathbb{N}^+)(|a_n| < M).$$

The structure of convergent sequences is relatively simple. Namely, "most" of the members of a convergent sequence accumulate "near" the limit. What can we say about divergent sequences ? Consider the simplest example of a divergent sequence

$$(-1, 1, -1, 1, -1, 1, -1, 1, \dots). \tag{1}$$

Intuitively, this sequence consists of two "parts". One is the sequence $(1, 1, 1, \dots)$ and another is $(-1, -1, -1, \dots)$. Each of this parts converges to its own limit.

Definition 5.1 Let $(a_n)_{n\in\mathbb{N}^+}$ be a sequence. A *subsequence* of a sequence $(a_n)_{n\in\mathbb{N}^+}$ is a sequence

$$(a_{m(k)})_{k\in\mathbb{N}^+} = (a_{m(1)}, a_{m(2)}, a_{m(3)}, \dots, a_{m(k)}, \dots),$$

where $m : \mathbb{N}^+ \to \mathbb{N}^+$ is a *monotone* function, that is

$$m(1) < m(2) < m(3) < \dots < m(k) < \dots.$$

Remark 5.2 (a) The sequence $(a_n)_{n\in\mathbb{N}^+}$ is a subsequence of itself (take $m(k) = k$).
(b) The monotonicity of the function m implies that $(\forall k \in \mathbb{N}^+)(m(k) \geq k)$. Therefore

$$m(k) \to \infty \quad \text{as} \quad k \to \infty.$$

(c) Every sequence has infinitely many subsequences. To see this, define, for example,

$$m_1(k) = k, \quad m_2(k) = 2k, \quad m_3(k) = 3k, \quad \dots.$$

Example 5.3 Consider a sequence $(a_n)_{n\in\mathbb{N}^+}$ from (1) defined by the formula $a_n = (-1)^n$. Then

$$(a_{2k+1})_{k\in\mathbb{N}^+} = (-1, -1, -1, -1 \dots),$$
$$(a_{2k})_{k\in\mathbb{N}^+} = (1, 1, 1, 1 \dots),$$
$$(a_{3k})_{k\in\mathbb{N}^+} = (-1, 1, -1, 1 \dots)$$

are subsequences of $(a_n)_{n\in\mathbb{N}^+}$. We see that $a_{2k+1} \to -1$, $a_{2k} \to 1$ and a_{3k} diverges

Example 5.4 Let $X = \{x_1, x_2, \dots, x_s\} \subset \mathbb{R}$ be a finite set of real numbers. Consider a sequence

$$(a_n)_{n\in\mathbb{N}^+} = (\underbrace{x_1, x_2, \dots, x_s}, \underbrace{x_1, x_2, \dots, x_s}, \underbrace{x_1, x_2, \dots, x_s}, \dots).$$

Such sequence can be written by the formula

$$a_n = \begin{cases} x_1 & \text{if } n \equiv 1 \pmod{s} \\ x_2 & \text{if } n \equiv 2 \pmod{s} \\ \vdots & \vdots \\ x_s & \text{if } n \equiv 0 \pmod{s} \end{cases}$$

Then $a_{sn+1} \to x_1$, $a_{sn+2} \to x_2$, ..., and $a_{sn} \to x_s$ as $n \to \infty$. So $(a_n)_{n\in\mathbb{N}^+}$ is a sequence that has subsequences tending to every element of X.

Example 5.5 Consider a sequence

$$(a_n)_{n\in\mathbb{N}^+} = (\underbrace{1}, \underbrace{1,2}, \underbrace{1,2,3}, \underbrace{1,2,3,4}, \underbrace{1,2,3,4,5}, \dots).$$

Such sequence contains every natural number an infinite number of times (there are many other sequences with this property which would work just as well), so for any $m \in \mathbb{N}^+$ there is a subsequence

$$(m, m, m, m, \dots)$$

of $(a_n)_{n\in\mathbb{N}^+}$ which tends to m as $n \to \infty$.

Lemma 5.6 *Let a sequence $(a_n)_{n\in\mathbb{N}^+}$ converges to a, that is*

$$(\forall\varepsilon > 0)(\exists N \in \mathbb{N}^+)(\forall n \in \mathbb{N}^+)\,[(n > N) \Rightarrow (|a_n - a| < \varepsilon)]\,. \tag{2}$$

Then any subsequence $(a_{m(k)})_{k\in\mathbb{N}^+}$ of $(a_n)_{n\in\mathbb{N}^+}$ converges to a.

Proof. Let $(a_{m(k)})_{k\in\mathbb{N}^+}$ be a subsequence of $(a_n)_{n\in\mathbb{N}^+}$. Fix $\varepsilon > 0$. From the monotonicity of the function m it follows that $m(N) \geq N$. Then

$$(\forall k \in \mathbb{N}^+)\left[(k > N) \Rightarrow (|a_{m(k)} - a| < \varepsilon)\right],$$

where $N \in \mathbb{N}^+$ is taken from (2). This means that $a_{m(k)} \to a$ as $k \to \infty$. □

Lemma 5.7 *Let $(a_n)_{n\in\mathbb{N}^+}$ be a bounded sequence, that is*

$$(\exists M > 0)(\forall n \in \mathbb{N}^+)(|a_n| < M)]. \tag{3}$$

Then any subsequence of $(a_n)_{n\in\mathbb{N}^+}$ is bounded.

Proof. Let $(a_{m(k)})_{k\in\mathbb{N}^+}$ be a subsequence of $(a_n)_{n\in\mathbb{N}^+}$. By the definition of a subsequence, any member of $(a_{m(k)})_{k\in\mathbb{N}^+}$ is a member of the sequence $(a_n)_{n\in\mathbb{N}^+}$. Therefore

$$(\forall k \in \mathbb{N}^+)\left[|a_{m(k)}| < M\right],$$

where $M > 0$ is taken from (3). This means that the subsequence $(a_{m(k)})_{k\in\mathbb{N}^+}$ is bounded. □

Definition 5.8 We say that $a \in \mathbb{R}$ is an *accumulation point* of a sequence $(a_n)_{n\in\mathbb{N}^+}$ if $(a_n)_{n\in\mathbb{N}^+}$ has a subsequence $(a_{m(k)})_{k\in\mathbb{N}^+}$ that converges to a.

Example 5.9 (a) Consider the sequence

$$(-1, 1, -1, 1, -1, 1, -1, 1, \dots).$$

from Example 5.3. Then the set of accumulation points of $(a_n)_{n\in\mathbb{N}^+}$ is

$$\{-1, 1\}.$$

(b) The set of accumulation points of the sequence

$$(a_n)_{n\in\mathbb{N}^+} = (\underbrace{x_1, x_2, \dots, x_s}, \underbrace{x_1, x_2, \dots, x_s}, \underbrace{x_1, x_2, \dots, x_s}, \dots)$$

from Example 5.4 is the set

$$X = \{x_1, x_2, \ldots, x_s\}.$$

(c) Consider a sequence

$$(a_n)_{n\in\mathbb{N}^+} = (\underbrace{1}, \underbrace{1,2}, \underbrace{1,2,3}, \underbrace{1,2,3,4}, \underbrace{1,2,3,4,5}, \ldots)$$

from Example 5.5. The set of accumulation points of $(a_n)_{n\in\mathbb{N}^+}$ is the set of natural numbers

$$\mathbb{N}^+ = \{1, 2, 3, 4, 5, \ldots\}.$$

Exercise 5.10 Let $(a_n)_{n\in\mathbb{N}^+}$ be a sequence that converges to a. Then the set of all accumulation points of $(a_n)_{n\in\mathbb{N}^+}$ is $\{a\}$.

Hint. Apply Lemma 5.6.

Exercise 5.11 Let $(a_n)_{n\in\mathbb{N}^+}$ be a bounded sequence. Then the set of all accumulation points of $(a_n)_{n\in\mathbb{N}^+}$ is bounded.

Hint. Apply Lemma 5.7.

Characterization of the set of all accumulation points of a sequence. How can we describe the set of *all* accumulation points of a sequence ? Consider the sequence $(a_n)_{n\in\mathbb{N}^+}$ defined by the formula $a_n = (-1)^n$. In the Example 5.9, a) we claimed that the set of accumulation points of $(a_n)_{n\in\mathbb{N}^+}$ is $\{-1, 1\}$. Can we make sure that this sequence has no other accumulation points ? Or, in other words, can we make sure that $\{-1, 1\}$ is the set of *all* accumulation points of our sequence ? Formally to do this we need to check that every point in $\mathbb{R} \setminus \{-1, 1\}$ is not an accumulation point of $(a_n)_{n\in\mathbb{N}^+}$. The following characterization of the set of all accumulation points of a sequence can be extremely useful.

Proposition 5.12 *A number $a \in \mathbb{R}$ is an accumulation point of a sequence $(a_n)_{n\in\mathbb{N}^+}$ if and only if*

(4) *for any $\varepsilon > 0$ the set $\{n \in \mathbb{N}^+ : a - \varepsilon < a_n < a + \varepsilon\}$ is infinite.*

Intuitively, Proposition 5.12 says that $a \in \mathbb{R}$ is an accumulation point of a sequence $(a_n)_{n\in\mathbb{N}^+}$ if and only if one can find *infinitely many* members of $(a_n)_{n\in\mathbb{N}^+}$ arbitrary close to a. We also formulate the negation of Proposition 5.12.

Negation of 5.12 *A number $a \in \mathbb{R}$ is not an accumulation point of a sequence $(a_n)_{n\in\mathbb{N}^+}$ if and only if there is an $\varepsilon > 0$ such that the set $\{n \in \mathbb{N}^+ : a - \varepsilon < a_n < a + \varepsilon\}$ is finite.*

Proof of Proposition 5.12.[7] It is easy to see that (4) is equivalent to the following statement:

$$(4') \qquad (\forall \varepsilon > 0)(\forall N \in \mathbb{N}^+)(\exists n \in \mathbb{N}^+)[(n \geq N) \wedge (|a_n - a| < \varepsilon)].$$

Assume that $a \in \mathbb{R}$ be an accumulation point of a sequence $(a_n)_{n\in\mathbb{N}^+}$. We shall prove that (4′) holds.

Since $a \in \mathbb{R}$ is an accumulation point of $(a_n)_{n\in\mathbb{N}^+}$, there is a subsequence $(a_{m(k)})_{k\in\mathbb{N}^+}$ such that $\lim_{k\to\infty} a_{m(k)} = a$. Let $\varepsilon > 0$ and $N \in \mathbb{N}^+$ be given. By the definition of the limit

$$(\exists K > 0)(\forall k \in \mathbb{N}^+)[(k \geq K) \Rightarrow (|a_{m(k)} - a| < \varepsilon)].$$

[7] This proof is not included into Exam Paper.

From the definition of a subsequence it follows that

$$(\forall k \in \mathbb{N}^+)[(k \geq N) \Rightarrow (m(k) \geq N)].$$

Set $k := \max\{N, K\}$ and $n := m(k)$. Then we have

$$[(n \geq N) \wedge (|a_n - a| < \varepsilon)],$$

that is (4′) holds.

Now assume that (4′) holds. We need to prove that a is an accumulation point of $(a_n)_{n\in\mathbb{N}^+}$. Set $\varepsilon = 1$ and $N = 1$. From (4′) we conclude that

$$(\exists m(1) \geq 1)(|a_{m(1)} - a| < 1).$$

Set $\varepsilon = 1/2$ and $N = m(1) + 1$. From (4′) we conclude that

$$(\exists m(2) \geq m(1) + 1)(|a_{m(2)} - a| < 1/2).$$

Set $\varepsilon = 1/3$ and $N = m(2) + 1$. From (4′) we conclude that

$$(\exists m(3) \geq m(2) + 1)(|a_{m(3)} - a| < 1/3).$$

$$\ldots\ldots$$

In such a way we have constructed a subsequence $(a_{m(k)})_{k\in\mathbb{N}^+}$ such that

$$(\forall k \in \mathbb{N}^+)(|a_{m(k)} - a| < 1/k).$$

This means that $\lim_{k\to\infty} a_{m(k)} = a$, that is a is an accumulation point of $(a_n)_{n\in\mathbb{N}^+}$. □

Example 5.13 Find the sets of all accumulation points of the sequences $(a_n)_{n\in\mathbb{N}^+}$ defined by

$$a_n = (-1)^n + \frac{1}{n}.$$

Solution. One can see immediately that $(a_n)_{n\in\mathbb{N}^+}$ contains two convergent subsequences $(a_{2k-1})_{k\in\mathbb{N}^+}$ and $(a_{2k})_{k\in\mathbb{N}^+}$, where

$$a_{2k-1} \to -1 \quad \text{and} \quad a_{2k} \to 1.$$

So -1 and 1 are accumulation points of $(a_n)_{n\in\mathbb{N}^+}$. We are going to show that $(a_n)_{n\in\mathbb{N}^+}$ has no other accumulation points.

Fix $b \notin \{-1, 1\}$. To prove that b is not an accumulation point of $(a_n)_{n\in\mathbb{N}^+}$, we have to check the negation of (4′), that is

$$(\exists \varepsilon > 0)(\exists N \in \mathbb{N}^+)(\forall n \in \mathbb{N}^+)\,[(n \geq N) \Rightarrow (|a_n - b| > \varepsilon|]$$

Fix $\varepsilon = \frac{1}{2}\min\{|1-b|, |-1-b|\}$. Then by the inequality $|x - y| \geq ||x| - |y||$ we obtain

$$\left|\left((-1)^n + \frac{1}{n}\right) - b\right| \geq |(-1)^n - b| - \frac{1}{n} \geq \min\{|1-b|, |-1-b|\} - \frac{1}{n} = 2\varepsilon - \frac{1}{n}.$$

Choose $N \in \mathbb{N}^+$ such that $\frac{1}{N} < \epsilon$. Then

$$(\forall n \in \mathbb{N}^+)\left[(n \geq N) \Rightarrow (|((-1)^n + \frac{1}{n}) - b| > \varepsilon|\right].$$

This means that b is not an accumulation point of $(a_n)_{n\in\mathbb{N}^+}$, that is the set of *all* accumulation points is $\{-1, 1\}$.

6 The Bolzano–Weierstrass Theorem

The following theorem has been proved in Analysis.

Theorem A 3.1.9 *Any convergent sequence is bounded.*

Proof. See Analysis Lecture Notes, p.44.[8]

The converse to this theorem is not true. Namely, bounded sequence may not converge.

Example 6.1 This sequence $a_n = (-1)^n$ is bounded, but it does not converges.

What can we say about "convergence properties" of bounded sequences ? The answer is given by the Bolzano–Weierstrass Theorem, which is one of the fundamental theorems of analysis.

Theorem 6.2 (Bolzano–Weierstrass Theorem) *Any bounded sequence has a convergent subsequence.*

In other words, the Bolzano–Weierstrass Theorem says that the set of accumulation points of a bounded sequence is nonempty. The proof of it is based on the "nested intervals" theorem from Analysis.

Theorem A 3.2.5. *Let* $([b_k, c_k])_{k\in\mathbb{N}^+}$ *be a sequence of closed intervals such that:*

i) $(\forall k \in \mathbb{N}^+)([b_{k+1}, c_{k+1}] \subset [b_k, c_k])$*;*

ii) $\lim_{k\to\infty}(c_k - b_k) = 0$.

Then there exists a unique point $A \in \cap_{k\in\mathbb{N}^+}[b_k, c_k]$*, i.e.* A *belongs to all intervals.*

Proof of the Bolzano–Weierstrass Theorem. Let $(a_n)_{n\in\mathbb{N}^+}$ be a bounded sequence. This simply means that

$$(\exists M > 0)(\forall n \in \mathbb{N}^+)(|a_n| < M)].$$

In other words, $(\forall n \in \mathbb{N}^+)(a_n \in [-M, M])$.

We construct a sequence of intervals $[b_n, c_n]$ as follows:

Step 1	Denote $[b_1, c_1] := [-M, M]$	$c_1 - b_1 = 2M$
Step 2	Divide $[b_1, c_1]$ into two equal parts. At least one part has infinitely many members of $(a_n)_{n\in\mathbb{N}^+}$. Denote this part $[b_2, c_2]$.	$[b_2, c_2] \subset [b_1, c_1]$ $c_2 - b_2 = M$
Step 3	Divide $[b_2, c_2]$ into two equal parts. At least one part has infinitely many members of $(a_n)_{n\in\mathbb{N}^+}$. Denote this part $[b_3, c_3]$.	$[b_3, c_3] \subset [b_2, c_2]$ $c_3 - b_3 = M/2$
	......	...
Step k	Divide $[b_{k-1}, c_{k-1}]$ into two equal parts. At least one part has infinitely many members of $(a_n)_{n\in\mathbb{N}^+}$. Denote this part $[b_k, c_k]$.	$[b_k, c_k] \subset [b_{k-1}, c_{k-1}]$ $c_k - b_k = M/2^{k-2}$
	......	...

[8] The proof of this theorem might be included into FTA Exam Paper

In such a way we constructed a sequence of intervals $([b_k, c_k])_{k\in\mathbb{N}^+}$ such that:

i) $(\forall k \in \mathbb{N}^+)([b_{k+1}, c_{k+1}] \subset [b_k, c_k])$;

ii) $\lim_{k\to\infty}(c_k - b_k) = \frac{M}{2^{k-2}} = 0$.

Thus by Theorem A 3.4.5 there exists a unique point $A \in \cap_{k\in\mathbb{N}^+}[b_k, c_k]$, or, in other words,

$$\lim_{k\to\infty} b_k = \lim_{k\to\infty} c_k = A. \tag{5}$$

Now we construct a subsequence $(a_{m(k)})_{k\in\mathbb{N}^+}$ of the sequence $(a_n)_{n\in\mathbb{N}^+}$ as follows.

$k = 1$: Take $a_{m(1)} = a_1$.

$k = 2$: Take any $a_j \in [b_2, c_2]$ such that $j > m(1)$.
(This is possible since $[b_2, c_2]$ has ∞–many members of $(a_n)_{n\in\mathbb{N}^+}$).
Denote $a_{m(2)} = a_j$.

$k = 3$: Take any $a_j \in [b_3, c_3]$ such that $j > m(2)$.
(This is possible since $[b_3, c_3]$ has ∞–many members of $(a_n)_{n\in\mathbb{N}^+}$).
Denote $a_{m(3)} = a_j$.

......

We have constructed a subsequence $(a_{m(k)})_{k\in\mathbb{N}^+}$ of the sequence $(a_n)_{n\in\mathbb{N}^+}$ such that

$$(\forall k \in \mathbb{N}^+)[b_k \le a_{m(k)} \le c_k].$$

Since $\lim_{k\to\infty} b_k = \lim_{k\to\infty} c_k = A$ by the Sandwich Rule we conclude that

$$\lim_{k\to\infty} a_{m(k)} = A.$$

Thus $(a_{m(k)})_{k\in\mathbb{N}^+}$ is a convergent subsequence of $(a_n)_{n\in\mathbb{N}^+}$. □

Remark 6.3 The following examples show that the Bolzano–Weierstrass Theorem does not hold for unbounded sequences:
(a) the sequence $a_n = n$ diverges to $+\infty$ and has no convergent subsequences;
(b) the sequence $a_n = (-n)^n$ diverges and has no convergent subsequences;
(c) the sequence $a_n = n + (-1)^n n$ diverges but has a convergent subsequence $a_{2k+1} = 0$.

Example 6.4 Consider the sequence $a_n = \cos(n)$. Observe that the sequence is bounded, as $|\cos(n)| \le 1$ for all $n \in \mathbb{N}^+$. The Bolzano–Weierstrass Theorem guarantees that $\cos(n)$ has a convergent subsequence, even though it would be difficult to see what that convergent subsequence is.

Exercise 6.5 Prove that any unbounded sequence has a subsequence that diverges to $+\infty$ or to $-\infty$.

Hint. Imitate the proof of the Bolzano–Weierstrass Theorem by selecting a sequence of unbounded intervals of the type $(-\infty, n]$ or $[n, +\infty)$ that contain infinitely many members of the sequence.

Bolzano–Weierstrass Theorem – another proof. One can give a different proof of Bolzano–Weierstrass Theorem, which instead of the "nested intervals" lemma, uses the following.

Lemma 6.6 *Every sequence has a monotone subsequence.*

Proof. [9] Let $(x_n)_{n\in\mathbb{N}^+}$ be a sequence. Call an integer $m \in \mathbb{N}^+$ a 'far seeing integer' if

$$(\forall n \in \mathbb{N}^+)[(n \geq m) \Rightarrow (x_m \geq x_n)].$$

There are two possibilities:

(i) There exists infinitely many far seeing integers. Thus we can find integers $m(1) < m(2) < \cdots < m(j) < \ldots$ such each $m(j)$ is far seeing and so

$$x_{m(1)} \geq x_{m(2)} \geq \cdots \geq x_{m(j)} \geq \ldots$$

is a *monotone decreasing* subsequence of $(x_n)_{n\in\mathbb{N}^+}$.

(i) There are only finitely many far seeing integers, that is

$$(\exists N \in \mathbb{N}^+)(\forall n \in \mathbb{N}^+)[(n \geq N) \Rightarrow (\exists m \in \mathbb{N}^+)[(m > n) \wedge (x_m \leq x_n)]].$$

Thus, given $n(j) \geq N$ we can find $n(j+1) > n(j)$ with $x_{n(j)} \leq x_{n(j+1)}$. Proceeding inductively, we obtain

$$x_{n(1)} \leq x_{n(2)} \leq \cdots \leq x_{n(j)} \leq \ldots,$$

which is a *monotone increasing* subsequence of $(x_n)_{n\in\mathbb{N}^+}$.

Remark 6.7 Note that the sequence in Lemma 6.6 might be unbounded.

Exercise 6.8 Prove Bolzano–Weierstrass Theorem using Lemma 6.6.

[9] See T. W. Körner *"A Companion to Analysis"*, American Mathematical Society, Providence, 2003; p.38.

7 Limit Superior and Limit Inferior

Recall the definitions of supremum and infimum of a set from Analysis (**A2.4.1, A2.4.2**).

Let $X \subset \mathbb{R}$ be a set. We say that a number $a \in \mathbb{R}$ is the *supremum* of X ("least upper bound of X") and we write $\sup X = a$ if

i) $(\forall x \in X)(x \leq a)$ ("a is an upper bound of X")

ii) $(\forall \varepsilon > 0)(\exists x \in X)[x > a - \varepsilon]$ ("there is no upper bound less then a")

We say that a number $b \in \mathbb{R}$ is the *infimum* of X ("greatest lower bound of X") and we write $\inf X = b$ if

i) $(\forall x \in X)(x \geq b)$ ("a is a lower bound of X")

ii) $(\forall \varepsilon > 0)(\exists x \in X)[x < b + \varepsilon]$ ("there is no lower bound bigger then a")

Recall the Completeness Axiom of real numbers and its immediate consequence.

Completeness Axiom. *Let $X \subset \mathbb{R}$ be nonempty and bounded above. Then $\sup X$ exists.*

Theorem A 2.4.1 *Let $X \subset \mathbb{R}$ be nonempty and bounded below. Then $\inf X$ exists.*

Example 7.1 *a)* Let $X = (0, 1)$. Then $\sup X = 1$, $\inf X = 0$.
b) Let $X = [0, 1]$. Then $\sup X = 1$, $\inf X = 0$.
c) Let $X = \mathbb{N}^+$. Then $\inf X = 1$ but supremum of X does not exists (because X is unbounded above).
d) Let $X = \{\frac{1}{n} | n \in \mathbb{N}^+\}$. Then $\sup X = 1$, $\inf X = 0$.

According to the Bolzano–Weierstrass Theorem the set of all accumulation points of a bounded sequence is nonempty. Our purpose in this section is to investigate the supremum and the infimum of this set. So, throughout this section we shall suppose that *all sequences we are talking about are bounded.*

Definition 7.2 Let $(a_n)_{n\in\mathbb{N}^+}$ be a bounded sequence and $\mathcal{A}$ denotes the set of all accumulation points of $(a_n)_{n\in\mathbb{N}^+}$. We say that a number $a \in \mathbb{R}$ is the *limit superior* of the sequence $(a_n)_{n\in\mathbb{N}^+}$ if a is the supremum of the set $\mathcal{A}$ of all accumulation points of $(a_n)_{n\in\mathbb{N}^+}$. We write

$$a = \limsup_{n\to\infty} a_n := \sup \mathcal{A}.$$

We say that a number $b \in \mathbb{R}$ is the *limit inferior* of the sequence $(a_n)_{n\in\mathbb{N}^+}$ if b is the infimum of the set $\mathcal{A}$ of all accumulation points of $(a_n)_{n\in\mathbb{N}^+}$. We write

$$b = \liminf_{n\to\infty} a_n := \inf \mathcal{A}.$$

Remark 7.3 The limit superior and the limit inferior of a bounded sequence are always exist. Indeed, let $\mathcal{A}$ be the set of all accumulation points of a bounded sequence $(a_n)_{n\in\mathbb{N}^+}$. By the Bolzano–Weierstrass Theorem the set $\mathcal{A}$ is nonempty and by Exercise 5.11 the set $\mathcal{A}$ is bounded. Therefore from Theorems **A 2.4.1** and **A 2.4.2** we conclude that both $\sup \mathcal{A}$ and $\inf \mathcal{A}$ exist.

Example 7.4 *a)* Let $a_n = \frac{1}{n}$. This sequence converges to zero. Hence the set of all accumulation points is $\mathcal{A} = \{0\}$. Therefore

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = 0.$$

b) Let $a_n = (-1)^n$. The set of all accumulation points of this sequence is $\mathcal{A} = \{-1, 1\}$. Hence

$$\limsup_{n\to\infty} a_n = 1, \qquad \liminf_{n\to\infty} a_n = -1.$$

c) Let $a_n = (-1)^n + \frac{1}{n}$. The set of all accumulation points of this sequence is $\mathcal{A} = \{-1, 1\}$ (see Example 5.13). Hence

$$\limsup_{n\to\infty} a_n = 1, \qquad \liminf_{n\to\infty} a_n = -1.$$

d) Let $a_n = n$. This sequence is unbounded. We do not define the limit superior and the limit inferior of an unbounded sequence !

Exercise 7.5 Let $(a_n)_{n\in\mathbb{N}^+}$ be a convergent sequence and $\lim_{n\to\infty} a_n = a$. Prove that

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a.$$

Hint. Use Exercise 5.10.

Theorem 7.6 *Let $(a_n)_{n\in\mathbb{N}^+}$ be a bounded sequence and*

$$\limsup_{n\to\infty} a_n = \alpha, \quad \liminf_{n\to\infty} a_n = \beta.$$

Then α and β are accumulation points of $(a_n)_{n\in\mathbb{N}^+}$.

Proof. Let $\mathcal{A}$ denote the set of all accumulation points of $(a_n)_{n\in\mathbb{N}^+}$. By the definition of the limit superior $\alpha = \sup \mathcal{A}$. We shall proof that $\alpha \in \mathcal{A}$. The proof of the fact that $\beta \in \mathcal{A}$ is similar.

Fix $\varepsilon > 0$ and $N \in \mathbb{N}^+$. Since $\alpha = \sup \mathcal{A}$ by the definition of the supremum of a set (Definition **A 2.4.1**)

$$(\exists a \in \mathcal{A})(a > \alpha - \frac{\varepsilon}{2}).$$

Since $a \in \mathcal{A}$ by Theorem 5.12 ("Characterisation of the set of all accumulation point of a sequence")

$$(\exists k \in \mathbb{N}^+)[(k \geq N) \wedge (|a_k - a| < \frac{\varepsilon}{2})].$$

Then by the triangle inequality

$$|\alpha - a_k| = |(\alpha - a) + (a - a_k)| \leq \underbrace{(\alpha - a)}_{<\varepsilon/2} + \underbrace{|a - a_k|}_{<\varepsilon/2} < \varepsilon.$$

Now Theorem 5.12 implies that α is an accumulation point of the sequence $(a_n)_{n\in\mathbb{N}^+}$. $\square$

Characterization of the limit superior and limit inferior. Our definition of the limit superior and limit inferior of a bounded sequence assumes that we know the set of all accumulation points of a sequence. In many cases it might be useful to have an intrinsic characterization of the limit superior and limit inferior which uses only members of the sequence and does not involves the set of accumulation points. To obtain such characterization we need the following proposition.

Proposition 7.7 *Let $(a_n)_{n\in\mathbb{N}^+}$ be a bounded sequence. Then $\alpha = \limsup_{n\to\infty} a_n$ if and only if for any $\varepsilon > 0$:*

i) the set $\{n \in \mathbb{N}^+ : a_n > \alpha + \varepsilon\}$ is finite,

ii) the set $\{n \in \mathbb{N}^+ : a_n > \alpha - \varepsilon\}$ is infinite;

and $\beta = \liminf_{n\to\infty} a_n$ if and only if for any $\varepsilon > 0$:

j) the set $\{n \in \mathbb{N}^+ : a_n < \beta - \varepsilon\}$ is finite,

jj) the set $\{n \in \mathbb{N}^+ : a_n < \beta + \varepsilon\}$ is infinite.

Proof. [10] Let $\mathcal{A}$ be the set of all accumulation points of $(a_n)_{n\in\mathbb{N}^+}$ and $\alpha = \limsup_{n\to\infty} a_n := \sup \mathcal{A}$. We shall prove that (i) and (ii) hold.

Fix $\varepsilon > 0$. Note that $\alpha \in \mathcal{A}$ by Theorem 7.6. Then there exists a subsequence $(a_{m(k)})_{k\in\mathbb{N}^+}$ such that $\lim_{k\to\infty} a_{m(k)} = \alpha$. By the definition of the limit this means that

$$(\exists K \in \mathbb{N}^+)(\forall k \in \mathbb{N}^+)[(k > K) \Rightarrow (|a_{m(k)} - \alpha| < \varepsilon),$$

where ε is as fixed above. In particular, this implies that

$$(\forall k > K)(a_{m(k)} > \alpha - \varepsilon),$$

so the set $\{n \in \mathbb{N}^+ : a_n > \alpha - \varepsilon\}$ is infinite and the condition (ii) holds.

Assume, by a contradiction, that (i) does not hold. This means that there exists $\varepsilon^* >$ and a subsequence $(a_{m^*(k)})_{k\in\mathbb{N}^+}$ such that

$$(\forall k \in \mathbb{N}^+)(a_{m^*(k)} > \alpha + \varepsilon^*).$$

Since $(a_{m^*(k)})_{k\in\mathbb{N}^+}$ is bounded by the Bolzano–Weierstrass Theorem and Theorem 5.12 we conclude that $(a_{m^*(k)})_{k\in\mathbb{N}^+}$ has an accumulation point $\alpha^* \geq \alpha + \varepsilon^*$. This is a contradiction. We conclude that the condition (i) holds.

Now let $\alpha \in \mathbb{R}$ be such that for any $\varepsilon > 0$ properties (i) and (ii) hold. We shall prove that $\alpha = \limsup_{n\to\infty} a_n := \sup \mathcal{A}$.

First we prove that $\alpha \in \mathcal{A}$. Indeed, from (i) and (ii) it follows that for any $\varepsilon > 0$ the set $\{n \in \mathbb{N}^+ : \alpha - \varepsilon < a_n < \alpha + \varepsilon\}$ is infinite. Thus by Theorem 5.12 we conclude that $\alpha \in \mathcal{A}$.

We prove that $\alpha = \sup \mathcal{A}$. By a contradiction, assume there exists $\gamma \in \mathcal{A}$ such that $\gamma > \alpha$. Set $\varepsilon := \frac{1}{2}(\gamma - \alpha)$. Since $\gamma \in \mathcal{A}$ by Theorem 5.12 the set $\{n \in \mathbb{N}^+ : \gamma - \varepsilon < a_n < \gamma + \varepsilon\}$ is infinite. However $\gamma - \varepsilon > \alpha$, so this contradicts to (i). We conclude that $\alpha = \sup \mathcal{A}$.

The consideration of the limit inferior is similar. □

Using Proposition 7.7 we can obtain explicit formulae for the limit superior and limit inferior.

[10] This proof is not included into Exam Paper

Theorem 7.8 (Formulae for the limit superior and limit inferior) *Let $(a_n)_{n\in\mathbb{N}^+}$ be a bounded sequence. Then*

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sup_{k\geq n} a_k\right), \qquad \liminf_{n\to\infty} a_n = \lim_{n\to\infty}\left(\inf_{k\geq n} a_k\right). \tag{6}$$

Proof. [11] Let $\alpha = \lim_{n\to\infty}\left(\sup_{k\geq n} a_k\right)$. This means that

$$(\forall\varepsilon>0)(\exists N\in\mathbb{N}^+)[(n\geq N)\Rightarrow(|\alpha-\sup_{k\geq n} a_k|)<\varepsilon)], \tag{7}$$

or, equivalently,

$$(\forall\varepsilon>0)(\exists N\in\mathbb{N}^+)[(n\geq N)\Rightarrow(\alpha-\varepsilon<\sup_{k\geq n} a_k<\alpha+\varepsilon)], \tag{8}$$

Therefore

i) the set $\{n>N : a_n>\alpha+\varepsilon\}$ is empty and hence $\{n\in\mathbb{N}^+ : a_n>\alpha+\varepsilon\}$ has at most N elements;

ii) by definition of the supremum

$$(\forall n\geq N)(\exists k\geq n)[a_k\geq(\alpha-\varepsilon)-\varepsilon=\alpha-2\varepsilon]$$

and hence the set $\{n\in\mathbb{N}^+ : a_n>\alpha-2\varepsilon\}$ is infinite.

By Proposition 7.7 we conclude that $\alpha=\limsup_{n\to\infty} a_n$.

The case of the limit inferior is similar. □

Example 7.9 Let $a_n=(-1)^n+\frac{1}{n}$. Find $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ by using (6).

Solution. Set $A_n:=\sup_{k\geq n} a_k$ and $B_n:=\inf_{k\geq n} a_k$. Then we obtain

$$A_n=\sup\{a_n,a_{n+1},a_{n+2},a_{n+3},\dots\}=\begin{cases}1+\frac{1}{n}, & \text{if } n \text{ is even},\\ 1+\frac{1}{n+1}, & \text{in } n \text{ is odd.}\end{cases}$$

so $\limsup_{n\to\infty} a_n=\lim_{n\to\infty} A_n=1$. Similarly,

$$B_n=\inf\{a_n,a_{n+1},a_{n+2},a_{n+3},\dots\}=-1.$$

so $\liminf_{n\to\infty} a_n=\lim_{n\to\infty} B_n=-1$.

Exercise 7.10 *(a)* Give an example of bounded sequences $(a_n)_{n\in\mathbb{N}^+}$ and $(b_n)_{n\in\mathbb{N}^+}$ such that

$$\limsup_{n\to\infty}(a_n+b_n)\neq\limsup_{n\to\infty} a_n+\limsup_{n\to\infty} b_n.$$

(b) Prove that for any bounded sequences $(a_n)_{n\in\mathbb{N}^+}$ and $(b_n)_{n\in\mathbb{N}^+}$ one has

$$\limsup_{n\to\infty}(a_n+b_n)\leq\limsup_{n\to\infty} a_n+\limsup_{n\to\infty} b_n,$$

$$\liminf_{n\to\infty}(a_n+b_n)\geq\liminf_{n\to\infty} a_n+\liminf_{n\to\infty} b_n.$$

Hint. See Home Exercises 6, Q3.

[11] This proof is not included into Exam Paper

8 Cauchy sequences

In this section we are going to derive an intrinsic criterion for convergence of sequences, i.e. the criterion which does not use the value of the limit and is expressed in terms of a condition on the members of a sequence.

Definition 8.1 A sequence $(a_n)_{n\in\mathbb{N}^+}$ is called a *Cauchy sequence* if

$$(\forall\varepsilon > 0)(\exists N \in \mathbb{N}^+)(\forall n, m \in \mathbb{N}^+)[(n \geq N) \wedge (m \geq N) \Rightarrow (|a_m - a_n| < \varepsilon)].$$

Remark 8.2 Sometimes Cauchy sequences are called also *fundamental sequences.*

Example 8.3 The sequence $a_n = \frac{1}{n}$ is a Cauchy sequence.

Solution. Fix $\varepsilon > 0$ and take $N \in \mathbb{N}^+$ such that $N > \frac{1}{\varepsilon}$. Let $m, n \geq N$. Then

$$|a_m - a_n| = \left|\frac{1}{m} - \frac{1}{n}\right| \leq \max\left(\frac{1}{m}, \frac{1}{n}\right) \leq \frac{1}{N} < \varepsilon.$$

This means that $(a_n)_{n\in\mathbb{N}^+}$ is a Cauchy sequence.

Example 8.4 The sequence $a_n = \frac{n^2+1}{n^2}$ is a Cauchy sequence.

Solution. Fix $\varepsilon > 0$ and take $N \in \mathbb{N}^+$ such that $N > \frac{1}{\sqrt{\varepsilon}}$. Let $m, n \geq N$. Then

$$|a_m - a_n| = \left|\frac{m^2+1}{m^2} - \frac{n^2+1}{n^2}\right| = \left|\frac{1}{m^2} - \frac{1}{n^2}\right| \leq \max\left\{\frac{1}{m^2}, \frac{1}{n^2}\right\} \leq \frac{1}{N^2} < \varepsilon.$$

This means that $(a_n)_{n\in\mathbb{N}^+}$ is a Cauchy sequence.

Theorem 8.5 *Let $(a_n)_{n\in\mathbb{N}^+}$ be a convergent sequence. Then $(a_n)_{n\in\mathbb{N}^+}$ is a Cauchy sequence.*

Proof. [12] Let $(a_n)_{n\in\mathbb{N}^+}$ be a convergent sequence and $a = \lim_{n\to\infty} a_n$. Fix $\varepsilon > 0$. Then

$$(\exists N \in \mathbb{N}^+)(\forall n \in \mathbb{N}^+)[(n \geq N) \Rightarrow (|a_n - a| < \frac{\varepsilon}{2})].$$

Let $m, n \geq N$. Then by the triangle inequality

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < \varepsilon.$$

Therefore $(a_n)_{n\in\mathbb{N}^+}$ is a Cauchy sequence. □

It turns out that the converse to Theorem 8.5 is true. This is expressed in the following fundamental theorem.

Theorem 8.6 (Cauchy Theorem – The general principle of convergence) *Let $(a_n)_{n\in\mathbb{N}^+}$ be a Cauchy sequence. Then $(a_n)_{n\in\mathbb{N}^+}$ converges.*

[12] See also Proposition **A 3.5.1**

Proof. Let $(a_n)_{n\in\mathbb{N}^+}$ be a Cauchy sequence. First we show that $(a_n)_{n\in\mathbb{N}^+}$ is bounded. Then we prove that $(a_n)_{n\in\mathbb{N}^+}$ converges.

Fix $\varepsilon > 0$. Then

$$(\exists N \in \mathbb{N}^+)(\forall n, m \in \mathbb{N}^+)[(n \geq N) \wedge (m \geq N) \Rightarrow (|a_m - a_n| < \varepsilon)].$$

Set $M := \max\{|a_1|, |a_2|, \dots, |a_N|\}$. Then by the triangle inequality

$$(\forall n \in \mathbb{N}^+)[(n \geq N) \Rightarrow (|a_n| \leq \underbrace{|a_N - a_n|}_{<\varepsilon} + \underbrace{|a_N|}_{<M} < M + \varepsilon)].$$

Therefore $(a_n)_{n\in\mathbb{N}^+}$ is bounded.

Since $(a_n)_{n\in\mathbb{N}^+}$ is bounded by the Bolzano–Weierstrass Theorem $(a_n)_{n\in\mathbb{N}^+}$ has a convergent subsequence $(a_{m(k)})_{k\in\mathbb{N}^+}$ such that $\lim_{k\to\infty} a_{m(k)} = a$. We shall prove that the original sequence $(a_n)_{n\in\mathbb{N}^+}$ also converges to a.

Fix $\varepsilon > 0$. Then

$$(\exists N \in \mathbb{N}^+)(\forall n, m \in \mathbb{N}^+)[(n \geq N) \wedge (m \geq N) \Rightarrow (|a_n - a_m| < \frac{\varepsilon}{2})],$$

$$(\exists K \in \mathbb{N}^+)(\forall k \in \mathbb{N}^+)[(k \geq K) \Rightarrow (|a_{m(k)} - a| < \frac{\varepsilon}{2})].$$

Choose $l \in \mathbb{N}^+$ such that $m(l) \geq l \geq \max\{N, K\}$. Then

$$(\forall n \in \mathbb{N}^+)[(n \geq N) \Rightarrow (|a_n - a| \leq \underbrace{|a_n - a_{m(l)}|}_{<\varepsilon/2} + \underbrace{|a_{m(l)} - a|}_{<\varepsilon/2} < \varepsilon)].$$

This means that $\lim_{n\to\infty} a_n = a$. □

Remark 8.7 Combining Theorems 8.5 and 8.6 we see that

a sequence $(a_n)_{n\in\mathbb{N}^+}$ converges if and only if $(a_n)_{n\in\mathbb{N}^+}$ is a Cauchy sequence.

The give us an intrinsic criterion of convergence. Indeed, to say that a sequence converges we need to know its limit. On the other hand the definition of a Cauchy sequence is intrinsic in the sense that it involves only the members of a sequence and does not involve any information about its limit !

Remark 8.8 Theorem 8.6 may be useful when we need to show that a sequence diverges. Indeed, combining Theorems 8.5 and 8.6 we see that

a sequence $(a_n)_{n\in\mathbb{N}^+}$ diverges if and only if $(a_n)_{n\in\mathbb{N}^+}$ is not a Cauchy sequence.

Example 8.9 Prove that the sequence $a_n = (-1)^n$ diverges.

Solution. Using Theorem 8.6 we can proceed as follows. Let $N \in \mathbb{N}^+$ be any positive integer. Then for any integers $n \geq N$ we have

$$|a_n - a_{n+1}| = 2.$$

So, if we fix any $\varepsilon < 2$, say $\varepsilon := 1$, and choose $m := n + 1$ then

$$|a_n - a_m| > \varepsilon.$$

So $(a_n)_{n\in\mathbb{N}^+}$ is not a Cauchy sequence and therefore $(a_n)_{n\in\mathbb{N}^+}$ diverges. □

9 Uniformly Continuous Functions

Let A be a subset of the real line and $f : A \to \mathbb{R}$ be a function on A. Recall (see Definition **A 4.2.1**) that the function f is called *continuous at a point* $x \in A$ if

$$\lim_{y \to x} f(y) = f(x).$$

We say that a function f is *continuous on the set* A if f is continuous at every $x \in A$. One can reformulate the last definition using the Cauchy's definition of the limit in the following way. We say that f is *continuous on* A if

$$(\forall x \in A)(\forall \varepsilon > 0)(\exists \delta = \delta(\varepsilon, x) > 0)(\forall y \in A)[(|x - y| < \delta) \Rightarrow (|f(x) - f(y)| < \varepsilon)]. \tag{9}$$

In this definition continuity is regarded as a local property; f is continuous if it is continuous at each point x of its domain A, and the formal definition (9) involves a number δ which *depends on ε and on x*. In this section we shall develop a "global" concept of continuity, in that a common δ applies to all the points x of the domain.

Definition 9.1 Let A be a subset of the real line and $f : A \to \mathbb{R}$ be a function on A. We say that f is *uniformly continuous on* A if

$$(\forall \varepsilon > 0)(\exists \delta = \delta(\varepsilon) > 0)(\forall x, y \in A)[(|x - y| < \delta) \Rightarrow (|f(x) - f(y)| < \varepsilon)]. \tag{10}$$

In this definition δ *depends only on ε and should not depend on x*! In other words, a function f is uniformly continuous on A if in (10) for a given $\varepsilon > 0$ one can choose a *common* value of $\delta > 0$ which applies uniformly to *all* $x \in A$.

Lemma 9.2 *Let $f : A \to \mathbb{R}$ be a uniformly continuous function on a set $A \subset \mathbb{R}$. Then f is continuous on A.*

Proof. We need to prove that f is continuous at every point $x \in A$. Fix $x \in A$. Since f is uniformly continuous on A and x is fixed we obtain from (10) that

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in A)[(|x - y| < \delta) \Rightarrow (|f(x) - f(y)| < \varepsilon)].$$

This means that f is continuous at $x \in A$. □

Example 9.3 Let $-\infty \le a < b \le +\infty$. The function $f(x) = x$ is uniformly continuous on the interval $[a, b]$.

Solution. Given $\varepsilon > 0$, choose $\delta = \varepsilon$. Then for any $x, y \in [a, b]$ we obtain

$$(|x - y| < \delta) \Rightarrow (|f(x) - f(y)| = \underbrace{|x - y|}_{<\delta} < \delta = \varepsilon).$$

This means that f is uniformly continuous on $[0, 1]$.

Example 9.4 Let $-\infty < a < b < +\infty$. The function $f(x) = x^2$ is uniformly continuous on the interval $[a, b]$, *Solution.* Set $c = \max\{|a|, |b|\}$ and note that $|x + y| \le 2c$ for any $x, y \in [a, b]$. Hence

$$|x^2 - y^2| = |x + y||x - y| \le 2c|x - y| \quad \text{for any } \ x, y \in [a, b].$$

For a given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{2c}$. Then for any $x, y \in [a, b]$ we obtain

$$(|x - y| < \delta) \Rightarrow (|f(x) - f(y)| = |x^2 - y^2| = \underbrace{|x + y|}_{\le 2c}\underbrace{|x - y|}_{<\delta} < 2c\delta = \varepsilon).$$

This means that f is uniformly continuous on $[0, 1]$.

Remark 9.5 Note that in this example $\delta(\varepsilon) = \frac{\varepsilon}{2c}$ explicitly depends on ε and the "size" of the interval $[a, b]$, however δ does not depend on the particular choice of $x, y \in [a, b]$. Note also that if $a = -\infty$ or $b = +\infty$ then $c = +\infty$ and hence δ in the above argument is not defined. This suggests that the function $f(x) = x^2$ may be not uniformly continuous on the unbounded intervals $(-\infty, b)$, $(a, +\infty)$ or $(-\infty, +\infty)$. Compare this with Example 9.3, where the function $f(x) = x$ is, in fact, uniformly continuous on the entire real line $\mathbb{R}$.

Example 9.6 Let $f : (a, b) \to \mathbb{R}$ be a differentiable function on the open interval (a, b). Assume that the derivative of f is bounded on (a, b), i.e.

$$(\exists M > 0)(\forall x \in (a, b))(|f'(x)| < M).$$

Then f is uniformly continuous on (a, b).

Solution. Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{M}$. By the Mean Value Theorem (Theorem **A 5.2.3**) we know that

$$(\forall x, y \in (a, b))(\exists z \in (x, y))[f(x) - f(y) = f'(z)(x - y)].$$

Then for any $x, y \in [a, b]$ we obtain

$$(|x - y| < \delta) \Rightarrow (|f(x) - f(y)| = \underbrace{|f'(z)|}_{\le M}\underbrace{|x - y|}_{<\delta} < M\delta = \varepsilon).$$

Thus f is uniformly continuous on (a, b).

In order to consider an example of a continuous function which is not uniformly continuous we state the negation of Definition 9.1. Formally, a function $f : A \to \mathbb{R}$ is *not uniformly continuous* on A if

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x, y \in A)[(|x - y| < \delta) \wedge (|f(x) - f(y)| \ge \varepsilon)]. \tag{11}$$

However in practice it is more convenient to consider a sequence of arbitrary small $\delta_n = \frac{1}{n}$ instead of dealing with all $\delta > 0$. In what follows we will use the following

Negation of Definition 9.1. A function $f : A \to \mathbb{R}$ is *not uniformly continuous* on A if

$$(\exists \varepsilon > 0)(\forall n \in \mathbb{N}^+)(\exists\, x_n, y_n \in A)[(|x_n - y_n| < \delta_n = \frac{1}{n}) \wedge (|f(x_n) - f(y_n)| \ge \varepsilon)]. \tag{12}$$

Exercise 9.7 Prove that (12) is equivalent to (11).

Example 9.8 The function $f(x) = \frac{1}{x}$ is continuous on $(0, 1)$ but is not uniformly continuous on $(0, 1)$.

Solution. The fact that $f(x) = \frac{1}{x}$ is continuous on $(0, 1)$ follows from Theorem **A 4.2.3**. We shall prove that f is not uniformly continuous on $(0, 1)$.

Fix $\varepsilon = 1$. Then for each $\delta_n = \frac{1}{n}$ $(n \in \mathbb{N}^+)$ choose $x_n = \frac{1}{2n}$ and $y_n = \frac{1}{4n}$. Hence

$$|x_n - y_n| = \left|\frac{1}{4n}\right| < \delta_n = \frac{1}{n}$$

and

$$|f(x_n) - f(y_n)| = \left|\frac{1}{x_n} - \frac{1}{y_n}\right| = |2n - 4n| = 2n > \varepsilon = 1.$$

This means that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

Remark 9.9 Observe that the function $f(x) = \frac{1}{x}$ is differentiable on $(0, 1)$ with $f'(x) = -\frac{1}{x^2}$. But the derivative $f'(x)$ is unbounded on $(0, 1)$. Hence we can not apply the result of Example 9.6. On the other hand we see that for any fixed $a \in (0, 1)$ the derivative $f'(x)$ is bounded on the interval $(a, 1)$ and by Example 9.6 the function f is uniformly continuous on $(a, 1)$. So the uniform continuity of f is "violated" near zero. Roughly speaking f grows "too fast" near zero, in such a way that we can not control the "variation" $|f(x) - f(y)|$ when both x and y are getting close to zero.

Example 9.10 The function $f(x) = x^2$ is not uniformly continuous on $(0, \infty)$.

Solution. Fix $\varepsilon = 1$. Then for each $\delta_n = \frac{1}{n}$ $(n \in \mathbb{N}^+)$ choose $x_n = n$ and $y_n = n + \frac{1}{2n}$. Hence $|x_n - y_n| = \frac{1}{2n} < \delta_n$ and

$$|f(x_n) - f(y_n)| = \left|n^2 - \left(n + \frac{1}{2n}\right)^2\right| = 1 + \frac{1}{4n^2} > 1 = \varepsilon.$$

This means that $f(x) = x^2$ is not uniformly continuous on $(0, \infty)$.

The following is one of the fundamental theorems of Analysis.

Theorem 9.11 (Cantor's theorem on Uniform Continuity) *Let $-\infty < a < b < +\infty$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$. Then f is uniformly continuous on $[a, b]$.*

Proof. By contradiction, assume that f is not uniformly continuous on the interval $[a, b]$. Then, as in (12),

$$(\exists \varepsilon > 0)(\forall n \in \mathbb{N}^+)(\exists\, x_n, y_n \in [a, b])[(|x_n - y_n| < \delta_n = \frac{1}{n}) \wedge (|f(x_n) - f(y_n)| \geq \varepsilon)].$$

Now, observe that the sequence $(x_n)_{n\in\mathbb{N}^+} \subset [a, b]$ is bounded. Therefore, by the Bolzano–Weierstrass Theorem $(x_n)_{n\in\mathbb{N}^+}$ has a convergent subsequence $(x_{m(k)})_{n\in\mathbb{N}^+}$ such that

$$x_{m(k)} \to x_0 \quad \text{as} \quad k \to \infty.$$

Since

$$|x_{m(k)} - y_{m(k)}| < \delta_{m(k)} = \frac{1}{m(k)} \to 0 \quad \text{as} \ \ k \to \infty$$

by the triangle inequality we conclude that

$$|y_{m(k)} - x_0| \le \underbrace{|x_{m(k)} - y_{m(k)}|}_{\to 0} + \underbrace{|x_{m(k)} - x_0|}_{\to 0} \to 0 \quad \text{as} \ \ k \to \infty.$$

Therefore

$$y_{m(k)} \to x_0 \quad \text{as} \ \ k \to \infty.$$

Now we observe that f is continuous at x_0 by the assumption of the theorem. Thus (using Heine's definition of the limit) we conclude that

$$f(x_{m(k)}) \to f(x_0) \quad \text{and} \quad f(y_{m(k)}) \to f(x_0) \quad \text{as} \ \ k \to \infty.$$

Therefore by the triangle inequality

$$|f(x_{m(k)}) - f(y_{m(k)})| \le \underbrace{|f(x_{m(k)}) - f(x_0)|}_{\to 0} + \underbrace{|f(y_{m(k)}) - f(x_0)|}_{\to 0} \to 0 \quad \text{as} \ \ k \to \infty.$$

However

$$|f(x_{m(k)}) - f(y_{m(k)})| \ge \varepsilon$$

by the assumption. So we arrived to a contradiction.

We conclude that $f(x)$ is uniformly continuous on $[a, b]$. □

Example 9.12 The function $f(x) = \sqrt{x}$ is uniformly continuous on the closed interval $[0, 1]$.

Solution. It is known (see, e.g., Theorem **A 4.5.1**) that the function $f(x) = \sqrt{x}$ is continuous on the closed interval $[0, 1]$. Then by Theorem 9.11 we conclude that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$.

Remark 9.13 Note that $f(x) = \sqrt{x}$ is not differentiable at $x = 0$ and the derivative $f'(x) = -\frac{1}{2\sqrt{x}}$ is unbounded on $(0, 1]$. This shows that converse to the statement from Example 9.6 is false.

10 Pointwise and Uniform convergence of sequences of functions

In this section we discuss two different types of convergences of sequences of functions. The first one is the pointwise convergence. Another one, the most important, is the uniform convergence.

Let A be a subset of the real line and $(f_n)_{n\in\mathbb{N}^+}$ be a sequence of functions $f_n(x)$ from A to $\mathbb{R}$. If we fix $x \in A$ then $(f_n(x))_{n\in\mathbb{N}^+}$ is a sequence of real numbers. If for each fixed $x \in A$ the limit

$$\lim_{n\to\infty} f_n(x) = f(x)$$

exists, we say that the sequence of functions $(f_n)_{n\in\mathbb{N}^+}$ converges pointwise on A to the function f. The formal way to define the *pointwise convergence* is the following.

Definition 10.1 A sequence of functions $(f_n)_{n\in\mathbb{N}^+}$ *converges pointwise on* A to a function $f : A \to \mathbb{R}$ if

$$(\forall x \in A)(\forall \varepsilon > 0)(\exists N = N(\varepsilon, x) \in \mathbb{N}^+)(\forall n \in \mathbb{N}^+)[(n \geq N) \Rightarrow (|f_n(x) - f(x)| < \varepsilon)], \tag{13}$$

or, in other words,

$$(\forall x \in A)[\lim_{n\to\infty} f_n(x) = f(x)].$$

Remark 10.2 In this definition the number N *depends both on* ε *and* x!

Remark 10.3 Pointwise convergence of sequences of functions inherits many properties of the convergence of real numbers. In particular, since a sequence of real numbers have at most one limit (Theorem **A 3.3.3**), we conclude that a sequence of functions $(f_n)_{n\in\mathbb{N}^+}$ has at most one limiting function f.

Definition 10.4 A sequence of functions $(f_n)_{n\in\mathbb{N}^+}$ *converges uniformly on* A to a function $f : A \to \mathbb{R}$ if

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N}^+)(\forall x \in A)(\forall n \in \mathbb{N}^+)[(n \geq N) \Rightarrow (|f_n(x) - f(x)| < \varepsilon)]. \tag{14}$$

Remark 10.5 In this definition the number N *depends only on* ε and *does not depend on* $x \in A$! In other words in (14), for a given $\varepsilon > 0$ we must be able to choose a *common* number $N \in \mathbb{N}^+$ which applies uniformly to *all* $x \in A$, in such a way that for any $n \geq N$ one has $\sup_A |f_n(x) - f(x)| < \varepsilon$.

Remark 10.6 It is clear that if a sequence of functions $(f_n)_{n\in\mathbb{N}^+}$ converges *uniformly* on A to a function $f : A \to \mathbb{R}$ then $(f_n)_{n\in\mathbb{N}^+}$ converges *pointwise* on A to $f : A \to \mathbb{R}$. The converse is not true in general, as shows Example 10.7 below.

Negation of Definition 10.4 A sequence of functions $(f_n)_{n\in\mathbb{N}^+}$ *does not converges uniformly on* A to a function $f : A \to \mathbb{R}$ if

$$(\exists \varepsilon > 0)(\forall N \in \mathbb{N}^+)(\exists x \in A)(\exists n \in \mathbb{N}^+)[(n \geq N) \wedge (|f_n(x) - f(x)| \geq \varepsilon)].$$

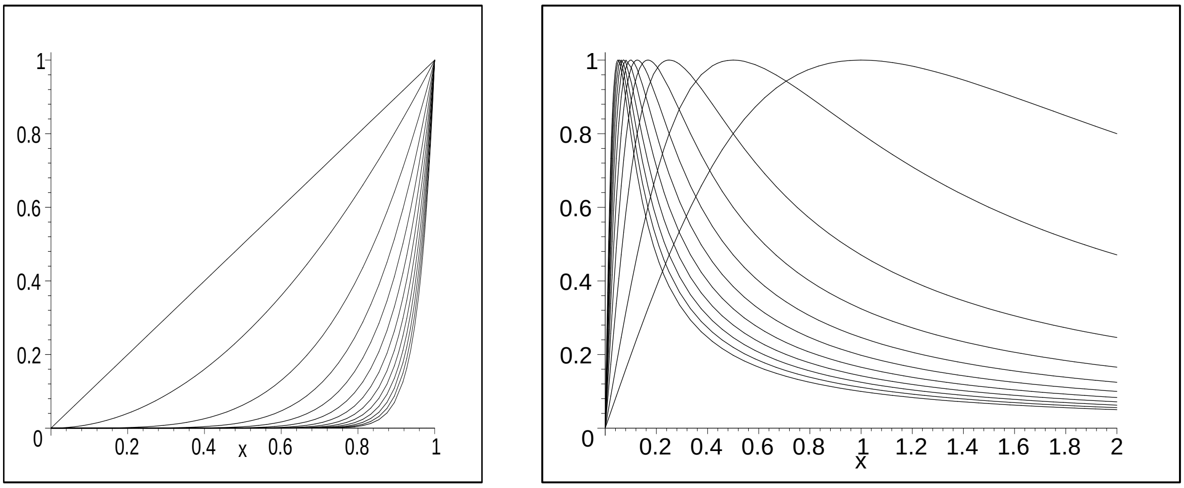

Figure 1: The sequences $f_n(x) = x^n$ (left) and $f_n(x) = \frac{2nx}{1+n^2x^2}$ (right).

Example 10.7 Let $f_n(x) = x^n$ $(n \in \mathbb{N}^+)$ on the interval $[0, 1]$. The sequence of functions $(f_n)_{n\in\mathbb{N}^+}$ converges pointwise on $[0, 1]$ to the discontinuous function

$$f_0(x) = \begin{cases} 0, & \text{if } x \in [0,1), \\ 1, & \text{if } x = 1. \end{cases}$$

For any $a \in (0, 1)$ the sequence $(f_n)_{n\in\mathbb{N}^+}$ converges uniformly to f_0 on the interval $[0, a]$. The sequence $(f_n)_{n\in\mathbb{N}^+}$ does not converges uniformly to f_0 on the interval $[0, 1]$.

Solution. We see immediately that

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & \text{if } x \in [0,1), \\ 1, & \text{if } x = 1. \end{cases}$$

This means that $f_n(x) = x^n$ $(n \in \mathbb{N}^+)$ converges pointwise on $[0, 1]$ to the function $f_0(x)$.

We are going to show that for any $a \in (0, 1)$ the sequence $f_n(x) = x^n$ converges uniformly on $[0, a]$ to $f_0(x) = 0$.

Indeed, fix $a \in (0, 1)$. Fix $\varepsilon > 0$. Choose $N \in \mathbb{N}^+$ such that $a^N < \varepsilon$ (one can choose such N by using the Archimedian Principle, see Theorem **A 2.4.7**). Then for any $x \in [0, a]$ and any $n \geq N$ we obtain

$$|f_n(x) - f_0(x)| = |x^n - 0| \leq a^n \leq a^N < \varepsilon.$$

This means that the sequence $f_n(x) = x^n$ converges uniformly on $[0, a]$ to $f_0(x) = 0$.

Now we prove that $f_n(x) = x^n$ does not converges uniformly on $[0, 1]$ to $f_0(x)$. To do this, we need to verify that

$$(\exists \varepsilon > 0)(\forall N \in \mathbb{N}^+)(\exists x_N \in [0, 1])(\exists n \in \mathbb{N}^+)[(n \geq N) \wedge (|f_n(x_N) - f_0(x_N)| \geq \varepsilon)].$$

Take $\varepsilon = \frac{1}{2}$. Fix $N \in \mathbb{N}^+$. Set $x_N = \frac{1}{\sqrt[N]{2}}$ and $n = N$. Obviously $x_N \in [0, 1]$. Then

$$|f_n(x_N) - f_0(x_N)| = |(x_N)^n - 0| = \frac{1}{2} = \varepsilon.$$

Thus $f_n(x) = x^n$ does not converges uniformly on $[0, 1]$ to $f_0(x)$. $\square$

Theorem 10.8 (Weierstrass's Theorem on Uniform Convergence) *Let A be a subset of the real line and $(f_n)_{n\in\mathbb{N}^+}$ be a sequence of continuous functions from A to $\mathbb{R}$. If $(f_n)_{n\in\mathbb{N}^+}$ converges uniformly on A to a function $f : A \to \mathbb{R}$ then f is a continuous function on A.*

Proof. Choose $x_0 \in A$. We are going to prove that the function $f(x)$ is continuous at the point x_0.

Fix $\varepsilon > 0$. Since the sequence $(f_n)_{n\in\mathbb{N}^+}$ converges uniformly on A to $f : A \to \mathbb{R}$, by definition there exists $N = N(\varepsilon) \in \mathbb{N}^+$ such that

$$(\forall x \in A)(\forall n \in \mathbb{N}^+)[(n \geq N) \Rightarrow (|f_n(x) - f(x)| < \varepsilon/3)]. \tag{15}$$

Since the function f_N is continuous at $x_0 \in A$, there exists $\delta = \delta(\varepsilon, x_0) > 0$ such that

$$(\forall x \in A)[(|x - x_0| < \delta) \Rightarrow (|f_N(x) - f_N(x_0)| < \varepsilon/3)]. \tag{16}$$

Now by the triangle inequality we see that if $x \in A$ and $|x - x_0| < \delta$ then

$$|f(x) - f(x_0)| \leq \underbrace{|f(x) - f_N(x)|}_{<\varepsilon/3 \text{ by } (15)} + \underbrace{|f_N(x) - f_N(x_0)|}_{<\varepsilon/3 \text{ by } (16)} + \underbrace{|f_N(x_0) - f(x_0)|}_{<\varepsilon/3 \text{ by } (15)} < \varepsilon.$$

This means that the function $f(x)$ is continuous at x_0. □

Remark 10.9 Theorem 10.8 can be used to give an easy proof that the sequence of functions $f_n(x) = x^n$ does not converge uniformly on $[0,1]$. Indeed, all the functions $f_n(x)$ are continuous on $[0,1]$. Assume that the sequence $(f_n)_{n\in\mathbb{N}^+}$ converges uniformly on $[0,1]$ to the limit function f. Then by Theorem 10.8 the function f must be continuous. But from Example 10.7 we know that the pointwise limit of the sequence $(f_n)_{n\in\mathbb{N}^+}$ is a discontinuous function. This is a contradiction. We conclude that the sequence $f_n(x) = x^n$ does not converge uniformly on $[0,1]$.

One can ask a question: is there a sequence of continuous functions $(f_n)_{n\in\mathbb{N}^+}$ that converges pointwise on a set A to a *continuous* function f, but does not converges to f uniformly on A ?

Example 10.10 The sequence of continuous functions $f_n(x) = \frac{2nx}{1+n^2x^2}$ converges pointwise on $[0,+\infty)$ to 0 but does not converges uniformly on $[0,+\infty)$ to 0.

Solution. It can be easily seen that $\lim_{n\to\infty} f_n(x) = 0$ for any fixed $x \in [0,+\infty)$. However the functions $f_n(x)$ possess a maximum of height $y = 1$ at $x = \frac{1}{n}$ (see Figure 1). So the convergence on $[0,+\infty)$ is not uniform.

Exercise 10.11 Analyze the behavior of the sequence $f_n(x) = \frac{x^n-1}{x^n+1}$ on $[0,2]$ and the sequence $f_n(x) = (1-x^2)^n$ on $[-1,1]$.

Hint. These are sequences of continuous functions that converge pointwise to a discontinuous limit (see Figure 2). By Theorem 10.8 both sequences does not converge uniformly.

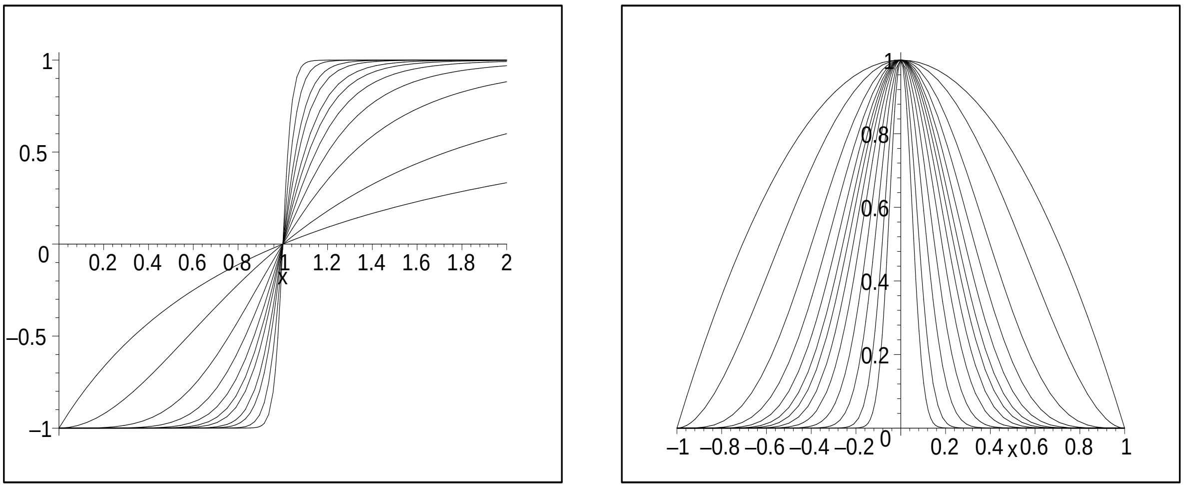

Figure 2: The sequences $f_n(x) = \frac{x^n - 1}{x^n + 1}$ (left) and $f_n(x) = (1 - x^2)^n$ (right).

Series of functions. Let A be a subset of the real line and $(f_n)_{n\in\mathbb{N}^+}$ be a sequence of functions from A to $\mathbb{R}$. Consider the partial sum

$$s_n(x) := \sum_{k=0}^{n} f_k(x).$$

We say that the series of functions

$$\sum_{k=0}^{\infty} f_k(x)$$

is (pointwise) *convergent* on A if the sequence of function $(s_n)_{n\in\mathbb{N}^+}$ converges pointwise on A to a function $S : A \to \mathbb{R}$. The function $S(x)$ is called the sum the series. We call the series *uniformly convergent* if the sequence of partial sums $(s_n)_{n\in\mathbb{N}^+}$ converges uniformly on A. An important example of series of functions is the series of the form

$$\sum_{k=0}^{\infty} c_k x^k.$$

Such series is called the *power series*.

Example 10.12 The *geometric series*

$$\sum_{k=0}^{\infty} x^k$$

converges on the open interval $(-1, 1)$ to the sum $S(x) = \frac{1}{1-x}$. For any $r \in (0, 1)$ this series converges uniformly on $[-r, r]$.

Example 10.13 The *exponential series*

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges on all real line $\mathbb{R}$. The sum $S(x)$ of this series is called the exponential function and is denoted by $\exp(x)$ or e^x. If we choose any $r > 0$ then this series converges uniformly on $[-r, r]$.

Series of functions will be studied in Analysis (Chapter 6), where they are used to define exponential, trigonometric and other special functions.

Part III
The Riemann Integral

RECOMMENDED TEXTS:

2. J. HOWIE "Real Analysis", Springer–Verlag, 2003.
2. S. KRANTZ "Real Analysis and Foundations", Second Edition. Chapman and Hall/CRC Press, 2005.
3. G. WANNER "Analysis by its History", Springer–Verlag, 1996.

This part of the notes is taken with minor modifications from the last chapter of "*Lecture Notes in Analysis*" by VITALI LISKEVICH. Full text of Vitali Liskevich's notes is available at: `http://www.maths.bris.ac.uk/~maval/EX1/an1.html`

11 Definition of integral

Let us consider a function $f : [a, b] \to \mathbb{R}$ defined on a closed interval $[a, b] \subset \mathbb{R}$. We are going to measure the area under the graph of the curve $y = f(x)$ on $\mathbb{R} \times \mathbb{R}$ between the vertical lines $x = a$ and $x = b$. The idea is to divide the interval $[a, b]$ into small subintervals and to approximate the area by a sum of small rectangles (see Figure 11). Such approach goes back at least to Cauchy (1823) and was rigorously completed by Riemann in 1854.

General assumption. Throughout this part of the notes we consider functions $f : [a, b] \to \mathbb{R}$, where $[a, b]$ is a *bounded* interval and f is a *bounded* function.

Definition 11.1 A *partition* $\mathcal{P}$ of the interval $[a, b]$ is a finite collection of points

$$\mathcal{P} = \{x_0, x_1, \ldots, x_n\} \subset [a, b],$$

such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

The length of the greatest subinterval of $[a, b]$ under the partition $\mathcal{P}$ is called *the norm* of the partition $\mathcal{P}$ and is denoted by $\|\mathcal{P}\|$, i.e.

$$\|\mathcal{P}\| := \max_{1 \le i \le n} (x_i - x_{i-1}).$$

Example 11.2 Let $n \in \mathbb{N}^+$. Define a partition $\mathcal{P}_n$ of the interval $[0, 1]$ by

$$\mathcal{P}_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}.$$

Such partition is called the *uniform partition* of the interval $[0,1]$, because all subintervals of the partition have equal length. Clearly $\|\mathcal{P}_n\| = \frac{1}{n}$.

Definition 11.3 Let $\mathcal{P} = \{x_0, \ldots, x_n\}$ be a partition of the interval $[a,b]$. Set

$$m_i = \inf\{f(x) \,|\, x_{i-1} \le x \le x_i\}, \qquad M_i = \sup\{f(x) \,|\, x_{i-1} \le x \le x_i\}.$$

The *lower sum* of the function f for a partition $\mathcal{P}$ is defined as

$$L(f,\mathcal{P}) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

The *upper sum* of the function f for a partition $\mathcal{P}$ is defined as

$$U(f,\mathcal{P}) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

Remark 11.4 Denote

$$m := \inf_{x\in[a,b]} f(x), \qquad M := \sup_{x\in[a,b]} f(x).$$

It is obvious that $m \le m_i \le M_i \le M$ for each $i \in \{1,2,\ldots,n\}$. Therefore we have

$$\begin{aligned} m(b-a) &= \sum_{i=1}^{n} m(x_i - x_{i-1}) \le \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\ &\le \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \le \sum_{i=1}^{n} M(x_i - x_{i-1}) = M(b-a). \end{aligned}$$

This implies that for every partition $\mathcal{P}$ of the interval $[a,b]$ the following inequality holds

$$m(b-a) \le L(f,\mathcal{P}) \le U(f,\mathcal{P}) \le M(b-a).$$

In other words, the sets of real numbers $\{L(f,\mathcal{P}) \,|\, \mathcal{P}\}$ and $\{U(f,\mathcal{P}) \,|\, \mathcal{P}\}$ are bounded, and hence, have infimum and supremum.

Definition 11.5 The *upper integral* of f over $[a,b]$ is defined as

$$J := \inf_{\mathcal{P}} U(f,\mathcal{P}),$$

the *lower integral* of f over $[a,b]$ is defined as

$$j := \sup_{\mathcal{P}} L(f,\mathcal{P}),$$

where the infimum and the supremum are taken over all partitions $\mathcal{P}$ of the interval $[a,b]$.

Definition 11.6 A function $f : [a,b] \to \mathbb{R}$ is called *Riemann integrable* over $[a,b]$ if

$$J = j.$$

In this case the common value is called integral of f over $[a,b]$ and is denoted by

$$\int_a^b f(x)dx := J = j.$$

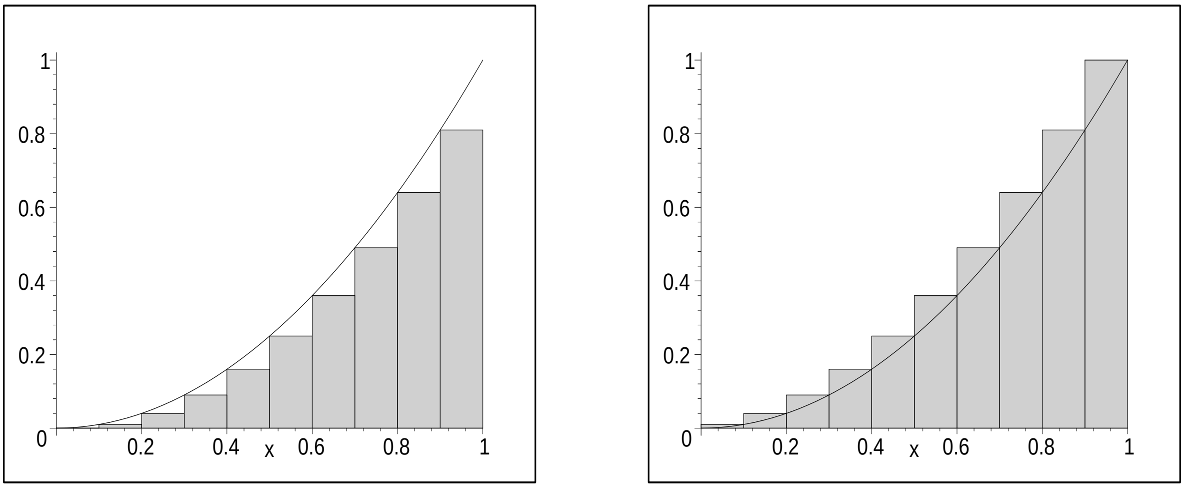

Figure 3: Lower sum $L(x^2, \mathcal{P}_n)$ (left) and upper sum $U(x^2, \mathcal{P}_n)$ (right)

Example 11.7 Let $f : [a, b] \to \mathbb{R}$ be a constant function $f(x) = C$. Then for any partition $\mathcal{P}$ of the interval $[a, b]$ we have

$$L(f, \mathcal{P}) = U(f, \mathcal{P}) = C(b - a).$$

Hence $J = j = C(b - a)$. We conclude that $f(x) = C$ is Riemann integrable over $[a, b]$ and

$$\int_a^b f(x)dx = C(b - a).$$

Remark 11.8 Integrability of f over $[a, b]$ means by definition $j = J$. If $j \neq J$ we say that f is *not Riemann integrable*.

Example 11.9 (An example of a function which is not Riemann integrable) The *Dirichlet function* $D : [0, 1] \to \mathbb{R}$ is defined as

$$D(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

It is discontinuous at all rational $x \in [0, 1]$. For any partition $\mathcal{P}$ of the interval $[0, 1]$ we have

$$L(D, \mathcal{P}) = 0 \quad \text{and} \quad U(D, \mathcal{P}) = 1.$$

Thus $j = 0$ and $J = 1$. Hence the Dirichlet function $D(x)$ is not Riemann integrable.

Example 11.10 The function $f(x) = x^2$ is Riemann integrable over the interval $[0, 1]$ and

$$\int_0^1 f(x)dx = \frac{1}{3}.$$

Solution. Let $\mathcal{P}_n$ be the uniform partition of $[0, 1]$, which is

$$0 < \frac{1}{n} < \frac{2}{n} \cdots < \frac{n-1}{n} < 1.$$

The function $f(x) = x^2$ is monotone increasing on $[0, 1]$, hence

$$m_i = \inf_{[x_{i-1},x_i]} f(x) = f(x_{i-1}) = \frac{(i-1)^2}{n^2} \quad \text{and} \quad M_i = \sup_{[x_{i-1},x_i]} f(x) = f(x_i) = \frac{i^2}{n^2}.$$

Then (see Figure 11)

$$L(f, \mathcal{P}_n) = \sum_{i=1}^{n} \frac{(i-1)^2}{n^2}\frac{1}{n} = \frac{1}{n^3}\sum_{i=0}^{n-1} i^2, \qquad U(f, \mathcal{P}_n) = \sum_{i=1}^{n} \frac{i^2}{n^2}\frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2.$$

One can prove *by induction* the formula

$$(*) \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Using $(*)$ we obtain

$$L(f, \mathcal{P}_n) = \frac{n(n-1)(2n-1)}{6n^3} = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2},$$

$$U(f, \mathcal{P}_n) = \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$$

Therefore

$$(17) \qquad \lim_{n\to\infty} L(f, \mathcal{P}_n) = \frac{1}{3}, \qquad \lim_{n\to\infty} U(f, \mathcal{P}_n) = \frac{1}{3}$$

and

$$(18) \qquad \lim_{n\to\infty} [U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n))] = 0.$$

Note that in the above computations we considered only a specific family of uniform partitions $\mathcal{P}_n$, so we can not use directly definition of the Riemann integral, which involves *all* partitions. However, by Corollary 12.2 from the Criterion of Integrability [13], (18) and (17) together imply that the function $f(x) = x^2$ is Riemann integrable over the interval $[0, 1]$ with

$$j = \lim_{n\to\infty} L(f, \mathcal{P}_n) = \lim_{n\to\infty} U(f, \mathcal{P}_n) = J,$$

and

$$\int_0^1 x^2\, dx := j = J = \frac{1}{3}.$$

This completes the argument. □

[13] It will be proved later

Properties of upper and lower sums. We shall prove several important properties of upper and lower integral sums which will be used further in studying properties of the Riemann integral.

Proposition 11.11 *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of $[a, b]$ such that $\mathcal{P} \subset \mathcal{Q}$. Then*

$$L(f, \mathcal{P}) \leq L(f, \mathcal{Q}), \qquad U(f, \mathcal{P}) \geq U(f, \mathcal{Q}).$$

Remark 11.12 If $\mathcal{P} \subset \mathcal{Q}$.We say that a partition $\mathcal{Q}$ is a *refinement* of the partition $\mathcal{P}$ For example, if $\mathcal{P}_n$ is the uniform partition of $[a, b]$ then $\mathcal{P}_{2n}$ is a refinement of $\mathcal{P}_n$.

Proof. First let us consider a particular case. Let $\mathcal{P}'$ be a partition formed from $\mathcal{P}$ by adding one extra point, say $c \in [x_{k-1}, x_k]$. Let

$$m'_k = \inf_{x \in [x_{k-1}, c]} f(x), \quad m''_k = \inf_{x \in [c, x_k]} f(x).$$

Then $m'_k \geq m_k$, $m''_k \geq m_k$, and we have

$$\begin{aligned} L(f, \mathcal{P}') &= \sum_{i=1}^{k-1} m_i(x_i - x_{i-1}) + m'_k(c - x_{k-1}) + m''_k(x_k - c) + \sum_{i=k+1}^{n} m_i(x_i - x_{i-1}) \\ &\geq \sum_{i=1}^{k-1} m_i(x_i - x_{i-1}) + m_k(x_k - x_{k-1}) + \sum_{i=k+1}^{n} m_i(x_i - x_{i-1}) = L(f, \mathcal{P}). \end{aligned}$$

Similarly one obtains that

$$U(f, \mathcal{P}') \leq U(f, \mathcal{P}).$$

Now to prove the assertion one has to add to $\mathcal{P}$ consequently a finite number of points in order to form $\mathcal{Q}$. □

Proposition 11.13 *Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and let $\mathcal{P}$ and $\mathcal{Q}$ be arbitrary partitions of $[a, b]$. Then*

$$L(f, \mathcal{P}) \leq U(f, \mathcal{Q}).$$

Proof. Consider the partition $\mathcal{P} \cup \mathcal{Q}$. Then by Proposition 11.11 we obtain

$$L(f, \mathcal{P}) \leq L(f, \mathcal{P} \cup \mathcal{Q}) \leq U(f, \mathcal{P} \cup \mathcal{Q}) \leq U(f, \mathcal{Q}),$$

which completes the proof. □

Proposition 11.14 *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let J and j be the upper and lower integrals of f over $[a, b]$. Then*

$$j \leq J.$$

Proof. Fix a partition $\mathcal{Q}$. Then by Proposition 11.13 for any partition $\mathcal{P}$ of $[a, b]$ we obtain

$$L(f, \mathcal{P}) \leq U(f, \mathcal{Q}).$$

Therefore

$$j = \sup\{L(f, \mathcal{P}) \mid \mathcal{P}\} \leq U(f, \mathcal{Q}).$$

And from the above, for any partition $\mathcal{P}$ we have

$$j \leq U(f, \mathcal{P}).$$

Hence

$$j \leq \inf\{U(f, \mathcal{P}) \mid \mathcal{P}\} = J,$$

which completes the proof. □

12 Criterion of integrability

In this section we prove an important Criterion of Integrability, which allows to consider only a suitable family of partitions instead of *all* partitions in definitions of lower and upper integrals j and J.

Theorem 12.1 (Criterion of Integrability) *A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if*

$$(\forall \varepsilon > 0)\,(\exists \text{ a partition } \mathcal{P} \text{ of } [a,b])\,[U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon]. \tag{19}$$

Proof. First, assume that f is integrable, that is $J = j$. Then, by definition of infimum, there exists a partition $\mathcal{P}_1$ such that

$$L(f,\mathcal{P}_1) > j - \varepsilon/2.$$

Also, by definition of supremum, there exists a partition $\mathcal{P}_2$ such that

$$U(f,\mathcal{P}_2) < J + \varepsilon/2.$$

Let $\mathcal{Q} = \mathcal{P}_1 \cup \mathcal{P}_2$, so $\mathcal{Q}$ is a refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$. Then by Propositions 11.11 and 11.13 we obtain

$$j - \varepsilon/2 < L(f,\mathcal{P}_1) \le L(f,\mathcal{Q}) \le U(f,\mathcal{Q}) \le U(f,\mathcal{P}_2) < J + \varepsilon/2.$$

Since $J = j$ by the assumption, we conclude that

$$U(f,\mathcal{Q}) - L(f,\mathcal{Q}) < \varepsilon,$$

that is (19) holds.

Conversely, assume that (19) holds. Fix $\varepsilon > 0$. Let $\mathcal{P}$ be a partition such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon.$$

Note that $J \le U(f,\mathcal{P})$ and $j \ge L(f,\mathcal{P})$ by definition of j and J, so

$$J - j \le U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon.$$

It follows that

$$(\forall \varepsilon > 0)\,[J - j < \varepsilon]\,.$$

Since $\varepsilon > 0$ can be chosen arbitrary small, we conclude that $J = j$, that is f is integrable. □

The following corollary is merely a reformulation of the Criterion of Integrability, which is convenient in computations. We already used it in Example 11.10.

Corollary 12.2 *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Assume that there exists a sequence of partitions $(\mathcal{P}_n)_{n\in\mathbb{N}^+}$ of $[a,b]$ such that*

$$\lim_{n\to\infty} \left(U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n)\right) = 0.$$

Then f is Riemann integrable over $[a,b]$ and

$$\int_a^b f(x)\,dx := \lim_{n\to\infty} U(f,\mathcal{P}_n) = \lim_{n\to\infty} L(f,\mathcal{P}_n).$$

Proof. Do it as an exercise. □

13 Classes of integrable functions

We have seen in Example 11.9 that there are bounded functions that are not integrable. In this section we prove that if a function is monotone or continuous then it is Riemann integrable.

Theorem 13.1 *Let $f : [a, b] \to \mathbb{R}$ be monotone. Then f is Riemann integrable over $[a, b]$.*

Proof. Without loss of generality assume that f is increasing, so that $f(a) < f(b)$. Fix $\varepsilon > 0$. Let us consider a partition $\mathcal{P}$ of $[a, b]$ such that

$$\|\mathcal{P}\| < \delta = \frac{\varepsilon}{f(b) - f(a)}.$$

For this partition we obtain

$$\begin{aligned} U(f, \mathcal{P}) - L(f, \mathcal{P}) &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1}) \\ &< \delta \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \delta(f(b) - f(a)) = \varepsilon. \end{aligned}$$

By the Criterion of Integrability we conclude that f is integrable. □

The following important theorem was asserted by Cauchy in 1823, but was proved rigorously only some 50 years later with the notion of uniform continuity.

Theorem 13.2 *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is Riemann integrable over $[a, b]$.*

Proof. Fix $\varepsilon > 0$. Since f is continuous on a closed interval, by Cantor's Theorem on uniform continuity, f is uniformly continuous on $[a, b]$. Therefore for $\frac{\varepsilon}{b-a}$ there exists $\delta > 0$ such that

$$(\forall x_1, x_2 \in [a, b])\big[(|x_1 - x_2| < \delta) \Rightarrow (|f(x_1) - f(x_2)| < \frac{\varepsilon}{b - a}.$$

Hence for every partition $\mathcal{P}$ with norm $\|\mathcal{P}\| < \delta$ we have

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < \frac{\varepsilon}{b - a} \sum_{i=1}^{n} (x_i - x_{i-1}) = \varepsilon.$$

By the Criterion of Integrability we conclude that f is integrable. □

The next example shows that there are not monotone and not continuous on $[a, b]$ functions which are Riemann integrable (compare it with Example 11.9).

Example 13.3 The function $f : [0, 1] \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational,} \\ \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ in its lowest terms,} \end{cases}$$

is discontinuous at all rational $x \in [0, 1]$. However, for any fixed $\varepsilon > 0$ there is only a finite number (say K) of $x \in [0, 1]$ for which $f(x) > \varepsilon$. Choose a partition $\mathcal{P}_\varepsilon$ with $\|\mathcal{P}_\varepsilon\| < \varepsilon/K$, such that every x for which $f(x) > \varepsilon$ lie in the interior of subinterval. Since $f(x) \leq 1$, this implies

$$U(f, \mathcal{P}) \leq \varepsilon + K\|\mathcal{P}\| < 2\varepsilon.$$

Since $L(f, \mathcal{P}) = 0$, we see that f is Riemann integrable and $\int_0^1 f(x)dx = 0$.

14 Inequalities and the Mean Value Property of integral

The following inequality is often useful for estimating integrals.

Theorem 14.1 *Let f, g be integrable on $[a, b]$ and*

$$(\forall x \in [a, b])\,(f(x) \leq g(x)).$$

Then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Proof. For any partition $\mathcal{P}$ of $[a, b]$ we have

$$L(f, \mathcal{P}) \leq L(g, \mathcal{P}) \leq \int_a^b g(x)dx.$$

The assertion follows by taking supremum over all partitions $\mathcal{P}$ of $[a, b]$. □

Exercise 14.2 Let f be integrable on $[a, b]$ and there are $M, m \in \mathbb{R}$ such that

$$(\forall x \in [a, b])\,(m \leq f(x) \leq M).$$

Prove that

$$m(b - a) \leq \int_a^b f(x)dx \leq M(b - a).$$

Hint. Apply Theorem 14.1.

Corollary 14.3 (The Mean Value Property of integral) *Let f be continuous on $[a, b]$. Then there exists $\theta \in [a, b]$ such that*

$$\int_a^b f(x)dx = f(\theta)(b - a).$$

Proof. Set $m = \min_{[a,b]} f(x)$, $M = \max_{[a,b]} f(x)$. From Exercise 14.2 it follows that

$$m \leq \frac{1}{b - a}\int_a^b f(x)dx \leq M,$$

Applying the Intermediate Value Theorem (Theorem **A 4.3.1**) to the function f we conclude that there exists $\theta \in [a, b]$ such that

$$f(\theta) = \frac{1}{b - a}\int_a^b f(x)dx,$$

which completes the proof. □

15 Further properties of integral.

In this section we present some elementary properties of integral which are familiar from Calculus, but have rather technical proofs.

Theorem 15.1 *Let $a < c < b$. Let f be integrable on $[a, b]$. Then f is integrable on $[a, c]$ and on $[c, b]$ and*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Conversely, if f is integrable on $[a, c]$ and on $[c, b]$ then it is integrable on $[a, b]$.

Proof. Suppose that f is integrable on $[a, b]$. Fix $\varepsilon > 0$. Then there exists a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

We can assume that $c \in P$ so that $c = x_j$ for some $j \in \{0, 1, \ldots, n\}$ (otherwise consider the refinement of P adding the point c). Then $P_1 = \{x_0, \ldots, x_j\}$ is a partition of $[a, c]$ and $P_2 = \{x_j, \ldots, x_n\}$ is a partition of $[c, b]$. Moreover,

$$L(f, P) = L(f, P_1) + L(f, P_2), \quad U(f, P) = U(f, P_1) + U(f, P_2).$$

Therefore we have

$$[U(f, P_1) - L(f, P_1)] + [U(f, P_2) - L(f, P_2)] = U(f, P) - L(f, P) < \varepsilon.$$

Since each of the terms on the left hand side is non-negative, each one is less than ε, which proves that f is integrable on $[a, c]$ and on $[c, b]$. Note also that

$$L(f, P_1) \leq \int_a^c f(x)dx \leq U(f, P_1), \qquad L(f, P_2) \leq \int_c^b f(x)dx \leq U(f, P_2),$$

so that

$$L(f, P) \leq \int_a^c f(x)dx + \int_c^b f(x)dx \leq U(f, P).$$

This is true for any partition of $[a, b]$. Therefore

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Now assume that f is integrable on $[a, c]$ and $[c, b]$. Fix $\varepsilon > 0$. Then there exists a partition P_1 of $[a, c]$ and a partition P_2 of $[c, b]$ such that

$$U(f, P_1) - L(f, P_1) < \varepsilon/2, \qquad U(f, P_2) - L(f, P_2) < \varepsilon/2.$$

Let $P = P_1 \cup P_2$. Then

$$U(f, P) - L(f, P) = [U(f, P_1) - L(f, P_1)] + [U(f, P_2) - L(f, P_2)] < \varepsilon.$$

By the Criterion of Integrability we conclude that f is integrable on $[a, b]$. $\square$

Remark 15.2 The integral $\int_a^b f(x)dx$ was defined only for $a < b$. We add by definition that

$$\int_a^a f(x)dx = 0 \text{ and } \int_a^b f(x)dx = -\int_b^a f(x)dx \text{ if } a > b.$$

With this convention we always have that

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Theorem 15.3 *Let f and g be integrable on $[a,b]$. Then $f+g$ is also integrable on $[a,b]$ and*

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

Proof. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$. Let

$$m_i' = \inf\{f(x) \,|\, x_{i-1} \le x \le x_i\}, \qquad m_i'' = \inf\{g(x) \,|\, x_{i-1} \le x \le x_i\},$$

$$m_i = \inf\{f(x)+g(x) \,|\, x_{i-1} \le x \le x_i\}.$$

Define M_i, M_i', M_i'' similarly. The following inequalities hold

$$m_i \ge m_i' + m_i'', \qquad M_i \le M_i' + M_i''.$$

Therefore we have

$$L(f,P) + L(g,P) \le L(f+g,P), \qquad U(f+g,P) \le U(f,P) + U(g,P).$$

Hence for any partition P

$$L(f,P) + L(g,P) \le L(f+g,P) \le U(f+g,P) \le U(f,P) + U(g,P),$$

or otherwise

$$U(f+g,P) - L(f+g,P) \le [U(f,P) - L(f,P)] + [U(g,P) - L(g,P)].$$

Fix $\varepsilon > 0$. Since f and g are integrable there are partitions P_1 and P_2 such that

$$U(f,P_1) - L(f,P_1) < \varepsilon/2, \qquad U(g,P_2) - L(g,P_2) < \varepsilon/2.$$

Thus for the partition $P = P_1 \cup P_2$ we obtain that

$$U(f+g,P) - L(f+g,P) < \varepsilon.$$

This proves that $f+g$ is integrable on $[a,b]$. Moreover,

$$L(f,P) + L(g,P) \le L(f+g,P) \le \int_a^b [f(x)+g(x)]dx \le U(f+g,P) \le U(f,P) + U(g,P),$$

and

$$L(f,P) + L(g,P) \le \int_a^b f(x)dx + \int_a^b g(x)dx \le U(f,P) + U(g,P).$$

Therefore it follows that

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

The proof is completed. □

Theorem 15.4 *Let f be integrable on $[a,b]$. Then, for any $c \in \mathbb{R}$, cf is also integrable on $[a,b]$ and*

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx.$$

Proof. The proof is left as an exercise. Consider separately two cases: $c \ge 0$ and $c \le 0$. □

Theorem 15.5 *Let f be integrable on $[a,b]$. Then $|f|$ is integrable on $[a,b]$ and*

$$\Big|\int_a^b f(x)dx\Big| \le \int_a^b |f(x)|dx.$$

Proof. Note that for any interval $[\alpha,\beta]$ we have

$$\sup_{[\alpha,\beta]} |f(x)| - \inf_{[\alpha,\beta]} |f(x)| \le \sup_{[\alpha,\beta]} f(x) - \inf_{[\alpha,\beta]} f(x). \tag{20}$$

Indeed,

$$(\forall x, y \in [\alpha,\beta]) \Big(f(x) - f(y) \le \sup_{[\alpha,\beta]} f(x) - \inf_{[\alpha,\beta]} f(x) \Big),$$

so that

$$(\forall x, y \in [\alpha,\beta]) \Big(|f(x)| - |f(y)| \le \sup_{[\alpha,\beta]} f(x) - \inf_{[\alpha,\beta]} f(x) \Big),$$

which proves (20) by passing to the supremum in x and y. It follows from (20) that for any partition of $[a,b]$

$$U(|f|,P) - L(|f|,P) \le U(f,P) - L(f,P),$$

which proves the integrability of $|f|$ by the Criterion of Integrability. The last assertion follows from Theorem 14.1. □

Theorem 15.6 *Let $f : [a,b] \to \mathbb{R}$ be an integrable function such that*

$$(\forall x \in [a,b])(m \le f(x) \le M).$$

Let $g : [m,M] \to \mathbb{R}$ be a continuous function. Then the composite function $h : [a,b] \to \mathbb{R}$ defined by $h(x) = g(f(x))$ is integrable.

Proof. Fix $\varepsilon > 0$. Since g is uniformly continuous on $[m,M]$, there exists $\delta > 0$ such that $\delta < \varepsilon$ and

$$(\forall t, s \in [m,M])\big[(|t-s| < \delta) \Rightarrow (|g(t) - g(s)| < \varepsilon)\big].$$

By integrability of f there exists a partition $P = \{x_0, \dots, x_n\}$ of $[a,b]$ such that

$$U(f,P) - L(f,P) < \delta^2. \tag{21}$$

Let $m_i = \inf_{[x_{i-1},x_i]} f(x)$, $M_i = \sup_{[x_{i-1},x_i]} f(x)$ and $m_i^* = \inf_{[x_{i-1},x_i]} h(x)$, $M_i^* = \sup_{[x_{i-1},x_i]} h(x)$. Decompose the set $\{1,\dots,n\}$ into two subset : $(i \in A) \Leftrightarrow (M_i - m_i < \delta)$ and $(i \in B) \Leftrightarrow (M_i - m_i \ge \delta)$.

For $i \in A$ by the choice of δ we have that $M_i^* - m_i^* < \varepsilon$.

For $i \in B$ we have that $M_i^* - m_i^* \le 2K$ where $K = \sup_{t\in[m,M]} |g(t)|$. By (21) we have

$$\delta \sum_{i\in B} (x_i - x_{i-1}) \le \sum_{i\in B} (M_i - m_i)(x_i - x_{i-1}) < \delta^2,$$

so that $\sum_{i\in B}(x_i - x_{i-1}) < \delta$. Therefore

$$\begin{aligned} U(h,P) - L(h,P) &= \sum_{i\in A} (M_i^* - m_i^*)(x_i - x_{i-1}) + \sum_{i\in B} (M_i^* - m_i^*)(x_i - x_{i-1}) \\ &< \varepsilon(b-a) + 2K\delta < \varepsilon[(b-a) + 2K], \end{aligned}$$

which proves the assertion since $\varepsilon > 0$ is arbitrary. □

Corollary 15.7 *Let f, g be integrable on $[a,b]$. Then the product fg is integrable on $[a,b]$.*

Proof. Since $f+g$ and $f-g$ are integrable on $[a,b]$, $(f+g)^2$ and $(f-g)^2$ are integrable on $[a,b]$ by the previous theorem. Therefore

$$fg = \frac{1}{4}[(f+g)^2 - (f-g)^2]$$

is integrable on $[a,b]$. □

16 Integration as the inverse to differentiation

For a given function $f : [a, b] \to \mathbb{R}$, let $F : [a, b] \to \mathbb{R}$ be a differentiable function such that

$$F'(x) = f(x), \qquad \forall x \in (a, b).$$

We then call $F(x)$ a *primitive*, or an *antiderivative* of $f(x)$ on $[a, b]$.

Remark 16.1 Primitives are not uniquely defined. Indeed, to each primitive $F(x)$ one can add an arbitrary constant C and then $F(x) + C$ is again a primitive. Note also that for $C = -F(a)$ we obtain the primitive $F(x) - F(a)$, which vanishes at $x = a$.

Example 16.2 Let $n \in \mathbb{N}^+$. For each $C \in \mathbb{R}$ the function $F(x) = \frac{x^{n+1}}{n+1} + C$ is a primitive of $f(x) = x^n$.

Let f be an integrable function on an interval $[a, b]$. Then we can define on $[a, b]$ the function

$$F(x) := \int_a^x f(t)dt. \tag{22}$$

We are going to show that the function $F(x)$ is a primitive of $f(x)$ on $[a, b]$. First we prove that if $f(x)$ is integrable then the function $F(x)$ is continuous.

Theorem 16.3 *Let f be integrable on $[a, b]$ and let F be defined on $[a, b]$ by*

$$F(x) = \int_a^x f(t)dt.$$

Then F is continuous on $[a, b]$.

Proof. By the definition of integrability f is bounded on $[a, b]$. Let $M = \sup_{[a,b]} |f(x)|$. Then for $x, y \in [a, b]$ we have

$$\big|F(x) - F(y)\big| = \Big|\int_x^y f(t)dt\Big| \leq M|x - y|,$$

which proves that F is uniformly continuous on $[a, b]$. □

Now we prove that if f is continuous then F defined by (22) is a primitive of f.

Theorem 16.4 (Existence of a primitive) *Let f be continuous on $[a, b]$ and let F be defined on $[a, b]$ by*

$$F(x) = \int_a^x f(t)dt.$$

Then F is differentiable on (a, b) and $F'(x) = f(x)$, so F is a primitive of f on $[a, b]$.

Proof. Let $x \in (a, b)$. Let $h > 0$. Then

$$\frac{F(x + h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt.$$

By Corollary 14.3 there exists $\theta \in [x, x+h]$ such that

$$\int_x^{x+h} f(t)dt = f(\theta)h.$$

Hence we have

$$\frac{F(x+h) - F(x)}{h} = f(\theta).$$

As $h \to 0$, $\theta \to x$, and due to continuity of f we conclude that $\lim_{h\to 0} f(\theta) = f(x)$. The assertion follows. The case $h < 0$ is similar. □

Now we are in a position to prove the formula which is often known as the *Fundamental Theorem of Calculus*. In particular, it shows *the uniqueness* of the primitive (*up to an additive constant*).

Theorem 16.5 (The Fundamental Theorem of Calculus) *Let f be continuous on $[a, b]$ and G be a primitive of of f on $[a, b]$. Then for $x \in [a, b]$*

$$\int_a^x f(t)dt = G(x) - G(a).$$

In particular,

$$\int_a^b f(x)dx = G(b) - G(a).$$

Proof. Let

$$F(x) = \int_a^x f(t)dt.$$

By Theorem 16.4 the function $F - G$ is differentiable on $[a, b]$ and

$$F' - G' = (F - G)' = 0.$$

Therefore by Corollary **A 5.2.2** there is a number $C \in \mathbb{R}$ such that

$$F = G + C.$$

Since $F(a) = 0$ we conclude that $G(a) = -C$. Thus for $x \in [a, b]$

$$\int_a^x f(t)dt = F(x) = G(x) - G(a),$$

which is required. □

The Fundamental Theorem of Calculus can be extended to Riemann integrable functions.

Theorem 16.6 (The Fundamental Theorem of Calculus for integrable functions) *Let f be integrable on $[a, b]$ and G be a primitive of of f on $[a, b]$. Then*

$$\int_a^b f(x)dx = G(b) - G(a).$$

Proof. We omit the proof. □