Differential Equations	Name
MAA121/MAG131 Sheet 5	Number
Hand in by – optional	Year
	Mark: /10
	date marked: / /2010

Please attach your working, with this sheet at the front.

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1. Find the general solution and the solution satisfying the specified initial condition of the following ODEs:

   (*a*) $\ddot{y} + \dot{y} - 2y = 0$ with $y(0) = 4$ and $\dot{y}(0) = -4$.
   (*b*) $\ddot{y} - 4\dot{y} = 0$ with $y(0) = 13$ and $\dot{y}(0) = 0$;
   (*c*) $\ddot{\theta} + 4\theta = 0$ with $\theta(0) = 0$ and $\dot{\theta}(0) = 10$.

2. Find the general solution to the following inhomogeneous ODEs:

   (*a*) $\ddot{y} - 4\dot{y} = t^2$;
   (*b*) $\ddot{\theta} + 4\theta = t^2$;
   (*c*) $\ddot{y} + \dot{y} - 2y = 3e^{-t}$;
   (*d*) $y'' + y = 2\sin(x)$.

3. Find the general solution of the following system of ODEs by reducing it to a 2nd order ODE:

$$\begin{cases} \dot{x} &= 2x + 5y, \\ \dot{y} &= -2x. \end{cases}$$

Differential Equations
MAA121/MAG131 Sheet 5 (Solutions)

1. (a) $\ddot{x} - 3\dot{x} + 2x = 0$ with $x(0) = 2$ and $\dot{x}(0) = 6$.

The characteristic equation is

$$k^2 - 3k + 2 = 0.$$

The discriminat $D = 1 > 0$ and the roots of the characteristic equation are $k_1 = 1$ and $k_2 = 2$. Then the general solution is

$$x(t) = Ae^t + Be^{2t}.$$

To solve the initial value problem we first compute the derivative of the general solution:

$$\dot{x}(t) = Ae^t + 2Be^{2t}.$$

Substituting $t = 0$ into the general solution $x(t)$ and its derivative $\dot{x}(t)$ we obtain

$$\underbrace{x(0)}_{=2} = Ae^0 + Be^0 = A + B,$$

$$\underbrace{\dot{x}(0)}_{=6} = Ae^0 + 2Be^0 = A + 2B.$$

Therefore $A = -2$, $B = 4$ and the required solution of the initial value problem is

$$\boxed{x(t) = -2e^t + 4e^{2t}.}$$

(b) $y'' - 4y' + 4y = 0$ with $y(0) = 0$ and $y'(0) = 3$.

The characteristic equation is

$$k^2 - 4k + 4 = 0.$$

The discriminat $D = 0$ and the repeated root of the characteristic equation is $k = 2$. Then the general solution is

$$y(t) = Ae^{2t} + Bte^{2t}.$$

To solve the initial value problem we first compute the derivative of the general solution:

$$\dot{y}(t) = (2A + B)e^{2t} + Bte^{2t}.$$

Substituting $t = 0$ into the general solution $y(t)$ and its derivative $\dot{y}(t)$ we obtain

$$\underbrace{y(0)}_{=0} = Ae^0 = A,$$

$$\underbrace{\dot{y}(0)}_{=3} = (2A + B)e^0 = 2A + B.$$

Therefore $A = 0$, $B = 3$ and the required solution of the initial value problem is

$$\boxed{y(t) = 3te^{2t}}.$$

$(c)\quad z'' - 4z' + 13z = 0$ with $z(0) = 7$ and $z'(0) = 42$.

The characteristic equation is

$$k^2 - 4k + 13 = 0.$$

The discriminat $D = -36 < 0$ and the complex roots of the characteristic equation are $k_{1,2} = 2 \pm 3i$. Then the general solution is

$$z(t) = e^{2t}\left(A\cos(3t) + B\sin(3t)\right).$$

To solve the initial value problem we first compute the derivative of the general solution:

$$\dot{z}(t) = e^{2t}\left((3B + 2A)\cos(3t) + (2B - 3A)\sin(3t)\right).$$

Substituting $t = 0$ into the general solution $z(t)$ and its derivative $\dot{z}(t)$ we obtain

$$\underbrace{z(0)}_{=7} = e^0 A = A,$$

$$\underbrace{\dot{z}(0)}_{=42} = e^0(3B + 2A) = 3B + 2A.$$

Therefore $A = 7$, $B = 28/3$ and the required solution of the initial value problem is

$$\boxed{z(t) = e^{2t}\left(7\sin(3t) + \frac{28}{3}\cos(3t)\right)}.$$

$(d)\quad \ddot{y} + \dot{y} - 2y = 0$ with $y(0) = 4$ and $\dot{y}(0) = -4$.

The characteristic equation is

$$k^2 + k - 2 = 0.$$

The discriminat $D = 9 > 0$ and the roots of the characteristic equation are $k_1 = -2$ and $k_2 = 1$. Then the general solution is

$$y(t) = Ae^{-2t} + Be^t.$$

To solve the initial value problem we first compute the derivative of the general solution:

$$\dot{y}(t) = -2Ae^{-2t} + Be^t.$$

Substituting $t = 0$ into the general solution $y(t)$ and its derivative $\dot{y}(t)$ we obtain

$$\underbrace{y(0)}_{=4} = Ae^0 + Be^0 = A + B,$$

$$\underbrace{\dot{y}(0)}_{=-4} = -2Ae^0 + Be^0 = -2A + B.$$

Therefore $A = 8/3$, $B = 4/3$ and the required solution of the initial value problem is

$$\boxed{y(t) = \frac{8}{3}e^{-2t} + \frac{4}{3}e^t.}$$

(e) $\quad \ddot{y} - 4\dot{y} = 0$ with $y(0) = 13$ and $\dot{y}(0) = 0$.

The characteristic equation is

$$k^2 - 4k = 0.$$

The discriminat $D = 16 > 0$ and the roots of the characteristic equation are $k_1 = 0$ and $k_2 = 4$. Then the general solution is

$$y(t) = A + Be^{4t}.$$

To solve the initial value problem we first compute the derivative of the general solution:

$$\dot{y}(t) = 4Be^t.$$

Substituting $t = 0$ into the general solution $y(t)$ and its derivative $\dot{y}(t)$ we obtain

$$\underbrace{y(0)}_{=13} = A + Be^0 = A + B,$$

$$\underbrace{\dot{y}(0)}_{=0} = 4Be^0 = 4B.$$

Therefore $A = 13$, $B = 0$ and the required solution of the initial value problem is

$$\boxed{y(t) = 13.}$$

(f) $\quad \ddot{\theta} + 4\theta = 0$ with $\theta(0) = 0$ and $\dot{\theta}(0) = 10$.

The characteristic equation is

$$k^2 + 4 = 0.$$

The discriminat $D = -16 < 0$ and the complex roots of the characteristic equation are $k_{1,2} = \pm 2i$. Then the general solution is

$$\theta(t) = A\cos(2t) + B\sin(2t).$$

To solve the initial value problem we first compute the derivative of the general solution:

$$\dot{\theta}(t) = 2B\cos(2t) - 2A\sin(2t).$$

Substituting $t = 0$ into the general solution $z(t)$ and its derivative $\dot{z}(t)$ we obtain

$$\underbrace{\theta(0)}_{=0} = A\cos(0) + B\sin(0) = A,$$

$$\underbrace{\dot\theta(0)}_{=10} = 2B\cos(0) - 2A\sin(0) = 2B.$$

Therefore $A = 0$, $B = 5$ and the required solution of the initial value problem is

$$\boxed{\theta(t) = 5\cos(2t)}.$$

2. (a) $\ddot y - 4\dot y = t^2$.

The complementary function (general solution of the homogeneous ODE) is already found in Q1(e):

$$y_c(t) = A + Be^{4t}.$$

To solve the inhomogeneous equation we need to find a particular integral. The function on the right hand side, $f(t) = t^2$, is a 2nd order polynomial. If we were to try a 2nd order polynomial as a particular integral then we would have the problem that the 2nd order derivative of a 2nd order polynomial would be a constant and hence a solution of the homogeneous equation. So we have to multiply by an extra factor t and try try to find a particular integral in the form

$$y_p(t) = t(at^2 + bt + c) = at^3 + bt^2 + ct.$$

Then

$$\dot y_p(t) = 3at^2 + 2bt + c,$$
$$\ddot y_p(t) = 6at + 2b.$$

Substituting we get

$$(6at + 2b) - 4(3at^2 + 2bt + c) = t^2.$$

This requires $a = -1/12$, $6a - 8b = 0$, so $b = -1/16$, and $2b - 4c = 0$, so that $c = -1/32$. Therefore the particular integral is

$$y_p(t) = -\frac{1}{12}t^3 - \frac{1}{16}t^2 - \frac{1}{32}t.$$

Hence the general solution of the inhomogeneous equation is

$$\boxed{y(t) = A + Be^{4t} - \frac{1}{12}t^3 - \frac{1}{16}t^2 - \frac{1}{32}t.}$$

(b) $\ddot\theta + 4\theta = t^2$.

The complementary function (general solution of the homogeneous ODE) is already found in Q1(f):

$$\theta_c(t) = A\cos(2t) + B\sin(2t).$$

To solve the inhomogeneous equation we need to find a particular integral. The function on the right hand side, $f(t) = t^2$, is a 2nd order polynomial. Neither

$f(t)$ nor any of its derivative are solutions of the homogeneous equation. So we try to find a particular integral in the form of a 2nd order polynomial

$$\theta_p(t) = at^2 + bt + c.$$

Then

$$\dot{\theta}_p(t) = 2at + b,$$
$$\ddot{\theta}_p(t) = 2a.$$

Substituting we get

$$(2a) + 4(at^2 + bt + c) = t^2.$$

This requires $a = 1/4$, $b = 0$ and $c = -1/8$. So the particular integral is

$$\theta_p(t) = \frac{1}{4}t^2 - \frac{1}{8}.$$

Hence the general solution of the inhomogeneous equation is

$$\boxed{\theta(t) = A\cos(2t) + B\sin(2t) + \frac{1}{4}t^2 - \frac{1}{8}.}$$

(*c*) $\quad \ddot{y} + \dot{y} - 2y = 3e^{-t}$.

The complementary function (general solution of the homogeneous ODE) is already found in Q1(d):

$$y_c(t) = Ae^t + Be^{-2t}.$$

To solve the inhomogeneous equation we need to find a particular integral. The function on the right hand side, $f(t) = 3e^{-t}$, is not a solution of the homogeneous equation. So we try to find a particular integral in the form

$$y_p(t) = ae^{-t}.$$

Then

$$\dot{y}_p(t) = -ae^{-t},$$
$$\ddot{y}_p(t) = ae^{-t}.$$

Substituting we get

$$ae^{-t} - ae^{-t} - 2ae^{-t} = 3e^{-t},$$

which requires $a = -3/2$. So the particular integral is

$$y_p(t) = -\frac{3}{2}e^{-t}.$$

Hence the general solution of the inhomogeneous equation is

$$\boxed{y(t) = Ae^t + Be^{-2t} - \frac{3}{2}e^{-t}.}$$

(*d*) $\quad y'' + y = 2\sin(x)$.

The complementary function is

$$y_c(x) = A\cos(x) + B\sin(x)$$

To solve the inhomogeneous equation we need to find a particular integral. The function on the right hand side of the equation, $f(t) = 2\sin(x)$, is a solution of the homogeneous equation. So we try to find a particular integral in the form

$$y_p(t) = ax\sin(x) + bx\cos(x).$$

A direct computation shows that the particular integral is

$$y_p(x) = -x\cos(x).$$

Hence the general solution of the inhomogeneous equation is

$$\boxed{y(x) = A\sin(x) + B\cos(x) - x\cos(x).}$$

3. From the 2nd equation we represent

$$x = -\frac{1}{2}\dot{y}, \qquad \dot{x} = -\frac{1}{2}\ddot{y}.$$

Substituting into the 2nd equation we obtain

$$-\frac{1}{2}\ddot{y} = -\dot{y} + 5y.$$

After rearrangement we obtain a 2nd order ODE

$$\ddot{y} - 2\dot{y} + 10y = 0. \tag{1}$$

The characteristic equation of (1) is

$$k^2 - 2k + 10 = 0,$$

with the complex conjugate roots $k_1 = 1 - 3i$, $k_2 = 1 + 3i$. Thus the general solution of (1) is

$$y(t) = e^t(A\sin(3t) + B\cos(3t)),$$

where $A, B \in \mathbb{R}$ are arbitrary constants. Substituting into $x(t)$ we also obtain

$$x(t) = \frac{1}{2}\dot{x}(t) = \frac{1}{2}e^t(3A\cos(3t) - 3B\sin(3t)).$$

The general solution of the system is therefore

$$\begin{cases} x(t) &= \frac{3}{2}e^t(A\cos(3t) - B\sin(3t)), \\ y(t) &= e^t(A\sin(3t) + B\cos(3t)). \end{cases}$$