# Swansea Summer School in Nonlinear PDEs

## Poster sessions - order of short talks

| **Monday Session, 15.30-16.30** | **Wednesday Session, 14.00-15.00** |
|---|---|
| • Marco Gallo<br>• Raffaele Grande<br>• Karol Hajduk<br>• Jacopo Schino<br>• Ye Zhang | • Damiano Greco<br>• Sho Katayama<br>• Jordan Marajh<br>• Prachi Sahjwani |

# Posters abstracts

## Matias Gomez Aedo

Imperial College London

**A approach to estimate solutions of Hamilton-Jacobi equations based on data-driven methods using second-order information.**

**Abstract.** From approximation theory, the problem of reconstructing a function from a dataset of images of points in an efficient way has been studied by Adcock and Sui. On the other hand, in the context of differential equations, it is sought to find solutions that cannot be explicitly determined using numerical analysis tools. We will present an approach to estimating solutions of Hamilton-Jacobi equations related to optimal control problems based on the work of Azmi, Kalise and Kunisch, where they reconstruct the function using polynomials in a way that avoids very high computational costs as the dimension grows. We will also compare, in a practical case, how the use of second-order information, which can be obtained from the structure of the optimal control problem, helps to achieve a good approximation with a smaller dataset.

## Asma Benhamida

University Paris Est Creteil Val De Marne

**Critical Sobolev  Problem with p-Laplacian operator and with weight**

**Abstract.** We consider the problem $-div(\alpha(x)|\nabla u|^{p-2}\nabla u) = \lambda|u|^{q-2}u + |u|^{p^\star-2}u, u > 0$ in $\Omega$, $u = 0$ on $\partial\Omega$,  where $\Omega$  is a bounded domain in $R^N$, $N > p$, $\alpha$ a positive continuous potential on $\bar{\Omega}$ , $p^*$ the critical Sobolev exponent and $2 \leq p \leq q < p^*$ and $\lambda$ is a real constant. We prove the existence of some positive solutions which depends, among others, on the behavior of the potential $\alpha(.)$ near its minima, the position of $p^2$ with respect to the dimension of the space and on the position of $q$ with respect to some precise values.

## James Coe

University of Edinburgh


**Sharp quasi-invariance threshold for the cubic Szegő equation**


**Abstract.** We consider the flow of Gaussian fields under the cubic Szegő equation, a toy model for dispersionless Hamiltonian dynamics. We show that for a class of Gaussian fields below a critical regularity, the induced measure at almost all times is singular with respect to the initial distribution, but above the critical regularity, the distribution is quasi-invariant under the flow. We introduce a new method to show singularity, first by exhibiting an instantaneous growth of Sobolev norm of the solution (at high frequencies), and then employing an abstract argument to show that such a property cannot hold with positive probability for uncountably many times.



## Abdalaziz Elhaj Bakhit Elkhwad

Vilnius University


**Discrete Sturm-Liouville Problem for Two-Dimensional Elliptic Equation with the Multiple Integral**


**Abstract.** We will consider finite difference approximation for solving the nonlocal boundary value problem for two-dimensional Poisson equation in a square domain with Dirichlet and double integral boundary conditions. Motivated by some theoretical results we will study the spectrum structure of the corresponding difference eigenvalue problem.



## Habib Fourti

King Faisal University


**Multispike Solutions for a slightly subcritical elliptic problem with non-power nonlinearity**


**Abstract.** "In this work, we are concerned with the following elliptic equation

$$\begin{cases} -\Delta u &= |u|^{4/(n-2)}u/[\ln(e+|u|)]^{\varepsilon} \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega$ is a smooth bounded open domain in $R^n$; $n \geq 4$ and $\varepsilon > 0$. By using a Ljapunov-Schmidt reduction method, Clapp, Pardo, Pistoia and Saldana proved in [1] that there exists a single-peak positive solution for small $\varepsilon$. This solution blows up at a non-degenerate critical point of the Robin function as $\varepsilon$ goes to 0. Here we construct positive as well as changing sign solutions concentrated at several points inside the domain $\Omega$ at the same time. More precisely, we build solutions which blow up (positively or negatively) at distinct points which form a non-degenerate critical point of a function defined explicitly in terms of the Green function and its regular part. Our proof follows the finite reduction method introduced by Bahri, Li and Rey in [2]. (Joint work with M. Ben Ayed and R. Ghoudi)


References:


[1] M. Clapp, R. Pardo, A. Pistoia, A. Saldana, A solution to a slightly subcritical elliptic problem with non-power nonlinearity, Journal of Diff. Eq. Vol 275, (2021).

[2] A. Bahri, YY. Li and O. Rey, On a variational problem with lack of compactness: The topological effect of the critical points at infinity, Calculus of Variation and Partial Diff. Equa. 3 (1995), 67-94.

## Marco Gallo

Università Cattolica del Sacro Cuore, Brescia


**A doubly nonlocal interaction in the asymptotic decay of solutions**


**Abstract.** In this poster session we will discuss the asymptotic decay of solutions for equations of the type

$$(-\Delta)^s u + u = \left( \frac{1}{|x|^{N-\alpha}} * u^{p+1} \right) u^p \quad \text{in } \mathbb{R}^N$$

where $(-\Delta)^s$ denotes the fractional Laplacian, $s \in (0,1)$, and $\frac{1}{|x|^{N-\alpha}}*$ denotes the convolution with the Riesz potential, $\alpha \in (0,N)$. Both the operator and the nonlinearity are thus nonlocal. Here the power $p$ varies in the range $[\frac{\alpha}{N}, \frac{\alpha+2s}{N-2s}]$. In the local case $s=1$ it has been shown that the asymptotic decay of $u(x)$ as $|x| \to +\infty$ hardly changes when $p$ is smaller or greater than $1$: in this doubly nonlocal case, instead, we will see that the interaction of the two nonlocalities generates a different threshold, $p_{s,\alpha} := \frac{\alpha+2s}{N+2s}$, which verifies $p_{1,\alpha} \neq 1$.



## Raffaele Grande

Czech Academy of Sciences


**Horizontal mean curvature flow and asymptotic rescaling in the Heisenberg group**


**Abstract.** We derive the geometric evolution by (horizontal) mean curvature flow of a hypersurface embedded in the 1-dimensional Heisenberg group from a formal asymptotic expansion of a nonlocal mean-field equation. This result is obtained by using the anisotropic rescaling induced by the Carnot group structure of the Heisenberg group. This is motivated by the aim of connecting mechanisms at a microscopic (i.e. cellular) level to macroscopic models of image processing through a multi-scale approach. This is a joint work with G.Citti (Università di Bologna), N.Dirr (Cardiff University) and F.Dragoni (Cardiff University)



## Damiano Greco

Swansea University


**On some Thomas–Fermi type variational problems: an attractive and a repulsive case**


**Abstract.** We consider two Thomas-Fermi type variational problems. The first one concerns studying existence and qualitative properties of the minimizers for a Thomas–Fermi type energy functional with non local repulsion involving a convolution with the Riesz kernel and interaction with an external potential. Under mild assumptions on the latter, we establish uniqueness and qualitative properties such as positivity, regularity and decay at infinity of the global minimizer. The second problem concerns the study of optimisers for a Gagliardo–Nirenberg type inequality. Such problem is well understood in connection with Keller-Segel models and appears in the study of Thomas-Fermi limit regimes for the Choquard equations with local repulsion. We establish optimal ranges of parameters for the validity of the inequality, discuss the existence and qualitative properties of the optimisers.

## Matthias Grutzner

Humboldt-Universität Berlin


**Using complex analysis in 2D Stokes flow**


**Abstract.** Researching the dynamics of microswimmers in Stokes flow has a variety of applications throughout multiple fields. In this work an approach to the low Reynolds number case of the Navier Stokes equation in two dimensions using complex analysis is studied. Specifically, a circular microswimmer modelled by the first two terms in the Blake squirmer model is considered in wedge boundary conditions and equivalent singularities describing the microswimmer are obtained. By transforming the boundary conditions, implicit analytical solutions for the flow field for an arbitrary wedge angle are obtained.



## Karol Hajduk

Institute of Mathematics, Polish Academy of Sciences


**Incompressible convective Brinkman-Forchheimer equations in a thin domain**


**Abstract.** The flow of a fluid through a porous medium is clasically described by Darcy's law. However, it typically applies for sufficiently slow viscous flows, e.g. for flows with small Reynolds number (laminar flows). When the flow is non-Darcian (e.g. turbulent flows), various modifications of Darcy's law are used to describe it. In the poster we will present one of such models, namely the convective Brinkman-Forchheimer equations (CBF). From the mathematical perspective, this model can be seen also as the Navier-Stokes equations with damping term $|u|^{r-1}u$, called the absorption term (or the Forchheimer term). We will give an overwiev of some available results for this model. We will also discuss stationary CBF flow through a 2D thin channel. We are interested in existence of solutions of such flows and asymptotic analysis of the model. This is an ongoing project in collaboration with Marko Radulović (University of Zagreb).



## Kazuya Hirose

Hokkaido University, Japan


**Lower gradient estimates for viscosity solutions of Hamilton-Jacobi equation depending on the unknown function**


**Abstract.** In this poster, we derive the lower gradient estimate for viscosity solutions of the Hamilton--Jacobi equation with the convex Hamiltonian depending on the unknown function. We obtain gradient estimates in two different ways.



## Sho Katayama

University of Tokyo


**Supercritical Lane-Emden equation with a forcing term**


**Abstract.** This study concerns the structure of positive solutions to the elliptic problem for the Lane--Emden equation on the whole space with a positive forcing term. Under a suitable assumption on the forcing term, we give a complete classification of the unique existence/multiple existence/nonexistence of positive solutions with respect to the size of the forcing term and the exponent of nonlinearity.

## Jordan Marajh
Queen Mary University London

**Controlled regularity at future null infinity from past asymptotic initial data: wave equation**

**Abstract.** In this work, we present the results obtained from a problem inspired by conformal scattering in a neighbourhood of spatial and null infinity for the conformal wave equation on the Minkowski spacetime. Moreover, we show how one can prescribe characteristic initial data on past null infinity to guarantee a certain regularity on future null infinity. Some key techniques used in this analysis to construct the estimates are Friedrich's cylinder at spatial infinity, a Gr¨onwall argument which is non-degenerate at the critical sets and some general theory of symmetric hyperbolic systems. This is work to appear soon on arXiv in collaboration with Grigalius Taujanskas and Juan A. Valiente Kroon (Marajh et al. 2024).

## Prachi Sahjwani
Cardiff University

**Stability of Alexandrov-Fenchel inequalities in hyperbolic space**

**Abstract.** In this talk, I will discuss the stability of Alexandrov-Fenchel inequalities in hyperbolic space. I will give a brief overview of the inequalities and their stability problems. To understand what I mean by stability, I will first discuss it for Isoperimetric inequality, which is a special case of Alexandrov-Fenchel inequalities. This is joint work with Prof. Dr. Julian Scheuer.

## Jacopo Schino
University of Warsaw

**About solutions with prescribed norm to a Schrödinger equation in a mixed regime**

**Abstract.** Schrödinger-type equations model a lot of natural phenomena and their solutions have interesting and important properties. This gives rise to the search for normalised solutions, i.e., when the $L^2$-norm is prescribed. Here, I will present results in a mixed regime, that is, when the non-linear term behaves differently at the origin and at infinity.

## Billy Sumners
Heriot-Watt University

**Triplets and Compatibility in Martensitic Phase Transformations**

**Abstract.** Recently, a nickel-titanium-based shape-memory alloy which displays good reversibility under thermal cycling was developed by the group of Tomonari Inamura. We attempt to continue the work started by themselves and Francesco Della Porta of modelling the microstructure that appears in the martensitic phase. The key factor distinguishing this alloy from ones with poorer reversibility are that its variants are highly compatible, represented by the satisfaction of a property of matrices known as a triplet condition. Also of interest are the lack of identifiable remnants in the martensite phase of the phase transition from the austenite phase. In light of this, we study the rigidity properties of Sobolev mappings whose gradient lives in a set of matrices satisfying a triplet condition. Furthermore, in order to model the phase transition, we discuss when such mappings are also allowed to have their gradient live in an additional set representing the austenite phase, and determine in particular at least some of the interfaces between the austenite and martensite phases.

Ye Zhang
Okinawa Institute of Science and Technology Graduate University

**Horizontal semiconcavity for the square of Carnot-Carath\'eodory distance on Carnot groups and applications to Hamilton-Jacobi equations**

**Abstract.** In our recent work, we establish the horizontal semiconcavity (h-semiconcavity) of the square of Carnot-Caratheodory distance from the origin, in a class of Carnot groups. Via Hopf-Lax formula, we apply this property to show h-semiconcavity for the solutions of a class of non-coercive evolutive Hamilton-Jacobi equations.

Erbol Zhanpeisov
Okinawa Institute of Science and Technology

**Liouville-type theorem for fully nonlinear elliptic and parabolic equations with boundary degeneracy**

**Abstract.** We study a general class of fully nonlinear boundary-degenerate elliptic or parabolic equations that admit a trivial solution. Although no boundary conditions are posed together with the equations, we show that the operator degeneracy actually generates an implicit boundary condition. Under appropriate assumptions on the degeneracy rate and regularity of the operator, we then prove that there exist no bounded solutions other than the trivial one. Our method is based on the uniqueness arguments for viscosity solutions of state constraint problems for Hamilton-Jacobi equations.