"BOUNDARY BLOWUP" TYPE SUB-SOLUTIONS TO SEMILINEAR ELLIPTIC EQUATIONS WITH HARDY POTENTIAL

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ABSTRACT. Semilinear elliptic equations which give rise to solutions blowing up at the boundary are perturbed by a Hardy potential $\mu/\delta(x,\partial\Omega)^2$. The size of this potential effects the existence of a certain type of solutions (large solutions): if μ is too small, then no large solution exists. The presence of the Hardy potential requires a new definition of large solutions, following the pattern of the associated linear problem. Nonexistence and existence results for different types of solutions will be given. Our considerations are based on a Phragmen-Lindelöf type theorem which enables us to classify the solutions and sub-solutions according to their behavior near the boundary. Nonexistence follows from this principle together with the Keller-Osserman upper bound. The existence proofs rely on sub- and super-solution techniques and on estimates for the Hardy constant derived in Marcus, Mizel and Pinchover [9].

1. INTRODUCTION

On bounded smooth domains $\Omega \subset \mathbb{R}^N, N \geq 2$ we study the existence and non-existence of positive solutions and sub-solutions to semilinear elliptic equations of the form

$$-\Delta u - \frac{\mu}{\delta^2}u + \frac{u^p}{\delta^s} = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $\mu, s \in \mathbb{R}$ and $p > 1$ are given constants and

$$\delta(x) := \operatorname{dist}(x, \partial\Omega).$$

There are two competing ingredients in (1.1), namely the nonlinear problem

$$-\Delta U + \frac{U^p}{\delta^s} = 0 \quad \text{in } \Omega \tag{N}$$

and the linear problem

$$-\Delta h - \frac{\mu}{\delta^2}h = 0 \text{ in } \Omega. \tag{L}$$

The nonlinear problem (N) has received a lot of attention in recent years, cf. [10] and the references cited therein. For $s < 2$ it possesses a maximal solution which is larger than any other solution in Ω. This solution behaves like $c_{p,s}\delta(x)^{\frac{s-2}{p-1}}$. Since it tends to $+\infty$ as

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x approaches the boundary, it became common to call such solutions *boundary blow-up* solutions or simply *large* solutions. For $s > 2$ only the trivial solution exists. It follows from the Keller-Osserman upper bound given in Section 3.2. A related nonexistence result is found in [13]. There problem (N) is considered in the unit ball of $\mathbb{R}^N$ with $N \geq 3$, $p = \frac{N+2}{N-2}$ and $s \geq 2$.

The linear problem (L) has been studied recently in [4] and in [9] in connection with Hardy's inequality. In this paper we are interested only in positive solutions of (L). We shall call them *harmonics*. The concept of *sub-* and *super-harmonics* is understood in the usual pointwise sense. It makes sense to extend the concept of (sub-/super-)harmonics to *local* (sub-/super-)harmonics, which are defined only in a neighbourhood of the boundary of Ω. For $\mu \leq 1/4$ the linear problem (L) shows a remarkable structural property for sub-harmonics, which we call Phragmen-Lindelöf Alternative: a given local sub-harmonic

(i) either dominates every local super-harmonic multiplied by a suitable positive constant
(ii) or is dominated by a multiple of any local super-harmonic.

The first type of sub-harmonic is called *large*, the second type is called *small*.

The key to our study is the observation that solutions and sub-solutions of (1.1) are sub-harmonics of (L). We can therefore classify them according to their behavior in a neighborhood of the boundary.

A local sub-solution of (1.1) will be called an *L-subsolution* if it is a large sub-harmonic and an *S-subsolution* if it is a small sub-harmonic. In the familiar case $s = \mu = 0$ large local sub-solutions are those with finite or infinite positive boundary values and small local sub-solutions attain zero boundary values. Note that in this paper the use of the word "large" for a sub-solution does not imply that this sub-solution has "infinite boundary values".

When both (L) and (N) are combined into problem (1.1), interesting threshold-phenomena with respect to existence or non-existence of local sub-solutions occur. Our first main result, given in Theorem 4.3, can be summarized as follows: if $p > 1$ and $\mu \leq 1/4$ then

$$\boxed{\text{local } L\text{-subsolutions of (1.1) exist if and only if } \frac{s-2}{p-1} < \frac{1}{2} - \sqrt{\frac{1}{4} - \mu}.}$$

The proof of the main result goes as follows:

(i) any local sub-solution u of (1.1) satisfies the bound $u(x) \leq \text{const.}\delta(x)^{\frac{s-2}{p-1}}$, which is known as the Keller-Osserman upper bound
(ii) if $\mu < 1/4$ then any local large sub-harmonic u of (L) satisfies $\limsup_{x\to\partial\Omega} \frac{u(x)}{\delta(x)^{\beta_-}} > 0$ where $\beta_- = \frac{1}{2} - \sqrt{\frac{1}{4} - \mu}$.

Both (i) and (ii) are compatible if $\frac{s-2}{p-1} < \beta_-$ and incompatible if $\frac{s-2}{p-1} > \beta_-$. The equality case belongs to the non-existence regime, but this requires a much more refined analysis. Likewise, the case $\mu = 1/4$ is more subtle and needs extra care.

Our second main result, which is also given in Theorem 4.3, shows that in the existence case, one can in fact prove the existence of two different L-solutions:

(i) an ML-solution to (1.1), which is large but still dominated by at least one super-harmonic

(ii) an XXL-solution, which dominates every super-harmonic and moreover grows as fast as the Keller-Osserman upper-bound $\delta(x)^{\frac{s-2}{p-1}}$.

As a consequence of the two main results we note that (1.1) has local sub-solutions blowing up near the boundary if and only if $s < 2$ and $\mu^* < \mu \leq \frac{1}{4}$. Here $\mu^* = \frac{1}{4} - \left(\frac{p-2s+3}{p-1}\right)^2$ is a negative value because $s < 2$. It is an open problem to determine the precise asymptotic behavior of an XXL-solution. We conjecture that the XXL-solution $U(x)$ is unique and that its correct asymptotic behaviour is given by $\lim_{x\to\partial\Omega} U(x)/\delta(x)^{\frac{s-2}{p-1}} = \text{const}$.

The paper is organized as follows. In Section 2 we analyse the linear problem (L). We explain the role played by the Hardy-constant and prove the Phragmen-Lindelöf Alternative. Moreover, we construct explicit sub- and super-harmonics and give estimates for the boundary-behaviour of large and small sub-harmonics. In Section 3 we prove a comparison principle, which plays an important role in our analysis, and we prove the Keller-Osserman upper bound. Section 4 contains the proof of the main result. In Section 5 we give some additional results about small sub-solutions of (1.1) and in the final Section 6 we pose some open problems.

2. Linear problem

2.1. **Definitions.** For $\rho > 0$ and $\varepsilon \in (0, \rho)$ we use the notation

$$\Omega_\rho := \{x \in \Omega : \delta(x) < \rho\}, \quad \Omega_{\varepsilon,\rho} := \{x \in \Omega : \varepsilon < \delta(x) < \rho\}$$
$$D_\rho := \{x \in \Omega : \delta(x) > \rho\}, \quad \Gamma_\rho := \{x \in \Omega : \delta(x) = \rho\}.$$

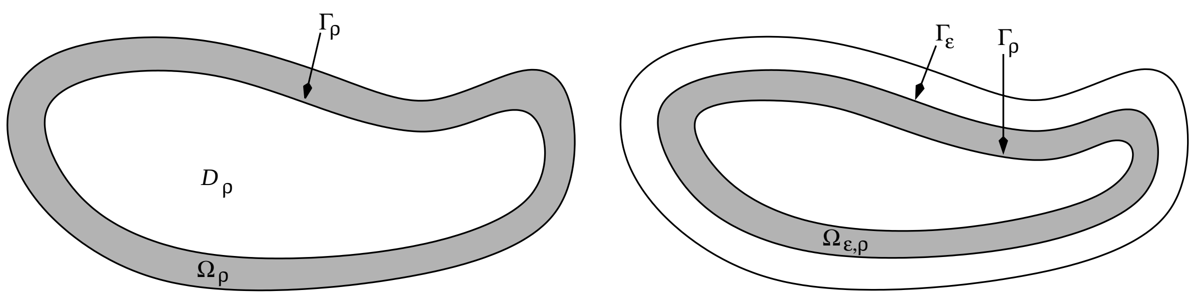

Figure 1. Illustration of $\Omega_\rho, D_\rho, \Gamma_\rho$ and $\Omega_{\epsilon,\rho}$

In this section we present several auxiliary facts concerning the linear problem (L). For simplicity set

$$\mathscr{L}_\mu := -\Delta - \frac{\mu}{\delta^2}.$$

Then (L) can be written in the form

$$\mathscr{L}_\mu h = 0 \quad \text{in } \Omega. \tag{2.1}$$

For convenience we call its solutions *harmonics.*

Definition 2.1. *Let $G \subset \Omega$ and let $H^1_c(G)$ denote the space functions from $H^1(G)$ with compact support. A sub-harmonic in G is a function $\underline{h} \in H^1_{loc}(G) \cap C(G)$ such that*

$$\int_G \nabla \underline{h} \cdot \nabla \varphi \, dx - \int_G \frac{\mu}{\delta^2} \underline{h} \varphi \, dx \leq 0, \qquad \forall\, 0 \leq \varphi \in H^1_c(G).$$

We say that $\underline{h}$ is a local sub-harmonic if there exists a parallel set Ω_ρ, $\rho > 0$ such that $\underline{h} \in H^1_{loc}(\Omega_\rho) \cap C(\Omega_\rho)$ is a sub-harmonic in Ω_ρ. Similarly, (local) super-harmonics $\overline{h}$ are defined with "$\geq$" in the above inequality.

Remark 2.2. By the classical maximum principle for the Laplacian, any nontrivial super-harmonic $\overline{h} \gneq 0$ in G is strictly positive in G,. Recall also that if $\underline{h}$ is a sub-harmonic in G then $\underline{h}_+$ is also a sub-harmonic in G, cf. [1, Lemma 2.10]

2.2. **The role of the Hardy constant.** The principal result of this section is given next.

Theorem 2.3. *Equation* (2.1) *admits a local positive super-harmonic if and only if $\mu \leq 1/4$. In particular no nontrivial harmonics exist if $\mu > \frac{1}{4}$.*

Its proof is accomplished via the following two lemmas which are intimately related to Hardy's inequality. Recall that the classical Hardy inequality reads as follows. There exists a constant $C_H(\Omega) > 0$ such that

$$\int_\Omega |\nabla u|^2 \, dx \geq C_H(\Omega) \int_\Omega \frac{u^2}{\delta^2} \, dx, \qquad \forall\, u \in H^1_0(\Omega). \tag{2.2}$$

The optimal constant will be denoted by $C_H(\Omega)$. For a bounded Lipschitz domain it is known that $C_H(\Omega) \in (0, 1/4]$. If Ω is convex then $C_H(\Omega) = 1/4$. In general, $C_H(\Omega)$ varies with the domain and could be arbitrary small (see, e.g. [9, Theorem I and Section 4]) for a discussion and examples, see also [5]).

The relation between Hardy inequalities and the existence of local positive super-harmonics in a neighborhood of the boundary is explained by the following classical result (cf. [1, Theorem 3.3]).

Lemma 2.4. *The following three statements are equivalent:*

(i) *Equation* (2.1) *admits a positive super-harmonic in Ω_ρ.*
(ii) *If $\underline{h}$ and $\overline{h}$ are sub- and super-harmonics of* (2.1) *in a subdomain G with $\overline{G} \subset \Omega_\rho$ and if $\underline{h} \leq \overline{h}$ on ∂G then $\underline{h} \leq \overline{h}$ a.e. in G.*

(iii) *The following inequality holds:*

$$\int_{\Omega_\rho} |\nabla u|^2\,dx \geq \mu \int_{\Omega_\rho} \frac{u^2}{\delta_\Omega^2}\,dx, \qquad \forall\, u \in H_0^1(\Omega_\rho). \tag{2.3}$$

Note that the above inequality (2.3) is not a particular case of (2.2) because $\operatorname{dist}(x, \partial\Omega) \neq \operatorname{dist}(x, \partial\Omega_\rho)$. Denote the optimal constant in (2.3) by

$$C_H^{loc}(\Omega_\rho) := \inf_{H_0^1(\Omega_\rho)} \frac{\int_{\Omega_\rho} |\nabla u|^2\,dx}{\int_{\Omega_\rho} \frac{u^2}{\delta^2}\,dx}.$$

The following result can be extracted from the arguments in [9, p.3246].

Lemma 2.5. (Local Hardy Inequality) *There exists $\bar\rho > 0$ such that for every $\rho \in (0, \bar\rho)$ one has $C_H^{loc}(\Omega_\rho) = 1/4$.*

Proof. It was already observed in [4] that $C_H^{loc}(\Omega_\rho) \geq 1/4$. (It follows also simply from the fact that the equation $\mathscr{L}_{1/4} h = 0$ admits positive super-solutions in $\Omega_{\bar\rho}$ for some $\bar\rho > 0$, see Lemma 2.8 below.) On the other hand, the proof of Theorem 5 in [9] implies that $C_H^{loc}(\Omega_\rho) \leq 1/4$ for all $\rho > 0$. □

Observe that in contrast to the "global" Hardy constant $C_H(\Omega)$ from (2.2), the value of $C_H^{loc}(\Omega_\rho)$ does not depend on the shape of domain Ω if ρ is sufficiently small.

2.3. **Phragmen–Lindelöf alternative.** We establish a version of the Phragmen–Lindelöf type comparison principle for sub-harmonics, which shows that sub-harmonics are in a certain sense "separated" by the the cone of positive super-harmonics. See [12, pp. 93-106] for a classical reference to the Phragmen–Lindelöf principle.

Theorem 2.6. (Phragmen–Lindelöf Alternative) *Let $\mu \leq 1/4$. Let $\underline{h}$ be a local sub-harmonic. Then the following alternative holds:*

(i) *either for every local super-harmonic $\overline{h} > 0$*

$$\limsup_{x \to \partial\Omega} \underline{h}/\overline{h} > 0, \tag{2.4}$$

(ii) *or for every local super-harmonic $\overline{h} > 0$*

$$\limsup_{x \to \partial\Omega} \underline{h}/\overline{h} < \infty. \tag{2.5}$$

Proof. Assume (i) does not hold, that is there exists a super-harmonic $\overline{h}_* > 0$ that

$$\lim_{x \to \partial\Omega} \underline{h}/\overline{h}_* = 0. \tag{2.6}$$

Let $\overline{h} > 0$ be an arbitrary super-harmonic in Ω_ρ. By Remark 2.2, there exists a constant $c > 0$ such that $\overline{h} \geq c\underline{h}$ on $\Gamma_{\rho/2}$. For $\tau > 0$, define a comparison function

$$v_\tau := c\underline{h} - \tau\overline{h}_*.$$

Then (2.6) implies that for every $\tau > 0$ there exists $\varepsilon = \varepsilon(\tau) \in (0, \rho)$ such that $v_\tau \le 0$ on Ω_ε. Applying the classical comparison principle in $\Omega_{\varepsilon/2,\rho/2}$, we conclude that $\overline{h} \ge v_\tau$ in $\Omega_{\varepsilon/2,\rho/2}$ and hence, in $\Omega_{\rho/2}$. So by considering arbitrary small $\tau > 0$, we conclude that for every super-harmonic $\overline{h} > 0$ in Ω_ρ there exist $c > 0$ such that $\overline{h} \ge c\underline{h}$ holds in Ω_ρ. This implies (2.5). □

Theorem 2.6 suggests the following classification of sub-harmonics.

Definition 2.7. *Let $\mu \le 1/4$ and let $\underline{h}$ be a local sub-harmonic in Ω_ρ. We say that $\underline{h}$ is* large *if it satisfies the first alternative (i). Otherwise, we say that $\underline{h}$ is a* small.

The classification of harmonics into small and large harmonics is included in the above definition. In the sequel we shall use the notation $\underline{h}$ for small and $\underline{H}$ for large sub-harmonics.

2.4. **Construction of local sub- and super-harmonics.** It is well known (cf. [7, Lemma 14.15]) that if Ω is of class C^k, $k \ge 2$, then there exists $\bar\rho > 0$ such that the distance function δ is in $C^k(\Omega_{\bar\rho})$ and the set Γ_ε is of class C^k for all $\varepsilon \in (0, \bar\rho)$. For every $x \in \Gamma_\varepsilon$ there exists a unique point $\sigma(x) \in \partial\Omega$ such that $|x - \sigma(x)| = \delta(x)$. Moreover,

$$|\nabla\delta(x)| = 1 + o(\delta(x)) \quad \text{as } \delta(x) \to 0, \tag{2.7}$$

while

$$\Delta\delta(x) = -(N-1)\mathcal{H}_0(\sigma(x)) + o(\delta(x)) \quad \text{as } \delta(x) \to 0, \tag{2.8}$$

where $\mathcal{H}_0(\sigma(x))$ denotes the mean curvature of $\partial\Omega$ at the point $\sigma(x)$. Note that the mean curvature of $\partial\Omega$ is bounded, since Ω is bounded and smooth.

In what follows, $\beta_- \le \beta_+$ denote the real roots of the scalar equation $\beta(1-\beta) = \mu$, i.e.

$$\beta_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \mu} \ \text{ provided } \ \mu \le 1/4. \tag{2.9}$$

Clearly, if $\mu = 1/4$ then $\beta_- = \beta_+ = 1/2$.

Lemma 2.8. (i) *Let $\mu < 1/4$. The function δ^β is a local super-harmonic of $\mathscr{L}_\mu$ if $\beta \in (\beta_-, \beta_+)$ and a local sub-harmonic of $\mathscr{L}_\mu$ if $\beta \notin [\beta_-, \beta_+]$. Moreover, if $\epsilon \in (0, \epsilon_*)$, where $\epsilon_* = \min\{1, \sqrt{1-4\mu}\}$ then*

$$\overline{h} := \delta^{\beta_+}(1 - \delta^\epsilon), \qquad \overline{H} := \delta^{\beta_-}(1 + \delta^\epsilon)$$

are positive local super-harmonics of $\mathscr{L}_\mu$, while

$$\underline{h} := \delta^{\beta_+}(1 + \delta^\epsilon), \qquad \underline{H} := \delta^{\beta_-}(1 - \delta^\epsilon)$$

are positive local sub-harmonics of $\mathscr{L}_\mu$.

(ii) *Let $\mu = 1/4$. The function $\delta^{1/2}\left(\log\frac{1}{\delta}\right)^\beta$ is a local super-harmonic of $\mathscr{L}_{1/4}$ if $\beta \in (0,1)$ and a local sub-harmonic of $\mathscr{L}_{1/4}$ if $\beta \notin [0,1]$. Moreover, if $\epsilon \in (0,1)$ then*

$$\overline{h} = \delta^{1/2}\left(1 - \left(\log\frac{1}{\delta}\right)^{-\epsilon}\right), \qquad \overline{H} := \delta^{1/2}\log\frac{1}{\delta}\left(1 + \left(\log\frac{1}{\delta}\right)^{-\epsilon}\right)$$

are positive local super-harmonics of $\mathscr{L}_{1/4}$, while

$$\underline{h} := \delta^{1/2}\left(1+\left(\log\frac{1}{\delta}\right)^{-\epsilon}\right), \qquad \underline{H} := \delta^{1/2}\log\frac{1}{\delta}\left(1-\left(\log\frac{1}{\delta}\right)^{-\epsilon}\right)$$

are positive local sub-harmonics of $\mathscr{L}_{1/4}$.

Proof. (i) Note that

$$\begin{aligned}\nabla\delta^\beta &= \beta\delta^{\beta-1}\nabla\delta,\\ -\Delta\delta^\beta &= \beta(1-\beta)\delta^{\beta-2}|\nabla\delta|^2-\beta\delta^{\beta-1}\Delta\delta.\end{aligned}$$

Thus a direct computation together with (2.7), (2.8) imply the result (cf. [9, Lemma 7]).

(ii) Observe that

$$\begin{aligned}\nabla\left(\delta^\gamma\log^\beta\frac{1}{\delta}\right) &= \left(\gamma\log^\beta\frac{1}{\delta}-\beta\log^{\beta-1}\frac{1}{\delta}\right)\delta^{\gamma-1}\nabla\delta,\\ -\Delta\left(\delta^\gamma\log^\beta\frac{1}{\delta}\right) &= \left(\gamma(1-\gamma)\log^\beta\frac{1}{\delta}+\beta(2\gamma-1)\log^{\beta-1}\frac{1}{\delta}+\beta(1-\beta)\log^{\beta-2}\frac{1}{\delta}\right)\delta^{\gamma-2}|\nabla\delta|^2\\ &\quad - \left(\gamma\log^\beta\frac{1}{\delta}-\beta\log^{\beta-1}\frac{1}{\delta}\right)\delta^{\gamma-1}\Delta\delta.\end{aligned}$$

Thus a direct computation together with (2.7), (2.8) imply the result. □

The following theorem, which is an immediate corollary of Theorem 2.6 and Lemma 2.8, summarises our results concerning the asymptotic behaviour of sub-harmonics at the boundary.

Theorem 2.9. *Let $\underline{h}$ be a small local sub-harmonic and $\underline{H}$ be a large local sub-harmonic of $\mathscr{L}_\mu$.*

(i) *If $\mu < 1/4$ then*

$$\limsup_{x\to\partial\Omega}\frac{\underline{h}}{\delta^{\beta_+}}<\infty, \qquad \limsup_{x\to\partial\Omega}\frac{\underline{H}}{\delta^{\beta_-}}>0.$$

(ii) *If $\mu = 1/4$ then*

$$\limsup_{x\to\partial\Omega}\frac{\underline{h}}{\delta^{1/2}}<\infty, \qquad \limsup_{x\to\partial\Omega}\frac{\underline{H}}{\delta^{1/2}\log\frac{1}{\delta}}>0.$$

The above leading order terms are sharp.

Lemma 2.8 and Theorem 2.9 have the following implications.

Corollary 2.10. *Let $\mu \le 1/4$.*

(i) *The small local sub-harmonics vanish on the entire boundary of Ω.*
(ii) *If $\mu < 0$ then the large sub-harmonics are unbounded at some points of $\partial\Omega$.*
(iii) *If $0 < \mu \le 1/4$ then there exist large sub-harmonics vanishing on $\partial\Omega$.*

Remark 2.11. (1) Observe that when $\mu = 0$ then $\underline{H} = const.$ is a large sub-harmonic in a neighbourhood of the boundary.

(2) For $\mu < 1/4$ large local sub-harmonics fail to belong to the subspace of functions in $H^1(\Omega_\rho)$ which vanish on $\partial\Omega$. Indeed, for $\mu \le 0$ large local sub-harmonics do not converge to zero near $\partial\Omega$. And for $0 < \mu < 1/4$, even if a large local sub-harmonic vanishes on $\partial\Omega$ then its gradient is not square-integrable near $\partial\Omega$. To see this, let $\underline{H}$ be a large sub-harmonic of $\mathscr{L}_\mu$ in Ω_ρ. For $\beta \in (1/2, \beta_+)$ the function δ^β is a super-harmonic in $H^1(\Omega_\rho)$ with vanishes on $\partial\Omega$. Hence $\underline{H}_\kappa := (\underline{H} - \kappa\delta^\beta)_+$ is a large sub-harmonic. By choosing a sufficiently large $\kappa > 0$ we can ensure that $\underline{H}_\kappa$ vanishes on Γ_ρ. Assume for contradiction that $\underline{H}_\kappa \in H^1_0(\Omega_\rho)$. Then

$$\int_{\Omega_\rho} |\nabla \underline{H}_\kappa|^2\, dx - \mu \int_{\Omega_\rho} \frac{\underline{H}_\kappa^2}{\delta_\Omega^2}\, dx \le 0$$

and by the local Hardy inequality we obtain $\underline{H}_\kappa \equiv 0$, i.e., $\underline{H} \le \kappa\delta^\beta$. This contradicts Theorem 2.9(i).

(3) If $\mu > 1/4$, then $\mathscr{L}_\mu$ has no positive local super-harmonics (cf. Theorem 2.3). However,

$$\delta^\gamma \log^\beta \frac{1}{\delta}$$

is a local sub-harmonic for arbitrary $\gamma, \beta \in \mathbb{R}$. This suggests that in the case $\mu > 1/4$ local sub-harmonics can not be naturally classified according to their asymptotic behaviour.

Another direct consequence of Theorem 2.6 and Lemma 2.8 is a two–sided bound on the asymptotic behaviour of positive super-harmonics at the boundary.

Theorem 2.12. *Let $\overline{h} > 0$ be a local super-harmonic of $\mathscr{L}_\mu$.*

(i) *If $\mu < 1/4$ then*

$$\liminf_{x\to\partial\Omega} \frac{\overline{h}}{\delta^{\beta_+}} > 0, \qquad \liminf_{x\to\partial\Omega} \frac{\overline{h}}{\delta^{\beta_-}} < \infty.$$

(ii) *If $\mu = 1/4$ then*

$$\liminf_{x\to\partial\Omega} \frac{\overline{h}}{\delta^{1/2}} > 0, \qquad \liminf_{x\to\partial\Omega} \frac{\overline{h}}{\delta^{1/2}\log\frac{1}{\delta}} < \infty.$$

The above leading order terms are sharp.

3. Estimates for the nonlinear problem

3.1. **Comparison principle.** We start with the definition of sub- and super-solutions to the nonlinear problem (1.1).

Definition 3.1. *A sub-solution to* (1.1) *in a subdomain* $G \subset \Omega$ *is a function* $\underline{u} \in H^1_{loc}(G) \cap C(G)$ *such that*

$$\int_G \nabla \underline{u} \cdot \nabla \varphi \, dx - \int_G \frac{\mu}{\delta^2} \underline{u} \varphi \, dx + \int_G \frac{\underline{u}^p}{\delta^s} \varphi \, dx \leq 0, \quad \forall \, 0 \leq \varphi \in H^1_c(G). \tag{3.1}$$

A super-solution $\overline{u}$ *is defined similarly by replacing "$\leq$" with "$\geq$". A function u which is both a sub- and super-solution will be called a* solution.

Lemma 3.2. (Comparison principle)

(*i*) *Let G be open with $G \subset \Omega$. Let $0 \leq \underline{u}, \overline{u} \in H^1_{loc}(G) \cap C(G)$ be a pair of sub- and super-solutions to* (1.1) *in G such that*

$$\limsup_{x \to \partial G} [\underline{u}(x) - \overline{u}(x)] < 0.$$

Then $\underline{u} \leq \overline{u}$ in G.

(*ii*) *Let G be open with $\overline{G} \subset \Omega$. Let $\underline{u}, \overline{u} \in H^1(G) \cap C(\overline{G})$ be a pair of sub- and super-solutions to* (1.1) *in G with $\overline{u} > 0$ in G and $\underline{u} \leq \overline{u}$ on ∂G. Then $\underline{u} \leq \overline{u}$ in G.*

Proof. (*i*) Subtracting one inequality from another we obtain

$$\int_G \nabla(\underline{u} - \overline{u}) \cdot \nabla \varphi \, dx - \int_G \frac{\mu}{\delta^2} (\underline{u} - \overline{u}) \varphi \, dx + \int_G \frac{W(x)}{\delta^s} (\underline{u} - \overline{u}) \varphi \, dx \leq 0,$$
$$\forall \varphi \in H^1_c(G), \varphi \geq 0,$$

where

$$W(x) := \frac{\underline{u}^p - \overline{u}^p}{\underline{u} - \overline{u}}.$$

Assume that $(\underline{u} - \overline{u})_+ \not\equiv 0$. Testing against $(\underline{u} - \overline{u})_+$ we conclude that

$$\int_G \left(|\nabla(\underline{u} - \overline{u})_+|^2 - \frac{\mu}{\delta^2} (\underline{u} - \overline{u})_+^2 + \frac{W(x)}{\delta^s} (\underline{u} - \overline{u})_+^2 \right) dx \leq 0. \tag{3.2}$$

Since $\overline{u} > 0$ we can write

$$(\underline{u} - \overline{u})_+ = \overline{u} \phi,$$

where $\phi \in H^1_c(G)$ due to the assumption that $\limsup_{x \to \partial G} [\underline{u}(x) - \overline{u}(x)] < 0$. Note that $\operatorname{supp} \phi = \overline{G}_+$, where $G_+ := \{x \in G : \underline{u}(x) > \overline{u}(x)\}$. We obtain

$$\begin{aligned}
\int_G |\nabla(\underline{u} - \overline{u})_+|^2 \, dx &= \int_G (\phi^2 |\nabla \overline{u}|^2 + 2\phi \overline{u} \nabla \overline{u} \cdot \nabla \phi + \overline{u}^2 |\nabla \phi|^2) \, dx \\
&= \int_G \phi^2 |\nabla \overline{u}|^2 \, dx + \int_G \nabla \overline{u} \cdot \nabla(\phi^2 \overline{u}) \, dx \\
&\geq \int_G \frac{\mu}{\delta^2} \phi^2 \overline{u}^2 - \frac{\phi^2 \overline{u}^{p+1}}{\delta^s} \, dx,
\end{aligned}$$

where we have used that $\overline{u}$ is a super-solution. Hence we conclude that

$$\int_G \left(|\nabla(\underline{u}-\overline{u})_+|^2 - \frac{\mu}{\delta^2}(\underline{u}-\overline{u})_+^2 + \frac{V(x)}{\delta^s}(\underline{u}-\overline{u})_+^2 \right) \geq 0, \tag{3.3}$$

where $V(x) = \overline{u}^{p-1}$. But by strict convexity we have $W(x) \gneq V(x)$ on G_+. Thus (3.3) and (3.2) imply that G_+ has zero measure, which contradicts the assumption $(\underline{u}-\overline{u})_+ \not\equiv 0$.

The proof of (ii) is similar if instead of ϕ one uses ϕ_ε defined by $(\underline{u} - (\overline{u}+\varepsilon))_+ = \overline{u}\phi_\varepsilon$ with $\varepsilon > 0$, so we omit it. □

Remark 3.3. Note that the above lemma is valid for any $\mu \in \mathbb{R}$. We do not require the assumption $\mu \leq 1/4$ which ensures positivity of the principal part $\mathscr{L}_\mu$ because for $\mu > 1/4$ the nonlinearity compensates for the loss of positivity.

3.2. **Keller–Osserman type bound.** By a simple computation analogous to Lemma 2.8 one finds that for $p > 1$ the function

$$\gamma\delta(x)^{\frac{2-s}{1-p}}$$

has the following properties:

	local sub-solution	local super-solution
$\beta_- \leq \frac{2-s}{1-p} \leq \beta_+$	–	γ arbitrary
$\frac{2-s}{1-p} < \beta_-$ or $\beta_+ < \frac{2-s}{1-p}$	γ small	γ large

TABLE 1. Properties of $\delta(x)^{\frac{2-s}{1-p}}$

In particular, this function is always a local super-solution if γ is sufficiently large. The next considerations show that in order to make it a global super-solution, one needs to replace the distance function δ by the regularized distance function $d : \Omega \to \mathbb{R}^+$ attributed to Whitney, cf. [14]. The regularized distance function is in $C^\infty(\Omega)$ regardless of the regularity of $\partial\Omega$ and has the following properties: there exists a positive constant c such that

$$\begin{aligned} c^{-1}\delta(x) \leq d(x) &\leq c\delta(x), \\ |\nabla d(x)| &\leq c, \\ |\Delta d(x)| &\leq c d^{-1}(x) \quad \text{for all} \quad x \in \Omega. \end{aligned} \tag{3.4}$$

Proposition 3.4. *Let $p > 1$. For γ sufficiently large, but independent of $\varepsilon \geq 0$, the function*

$$\overline{u} = \gamma d^{\frac{s}{p-1}}(d-\varepsilon)^{-\frac{2}{p-1}}$$

is a super-solution of (1.1) *in* $\{x \in \Omega, d(x) > \varepsilon\}$.

Proof. A straightforward computation together with (3.4) yields

$$\begin{aligned}
|\Delta d^{\frac{s}{p-1}}| &\leq \alpha_1 d^{\frac{s}{p-1}-2} \leq \alpha_1 d^{\frac{s}{p-1}}(d-\epsilon)^{-2},\\
|\Delta (d-\varepsilon)^{-\frac{2}{p-1}}| &\leq \alpha_2 (d-\varepsilon)^{-\frac{2}{p-1}-2} + \alpha_3 (d-\varepsilon)^{-\frac{2}{p-1}-1} d^{-1}\\
&\leq (\alpha_2+\alpha_3)(d-\epsilon)^{-\frac{2p}{p-1}},\\
|(\nabla d^{\frac{s}{p-1}}, \nabla (d-\varepsilon)^{-\frac{2}{p-1}})| &\leq \alpha_4 d^{\frac{s}{p-1}-1}(d-\varepsilon)^{-\frac{2}{p-1}-1} \leq \alpha_4 d^{\frac{s}{p-1}}(d-\epsilon)^{-\frac{2p}{p-1}},
\end{aligned}$$

where α_i, $i = 1\ldots 4$ depend only on c, p, s. In addition

$$|\frac{\mu}{\delta^2}\overline{u}| \leq \alpha_5 \gamma d^{\frac{s}{p-1}}(d-\varepsilon)^{\frac{-2p}{p-1}},$$

where again α_5 depends on c, μ, p, s. Collecting all the terms and keeping in mind that

$$\delta^{-s}(x) \geq c^{-|s|} d^{-s}(x)$$

we find

$$\begin{aligned}
\mathscr{L}_\mu \overline{u} &\geq -\gamma\alpha_6 d^{\frac{s}{p-1}}(d-\varepsilon)^{-\frac{2p}{p-1}} \geq -\gamma\alpha_6 c^{|s|} d^{\frac{s}{p-1}+s}(d-\varepsilon)^{-\frac{2p}{p-1}}\delta^{-s}\\
&= -\alpha_6 c^{|s|}\frac{\overline{u}^p}{\gamma^{p-1}}\delta^{-s} \geq -\overline{u}^p\delta^{-s},
\end{aligned}$$

for γ sufficiently large, but independent of $\varepsilon \geq 0$. □

Sub-solutions to the nonlinear equation (1.1) obey a universal upper bound given next. As a tool we use the comparison principle from Lemma 3.2.

Proposition 3.5. (Keller–Osserman Bound) *Assume $p > 1$. Let u be an arbitrary local sub-solution to* (1.1) *in Ω_ρ for some $\rho > 0$. Then there exists $\gamma_* > 0$ depending on u such that*

$$u(x) \leq \gamma_* \delta^{\frac{2-s}{1-p}}(x) \quad \text{in } \Omega_\rho. \tag{3.5}$$

If u is sub-solution in all of Ω, then γ_* can be chosen independently of u.

Proof. Let u be a local sub-solution of (1.1) in Ω_ρ. Thus

$$u(x) \leq \overline{u}(x) = \gamma d^{\frac{s}{p-1}}(x)(d(x)-\varepsilon)^{-\frac{2}{p-1}} \quad \text{in} \quad \{x \in \Omega_\rho : d(x) > \varepsilon\},$$

provided $0 < \epsilon < \rho/c$ with c as in (3.4) and provided γ is so large that $\overline{u} \geq u$ on Γ_ρ. Since the above inequality holds for arbitrary positive $\varepsilon < \rho/c$ it follows that

$$u(x) \leq \gamma d^{\frac{s-2}{p-1}}(x) \leq \gamma c^{\frac{s-2}{p-1}}\delta^{\frac{s-2}{p-1}}(x),$$

as required. If u is a sub-solution in all of Ω then the above construction works on the set $\{x \in \Omega : d(x) > \varepsilon\}$, which has only the boundary at $d(x) = \varepsilon$ and no second boundary Γ_ρ. □

4. The main results

Since every solution and sub-solution of (1.1) is a sub-harmonic of $\mathscr{L}_\mu$, we shall classify them in accordance with Definition 2.7.

Definition 4.1. *A solution of* (1.1) *is called an S–solution if it is a small sub-harmonic and it is called an L–solution if it is is a large sub-harmonic. Further, we introduce different classes of L–solutions:*

(ML): *U is an ML-solution*[1] *if there exists a super-harmonic H such that*

$$\limsup_{x\to\partial\Omega}\frac{U}{H}<+\infty;$$

(XL): *U is an XL–solution of* (1.1) *if for every super-harmonic H one has*

$$\liminf_{x\to\partial\Omega}\frac{U}{H}=+\infty;$$

(XXL): *U is an XXL–solution of* (1.1) *if one has*

$$\liminf_{x\to\partial\Omega}\frac{U}{\delta^{\frac{s-2}{p-1}}}>0.$$

The corresponding classes of sub-solutions and local (sub) solutions are defined accordingly.

Remark 4.2. Note that division of L-solutions into ML, XL, XXL solutions is not exhaustive. For example, the solution of the problem

$$\begin{cases} -\Delta u+u^p=0 \quad \text{in } \Omega, \\ u=0 \text{ on } \Gamma_0, \quad u=1 \text{ on } \Gamma_1, \quad u=+\infty \text{ on } \partial\Omega\setminus(\Gamma_0\cup\Gamma_1), \end{cases}$$

where $\Gamma_0,\Gamma_1\subset\partial\Omega$ are smooth submanifolds of $\partial\Omega$, is an L-solution which does not belong to the classes ML, XL, XXL.

Our main result in the paper reads as follows.

Theorem 4.3. *Let $\mu\le 1/4$, β_- be as in* (2.9) *and $p>1$.*

i) *If $s\ge\beta_-(p-1)+2$ then* (1.1) *admits no local L-subsolutions.*
ii) *If $s<\beta_-(p-1)+2$ then* (1.1) *admits ML- and XXL-solutions in Ω.*

The above result can be seen as a critical threshold phenomenon in two different ways by either taking p or μ as a parameter.

(a) *Critical value of p:* Let $p^*=1-\frac{2-s}{\beta_-}$ with the convention $p^*=+\infty$ if $\beta_-=0$ and $s<2$, $p^*=-\infty$ if $\beta_-=0$ and $s\ge 2$.

existence	nonexistence	
$p\ge p^*$	$1<p<p^*$	if $\mu<0$
$1<p\le p^*$	$p>\max\{1,p^*\}$	if $0<\mu\le 1/4$

[1] *Moderate solutions*, as introduced in [6]

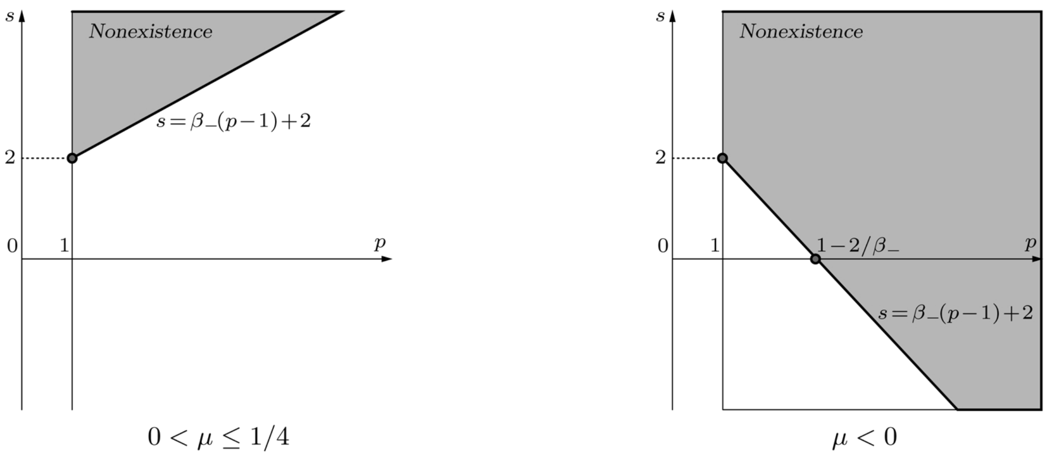

FIGURE 2. Nonexistence zones of equation (1.1) for typical values of μ.

(b) *Critical value of* μ*:* Let $\mu^* = \frac{1}{4} - \left(\frac{p-2s+3}{p-1}\right)^2$.

existence	nonexistence	
—	$\mu \leq 1/4$	if $s \geq (p+3)/2$
$\mu^* < \mu \leq 1/4$	$\mu \leq \mu^*$	if $s < (p+3)/2$

Remark 4.4. If $s \geq 2$, $\mu = 0$, $p = \frac{N+2}{N-2}$ and if Ω is the unit ball in $\mathbb{R}^N$, $N \geq 3$, Ratto et al. [13] proved that no global positive solution exists. Since such solutions are L-solutions our result (i) extends the non-existence result in [13].

In the remaining part of this section we prove Theorem 4.3. First we present the nonexistence part of the proof and after that, we consider the existence.

4.1. **Proof of Theorem 4.3.**

4.1.1. *Nonexistence.* Observe that in the *supercritical case* $s > \beta_-(p-1)+2$ the Keller–Osserman bound (3.5) is incompatible with the lower bound on large sub-harmonics in Theorem 2.9. As every L–subsolution to (2.1) is a large sub-harmonic of $\mathscr{L}_\mu$, this immediately implies the nonexistence of local L–subsolutions to (2.1).

In the *critical case* $s = \beta_-(p-1)+2$ the Keller–Osserman bound is comparable with the lower bound on large sub-harmonics, so different arguments must be used to prove the nonexistence.

Below we present a proof which covers both subcritical and critical cases. It consists of three parts:

(a) First we show that for every local L-subsolution $\underline{u}$ there exists a local L-subsolution $\underline{u}_*$, which vanishes on Γ_ρ and satisfies $\limsup_{x\to\partial\Omega} \frac{\underline{u}_*}{\underline{u}} = 1$.

(b) Then we construct a family of super-solutions u_ϵ in $\Omega_{\rho,\epsilon_\rho}$, converging to zero as $\epsilon \to 0$ and tending to $+\infty$ on the inner boundary and to zero on the outer boundary, cf. Figure 3.

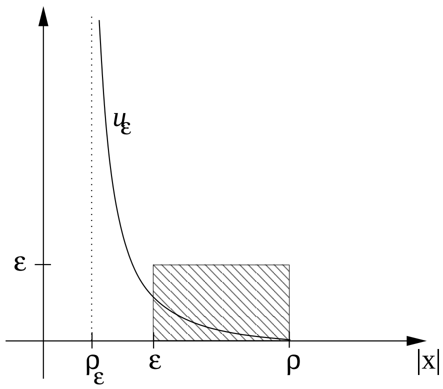

FIGURE 3. Graph of u_ϵ

(c) From this "improved upper bound" it follows from by Comparison Principle of Lemma 3.2 that zero is the only local L-subsolution.

Lemma 4.5. *Let $\mu \le 1/4$ and $p > 1$. Let $\underline{u} \ge 0$ be a local L–subsolution to* (1.1)*. Then* (1.1) *admits a local L–subsolution $\underline{u}_*$ such that for some $\rho > 0$*

$$\underline{u}_* = 0 \quad \text{on } \Gamma_\rho \quad \text{and} \quad \limsup_{x\to\partial\Omega} \frac{\underline{u}_*}{\underline{u}} = 1. \tag{4.1}$$

Proof. Let $\underline{u}$ be a local L–subsolution to (1.1). For $\kappa > 0$, set

$$h_* := \kappa\delta^{1/2}\log(1/\delta)^{1/2}.$$

According to Lemma 2.8, h_* is a local super-harmonic for $\mathscr{L}_{1/4}$ and hence for $\mathscr{L}_\mu$. Then $\underline{u} - h_*$ is also a local sub-harmonic of $\mathscr{L}_\mu$ and

$$\limsup_{x\to\partial\Omega} \frac{\underline{u} - h_*}{h_*} = +\infty,$$

by Theorem 2.9. In particular, this means that $\underline{u} - h_*$ is a large sub-harmonic of $\mathscr{L}_\mu$, according to Definition 2.7. Moreover, for a sufficiently small $\rho > 0$ we ensure $\underline{u} - h_* \le 0$ on Γ_ρ by choosing $\kappa > 0$ sufficiently big. Besides being a large local sub-harmonic the function $(\underline{u} - h_*)_+$ satisfies

$$\mathscr{L}_\mu(\underline{u} - h_*)_+ \le \mathscr{L}_\mu\underline{u} \le -\frac{C}{\delta^s}\underline{u}^p \le -\frac{C}{\delta^s}(\underline{u} - h_*)^p \quad \text{in } \{x \in \Omega_\rho : \underline{u} - h_* > 0\}. \tag{4.2}$$

Thus, setting $\underline{u}_* = (\underline{u} - h_*)_+$ we obtain a local L–subsolution in Ω_ρ with the required properties. Note that we have used the fact that the maximum of two sub-solutions is again a sub-solution. □

Remark 4.6. Note that the function $\underline{u}_*$ extended by zero to D_ρ is a sub-solution to (1.1) in the entire domain Ω.

Now we establish an "improved upper bound" on local sub-solutions of (1.1) vanishing on Γ_ρ for some $\rho > 0$, which immediately implies Theorem 4.3 (i) via the Comparison Principle (Lemma 3.2) and Lemma 4.5.

Lemma 4.7. *Let $\mu \le 1/4$, $p > 1$ and $s \ge \beta_-(p-1)+2$. Then there exists $\rho > 0$ such that for every $\varepsilon \in (0,\rho)$ there exists $\rho_\varepsilon \in (0,\varepsilon)$ and a positive super-solution z_ε of the nonlinear equation* (1.1) *in the ring-shaped domain $\Omega_{\rho_\varepsilon,\rho}$ such that*

$$z_\varepsilon = 0 \ \text{on } \Gamma_\rho, \qquad \liminf_{x\to\Gamma_{\rho_\varepsilon}} z_\varepsilon = +\infty \quad \text{and} \quad \sup_{\Omega_{\varepsilon,\rho}} z_\varepsilon \le \varepsilon. \tag{4.3}$$

Remark 4.8. In fact, Lemma 4.7 implies more then mere nonexistence. Consider a family of "large solution" problems

$$-\Delta u - \frac{\mu}{\delta^2}u + \frac{u^p}{\delta^s} = 0 \quad \text{in } D_\varepsilon, \qquad u = +\infty \quad \text{on } D_\varepsilon. \tag{4.4}$$

For each $\rho > 0$ such a problem is well-posed and admits a unique "large solution" u_ε, cf. [11]. Moreover, the family u_ε is monotone nonincreasing on compact subsets of Ω as $\varepsilon \to 0$. Thus for all sufficiently small $\varepsilon > 0$ one can find a super-harmonic h in Ω_ρ, where $\rho > 0$ is taken from Lemma 4.7, so that $u_\varepsilon - h \le 0$ on Γ_ρ. If $s \ge \beta_-(p-1)+2$ then Lemma 4.7 implies that $(u_\varepsilon - h)_+$ converges to zero as $\varepsilon \to 0$, uniformly on every compact subset of Ω. Thus, in the nonexistence regime $s \ge \beta_-(p-1)+2$, an attempt to approximate solutions of (1.1) by exhausting the domain Ω will lead to an S–solution (possibly trivial) in the limit.

Proof of Lemma 4.7. We are going to construct the super-solutions u_ε satisfying (4.3) using the solutions of an ODE initial value problem. Related arguments were previously used in [8].

Let as before $\sigma(x)$ be the projection of the point $x \in \Omega_\rho$ on $\partial\Omega$ and $\delta(x)$ be the distance of x to the boundary. Fix $\bar\rho > 0$ such that $\delta \in C^2(\Omega_{\bar\rho})$. If $\bar\rho$ is sufficiently small one can use (σ, δ) as new coordinates in Ω_ρ, for all $\rho \in (0,\bar\rho)$. In these coordinates the Laplacian becomes

$$\Delta_x = \frac{\partial^2}{\partial\delta^2} - (N-1)\mathcal{H}\frac{\partial}{\partial\delta} + \Delta_\sigma,$$

where Δ_σ is the Laplace-Beltrami operator on $\partial\Omega$ and $\mathcal{H} = \mathcal{H}(\cdot,\delta)$ is the mean curvature of Γ_δ (see [3] for a detailed discussion).

Let $\eta = \eta(\delta)$ be a positive super-harmonic of $\mathscr{L}_\mu$ in Ω_ρ, as constructed in Lemma 2.8. Set $\overline{\mathcal{H}} := (N-1)\sup_{\Omega_\rho}\mathcal{H}$. Consider the initial value problem

$$-\ddot v - \left(2\frac{\dot\eta}{\eta} - \overline{\mathcal{H}}\right)\dot v + \frac{\eta^{p-1}}{\delta^s}v^p = 0 \qquad v(\rho) = 0, \quad \dot v(\rho) = -\kappa, \tag{4.5}$$

where $\kappa > 0$. Let $v_\kappa = v_\kappa(r)$ be the maximal left solution of (4.5) defined on the maximal left interval of existence (R_κ, ρ) in the region $\{(r, v) \in (0, \rho) \times \mathbb{R}\}$ (cf. [12, pp. 10-12 and 24-36]).

Observe that $v_\kappa > 0$ and $\dot{v}_\kappa < 0$ for all $r \in (R_\kappa, \rho)$. Indeed, if $r_0 = \max\{r \in (R_\kappa, \rho) : \dot{v}_\kappa(r) = 0\}$ then $\ddot{v}_\kappa(r_0) > 0$. As $\kappa > 0$, we conclude that $\{r \in (R_\kappa, \rho) : \dot{v}_\kappa(r) = 0\} = \emptyset$ and $v_\kappa(r)$ is *strictly decreasing* on any interval. In particular,

$$\liminf_{r \to 0} v_\kappa(r) > 0. \tag{4.6}$$

An important consequence of the monotonicity of the solutions v_κ is that they can be used to construct super-solutions of (1.1).

Lemma 4.9. *Let $\eta = \eta(\delta)$ be a positive super-harmonic of $\mathscr{L}_\mu$ in Ω_ρ, and $v_\kappa : (R_\kappa, \rho) \to \mathbb{R}$ be the maximal left solution of* (4.5). *Then $z_\kappa(\delta) := v_\kappa(\delta)\eta(\delta)$ is a super-solution to* (1.1) *in $\Omega_{R_\kappa,\rho}$.*

Proof. A direct computation (cf. [12, p.8]) using the monotonicity of v_κ shows that

$$\begin{aligned}
\mathscr{L}_\mu z_\kappa &= -\ddot{v}_\kappa \eta - v_\kappa \ddot{\eta} - 2\dot{v}_\kappa \dot{\eta} + (N-1)\mathcal{H}\left(\dot{v}_\kappa \eta + v_\kappa \dot{\eta}\right) - \frac{\mu}{\delta^2} v_\kappa \eta \\
&\geq \left(-\ddot{v}_\kappa - \left(2\frac{\dot{\eta}}{\eta} - \overline{\mathcal{H}}\right)\dot{v}_\kappa\right)\eta + \left(\mathscr{L}_\mu \eta\right) v_\kappa \\
&\geq -\frac{\eta^p}{\delta^s} v_\kappa^p \quad \text{in } \Omega_{R_\kappa,\rho},
\end{aligned}$$

as required. □

Our analysis of (4.5) is be based on the following well known ODE comparison lemma, which we present here for reader's convenience.

Lemma 4.10. *Assume that $u > 0$ and $v > 0$ satisfy differential inequalities*

$$-\ddot{u} - a(r)\dot{u} + b(r)u^p \geq 0, \qquad -\ddot{v} - a(r)\dot{v} + b(r)v^p \leq 0 \qquad (r \in (R, \rho)),$$

where $a, b \in C(R, \rho)$, $b \geq 0$ and $p > 1$. Then

(i) (IVP)-case*: $u(\rho) \leq v(\rho)$ and $\dot{u}(\rho) > \dot{v}(\rho)$ imply $u(r) < v(r)$ for all $r \in (R, \rho)$;*
(ii) (BVP)-case*: $u(\rho) > v(\rho)$ and $u(R) > v(R)$ imply $u(r) > v(r)$ for all $r \in (R, \rho)$.*

Proof. Part (i) could be proved similarly to [12, pp. 26]. Part (ii) can be established following the arguments in the proof of Lemma 3.2. □

Lemma 4.7 follows via Lemma 4.9 from the following.

Lemma 4.11. (ODE Lemma) *Let $\mu \leq 1/4$, $p > 1$ and $s \geq \beta_-(p-1) + 2$. If v_κ is the maximal left solution of* (4.5) *on the maximal existence interval (R_κ, ρ) then*

(i) $R_\kappa > 0$ and $v_\kappa(r) \to +\infty$ as $r \searrow R_\kappa$;
(ii) $R_\kappa \to 0$ as $\kappa \to 0$;
(iii) for any $r_* \in (0, \rho)$ one has $\sup_{[r_*,\rho]} v_\kappa \to 0$ as $\kappa \to 0$.

Proof. To prove the lemma, one only has to show that $R_\kappa > 0$. As $v_\kappa(r)$ is decreasing in r this obviously implies $v_\kappa(r) \to +\infty$ as $r \searrow R_\kappa$.

Indeed, assume that (i) holds. Let $0 < \kappa_1 < \kappa_2$. Then $v_{\kappa_1} < v_{\kappa_2}$ for all $r \in (R_{\kappa_2}, \rho)$ by Lemma 4.10 (ii). In particular, this implies that $R_{\kappa_1} \le R_{\kappa_2}$.

Fix $r_* \in (0, \rho)$. For $\epsilon > 0$, let $v^{(\epsilon)}$ be the unique solution of the boundary value problem

$$-\ddot{v} - \left(2\frac{\dot{\eta}}{\eta} - \overline{\mathcal{H}}\right)\dot{v} + \frac{\eta^{p-1}}{r^s}v^p = 0, \qquad v(r_*) = \epsilon, \quad v(\rho) = 0. \tag{4.7}$$

Set $-\kappa(\epsilon) = \dot{v}^{(\epsilon)}(\rho)$. Thus $v_{\kappa(\epsilon)} = v^{(\epsilon)}$ for $r \in (r_*, \rho)$ in view of the uniqueness of solution for both (4.5) and (4.7). Moreover, $v_{\kappa(\epsilon)} \le \epsilon$ for $r \in (r_*, \rho)$ as $v_{\kappa(\epsilon)}$ is decreasing and $\kappa(\epsilon)$ is strictly decreasing in view of the BVP-comparison principle of Lemma 4.10 for equation (4.5). This proves (ii) and (iii).

Now we are going to show that $R_\kappa > 0$ for all $\kappa > 0$. To do this, we shall consider separately the cases $\mu < 1/4$ and $\mu = 1/4$, with different choices of the super-harmonics η.
Case $\mu < 1/4$. Here we choose a super-harmonic $\eta(r) := r^{\beta_-}(1 + r^\epsilon)$ and $\epsilon \in (0, 1)$ (see Lemma 2.8 (i)). Then (4.5) can be written as

$$-\ddot{v} - \frac{2\beta_-}{r}\left(1 + O(r^\epsilon)\right)\dot{v} + r^{(p-1)\beta_- - s}(1 + r^\epsilon)^{p-1}v^p = 0. \tag{4.8}$$

Assume that $R_\kappa = 0$ for some $\kappa > 0$. A direct computation (similar to the one in Proposition 3.4) shows that for a sufficiently large constant $\gamma > 0$ and all $R \in (0, \rho)$

$$\overline{v}_R = \gamma r^{\frac{s}{p-1} - \beta_-}(r - R)^{-\frac{2}{p-1}}$$

is a super-solution to (4.5) in (R, ρ), with γ independent of R. By Lemma 4.10 (i) we conclude that

$$v_\kappa \le \gamma r^{\frac{s-2}{p-1} - \beta_-} \quad \text{in } (0, \rho). \tag{4.9}$$

In the *subcritical case* $s > \beta_-(p-1) + 2$ this bound contradicts to (4.6), so we conclude that $R_\kappa > 0$.

In the *critical case* $s = \beta_-(p-1) + 2$, linearizing (4.5) on v_κ and taking into account (4.6) we conclude that v_κ is a sub-harmonic to the equation

$$-\ddot{v} - \frac{2\beta_-}{r}\left(1 + O(r^\epsilon)\right)\dot{v} + \frac{C(1 + r^\epsilon)^{p-1}}{r^2}v = 0 \quad \text{in } (0, \rho/2), \tag{4.10}$$

where $C := \inf_{(0,\rho/2)} v_\kappa^{p-1} > 0$. Let $\alpha_- < \alpha_+$ be the roots of the quadratic equation

$$\alpha(\alpha + 2\beta_- - 1) = C.$$

Note that $\alpha_- < 0$ as $\beta_- < 1/2$, and choose $\alpha'_- \in (\alpha_-, 0)$. A direct computation shows that for some $\rho_1 \in (0, \rho/2)$ the function $\overline{h} := Ar^{\alpha'_-}$ is a super-harmonic to (4.10) on $(0, \rho_1)$. Choose $A > 0$ in such a way that $\overline{h}(\rho_1) < v_\kappa(\rho_1)$ and $\dot{\overline{h}}(\rho_1) > \dot{v}_\kappa(\rho_1)$. Then

$$v_\kappa \ge \overline{h}$$

by Lemma 4.10 (ii). But this contradicts to (4.9), and we conclude that $R_\kappa > 0$.

Case $\mu = 1/4$. Choose a super-harmonic $\eta(r) := r^{1/2}\,(1-(\log(1/r)^{-\epsilon})$ and $\epsilon \in (0,1)$ as in Lemma 2.8 (i). Then (4.5) can be written as

$$-\ddot{v} - \frac{1}{r}\left(1 + O(\log(1/r)^{-\epsilon-1})\right)\dot{v} + r^{(p-1)/2-s}(1-\log(1/r)^{-\epsilon})^{p-1}v^p = 0. \tag{4.11}$$

Assume that $R_\kappa = 0$ for some $\kappa > 0$. A direct computation shows that for a sufficiently large constant $\gamma > 0$ and all $R \in (0,\rho)$

$$\overline{v}_R = \gamma r^{\frac{s}{p-1}-\frac{1}{2}}(r-R)^{-\frac{2}{p-1}}$$

is a super-solution to (4.11) in (R,ρ), with γ independent of R. As in (4.9), we obtain

$$v_\kappa \le \gamma r^{\frac{s-2}{p-1}-\frac{1}{2}} \quad \text{in } (R,\rho), \tag{4.12}$$

for all small $R > 0$. In the *subcritical case* $s > \frac{p+3}{2}$ this bound contradicts to (4.6), so we conclude that $R_\kappa > 0$.

In the *critical case* $s = \frac{p+3}{2}$, we simply observe that v_κ is a sub-harmonic to the homogeneous equation

$$-\ddot{v} - \frac{1}{r}\left(1 + O(\log(1/r)^{-\epsilon-1})\right)\dot{v} = 0 \quad \text{in } (0,\rho). \tag{4.13}$$

On the other hand, a direct computation shows that the function $\overline{h} := A\log^{1/2}(1/r)$ is a super-harmonic to (4.10) on $(0,\rho_1)$, for some $\rho_1 \in (0,\rho)$. Choose $A > 0$ in such a way that $\overline{h}(\rho_1) < v_\kappa(\rho_1)$ and $\dot{\overline{h}}(\rho_1) > \dot{v}_\kappa(\rho_1)$. Then

$$v_\kappa \ge \overline{h}$$

by Lemma 4.10 (*ii*). But this contradicts to (4.12), and we conclude that $R_\kappa > 0$. □

4.1.2. *Existence.* To prove the existence part of Theorem 4.3, we first establish the existence of a solution between ordered sub- and super-solutions.

Lemma 4.12. *Let $\mu \le 1/4$ and $p > 1$. Assume that* (1.1) *admits a sub-solution $\underline{u}$ and a super-solution $\overline{u}$ in Ω so that $0 \le \underline{u} \le \overline{u}$ in Ω. Then* (1.1) *has a solution U in Ω such that $\underline{u} \le U \le \overline{u}$ in Ω.*

Proof. For small $\varepsilon > 0$, let U_ε be a positive solution of

$$\mathscr{L}_\mu U_\varepsilon + \frac{U_\varepsilon^p}{\delta^s} = 0 \quad \text{in } D_\varepsilon, \qquad U_\varepsilon = \underline{u} \quad \text{on } \partial D_\varepsilon.$$

Such a solution is obtained, e.g., by minimization of the convex, coercive functional

$$\int_{D_\varepsilon} |\nabla U|^2 - \frac{\mu}{\delta^2}U^2 + \frac{|U|^{p+1}}{(p+1)\delta^s}\,dx$$

in $H^1(D_\varepsilon)$ with $U = \underline{u}$ on ∂D_ε. By applying the Comparison Principle of Lemma 3.2 (*ii*) we obtain $\underline{u} \le U_\varepsilon \le \overline{u}$ on D_ε. Applying interior regularity together with the usual diagonalization argument we conclude that $U = \lim_{\varepsilon\to 0} U_\varepsilon$ is the required solution of (1.1) in Ω. □

Now, we prove the existence of XXL-solution in all of Ω.

Lemma 4.13. *Let $\mu \le 1/4$, $p > 1$ and $s < \beta_-(p-1) + 2$. Then* (1.1) *admits an XXL-solution in Ω.*

Proof. Let $\mu \le 1/4$. Set

$$\underline{u}_\rho := \gamma \left(\delta^{\frac{2-s}{1-p}} - \kappa \delta^{1/2} \log(1/\delta)^{1/2} \right),$$

where $\kappa > 0$ is chosen in such a way that $\underline{u}_\rho(\rho) = 0$. For some $\rho > 0$ and sufficiently small $\gamma > 0$, the function $\underline{u}_\rho$ is a sub-solution to (1.1) in Ω_ρ, cf. Table 1 and the fact that $\delta^{1/2}\log(1/\delta)^{1/2}$ is a local super-harmonic to $\mathscr{L}_{1/4}$ and hence a local super-harmonic to $\mathscr{L}_\mu$ for all $\mu \le 1/4$ cf. Lemma 2.8(ii). Let $\underline{u}$ denote the function $\underline{u}_\rho$, extended by zero to D_ρ. Thus $\underline{u} \ge 0$ is a sub-solution to (1.1) in the entire domain Ω.

Set $\overline{u} := \gamma_* d^{\frac{2-s}{1-p}}$, where d is the Whitney-distance. Note that $\underline{u} \le \overline{u}$ in Ω in view of the Keller–Osserman bound of Proposition 3.5. Moreover, $\overline{u}$ is a super-solution to (1.1) in Ω, according to Proposition 3.4. By Lemma 4.12 we conclude that (1.1) admits a solution U in Ω so that $\underline{u} \le U \le \overline{u}$ in Ω, which is the required XXL-solution. □

Remark 4.14. The constructed XXL-solution U satisfies, for some $\gamma > 0$,

$$\gamma \le \liminf_{x\to\partial\Omega} \frac{U}{\delta^{\frac{2-s}{1-p}}} \le \limsup_{x\to\partial\Omega} \frac{U}{\delta^{\frac{2-s}{1-p}}} \le \gamma^{-1}. \tag{4.14}$$

Next, we prove the existence of an ML-solution in all of Ω.

Lemma 4.15. *Let $\mu \le 1/4$, $p > 1$ and $s < \beta_-(p-1) + 2$. Then* (1.1) *admits an ML-solution U in Ω.*

Proof. We consider separately the cases $\mu < 1/4$ and $\mu = 1/4$.
Case $\mu < 1/4$. Let $\alpha \in (\beta_-, \min\{\beta_- p + 2 - s, \beta_- + 1, \beta_+\})$ and $\kappa > 0$. Set

$$\underline{u}_\rho := \delta^{\beta_-} - \kappa \delta^\alpha,$$

where $\kappa > 0$ is chosen in such a way that $\underline{u}_\rho(\rho) = 0$. A direct computation shows that for a sufficiently small $\rho > 0$,

$$\mathscr{L}_\mu \underline{u}_\rho + \delta^{-s} \underline{u}_\rho^p \le -\kappa(\alpha(1-\alpha) - \mu)\delta^{\alpha-2}(1 + o(1)) + \delta^{-s}\left(\delta^{\beta_-} - \kappa\delta^\alpha\right)^p \le 0 \quad \text{in } \Omega_\rho,$$

that is $\underline{u}_\rho$ is a sub-solution of (1.1) in Ω_ρ. Let $\underline{u}$ denote the function $\underline{u}_\rho$, extended by zero to D_ρ. Hence $\underline{u} \ge 0$ is a sub-solution to (1.1) in the entire domain Ω.

Fix $\epsilon \in (0, \min\{1, \sqrt{1-4\mu}\})$. Then $\overline{H} := \delta^{\beta_-}(1 + \delta^\epsilon)$ is a large local super-harmonic of $\mathscr{L}_\mu$, as constructed in Lemma 2.8. We may assume that $\mathscr{L}_\mu \overline{H} \ge 0$ in Ω_ρ (otherwise we adjust ρ in the construction of $\underline{u}$). Let $R \in (0, \rho)$. Let $\overline{u}_R = \gamma_* d^{\frac{s}{p-1}}(d - R)^{-\frac{2}{p-1}}$, where $\gamma_* > 0$ is chosen in such a way that $\overline{u}_R$ is a super-solution to (1.1) in D_R, see Lemma 3.4. Choose $\tau_* > 1$ large enough, so that $\tau_* \overline{H} > \overline{u}_R$ on Γ_ρ. Then

$$\overline{u} := \min\{\tau_* \overline{H}, \overline{u}_R\}$$

is a super-solution to (1.1) in the entire Ω.

Note that $\underline{u} \le \overline{u}$ in Ω, in view of the Comparison Principle of Lemma 3.2 (i). By Lemma 4.12 we conclude that (1.1) has a solution U in Ω so that $\underline{u} \le U \le \overline{u}$ in Ω, which is the required ML-solution.

Case $\mu = 1/4$. Let $\alpha \in (0,1)$ and $\kappa > 0$. Set

$$\underline{u}_\rho := \delta^{1/2}\log\frac{1}{\delta} - \kappa\delta^{1/2}\log^\alpha\frac{1}{\delta},$$

where $\kappa > 0$ is chosen in such a way that $\underline{u}_\rho(\rho) = 0$. A direct computation shows that for a sufficiently small $\rho > 0$,

$$\begin{aligned}\mathscr{L}_{1/4}\underline{u}_\rho + \delta^{-s}\underline{u}_\rho^p \;&\le\; -\kappa\alpha(1-\alpha)\delta^{-3/2}\left(\log^{\alpha-2}\frac{1}{\delta}\right)(1+o(1))\\ &\quad+\delta^{-s}\left(\delta^{1/2}\log\frac{1}{\delta} - \kappa\delta^{1/2}\log^\alpha\frac{1}{\delta}\right)^p \le 0\end{aligned}$$

in Ω_ρ, that is $\underline{u}_\rho$ is a sub-solution of (1.1) in Ω_ρ.

To construct a super-solution to (1.1), fix $\epsilon \in (0,1)$ and set

$$\overline{H} := \delta^{1/2}\log\frac{1}{\delta}\left(1 + \log^{-\epsilon}\frac{1}{\delta}\right).$$

Thus $\overline{H}$ is a large local super-harmonic of $\mathscr{L}_{1/4}$, see Lemma 2.8. The rest of the proof is similar to the case $\mu < 1/4$ above, so we omit it. $\square$

Remark 4.16. The constructed ML-solution U satisfies the bound

$$1 \le \liminf_{x\to\partial\Omega}\frac{U}{\overline{H}} \le \limsup_{x\to\partial\Omega}\frac{U}{\overline{H}} \le \tau_*.$$

5. S–SOLUTIONS AND SOLUTIONS FOR ARBITRARY $\mu > 1/4$

It is easy to see that equation (1.1) admits local S-subsolutions for all $p > 1$, $s \in \mathbb{R}$ and $\mu \le 1/4$. Below we are going to show that the existence of global S-solutions is controlled by the global Hardy constant $C_H(\Omega)$ rather then by relations between p, s and μ.

Theorem 5.1. *Let $\mu \le C_H(\Omega)$, $p > 1$ and $s \in \mathbb{R}$. Then* (1.1) *has no nontrivial S–subsolution in Ω.*

Proof. Let $\underline{u} \ge 0$ be a nontrivial S–subsolution of (1.1) in Ω. Set $h_* := \delta^{1/2}\log^{1/2}(1/\delta)$. Note that for all $\mu \le 1/4$, h_* is a local super-harmonic of $\mathscr{L}_\mu$ and

$$\lim_{x\to\partial\Omega}\frac{u}{h_*} = 0,$$

cf. Theorem 2.9(i). For $\kappa > 0$, consider the family $\underline{v}_\kappa := (\underline{u} - \kappa h_*)_+$. Clearly, $\underline{v}_\kappa \in H^1_c(\Omega)$ and $\mathscr{L}_\mu\underline{v}_\kappa \le 0$ in Ω. Testing this inequality with $\underline{v}_\kappa$ yields

$$C_H(\Omega)\int_\Omega \frac{\underline{v}_\kappa^2}{\delta^2}\,dx \le \int_\Omega |\nabla\underline{v}_\kappa|^2\,dx < \mu\int_\Omega\frac{\underline{v}_\kappa^2}{\delta^2}\,dx,$$

which means $\underline{v}_\kappa = 0$ in Ω for every $\kappa > 0$. We conclude that $\underline{u} = 0$. $\square$

The following lemma is crucial in our construction of global solutions for $\mu > C_H(\Omega)$.

Lemma 5.2. *Let $\mu > C_H(\Omega)$, $p > 1$ and $s \in \mathbb{R}$. Then there exists $\rho > 0$ such that for every $\varepsilon \in (0, \rho)$ equation* (1.1) *in D_ε admits a positive solution $u_\varepsilon \in H^1_0(D_\varepsilon)$. Moreover, $u_\varepsilon \in H^1_0(D_\varepsilon)$ and $u_\varepsilon(x)$ is monotone nondecreasing as $\varepsilon \to 0$.*

Proof. For a small $\varepsilon > 0$, consider the problem

$$\mathscr{L}_\mu u_\varepsilon + \frac{u_\varepsilon^p}{\delta^s} = 0, \quad u_\varepsilon \in H^1_0(D_\varepsilon), \tag{5.1}$$

and the corresponding functional

$$J_\varepsilon(u) = \int_{D_\varepsilon} \frac{1}{2}|\nabla u|^2 - \frac{\mu}{2\delta^2}u^2 + \frac{u_+^{p+1}}{(p+1)\delta^s}\,dx$$

in $H^1_0(D_\varepsilon)$. It is standard to see that J_ε is coercive and weakly lower semicontinuous on $H^1_0(D_\varepsilon)$. Moreover, minimizers of J_ε are nonnegative and solve (5.1).

Let $u_\varepsilon \geq 0$ be a minimizer of J_ε. From the definition of Hardy's constant $C_H(\Omega)$, it follows that if $\mu > C_H(\Omega)$ then $u = 0$ is not a local minimum of J_ε for $\varepsilon > 0$ sufficiently small. Hence $u_\varepsilon \gneq 0$ is the required solution of (5.1).

Further, by applying the Comparison Principle of Lemma 3.2 (*ii*) we conclude that $u_\varepsilon(x)$ is monotone nondecreasing as $\varepsilon \to 0$. $\square$

Theorem 5.3. *Let Ω be such that $C_H(\Omega) < 1/4$. Let $\mu \in (C_H(\Omega), 1/4]$, $p > 1$ and $s \in \mathbb{R}$. Then equation* (1.1) *admits a positive S-solution in Ω.*

Proof. Let $\overline{h} > 0$ be a super-harmonic in Ω_ρ for some $\rho > 0$, as constructed in Lemma 2.8. For some fixed $R \in (0, \rho/2)$, let $\overline{u}_R = \gamma_* d^{\frac{s}{p-1}}(d-R)^{-\frac{2}{p-1}}$ be a super-solution to (1.1) in D_R, as constructed in Lemma 3.4. Choose $\tau_* \geq 1$ large enough, so that $\tau_*\overline{h} > \overline{u}_R$ on $\Gamma_{\rho/2}$. Then

$$\overline{u} := \min\{\tau_*\overline{h}, \overline{u}_R\}$$

is a super-solution to (1.1) in the entire Ω.

Let u_ε be the monotone increasing family of solutions (1.1) in D_ε, as constructed in Lemma 5.2. By applying the Comparison Principle of Lemma 3.2 (*ii*) we obtain $u_\varepsilon \leq \overline{u}$ on D_ε. Applying the usual diagonalization argument we conclude that $u := \lim_{\varepsilon\to 0} u_\varepsilon$ is the required S-solution of (1.1) in Ω. $\square$

Remark 5.4. Observe that if $s < \beta_+(p-1) + 2$ and $C_H(\Omega) < \mu \leq 1/4$ then

$$\lim_{x\to\partial\Omega} \frac{\delta^{\frac{s-2}{p-1}}}{\overline{h}} = 0$$

for every positive local super-harmonic $\overline{h}$ of $\mathscr{L}_\mu$, see Theorem 2.12. By Lemma 3.4 and the Comparison Principle of Lemma 3.2 (*ii*) we obtain that if $s < \beta_+(p-1) + 2$ then every

S-subsolution u of (1.1) satisfies an improved upper bound

$$u \le \gamma_* \delta^{\frac{s-2}{p-1}} \quad \text{in } \Omega,$$

which is stronger then the upper bound on S-subsolutions imposed by positive superharmonics.

Our classification of (sub) solutions to (1.1) is not applicable for $\mu > 1/4$. However, one can show that for all values of $\mu > 1/4$, equation (1.1) admits positive solutions which obey the Keller–Osserman bound.

Theorem 5.5. *Let $\mu > 1/4$ and $p > 1$. Then equation* (1.1) *admits a positive solution u in Ω such that $u \le \gamma_* \delta^{\frac{2-s}{1-p}}$ in Ω.*

Proof. Let u_ε be the monotone increasing family of solutions (1.1) in D_ε, as constructed in Lemma 5.2. By Lemma 3.4 and the Comparison Principle of Lemma 3.2 (*ii*) we obtain $u_\varepsilon \le \gamma_* d^{\frac{s-2}{p-1}}$ on D_ε. Applying the usual diagonalization argument we conclude that $u := \lim_{\varepsilon\to 0} u_\varepsilon$ is the required solution of (1.1) in Ω. □

6. Open problems

We finish our investigation with a list of open problems, which we consider as interesting:

Problem 1. If we assume $p > 1$, $\mu \le 1/4$ and $s < \beta_-(p-1)+2$ then in Theorem 4.3 we have proved the existence of an XXL-solution with boundary behaviour given by (4.14). What is the precise boundary behaviour of an XXL-solution? We conjecture that the correct asymptotic behaviour is given by $\lim_{x\to\partial\Omega} U(x)/\delta(x)^{\frac{s-2}{p-1}} = \text{const.}$, where the constant depends only on p, s and μ. In the case $s = 0$, $\mu = 0$ this was proved in [2], [3] and [11].

Problem 2. In the case $p > 1$, $\mu = s = 0$ every XL-solution is automatically an XXL-solution and moreover the XXL-solution is unique, see [2, 11]. Are these two statements true for every s, μ in the existence range?

Problem 3. What is the asymptotic behavior near the boundary of the solutions, constructed in Theorem 5.5 for arbitrary $\mu > 1/4$?

Problem 4. Is the existence and non-existence threshold phenomena similar to Theorem 4.3 valid for some (or maybe all) $p < 1$, or is there a natural reason, why the result can only be true for $p > 1$?

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