Complex Function Theory

Lecture Notes 2004/2005

Vitaly Moroz

School of Mathematics
University of Bristol
BS8 1TW Bristol, UK
v.moroz@bristol.ac.uk

Contents

Notations

$B_r(z_0)$	*open ball* $B_r(z_0) = \{z \in \mathbb{C} : \lvert z - z_0\rvert < r\}$ of radius $r > 0$, centered at $z_0 \in \mathbb{C}$
$\bar{B}_r(z_0)$	*closed ball* $\bar{B}_r(z_0) = \{z \in \mathbb{C} : \lvert z - z_0\rvert \le r\}$ of radius $r > 0$, centered at $z_0 \in \mathbb{C}$
$S_r(z_0)$	*circle* $S_r(z_0) = \{z \in \mathbb{C} : \lvert z - z_0\rvert = r\}$ of radius $r > 0$, centered at $z_0 \in \mathbb{C}$
$A_{r,R}(z_0)$	*open annulus* $A_{r,R}(z_0) = \{z \in \mathbb{C} : r < \lvert z - z_0\rvert < R\}$ with radii $0 \le r < R \le +\infty$, centered at $z_0 \in \mathbb{C}$
$\mathbb{C}_+$	open upper half plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$
$S_r^+(z_0)$	*positively oriented circle* parameterized by $\gamma(t) = z_0 + re^{it}$ $(0 \le t \le 2\pi)$
$[a, b]$	*line segment* between points $a, b \in \mathbb{C}$, parameterized by $\gamma(t) = (1-t)a + tb$ $(0 \le t \le 1)$

Recommended Texts

1. I. Stewart, D. Tall, *Complex Analysis*, Cambridge University Press, Cambridge, 1983.
2. J. E. Marsden, *Basic complex analysis*, Freeman, San Francisco, 1973.
3. S. Lang, *Complex analysis*, Springer–Verlag, New York – Berlin, 1977.
4. J. B. Conway, *Functions of one complex variable*, Springer–Verlag, New York – Berlin, 1978.

The *Schaum's outline of theory and problems of complex variables* by M. R. Spiegel (McGraw–Hill, London, New–York, 1974) is a good additional source of problems.

This Lecture Notes, Current Exercises and Solutions to Exercises are available on the Web:
`http://www.maths.bris.ac.uk/~mavbm/cft.html`

0 Preliminaries

Complex numbers. We define $\mathbb{C}$, the set of *complex numbers*, to be the set of all ordered pairs (x, y), where $x, y \in \mathbb{R}$ are real numbers and where *addition* and *multiplication* are defined by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$

$$(x_1, y_1).(x_2, y_2) = (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1).$$

It is easily checked that with these definitions $\mathbb{C}$ satisfies all the axioms for a field. That is, $\mathbb{C}$ satisfies associative, commutative and distributive laws for addition and multiplication, $(0, 0)$ and $(1, 0)$ are identities for addition and multiplication respectively, and there are multiplicative inverses for each nonzero element in $\mathbb{C}$.

The mapping $x \mapsto (x, 0)$ defines a field isomorphism of $\mathbb{R}$ into $\mathbb{C}$, so we may consider $\mathbb{R}$ as a subset of $\mathbb{C}$. If we write $i = (0, 1)$ then $(x, y) = x + iy$. Thus we may abandon the ordered pair notation for complex numbers and for $z = (x, y) \in \mathbb{C}$ write $z = x + iy$, where we call x the *real part* of the complex number z and y *imaginary part* of z, denoted by $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ respectively. Note that $i^2 = -1$.

Each complex number $z = x + iy \in \mathbb{C}$ can be identified with the unique point (x, y) in the plane $\mathbb{R}^2$. Because the points $(x, 0) \in \mathbb{R}^2$ correspond to real numbers, the x–axis is called the *real axis*. Similarly, the y–axis is called the *imaginary axis*. Note that the addition of numbers in $\mathbb{C}$ coincides with addition of the vectors in $\mathbb{R}^2$, while the multiplication of numbers in $\mathbb{C}$ has no straightforward interpretation in terms of vectors in $\mathbb{R}^2$.

We introduce two operations on $\mathbb{C}$ which are not field operations. For $z = x + iy \in \mathbb{C}$ we define

$$|z| = \sqrt{x^2 + y^2}$$

to be the *absolute value* or *modulus* of z, and

$$\bar{z} = x - iy$$

to be the *conjugate* of z. Note that $|z|^2 = z\bar{z}$ and, if $z \neq 0$ then

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$

The following elementary formulae can be easily verified

$$|zw| = |z||w|, \qquad \left|\frac{z}{w}\right| = \frac{|z|}{|w|}, \qquad |\bar{z}| = |z|,$$

$$(\bar{z} + \bar{w}) = \bar{z} + \bar{w}, \qquad \overline{zw} = \bar{z}\bar{w}, \qquad \mathrm{Re}(z) = \frac{\bar{z} + z}{2}, \qquad \mathrm{Im}(z) = \frac{\bar{z} - z}{2i}.$$

Exercise 0.1. Given $z, w \in \mathbb{C}$, prove the *triangle inequality*

$$|z + w| \leq |z| + |w|. \tag{0.1}$$

Hint. First show that $\mathrm{Re}(z\bar{w}) \leq |z||w|$, then use it to prove that $|z + w|^2 \leq (|z| + |w|)^2$.

Note that $|z - w|$ is exactly the Euclidean distance between vectors z and w on the plane $\mathbb{R}^2$. In particular, (0.1) is called the triangle inequality because, if we represent z and w in the plane, then (0.1) simply says that the length of one side of the triangle $[0, z, z + w]$ is less then the sum of the length of the other two sides.

Polar representation. Consider the nonzero point $z = x + iy \in \mathbb{C}$. This point has polar coordinates (r, φ) defined by

$$x = r\cos(\varphi), \qquad y = r\sin(\varphi). \tag{0.2}$$

Clearly $r = |z|$ and φ is the angle between the positive real axis and the line segment from 0 to z, measured in the anticlockwise direction. Notice that φ plus any multiple of 2π can be substituted for φ in the above equation. Any value of φ for which (0.2) holds is called the *argument* of z and is denoted by $\arg(z)$. Because of the ambiguity in the definition, $\arg(z)$ is not a function, or more precisely, $\arg(z)$ is a so–called multiple–valued function. For $\varphi \in \mathbb{R}$, introduce the notation

$$e^{i\varphi} := \cos(\varphi) + i\sin(\varphi). \tag{0.3}$$

Let $z_1 = r_1 e^{i\varphi_1}$, $z_2 = r_2 e^{i\varphi_2}$. Then by the formulae for the sine and cosine of the sum we obtain

$$z_1 z_2 = r_1 r_2\, e^{i(\varphi_1+\varphi_2)}. \tag{0.4}$$

Or alternatively,

$$|z_1 z_2| = |z_1||z_2|, \qquad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2). \tag{0.5}$$

In particular,

$$z^n = r^n \arg(n\varphi), \tag{0.6}$$

for every integer $n \geq 0$. Moreover, if $z \neq 0$ then $z(r^{-1}e^{-i\varphi}) = 1$, so that (0.6) also holds for all $n \in \mathbb{Z}$.

Exercise 0.2. For each nonzero $a = |a|e^{i\alpha} \in \mathbb{C}$ and an integer $n \geq 2$, prove that there are exactly n distinct roots of the equation

$$z^n = a,$$

and they are given by the formula

$$z_k = \sqrt[n]{|a|}\, e^{\frac{\alpha+2\pi k}{n}}, \qquad k = 0, 1 \ldots, n-1.$$

Exponential and trigonometric functions. Let $z = x + iy \in \mathbb{C}$. Define the complex *exponential* function by the formula

$$\exp(z) = e^x\, e^{iy},$$

where e^x is the real exponential function, and e^{iy} is defined by (0.3). Sometimes instead of $\exp(z)$ we use the notation e^z.

The complex *trigonometric* functions are defined via the exponential function by

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}.$$

It is easy to verify the following properties of the complex exponential function:

$$\exp(z_1 z_2) = \exp(z_1)\exp(z_2) \qquad (\forall z_1, z_2 \in \mathbb{C}),$$
$$\exp(z + 2\pi i k) = \exp(z) \qquad (\forall z \in \mathbb{C},\ k \in \mathbb{Z}),$$
$$|e^z| = e^x, \quad \arg(e^z) = y \qquad (\forall z = x + iy \in \mathbb{C}),$$
$$\exp(iz) = \cos(z) + i\sin(z) \qquad (\forall z \in \mathbb{C}).$$

Exercise 0.3. Prove that the equation

$$\exp(z) = a$$

has a solution if and only if $a \neq 0$, and for each nonzero $a = |a|e^{i\alpha} \in \mathbb{C}$ the roots of the equation are given by the formula

$$z_k = \log|a| + i(\alpha + 2k\pi), \qquad k \in \mathbb{Z}.$$

Sequences. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. We say that (z_n) *converges to the limit* $z \in \mathbb{C}$, and write $\lim_{n\to\infty} z_n = z$ or $z_n \to z$, iff

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})[(n \geq N) \implies (|z_n - z| < \varepsilon)].$$

Exercise 0.4. Let $z_n = x_n + iy_n$ $(n \in \mathbb{N})$ be a sequence of complex numbers. Prove that $z_n \to z$ if and only if $x_n \to x$ and $y_n \to y$.

The above exercise suggests that complex sequences inherit many properties of sequences of real numbers. For example, the limit of a complex sequence is unique if it exists.

A sequence (z_n) is called *Cauchy* iff

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})[(n, m \geq N) \implies (|z_n - z_m| < \varepsilon)].$$

Exercise 0.5. Prove that a sequence $(z_n) \subset \mathbb{C}$ is Cauchy if and only if it converges.

Hint. Use Exercise 0.4 and the corresponding property of real sequences.

A sequence $(z_n) \subset \mathbb{C}$ is *bounded* iff

$$(\exists M > 0)(\forall n \in \mathbb{N})[|z_n| \leq M].$$

Exercise 0.6. Prove that every bounded sequence $(z_n) \subset \mathbb{C}$ has a convergent subsequence.

Hint. Use Exercise 0.4 and the corresponding property of real sequences.

Series. Let $(z_n)_{n\in\mathbb{N}\cup\{0\}}$ be a complex sequences. We say that the series $\sum_{n=0}^{\infty} a_n$ converges to $A \in \mathbb{C}$ and write

$$A = \sum_{n=0}^{\infty} a_n$$

if the sequence of partial sums $S_m = \sum_{n=0}^{m} a_n$ converges to A. Otherwise we say that the series *diverges*.

We say that the series $\sum_{n=0}^{\infty} a_n$ *converges absolutely* if the series $\sum_{n=0}^{\infty} |a_n|$ converges. Thus all standard convergence tests of real analysis (such as comparison test, Weierstrass test, d'Alambert test etc.) are applicable for the study of absolute convergence of complex series.

Exercise 0.7. Prove that if the series $\sum_{n=0}^{\infty} a_n$ *converges absolutely* then $\sum_{n=0}^{\infty} a_n$ converges.

Hint. Show that the sequence of partial sums $S_m = \sum_{n=0}^{m} a_n$ is a Cauchy sequence.

Power Series. A *power series* about a point $z_0 \in \mathbb{C}$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n. \tag{0.7}$$

We say that power series (0.7) converges in a set $G \subseteq \mathbb{C}$ if it converges at every point $z \in G$.

The number R, $0 \leq R \leq +\infty$, defined by

$$R = \sup\{r \geq 0 : \text{power series (0.7) converges in a ball } B_r(z_0)\}$$

is called the *radius of convergence* of power series (0.7).

Example 0.8. The radius of convergence of the *complex geometric series* $\sum_{n=0}^{\infty} z^n$ is $R = 1$. To see this note that $1 - z^{n+1} = (1 - z)(1 + z + \cdots + z^n)$ and hence

$$1 + z + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

If $|z| < 1$ then $z^n \to 0$ and so the geometric series converges with

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z} \qquad (|z| < 1).$$

If $|z| > 1$ then $|z|^n \to \infty$ so the series diverges.

The following theorem shows that the ball $B_R(z_0)$ is indeed a natural region of convergence of power series with the radius of convergence R.

Theorem 0.9. *The radius of convergence R of power series* (0.7) *has the following properties:*
(*a*) *if $|z - z_0| < R$ then the series converges absolutely;*
(*b*) *if $|z - z_0| > R$ then the series diverges.*
Moreover, R can be computed by the formula

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}. \tag{0.8}$$

Proof. See Stewart and Tall, pp.56–58. $\square$

Exercise 0.10. Prove that the radius of convergence R of power series (0.7) is given by

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}, \tag{0.9}$$

if this limit exists.

Exercise 0.11. Prove that the radius of convergence of the *exponential series* $\sum_{n=0}^{\infty} \frac{1}{n!} z^n$ is $R = +\infty$.

Topology of the complex plane. A set $G \subset \mathbb{C}$ is said to be *open* if for every $z_0 \in G$ there exists $\varepsilon > 0$ such that $B_\varepsilon(z_0) \subset G$. We agree that both the empty set $\emptyset$ and the entire complex plane $\mathbb{C}$ are open.

Example 0.12. The open ball $B_r(z_0)$ is an open set. To prove this, fix a point $z_1 \in B_r(z_0)$. Let $\varepsilon = r - |z_1 - z_0|$. Assume $z \in B_\varepsilon(z_1)$. Then by the triangle inequality (0.1) we obtain $|z - z_0| \le |z - z_1| + |z_1 - z_0| < \varepsilon + |z_1 - z_0| = r$. Hence $B_\varepsilon(z_1) \subset B_r(z_0)$.

A set $F \subset \mathbb{C}$ is said to be *closed* if its complement $\mathbb{C} \setminus F$ is open. In particular, the empty set $\emptyset$ and the entire complex plane $\mathbb{C}$ are closed. We agree $\emptyset$ and $\mathbb{C}$ are both open and closed.

Exercise 0.13. Prove that the *closed ball* $\bar{B}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \le r\}$ and the sphere $S_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$ are closed sets.

Hint. Show that the complement of the set in question is open.

Exercise 0.14. Prove that a set G is open if and only if the complement $\mathbb{C} \setminus G$ is closed.

Note that the fact that a set is not open does not imply that it is closed.

Exercise 0.15. Let $a, b \in \mathbb{C}$. Prove that the line segment $[a, b] = \{ta + (1 - t)b \in \mathbb{C} | t \in [0, 1]\}$ is closed. Prove that the open line segment $(a, b) = \{ta + (1 - t)b | t \in (0, 1)\}$ is neither open nor closed.

A point $z_0 \in \mathbb{C}$ is called a *limit point* of a set $G \subset \mathbb{C}$ if for any $\varepsilon > 0$ the open ball $B_\varepsilon(z_0)$ contains points of G different with z_0.

Exercise 0.16. Prove that $z \in \mathbb{C}$ is a limit point of a set $G \in \mathbb{C}$ if and only if there is a sequence $(z_n) \subset G \setminus \{z\}$ such that $z_n \to z$.

Exercise 0.17. Prove that a set F is closed if and only if F contains all its limit points.

Hint. Show that a set $F \subseteq \mathbb{C}$ is closed if and only if for each convergent sequence $(z_n) \subset F$ we have $z_n \to z \in F$.

Let G be a subset of $\mathbb{C}$. A point $z \in \mathbb{C}$ is a *boundary point* of G if z is a limit point of G and also a limit point of the complement $\mathbb{C} \setminus G$. The *boundary* ∂G of the set G is the set of all boundary points of G. The *closure* $\bar{G}$ of the set G is defined to be the set $G \cup \partial G$. The *interior* $\operatorname{int} G$ of the set G is defined to be the set $G \setminus \partial G$.

Exercise 0.18. Prove that $\partial B_r(z_0) = S_r(z_0)$ and $\operatorname{int}\bar{B}_r(z_0) = B_r(z_0)$.

Exercise* 0.19. Let $G \subset \mathbb{C}$ be a set. Prove that the closure $\bar{G}$ and the boundary ∂G of the set G are closed sets. Prove that the interior $\operatorname{int} G$ is an open set.

A set $G \subset \mathbb{C}$ is said to be *compact* if every sequence $(z_n) \subset G$ contains a subsequence (z_{n_k}) that converges to a $z_0 \in G$. A set $G \subset \mathbb{C}$ is called *bounded* if there exists $R > 0$ such that $G \subset B_R(0)$.

Exercise 0.20. Prove that a set $G \subset \mathbb{C}$ is compact if and only if G is bounded and closed.

Hint. Use Exercises 0.6 and 0.17.

Complex Functions. Let $G \subset \mathbb{C}$ be a set and z_0 be a limit point of G. Let $f : G \to \mathbb{C}$ be a function. Similarly to the real case, we say that a complex number $a \in \mathbb{C}$ is the *limit* of a function f as z tends to z_0 and write

$$\lim_{z \to z_0} f(z) = a,$$

if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall z \in G)[(0 < |z - z_0| < \delta) \implies (|f(z) - a| < \varepsilon)].$$

Note that in the above definition a need not be a point of G.

Exercise 0.21. Prove that the limit of a function is unique, if exists.

A function $f : G \to \mathbb{C}$ is *continuous* at a point $z_0 \in G$ if

$$\lim_{z \to z_0} f(z) = f(z_0),$$

or, in other words, if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall z \in G)[(0 < |z - z_0| < \delta) \implies |f(z) - f(z_0)| < \varepsilon].$$

A function $f : G \to \mathbb{C}$ is *continuous* on the set G if f is continuous at every point $z_0 \in G$.

Exercise 0.22. Prove that a function $f : G \to \mathbb{C}$ is continuous at a point $z_0 \in G$ if and only if for every sequence $(z_n) \subset G$ such that $z_n \to z_0$ one has $f(z_n) \to f(z_0)$.

Exercise 0.23. Prove that a function $f = u + iv : G \to \mathbb{C}$ is continuous at a point $z_0 = x_0 + iy_0 \in G$ if and only if $u : G \to \mathbb{R}$ is continuous at (x_0, y_0) and $v : G \to \mathbb{R}$ is continuous at (x_0, y_0).

Hint. Use Exercises 0.4 and 0.22.

Theorem 0.24. *Let $G \subset \mathbb{C}$ be a compact set set and $f : G \to \mathbb{C}$ a continuous function. Then the image $f(G)$ is a compact subset of $\mathbb{C}$.*

Proof. Let $(w_n) \subset f(G)$ be a sequence. Then there exists a sequence $(z_n) \subset G$ such that $f(z_n) = w_n$. Since G is compact, the sequence (z_n) has a convergent subsequence (z_{n_k}), $z_{n_k} \to z_0 \in G$. Since $f : G \to \mathbb{C}$ is continuous, we have

$$\lim w_{n_k} = \lim f(z_{n_k}) = f(z_0) \in f(G).$$

Thus (w_n) has a subsequence that converges in $f(G)$. This means that $f(G)$ is compact. $\square$

Exercise 0.25. Let $\gamma : [0,1] \to \mathbb{C}$ be a continuous function. Prove that the image $\gamma([0,1])$ is a compact subset of $\mathbb{C}$.

Hint. Use the fact that the segment $[0,1]$ is a compact subset of the real line $\mathbb{R}$ and hence a compact subset of the complex plane $\mathbb{C}$.

Maxima and Minima of real functions. Let $G \subset \mathbb{C}$ and $g : G \to \mathbb{R}$ be a real valued function. We say that g is *bounded* on G if $\sup_G |g| < +\infty$. We say g attains its *minimum* (*maximum*) on G if there exists $z_0 \in G$ such that

$$g(z_0) = \inf_G g \qquad (g(z_0) = \sup_G g).$$

Theorem 0.26. *Let $G \subset \mathbb{C}$ be a compact set and $g : G \to \mathbb{R}$ a continuous function. Then g attains its minimum and maximum on G. In particular, g is bounded on G.*

Proof. It follows from Theorem 0.24, that $g(G)$ is a compact subset of $\mathbb{R}$. Let $m = \inf g(G)$. By definition of the infimum, there exists a sequence $(s_n) \subset g(G)$ such that $s_n \to m$. Then there exists a sequence $(z_n) \subset G$ such that $g(z_n) = s_n$. Since G is compact, the sequence (z_n) has a convergent subsequence (z_{n_k}), $z_{n_k} \to z_0 \in G$. Since $g : G \to \mathbb{R}$ is continuous,

$$m = \lim s_{n_k} = \lim g(z_{n_k}) = g(z_0).$$

Thus the assertion follows. $\square$

1 Differentiation of complex functions

Let $G \subseteq \mathbb{C}$ be an open set and $f : G \to \mathbb{C}$ a complex function on G.

Definition 1.1. The function f is (complex) *differentiable* at a point $z \in G$ iff the limit

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \qquad (h \neq 0,\ z+h \in G) \tag{1.1}$$

exists and finite. The function f is differentiable on the set G if f is differentiable at every point $z \in G$.

Exercise 1.2. Prove that the function f is (complex) differentiable at a point $z \in G$ iff it is represented as

$$f(z+h) - f(z) = f'(z)h + \omega(h), \tag{1.2}$$

where the reminder function $\omega : G \to \mathbb{C}$ satisfies the condition

$$\lim_{|h|\to 0} \frac{|\omega(h)|}{|h|} = 0 \qquad (h \neq 0,\ z+h \in G). \tag{1.3}$$

Observe, that (1.2) can be viewed as a "first" order Taylor expansion of f around z.

Example 1.3. Consider the function $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$. Then for any fixed $z \in \mathbb{C}$

$$f'(z) = \lim_{h\to 0} \frac{(z+h)^2 - z^2}{h} = \lim_{h\to 0} \frac{2zh + h^2}{h} = 2z + \lim_{h\to 0} h = 2z \qquad (h \neq 0,\ z+h \in G).$$

Thus $f(z) = z^2$ is differentiable at every point $z \in \mathbb{C}$ and $(z^2)' = 2z$.

Exercise 1.4. Prove that $\exp(z)$ is differentiable in $\mathbb{C}$ and $(\exp(z))' = \exp(z)$ for all $z \in \mathbb{C}$.

The exercises below describe some basic properties of differentiable functions.

Exercise 1.5. Prove that if f is differentiable at a point $z \in \mathbb{C}$ then the derivative $f'(z)$ is uniquely defined.

Exercise 1.6. Prove that if f is differentiable at a point $z \in G$ then f is continuous at z.

Exercise* 1.7. (Chain Rule) Let $G \subseteq \mathbb{C}$ be an open set. Suppose that $f : G \to \mathbb{C}$ is differentiable at $z \in G$, $B := f(G) \subseteq \mathbb{C}$ and $g : B \to \mathbb{C}$ is differentiable at $b = f(z) \in B$. Prove that then the composition $g \circ f : G \to \mathbb{C}$ is differentiable at z and

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

Exercise* 1.8. Suppose the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ has radius of convergence $R > 0$. Prove that the function $f : B_R(z_0) \to \mathbb{C}$ defined by

$$f(z) := \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is differentiable in $B_R(z_0)$ and for any $z \in B_R(z_0)$ the derivative is given by the formula

$$f'(z) := \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

Derive from this by induction that f is k times differentiable in $B_R(z_0)$ and

$$f^{(k)}(z) := \sum_{n=k}^{\infty} \{n(n-1)\dots(n-k+1)\}\, a_n (z - z_0)^{n-k}. \tag{1.4}$$

Relation between real and complex differentiability. Let $f : G \to \mathbb{C}$ be a complex function. Represent $f = u + iv$, where $u, v : G \to \mathbb{R}$ are real functions $u = u(x, y)$, $v = v(x, y)$. Recall that the *partial derivatives* of a real function $u : G \to \mathbb{R}$ at a point $(x, y) \in G$ are defined by

$$\frac{\partial u}{\partial x}(x,y) = \lim_{k\to 0} \frac{u(x+k,y) - u(x,y)}{k}, \qquad (k \neq 0,\ (x+k,y) \in G),$$

$$\frac{\partial u}{\partial y}(x,y) = \lim_{l\to 0} \frac{u(x,y+l) - u(x,y)}{l}, \qquad (l \neq 0,\ (x,y+l) \in G).$$

Similarly for $v = v(x, y)$. The Cauchy–Riemann equations show how the complex derivative of f (if exists!) is related to the partial derivatives of $f = u + iv$ as a function of real variables.

Theorem 1.9. (CAUCHY–RIEMANN EQUATIONS) *Let the function $f = u + iv$ be (complex) differentiable at a point $z = x + iy \in G$. Then all partial derivatives of u and v exist at (x, y) and the following Cauchy–Riemann equations hold:*

$$\left\{ \begin{array}{rcl} \frac{\partial u}{\partial x}(x,y) & = & \frac{\partial v}{\partial y}(x,y), \\ \frac{\partial u}{\partial y}(x,y) & = & -\frac{\partial v}{\partial x}(x,y). \end{array} \right. \tag{1.5}$$

In this case, the derivative of f at z can be represented by the formula

$$f'(z) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y) = \frac{\partial v}{\partial y}(x,y) + i\frac{\partial v}{\partial x}(x,y). \tag{1.6}$$

Proof. Let $h := k + i\,0$ $(k \in \mathbb{R})$. Then

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{k\to 0} \frac{u(x+k,y) + iv(x+k,y) - u(x,y) - iv(x,y)}{k} \\ &= \lim_{k\to 0} \frac{u(x+k,y) - u(x,y)}{k} + i\frac{v(x+k,y) - v(x,y)}{k} \\ &= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y). \end{aligned}$$

Now let $h := 0 + il$ $(l \in \mathbb{R})$. Then

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{l\to 0} \frac{u(x,y+l) + iv(x,y+l) - u(x,y) - iv(x,y)}{il} \\ &= \lim_{l\to 0} \frac{v(x,y+l) - v(x,y)}{l} + \frac{1}{i}\frac{u(x,y+l) - u(x,y)}{l} \\ &= \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y). \end{aligned}$$

Therefore we obtain

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y). \tag{1.7}$$

Equating the real and imaginary parts of $f'(z)$ we derive Cauchy–Riemann equations (1.5). Then representation formula (1.6) follows from (1.7) and (1.5). □

Example 1.10. Consider $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} 0, & xy = 0, \\ 1, & xy \neq 0, \end{cases}$$

Note that $f(z) = \mathrm{Re}(f(z))$. Representing $f = u + iv$ with $u(x, y) = f(x + iy)$ and $v(x, y) = 0$, we compute that

$$\frac{\partial u}{\partial x}(0,0) = \frac{\partial u}{\partial y}(0,0) = \frac{\partial v}{\partial x}(0,0) = \frac{\partial v}{\partial y} v(0,0) = 0.$$

So f satisfies the Cauchy–Riemann equations at $z = 0$. However f is not continuous at $z = 0$ and therefore f is not differentiable at $z = 0$.

To understand the above example we need to revise the definition of real differentiability. Let $f = u + iv : G \to \mathbb{C}$ be a complex function. Represent f as a real function (vector field) $f : G \to \mathbb{R}^2$,

$$f(x, y) = (u(x, y), v(x, y)).$$

Recall that the *Jacobian matrix* of f at a point $(x, y) \in G$ is defined as the matrix of partial derivatives given by

$$Df(x,y) = \begin{pmatrix} \frac{\partial u}{\partial x}(x,y) & \frac{\partial u}{\partial y}(x,y) \\ \frac{\partial v}{\partial x}(x,y) & \frac{\partial v}{\partial y}(x,y) \end{pmatrix}.$$

The function f is *real differentiable* at a point $(x, y) \in G$ iff the Jacobian matrix $Df(x, y)$ exists and

$$f(x + k, y + l) - f(x, y) = Df(x, y) \begin{pmatrix} k \\ l \end{pmatrix} + \omega(k, l), \tag{1.8}$$

where the reminder (vector) function $\omega : G \to \mathbb{R}^2$ satisfies the condition

$$\lim_{|(k,l)| \to 0} \frac{|\omega(k,l)|}{|(k,l)|} = 0, \tag{1.9}$$

here $|(k, l)| = \sqrt{k^2 + l^2}$ stands for the length of a vector (k, l). Thus real differentiability stronger then existence of partial derivatives!

Example 1.10 (continued). The function $f = u + iv$ from Example 1.10 has vanishing partial derivatives of u and v at $z = 0$. Therefore $Df(0, 0) = \mathbf{0}$ is the null matrix and

$$f(k, l) - f(0, 0) = \omega(k, l).$$

Hence $\omega(k, l) = f(k, l)$ (note that $f(0, 0) = (0, 0)$). Let $l = k$. Then

$$\lim_{|(k,l)| \to 0} \frac{|\omega(k,l)|}{|(k,l)|} = \lim_{k \to 0} \frac{1}{k\sqrt{2}}. \tag{1.10}$$

This limit does not exist. Therefore f is not real differentiable at zero. To explain this example simply notice that the limit in (1.9) is taken for an arbitrary (k, l) approaching zero (e.g., for $l = k$ as in (1.9)), but not along two particular ($k = 0$ and $l = 0$) directions as in the definitions of partial derivatives.

The main result relating real and complex differentiability is the following theorem.

Theorem 1.11. *A function $f = u + iv : G \to \mathbb{C}$ is complex differentiable at a point $z = x + iy \in G$ if and only if f is real differentiable at (x, y) and partial derivatives of u and v at (x, y) satisfy the Cauchy–Riemann equations.*

Proof. Assume that f is complex differentiable at $z = x + iy$. Then from Exercise 1.2 and Theorem 1.9 immediately follows that f is real differentiable at (x, y) and satisfies the Cauchy–Riemann equations.

Assume f is real differentiable at (x, y) and partial derivatives of u and v satisfy the Cauchy–Riemann equations. Then

$$f(x+k, y+l) - f(x, y) = \begin{pmatrix} \frac{\partial u}{\partial x}(x,y) & \frac{\partial u}{\partial y}(x,y) \\ -\frac{\partial u}{\partial y}(x,y) & \frac{\partial u}{\partial x}(x,y) \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} + \omega(k, l), \tag{1.11}$$

where $\omega(k, l)$ satisfies (1.9). Set

$$f'(z) := \frac{\partial u}{\partial x}(x, y) - i\frac{\partial u}{\partial y}(x, y)$$

and $h := k + il$. Thus (1.11) can be rewritten as

$$f(z + h) - f(z) = f'(z)h + \omega(h),$$

where $\omega(h)$ satisfies (1.3). This means that f is complex differentiable at z. $\square$

Example 1.10 describes a complex function that satisfies the Cauchy–Riemann equations but is not real differentiable. The exercise below gives an example of a function that is real differentiable but does not satisfy the Cauchy–Riemann equations.

Exercise 1.12. Consider $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(x + iy) = (x^2 - y^2) + ic\,xy,$$

where $c \in \mathbb{R}$ is a real parameter. Prove that f is complex differentiable at $z \neq 0$ iff $c = 2$ (and $f(z) = z^2$ in this case).

Exercise 1.13. Let $f(z) = u(r, \varphi) + iv(r, \varphi)$ be differentiable at a point $z = (r, \varphi) \neq 0$, where (r, φ) are polar coordinates $r = |z|$, $\varphi = \mathrm{Arg}(z)$ (and we agree that $\mathrm{Arg}(z) \in [0, 2\pi)$). Show, by changing variables, that the Cauchy–Riemann equations at point z in terms of polar coordinates transform into

$$\frac{\partial u}{\partial r}(r, \varphi) = \frac{1}{r}\frac{\partial v}{\partial \varphi}(r, \varphi), \qquad \frac{\partial v}{\partial r}(r, \varphi) = -\frac{1}{r}\frac{\partial u}{\partial \varphi}(r, \varphi). \tag{1.12}$$

2 Integration of complex functions

2.1 Paths and contours on the plane

Let $[\alpha, \beta] \subset \mathbb{R}$ be a closed interval of real numbers and $G \subset \mathbb{C}$ an open set.

Definition 2.1. A *path* is a continuous map

$$\gamma : [\alpha, \beta] \to G.$$

We call $\gamma(\alpha)$ the *initial point*, and $\gamma(\beta)$ the *final point* of the path γ. We say that γ is a *closed* path if $\gamma(\alpha) = \gamma(\beta)$.

Example 2.2. (a) Let $a, b \in \mathbb{C}$ be points on the complex plane. Then

$$\gamma(t) = (1-t)a + tb, \qquad t \in [0,1]$$

is called the *line segment* from a to b. Observe that the path $\tilde{\gamma}(t) = ta + (1-t)b$ $(t \in [0,1])$ is the same line segment having the "opposite direction" from b to a. This is frequently denoted by $[a, b]$.
(b) Let $a \in \mathbb{C}$ and $r > 0$. The path

$$\gamma(t) = a + re^{it}, \qquad t \in [0, 2\pi]$$

is called the positively oriented *circle of radius r centered at a*. This is frequently denoted by $S_r^+(a)$ or simply $S_r(a)$. Observe that, e.g., the path $\gamma_1(t) = a + r(\cos(2\pi t) + i\sin(2\pi t))$ $(t \in [0,1])$ represents the same "geometrical" circle with the same "positive orientation". The path $\gamma_2(t) = a + re^{it}$ $(t \in [0,2])$ is the same circle again, however every point of the circle is the image of two points of the segment $[0, 4\pi]$.

Remark 2.3. (a) Paths $\gamma_1 : [\alpha_1, \beta_1] \to G$ and $\gamma_2 : [\alpha_2, \beta_2] \to G$ are said to be *equivalent* if there is a strictly increasing continuous function $\phi : [\alpha_1, \beta_1] \to [\alpha_2, \beta_2]$ such that $\gamma_1 = \gamma_2 \circ \phi$. This is frequently denoted by $\gamma_1 \sim \gamma_2$. It is easy to see that such notion of equivalence is an equivalence relation. In particular, taking $\phi(t) = \frac{t-\alpha}{\beta-\alpha}$, every path is equivalent to one whose parameter interval is $[0,1]$. Paths γ and γ_1 in Example 2.2 (b) are equivalent.
(b) For a path γ, the range of γ is a subset of G, called the *trace* of γ, and denoted by

$$\gamma^* = \{z \in G : z = \gamma(t) \quad \text{for some } \ t \in [\alpha, \beta]\}.$$

Paths γ and $\tilde{\gamma}$ in Example 2.2 (a) and paths γ, γ_1, γ_2 in Example 2.2 (b) have the same trace.
(c) For any path $\gamma : [\alpha, \beta] \to G$ we may define the *reverse path* $-\gamma : [\alpha, \beta] \to G$ by

$$-\gamma(t) = \gamma(\alpha + \beta - t).$$

The reverse path $-\gamma$ has initial point $\gamma(\beta)$ and final point $\gamma(\alpha)$. Note that $(-\gamma)^* = \gamma^*$.
(d) Suppose that $\gamma_1 : [\alpha_1, \beta_1] \to G$ and $\gamma_2 : [\alpha_2, \beta_2] \to G$ are paths with $\gamma_1(\beta_1) = \gamma_2(\alpha_2)$. Then we may define the *join* $\gamma_1 + \gamma_2 : [\alpha_1, \beta_1 + \beta_2 - \alpha_2]$ by

$$\gamma_1 + \gamma_2 = \begin{cases} \gamma_1(t), & \alpha_1 \le t \le \beta_1, \\ \gamma_2(t + \alpha_2 - \beta_1), & \beta_1 \le t \le \beta_1 + \beta_2 - \alpha_2, \end{cases}$$

Similarly (or by induction), one can define a multiple join of n paths $\gamma_1 + \gamma_2 + \cdots + \gamma_n$. We agree that $\gamma_1 - \gamma_2 := \gamma_1 + (-\gamma_2)$.
(e) A path $\gamma : [\alpha, \beta] \to G$ is *smooth* if the derivative γ' exists and continuous on $[\alpha, \beta]$, where

$$\gamma'(t) = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}, \qquad \gamma'(\alpha) = \lim_{t \to \alpha} \gamma'(t), \quad \gamma'(\beta) = \lim_{t \to \beta} \gamma'(t).$$

Definition 2.4. A *contour* (a piecewise smooth path) γ is the join of a finite number of smooth paths, e.g. $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$.

Example 2.5. Let $\gamma_1 = \cos(t) + i\sin(t)$ ($t \in [0, \pi]$) and $\gamma_2 = 2t - 1$ ($t \in [0, 1]$). Then $\gamma = \gamma_1 + \gamma_2$ is a closed semicircular contour.

Definition 2.6. Let $\gamma_1 : [\alpha_1, \beta_1] \to G$ be a contour. A contour $\gamma_2 : [\alpha_2, \beta_2] \to G$ is called a *reparametrization* of γ_1 iff there is a continuously differentiable function $\phi : [\alpha_1, \beta_1] \to [\alpha_2, \beta_2]$ with $\phi'(t) > 0$, $\phi(\alpha_1) = \alpha_2$, $\phi(\beta_1) = \beta_2$, such that $\gamma_1 = \gamma_2 \circ \phi$.

Remark 2.7. If γ_2 is a reparametrization of γ_1 then γ_1 and γ_2 are equivalent. In particular $\gamma_1^* = \gamma_2^*$. The condition $\phi'(t) > 0$ (hence ϕ is increasing) and $\phi(\alpha_1) = \alpha_2$, $\phi(\beta_1) = \beta_2$ means that γ_2 has the same orientation, initial and final points as γ_1.

Definition 2.8. The *length* of the contour $\gamma : [\alpha, \beta] \to G$ is defined by

$$L(\gamma) = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} |\gamma'(t)| dt,$$

where $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ is the partition of the segment $[\alpha, \beta]$, associated to $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$.

Exercise 2.9. (a) Let $a, b \in \mathbb{C}$ and $\gamma(t) = (1-t)a + tb$ ($t \in [0, 1]$) be the line segment from a to b. Prove that $L(\gamma) = |b - a|$.
(b) Let γ_2 be a reparametrization of the contour γ_1. Prove that $L(\gamma_1) = L(\gamma_2)$.

2.2 Contour integration of complex functions

Integral over the real segment. Let $F : [\alpha, \beta] \to \mathbb{C}$ be a continuous function, $F(t) = u(t) + iv(t)$. Define the *integral of* F *over the real segment* $[\alpha, \beta]$ to be

$$\int_\alpha^\beta F(t) dt := \int_\alpha^\beta u(t) dt + i \int_\alpha^\beta v(t) dt,$$

where integrals of u and v are the usual (real) Riemann integrals. Such defined integral respects the usual properties of the Riemann integral. Note however that the integral of F over $[\alpha, \beta]$ is a complex number!

Exercise 2.10. (a) Prove that if F is continuously differentiable on $[\alpha, \beta]$ then

$$\int_\alpha^\beta F'(t) dt = F(\beta) - F(\alpha). \tag{2.1}$$

(b) Prove that if F is continuous on $[\alpha, \beta]$ then

$$\left| \int_\alpha^\beta F(t) dt \right| \le \int_\alpha^\beta |F(t)| dt. \tag{2.2}$$

Contour integral. Now we are ready to define the contour integral of a complex function.

Definition 2.11. Let $G \subset \mathbb{C}$ be an open set, $f : G \to \mathbb{C}$ a continuous function and $\gamma : [\alpha, \beta] \to G$ a contour. Define the *contour integral of f along γ* to be

$$\int_\gamma f = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(\gamma(t))\gamma'(t)dt,$$

where $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ is the partition of $[\alpha, \beta]$, associated to $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$. This is also frequently written $\int_\gamma f(z)dz$.

Example 2.12. Let $\gamma = S_r^+(a)$. Then

$$\int_\gamma (z-a)^n dz = \begin{cases} 2\pi i & \text{if } n = -1, \\ 0 & \text{if } n \in \mathbb{Z},\ n \neq -1. \end{cases}$$

Proof. We may write $\gamma(t) = a + re^{it}$ $(t \in [0, 2\pi])$. Then $\gamma'(t) = ire^{it}$. We obtain

$$\begin{aligned} \int_\gamma (z-a)^n dz &= \int_0^{2\pi} \left(re^{it}\right)^n ire^{it} dt = ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt \\ &= \begin{cases} i \int_0^{2\pi} dt & = \ 2\pi i \quad \text{if } n = -1, \\ ir^{n+1} \left. \frac{e^{i(n+1)t}}{i(n+1)} \right|_0^{2\pi} & = \ 0 \quad \text{if } n \in \mathbb{Z},\ n \neq -1, \end{cases} \end{aligned}$$

by taking into account the periodicity $e^z = e^{z+2\pi ni}$ $(z \in \mathbb{C},\ n \in \mathbb{Z})$. □

Exercise 2.13. Prove the following properties [1] of contour integrals:

$$\int_\gamma (\lambda f + \mu g) = \lambda \int_\gamma f + \mu \int_\gamma g \qquad (\lambda, \mu \in \mathbb{C}); \tag{2.3}$$

$$\int_{\gamma_1 + \gamma_2 + \cdots + \gamma_n} f = \int_{\gamma_1} f + \int_{\gamma_2} f + \cdots + \int_{\gamma_n} f; \tag{2.4}$$

$$\int_{-\gamma} f = -\int_\gamma f. \tag{2.5}$$

Exercise 2.13 (iii) says that the value of the contour integral does depend on the orientation of the contour. We are going to show that however it does not depend on the particular choice of parametrization.

Lemma 2.14. *Let $G \subset \mathbb{C}$ be an open set, $f : G \to \mathbb{C}$ a continuous function and $\gamma : [\alpha, \beta] \to G$ a contour. Let $\tilde{\gamma} : [\tilde{\alpha}, \tilde{\beta}] \to G$ be a reparametrization of γ. Then*

$$\int_\gamma f = \int_{\tilde{\gamma}} f.$$

Proof. Assume for simplicity that γ is smooth. By definition,

$$\int_\gamma f = \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt.$$

[1] Taking into account property (ii) we shall present most of the proofs only for the case of smooth contours.

By the real Chain Rule applied separately to real and imaginary parts of γ we obtain

$$\gamma'(t) = (\tilde{\gamma}(\phi(t)))' = \tilde{\gamma}'(\phi(t))\phi'(t).$$

Set $s = \phi(t)$ be a new variable. Then

$$\int_\alpha^\beta f(\gamma(t))\gamma'(t)dt = \int_\alpha^\beta f(\tilde{\gamma}(\phi(t)))\left\{\tilde{\gamma}'(\phi(t))\phi'(t)\right\}dt = \int_{\tilde{\alpha}}^{\tilde{\beta}} f(\tilde{\gamma}(s))\tilde{\gamma}'(s)ds.$$

The changing of variables in a complex integral over a real segment is justified by applying the usual rule to its real and imaginary parts. □

Lemma 2.15. (ESTIMATION LEMMA) *Let $G \subset \mathbb{C}$ be an open set, $f : G \to \mathbb{C}$ a continuous function and $\gamma : [\alpha, \beta] \to G$ a contour. If $\sup_{\gamma^*} |f| \leq M$ then*

$$\left|\int_\gamma f\right| \leq ML(\gamma).$$

Proof. Assume for simplicity that γ is smooth. Then

$$\left|\int_\gamma f\right| = \left|\int_\alpha^\beta f(\gamma(t))\gamma'(t)dt\right| \leq \int_\alpha^\beta |f(\gamma(t))||\gamma'(t)|dt \leq M\int_\alpha^\beta |\gamma'(t)|dt = ML(\gamma),$$

by using (2.2). □

Lemma 2.16. (CONTOUR CHAIN RULE) *Let $G \subset \mathbb{C}$ be an open set, $F : G \to \mathbb{C}$ a differentiable function and $\gamma : [\alpha, \beta] \to G$ a contour. Then*

$$(F(\gamma(t))' = F'(\gamma(t))\gamma'(t).$$

Proof. Set $F = u + iv$ and $\gamma = \gamma_1 + i\gamma_2$. Applying the real Chain Rule to real and imaginary parts of $F(\gamma(t))$ and using the Cauchy–Riemann equations we obtain

$$\begin{aligned}(F(\gamma(t)))' &= (u(\gamma_1(t), \gamma_2(t)) + iv(\gamma_1(t), \gamma_2(t)))' \\ &= \frac{\partial u}{\partial x}(\gamma(t))\gamma_1'(t) + \frac{\partial u}{\partial y}(\gamma(t))\gamma_2'(t) + i\frac{\partial v}{\partial x}(\gamma(t))\gamma_1'(t) + i\frac{\partial v}{\partial y}(\gamma(t))\gamma_2'(t) \\ &= \left(\frac{\partial u}{\partial x}(\gamma(t)) - i\frac{\partial u}{\partial y}(\gamma(t))\right)(\gamma_1'(t) + i\gamma_2'(t)) = F'(\gamma(t))\gamma'(t).\end{aligned}$$

□

Theorem 2.17. (FUNDAMENTAL THEOREM) *Let $G \subset \mathbb{C}$ be an open set, $f : G \to \mathbb{C}$ a continuous function and $\gamma : [\alpha, \beta] \to G$ a contour with initial point $a = \gamma(\alpha)$ and final point $b = \gamma(\beta)$. Suppose there is a differentiable function $F : G \to \mathbb{C}$ such that*

$$F'(z) = f(z) \quad \text{in } G.$$

Then

$$\int_\gamma f = F(b) - F(a).$$

In particular, if γ is a closed contour then $\int_\gamma f = 0$.

Proof. By the Contour Chain Rule we obtain

$$\int_\gamma f = \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt = \int_\alpha^\beta \left\{F'(\gamma(t))\gamma'(t)\right\} dt = \int_\alpha^\beta (F(\gamma(t)))'dt = F(b) - F(a),$$

which is required. □

Application of the Fundamental Theorem can save a lot of effort in computations.

Example 2.18. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a contour. Then

$$\int_\gamma (z-a)^n = \frac{(\gamma(\beta)-a)^{n+1}}{n+1} - \frac{(\gamma(\alpha)-a)^{n+1}}{n+1} \qquad (n \in \mathbb{N} \cup 0).$$

In particular,

$$\int_{S_r^+(a)} (z-a)^n dz = 0 \qquad (n \in \mathbb{N} \cup 0).$$

Proof. We merely note that if $n \in \mathbb{Z}$ and $n \geq 0$ then $(z-a)^n = \left(\frac{(z-a)^{n+1}}{n+1}\right)'$ in $\mathbb{C}$. Observe that the value of integral depends only on the initial and final point of γ but not on γ itself. □

2.3 Antiderivatives and converse to the Fundamental Theorem

Connected sets. Let $G \subseteq \mathbb{C}$ be an open set. We want to give precise meaning to the assertion that "G consists of one piece". For our purposes the following definition can be used.

Definition 2.19. We say that an open set $G \subseteq \mathbb{C}$ is *connected* if, for every $a, b \in G$, there is a smooth path $\gamma : [\alpha, \beta] \to G$ with $\gamma(\alpha) = a$ and $\gamma(\beta) = b$. In this case we say that γ is a contour from a to b. We say that $D \subseteq \mathbb{C}$ is a *domain* if D is an open connected set.

Example 2.20. A set $C \subset \mathbb{C}$ is called *convex* iff, for every $a, b \in G$, the line segment $[a, b]$ is contained in C. It follows that every open convex set is connected.

Theorem 2.21. *Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ a differentiable function. If $f'(z) = 0$ on D then f is constant on D.*

Proof. Let $a, b \in D$ and $\gamma : [\alpha, \beta] \to D$ be a smooth path with $\gamma(\alpha) = a$ and $\gamma(\beta) = b$. By the Contour Chain Rule

$$(f(\gamma(t)))' = f'(\gamma(t))\gamma'(t) = 0,$$

since $f'(z) = 0$. Thus writing $f = u + iv$ we conclude that $(u(\gamma(t)))' = 0$ and $(v(\gamma(t)))' = 0$. From Analysis, we know this implies that $u(\gamma(t))$ and $v(\gamma(t))$ are constant functions of t. Comparing the values at $t = a$ and $t = b$ we conclude that $f(a) = f(b)$. □

Remark 2.22. Clearly connectedness is needed. If a set $G \subset \mathbb{C}$ consists of "two pieces" then we could let f to be equal 1 on one "piece" and 0 on another. Thus f' vanishes on G but f would not be constant on G.

Antiderivatives. The Fundamental Theorem suggests the following definition.

Definition 2.23. Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ a continuous function. We say that f *has an antiderivative in* D iff there is a differentiable function $F : D \to \mathbb{C}$ such that

$$F'(z) = f(z) \quad \text{in } D.$$

Exercise 2.24. Let $f : D \to \mathbb{C}$ be a continuous function on a domain $D \subseteq \mathbb{C}$ and $F, G : D \to \mathbb{C}$ antiderivatives of f in D. Prove that $F - G = \mathit{const}$ in D.

Hint. Apply Theorem 2.21 to the difference $F - G$.

Example 2.25. Let $n \in \mathbb{N}$, $n \geq 2$. The function $F(z) = z^{-n}$ is an antiderivative of the function $f(z) = -nz^{-n-1}$ in any annulus $A_{r,R}(0) := \{z \in \mathbb{C} : r < |z| < R\}$ $(0 < r < R < +\infty)$.

Example 2.26. The function $f(z) = z^{-1}$ has no antiderivatives in an annulus $A_{r,R}(0) := \{z \in \mathbb{C} : r < |z| < R\}$ $(0 < r < R < +\infty)$.

Proof. Follows by the Fundamental Theorem from the fact that $\int_{S^+_\rho(0)} z^{-1} = 2\pi i \neq 0$ for any $\rho \in (r, R)$ (see Example 2.12). □

Exercise 2.27. Suppose the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has radius of convergence $R > 0$. Prove that the function $f : B_R(z_0) \to \mathbb{C}$ defined by

$$f(z) := \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

has an antiderivative in $B_R(z_0)$ which is given by the formula

$$F(z) := \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}.$$

Hint. Use Exercise 1.8.

Converse to the Fundamental Theorem. According to Fundamental Theorem, if a function has an antiderivative, then the integral does not depend on the path of integration. We are going to show the converse, namely if the integral of a function does not depend on the contour of integration then the function has an antiderivative. The construction of an antiderivative is described in the following lemma.

Lemma 2.28. *Let $D \subseteq \mathbb{C}$ be a domain, $f : D \to \mathbb{C}$ a continuous function and $\gamma : [\alpha, \beta] \to G$ a contour. Assume that $\int_\gamma f$ depends only on the initial and final points of γ. Fix a point $a \in D$ and define a function $F_a : D \to \mathbb{C}$ by*

$$F_a(z) := \int_{\gamma_{a,z}} f, \tag{2.6}$$

where $\gamma_{a,z} : [\alpha, \beta] \to D$ is a smooth contour with initial point a and final point z. Then

$$F_a'(z) = f(z) \quad \text{in} \quad D,$$

that is F_a is an antiderivative of f in D.

Remark 2.29. The integral in (2.6) correctly defines a (single–valued) function $F_a(z)$ from D to $\mathbb{C}$ because, according to the assumption, $\int_{\gamma_{a,z}} f$ depends only on the initial and final points of the contour $\gamma_{a,z}$ where the initial point $a \in D$ is fixed.

Proof. Since D is open there exists $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(z) \subset D$. Hence for any $z+h \in B_{\varepsilon_0}(z)$ the line segment $[z, z+h]$ lies in D. Parameterize $[z, z+h]$ as $\lambda(t) = z + th$ $(0 \le t \le 1)$. Then

$$F_a(z+h) := \int_{\gamma_{a,z}} f + \int_{[z,z+h]} f.$$

Thus

$$\frac{F_a(z+h) - F_a(z)}{h} = \frac{1}{h}\int_{[z,z+h]} f.$$

From Example 2.18 we conclude that

$$\int_{[z,z+h]} \frac{f(z)}{h} dw = \frac{f(z)}{h}((z+h)-z) = f(z).$$

Therefore
$$\frac{F_a(z+h)-F_a(z)}{h}-f(z)=\int_{[z,z+h]}\frac{f(w)-f(z)}{h}dw.$$

Then by the Estimation Lemma and continuity of f we obtain

$$\left|\frac{F_a(z+h)-F_a(z)}{h}-f(z)\right|\le\frac{1}{|h|}\int_{[z,z+h]}|f(w)-f(z)|\,dw\le$$

$$\le\frac{1}{|h|}L([z,z+h])\sup_{w\in[z,z+h]}|f(w)-f(z)|=$$

$$=\sup_{w\in[z,z+h]}|f(w)-f(z)|\to 0\quad\text{as }\ h\to 0.$$

We conclude that $F_a'(z)=f(z)$ in D, as required. □

Exercise 2.30. Let $a,b\in D$ and define $F_a,F_b:D\to\mathbb{C}$ as in (2.6). Show that

$$F_a(z)-F_b(z)=F_a(b)\qquad(z\in D).$$

Hint. Use the fact that $F_b(b)=0$.

The theorem below summarizes relations between the existence of antiderivatives and independence of the integral on the contour of integration.

Theorem 2.31. (Converse to the Fundamental Theorem) *Let $D\subseteq\mathbb{C}$ be a domain and $f:D\to\mathbb{C}$ a continuous function. Let $\gamma:[\alpha,\beta]\to G$ be a contour. The following statements are equivalent:*

(*a*) *f has an antiderivative in D;*

(*b*) *$\int_\gamma f$ depends only on the initial and final points of γ;*

(*c*) *if γ is closed then $\int_\gamma f=0$.*

Proof. The implications $(a)\Rightarrow(b)\Rightarrow(c)$ are already proved in the Fundamental Theorem.

In order to prove the implication $(c)\Rightarrow(b)$, let $a,b\in D$. Suppose $\gamma_1,\gamma_2:[\alpha,\beta]\to D$ are two contours from a to b. Then $\gamma:=\gamma_1-\gamma_2$ is a closed contour. Hence

$$0=\int_\gamma f=\int_{\gamma_1}-\int_{\gamma_2}f.$$

So $\int_{\gamma_1}f=\int_{\gamma_2}f$, which is (b).

Finally, the implication $(b)\Rightarrow(a)$ has been proved in Lemma 2.28. □

3 Cauchy's Theorems

According to Theorem 2.31 a *continuos* function f has an antiderivative iff $\int_\gamma f = 0$ for any closed contour γ. Cauchy's Theorems put forward conditions under which $\int_\gamma f = 0$ when there is no initial reason to have an antiderivative. Instead, *differentiability* of f will be assumed. We first prove Cauchy's Theorem for triangles, then for starshaped domains and finally state the most general Cauchy's Theorem for simply connected domains.

3.1 Cauchy Theorem for a triangle

We prove the first version of the Cauchy Theorem for the interior of a triangle.

Definition 3.1. Let $(a, b, c) \in \mathbb{C}$ be an (ordered) triple of disjoint points. The (open) *triangle*

$$T = T(a, b, c) = \{\lambda a + \mu b + \nu c : \lambda, \mu, \nu > 0,\ \lambda + \mu + \nu = 1\}$$

is the set of points inside of the "triangle" with vertices a, b, c. The boundary of the triangle T is defined as $\partial T = [a, b] \cup [b, c] \cup [c, a]$ and the closed triangle $\bar{T}$ is defined as $\bar{T} = T \cup \partial T$. The boundary ∂T as a contour is defined to be $\partial T = [a, b] + [b, c] + [c, a]$.

Theorem 3.2. (Cauchy Theorem for a triangle) *Let $D \subseteq \mathbb{C}$ be a domain, $f : D \to \mathbb{C}$ a differentiable function. Let T be a triangle such that $\bar{T} \subset D$. Then*

$$\int_{\partial T} f = 0.$$

Proof. Assume

$$\left| \int_{\partial T} f \right| = c \geq 0.$$

We will show that $c = 0$ by an indirect argument.

First, subdivide T into four triangles $T^{(1)}$, $T^{(2)}$, $T^{(3)}$, $T^{(4)}$ by joining the midpoints on the sides. Then

$$\int_{\partial T} f = \sum_{r=1}^{4} \int_{T^{(r)}} f.$$

We obtain

$$c = \left| \int_{\partial T} f \right| \leq \sum_{r=1}^{4} \left| \int_{T^{(r)}} f \right|.$$

Hence we can choose $r \in \{1, 2, 3, 4\}$ such that

$$\left| \int_{\partial T^{(r)}} f \right| \geq \frac{1}{4} c.$$

Define $T_1 := T^{(r)}$. Then

$$\left| \int_{\partial T_1} f \right| \geq \frac{1}{4} c \qquad \text{and} \qquad L(\partial T_1) = \frac{1}{2} L(\partial T).$$

We repeat the process of subdivision to get a sequence of triangles

$$T \supset T_1 \supset T_2 \supset \cdots \supset T_n \supset \cdots,$$

satisfying

$$\left|\int_{\partial T_n} f\right| \geq \frac{1}{4^n}c \qquad \text{and} \qquad L(\partial T_n) = \frac{1}{2^n}L(\partial T). \tag{3.1}$$

We claim that the nested sequence $\bar{T} \supset \bar{T}_1 \supset \bar{T}_2 \supset \cdots \supset \bar{T}_n \supset \cdots$ contains a point $z_0 \in \cap_{n=1}^{\infty} \bar{T}_n$. Indeed, on each step choose a point $z_n \in T_n$. It is easy to see that (z_n) is a Cauchy sequence. Then (z_n) converges to a point $z_0 \in \cap_{n=1}^{\infty} \bar{T}_n$ since each of $\bar{T}_n$ is closed. This proves the claim.

Now we obtain another estimate on c using the fact that f is differentiable. Since f is differentiable at z_0, for a given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \qquad \text{implies} \qquad \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon,$$

that is

$$0 < |z - z_0| < \delta \qquad \text{implies} \qquad |f(z) - f(z_0) - f'(z_0)(z - z_0)| < \varepsilon|z - z_0|.$$

For $z \in T_n$ we obviously have $|z - z_0| < L(\partial T_n)$, so the Estimation Lemma implies that

$$\left|\int_{\partial T_n} \left\{ f(z) - \underbrace{(f(z_0) + f'(z_0)(z - z_0))}_{g(z)} \right\} dz\right| \leq \varepsilon L(\partial T_n).L(\partial T_n). \tag{3.2}$$

Now observe that the function $g(z) := f(z_0) + f'(z_0)(z - z_0)$ is of the form $Az + B$ and has an antiderivative in D. Thus $\int_{\partial T_n} g = 0$ and (3.2) reduces to

$$\left|\int_{\partial T_n} f(z)dz\right| \leq \varepsilon L^2(\partial T_n).$$

Comparing this with (3.1) we see that

$$\frac{1}{4^n}c \leq \left|\int_{\partial T_n} f(z)dz\right| \leq \varepsilon L^2(\partial T_n) = \frac{1}{4^n}\varepsilon L^2(\partial T).$$

This gives

$$c \leq \varepsilon L^2(\partial T).$$

Since $\varepsilon > 0$ can be chosen arbitrary small we conclude that $c = 0$. □

Exercise 3.3. (Cauchy Theorem for a rectangle) Let $D \subseteq \mathbb{C}$ be a domain, $f : D \to \mathbb{C}$ a differentiable function. Let

$$R = \{x + iy \in \mathbb{C} : a < x < b,\ c < y < d\}$$

be a rectangle with the boundary $\partial R = [z_1, z_2] \cup [z_2, z_3] \cup [z_3, z_4] \cup [z_4, z_1]$, where $z_1 = a + ic$, $z_2 = b + ic$, $z_3 = b + id$, $z_4 = a + id$. Assume that $\bar{R} = \partial R \cup R \subset D$. Prove that

$$\int_{\partial R} f = 0.$$

Hint. Insert the opposite contours $[z_1, z_3]$ and $[z_3, z_1]$ and use the Cauchy Theorem for a Triangle twice, for triangles $T(z_1, z_2, z_3)$ and $T(z_1, z_3, z_4)$.

3.2 Cauchy Theorem for starshaped domains

Definition 3.4. We say that a set $D \subseteq \mathbb{C}$ is a *starshaped domain* iff D is open and there exists a point $a \in D$, called a *star–center of* D, such that for any $z \in D$ one has $[a, z] \subset D$.

Exercise 3.5. Prove that:
(a) a starshaped domain is a domain (i.e. prove that any starshaped domain is connected);
(b) $\mathbb{C}$ is a starshaped domain;
(c) a convex domain is a starshaped domain (cf. Example 2.20);
(d) $\mathbb{C} \setminus (-\infty, 0]$ is a starshaped domain with a star–center 1;
(e) $\mathbb{C} \setminus \{0\}$ is not a starshaped domain.

Below we prove a version of Cauchy's theorem for starshaped domains.

Theorem 3.6. (Cauchy Theorem for a starshaped domain) *Let $D \subset \mathbb{C}$ be a starshaped domain with a star-center $a \in D$ and $f : D \to \mathbb{C}$ a differentiable function. Then f has an antiderivative in D defined by*

$$F_a(z) = \int_{[a,z]} f.$$

In particular, for any closed contour $\gamma : [\alpha, \beta] \to D$ one has

$$\int_\gamma f(z)dz = 0.$$

Proof. [2] Define a function $F_a : D \to \mathbb{C}$ by

$$F_a(z) := \int_{[a,z]} f(w)dw.$$

We shall prove that $F_a'(z) = f(z)$ in D. Fix $z \neq a \in D$. Since D is open there exists $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(z) \subset D$. Since D is starshaped, for $|h| < \varepsilon_0$ the closed triangle $\bar{T}$ with vertices $a, z, z + h$ is contained in D. By the Cauchy Theorem for a triangle

$$\int_{[a,z]} f + \int_{[z,z+h]} f + \int_{[z+h,a]} f = 0.$$

Therefore

$$\frac{F_a(z + h) - F_a(z)}{h} = \frac{1}{h} \int_{[z,z+h]} f(w)dw.$$

Notice that $\int_{[z,z+h]} f(z)dw = f(z)h$. Then

$$\frac{F_a(z + h) - F_a(z)}{h} - f(z) = \frac{1}{h} \int_{[z,z+h]} (f(w) - f(z))\, dw.$$

Then by the Estimation Lemma and continuity of f we conclude that

$$\left| \frac{F_a(z + h) - F_a(z)}{h} - f(z) \right| \le \frac{1}{|h|} \int_{[z,z+h]} |f(w) - f(z)|\, dw \le$$

[2] Compare this proof with the proof of Lemma 2.28.

$$\leq \frac{1}{|h|} L([z, z+h]) \sup_{w\in[z,z+h]} |f(w) - f(z)| =$$

$$= \sup_{w\in[z,z+h]} |f(w) - f(z)| \to 0 \quad \text{as} \;\; h \to 0.$$

We conclude that $F_a'(z) = f(z)$ in D. Thus the required statement follows by the Fundamental Theorem of contour integration. □

Principal branch of the logarithm. The function $f(z) = z^{-1}$ is differentiable in $\mathbb{C} \setminus \{0\}$ with the derivative $f'(z) = -z^{-2}$. We know that z^{-1} has no antiderivative in $\mathbb{C} \setminus \{0\}$ and $\mathbb{C} \setminus \{0\}$ is not a starshaped domain). However if we restrict the function to a starshaped domain then Theorem 3.6 must apply to z^{-1}. Clearly $\mathbb{C} \setminus (-\infty, 0]$ is a starshaped domain with the starcenter 1. By Theorem 3.6, z^{-1} has an antiderivative in $\mathbb{C} \setminus (-\infty, 0]$, which we denote by Log, given by the formula

$$\text{Log}(z) := \int_{[1,z]} \frac{1}{z} dz. \tag{3.3}$$

We call such defined function $\text{Log} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ the *principal branch of the logarithm*. Clearly $\text{Log}'(z) = z^{-1}$ in $\mathbb{C} \setminus (-\infty, 0]$ and $\text{Log}(x) = \log(x)$ for $x > 0$, where $\log(x)$ denotes the usual real logarithm.

Exercise 3.7. Prove that the value of the principal branch of the logarithm can be evaluated by the formula

$$\text{Log}(z) = \log(|z|) + i\text{Arg}(z), \tag{3.4}$$

where $\text{Arg}_{-\pi}(z) \in (-\pi, \pi)$ and $\log : \mathbb{R}_+ \to \mathbb{R}$ denotes the usual real logarithm.

Hint. Represent $z \in \mathbb{C} \setminus (-\infty, 0]$ in polar form $z = re^{i\varphi}$, where $r = |z|$ and $\varphi = \text{Arg}_{-\pi}(z)$. If $\varphi > 0$, let $\gamma = [1, r] + \gamma_2$, where $\gamma_2 = re^{it}$ with $t \in [0, \varphi]$. Prove that

$$\int_{[1,z]} \frac{1}{z} dz = \int_\gamma \frac{1}{z} dz$$

and compute the latter integral. Similarly for $\varphi < 0$.

Exercise 3.8. Prove that $e^{\text{Log}(z)} = z$ for each $z \in \mathbb{C} \setminus (-\infty, 0]$.

Hint. By (3.4) we obtain $e^{\text{Log}(z)} = e^{\log(|z|)} e^{i\text{Arg}_{-\pi}(z)} = z$.

Exercise 3.9. Prove that if $z_1, z_2, z_1 z_2 \in \mathbb{C} \setminus (-\infty, 0]$ then $\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2)$.

Hint. Use (3.4) and the identity $\text{Arg}_{-\pi}(z_1 z_2) = \text{Arg}_{-\pi}(z_1) + \text{Arg}_{-\pi}(z_2)$.

Complex powers. Once $\text{Log} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ is defined, for any $a \in \mathbb{C}$ the *principal branch* of complex fractional powers $z^a :\in \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ can be defined via

$$z^a := e^{a \,\text{Log}(z)}.$$

It is easy to see that when $a \in \mathbb{Z}$ such definition agrees with the usual integer powers of z.

3.3 Homotopy. Deformation Theorem

Definition 3.10. Let $D \subseteq \mathbb{C}$ be a domain. Two closed paths $\gamma_0, \gamma_1 : [\alpha, \beta] \to D$ are called *homotopic in* D if there exists a continuous map $H = H(t,s) : [\alpha, \beta] \times [0,1] \to D$ such that

(a) $H(t,0) = \gamma_0(t)$ and $H(t,1) = \gamma_1(t)$ $(t \in [\alpha, \beta])$;

(b) $H(\alpha, s) = H(\beta, s)$ $(s \in [0,1])$.

We write $\gamma_1 \sim \gamma_2$ in D.

The idea of this definition is that the path γ_0 can be continuously deformed to γ_1 without passing outside of the domain D. Or, in other words, as s changes from 0 to 1, we have a family of paths that continuously ("without cutting") change its shape from γ_0 to γ_1 without leaving D. Note that the paths can be self–intersecting!

Exercise 3.11. Prove that the the homotopy of two closed paths in a domain D is an equivalence relation.

Example 3.12. Let $C \subset \mathbb{C}$ be a convex domain. Then any two closed paths $\gamma_0, \gamma_1 : [\alpha, \beta] \to D$ are homotopic in C.

Proof. Define the *linear homotopy* between γ_0 and γ_1 by

$$H(t,s) := (1-s)\gamma_0(t) + s\gamma_1(t).$$

Since C is convex, H maps $[\alpha, \beta] \times [0,1]$ into C. Since $\gamma_0, \gamma_1 : [\alpha, \beta] \to D$ are continuous, so is H. Clearly $H(t,0) = \gamma_0(t)$, $H(t,1) = \gamma_1(t)$ $(t \in [\alpha, \beta])$ and $H(\alpha, s) = H(\beta, s)$ as $s \in [0,1]$. Thus $\gamma_0 \sim \gamma_1$ in C. □

Exercise 3.13. Let $D \subset \mathbb{C}$ be a domain. Let $a, b \in \mathbb{C}$ and $r, R > 0$ be such that:

(a) $B_r(a) \subset B_R(b)$ (but not necessarily $B_r(a) \subset D$);

(b) $A := \{z \in \mathbb{C} : |z-b| \leq R, |z-a| \geq r\} \subset D$.

Prove that the positively oriented circles $S_R^+(b)$ and $S_r^+(a)$ are homotopic in D.

Hint. Consider the linear homotopy between $S_R^+(b)$ and $S_r^+(a)$.

Theorem 3.14. (Deformation Theorem) *Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ differentiable function. Let $\gamma_0, \gamma_1 : [\alpha, \beta] \to D$ be to closed contours that are homotopic in D. Then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

Proof. See Marsden, pp.99–113.

Example 3.15. Let $\gamma : [\alpha, \beta] \to \mathbb{C} \setminus \{0\}$ be a closed contour homotopic to the unit circle $S_1^+(0)$ in $\mathbb{C} \setminus \{0\}$. Then

$$\int_\gamma \frac{1}{z} dz = \int_{S_1^+(0)} \frac{1}{z} dz = 2\pi i$$

by the Deformation Theorem.

3.4 Cauchy Theorem for simply connected domains

Let $D \subseteq \mathbb{C}$ be a domain and $a \in D$ a fixed point. Denote by $\gamma_a : [\alpha, \beta] \to D$ the constant path

$$\gamma_a(t) := \{a\}.$$

Obviously γ_a is a closed smooth path in D with $|\gamma'(t)| = 0$.

Definition 3.16. We say that a closed path $\gamma : [\alpha, \beta] \to D$ is *homotopic to a point in D* if there is a point $a \in D$ such that $\gamma \sim \gamma_a$ in D.

Exercise 3.17. Let $D \subseteq \mathbb{C}$ be a domain and $\gamma : [\alpha, \beta] \to D$ a closed path. Let $a \in D$ be a given point. Assume that $\gamma \sim \gamma_a$ in D. Prove that $\gamma \sim \gamma_b$ in D for any other point $b \in D$.

Definition 3.18. A domain $D \subseteq \mathbb{C}$ is called *simply connected* if every closed path $\gamma : [\alpha, \beta] \to D$ is homotopic to a point in D.

Intuitively *simply connected* means that the domain "has no holes".

Exercise 3.19. Prove that:
(a) a convex domain is simply connected;
(b) a starshaped domain is simply connected.

Theorem 3.20. (Cauchy Theorem for simply connected domains) *Let $D \subseteq \mathbb{C}$ be a simply connected domain and $f : D \to \mathbb{C}$ differentiable function. Let $\gamma : [\alpha, \beta] \to D$ be a closed contour. Then*

$$\int_\gamma f = 0.$$

Proof. Let $\gamma : [\alpha, \beta] \to D$ be a closed contour. Since D is simply connected, there exists $a \in D$ such that $\gamma \sim \gamma_a$ in D. By the Deformation Theorem

$$\int_\gamma f = \int_{\gamma_a} f = 0,$$

since $|\gamma_a'(t)| = 0$. □

Corollary 3.21. *Let $D \subseteq \mathbb{C}$ be a simply connected domain and $f : D \to \mathbb{C}$ differentiable function. Fix a point $a \in D$. Let $\gamma_{a,z} : [\alpha, \beta] \to D$ be a contour from a to $z \in D$. Then the function*

$$F(z) = \int_{\gamma_{a,z}} f,$$

is an antiderivative of f in D.

Proof. Follows from Theorem 3.20 and Lemma 2.28. □

Exercise 3.22. Prove that $\int_{S_1(0)} e^{z^2} dz = 0$.

Exercise 3.23. Prove that $\mathbb{C} \setminus \{0\}$ is not simply connected.

Hint. Use Theorem 3.20 and Example 3.15.

4 Cauchy Integral Formula

The Cauchy Integral Formula shows that the values of a differentiable function on a ball is determined completely by its values on the boundary circle of the ball.

Theorem 4.1. (Cauchy Integral Formula) *Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ differentiable function. Let $\bar{B}_R(a) \subset D$. Then for any $z \in B_R(a)$ one has*

$$f(z) = \frac{1}{2\pi i} \int_{S_R^+(a)} \frac{f(w)}{w-z} dw. \tag{4.1}$$

Proof. Fix $z \in B_R(a)$. Observe that the function

$$g(w) := \frac{f(w)}{w-z}$$

is differentiable in $D \setminus \{z\}$. Choose $0 < \varepsilon < R - |z-a|$. By the Deformation Theorem and Exercise 3.13

$$\begin{aligned} \int_{S_R^+(a)} \frac{f(w)}{w-z} dw &= \int_{S_\varepsilon^+(z)} \frac{f(w)}{w-z} dw \\ &= \int_{S_\varepsilon^+(z)} \frac{f(z)}{w-z} dw + \int_{S_\varepsilon^+(z)} \left\{ \frac{f(w)-f(z)}{w-z} - f'(z) \right\} dw + \int_{S_\varepsilon^+(z)} f'(z) dw. \end{aligned}$$

Now by Example 2.12 we know that

$$\int_{S_\varepsilon^+(z)} \frac{f(z)}{w-z} dw = 2\pi i f(z) \qquad \text{and} \qquad \int_{S_\varepsilon^+(z)} f'(z) dw = 0.$$

Next, by the differentiability of f for any $M > 0$ there exists $\delta > 0$ such that

$$0 < |z-w| < \delta \quad \Rightarrow \quad \left| \frac{f(w)-f(z)}{w-z} - f'(z) \right| \le M.$$

Therefore for any $\varepsilon \in (0, \delta)$ by the Estimation Lemma we obtain

$$\left| \int_{S_\varepsilon^+(z)} \left\{ \frac{f(w)-f(z)}{w-z} - f'(z) \right\} dw \right| \le ML(S_\varepsilon^+(z)) = 2\pi\varepsilon M.$$

Letting $\varepsilon \to 0$ we get

$$\int_{S_R^+(a)} \frac{f(w)}{w-z} dw = 2\pi i f(z),$$

which is required. □

Remark 4.2. Observe that if $z \in D \setminus \bar{B}_R(a)$ then

$$\frac{1}{2\pi i} \int_{S_R^+(a)} \frac{f(w)}{w-z} dw = 0.$$

This follows from the Cauchy Theorem because $\frac{f(w)}{w-z}$ is differentiable in $B_{R+\varepsilon}(a)$, for sufficiently small $\varepsilon > 0$.

The Cauchy Integral Formula might be extremely useful in computations.

Example 4.3. In order to evaluate the integral

$$\int_{S_1^+(0)} \frac{e^w}{w} dw$$

we may use (4.1) with $f(w) := e^w$ and $z := 0$ Clearly f is differentiable in $\mathbb{C}$. Then

$$\int_{S_1^+(0)} \frac{e^w}{w} dw = 2\pi i \, e^0 = 2\pi i.$$

Example 4.4. Evaluate

$$\int_\gamma \frac{e^w + z}{z+2} dw,$$

where (a) $\gamma = S_1^+(0)$; (b) $\gamma = S_3^+(0)$.

Solution. Note that the integrand $\frac{e^w+z}{z+2}$ is differentiable in $\mathbb{C} \setminus \{-2\}$.
(a) It is clear that $S_1^+(0)$ is contractible into a point in $\mathbb{C} \setminus \{-2\}$. Thus by the Deformation Theorem

$$\int_{S_1^+(0)} \frac{e^w + z}{z+2} dw = 0.$$

(b) We may use (4.1) with $f(w) := e^w + w$ and $z := -2$. Clearly f is differentiable in $\mathbb{C}$ and we obtain

$$\int_{S_3^+(0)} \frac{e^w + z}{z+2} dw = 2\pi i(e^{-2} - 2).$$

Example 4.5. Evaluate

$$\int_{S_1^+(i)} \frac{dw}{w^2+1}.$$

Solution. Observe that

$$\frac{1}{w^2+1} = \frac{1}{(w-i)(w+i)}.$$

Set $f(w) := \frac{1}{w+i}$ and let $z := i$. Thus f is differentiable in $\mathbb{C} \setminus \{-i\}$. Hence by the Cauchy Integral Formula

$$\int_{S_1^+(i)} \frac{dw}{w^2+1} = \int_{S_1^+(i)} \frac{f(w)}{w-i} = 2\pi i f(i) = \pi.$$

In the same way we compute that

$$\int_{S_1^+(-i)} \frac{dw}{w^2+1} = -\pi.$$

Remark 4.6. Note that in Cauchy's Integral Formula the function f, not the integrand $f(w)/(w-z)$, is analytic. If $z \in B_R(a)$ then the integrand is analytic only on $B_R(a) \setminus \{z\}$, so we can not use Cauchy's Theorem to conclude that the integral is zero.

5 Analytic functions

5.1 Analytic functions

Definition 5.1. Let $D \subseteq \mathbb{C}$ be a domain. A function $f : D \to \mathbb{C}$ is said to be *analytic* at a point $z_0 \in D$ iff for some ball $B_r(z_0) \subseteq D$ a power series expansion

$$f(z_0 + h) = \sum_{n=0}^{\infty} a_n h^n \qquad (|h| < r) \tag{5.1}$$

is valid. A function $f : D \to \mathbb{C}$ is said to be *analytic* in a domain D iff f is analytic at every point $z \in D$.

Remark 5.2. Recall, that the number R, $0 \le R \le +\infty$, defined by

$$R = \sup\{r \ge 0 : \text{power series (5.1) converges in } B_r(z_0)\}$$

is called the *radius of convergence* of power series (5.1). The radius of convergence R of power series (5.1) has the following properties:

(a) if $|h| < R$ then the series converges absolutely;

(b) if $|h| > R$ then the series diverges.

Moreover, R can be computed by the formula

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n},$$

where we agree that $+\infty^{-1} = 0$, $0^{-1} = +\infty$. In view of the properties (a) and (b) we may say that the natural domain of convergence of power series is the ball of radius R.

Example 5.3. Recall, that a (complex) *polynomial of degree* n is a function of the form

$$p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z_1 + a_0,$$

where $a_1, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$. Clearly every polynomial is analytic in $\mathbb{C}$. Recall also the power series at $z_0 = 0$ of some common elementary functions:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1), \qquad \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad (|z| < \infty).$$

Exercise* 5.4. Prove that the coefficients a_n of a power series expansion of an analytic function are uniquely defined. Namely, prove that if

$$f(z_0 + h) = \sum_{n=0}^{\infty} a_n h^n = \sum_{n=0}^{\infty} b_n h^n \qquad (|h| < R)$$

then $a_n = b_n$ for all $n \in \mathbb{N} \cup \{0\}$.

It is easy to see that every analytic function is is infinitely many times differentiable.

Lemma 5.5. *Suppose $f : B_R(z_0) \to \mathbb{C}$ is analytic at z_0 with a power series expansion*

$$f(z_0 + h) = \sum_{n=0}^{\infty} a_n h^n \qquad (|h| < R).$$

Then f is infinitely many times differentiable in $B_R(z_0)$ and $f^{(n)}(z_0) = n! a_n$.

Proof. Set $z := z_0 + h$. From Exercise 1.8 we know that a power series can be differentiated infinitely many times and

$$f^{(n)}(z) := \sum_{k=n}^{\infty} \{k(k-1)\dots(k-n+1)\}\ a_k (z - z_0)^{k-n}.$$

Substituting z with z_0 gives $f^{(n)}(z_0) = n! a_n$. □

5.2 Taylor Series Theorem

Using the Cauchy Integral Formula we can now prove that *every differentiable complex function is infinitely many times differentiable and analytic.*

Theorem 5.6. (Taylor Series Theorem) *Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ differentiable function. Then f is analytic in D and for any ball $B_R(z_0) \subseteq D$ the power series expansion*

$$f(z_0 + h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} h^n \qquad (|h| < R) \tag{5.2}$$

is valid. Further, if $r \in (0, R)$ then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{S_r^+(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} dw. \tag{5.3}$$

Remark 5.7. *i*) Power series expansion (5.2) is called the *Taylor series expansion* of f at z_0.
ii) By the Deformation Theorem the integral in the right hand side of (5.3) does not depend on $r \in (0, R)$.

Proof. Fix $z_0 \in D$ and a ball $B_R(z_0) \subseteq D$. Choose h such that $z_0 + h \in B_R(z_0)$ and r such that $|h| < r < R$. By the Cauchy Integral Formula applied at the point $z_0 + h \in B_r(z_0)$ we know that

$$f(z_0 + h) = \frac{1}{2\pi i} \int_{S_r^+(z_0)} \frac{f(w)}{w - (z_0 + h)} dw.$$

By the (complex) geometric sum formula

$$\frac{1}{1-q} = 1 + q + q^2 + \dots + q^m + \frac{q^{m+1}}{1-q} \tag{5.4}$$

applied with $q := \frac{h}{w - z_0}$ we obtain

$$\begin{aligned} \frac{1}{w - (z_0 + h)} &= \frac{1}{w - z_0} \frac{1}{1 - \frac{h}{w - z_0}} \\ &= \frac{1}{w - z_0} \left\{ 1 + \frac{h}{w - z_0} + \frac{h^2}{(w - z_0)^2} + \dots + \frac{h^m}{(w - z_0)^m} + \frac{h^{m+1}}{(w - z_0)^m (w - (z_0 + h))} \right\}. \end{aligned}$$

Therefore

$$\begin{aligned}
f(z_0+h) &= \frac{1}{2\pi i}\int_{S_r^+(z_0)} \frac{f(w)}{w-(z_0+h)}dw \\
&= \frac{1}{2\pi i}\int_{S_r^+(z_0)} f(w)\Big\{\tfrac{1}{w-z_0} + \tfrac{h}{(w-z_0)^2} + \cdots + \tfrac{h^m}{(w-z_0)^{m+1}} + \tfrac{h^{m+1}}{(w-z_0)^{m+1}(w-(z_0+h))}\Big\}dw \\
&= \sum_{n=0}^{m} a_n h^n + A_m,
\end{aligned}$$

where

$$a_n = \frac{1}{2\pi i}\int_{S_r^+(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}dw \quad \text{and} \quad A_m = \frac{1}{2\pi i}\int_{S_r^+(z_0)} \frac{f(w)h^{m+1}}{(w-z_0)^{m+1}(w-(z_0+h))}dw.$$

We are going to show that $\lim_{m\to\infty} A_m = 0$.

Indeed, f is differentiable in $B_R(z_0)$ and hence continuous on $S_r(z_0)$. So f is bounded on $S_r(z_0)$ (because $S_r(z_0)$ is compact). This means that there exists $M > 0$ such that

$$|f(w)| \le M \qquad (w \in S_r(z_0)).$$

Now $|h| < r$, $|w - z_0| = r$ and $|w-(z_0+h)| \ge ||w-z_0|-|h|| = r-|h|$. Then by the Estimation Lemma

$$|A_m| \le \frac{1}{2\pi}\frac{M|h|^{m+1}}{r^{m+1}(r-|h|)}\underbrace{L(S_r(z_0))}_{2\pi r} = \frac{Mh}{r-|h|}\left(\frac{|h|}{r}\right)^m.$$

Since $|h| < r$ we conclude that $\lim_{m\to\infty} A_m = 0$.

Therefore

$$\lim_{m\to\infty}\left(f(z_0+h) - \sum_{n=0}^{m} a_n h^n\right) = 0.$$

This means that f is analytic at z_0 with the power series expansion

$$f(z_0+h) = \sum_{n=0}^{\infty} a_n h^n,$$

and this expansion is valid for all $|h| < R$ and

$$a_n = \frac{1}{2\pi i}\int_{S_r^+(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}dw$$

for $|h| < r$. However the restriction $|h| < r$ is unnecessary by the Remark 5.7, so we can let $|h| < R$.

To complete the proof we simply observe that $a_n = \frac{f^{(n)}(z_0)}{n!}$ by Lemma 5.5. $\square$

Remark 5.8. The Taylor Series Theorem states, in particular, that *every differentiable complex function is infinitely many times differentiable and analytic*. A corresponding notion of analyticity can be similarly introduced for real functions. However, the following example demonstrates that for real functions existence off all higher order derivatives does not imply analyticity. Let $f : \mathbb{R} \to \mathbb{R}$ be a real function, $f(x) = e^{-\frac{1}{x^2}}$. Then f is infinitely many times differentiable and

$$f^{(n)}(0) = 0 \qquad (n \in \mathbb{N} \cup \{0\}).$$

However $f \not\equiv 0$ in a neighborhood of $x = 0$ and therefore f is not analytic.

Example 5.9. The Taylor series of $\exp(z)$ about $z_0 = 0$ is given by

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \qquad (z \in \mathbb{C}).$$

Proof. The required expansion can be obtained by using (5.2) and $\exp^{(n)}(0) = 1$. Since $\exp(z)$ is differentiable for every $z \in \mathbb{C}$, the radius of convergence of the series is $R = +\infty$. $\square$

In practice, we determine the radius of convergence of the Taylor series of a differentiable function as *the radius of the largest ball contained in the domain differentiability of the function.*

Example 5.10. Find the Taylor expansion of the function $f(z) = \frac{1}{1+z^2}$ about $z_0 = 3i$.

Solution. One can obtain the required expansion by computing the derivatives $f'(n)(3i)$ explicitly. However it might be more efficient to represent f as partial fractions

$$\frac{1}{1+z^2} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right).$$

Then we compute

$$f^{(n)}(z) = \frac{1}{1+z^2} = \frac{1}{2i}\left(\frac{(-1)^n n!}{(z-i)^{n+1}} - \frac{(-1)^n n!}{(z+i)^{n+1}}\right).$$

hence

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2i}\left(\frac{1}{(2i)^{n+1}} - \frac{1}{(4i)^{n+1}}\right) h^n.$$

In order to determine the radius of convergence of the series observe that $1+z^2 = 0$ iff $z = \pm i$. Thus the domain of differentiability of f is $\mathbb{C} \setminus \{i, -i\}$. Since $B_2(3i) \subset \mathbb{C} \setminus \{i, -i\}$ is the largest ball centered at $z_0 = 3i$ contained in $\mathbb{C} \setminus \{i, -i\}$, we conclude that the series converges for $|h| < 2$.

5.3 Morera Theorem

The following theorem gives a partial "converse" to Cauchy's type theorems.

Theorem 5.11. (MORERA THEOREM) *Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ continuous function. Assume that for any closed contour $\gamma : [\alpha, \beta] \to D$ one has*

$$\int_\gamma f = 0.$$

Then f is differentiable in D.

Proof. If $\int_\gamma f = 0$ for any closed contour γ in D then, by Theorem 2.31, the function f has an antiderivative in D. In other words, there exists a *differentiable* function $F : D \to \mathbb{C}$ such that

$$F'(z) = f(z) \qquad (z \in D).$$

But then, by the Taylor Series Theorem, F is infinitely many times differentiable in D. In particular,

$$F''(z) = (F'(z))' = f(z)' \qquad (z \in D),$$

that is f is differentiable in D. □

Remark 5.12. It follows that if a function f is not differentiable in D then f has no antiderivative in D. For example, the functions $|z|$, $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ have no antiderivative in any domain $D \subseteq \mathbb{C}$.

5.4 Cauchy Estimates and the corollaries

The next lemma is an immediate consequence of the formula (5.3) but is stated separately in view of its importance.

Lemma 5.13. (CAUCHY ESTIMATE) *Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ analytic function. Suppose that $\bar{B}_R(z_0) \subset D$ and*

$$|f(z)| \leq M \qquad (z \in S_R(z_0)).$$

Then for any $n \in \mathbb{N} \cup \{0\}$

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}. \tag{5.5}$$

Proof. From the Taylor Series Theorem we know that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{S_R^+(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} dw.$$

Hence, by the Estimation Lemma,

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} \underbrace{L(S_R^+(z_0))}_{2\pi R} = \frac{n!M}{R^n},$$

as required. □

Definition 5.14. An *entire function* is a function which is defined and analytic on the whole complex plane $\mathbb{C}$.

Example 5.15. Clearly every polynomial is an entire function. Functions $\exp(z)$, $\sin(z)$, $\cos(z)$ are entire. By the Taylor Series Theorem every entire function f has a power series expansion $\sum_{n=0}^{\infty} a_n z^n$ with infinite radius of convergence, so entire functions can be viewed as polynomials of "infinite" degree.

Theorem 5.16. (Liouville Theorem) *A bounded entire function is constant.*

Proof. Let f be an entire function. Suppose $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Fix $z \in \mathbb{C}$. Since f is analytic in every ball $B_R(z)$, by the Cauchy Estimate we have

$$|f'(z)| \leq \frac{M}{R}.$$

Since M is independent of R and R can be chosen arbitrary large, we conclude that $|f'(z)| = 0$.

Since $z \in \mathbb{C}$ was arbitrary and $\mathbb{C}$ is connected by Theorem 2.21 we conclude that f is constant in $\mathbb{C}$. □

As a simple corollary of the Liouville Theorem we prove that every polynomial has a root in the complex plane.

Theorem 5.17. (Fundamental Theorem of Algebra) *Let p_n be a polynomial of degree $n \geq 1$. Then there is a point $z_0 \in \mathbb{C}$ such that $p_n(z_0) = 0$.*

Proof. Suppose that $p_n(z) \neq 0$ for all $z \in \mathbb{C}$. Then

$$g(z) := \frac{1}{p_n(z)}$$

is an entire function.

On the other hand, writing

$$p_n(z) = a_n z^n \left(1 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots + \frac{b_n}{z^n}\right)$$

with appropriate constants $b_1, \ldots, b_n$, we see that[3]

$$\lim_{|z|\to\infty} |p_n(z)| = \infty.$$

Hence

$$\lim_{|z|\to\infty} |g(z)| = 0.$$

In particular, there is a radius $R > 0$ such that $|g(z)| < 1$ if $|z| > R$. But $|g(z)|$ is continuous on $\bar{B}_R(0)$ so there is a constant $M > 0$ such that $|g(z)| < M$ if $|z| \leq R$. Therefore g is a bounded entire function. By the Liouville Theorem g must be constant. This is obviously a contradiction. □

Exercise 5.18. *a*) Let p_n be a polynomial of degree $n \geq 1$. Prove that $p_n(\mathbb{C}) = \mathbb{C}$.

b) Give an example of an entire function f such that $f(\mathbb{C}) \subsetneq \mathbb{C}$.

[3] Prove this as an exercise.

6 Laurent Series

By a *Laurent series* we mean a series

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n. \tag{6.1}$$

This is to be thought of as a compact notation for the sum of two series

$$\underbrace{\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n}_{\text{Laurent series}} := \underbrace{\sum_{n=0}^{\infty} a_n(z-z_0)^n}_{\text{regular part}} + \underbrace{\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}}_{\text{singular part}}.$$

Laurent series (absolutely) *converges* in a set $A \subset \mathbb{C}$ iff both regular and singular parts (absolutely) converge at every $z \in A$. Note that the singular part of a Laurent series is not defined at $z = z_0$.

We know that the natural domain of convergence of a power series is a ball, see Remark 5.2. Consequently, the singular part of a Laurent series converges outside a ball.

Exercise 6.1. Let $\sum_{n=1}^{\infty} a_n(z-z_0)^{-n}$ be a "singular" series. Prove that:

(a) if the series converges at z_1 then it absolutely converges for all z such that $|z-z_0| > |z_1-z_0|$,
(b) if the series diverges at z_2 then it diverges for all z such that $|z-z_0| < |z_2-z_0|$.

Hint. Set $w := \frac{1}{z-z_0}$ and consider the power series $\sum_{n=1}^{\infty} a_n w^n$.

Exercise 6.2. Prove that Laurent series (6.1) converges for all z such that $r < |z-z_0| < R$, where

$$r = \limsup_{n\to\infty} |a_{-n}|^{\frac{1}{n}}, \qquad R^{-1} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}, \tag{6.2}$$

with $0 \le r, R \le +\infty$, and where we agree that $+\infty^{-1} = 0$, $0^{-1} = +\infty$.

Hint. Apply the formula for the radius of convergence of power series separately to the regular and singular parts of the Laurent series.

In view of Exercises 6.1, 6.2 we may say that the natural domain of convergence of a Laurent series is an (open) annulus

$$A_{r,R}(z_0) = \{z \in \mathbb{C} : r < |z-z_0| < R\},$$

where r and R are defined in (6.2), and we assume that if $r \ge R$ then the annulus of convergence is empty.

Theorem 6.3. (Laurent Series Theorem) *Let $f : A_{r,R}(z_0) \to \mathbb{C}$ be an analytic function. Then f has a Laurent series expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n, \qquad (z \in A_{r,R}(z_0)).$$

Further, if $\rho \in (r, R)$ then

$$a_n = \frac{1}{2\pi i}\int_{S_\rho^+(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}dw. \tag{6.3}$$

Remark 6.4. *i)* By the Deformation Theorem the integral in the right hand side of (6.3) does not depend on $\rho \in (r, R)$.

ii) Note that in contrast to the Taylor Series we can not longer assert that $a_n = \frac{f^{(n)}(z_0)}{n!}$ even for $n \geq 0$, since f need not be differentiable at z_0 under the hypothesis of the Laurent Series Theorem.

Exercise* 6.5. Prove that the coefficients a_n of a Laurent series expansion of a function f are uniquely defined. Namely, prove that if

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n \qquad (z \in A_{r,R}(z_0))$$

then $a_n = b_n$ for all $n \in \mathbb{Z}$.

Formula (6.3) for the coefficients a_n is not very practical for computing the Laurent series of a given function. Instead, tricks can be used to obtain an expansion of the required form. Then the uniqueness of the expansion indicates that this is the desired one.

Example 6.6. Let $f(z) = \exp(z) + \exp(\frac{1}{z})$. We have

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \quad (|z| < \infty); \qquad \exp(\frac{1}{z}) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} \quad (|z| > 0).$$

Therefore $f(z)$ has the Laurent series expansion at $z_0 = 0$ given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where $a_n = \frac{1}{n!}$ for $n \geq 0$, $a_0 = 2$ and $a_n = \frac{1}{(-n)!}$ for $n \leq -1$.

Example 6.7. Let $f(z) = \frac{1}{z} + \frac{1}{1-z}$. Then, by using the geometric series $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ $(0 < |z| < 1)$ we see that

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} z^n \quad (0 < |z| < 1) \tag{6.4}$$

is a Laurent expansion of f in $A_{0,1}(0)$. On the other hand, by using transformation $w := \frac{1}{z}$ we obtain

$$f(w) = w + w\left(-\frac{1}{1-w}\right) = w - w\sum_{n=0}^{\infty} w^n = -\sum_{n=2}^{\infty} w^n \quad (|w| < 1).$$

Therefore

$$f(z) = -\sum_{n=-\infty}^{-2} z^n \quad (|z| > 1) \tag{6.5}$$

is a Laurent expansion of f in $A_{1,\infty}(0)$.

Remark 6.8. Observe that both (6.4) and (6.5) are Laurent expansions of $f(z)$ at $z_0 = 0$. This does not contradict to the uniqueness of the Laurent expansion since (6.4) and (6.5) are valid in *different annuli* !

Example 6.9. Consider

$$f(z) = \frac{1}{(z-1)(z-2)}.$$

This function has three Laurent expansions about $z_0 = 0$, given by

$$f(z) = \sum_{n=0}^{\infty} \left(1 - 2^{-(n+1)}\right) z^n \quad (z \in B_1(0)), \tag{6.6}$$

$$f(z) = \sum_{n=-\infty}^{-1} z^n - \sum_{n=0}^{\infty} 2^{-(n+1)} z^n \quad (z \in A_{1,2}(0)), \tag{6.7}$$

$$f(z) = \sum_{n=-\infty}^{-1} \left(2^{-(n+1)} - 1\right) z^n \quad (z \in A_{2,\infty}(0)). \tag{6.8}$$

Each of these expansions may be obtained by writing f in partial fractions

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

and using the geometric series. Note that (6.6) is simply the Taylor expansion at $z_0 = 0$ while the others are genuinely Laurent expansions involving the singular part.

In order to prove the Laurent Series Theorem we need a modification of the Cauchy Integral Formula for annular domains.

Lemma 6.10. *Let $f : A_{r,R}(z_0) \to \mathbb{C}$ be an analytic function. Let $z \in A_{r,R}(z_0)$. Let ρ_1, ρ_2 and ε be such that $r < \rho_1 < \rho_2 < R$ and $\bar{B}_\varepsilon(z) \subset A_{\rho_1,\rho_2}(z_0)$. Then*

$$f(z) = \frac{1}{2\pi i} \int_{S^+_{\rho_2}(z_0)} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \int_{S^+_{\rho_1}(z_0)} \frac{f(w)}{w-z} dw.$$

Proof. Join $S^+_{\rho_1}(z_0)$ with $S^+_\varepsilon(z)$ and $S^+_\varepsilon(z)$ with $S^+_{\rho_2}(z_0)$ via straight line segments γ_2 and γ_2, traversed in opposite directions each (see Figure 1). The obtained closed contour is contractible into a point in $A_{r,R}(z_0)$. Hence

$$\int_{S^+_\varepsilon(z)} \frac{f(w)}{w-z} dw + \int_{S^+_{\rho_1}(z_0)} \frac{f(w)}{w-z} dw - \int_{S^+_{\rho_2}(z_0)} \frac{f(w)}{w-z} dw = 0$$

by the Deformation Theorem. Therefore

$$\int_{S^+_{\rho_2}(z_0)} \frac{f(w)}{w-z} dw - \int_{S^+_{\rho_1}(z_0)} \frac{f(w)}{w-z} dw = \int_{S^+_\varepsilon(z)} \frac{f(w)}{w-z} dw = 2\pi i f(z)$$

by the Cauchy Integral formula, applied at the point $z \in \bar{B}_\varepsilon(z)$. $\square$

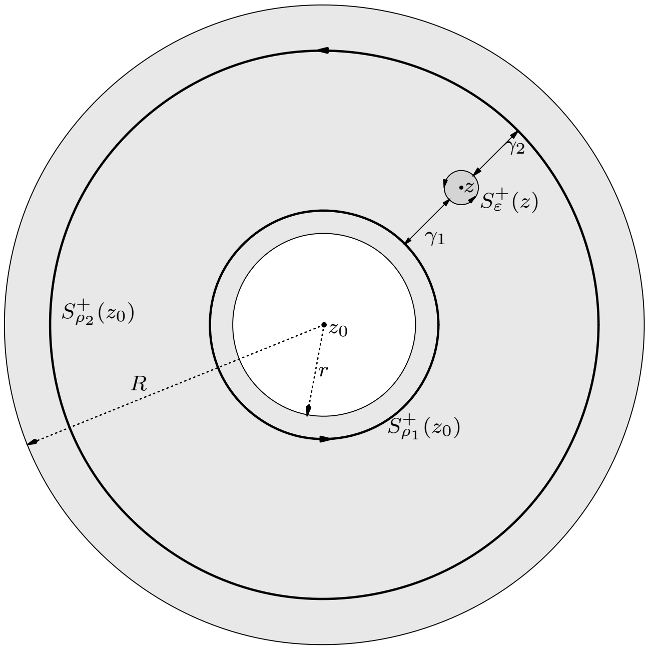

Figure 1: Contour of integration in the proof of Lemma 6.10.

Proof of the Laurent Series Theorem. Fix $z \in A_{r,R}(z_0)$. Choose ρ_1, ρ_2 and ε such that $r < \rho_1 < \rho_2 < R$ and $\bar{B}_\varepsilon(z) \subset A_{\rho_1,\rho_2}(z_0)$. Then by Lemma 6.10,

$$f(z) = \underbrace{\frac{1}{2\pi i}\int_{S_{\rho_2}^+(z_0)} \frac{f(w)}{w-z}dw}_{\text{regular part}} - \underbrace{\frac{1}{2\pi i}\int_{S_{\rho_1}^+(z_0)} \frac{f(w)}{w-z}dw}_{\text{singular part}}.$$

All we need to do now is to work out the two integrals as power series and calculate the coefficients.

As in the proof of the Taylor Series Theorem we obtain

$$\frac{1}{2\pi i}\int_{S_{\rho_2}^+(z_0)} \frac{f(w)}{w-z}dw = \sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i}\int_{S_{\rho_2}^+(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}dw.$$

The treatment of the second integral is similar. Using the geometric sum (5.4) with $q := \frac{w-z_0}{z-z_0}$ we obtain

$$\begin{aligned} -\frac{1}{w-z} &= \frac{1}{z-z_0}\frac{1}{1-\frac{w-z_0}{z-z_0}} \\ &= \frac{1}{z-z_0}\Big\{1 + \tfrac{w-z_0}{z-z_0} + \cdots + \tfrac{(w-z_0)^{n-1}}{(z-z_0)^{n-1}} + \tfrac{(w-z_0)^n}{(z-z_0)^{n-1}(z-w)}\Big\}. \end{aligned}$$

Therefore

$$\begin{aligned}
-\frac{1}{2\pi i}&\int_{S^+_{\rho_1}(z_0)}\frac{f(w)}{w-z}dw\\
&=\frac{1}{2\pi i}\int_{S^+_{\rho_1}(z_0)}f(w)\Big\{\tfrac{1}{z-z_0}+\tfrac{w-z_0}{(z-z_0)^2}+\cdots+\tfrac{(w-z_0)^{n-1}}{(z-z_0)^n}+\tfrac{(w-z_0)^n}{(z-z_0)^n(z-w)}\Big\}dw\\
&=\sum_{m=1}^{n}b_m(z-z_0)^{-m}+B_n,
\end{aligned}$$

where

$$b_m=\frac{1}{2\pi i}\int_{S^+_{\rho_1}(z_0)}f(w)(w-z_0)^{m-1}dw \quad\text{and}\quad B_n=\frac{1}{2\pi i}\int_{S^+_{\rho_1}(z_0)}f(w)\frac{(w-z_0)^n}{(z-z_0)^n(z-w)}dw.$$

We are going to show that $\lim_{n\to\infty}B_n=0$.

Indeed, f is analytic in $A_{r,R}(z_0)$ and hence bounded on $S_{\rho_1}(z_0)$ (because $S_{\rho_1}(z_0)$ is compact). This means that there exists $M>0$ such that

$$|f(w)|\le M \qquad (w\in S_{\rho_1}(z_0)).$$

Set $\delta:=|z-z_0|-\rho_1>0$. Now $|w-z_0|=\rho_1$, $|z-z_0|=\rho_1+\delta$ and $|z-w|\ge||z-z_0|-|w-z_0||=\delta$. Then by the Estimation Lemma

$$|B_n|\le\frac{1}{2\pi}\frac{M\rho_1^n}{(\rho_1+\delta)^n\delta}\underbrace{L(S_{\rho_1}(z_0))}_{2\pi\rho_1}=\frac{M\rho_1}{\delta}\left(\frac{\rho_1}{\rho_1+\delta}\right)^n.$$

We conclude that $\lim_{n\to\infty}B_n=0$.

Therefore

$$-\frac{1}{2\pi i}\int_{S^+_{\rho_1}(z_0)}\frac{f(w)}{w-z}dw=\sum_{m=1}^{\infty}b_m(z-z_0)^{-m}$$

and

$$f(z)=\underbrace{\sum_{n=0}^{\infty}a_n(z-z_0)^n}_{\text{regular part}}+\underbrace{\sum_{n=1}^{\infty}b_n(z-z_0)^n}_{\text{singular part}} \qquad (z\in A_{r,R}(z_0)),$$

which the required Laurent expansion. To finish the proof we observe that if $\rho\in(r,R)$ then $S^+_\rho(z_0)\sim S^+_{\rho_1}(z_0)\sim S^+_{\rho_2}(z_0)$ in $A_{r,R}(z_0)$. Thus the integrals in the expressions for a_n and b_n do not depend on the particular choice of $\rho\in(r,R)$. □

7 Isolated singularities

Let $f : A_{0,R}(z_0) \to \mathbb{C}$ be an analytic function. We say that z_0 is an *isolated singularity* of f if f is not analytic (or is not defined) at z_0. For example, $f(z) = \frac{1}{z}$ has an isolated singularity at $z_0 = 0$.

7.1 Classification of isolated singularities

Observe that if z_0 is an isolated singularity of a function f then f has a Laurent series expansion

$$f(z) = \underbrace{\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}}_{\text{singular part}} + \underbrace{\sum_{m=0}^{\infty} a_m (z-z_0)^m}_{\text{regular part}} \qquad (z \in A_{0,R}(z_0)).$$

We classify isolated singularities according to the number of nonzero coefficients in the singular part of the Laurent expansion of f at z_0. Three different situations are possible.

Removable singularity. We say that z_0 is a *removable singularity* of f if for all $n > 0$ one has $a_{-n} = 0$. In this case the Laurent expansion of f at z_0 consists only of the regular part, e.g.

$$f(z) = \underbrace{\sum_{m=0}^{\infty} a_m (z-z_0)^m}_{\text{Taylor series}} \qquad (z \in A_{0,R}(z_0)).$$

Then the singularity at z_0 can be *removed*, by defining $f(z_0) := a_0$, and we obtain a function which is analytic on $B_R(z_0)$. For example, consider

$$f(z) = \frac{\sin(z)}{z} \qquad (z \neq 0).$$

Clearly f is analytic on $A_{0,\infty}(0)$ and z_0 is an isolated singularity of f, because f is not defined at $z_0 = 0$. The Laurent expansion of f at $z_0 = 0$ has the form

$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots \qquad (z \in A_{0,\infty}(0)).$$

So z_0 is a removable singularity. By defining $f(0) := 1$ we get a function which is analytic on $\mathbb{C}$.

Pole. We say that z_0 is a *pole* [4] *of order* k of f if there exists $k \in \mathbb{N}$ such that $a_{-k} \neq 0$ and for all $n > k$ one has $a_{-n} = 0$. In this case the Laurent expansion of f at z_0 has the form

$$f(z) = \underbrace{\sum_{n=1}^{k} \frac{a_{-n}}{(z-z_0)^n}}_{\text{"finite" singular part}} + \underbrace{\sum_{m=0}^{\infty} a_m (z-z_0)^m}_{\text{regular part}} \qquad (z \in A_{0,R}(z_0)).$$

For example, consider

$$f(z) = \frac{\sin(z)}{z^4} \qquad (z \neq 0).$$

[4] Sometimes a pole of order 1 is called a *simple* pole.

Clearly f is analytic on $A_{0,\infty}(0)$ and z_0 is an isolated singularity of f. The Laurent expansion of f at $z_0 = 0$ has the form

$$f(z) = \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{z}{5!} - \frac{z^3}{7!} + \dots \qquad (z \in A_{0,\infty}(0)).$$

So z_0 is a pole of order 3.

Essential singularity. We say that z_0 is an *essential singularity* of f if for any $k \in \mathbb{N}$ there exists $n > k$ such that $a_{-n} \neq 0$. In this case the Laurent expansion of f at z_0 has the form

$$f(z) = \underbrace{\sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n}}_{\text{"infinite" singular part}} + \underbrace{\sum_{m=0}^{\infty} a_m (z - z_0)^m}_{\text{regular part}} \qquad (z \in A_{0,R}(z_0)).$$

For example, consider

$$f(z) = \sin(\frac{1}{z}) \qquad (z \neq 0).$$

Clearly f is analytic on $A_{0,\infty}(0)$ and z_0 is an isolated singularity of f. The Laurent expansion of f at $z_0 = 0$ has the form

$$f(z) = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \frac{1}{7!}\frac{1}{z^7} + \dots \qquad (z \in A_{0,\infty}(0)).$$

So z_0 is an essential singularity.

7.2 Local behavior of analytic functions

Removable singularities are simple because they always can be removed. The next proposition allows to recognize removable singularities.

Proposition 7.1. *Let $f : A_{0,R}(z_0) \to \mathbb{C}$ be an analytic function. Assume that z_0 is an isolated singularity of f. Then the following are equivalent:*

(a) z_0 is a removable singularity;

(b) $\lim_{z \to z_0} f(z)$ exists and finite;

(c) $|f(z)|$ is bounded on $A_{0,r}(z_0)$ for every $r \in (0, R)$.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are obvious.

$(c) \Rightarrow (a)$. Let $r \in (0, R)$. Suppose that $|f(z)|$ is bounded on $A_{0,r}(z_0)$, e.g. there exists $M > 0$ such that

$$|f(z)| \leq M \qquad (z \in A_{0,r}(z_0)).$$

Consider the Laurent expansion

$$f(z) = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n} + \sum_{m=0}^{\infty} a_m (z - z_0)^m \qquad (z \in A_{0,r}(z_0)).$$

We are going to prove that $a_{-n} = 0$ for all $n > 0$. Indeed, for $n > 0$ and $\rho \in (0, r)$ we have

$$a_{-n} = \frac{1}{2\pi i} \int_{S^+_\rho(z_0)} f(w)(w - z_0)^{n-1} dw.$$

By the Estimation Lemma

$$|a_{-n}| \leq \frac{1}{2\pi} M\rho^{n-1} L(S_\rho^+(z_0)) = M\rho^n.$$

If we let $\rho \to 0$ it follows that $|a_{-n}| = 0$. $\square$

Exercise 7.2. Let z_0 be a non removable isolated singularity of a function f. Prove that

$$\limsup_{z \to z_0} |f(z)| = +\infty.$$

There is a similar criterion for recognizing poles.

Proposition 7.3. *Let $f : A_{0,R}(z_0) \to \mathbb{C}$ be an analytic function. Assume that z_0 is an isolated singularity of f. Then the following are equivalent:*
(a) z_0 is a pole of order k;
(b) $\lim_{z\to z_0}(z - z_0)^k f(z) = w \neq 0$.

Proof. The implications $(a) \Rightarrow (b)$ is obvious.
$(b) \Rightarrow (a)$. Suppose $\lim_{z\to z_0}(z - z_0)^k f(z) = w \neq 0$. Then

$$g(z) := (z - z_0)^k f(z)$$

has a removable singularity at z_0 by Proposition 7.1. Set $g(z_0) := w$. Hence g is analytic on $B_R(z_0)$ and has a Taylor expansion

$$g(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m \qquad (z \in B_R(z_0)),$$

where $a_0 = w$. Therefore

$$f(z) = \sum_{n=1}^{k} \frac{a_{k-n}}{(z - z_0)^n} + \sum_{m=0}^{\infty} a_{k+m}(z - z_0)^m \qquad (z \in A_{0,R}(z_0))$$

where $a_0 \neq 0$, so f has a pole of order k at z_0. $\square$

Exercise 7.4. Let z_0 be a pole of a function f. Then

$$\lim_{z\to z_0} |f(z)| = +\infty.$$

Exercise 7.5. A *rational function* is one of the form

$$f(z) = \frac{q_m(z)}{p_n(z)},$$

where $q_m(z)$ and $p_n(z)$ are polynomials of orders m and n. Assume that $z_0 \in \mathbb{C}$ is a root of p_n of multiplicity $k > 0$, i.e. there exists a polynomial $p_{n-k}(z)$ of order $n - k$ such that

$$p_n(z_0) = (z - z_0)^k p_{n-k}(z).$$

Prove that if $q_m(z_0) \neq 0$ then z_0 is a pole of order k of the rational function $f(z)$.

Remark 7.6. A function $f : D \to \mathbb{C}$ is said to be *meromorphic* in the domain D, if f is analytic in D except for points at which f has poles. It follows from the previous exercise that rational functions are meromorphic in $\mathbb{C}$. An example of a non rational meromorphic function is $f(z) = \frac{1}{\sin(z)}$.

The behavior of a function f near a pole z_0 is relatively simple. In particular, we know that $\lim_{z \to z_0} |f(z)| = \infty$. However in the case of an essential singularity $|f(z)|$ will not have a limit as $z \to z_0$. The following result is classical.

Theorem 7.7. (CASORATI–WEIERSTRASS THEOREM) *Let $f : A_{0,R}(z_0) \to \mathbb{C}$ be an analytic function. Assume that z_0 is an essential singularity of f. Then for any $w \in \mathbb{C}$ there exists a sequence $(z_n)_{n \in \mathbb{N}}$ such that*

$$\lim_{n \to \infty} z_n = z_0 \qquad \textit{and} \qquad \lim_{n \to \infty} f(z_n) = w.$$

Proof. Suppose the conclusion of the theorem is false. Then[5] there exist $w \in \mathbb{C}$, $r \in (0, R)$ and $\varepsilon > 0$ such that

$$|f(z) - w| \geq \varepsilon \qquad (z \in A_{0,r}(z_0)).$$

Let

$$g(z) := \frac{1}{f(z) - w}.$$

Then g is analytic on $A_{0,r}(z_0)$. Moreover,

$$|g(z)| \leq \frac{1}{\varepsilon} \qquad (z \in A_{0,r}(z_0)).$$

Hence z_0 is a removable singularity of function g by Proposition 7.1. Let

$$g(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m \qquad (z \in B_r(z_0))$$

be the Taylor expansion of g and a_k the first nonzero coefficient in this expansion. Then

$$\lim_{z \to z_0} (z - z_0)^k (f(z) - w) = \lim_{z \to z_0} \frac{(z - z_0)^k}{g(z)} = \frac{1}{a_k} \neq 0.$$

By Proposition 7.3 we conclude that z_0 is a pole of order k of the function $f(z) - w$. Observe that the Laurent series of $f(z) - w$ differs from the Laurent series of $f(z)$ only in the zero–order coefficient. So z_0 is a pole of order k of $f(z)$, which is a contradiction. □

In fact a much stronger result, known as Picard Theorem, is true. The proof requires techniques considerably beyond the reach of this course.

Theorem 7.8. (PICARD THEOREM) *Let $f : A_{0,R}(z_0) \to \mathbb{C}$ be an analytic function. Assume that z_0 is an essential singularity of f. Then in each neighborhood of z_0 the function f attains each complex value, with one possible exception, an infinite number of times.*

Proof. See CONWAY, pp.302–303. □

Remark 7.9. The exception mentioned in the theorem can really occur. For example, the function $f(z) = \exp(\frac{1}{z})$ misses the value 0 but attains all others. In other words, for any $\varepsilon > 0$ one has $f(A_{0,\varepsilon}(0)) = \mathbb{C} \setminus \{0\}$. On the other hand the functions $f(z) = \sin(\frac{1}{z})$ attains every value, that is for any $\varepsilon > 0$ one has $f(A_{0,\varepsilon}(0)) = \mathbb{C}$, as can easily be verified.

[5]Prove the following statement as an exercise

7.3 Singularities at infinity

Let $f : A_{R,\infty}(0) \to \mathbb{C}$ be an analytic function. Define

$$\tilde{f}(z) := f(z^{-1}).$$

Thus the function $\tilde{f} : A_{0,\frac{1}{R}}(0) \to \mathbb{C}$ is analytic and has an isolated singularity at $z_0 = 0$. We classify the behavior of f at infinity according to that of $\tilde{f}$ at zero.

Removable singularity at infinity. We say that f has a *removable singularity at infinity* if $\tilde{f}$ has a removable singularity at zero. For example, the function $f(z) = z^{-m}$ has a removable singularity at infinity for any $m \in \mathbb{N} \cup \{0\}$.

Exercise 7.10. Let $f : A_{R,\infty}(0) \to \mathbb{C}$ be an analytic function. Prove that the following statements are equivalent:
(a) f has a removable singularity at infinity;
(b) $\lim_{z\to\infty} f(z)$ exists and finite;
(c) $|f(z)|$ is bounded on $\bar{A}_{r,\infty}(0)$ for every $r \in (R, \infty)$.

Pole. We say that f has a *pole of order k at infinity* if $\tilde{f}$ has a pole of order k at zero. For example, a polynomial of degree $k > 1$ has a pole of order k at infinity.

Exercise 7.11. Let $f : A_{R,\infty}(0) \to \mathbb{C}$ be an analytic function. Prove that the following statements are equivalent:
(a) f has a pole of order k at infinity;
(b) $\lim_{z\to\infty} \frac{f(z)}{z^k} = w \neq 0$.

Essential singularity. We say that f has an *essential singularity at infinity* if $\tilde{f}$ has an essential singularity at zero. For example, $\exp(z)$, $\sin(z)$, $\cos(z)$ have essential singularities at infinity.

Exercise 7.12. State and prove a version of the Casorati–Weierstrass Theorem "at infinity".

Extended complex plane. Sometimes it is convenient to extend the complex plane by adjoining to $\mathbb{C}$ a single "point" ∞, much as the real line can be extended by adjoining $+\infty$ and $-\infty$. Denote by $\mathbb{C}^*$ the set $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$. We call $\mathbb{C}^*$ the *extended complex plane.* We say that a sequence $(z_n)_{n\in\mathbb{N}}$ *converges to* ∞ in $\mathbb{C}^*$ if

$$(\forall R > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[(n \geq N) \Rightarrow (|z_n| > R)].$$

With such definitions, infinity may be thought as an "isolated singularity" of a complex function in a very natural way.

The points of $\mathbb{C}^*$ could be identified with the points of the unit sphere S^2 in $\mathbb{R}^3$ via the *stereographic projection.* Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. Let $z = e^{i\varphi}$. Then the map σ, defined by

$$\infty \xrightarrow{\sigma} (0, 0, 1) = N \in \mathbb{R}^3, \qquad \mathbb{C} \ni z = re^{i\varphi} \xrightarrow{\sigma} \left(\frac{2r\cos(\varphi)}{r^2+1}, \frac{2r\sin(\varphi)}{r^2+1}, \frac{r^2-1}{r^2+1}\right) \in S^2 \subset \mathbb{R}^3,$$

where N denotes "the northern pole" of the sphere, establishes a one–to–one correspondence between $\mathbb{C}^*$ and S^2. Observe, that the correspondence is continuous in the sense that if the sequence $(z_n)_{n\in\mathbb{N}} \in \mathbb{C}^*$ converges to a point $z \in \mathbb{C}^*$ then the sequence $(\sigma(z_n))_{n\in\mathbb{N}} \in S^2$ of the corresponding points on the sphere converges to the point $\sigma(z) \in S^2$. Because of this correspondence, which was introduced by Riemann, the extended complex plane sometimes is called also the *Riemann sphere.*

8 Winding number of a curve

In Example 2.12 we found that

$$\frac{1}{2\pi i}\int_{S_\rho^+(a)}\frac{1}{z-a}dz = 1.$$

If $S_\rho^-(a) := -S_\rho^+(a)$ is the circle traversed in the negative (clockwise) direction then obviously

$$\frac{1}{2\pi i}\int_{S_\rho^-(a)}\frac{1}{z-a}dz = -1.$$

If $\gamma_n(t) = a + \rho e^{it}$ $(0 \le t \le 2\pi n)$ then $\gamma_n = S_\rho(a) + \cdots + S_\rho(a)$ winds n times around the point a and we find that

$$\frac{1}{2\pi i}\int_{\gamma_n}\frac{1}{z-a}dz = n.$$

Now suppose that $\tilde\gamma : [0, 2\pi n] \to \mathbb{C}\setminus\{a\}$ is a closed contour that homotopic to γ_n in $\mathbb{C}\setminus\{a\}$. Then by the Deformation Theorem again

$$\frac{1}{2\pi i}\int_{\tilde\gamma}\frac{1}{z-a}dz = n.$$

Exercise 8.1. Prove that

$$\frac{1}{2\pi i}\int_{S_\rho^+(a)}\frac{1}{z-z_0}dz = \begin{cases} 1, & \text{if } z_0 \text{ lies } \textit{inside } S_\rho^+(a), \text{ i.e. } |z_0 - a| < \rho, \\ 0, & \text{if } z_0 \text{ lies } \textit{outside } S_\rho^+(a), \text{ i.e. } |z_0 - a| > \rho. \end{cases}$$

Hint. Use Deformation Theorem and Exercise 3.13 when z_0 is inside $S_\rho^+(a)$. Use Cauchy Theorem when z_0 is outside $S_\rho^+(a)$.

These observations lead to the following definition.

Definition 8.2. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed contour and $z_0 \notin \gamma^*$. The the *winding number of* γ *with respect to* z_0 (also called the *index* of γ with respect to z_0) is defined by

$$W(\gamma, z_0) := \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}dz.$$

Geometrically, this means that the contour γ winds $W(\gamma, z_0)$ times around the point z_0. The definition of the winding number would be improper if the following result were not true.

Theorem 8.3. *Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed contour and $z_0 \notin \gamma^*$. Then $W(\gamma, z_0) \in \mathbb{Z}$.*

Proof. Assume for simplicity that γ is a smooth contour. Since $z_0 \notin \gamma^*$ the function θ : $[\alpha, \beta] \to \mathbb{C}$,

$$\theta(t) = \int_\alpha^t \frac{\gamma'(s)}{\gamma(s) - z_0}ds$$

is well defined, continuous and differentiable. Moreover, $\theta(\alpha) = 0$, $\theta(\beta) = 2\pi i W(\gamma, z_0)$ and

$$\theta'(t) = \frac{\gamma'(t)}{\gamma(t) - z_0}.$$

Rewrite this as

$$\gamma'(t) = \theta'(t)(\gamma(t) - z_0).$$

Hence

$$\frac{d}{dt}\left\{(\gamma(t) - z_0)e^{-\theta(t)}\right\} = \gamma'(t)e^{-\theta(t)} - \theta'(t)e^{-\theta(t)}(\gamma(t) - z_0) = 0.$$

Therefore $(\gamma(t) - z_0)e^{-\theta(t)}$ is constant on $[\alpha, \beta]$. In particular,

$$(\gamma(\alpha) - z_0)e^{-\theta(\alpha)} = (\gamma(\beta) - z_0)e^{-\theta(\beta)}.$$

Since $\gamma(\alpha) = \gamma(\beta) \neq z_0$, we have $e^{-\theta(\alpha)} = e^{-\theta(\beta)}$. By the periodicity property of the complex exponential function we conclude that $\theta(\beta)$ is an integer multiple of $2\pi i$, as required. □

Exercise 8.4. For each given $n \in \mathbb{Z}$, construct a closed contour γ_n such that $W(\gamma, 0) = n$.

The following summarizes the most important properties of the winding number.

Exercise 8.5. Prove the following properties of the winding number:

(a) $W(-\gamma, z_0) = -W(\gamma, z_0)$;

(b) $W(\gamma_1 + \gamma_2, z_0) = W(\gamma_1, z_0) + W(\gamma_2, z_0)$;

(c) if $\gamma_1 \sim \gamma_2$ in $\mathbb{C} \setminus \{z_0\}$ then $W(\gamma_1, z_0) = W(\gamma_2, z_0)$.

Hint. (a) and (b) follows from Exercise 2.13, (c) follows from the Deformation Theorem.

Remark 8.6. In fact, the converse to the property (c) above is valid. Namely, if $W(\gamma_1, z_0) = W(\gamma_2, z_0)$ then $\gamma_1 \sim \gamma_2$ in $\mathbb{C} \setminus \{z_0\}$ (for the proof of this deep topological result, see CONWAY, p.90 and p.252). So the winding number completely describes the classes of equivalence of contours which are homotopic in $\mathbb{C} \setminus \{z_0\}$.

Inside and outside of a closed contour. By definition, a closed paths $\gamma : [\alpha, \beta] \to \mathbb{C}$ is a continuous map of the segment $[\alpha, \beta]$ into $\mathbb{C}$. Hence the trace $\gamma^* \subset \mathbb{C}$ is a compact subset of $\mathbb{C}$. In particular, γ^* is a bounded subset of $\mathbb{C}$, that is there exists $R > 0$ such that $\gamma^* \subset B_R(0)$.

Exercise 8.7. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed contour such that $\gamma_* \subset B_R(0)$ and $z_0 \notin B_R(0)$. Prove that $W(\gamma, z_0) = 0$.

Hint. Observe that the function $f(z) = \frac{1}{z - z_0}$ is analytic in $B_R(0)$. Then apply Cauchy's Theorem.

Consider the open set $G = \mathbb{C} \setminus \gamma^*$. Clearly $\mathbb{C} \setminus B_R(0) \subset G$. Thus G has one, and only one unbounded component, say G_∞. Geometrically, it is natural to say that G_∞ is the outside of the path γ. However, Exercise 8.7 suggests the following analytical definition of outside and inside of contours.

Definition 8.8. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed contour. The *outside* of the contour γ is defined by $\{z \in \mathbb{C} \setminus \gamma^* : W(\gamma, z) = 0\}$. The *inside* of γ is defined, accordingly, by $\{z \in \mathbb{C} \setminus \gamma^* : W(\gamma, z) \neq 0\}$.

We say that a closed path $\gamma : [\alpha, \beta] \to \mathbb{C}$ is a *simple contour* if γ is an injection. Geometrically, this means that γ has no self–intersections. The proof of the following fundamental topological result is far beyond the scope of this course.

Theorem 8.9. (JORDAN CURVE THEOREM) *Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a simple closed contour. Then the inside of γ is a simply connected domain and for any points z_1 and z_2 inside of γ one has $W(\gamma, z_1) = W(\gamma, z_2) = \pm 1$.*

In particular, the Jordan Curve Theorem implies that if $D \subset \mathbb{C}$ is a bounded domain and its boundary ∂D is a simple contour, then D is a simply connected domain.

9 Residue Theorem

Let z_0 be an isolated singularity of a function f. By the Laurent Series Theorem f has a Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad (z \in A_{0,R}(z_0)). \tag{9.1}$$

Definition 9.1. The *residue* of a function f at an isolated singularity z_0 is the coefficient a_{-1} of the Laurent expansion (9.1) of f about z_0. This is denoted by

$$\mathrm{Res}(f, z_0) := a_{-1}.$$

Remark 9.2. By the formula (6.3) for the Laurent coefficients we know that

$$a_{-1} = \frac{1}{2\pi i} \int_{S_\rho^+(z_0)} f(w)dw \qquad (\rho \in (0, R)).$$

Thus if we know the Laurent coefficients we can evaluate certain integrals and vice versa. We will see later that in some cases, however, one can compute the residue of a function without having to find the full Laurent expansion or computing integrals.

9.1 Residue Theorem

The Residue Theorem, which is proved in this section, is one of the main results of Complex Analysis. It includes Cauchy's Theorem and Cauchy's Integral Formula as special cases and leads quickly to important applications. In particular, it becomes one of the most powerful tools of Analysis for evaluation of definite integrals.

Theorem 9.3. (Residue theorem) *Let $D \subseteq \mathbb{C}$ be a simply connected domain, and $\mathcal{S} = \{z_1, z_2, \cdots, z_m\} \subset D$ be a finite subset of D. Let $f : D \setminus \mathcal{S} \to \mathbb{C}$ be an analytic function, and $\gamma : [\alpha, \beta] \to D \setminus \mathcal{S}$ be a closed contour. Then*

$$\int_\gamma f = 2\pi i \sum_{z_k \in \mathcal{S}} W(\gamma, z_k)\mathrm{Res}(f, z_k). \tag{9.2}$$

Proof. Since $z_k \in \mathcal{S}$ is an isolated singularity of f, we can write the Laurent expansion at z_k of the form

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z - z_k)^n}_{\text{regular part}} + \underbrace{\sum_{n=1}^{\infty} a_{-n} (z - z_k)^{-n}}_{\text{singular part } S_k(z)} \qquad (z \in A_{0,R_k}(z_k)),$$

for some $R_k > 0$ such that $A_{0,R_k}(z_k) \subset D \setminus \mathcal{S}$. Recall from the proof of the Laurent Series Theorem that the singular part of this expansion converges on $\mathbb{C} \setminus \{z_k\}$. Hence the function

$$S_k(z) := \sum_{n=1}^{\infty} a_{-n} (z - z_k)^{-n} \qquad (z \in A_{0,\infty}(z_k)),$$

is analytic on $\mathbb{C} \setminus \{z_k\}$. It follows that the function

$$g(z) := f(z) - \sum_{z_k \in \mathcal{S}} S_k(z).$$

is analytic on $D \setminus \mathcal{S}$.

Observe that each $z_k \in \mathcal{S}$ is a removable singularity of g. Indeed, for a given k we obtain

$$g(z) = \{f(z) - S_k(z)\} - \sum_{\substack{z_l \in \mathcal{S} \\ l \neq k}} S_l(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z - z_k)^n}_{f(z) - S_k(z)} - \underbrace{\sum_{\substack{z_l \in \mathcal{S} \\ l \neq k}} S_l(z)}_{\text{analytic at } z_k} \qquad (z \in A_{0,R_k}(z_k)).$$

So singularities at $\mathcal{S}$ can be removed, that is g can be defined at points of $\mathcal{S}$ in such a way that g is analytic on D. Then we may apply the Cauchy Theorem to g to obtain

$$0 = \int_\gamma g = \int_\gamma f - \sum_{z_k \in \mathcal{S}} \int_\gamma S_k(z). \tag{9.3}$$

Next we evaluate the integrals $\int_\gamma S_k$. We have

$$\int_\gamma S_k(z) dz = \int_\gamma \left\{ \sum_{n=1}^{\infty} a_{-n} (z - z_k)^{-n} \right\} dz.$$

One can verify[6] that in order to evaluate the last integral we may integrate term by term, so

$$\int_\gamma S_k(z) dz = \sum_{n=1}^{\infty} \left\{ a_{-n} \int_\gamma (z - z_k)^{-n} dz \right\}.$$

If $n > 1$ then the function $(z - z_k)^{-n}$ has an antiderivative $\frac{(z-z_k)^{1-n}}{1-n}$ in $A_{0,\infty}(z_k)$. By the Fundamental Theorem we conclude that

$$\int_\gamma (z - z_k)^{-n} dz = 0 \qquad (n > 1).$$

When $n = 1$, by definition of the winding number

$$\int_\gamma (z - z_k)^{-1} dz = 2\pi i \, W(\gamma, z_k).$$

Hence $\int_\gamma S_k = a_{-1} \, 2\pi i \, W(\gamma, z_k)$. By (9.3) we conclude that

$$\int_\gamma f = 2\pi i \sum_{z_k \in \mathcal{S}} W(\gamma, z_k) \operatorname{Res}(f, z_k),$$

as required. □

[6] See Problem Sheet 6

The Residue Theorem includes Cauchy's Theorem as a special case if we assume that $\mathcal{S} = \emptyset$ (note however that the proof of the Residue Theorem relies on Cauchy's theorem). An interesting application of the Residue Theorem is the following extension of the Cauchy Integral Formula to a general closed contour (compare with the Cauchy Integral Formula (4.1)).

Corollary 9.4. (Generalized Cauchy Integral Formula) *Let $D \subseteq \mathbb{C}$ be a simply connected domain. Let $f : D \to \mathbb{C}$ be an analytic function, $z_0 \in D$ and $\gamma : [\alpha, \beta] \to D \setminus \{z_0\}$ a closed contour. Then*

$$W(\gamma, z_0)\, f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z_0} dw. \tag{9.4}$$

Proof. Apply Residue's Theorem to $g(z) := \frac{f(z)}{z-z_0}$. □

Remark 9.5. One can formulate the Residue Theorem without mentioning the winding number as follows. *Let $D \subseteq \mathbb{C}$ be a domain, $\gamma : [\alpha, \beta] \to D$ a simple* positively oriented *closed contour and $\mathcal{S} = \{z_1, z_1, \cdots, z_m\}$ a finite subset* inside *γ. Let $f : D \setminus \mathcal{S} \to \mathbb{C}$ be an analytic function. Then*

$$\int_\gamma f = 2\pi i \sum_{z_k \in \mathcal{S}} \operatorname{Res}(f, z_k).$$

However the notions *positively oriented* and *inside* assume implicitly the use of the winding numbers and/or Jordan Curve Theorem (see Theorem 8.9, which we did not prove in this course).

9.2 Calculating residues

It is important that in many cases residues can be calculated without finding the full Laurent expansion of a function.

Example 9.6. Find the residue of $f(z) = \frac{\sin(z)}{z^2}$ at $z_0 = 0$.

Solution. We have

$$\frac{\sin(z)}{z^2} = \frac{1}{z^2} \underbrace{\left(z - \frac{z^3}{3!} + \dots \right)}_{\sin(z)} = \frac{1}{z} + \dots.$$

Hence $\operatorname{Res}(f, 0) = 1$.

In general, if we are given a function f with an isolated singularity at z_0 then we proceed in the following way. First we decide whether we can find easily the first few terms of the Laurent expansion in $A_{0,R}(z_0)$. If so, the residue of f at z_0 will be the coefficient a_{-1} in the expansion. If not some other methods or rules can be used, according to the type of singularity. Consider several examples.

Removable singularity. If z_0 is a removable singularity of a function f, then the Laurent expansion of f about z_0 has no singular part. Hence $\operatorname{Res}(f, z_0) = 0$, so removable singularities do not contribute to the value of integral in (9.2).

Pole. In the case of a simple pole one can develop easy criteria for calculating residues.

Lemma 9.7. *Let z_0 be a pole of order one of a function f. Then*

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Proof. If z_0 is a pole of order one of f then

$$f(z) = \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (z \in A_{0,R}(z_0)),$$

for some $R > 0$. Hence

$$\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} \left\{ a_{-1} + \sum_{n=0}^{\infty} a_n (z - z_0)^{n+1} \right\} = a_{-1},$$

as required. □

Lemma 9.7 can be easily extended to the case of higher order poles.

Lemma 9.8. *Let z_0 be a pole of order k of a function f. Set $\varphi(z) := (z - z_0)^k f(z)$. Then*

$$\mathrm{Res}(f, z_0) = \frac{1}{(k-1)!} \lim_{z \to z_0} \varphi^{(k-1)}(z).$$

Proof. If z_0 is a pole of order k of f then

$$f(z) = \frac{a_{-k}}{(z - z_0)^k} + \frac{a_{-k+1}}{(z - z_0)^{k-1}} + \cdots + \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (z \in A_{0,R}(z_0)),$$

for some $R > 0$. Hence

$$\varphi(z) = a_{-k} + a_{-k+1}(z - z_0) + \cdots + a_{-1}(z - z_0)^{k-1} + \sum_{n=0}^{\infty} a_n (z - z_0)^{k+n} \qquad (z \in A_{0,R}(z_0)).$$

Therefore

$$\varphi^{(k-1)}(z) = (k-1)! a_{-1} + \sum_{n=0}^{\infty} a_n \frac{(k+n)!}{(1+n)!} (z - z_0)^{1+n} \qquad (z \in A_{0,R}(z_0))$$

and the result follows on taking limits. □

Example 9.9. Find the residues of $f(z) = \frac{z^2}{(z-1)^3(z+1)}$ at $z_0 = \pm 1$.

Solution. Clearly $z_0 = -1$ is a pole of order one. By Lemma 9.7 we compute

$$\mathrm{Res}(f, -1) = \lim_{z \to -1} (z + 1) f(z) = \lim_{z \to -1} \frac{z^2}{(z-1)^3} = -\frac{1}{8}.$$

We see also that $z_0 = 1$ is a pole of order $k = 3$. Thus

$$\varphi(z) := (z - z_0)^k f(z) = \frac{z^2}{z+1}, \qquad \varphi''(z) = \frac{2}{z+1} - \frac{4z}{(z+1)^2} + \frac{2z^2}{(z+1)^3}.$$

By Lemma 9.8 we compute

$$\mathrm{Res}(f, 1) = \frac{1}{2!} \lim_{z \to 1} \varphi''(z) = \frac{1}{8}.$$

Exercise 9.10. Let $D \subseteq \mathbb{C}$ be a simply connected domain. Let $f : D \to \mathbb{C}$ be an analytic function, $z_0 \in D$ and $\gamma : [\alpha, \beta] \to D \setminus \{z_0\}$ a closed contour. Prove that

$$W(\gamma, z_0) f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{n+1}} dw. \tag{9.5}$$

Compare this formula with (5.3).

Hint. Apply Lemma 9.8 to $g(z) = \frac{f(z)}{(z-z_0)^{n+1}}$.

Essential singularity. In the case of an essential singularity there are no simple rules like for poles, so we must rely on our ability to find the Laurent expansion.

Example 9.11. Find the residue of $f(z) = \exp(z + z^{-1})$ at $z_0 = 0$.

Solution. We have

$$f(z) = \exp(z + z^{-1}) = \exp(z)\exp(z^{-1}) = \left(1 + z + \frac{z^2}{2!} + \ldots\right)\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots\right).$$

Gathering terms involving $\frac{1}{z}$ we get

$$\frac{1}{z}\left\{1 + \frac{1}{2!} + \frac{1}{2!3!} + \frac{1}{3!4!} + \ldots\right\}.$$

Thus the residue is

$$\operatorname{Res}(f, 0) = 1 + \frac{1}{2!} + \frac{1}{2!3!} + \frac{1}{3!4!} + \ldots.$$

One can not sum the series explicitly.

9.3 Evaluating Integrals

We now illustrate the practical use of the Residue Theorem in evaluating some integrals.

Example 9.12. Evaluate

$$\int_{S_2(0)} \frac{1}{(z+1)(z+3)} dz.$$

Solution. The only pole of the integrand $f(z) = \frac{1}{(z+1)(z+3)}$ inside $S_1^+(0)$ is a simple pole at $z_1 = -1$. By Lemma 9.7, we obtain

$$\operatorname{Res}(f, z_1) = \lim_{z \to -1} (z+1) f(z) = \frac{1}{2}.$$

Further, it is geometrically clear that $W(S_2(0), -1) = 1$. Thus by the Residue Theorem,

$$\int_{S_2(0)} f(z)dz = 2\pi i\, W(S_2(0), -1)\operatorname{Res}(f, -1) = \pi i,$$

Example 9.13. Evaluate

$$\int_\gamma \frac{z^2}{(z-1)^3(z+1)} dz,$$

where γ is a square $[2, 2i] + [2i, -2] + [-2, -2i] + [-2i, 2]$.

Solution. The singularities of the integrand $f(z) = \frac{z^2}{(z-1)^3(z+1)}$ occur at ± 1. Thus, by the Residue Theorem

$$\int_\gamma f(z)dz = 2\pi i \left\{ W(\gamma,-1)\mathrm{Res}(f,-1) + W(\gamma,1)\mathrm{Res}(f,1) \right\}.$$

It is geometrically evident that $W(\gamma, \pm 1) = 1$. Using calculations from Example 9.9 we evaluate

$$\int_\gamma f(z)dz = 2\pi i \left\{ -\frac{1}{8} + \frac{1}{8} \right\} = 0.$$

It is remarkable that in some cases systematic methods for evaluating *real* integrals can be developed using the Residues Theorem of *complex* analysis. We consider only simplest results of this type. For other applications of the Residue Theorem in evaluation of real integrals see Stewart and Tall, Marsden, or any other textbook on Complex Analysis.

Integrals of type $\int_{\mathbb{R}} f(x)dx$. Consider integrals of the form

$$\int_{-\infty}^{+\infty} f(x)dx.$$

In what follows, $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ denotes the upper half–plane in $\mathbb{C}$, while $\bar{\mathbb{C}}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$ stands for the closed half–plane.

Theorem 9.14. *Let f be a complex function such that $f(\mathbb{R}) \subseteq \mathbb{R}$. Suppose that f is analytic in the closed upper half–plane $\bar{\mathbb{C}}_+$, except for a finite number of isolated singularities $\{z_1, \dots, z_m\} \subset \mathbb{C}_+$. Suppose also that there exist $R > 0$, $M > 0$ and $\epsilon > 0$ such that*

$$|f(z)| \leq \frac{M}{|z|^{1+\epsilon}} \qquad (|z| > R,\ z \in \bar{\mathbb{C}}_+). \tag{9.6}$$

Then

$$\int_{-\infty}^{+\infty} f(x)dx = 2\pi i \sum_{k=1}^{m} \mathrm{Res}(f, z_k). \tag{9.7}$$

Proof. Let $r > R$. Consider the contour $\gamma_r = \tilde{\gamma}_r + [-r, r]$, where $\tilde{\gamma}_r = re^{it}$ ($t \in [0, \pi]$) is the "upper" half–circle. Assumption (9.6) implies that all the singularities of f lie inside γ_r. It is clear also that $W(\gamma_r, z_k) = 1$, for each k. Hence

$$\int_{\gamma_r} f = 2\pi i \sum_{k=1}^{m} \mathrm{Res}(f, z_k)$$

by the Residue Theorem. But

$$\int_{\gamma_r} f(z)dz = \int_{[-r,r]} f(z)dz + \int_{\tilde{\gamma}_r} f(z)dz = \int_{-r}^{r} f(x)dx + \int_0^\pi f(re^{is})ire^{is}ds.$$

By (2.2) and (9.6) we obtain

$$\left| \int_0^\pi f(re^{is})ire^{is}ds \right| \leq r \int_0^\pi \underbrace{|f(re^{is})|}_{\leq M/r^{1+\epsilon}} \underbrace{|e^{is}|}_{=1} ds \leq \frac{\pi M}{r^\epsilon} \to 0 \quad \text{as } r \to \infty,$$

since $\epsilon > 0$. Therefore

$$\lim_{r\to\infty}\int_{-r}^{r} f(x)dx = \underbrace{\lim_{r\to\infty}\int_{\gamma_r} f(z)dz}_{=2\pi i\sum_{k=1}^{m}\mathrm{Res}(f,z_k)} - \underbrace{\lim_{r\to\infty}\int_0^{\pi} f(re^{is})ire^{is}ds}_{\to 0} = 2\pi i\sum_{k=1}^{m}\mathrm{Res}(f,z_k).$$

On the other hand, from Calculus we know that $f(x)$ is integrable on $\mathbb{R}$ and

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{r\to\infty}\int_{-r}^{r} f(x)dx,$$

because f is continuous on $\mathbb{R}$ and because of the condition (9.6). Thus (9.7) follows. □

Example 9.15. Evaluate

$$\int_{-\infty}^{+\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)} \qquad (a, b > 0,\ a \neq b).$$

Solution. The singularities of the integrand

$$f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$$

occur at $\pm ia$ and $\pm ib$. Let $\delta = \min\{a,b\}/2$. In particular, f is analytic in $\bar{\mathbb{C}}_+$, except for poles of order one $\{ia, ib\} \subset \mathbb{C}_+$. Obviously, f satisfies the assumption (9.6). So all the conditions of Theorem 9.14 are verified and hence

$$\int_{-\infty}^{+\infty} f(x)dx = 2\pi i\left\{\mathrm{Res}(f,ia) + \mathrm{Res}(f,ib)\right\}.$$

By Lemma 9.7 we compute

$$\mathrm{Res}(f,ia) = \lim_{z\to ia}\frac{z-ia}{(z^2+a^2)(z^2+b^2)} = \frac{1}{2ia(b^2-a^2)}.$$

In the similar way,

$$\mathrm{Res}(f,ib) = \lim_{z\to ib}\frac{z-ib}{(z^2+a^2)(z^2+b^2)} = \frac{1}{2ib(a^2-b^2)}.$$

Therefore

$$\int_{-\infty}^{+\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}.$$

Exercise 9.16. (Cauchy integral formula for the half–space) Let f be a complex function such that $f(\mathbb{R}) \subseteq \mathbb{R}$. Suppose that f is analytic in the closed upper half–plane $\bar{\mathbb{C}}_+$ and that there exist $M > 0$ and $\epsilon > 0$ such that

$$|f(z)| \leq \frac{M}{|z|^{\epsilon}} \qquad (z \in \mathbb{C}_+).$$

Prove the integral formula

$$f(z) = \int_{-\infty}^{+\infty}\frac{f(x)}{x-z}dx \qquad (z \in \mathbb{C}_+).$$

Hint. Use Theorem 9.14.

Trigonometric Integrals. Consider integrals of the form

$$\int_0^{2\pi} Q(\cos(\varphi), \sin(\varphi))\, d\varphi,$$

where $Q = Q(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is a *rational function*, that is

$$Q(x, y) = \frac{q_m(x, y)}{p_n(x, y)},$$

where $q_m(x, y)$ and $p_n(x, y)$ are real polynomials of orders m and n.

Assume that $p_n(x, y)$ has no roots on the unit circle $S_1(0)$. Then $Q(x, y)$ is continuous on the unit circle. The substitution $z = e^{i\varphi}$ may be used to convert such trigonometric integrals to those involving rational complex functions. Indeed, after the substitution $z = e^{i\varphi}$ we obtain

$$\cos(\varphi) = \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad \sin(\varphi) = \frac{1}{2i}\left(z - \frac{1}{z}\right) \qquad (z = e^{i\varphi},\ \varphi \in [0, 2\pi]).$$

Introduce the function

$$f(z) = \frac{Q\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)}{iz}. \tag{9.8}$$

Thus $f : \mathbb{C} \to \mathbb{C}$ is a rational function (cf. Exercise 7.5) that has no poles on $S_1(0)$ and

$$f(x + iy) = Q(x, y) \qquad (z = x + iy \in S_1(0)).$$

Then by the Residue Theorem we obtain the formula

$$\int_0^{2\pi} Q(\cos(\varphi), \sin(\varphi)) d\varphi = \int_0^{2\pi} f(e^{i\varphi}) i e^{i\varphi} d\varphi = \int_{S_1^+(0)} f(z) dz = 2\pi i \sum_{z_k \in \mathcal{S}} \operatorname{Res}(f, z_k), \tag{9.9}$$

where $\mathcal{S} = \{z_1, \ldots, z_m\}$ is the set of poles of f inside of the unit circle $S_1(0)$.

Example 9.17. For $a > 1$, evaluate

$$\int_0^{2\pi} \frac{d\varphi}{a + \sin(\varphi)}.$$

Solution. From (9.8) we obtain

$$f(z) = \frac{2}{z^2 + 2iaz - 1}.$$

The only pole inside $S_1^+(0)$ is a pole of order one at $z_1 = -ia + i\sqrt{a^2 - 1}$. By Lemma 9.7, we obtain

$$\operatorname{Res}(f, z_1) = \lim_{z \to z_1} (z - z_1) f(z) = \frac{1}{i\sqrt{a^2 - 1}}.$$

Thus by (9.9) we conclude that

$$\int_0^{2\pi} \frac{d\varphi}{a + \sin(\varphi)} = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

10 Zeros. Unique Continuation

Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ analytic function. We say that a point $z_0 \in D$ is a *zero* of f iff $f(z_0) = 0$.

Classification of zeros. Let z_0 be a zero of an analytic function f. Expanding f in Taylor series about z_0 we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (z \in B_R(z_0))$$

where $B_R(z_0) \subset D$. Then $a_0 = f(z_0) = 0$ and two different possibility can occur:

i) all the coefficients a_n are zero, and then $f(z) \equiv 0$ in $B_R(z_0)$,

ii) there exists $m \in \mathbb{N}$ such that $a_0 = a_1 = \cdots = a_{m-1} = 0$, but $a_m \neq 0$.

In the case (ii) we say that z_0 is a zero of f of *order* m. Sometime zeros of order 1 are called *simple* zeros. For example, the function $f(z) = z^m$ has a zero of order m at $z_0 = 0$.

Exercise 10.1. Let $p_n(z)$ be a polynomial and z_0 be a root of p_n of multiplicity m (see Exercise 7.5). Prove that z_0 is a zero of p_n of order m.

Exercise 10.2. Prove that the following conditions are equivalent:

a) z_0 is a zero of a function f of order m;

b) $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$;

c) f can be represented as

$$f(z) = (z - z_0)^m g(z) \qquad (z \in B_R(z_0)),$$

where g is analytic in $B_R(z_0)$ and $g(z_0) \neq 0$.

d) z_0 is a pole of order m of the function $\frac{1}{f(z)}$.

Remark 10.3. Proposition 7.3 implies that if z_0 is a pole of order m of the function $g(z)$ then the function $\frac{1}{g(z)}$ is analytic at z_0, and z_0 is its zero of order m. However, if z_0 is an essential singularity of $g(z)$ then $\frac{1}{g(z)}$ can not be analytic at z_0. Moreover z_0 is an essential singularity of $\frac{1}{g(z)}$ too. This follows from the Casorati–Weierstrass Theorem (explain how).

Example 10.4. Find zeros and their orders of the function

$$f(z) = \frac{(z^2 + 1)^2}{z}.$$

Solution. It is clear that zeros of f occur at the roots of $(z^2 + i)^2 = 0$, that is at $z = \pm i$. To find the order of zeros we represent f as follows:

$$\frac{(z^2+1)^2}{z} = \begin{cases} (z-i)^2 g(z), & \text{where } g(z) = \frac{(z+i)^2}{z} \text{ is analytic at } z = i \text{ and } g(i) = 4i \neq 0, \\ (z+i)^2 g(z), & \text{where } g(z) = \frac{(z-i)^2}{z} \text{ is analytic at } z = -i \text{ and } g(-i) = 4i \neq 0. \end{cases}$$

So by Exercise 10.2 (c) we conclude that $z = i$ and $z = -i$ are both zeros of f of order 2.

Example 10.5. Find zeros and their orders of the function

$$f(z) = \cos(z).$$

Solution. It is clear that zeros of f occur at $z_n = (n + \frac{1}{2})\pi$, $(n \in \mathbb{Z})$. Observe that

$$\cos'(z_n) = -\sin(z_n) = (-1)^{n+1} \neq 0.$$

So by Exercise 10.2 (b) we conclude that $z_n = (n + \frac{1}{2})\pi$ are zeros of f of order 1.

Unique continuation. A remarkable property of analytic functions is that if two functions agree on a small portion of the domain then they agree on the whole domain on which they both analytic. We start with the following definition.

Definition 10.6. We say that a zero z_0 of an analytic function f is *isolated* iff there exists $r > 0$ such that $f(z) \neq 0$ on $A_{0,r}(z_0)$.

Lemma 10.7. *A zero of finite order of an analytic function is isolated.*

Proof. Write

$$f(z) = (z - z_0)^m g(z) \qquad (z \in B_R(z_0)),$$

where g is analytic in $B_R(z_0)$ and $g(z_0) \neq 0$. In particular, g is continuous at z_0. Set $\varepsilon = |g(z_0)|/2$. Hence there exists $r > 0$ such that for any $z \in B_r(z_0)$ one has $|g(z) - g(z_0)| < \varepsilon$. Therefore for $z \in B_r(z_0)$ we obtain

$$|g(z)| \geq \big| \underbrace{|g(z_0)|}_{=2\varepsilon} - \underbrace{|g(z) - g(z_0)|}_{<\varepsilon} \big| > \varepsilon.$$

In particular, $g(z) \neq 0$ on $B_r(z_0)$. But then $f(z) = (z - z_0)^m g(z) \neq 0$ on $A_{0,r}(z_0)$, since $(z - z_0)^m g(z) \neq 0$ on $A_{0,r}(z_0)$. ☐

Remark 10.8. Consider a real function $f(x) = x^2 \sin(\frac{1}{x})$. Obviously $z_0 = 0$ is a non isolated zero of f. Note that f is differentiable on $\mathbb{R}$!

Recall, that a point $z_0 \in \mathbb{C}$ is called a *limit point* of a set $S \subset D$ iff for any $r > 0$ one has $S \cap A_{0,r}(z_0) \neq \emptyset$. In other words, there exists a sequence $(z_n) \subset S$, such that $z_n \neq z_0$ and $\lim_{n\to\infty} z_n = z_0$.

Lemma 10.9. *Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ analytic function. Let $\mathcal{N} \subseteq D$ be a set of zeros of f. If $\mathcal{N}$ has a limit point $z_0 \in D$ then $f(z) \equiv 0$ in any ball $B_R(z_0) \subset D$.*

Proof. Let $(z_n) \subset \mathcal{N}$ be a sequence such that $\lim_{n\to\infty} z_n = z_0 \in D$. Since f is analytic and, in particular, continuous in D, we conclude that

$$f(z_0) = \lim_{n\to\infty} f(z_n) = 0.$$

Therefore z_0 is a zero of f, which is not isolated. Hence z_0 is a zero f of infinite order, that is

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n \qquad (z \in B_R(z_0)),$$

for any ball $B_R(z_0) \subset D$ with all coefficients $a_n = 0$. ☐

Theorem 10.10. (Identity Theorem) *Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ analytic function. Let $\mathcal{N} \subseteq D$ be a set of zeros of f. If $\mathcal{N}$ has a limit point in D then $f(z) \equiv 0$ in D.*

Proof. By Lemma 10.9 we know that $f(z) \equiv 0$ in any ball $B_R(z_0) \subset D$. For any other point $z \in D$ choose a contour $\gamma : [\alpha, \beta] \to D$ from z_0 to z. We are going to show that $f(\gamma(t)) = 0$ for all $t \in [\alpha, \beta]$.

By continuity of the map γ one can find $\delta > 0$ such that $\gamma(t) \in B_R(z_0)$ for each $t \in [\alpha, \alpha + \delta]$. Hence $f(\gamma(t)) = 0$ for all $t \in [\alpha, \alpha + \delta]$. Set

$$T := \sup\{\tau \in [\alpha, \beta] : f(\gamma(t)) = 0 \text{ for all } t \in [\alpha, \alpha + \tau]\}.$$

Clearly $\alpha + \delta \leq T \leq \beta$. We need to prove that $T = \beta$.

Note that $f(\gamma(T)) = 0$ by continuity of γ. Assume that $T < \beta$. Then $z_1 = \gamma(T)$ is a non isolated zero of f. By Lemma 10.9 we know that $f(z) \equiv 0$ in a ball $B_{R_1}(z_1) \subset D$. Hence, as before, we can find $\delta_1 > 0$ such that $f(\gamma(t)) = 0$ for all $t \in [\alpha, T + \delta_1]$. But this contradicts the definition of T. Thus we conclude that $T = \beta$. □

Corollary 10.11. (Unique Continuation Theorem) *Let $D \subset \mathbb{C}$ be a domain and $f, g : D \to \mathbb{C}$ analytic functions. Let $\mathcal{S} \subset D$ be a subset of D and suppose that*

$$f(z) = g(z) \quad \textit{for all} \quad z \in \mathcal{S}.$$

If $\mathcal{S}$ has an accumulation point in D then $f(z) \equiv g(z)$ in D.

Proof. Apply the Identity Theorem to the function $f - g$. □

Analytic continuation. Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ an analytic function. We say that f is an *extension* (or a *continuation*) of a function $g : \mathcal{S} \to \mathbb{C}$ if $\mathcal{S} \subset D$ and $f(z) = g(z)$ for all $z \in \mathcal{S}$. According to the Unique Continuation Theorem, if $\mathcal{S}$ has an accumulation point in D then there exists *at most one* analytic extension of g from $\mathcal{S}$ to D. An important application is that real functions $f : \mathbb{R} \to \mathbb{R}$ have at most one analytic extension. For example, the complex exponential function $(\exp, \mathbb{C})$ defined on $\mathbb{C}$ is the unique analytic extension the real exponential function $(\exp, \mathbb{R})$, defined on $\mathbb{R}$. Note however that not every real (differentiable) function admits an analytic extension.

Example 10.12. Consider $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x|x|$. Assume that $(g, \mathbb{R})$ has an analytic extension (f, D) defined on a domain $D \subseteq \mathbb{C}$. Then such extension must be unique and, in particular, it must agree with f on $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$. Clearly $f(z) = z^2$ is the unique analytic extension of $(g, \mathbb{R}_+)$ from $\mathbb{R}_+$ to $\mathbb{C}$. However $f(z) \neq g(z)$ for $x < 0$! Observe that g is (real) differentiable on $\mathbb{R}$ and $g'(x) = |x|$, but $g''(0)$ does not exist.

Another important consequence of the Unique Continuation Theorem is that any convergent power series defines uniquely an analytic function in the entire domain of analyticity.

Example 10.13. Consider $g : B_1(0) \to \mathbb{C}$ given by $g(z) = \sum_{n=0}^{\infty} z^n$. The function $f(z) = \frac{1}{1-z}$ on $D = \mathbb{C} \setminus \{1\}$ is the unique analytic extension of g. Observe that f is the maximal extension of g in a sense that f can not be extended analytically beyond $\mathbb{C} \setminus \{1\}$.

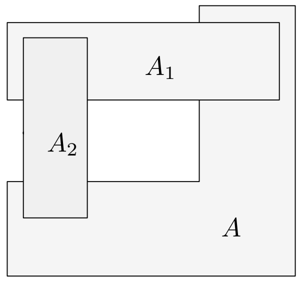

Figure 2: Analytic continuation.

The Unique Continuation Theorem provides a method for making the domain of an analytic function as large as possible, e.g., starting from a given real differentiable function, or starting from a convergent power series. However, the following phenomena can occur. Let a function $f : A \to \mathbb{C}$ be continued to a region A_1 and A_2, as pictured in Figure 2. If we continue f to be analytic on A_1 and then continue this new function from A_1 to A_2, the result need not agree on $A_2 \cap A$ with the original function on A.

Example 10.14. Recall, that the *principal branch of the logarithm* $\mathrm{Log} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ was defined (see 3.3) by

$$\mathrm{Log}(z) := \int_{[1,z]} \frac{1}{z} dz \qquad (z \in \mathbb{C} \setminus (-\infty, 0]). \tag{10.1}$$

Observe that such defined function $\mathrm{Log}(z)$ is the unique analytic extension to $\mathbb{C} \setminus (-\infty, 0]$ of the real logarithm $\log(x)$, or of the convergent power series

$$g(z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n} \qquad (z \in B_1(1)),$$

simply because $\mathrm{Log}(x) = \log(x) = g(x)$ $(x \in (0, 2))$.

However $\mathrm{Log}(z)$ can be extended analytically beyond $\mathbb{C} \setminus (-\infty, 0]$. Indeed, consider $\mathrm{Log} : A \to \mathbb{C}$, where $A = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\} \cup \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$ is the union of the right half plane and the lower half plane. Then Log can be continued from A uniquely so as to include the upper half plane $A_1 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Such extension, say $\mathrm{Log}_{(1)}$, can be defined, e.g. by the formula

$$\mathrm{Log}_{(1)}(z) := \mathrm{Log}(1 + i) + \int_{[1+i,z]} \frac{1}{z} dz \qquad (z \in A_1).$$

Similarly, we may continue the $\mathrm{Log}_{(1)}$ from the upper half plane A_1 so as to include the left half plane $A_2 = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$. But this extension, say $\mathrm{Log}_{(2)}$, does not agree with the "original" function $\mathrm{Log}(z)$ on $A_2 \cap A$, the values differ by $2\pi i$ (this can be verified by direct computation). So analytic continuation of Log turned out to be a "multiple valued" function.

Remark 10.15. To obtain a genuine (single–valued) function, we agree that Log always denotes the *principal value* of the logarithm, defined via (10.1) on $\mathbb{C} \setminus (-\infty, 0]$. Once Log is defined, the *principal value* of complex fractional powers z^a $(z \in \mathbb{C} \setminus (-\infty, 0], a \in \mathbb{C})$ can be defined via

$$z^a := \exp(a \, \mathrm{Log}(z)).$$

Obviously, if $a \in \mathbb{R}$ then $z^a : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ is the analytic extension of the real fractional power $x^a : \mathbb{R}_+ \to \mathbb{R}$.

Further investigation of *analytic continuation* leads naturally to the concept of *multivalued functions* and *Riemann surfaces* as maximal domains of multivalued analytic functions. We do not develop this subject in our course.

11 Principle of the Argument and applications

We now turn to an application of the Residue Theorem which gives information concerning the number of zeros and poles of a complex function. It is convenient to introduce the notation

$$\operatorname{ind}(f, z_0) = \begin{cases} m, & \text{if } f \text{ has a zero of order } m \ (m \in \mathbb{N}) \text{ at } z_0, \\ -m, & \text{if } f \text{ has a pole of order } m \ (m \in \mathbb{N}) \text{ at } z_0, \\ 0, & \text{if } f \text{ is analytic (or has a removable singularity) at } z_0, \text{ and } f(z_0) \neq 0. \end{cases}$$

We say that $\operatorname{ind}(f, z_0)$ is the *index* of f at z_0.

Theorem 11.1. (Principle of the Argument) *Let $D \subseteq \mathbb{C}$ be a simply connected domain and $\mathcal{S} = \{z_1, z_2, \cdots, z_m\} \subset D$. Let $f : D \setminus \mathcal{S} \to \mathbb{C}$ be an analytic function which has no zeros in $D \setminus \mathcal{S}$ and has a zero or a pole at each point of $\mathcal{S}$. Let $\gamma : [\alpha, \beta] \to D \setminus \mathcal{S}$ be a closed contour. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{z_k \in \mathcal{S}} W(\gamma, z_k) \operatorname{ind}(f, z_k). \tag{11.1}$$

Proof. By the Residue Theorem,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{z_k \in \mathcal{S}} W(\gamma, z_k) \operatorname{Res}(f'/f, z_k).$$

We are going to show that:
(a) If $\operatorname{ind}(f, z_k) = m > 0$ then f'/f has a pole of order 1 at z_k and $\operatorname{Res}(f'/f, z_k) = m$;
(b) If $\operatorname{ind}(f, z_k) = -m < 0$ then f'/f has a pole of order 1 at z_k and $\operatorname{Res}(f'/f, z_k) = -m$.

Case (a). To prove (a) note that by Exercise 10.2 (c)

$$f(z) = (z - z_k)^m g(z) \qquad (z \in B_R(z_k)),$$

where $g : B_R(z_k) \to \mathbb{C}$ is analytic and $g(z_k) \neq 0$. Therefore

$$f'(z) = m(z - z_k)^{m-1} g(z) + (z - z_k)^m g'(z),$$

and hence

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_k} + \frac{g'(z)}{g(z)}$$

has a simple pole with residue m at z_k, because g'/g is analytic at z_k.

Case (b). In this case, similarly as in Proposition 7.3,

$$f(z) = (z - z_k)^{-m} g(z) \qquad (z \in A_{0,R}(z_k)),$$

where $g : B_R(z_k) \to \mathbb{C}$ is analytic and $g(z_k) \neq 0$. Therefore

$$f'(z) = -m(z - z_k)^{-m-1} g(z) + (z - z_k)^{-m} g'(z),$$

and hence

$$\frac{f'(z)}{f(z)} = \frac{-m}{z - z_k} + \frac{g'(z)}{g(z)},$$

which has a simple pole with residue $-m$ at z_k, because g'/g is analytic at z_k. □

In what follows, for an analytic function f defined on a domain $D \subseteq \mathbb{C}$ we denote $\mathcal{N}_f := \{z \in D : f(z) = 0\}$. Observe that by Lemma 10.9 the set N_f has no limit points in D, except for the case $f \equiv 0$ in D.

Corollary 11.2. *Let $D \subseteq \mathbb{C}$ be a simply connected bounded domain and $f : D \to \mathbb{C}$ an analytic function. Let $\gamma : [\alpha, \beta] \to D$ be a closed contour such that $\gamma^* \cap \mathcal{N}_f = \emptyset$. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{z_k \in \mathcal{N}_f} W(\gamma, z_k)\mathrm{ind}(f, z_k). \tag{11.2}$$

Hint. The set of zeros of f inside γ^* is finite.

The theorem below is frequently used to locate zeros of a function by comparing it with another function whose zeros could be located more easily.

Theorem 11.3. (Rouché Theorem) *Let $D \subseteq \mathbb{C}$ be a simply connected bounded domain and $f, g : D \to \mathbb{C}$ analytic functions. Let $\gamma : [\alpha, \beta] \to D$ be a closed contour. Assume that*

$$|f(z) - g(z)| < |f(z)| \qquad (z \in \gamma^*). \tag{11.3}$$

Then

$$\sum_{z_k \in \mathcal{N}_f} W(\gamma, z_k)\mathrm{ind}(f, z_k) = \sum_{z_k \in \mathcal{N}_g} W(\gamma, z_k)\mathrm{ind}(g, z_k). \tag{11.4}$$

Remark 11.4. Condition (11.4) implies that neither f nor g can be zero on γ^*.

Proof. Let $F(z) := \frac{g(z)}{f(z)}$. Then from (11.3) we have

$$|1 - F(z)| < 1 \qquad (z \in \gamma^*). \tag{11.5}$$

Let $\Gamma(t) = F(\gamma(t))$. Clearly $\Gamma : [\alpha, \beta] \to \mathbb{C}$ is a closed contour. Moreover, $\Gamma^* \subset B_1(1)$ by (11.5). Hence $W(\Gamma, 0) = 0$ and therefore

$$\begin{aligned} \int_\gamma \frac{F'(z)}{F(z)}dz &= \int_\alpha^\beta \frac{F'(\gamma(t))}{F(\gamma(t))}\gamma'(t)dt = \int_\alpha^\beta \frac{\Gamma'(t)}{\Gamma(t)}dt \\ &= \int_\Gamma \frac{1}{z}dz = 2\pi i\, W(\Gamma, 0) = 0. \end{aligned}$$

But by direct computation

$$\frac{F'(z)}{F(z)} = \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)},$$

which implies (11.4). □

Example 11.5. Let $p(z) = z^8 - 5z^3 + z - 2$. Prove that $p(z)$ has exactly 3 roots (counted with multiplicities) inside the unit circle.

Solution. Let $q(z) = -5z^3$. For $|z| = 1$ it is immediate that

$$|q(z) - p(z)| = |-z^8 - z + 2| < |q(z)| = 5.$$

By Rouché's Theorem p and q have the same number of zeros (counted with multiplicities) inside the unit circle, and the equation $5z^3 = 0$ obviously has one zero with multiplicity 3 inside the unit circle. □

Exercise 11.6. (Fundamental Theorem of Algebra) Let p_n be a polynomial of degree $n \geq 1$. Prove that p_n has exactly n zeros (counted with multiplicities).

Hint. Let $q(z) = a_n z^n$ and estimate $|p_n(z) - q(z)|$ on $\mathcal{S}_R^+(0)$ for large enough $R > 0$.

12 Maximum Modulus Principle

The Maximum Modulus Principle states that if f is analytic in a domain D then $|f(z)|$ can have its maximum only the boundary of D. This property of analytic functions becomes crucial in the study of harmonic functions and boundary value problems. We start with a preliminary lemma.

Lemma 12.1. (Mean Value Property) *Let $f : B_R(z_0) \to \mathbb{C}$ be an analytic function. Then for any $r \in (0, R)$ one has*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\varphi})d\varphi. \tag{12.1}$$

Proof. By the Cauchy Integral Formula

$$f(z_0) = \frac{1}{2\pi i}\int_{S_r^+(z_0)} \frac{f(z)}{z-z_0}dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{i\varphi})}{re^{i\varphi}}rie^{i\varphi}d\varphi = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\varphi})d\varphi,$$

which is required. □

Remark 12.2. Note that the integral in (12.1) is the integral over the real segment. It has the meaning of the mean value of f on $S_r(z_0)$, so the Mean Value Property states that the value of f at the center of a ball $B_r(z_0)$ is the average its values on the spheres $S_r(z_0)$.

Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ an analytic function. We say that $z_0 \in D$ is a *local maximum* point of $|f|$ iff there exists $r > 0$ such that $|f(z)| \le |f(z_0)|$ for all $z \in B_r(z_0) \subset D$. We say that $z_0 \in D$ is a *local minimum* point of $|f|$ iff z_0 is a local maximum of $-|f|$.

Theorem 12.3. (Maximum Modulus Theorem) *Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ an analytic function. If $|f|$ has a local maximum point in D then f is constant in D.*

Proof. Assume that $z_0 \in D$ is a local maximum point of $|f|$. Choose $R > 0$ such that $B_R(z_0) \subset D$. By the Mean Value Property applied at $z_0 \in D$, for any $r \in (0, R)$ one has

$$|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\varphi})|d\varphi.$$

On the other hand

$$\frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(z_0+re^{i\varphi})|}_{\le |f(z_0)|}d\varphi \le |f(z_0)|,$$

since z_0 is a local maximum point of $|f|$. We conclude that

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\varphi})|d\varphi = |f(z_0)|.$$

Rewrite the last equality as

$$\frac{1}{2\pi}\int_0^{2\pi} \underbrace{\left\{|f(z_0)| - |f(z_0+re^{i\varphi})|\right\}}_{\ge 0,\text{ since } z_0 \text{ is a local maximum}}d\varphi = 0.$$

Therefore $|f(z_0)| - |f(z_0 + re^{i\varphi})| = 0$ on $S_r(z_0)$. Hence

$$\forall r \in (0, R) : \quad |f(z_0)| \equiv |f(z_0 + re^{i\varphi})| \quad \text{on } S_r(z_0).$$

We conclude that $|f(z_0)| \equiv |f(z_0 + re^{i\varphi})|$ in $B_R(z_0)$. As in Problem Sheet 2 Q8, this implies that $f(z) \equiv const$ in $B_R(z_0)$. Thus $f(z)$ is constant in D by the Identity Theorem. □

Remark 12.4. In other words, if f is non constant in D then any local maximum point of the modulus $|f|$ may occur only on the boundary of D. Observe that f may have no local maxima on the boundary of D. For example, if $f(z) = z$ and $D = \mathbb{C}_+$ is the upper half plane, then $|f(z)| = |z|$ has no local maximum points neither in D nor on the boundary of D. Another example is the function $f(z) = z^2$ on $D = \{z \in \mathbb{C} : |z| > 1\}$. Here every point of the boundary of D is a minimum point of $|f|$! Note also that in the trivial case when f is constant in D, the maximum of $|f|$ is attained at every point of D.

Example 12.5. If $f(z) = \exp(z)$ and $D = \bar{B}_1(0)$ then $|f(z)| = e^{\mathrm{Re}(z)}$. Hence $|f|$ has its maximum on $\bar{B}_1(0)$ at $z_0 = 1$.

Exercise 12.6. (Minimum Modulus Theorem) Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ an analytic function. If $|f|$ has a local minimum point $z_0 \in D$ then either $f(z_0) = 0$ or f is constant in D.

Hint. Apply the Maximum Modulus Theorem to the function $\frac{1}{f(z)}$.

Remark 12.7. In other words, if f is non constant in D then any local minimum point of the modulus $|f|$ may occur only at zeros of f or on the boundary of D. For example, $f(z) = z$ has minimum of its modulus at the origin.

Example 12.8. If $f(z) = z^2$ and $D = \bar{B}_1(0)$ then $|f(z)| = |z|^2$. Hence every $z \in S_1(0)$ is a maximum point of f and $z_0 = 0$ is the minimum point of $|f|$.

13 Harmonic functions and harmonic conjugates

Let $G \subseteq \mathbb{R}^2$ be an open set and $u = u(x, y) : G \to \mathbb{R}$ a real valued function. A natural question to ask is: *when u is a real (or imaginary) part of an analytic function $f : G \to \mathbb{C}$?* Recall, that for a given real valued function $u = u(x, y)$, the differential expression

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

is called the *Laplacian* of a u.

Definition 13.1. Let $G \subseteq \mathbb{R}^2$ be an open set. A function $u : G \to \mathbb{R}$ is called *harmonic* in G iff u has continuous second partial derivatives in G and satisfies the Laplace equation

$$\Delta u = 0 \qquad (u \in G).$$

Example 13.2. The constant functions and the functions $x + y$, $x^2 - y^2$, xy, $\exp(x)\cos(y)$ are harmonic in $\mathbb{R}^2$.

Example 13.3. If u and v are harmonic in G and $\lambda, \mu \in \mathbb{R}$ then $\lambda u + \mu v$ is harmonic in G.

In what follows we shall frequently use the polar coordinates (r, φ) on $\mathbb{R}^2 \setminus \{0\}$, where $x = r\cos(\varphi)$, $y = r\sin(\varphi)$ and $\varphi \in [0, 2\pi)$.

Exercise 13.4. Let (r, φ) be the polar coordinates on the plane. Prove that in polar coordinates the Laplacian of a function $u = u(x, y)$ is represented by

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}.$$

Hint. Use the (real) Chain Rule and direct computations of partial derivatives.

Exercise 13.5. Verify that $r^{\pm k}\cos(k\varphi)$ $(k \in \mathbb{N})$ is harmonic in $\mathbb{R}^2$ and $\log(r)$ is harmonic in $\mathbb{R}^2 \setminus \{0\}$.

The following lemma is an easy consequence of the Cauchy–Riemann equations.

Lemma 13.6. *Let $G \subseteq \mathbb{C}$ be an open set and $f = u + iv : G \to \mathbb{C}$ an analytic function. Then* $\operatorname{Re}(f) = u$ *and* $\operatorname{Im}(f) = v$ *are harmonic in G.*

Proof. See Problem Sheet 1, Q13.

The theorem below combined with the previous lemma states that a function is harmonic if and only if it is a real part of an analytic function.

Theorem 13.7. (Harmonic Conjugate Theorem) *Let $D \subset \mathbb{R}^2$ be a simply connected domain and $u : D \to \mathbb{R}$ a harmonic function. Then*

(a) u is infinitely many times differentiable in G;

(b) there exists a function $v : D \to \mathbb{R}$, called a harmonic conjugate to u in D, such that the function $f = u + iv : D \to \mathbb{C}$ is complex analytic.

Remark 13.8. If v is a harmonic conjugate to u in D and $c \in \mathbb{R}$, then $v + c$ is also a harmonic conjugate to u. It follows from the Cauchy–Riemann equations that the converse is also true. Namely, if v and w are harmonic conjugates to u in D, then $v - w = const$ in D.

Proof. We prove (b) first. Consider the function

$$g(z) = \underbrace{\frac{\partial u}{\partial x}(x+iy)}_{U(x+iy)} - i\underbrace{\frac{\partial u}{\partial y}(x+iy)}_{V(x+iy)}. \tag{13.1}$$

We show that g is analytic in D.

Set $g(z) = U(x+iy)+iV(x+iy)$. Observe that U and V have continuous partial derivatives by assumption. Therefore g is continuously differentiable in D with

$$\frac{\partial U}{\partial x} = \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial U}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}; \qquad -\frac{\partial V}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}, \quad \frac{\partial V}{\partial y} = \underbrace{-\frac{\partial^2 u}{\partial y^2}}_{=\partial^2 u/\partial x^2}$$

So g has continuous partial derivatives which satisfy the Cauchy-Riemann equations in D. Therefore g is differentiable in D and hence analytic in D. Furthermore, Cauchy's Theorem implies that g has an antiderivative G in D. Set $G = \tilde{u} + i\tilde{v}$. Then

$$\underbrace{G'}_{=g} = \underbrace{\frac{\partial \tilde{u}}{\partial x}}_{=\partial u/\partial x} - i \underbrace{\frac{\partial \tilde{u}}{\partial y}}_{=\partial u/\partial y}.$$

Thus $\tilde{u} - u = const$ in D. Set $f = G - const$. Then $\mathrm{Re}(f) = u$, as required.

To prove (a) simply observe that the real part of a complex analytic function is infinitely many times differentiable. $\square$

Remark 13.9. The proof of the Harmonic Conjugate Theorem suggests a method of finding a harmonic conjugate to a given harmonic function. Indeed, let $u(x,y)$ be a given harmonic in a simply connected domain $D \subset \mathbb{C}$. As in (13.1), define the function

$$g(z) = \frac{\partial u}{\partial x}(x+iy) - i\frac{\partial u}{\partial y}(x+iy).$$

Thus g is analytic in D and has an antiderivative $G : D \to \mathbb{C}$ that can be defined by

$$G(z) = \int_{\gamma_{a,z}} g(w)dw,$$

where $a \in D$ is a fixed point, $z \in D$ and $\gamma_{a,z} : [\alpha, \beta] \to D$ is a contour from a to z (see Corollary 3.21). Taking the imaginary part of G and adjusting the constants if necessary, we obtain a harmonic conjugate to u.

Example 13.10. Find harmonic conjugates to the harmonic function $u(x,y) = x^2 - y^2$.

Solution. By direct computation we see that u is harmonic on $\mathbb{R}^2$. Then a harmonic conjugate to u in $\mathbb{R}^2$ can constructed as the imaginary part of the function

$$G(z) = \int_{\gamma_{0,z}} \underbrace{2(x+iy)}_{g(w)} \underbrace{dw}_{w=x+iy} = z^2,$$

that is $v(x,y) = \mathrm{Im}(z^2) = 2xy$. $\square$

Remark 13.11. Another method of finding a harmonic conjugate relies on Green's Theorem from Calculus. Let $u(x,y)$ be a given harmonic in a simply connected domain D and $v = v(x,y)$ its harmonic conjugate. Then by the Cauchy–Riemann equations

$$\underbrace{\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)}_{=\nabla v} = \underbrace{\left(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}\right)}_{\Delta u = 0} \quad \text{in } D.$$

Note that $(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y})$ is a gradient field and $(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x})$ is a conservative field. Thus

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy, \tag{13.2}$$

where (x_0, y_0) is a fixed point in D, and both integrals do not depend on the path of integration by the Green Theorem from Calculus. This gives another method of finding a harmonic conjugate to a given harmonic function. For example, by virtue of (13.2) a harmonic conjugate to the harmonic function $u(x,y) = x^2 - y^2$ can be defined as

$$v(x,y) = \int_{(0,0)}^{(x,y)} -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy = \int_{(0,0)}^{(x,y)} 2y\,dx + 2x\,dy = 2xy.$$

One can recognize in this case, that u and v are real and imaginary parts of $f(z) = z^2$.

Remark 13.12. We saw in Harmonic Conjugate Theorem that harmonic functions are infinitely many times differentiable. In fact, harmonic functions are also *real analytic* (as the real parts of complex analytic functions). Roughly speaking, a function is real analytic if it is expressed locally as a power series in the variables x_1, x_2. More precisely, a real valued function $u(x,y)$ is real analytic at a point (x_0, y_0) if

$$u(x,y) = \sum_{n=0}^{\infty} \sum_{\substack{\alpha_1+\alpha_2=n \\ \alpha_1,\alpha_2 \in \mathbb{N}\cup\{0\}}} c_{\alpha_1,\alpha_2} (x - x_0)^{\alpha_1} (y - y_0)^{\alpha_2},$$

where the series converges absolutely in a neighborhood of (x_0, y_0). We do not study real analytic functions in this course.

Exercise 13.13. Prove that if u is harmonic in a simply connected domain $D \subset \mathbb{R}^2$ then $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are harmonic in D.

Hint. By the Harmonic Conjugate Theorem, u is infinitely many times differentiable in D.

Exercise 13.14. If f is analytic in a simply connected domain $D \subseteq \mathbb{C}$ and $f(z) \notin (-\infty, 0]$ for any $z \in D$, show that $u = \log|f|$ is harmonic in D.

Properties of Harmonic Functions. One reason why Harmonic Conjugate Theorem is important is that it enables us to deduce properties of harmonic functions from corresponding properties of analytic functions. In what follows, if convenient, we identify a complex number $z = x + iy \in \mathbb{C}$ with the point $(x,y) \in \mathbb{R}^2$ on the real plane, without further notices.

Lemma 13.15. (Mean Value Property) *Let $u : B_R(z_0) \to \mathbb{R}$ be a harmonic function. Then for any $r \in (0, R)$ one has*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\varphi}) d\varphi. \tag{13.3}$$

Proof. Let $v : B_R(z_0) \to \mathbb{R}$ be a harmonic conjugate to u and $f = u + iv : B_R(z_0) \to \mathbb{C}$ the corresponding analytic function. By the Mean Value Property for complex analytic functions, for any $r \in (0, R)$ one has

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\varphi}) d\varphi.$$

Taking the real part of both sides of the equation gives the required result. □

Remark 13.16. In fact, one can prove the converse. Namely, if a continuous real function $u : B_R(z_0) \to \mathbb{R}$ satisfies for every $r \in (0, R)$ the Mean Value Property (13.3) then u is harmonic in $B_R(z_0)$.

Exercise 13.17. Prove that a zero of a harmonic function is never isolated. More precisely, let $u : B_R(z_0) \to \mathbb{R}$ be a harmonic function such that $u(z_0) = 0$. Show that for any $r \in (0, R)$ there exists $z_r \in S_r(z_0)$ such that $u(z_r) = 0$.

Hint. Use the Mean Value Property.

Next we deduce a Maximum Principle for harmonic functions.

Theorem 13.18. (Maximum Principle) *Let $D \subset \mathbb{R}^2$ be a simply connected domain and $u : D \to \mathbb{R}$ a harmonic function. If u has a local maximum or a local minimum point in D then u is constant in D.*

Proof. Let $v : D \to \mathbb{R}$ be a harmonic conjugate to u and $f = u+iv : D \to \mathbb{C}$ the corresponding analytic function. Consider the function $g := \exp(f)$. Obviously $g : D \to \mathbb{C}$ is analytic and $|g| = e^{u(z)}$. Since e^s is strictly increasing, the local maxima of u are the same as those of $|g|$. By the Maximum Modulus Theorem, if $|g|$ has a local maximum in D then $|g|$ is constant in D. Hence, u is constant in D, again because e^s is strictly increasing.

Finally, if u has a local minimum in D then $-u$ has a local maximum in D. Thus we can repeat the previous argument for the function $-u$. □

The following refinement of the Maximum Principle for bounded domains is crucial in the study of boundary value problems. In what follows, $\bar{D}$ denotes the *closure* of a domain $D \subset \mathbb{R}^2$ and ∂D denotes the *boundary* of D. Then the set D is called the *interior* of $\bar{D}$.

Theorem 13.19. (Maximum Principle for bounded domains) *Let $D \subset \mathbb{R}^2$ be a bounded simply connected domain and $u : \bar{D} \to \mathbb{R}$ a continuous functions that is harmonic in D. Then u attains its maximum and minimum values on ∂D.*

Proof. It is known from Analysis that every continuous function on a bounded closed set attains its maximum and minimum values. However a harmonic function u can not have local maxima or minima in the interior of $\bar{D}$. Thus the result follows. □

Remark 13.20. A harmonic function on a bounded domain D which is not continuous up to the boundary may not attain its maximum or minimum values on the boundary ∂D Consider, for example $u(r, \varphi) = \log(r)$ on the punctured ball $A_{0,1}$.

Remark 13.21. Harmonic functions on an unbounded domain D may not attain its maximum or minimum values on the boundary ∂D (and hence on the entire domain $\bar{D}$). Consider, for example, $u(x, y) = \exp(x)\cos(y)$ on the strip $\Pi = \{x \in \mathbb{R}, y \in (-\pi, \pi)\}$ or $u(r, \varphi) = r\cos(\varphi)$ on the upper half–plane $\mathbb{R}^2_+ = \{x \in \mathbb{R}, y > 0\}$.

14 Dirichlet problem for harmonic functions

Let $D \subset \mathbb{C}$ be a *bounded* domain, ∂D the boundary and $\bar{D}$ the closure of D. The Dirichlet Problem for harmonic functions is: *given a continuous function $g : \partial D \to \mathbb{R}$, find a continuous function $u : \bar{D} \to \mathbb{R}$, that is harmonic in D and that equals g on ∂D*; or in other words, that satisfies

$$\left\{ \begin{array}{rcllr} \Delta u & = & 0 & \text{in} & D, \\ u & = & g & \text{on} & \partial D. \end{array} \right. \tag{14.1}$$

Observe that if $g \equiv const$ then $u \equiv const$ is a solution the Dirichlet problem (14.1). One can easily see that the solution of the Dirichlet problem is always unique (if exists).

Theorem 14.1. *The Dirichlet Problem* (14.1) *has at most one solution.*

Proof. Let u_1 and u_2 be two solutions of (14.1). Let $w = u_1 - u_2$. Then w is harmonic in D and $w = 0$ on ∂D. By the Maximum Principle w attains its maximum and minimum values on the boundary ∂D. But since $w = 0$ on ∂D, we conclude that $w \equiv 0$ in $\bar{D}$. Thus $u_1 = u_2$. □

Remark 14.2. If the domain D is unbounded then the statement of Theorem 14.1 failed. For example, let $D = A_{1,\infty}(0)$ and $g \equiv 0$. Then $u_1 \equiv 0$ and $u_2 = \log(r)$ are two different solutions of (14.1).

Exercise 14.3. Prove that if $g \gneq 0$ on ∂D and u is a solution to (14.1) then $u > 0$ in D.

The Dirichlet problem on general bounded domains is rather complicated. We want to find a solution for the case where the domain D is the ball. To do this we derive an integral formula that explicitly expresses the values of a harmonic function on a ball in terms of its values on the boundary of the ball.

Theorem 14.4. (Poisson Formula) *Let $u : \bar{B}_R(0) \to \mathbb{R}$ be a continuous function that is harmonic in $B_R(0)$. Then for any $r \in (0, R)$ and $\varphi \in [0, 2\pi)$ one has*

$$u(re^{i\varphi}) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{u(Re^{i\theta})}{|Re^{i\theta} - re^{i\varphi}|^2} d\theta. \tag{14.2}$$

Remark 14.5. If $r = 0$ then (14.2) is simply the Mean Value Property for u.

Sketch of the proof. Let $v : B_R(0) \to \mathbb{R}$ be a harmonic conjugate to u and $f = u + iv : B_R(0) \to \mathbb{C}$ the corresponding analytic function. Then by the Cauchy Integral Formula for any $s \in (0, R)$ we have

$$f(z) = \frac{1}{2\pi i} \int_{S_s^+(0)} \frac{f(w)}{w - z} dw \qquad (z \in B_s(0)).$$

Let

$$\tilde{z} = \frac{s^2}{\bar{z}},$$

which is called the *reflection* of z in the sphere $S_s^+(0)$. Thus if $z \in A_{0,s}(0)$ then $\tilde{z} \in A_{s,\infty}(0)$, and therefore

$$\frac{1}{2\pi i} \int_{S_s^+(0)} \frac{f(w)}{w - \tilde{z}} dw = 0 \qquad (z \in A_{0,s}(0))$$

Thus we may subtract this integral to obtain

$$f(z) = \frac{1}{2\pi i}\int_{S_s^+(0)} f(w)\left\{\frac{1}{w-z} - \frac{1}{w-\tilde{z}}\right\}dw \qquad (z \in A_{0,s}(0)).$$

Observing that $|w| = s$ we can simplify

$$\frac{1}{w-z} - \frac{1}{w-\tilde{z}} = \frac{1}{w-z} + \frac{1}{w - \frac{|w|^2}{\bar{z}}} = \frac{1}{w-z} - \frac{\bar{z}}{w(\bar{z}-\bar{w})} = \frac{|w|^2 - |z|^2}{w|w-z|^2}.$$

Hence we have

$$f(z) = \frac{1}{2\pi i}\int_{S_s^+(0)} f(w)\frac{|w|^2-|z|^2}{w|w-z|^2}dw \qquad (z \in A_{0,s}(0)),$$

or, in polar coordinates $w = se^{i\theta}$, $z = re^{i\varphi}$ we get

$$f(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} f(se^{i\theta})\frac{s^2-r^2}{|se^{i\theta}-re^{i\varphi}|^2}d\theta \qquad (0 < r < s,\ \varphi \in [0,2\pi)).$$

Taking the real parts on both sides of the equation, we obtain

$$u(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} u(se^{i\theta})\frac{s^2-r^2}{|se^{i\theta}-re^{i\varphi}|^2}d\theta \qquad (0 < r < s,\ \varphi \in [0,2\pi)).$$

Observe that this formula is true for any $s < R$.

To complete the proof we only need to show that the formula still true for $s = R$ *and* $r = 0$, hence for any $z \in B_R(0)$.

Since u is continuous on $\bar{B}_R(0)$ and since $|se^{i\theta} - re^{i\varphi}|$ is never zero when $r < s$, we conclude, that for a fixed $r \in (0,s)$ and $\varphi \in [0,2\pi)$, the function

$$U(s,\theta) := u(se^{i\theta})\frac{s^2-r^2}{|se^{i\theta}-re^{i\varphi}|^2}$$

is continuous on the compact set $\bar{A}_{\frac{R+r}{2},R}(0)$. Thus, by a result from Analysis, $U(s,\theta)$ is uniformly continuous on $\bar{A}_{\frac{R+r}{2},R}(0)$. Consequently,

$$\lim_{s\to R} U(s,\theta) = U(R,\theta),$$

uniformly in θ. By a result from Analysis, this implies that

$$\lim_{s\to R}\frac{1}{2\pi}\int_0^{2\pi} u(se^{i\theta})\frac{s^2-r^2}{|se^{i\theta}-re^{i\varphi}|^2}d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\frac{R^2-r^2}{|Re^{i\theta}-re^{i\varphi}|^2}d\theta.$$

Finally, if $r = 0$ then (14.2) simply expresses the Mean Value Property (13.3) for u. □

Exercise 14.6. Let $D \subset \mathbb{R}^2$ be a domain and $u : \bar{D} \to \mathbb{R}$ a continuous functions. Show that if u satisfies the Mean Value Property (13.3) at every point of D then u is harmonic in D.

Hint. Let $z_0 \in D$ and $B_R(z_0) \subset D$. Then there exists a continuous function $v : \bar{B}_R(z_0) \to \mathbb{R}$ that is harmonic in $B_R(z_0)$ and coincides with u on $S_R(z_0)$. Set $w := u - v$. Thus $w = 0$ on $S_R(z_0)$ and w satisfies the Mean Value Property in $B_R(z_0)$. Show that then $w = 0$ in $B_R(z_0)$, and therefore u is harmonic in D.

The Poisson Formula enables to settle the Dirichlet Problem for the case when the domain D is an open ball. Suppose that we are given a continuous function $g : S_R(0) \to \mathbb{R}$. Then we can plug in g instead of u in the right hand side of (14.2). It is relatively simple to show that such defined function u is harmonic in $B_R(0)$. It is more difficult to prove that u is continuous on $\bar{B}_R(0)$. The expression must be examined in the critical case in which $r \to R$ and the integrand takes the values "0/0" near $\theta = \varphi$. We only give the statement of the theorem.

Theorem 14.7. (Solution of the Dirichlet Problem for a ball) *Let $g : S_R(0) \to \mathbb{R}$ be a continuous function. Then there exists a continuous function $u : \bar{B}_R(0) \to \mathbb{R}$, that is harmonic in $B_R(0)$ and that equals g on $S_R(0)$. Moreover, such u is unique and is defined by the formula*

$$u(re^{i\varphi}) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{g(Re^{i\theta})}{|Re^{i\theta} - re^{i\varphi}|^2} d\theta \qquad (r \in (0,R),\ \varphi \in [0,2\pi)) \tag{14.3}$$

Proof. See, e.g., Lang, pp.244–251 or Conway, pp.258–262.

Exercise 14.8. (Harnack Inequality) Let $u : \bar{B}_R(z_0) \to \mathbb{R}$ be a continuous functions that is harmonic in $B_R(z_0)$. Suppose $u \geq 0$ in $B_R(z_0)$. Prove that

$$\frac{R-r}{R+r} u(z_0) \leq u(z_0 + re^{i\varphi}) \leq \frac{R+r}{R-r} u(z_0) \qquad (r < R,\ \varphi \in [0,2\pi)).$$

Hint. Note that $R - r \leq |Re^{i\theta} - re^{i\varphi}| \leq R + r$ and insert this bound into (14.2).

Exercise 14.9. (Liouville Theorem) Prove that every bounded harmonic function on $\mathbb{R}^2$ is constant.

Hint. Use Harnack's inequality.

The basic method for solving the Dirichlet problem on general domains is as follows. Take the given domain D and transfer it by a *conformal map* to a ball, where the problem can be explicitly solved. This procedure is justified by the fact that under a conformal mappings, harmonic functions are transformed again into harmonic functions. When we have solved the problem on the ball, we can conformally transform the answer back to D.

15 Conformal mappings

We wish to define an equivalence relation between domains in $\mathbb{C}$. After doing this it will be shown that all simply connected domains in $\mathbb{C}$ are equivalent to the open ball $B_1(0)$, and hence are equivalent to one another.

15.1 Mapping properties of analytic functions

We start with the following topological lemma, which says that the number of zeros of an analytic function is stable under "small perturbations".

Lemma 15.1. *Let $G \subseteq \mathbb{C}$ be an open set and $f : G \to \mathbb{C}$ a nonconstant analytic function. Let $z_0 \in G$, $w_0 = f(z_0)$ and $m \geq 1$ be the order of zero of the function $f(z) - w_0$ at $z = z_0$. Then there is $r > 0$ that for any $\varepsilon \in (0, r)$ there exists $\delta = \delta(\varepsilon) > 0$ such that for each $w \in B_\delta(w_0)$ the function $f(z) - w$ has exactly m zeros of order one in $B_\varepsilon(z_0)$.*

Remark 15.2. The order of zero of the function $f(z) - w_0$ at $z = z_0$ is necessarily finite, otherwise $f(z)$ would be constant by the Unique Continuation Theorem.

Proof. By Lemma 10.7, zeros of a nonconstant analytic function are isolated. Thus we can choose $r_0 > 0$ such that $B_{r_0}(z_0) \subset G$ and $f(z) - w_0 \neq 0$ in $A_{0,r_0}(z_0)$. Moreover, if $m = 1$ then $f'(z_0) \neq 0$, while if $m \geq 2$ then $f'(z_0) = 0$ and $f'(z)$ is nonconstant. In either case there is $r_1 > 0$ such that $B_{r_1}(z_0) \subset G$ and $f'(z) \neq 0$ in $A_{0,r_1}(z_0)$.

Let $r = \min\{r_0, r_1\}$. Choose $\varepsilon \in (0, r)$. Since $S_\varepsilon^+(z_0)$ is compact and, by the previous consideration, $|f(z) - w_0|$ is a continuous nonnegative function which does not vanish on $S_\varepsilon^+(z_0)$, we conclude that

$$\delta = \inf_{S_\varepsilon^+(z_0)} |f(z) - f(z_0)| > 0.$$

Suppose that $w \in A_{0,\delta}(w_0)$. Then

$$|(f(z) - w_0) - (f(z) - w)| = |w - w_0| < \delta \leq |f(z) - f(z_0)| = |f(z) - w_0| \qquad (z \in S_\varepsilon^+(z_0)).$$

By the Rouché Theorem we conclude that $f(z) - w_0$ and $f(z) - w$ have the same number of zeros in $B_\varepsilon(z_0)$. Moreover, we have ensured that $f'(z) \neq 0$ in $A_{0,r}(z_0)$, So, zeros of $f(z) - w$ are necessarily simple (note that w_0 is not a zero of $f(z) - w$, since $w \neq w_0$). □

Example 15.3. (a) Consider the function $f(z) = z^2$. Let $z_0 = 0$. Then $f(z_0) = 0$ and $z_0 = 0$ is a zero of f of order $m = 2$. We may take $r = \infty$, since $f'(z) \neq 0$ in $A_{0,\infty}(0)$. Then, for any $\varepsilon > 0$ set $\delta = \varepsilon^2$. For each $w \in B_\delta(0)$ the equation $f(z) = w$ has two solutions $z_1 = \sqrt{w}$, $z_2 = -\sqrt{w}$, and $z_1, z_2 \in B_\varepsilon(0)$.

(b) Consider the function $f(z) = z^2 - z^3$. Let $z_0 = 0$. Then $f(z_0) = 0$ and $z_0 = 0$ is a zero of f of order $m = 2$. Put $r = 1/2$ (in fact, we may take any $r \in (0, 2/3)$, since $f'(z) \neq 0$ in $A_{0,2/3}(0)$). Then, for any $\varepsilon \in (0, 1/2)$ let $\delta = \varepsilon^2/2$. One can verify that for each $w \in B_\delta(0)$ the equation $f(z) = w$ has two solutions in the ball $B_\varepsilon(0)$ (and the third solution of the equation is located away from $B_\varepsilon(0)$).

Lemma 15.1 has several important consequences.

Theorem 15.4. (Open Mapping Theorem) *Let $G \subseteq \mathbb{C}$ be an open set and $f : G \to \mathbb{C}$ a nonconstant analytic function. Then the image $f(G)$ is open.*

Proof. Let $w_0 \in f(G)$. By Lemma 15.1, there is $\delta > 0$ such that $B_\delta(w_0) \subset f(G)$. This means that $f(G)$ is open. □

Exercise 15.5. Derive the Maximum Modulus Theorem from the Open Mapping Theorem.

Definition 15.6. Let $G \subseteq \mathbb{C}$ be an open set. We say that a function $f : G \to \mathbb{C}$ is *one–to–one* iff f is a bijection onto its image $f(G)$.

Remark 15.7. It follows immediately from Lemma 15.1 that if $f : G \to \mathbb{C}$ is a one–to–one analytic function then $f'(z) \neq 0$ in G. The converse is true only "locally", see below.

Theorem 15.8. (Inverse Mapping Theorem) *Let $G \subseteq \mathbb{C}$ be an open set and $f : G \to \mathbb{C}$ a one-to-one analytic function. Then $f^{-1} : f(G) \to G$ is analytic. Moreover, if $z_0 \in G$ and $w_0 = f(z_0)$ then $(f^{-1})'(w_0) \neq 0$ and $(f^{-1})'(w_0) = (f'(z_0))^{-1}$.*

Proof. Since f a bijection between G and $f(G)$, the inverse function $f^{-1} : f(G) \to G$ is well–defined.

We show first that $f^{-1} : f(G) \to G$ is continuous at w_0. Let $z_0 \in G$ and $w_0 = f(z_0)$. According to Lemma 15.1, for any $\varepsilon \in (0, r)$ there exists $\delta > 0$ such that for each $w \in B_\delta(w_0)$ there is (necessarily unique) $z \in B_\varepsilon(z_0)$ such that $f(z) = w$. Or, in other words,

$$w \in B_\delta(w_0) \quad \Rightarrow \quad f^{-1}(w) \in B_\varepsilon(f^{-1}(w_0)).$$

This means that f^{-1} is continuous at w_0.

Next we prove that $f^{-1} : f(G) \to G$ is differentiable at w_0. Note that $f(G)$ is open by the Open Mapping Theorem. Let $w, w_0 \in f(G)$, $z_0 = f^{-1}(w_0)$ and $z = f^{-1}(w)$. Since f^{-1} is continuous at w_0 and $f'(z_0) \neq 0$ (see Remark 15.7), we have

$$(f^{-1})'(w_0) = \lim_{w \to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{f'(z_0)}.$$

This completes the proof. □

The following example shows that the converse to Inverse Mapping Theorem is false.

Example 15.9. Consider $f(z) = \exp(z)$ on $\mathbb{C}$. Thus $f'(z) \neq$ for each $z \in \mathbb{C}$. However f is not one–to–one, e.g. $f(z) = f(z + 2\pi i)$ for any $z \in \mathbb{C}$.

However one can easily prove a local version of converse to Theorem 15.8.

Exercise 15.10. (Local Inverse Function Theorem) Let $G \subseteq \mathbb{C}$ be an open set and $f : G \to \mathbb{C}$ a nonconstant analytic function. Let $z_0 \in G$, $w_0 = f(z_0)$ and $f'(z_0) \neq 0$. Then there is $r > 0$ such that $f : B_r(z_0) \to \mathbb{C}$ is one–to–one.

Hint. Use Lemma 15.1.

An important corollary of the Inverse Mapping Theorem is the following.

Exercise 15.11. Let $D \subseteq \mathbb{C}$ be a (simply connected) domain. Suppose that $f : D \to \mathbb{C}$ is a one–to–one analytic function. Then the image $f(D)$ is a (simply connected) domain.

Hint. Let $\gamma_0, \gamma_1 : [\alpha, \beta] \to D$ be two closed paths in D and $H = H(t, s) : [\alpha, \beta] \times [0, 1] \to D$ a homotopy between γ_1 and γ_2. Verify that $f \circ \gamma_1$ and $f \circ \gamma_2$ are closed paths in $f(D)$ and $f \circ H$ is a homotopy between $f \circ \gamma_1$ and $f \circ \gamma_2$.

15.2 Conformal mappings

Let $D \subset \mathbb{C}$ be a domain. Let $\gamma : [\alpha, \beta] \to D$ be a smooth path and $\gamma'(t_0) \neq 0$ for some $t_0 \in (\alpha, \beta)$. Then γ has a tangent line at the point $z_0 = \gamma(t_0)$. This line goes through the point z_0 in the direction of the vector $\gamma'(t_0)$. If $\gamma_1, \gamma_2 : [\alpha, \beta] \to D$ are smooth paths with $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ and $\gamma_1'(t_1) \neq 0$, $\gamma_2'(t_2) \neq 0$, then we define *the angle between the paths* γ_1 *and* γ_2 *at* z_0 to be

$$\operatorname{Arg}(\gamma_2'(t_2)) - \operatorname{Arg}(\gamma_1'(t_1)).$$

Now let $f : D \to \mathbb{C}$ be an analytic function. Also suppose that the paths γ_1 and γ_2 are not tangent to each other at z_0, that is the angle between the paths γ_1 and γ_2 at z_0 is nonzero. Let $\Gamma_1(t) = f(\gamma_2(t))$ and $\Gamma_2(t) = f(\gamma_2(t))$. Then

$$\Gamma_1'(t_1) = f'(\underbrace{\gamma_1(t_1)}_{z_0})\gamma_1'(t_1), \qquad \Gamma_2'(t_2) = f'(\underbrace{\gamma_2(t_2)}_{z_0})\gamma_2'(t_2).$$

Therefore [7]

$$\underbrace{\operatorname{Arg}(\Gamma_2'(t_2))}_{\operatorname{Arg}(f'(z_0))+\operatorname{Arg}(\gamma_2'(t_2))} - \underbrace{\operatorname{Arg}(\Gamma_1'(t_1))}_{\operatorname{Arg}(f'(z_0))+\operatorname{Arg}(\gamma_1'(t_1))} = \operatorname{Arg}(\gamma_2'(t_2)) - \operatorname{Arg}(\gamma_1'(t_1)).$$

This says that given any two paths through z_0, f maps these paths onto two paths through $w_0 = f(z_0)$ and, when $f'(z_0) \neq 0$, the angles between the paths are preserved both in magnitude and direction. We say that f *preserves angles at point* z_0.

Definition 15.12. Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ an analytic function. If f preserves angles at a point z_0 then f is said to be *conformal at* z_0. If f is conformal at every $z_0 \in D$ then f we say that f is *conformal in* D. In the latter case we also say that f is a *conformal mapping of* D *onto* $f(D)$.

Therefore, we have proved the following.

Theorem 15.13. *Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ an analytic function. Then f is conformal at each point $z_0 \in D$ such that $f'(z_0) \neq 0$.*

Remark 15.14. If $f'(z_0) = 0$ then angles at z_0 need not be preserved. For example, if $f(z) = z^2$ and $z_0 = 0$ then x– and y–axes intersect at an angle $\pi/2$ but the images intersect at an angle π.

Example 15.15. Consider $f(z) = \exp(z)$. Thus f is conformal on $\mathbb{C}$. Let us look on the mapping properties of the exponential more closely. Fix $c \in \mathbb{R}$. If $z = c + iy$, then $f(z) = re^{iy}$ for $r = e^c$. That is, f maps the line $x = c$ onto the circle with center at the origin and of radius e^c. Also, f maps the line $y = d$ onto the infinite ray re^{id} $(r \in (0, \infty))$.

Let $D = \{z \in \mathbb{C} : -\pi < \operatorname{Im}(z) < \pi\}$ be a horizontal strip. Periodicity the exponential implies that f is one–to–one on D. Thus $f(D) = \mathbb{C} \setminus (-\infty, 0)$.

Similar analysis could be carried out for $\sin(z)$, $\cos(z)$ and other analytic functions.

[7] Note that for $0 \neq a, b \in \mathbb{C}$ one has $\operatorname{Arg}(ab) = \operatorname{Arg}(a) + \operatorname{Arg}(b)$.

15.3 Riemann Mapping Theorem

Now we are in a position to define an equivalence relation between domains in $\mathbb{C}$.

Definition 15.16. Let $D_1, D_2 \subseteq \mathbb{C}$ be domains. We say that D_1 is *conformally equivalent* to D_2 iff there is a one–to–one conformal mapping of D_1 onto D_2. Clearly this is an equivalence relation.

Remark 15.17. In fact, according the Open Mapping Theorem and Theorem 15.13, every one–to–one analytic function is conformal. The converse is not true, for example $f(z) = z^2$ is a conformal mapping of $\mathbb{C} \setminus \{0\}$ onto itself, but is not one–to–one, e.g. $(\pm i)^2 = 1$. The use of the term *one–to–one conformal* instead of the equivalent term *one–to–one analytic* is a traditional terminology.

Example 15.18. Let $f(z) = z^2$. Thus f is a one–to–one conformal mappings between $D = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$ and the upper half plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Also, f is a one–to–one conformal mappings between the upper half plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and $\mathbb{C} \setminus (0, \infty)$.

Exercise 15.19. Prove that $\mathbb{C}$ is not conformally equivalent to any bounded domain.

Hint. Use Liouville's Theorem.

Remark 15.20. Let $D \subseteq \mathbb{C}$ be a domain. Obviously, D is conformally equivalent to itself, the corresponding one–to–one conformal mapping is the identity mapping $f(z) = z$. Also, if $f, g : D \to D$ are one–to–one conformal mappings then the composition $g \circ f : D \to D$ is one–to–one and conformal. Finally, if $f : D \to D$ is a one–to–one and conformal then $f^{-1} : D \to D$ is one–to–one and conformal, by the Inverse Mapping Theorem. Because of these properties, the set of all one–to–one conformal mappings from D onto D forms a group (with respect to compositions), which is called the *group of conformal transformations of D.*

Exercise 15.21. Prove that the group of conformal transformations of the ball $B_1(0)$ is $\{cz : c \in \mathbb{C}, |c| = 1\}$.

Hint. Consider Schwarz's Lemma (Problem Sheet 7 Q9).

The theorem below is one of the fundamental results in the theory of conformal mappings.

Theorem 15.22. (Riemann Mapping Theorem) *Let $D \subset \mathbb{C}$ be a simply connected domain such that $D \neq \mathbb{C}$ and let $z_0 \in D$. Then there is a unique one–to–one conformal mapping f of D onto the unit ball $B_1(0)$, having the properties $f(z_0) = 0$ and $\mathrm{Re}(f'(z_0)) > 0$, $\mathrm{Im}(f'(z_0)) = 0$.*

Proof of the uniqueness. See Problem Sheet 8 (consider also Exercise 15.21).

Proof of the existence. We do not proof the existence part of the theorem, which is far beyond the scope of this course (see Conway, pp.156–159).

Corollary 15.23. *Let $D_1, D_2 \subset \mathbb{C}$ be simply connected domains such that $D_1, D_2 \neq \mathbb{C}$. Then D_1 and D_2 are conformally equivalent.*

Proof. We merely observe that D_1 is conformally equivalent to the unit ball $B_1(0)$, and $B_1(0)$ is conformally equivalent to D_2, via the inverse of the conformal mapping $f : D_2 \to B_1(0)$. $\square$

Remark 15.24. In other words, among the simply connected domains there are only two equivalence classes; one consisting of $\mathbb{C}$ alone and another containing all proper (different from $\mathbb{C}$) simply connected domains.

Observe, that classification of conformally equivalent *non* simply connected domains becomes far more complicated. For example, an annulus $A_{1,2}(0)$ is not conformally equivalent to the punctured unit ball $A_{0,1}(0)$. We do not consider this topic in the course.

15.4 Dirichlet Problem on Jordan domains

Theorem 15.25. *Let $D_1, D_2 \subseteq \mathbb{C}$ be conformally equivalent domains and $f : D_1 \to \mathbb{C}$ a one–to–one conformal mapping of D_1 onto D_2. Let $u : D_2 \to \mathbb{R}$ be a harmonic function. Then the composition $u \circ f : D_1 \to \mathbb{R}$ is a harmonic function.*

Proof. Let $z_0 \in D_1$ and $w_0 = f(z_0) \in D_2$. Let $B_r(w_0) \subseteq D_2$ and $V = f^{-1}(B_r(w_0))$. Clearly $V \subseteq D_1$.

Since $B_r(w_0)$ is simply connected, by the Harmonic Conjugate Theorem, there is an analytic function $g : B_r(w_0) \to \mathbb{C}$ such that $u = \text{Re}(g)$. Hence $g \circ f : V \to \mathbb{C}$ is analytic as a composition of analytic functions. Therefore $u \circ f = \text{Re}(g \circ f)$, as one easily sees. Thus $u \circ f : V \to \mathbb{C}$ is harmonic as the real part of an analytic function. Since $z_0 \in D_1$ was arbitrary, we conclude that $u \circ f$ is harmonic in D_1. □

Remark 15.26. One can think of a one–to–one conformal mapping of D_1 onto D_2 as a change of coordinates. Note, that definition of the Laplace operator Δ involves partial derivatives of u. So, *a priory*, definition of a harmonic function depends on the choice of coordinates. Theorem 15.25 says however, that a harmonic function remains harmonic after a conformal change of coordinates.

Theorem 15.25, together with the Riemann Mapping Theorem, suggests a method of solving the Dirichlet Problem on the so–called Jordan Domains. Recall, that a path $\gamma : [\alpha, \beta] \to \mathbb{C}$ is *simple* if γ is an injection, that is γ has no self intersections (compare also Theorem 8.9).

Definition 15.27. Let $D \subset \mathbb{C}$ be a bounded simply connected domain whose boundary ∂D is a simple closed path. Then D is said to be a *Jordan domain*.

Let $D \subset \mathbb{C}$ be a Jordan domain and $g : \partial D \to \mathbb{R}$ a continuous function. The *Dirichlet Problem* is to find a continuous function $u : \bar{D} \to \mathbb{R}$ that satisfies

$$\left\{ \begin{array}{rcll} \Delta u & = & 0 & \text{in } D, \\ u & = & g & \text{on } \partial D. \end{array} \right. \tag{15.1}$$

Since D is simply connected and $D \neq \mathbb{C}$, by the Riemann Mapping Theorem, there is a one–to–one conformal mapping f of D onto the unit ball $B_1(0)$. Furthermore, it is known that *if D is a Jordan domain, then f is continuous up to the boundary ∂D (that is $f : \bar{D} \to \bar{B}_1(0)$ is continuous) and f is a one–to–one mapping of ∂D onto $S_1(0)$.* [8]

[8] The proof of this deep result is far beyond the scope of the course. See, e.g., Lang, pp.351–358.

Thus the function g is transformed via the conformal mapping f^{-1} into continuous function $G = g \circ f^{-1} : S_1(0) \to \mathbb{R}$. Let $U : \bar{B}_1(0) \to \mathbb{R}$ be the solution of the Dirichlet problem

$$\begin{cases} \Delta U &= 0 \quad \text{in} \quad B_1(0), \\ U &= G \quad \text{on} \quad S_1(0). \end{cases}$$

Such solution is given by the Poisson Formula

$$U(re^{i\varphi}) = \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{G(e^{i\theta})}{|e^{i\theta} - re^{i\varphi}|^2} d\theta \qquad (r \in (0,1),\ \varphi \in [0, 2\pi)).$$

Then the function $u := U \circ f : D \to \mathbb{R}$ is the solution of the original Dirichlet Problem (15.1). Therefore, we outlined the proof of the following result. [9]

Theorem 15.28. (Solution of the Dirichlet Problem for a Jordan domain) *Let $D \subset \mathbb{C}$ be a Jordan domain and $g : \partial D \to \mathbb{R}$ a continuous function. Then there exists a unique continuous function $u : \bar{D} \to \mathbb{R}$, that is harmonic in D and that equals g on ∂D.*

[9] For a complete proof see, e.g., R. Nevanlinna and V. Paatero, *Introduction to complex analysis*, Addison–Wesley, 1969; pp.305–324.