

# POSITIVITY PRINCIPLES AND DECAY OF SOLUTIONS IN SEMILINEAR ELLIPTIC PROBLEMS

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ABSTRACT. This is an outline of the lectures given at Roma Tor Vergata in May-June 2024. Please, email me if you spot any typos or mistakes.

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Consider the linear equation

$$(*) \quad -\Delta u + Vu = f \quad \text{in } \Omega,$$

where  $\Omega$  is a domain (open connected set) in  $\mathbb{R}^N$  with  $N \geq 2$  and  $V$  is a potential and  $0 \leq f \in L^1_{loc}(\Omega)$  is a nonnegative right hand side. We assume that  $V = V^+ - V^-$  and

$$V^+ \in L^\infty_{loc}(\Omega), \quad V^- \in L^1_{loc}(\Omega).$$

The natural quadratic form associated to the Schrödinger operator  $-\Delta + V$  is given by

$$\mathcal{E}_V(\varphi) := \int_{\Omega} |\nabla \varphi|^2 dx + \int_{\Omega} V \varphi^2 dx \quad (\varphi \in H^1_c(\Omega) \cap L^\infty(\Omega)),$$

where the subscript  $c$  denotes the *compact support*. We say that  $\mathcal{E}_V$  is non-negative if

$$\mathcal{E}_V(\varphi) \geq 0, \quad \forall \varphi \in H^1_c(\Omega) \cap L^\infty(\Omega).$$

We are going to study the relationship between the existence and some properties of positive supersolutions to  $(*)$  and non-negativity of the quadratic form  $\mathcal{E}_V$ . Such a relationship is commonly referred to as Agmon–Allegretto–Piepenbrink’s (AAP) Principle or “Ground

State transformation". Our exposition is inspired and largely based on [4–7, 15] but is adapted with applications to semilinear equations in mind, as developed in [12–14].

### 1. THE AAP POSITIVITY PRINCIPLE

A *weak* supersolution to  $(*)$  is a function  $u \in H_{loc}^1(\Omega) \cap L_{loc}^1(\Omega, V dx)$  such that

$$(1.1) \quad \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx + \int_{\Omega} V u \varphi \, dx \geq \int_{\Omega} f \varphi \, dx, \quad \forall 0 \leq \varphi \in H_c^1(\Omega) \cap L_c^\infty(\Omega).$$

The notions of a weak sub-solution and weak solution are defined similarly by replacing " $\geq$ " with " $\leq$ " and " $=$ " respectively. Note that if  $u \geq 0$  is a weak supersolution to  $(*)$  with  $f \geq 0$  then  $u > 0$  in  $\Omega$ , in the sense that  $u^{-1} \in L_{loc}^\infty(\Omega)$ . This follows from the weak Harnack inequality which is ensured by the assumption  $V^+ \in L_{loc}^\infty(\Omega)$ .

In what follows we often omit the word *weak* and use simply solution, super and sub-solution. We say that  $u$  is a solution to  $-\Delta + V$  if  $u$  is a solution for  $(*)$  with  $f = 0$ . Similarly for sub and supersolutions.

When  $V \geq 0$ , the quadratic form  $\mathcal{E}_V$  is non-negative, and any positive constant is a supersolution. Thus, the interesting case to consider is when  $V$  is negative or changes sign. The fundamental relation between the existence of positive supersolutions to  $(*)$  and non-negativity of the quadratic form  $\mathcal{E}_V$  is described in the following result, which was originally proved by W. Allegretto [2] and J. Piepenbrink [3] in 1974, and later became the foundation of S. Agmon's Criticality Theory [4].

**Theorem 1.1** (AAP Positivity Principle). *Assume that  $(*)$  admits a weak positive (super)solution  $u_* > 0$ . Then*

$$(1.2) \quad \mathcal{E}_V(\varphi) (\geq) = \int_{\Omega} \left| \nabla \left( \frac{\varphi}{u_*} \right) \right|^2 u_*^2 dx + \int_{\Omega} \varphi^2 \frac{f}{u_*} dx, \quad \forall 0 \leq \varphi \in H_c^1(\Omega) \cap L_c^\infty(\Omega).$$

*Proof.* Let  $0 \leq \varphi \in H_c^1(\Omega) \cap L_c^\infty(\Omega)$ . Then  $\psi := \frac{\varphi^2}{u_*} \in H_c^1(\Omega) \cap L_c^\infty(\Omega)$ . Testing  $(*)$  against  $\psi$  we arrive at

$$2 \int_{\Omega} \varphi \nabla \varphi \frac{\nabla u_*}{u_*} dx (\geq) = \int_{\Omega} \varphi^2 \frac{|\nabla u_*|^2}{u_*^2} dx - \int_{\Omega} V \varphi^2 dx + \int_{\Omega} \frac{f}{u_*} \varphi^2 dx.$$

Direct computation yields

$$\begin{aligned} & \int_{\Omega} (|\nabla \varphi|^2 + V \varphi^2) dx - \int_{\Omega} \left| \nabla \left( \frac{\varphi}{u_*} \right) \right|^2 u_*^2 dx \\ &= \int_{\Omega} (|\nabla \varphi|^2 + V \varphi^2) dx - \int_{\Omega} \left( \frac{|\nabla \varphi|^2}{u_*^2} - 2 \varphi \nabla \varphi \frac{\nabla u_*}{u_*^3} + \varphi^2 \frac{|\nabla u_*|^2}{u_*^4} \right) u_*^2 dx \\ &= 2 \int_{\Omega} \varphi \nabla \varphi \frac{\nabla u_*}{u_*} dx - \int_{\Omega} \varphi^2 \frac{|\nabla u_*|^2}{u_*^2} dx + \int_{\Omega} V \varphi^2 dx (\geq) = \int_{\Omega} \varphi^2 \frac{f}{u_*} dx. \end{aligned}$$

This proves (1.2) on  $H_c^1(\Omega) \cap L_c^\infty(\Omega)$ .  $\square$

The following straightforward corollary of Theorem 1.1 is crucial in the analysis of nonexistence of positive solutions to semilinear equations.

**Corollary 1.2** (Nonexistence principle). *Assume there exists  $\varphi \in C_c^\infty(\Omega)$  such that  $\mathcal{E}_V(\varphi) < 0$ . Then  $-\Delta + V$  has no positive weak supersolutions in  $\Omega$ .*

**Remark 1.3.** The computation in the proof of the AAP is often known as the *Picone's identity*:

$$|\nabla \varphi|^2 - \nabla \left( \frac{\varphi^2}{u} \right) \cdot \nabla u = \left| \nabla \left( \frac{\varphi}{u} \right) \right|^2 u^2 \geq 0,$$

for any  $\varphi, v \in H_{loc}^1(\Omega)$  such that  $\varphi \geq 0$ ,  $u > 0$  in  $\Omega$ .

**Remark 1.4.** It is simple to see that the " $u_*$ -ground state transformed" quadratic form

$$\mathcal{E}_{u_*}(\varphi) = \int_{\Omega} \left| \nabla \left( \frac{\varphi}{u_*} \right) \right|^2 u_*^2 dx$$

is generated by the nonnegative selfadjoint operator

$$-\Delta + 2\nabla \log(u_*) \nabla$$

in  $L^2(\Omega, u_*^2 dx)$ .

**Remark 1.5.** (a) We do not assume any boundary conditions on the reference supersolution  $u_*$  in Lemma 1.1. We also do not require any boundary regularity of  $u_*$ . All we need is that  $u_* \in H_{loc}^1(\Omega)$  is a positive weak supersolution of  $-\Delta + V$ . In many examples  $u_*$  is positive or even singular on parts of the boundary (Example 2.1), but  $u_*$  also may be zero on the boundary (Examples 2.4 and 2.6).

(b) We do not assume any boundary regularity of the potential  $V$ , which could be 'very' singular on  $\partial\Omega$  (Examples 2.1, 2.4).

(c) Domain  $\Omega$  need not be smooth (e.g.  $\mathbb{R}^N \setminus \{0\}$  in Example 2.1).

## 2. HARDY TYPE INEQUALITIES

In this section we will use the AAP positivity principle to prove several Hardy type inequalities. We start with the hardy inequality in  $\mathbb{R}^N$ .

**Example 2.1** (Hardy inequality in  $\mathbb{R}^N$ ). Consider

$$(2.1) \quad -\Delta u - \frac{C_H}{|x|^2} u = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

where  $N \geq 3$  and  $C_H := (N-2)^2/4$ . Clearly,

$$u_*(|x|) = |x|^{\frac{2-N}{2}}$$

is a positive weak solution of (2.1). Hence, by Lemma 1.1 we obtain Hardy's inequality

$$(2.2) \quad \int_{\mathbb{R}^N} |\nabla \varphi|^2 \geq C_H \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}).$$

**Remark 2.2.** We had to remove the origin in (2.1) because  $|x|^{\frac{2-N}{2}} \notin H_{loc}^1(\mathbb{R}^N)$  – the singularity at the origin is too strong. To prove (2.1) in  $\mathbb{R}^N$ , first prove

$$(2.3) \quad \int_{\mathbb{R}^N} |\nabla \varphi|^2 \geq c \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N).$$

for any  $c \in (0, C_H)$ . To do this, use  $u_* = |x|^{\alpha_+}$ , where  $\alpha_+$  is the biggest root of the scalar equation  $-\alpha(\alpha + N - 2) = c$ . Since  $\alpha_+ > -\frac{N-2}{2}$ , we have  $|x|^{\alpha_+} \in H_{loc}^1(\mathbb{R}^N)$  and the AAP principle is valid. Since (2.3) is valid for any  $c < C_H$ , it is also valid for  $c = C_H$ .

**Exercise 2.3.** Show that  $C_H = \frac{(N-2)^2}{4}$  ( $N \geq 3$ ) is the *best constant* in the Hardy inequality (2.2) i.e. the inequality fails for any  $c > C_H$ .

*Hint.* Take a family of trial functions  $\varphi_R = u_* \eta_R$  where  $u_* = |x|^{\frac{2-N}{2}}$  and  $\eta_R(r)$  is a non-negative  $C^1$ -cut-off function such that  $\eta_R = 1$  for  $r \in [0, R]$ ,  $\eta_R = 0$  for  $r > 2R$  and  $|\eta'(r)| \leq 2/R$ . Then use representation (1.2) to show that for any  $\varepsilon > 0$ ,

$$\int_{\mathbb{R}^N} |\nabla \varphi_R|^2 - C_H \int_{\mathbb{R}^N} \frac{\varphi_R^2}{|x|^2} - \varepsilon \int_{\mathbb{R}^N} \frac{\varphi_R^2}{|x|^2} \rightarrow -\infty$$

as  $R \rightarrow \infty$ .

**Example 2.4** (Hardy inequality in  $\mathbb{R}^2$ ). Consider

$$(2.4) \quad -\Delta u - \frac{1/4}{|x|^2 \left( \log \frac{|x|}{\rho} \right)^2} u = 0 \quad \text{in } \mathbb{R}^2 \setminus B_\rho(0).$$

where  $\rho > 0$ . Clearly,

$$u_*(|x|) = \left( \log \frac{|x|}{\rho} \right)^{1/2}$$

is a positive weak solution of (2.4). Hence, by Lemma 1.1 we obtain a Hardy type inequality

$$\int_{\mathbb{R}^2} |\nabla \varphi|^2 \geq \frac{1}{4} \int_{\mathbb{R}^2} \frac{\varphi^2}{|x|^2 \left( \log \frac{|x|}{\rho} \right)^2}, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^2 \setminus B_\rho(0)).$$

**Exercise 2.5** (Improved Hardy inequality). For  $N \geq 2$  prove the following inequality

$$\int_{\mathbb{R}^N} |\nabla \varphi|^2 \geq C_H \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} + \frac{1}{4} \int_{\mathbb{R}^2} \frac{\varphi^2}{|x|^2 (\log |x|)^2}, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus B_1).$$

*Hint:* Take  $u_*(|x|) = |x|^{-\frac{N-2}{2}} (\log |x|)^{1/2}$

**Example 2.6** (Principal Dirichlet eigenvalue). Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain,  $\lambda_1 > 0$  be the principal Dirichlet eigenvalue of  $-\Delta$  on  $\Omega$  and  $u_* := \phi_1 > 0$  be the corresponding principal eigenfunction. By Lemma 1.1 we obtain the inequality

$$(2.5) \quad \int_{\Omega} |\nabla \varphi|^2 \geq \lambda_1 \int_{\Omega} \varphi^2, \quad \forall \varphi \in C_c^\infty(\Omega).$$

**Example 2.7** (Torsional Hardy inequality). Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain and  $\psi_\Omega > 0$  be a torsion function of  $\Omega$ , that is the unique solution of

$$-\Delta\psi = 1 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Then we obtain “torsional Hardy inequality”

$$\int_{\Omega} |\nabla\varphi|^2 \geq \int_{\Omega} \frac{\varphi^2}{\psi_\Omega} \quad \forall \varphi \in C_c^\infty(\Omega).$$

Now take  $\varphi = \varphi_1$ , the principal Dirichlet eigenvalue of  $\Omega$  such that  $\|\varphi_1\|_2 = 1$ . Then

$$(2.6) \quad \lambda_1 = \int_{\Omega} |\nabla\varphi|^2 \geq \int_{\Omega} \frac{\varphi_1^2}{\psi_\Omega} \geq \frac{\|\varphi_1\|_2^2}{\|\psi_\Omega\|_\infty} = \frac{1}{\|\psi_\Omega\|_\infty}.$$

Hence we deduce a lower bound

$$\|\psi_\Omega\|_\infty \geq \frac{1}{\lambda_1}.$$

**Exercise 2.8** (Barta’s inequality). Prove the following inequality

$$(2.7) \quad \inf_{x \in \Omega} \frac{(-\Delta - V)\phi}{\phi} \leq \inf_{0 \neq u \in C_c^\infty(\Omega)} \frac{\mathcal{E}_V(u, u)}{\|u\|_{L^2}^2}.$$

*Hint:* Similar to (2.6).

**Exercise 2.9.** For  $R > 0$  and  $B_R \subset \mathbb{R}^N$ , prove the inequality

$$\int_{B_R} |\nabla\varphi|^2 \geq 2N \int_{B_R} \frac{\varphi^2}{R^2 - |x|^2} \quad \forall \varphi \in C_c^\infty(B_R).$$

### 3. THE ENERGY SPACE

**3.1. The  $\lambda$ -property and the energy space.** Following [5], we say that the form  $\mathcal{E}_V$  satisfies the  $\lambda$ -property if there exists a function  $0 < \lambda \in L_{loc}^1(\Omega)$  such that  $\lambda^{-1} \in L_{loc}^\infty(\Omega)$  and

$$(3.1) \quad \mathcal{E}_V(u) \geq \int_{\Omega} u^2 \lambda(x) dx, \quad \forall u \in C_c^\infty(\Omega).$$

It is elementary to see that if  $\mathcal{E}_V$  satisfies the  $\lambda$ -property then  $\|\cdot\|_V := \sqrt{\mathcal{E}_V(\cdot)}$  is a norm on  $C_c^\infty(\Omega)$ . Denote by  $D_V^1(\Omega)$  the family of measurable a.e. finite functions  $u : \Omega \rightarrow \mathbb{R}$  such that there exists an  $\|\cdot\|_V$ -Cauchy sequence  $(u_n) \subset C_c^\infty(\Omega)$  that converges to  $u$  a.e. in  $\Omega$ . This sequence  $(u_n)$  is called an *approximating sequence* for  $u \in D_V^1(\Omega)$ . Then the limit

$$\mathcal{E}_V(u) := \lim_{n \rightarrow \infty} \mathcal{E}_V(u_n)$$

exists and is independent of the choice of the approximating sequence. Thus  $\mathcal{E}_V$  is extended uniquely to a nonnegative definite quadratic form on  $D_V^1(\Omega)$ . The family  $D_V^1(\Omega)$  is called the *energy space* of  $\mathcal{E}_V$ .

**Theorem 3.1.** *The energy space  $D_V^1(\Omega)$  is a Hilbert space with the inner product*

$$\langle u, v \rangle_V := \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Omega} V u v \, dx$$

*Moreover,  $D_V^1(\Omega)$  is continuously embedded into  $L^2(\Omega, \lambda \, dx)$ . Additionally, if  $u_*$  is locally Lipschitz in  $\Omega$  then  $D_V^1(\Omega)$  is continuously embedded in  $H_{loc}^1(\Omega)$ .*

*Proof.* Clearly, (3.1) implies that  $(u_n)$  converges to  $u$  in  $L^2(\Omega, \lambda \, dx)$  and hence  $D_V^1(\Omega)$  is continuously embedded into  $L^2(\Omega, \lambda \, dx)$ .

To prove the embedding  $D_V^1(\Omega) \subset H_{loc}^1(\Omega)$ , following [10] we first observe that

$$\begin{aligned} \int_{\Omega} \left| \nabla \left( \frac{\varphi}{u_*} \right) \right|^2 u_*^2 \, dx &= \int_{\Omega} (|\nabla \varphi|^2 - 2 \nabla \varphi \cdot \frac{\varphi}{u} \nabla u + \frac{\varphi^2}{u^2} |\nabla u|^2) \, dx \\ &\geq \frac{1}{2} \int_{\Omega} (|\nabla \varphi|^2 - |\nabla \log(u)|^2 \varphi^2) \, dx, \end{aligned}$$

where we used the Cauchy–Schwartz to estimate

$$\nabla \varphi \cdot \frac{\varphi}{u} \nabla u \leq \frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{\varepsilon^{-1}}{2} \frac{\varphi^2}{u^2} |\nabla u|^2$$

and set  $\varepsilon = \frac{1}{2}$ . Then using (1.2) we obtain

$$\begin{aligned} \frac{1}{2} \int_{\Omega'} |\nabla \varphi|^2 &\leq \int_{\Omega} \left| \nabla \left( \frac{\varphi}{u_*} \right) \right|^2 u_*^2 \, dx + \frac{1}{2} \int_{\Omega'} |\nabla \log(u)|^2 \varphi^2 \, dx \\ &\leq \mathcal{E}_V(\varphi) + c' \int_{\Omega'} \varphi^2 \lambda(x) \, dx \leq (1 + c') \mathcal{E}_V(\varphi), \end{aligned}$$

where the constant  $c' > 0$  depends on the choice of subdomain  $\Omega'$ .  $\square$

**Remark 3.2.** If  $\mathcal{E}_V$  satisfies the  $\lambda$ -property in  $\Omega$  and  $\Omega'$  is a bounded subdomain such that  $\bar{\Omega}' \subset \Omega$ , then  $\mathcal{E}_V$  satisfies the  $\lambda$ -property in  $\Omega'$  with  $\lambda^{-1} \in L^\infty(\Omega')$ . Hence  $D_V^1(\Omega') = H_0^1(\Omega')$ .

Denote by  $D_V^{-1}(\Omega)$  the dual space of all linear continuous functionals on  $D_V^1(\Omega)$ . The following lemma is a standard consequence of the Riesz Representation Theorem.

**Lemma 3.3.** *Assume that  $\mathcal{E}_V$  satisfies the  $\lambda$ -property. Let  $l \in D_V^{-1}(\Omega)$ . Then there exists a unique  $u_* \in D_V^1(\Omega)$  such that*

$$(3.2) \quad \langle u_*, \varphi \rangle = l(\varphi), \quad \forall \varphi \in D_V^1(\Omega).$$

Since  $D_V^1(\Omega) \subset L^2(\Omega, \lambda \, dx)$ , by duality we conclude that  $L^2(\Omega, \lambda^{-1} \, dx) \subset D_V^{-1}(\Omega)$ , in the sense that if  $f \in L^2(\Omega, \lambda^{-1} \, dx)$  then

$$l_f(\varphi) := \int_{\Omega} f \varphi \, dx \in D_V^{-1}(\Omega).$$

Thus Lemma 3.3 implies that for any  $f \in L^2(\Omega, \lambda^{-1} \, dx)$  the equation

$$(3.3) \quad -\Delta u + V u = f \quad \text{in } D_V^1(\Omega)$$

has a unique solution  $u_f \in D_V^1(\Omega)$ .

**Exercise 3.4** (The space  $H^1(\mathbb{R}^N)$ ). Show that  $-\Delta + 1$  satisfies the  $\lambda$ -property in  $\mathbb{R}^N$  and that the corresponding energy space  $D_1^1(\mathbb{R}^N)$  coincides with the standard Sobolev space  $H^1(\mathbb{R}^N)$ .

**Example 3.5** (The space  $D_0^1(\mathbb{R}^N)$  for  $N \geq 3$ ). Consider  $-\Delta$  on  $\mathbb{R}^N$  with  $N \geq 3$ . Take

$$u_*(|x|) := (1 + |x|^2)^{-\frac{N-2}{2}},$$

which is known as a Talenti function<sup>1</sup>. Then

$$-\Delta u_* = N(N-2)(1 + |x|^2)^{-\frac{N+2}{2}}$$

and

$$\lambda(x) = \frac{-\Delta u_*(x)}{u_*(x)} = N(N-2)(1 + |x|^2)^{-2}.$$

Hence the space  $D_0^1(\mathbb{R}^N)$ , a completion of  $C_c^\infty(\mathbb{R}^N)$  with respect to the homogenous gradient norm  $\|\nabla u\|_{L^2(\mathbb{R}^N)}$ , is a well-defined Hilbert space and moreover,

$$D_0^1(\mathbb{R}^N) \subset L^2(\mathbb{R}^N, (1 + |x|^2)^{-2} dx).$$

It is easy to see that  $D_0^1(\mathbb{R}^N) \not\subset L^2(\mathbb{R}^N)$  and hence  $D_0^1(\mathbb{R}^N) \neq H^1(\mathbb{R}^N)$ . To see this when  $N = 3, 4$ , check that  $\|\nabla u_*\|_{L^2(\mathbb{R}^N)} < \infty$  but  $u_* \notin L^2(\mathbb{R}^N)$  when  $N = 3, 4$ . In fact,  $D_0^1(\mathbb{R}^N) \subset L^{\frac{2N}{N-2}}(\mathbb{R}^N)$  – this follows from the Sobolev inequality.<sup>2</sup>

**Example 3.6** (The space  $D^1(\mathbb{R}^2)$  is not well-defined). According to the Liouville's theorem, every positive super solution on  $\mathbb{R}^2$  is constant. This means that the Laplacian  $-\Delta$  does not satisfy  $\lambda$ -property on  $\mathbb{R}^2$ . Hence we can not apply Theorem 3.1 to construct the energy space  $D_0^1(\mathbb{R}^2)$ . In fact, such a space is not well defined as the space of functions, see [15] for details.

**Example 3.7** (The space  $D^1(\mathbb{R}^2 \setminus \bar{B}_\rho)$  in  $\mathbb{R}^2$ ). Consider  $-\Delta$  on  $\Omega = \mathbb{R}^2 \setminus \bar{B}_\rho$ , for a  $\rho > 0$ . Take

$$u_*(|x|) = \left( \log \frac{|x|}{\rho} \right)^{1/2}.$$

As in the Example 2.4, we conclude that  $-\Delta$  in  $\Omega$  satisfies the  $\lambda$ -property with

$$\lambda(x) = \frac{1/4}{|x|^2 \left( \log \frac{|x|}{\rho} \right)^2}.$$

Hence the space  $D_0^1(\mathbb{R}^2 \setminus \bar{B}_\rho)$ , the completion of  $C_c^\infty(\mathbb{R}^2 \setminus \bar{B}_\rho)$  with respect to the homogenous gradient norm  $\|\nabla u\|_{L^2(\mathbb{R}^2)}$  is a well-defined Hilbert space.

**Exercise 3.8.** Use the improved Hardy Inequality 2.5 to show that for  $N \geq 3$  the critical Hardy operator

$$-\Delta - \frac{C_H}{|x|^2}, \quad C_H := \left( \frac{N-2}{2} \right)^2,$$

<sup>1</sup>Up to a rescaling, it is a minimizer of the Sobolev inequality

<sup>2</sup>The space  $D_0^1(\mathbb{R}^N)$  from this example is also often denoted as  $D^1(\mathbb{R}^N)$  or  $\dot{H}^1(\mathbb{R}^N)$ .

satisfies the  $\lambda$ -property in  $\mathbb{R}^N \setminus \bar{B}_1$  and define the energy space  $D^1_{-\frac{C_H}{|x|^2}}(\mathbb{R}^N \setminus \bar{B}_1)$ . Then use the arguments in Exercise 2.3 to show that

$$u_0 = |x|^{\frac{2-N}{2}} \in D^1_{-\frac{C_H}{|x|^2}}(\mathbb{R}^N \setminus \bar{B}_1).$$

Note that  $u_0 \notin H^1_{loc}(\mathbb{R}^N)$ !

**3.2. Maximum and comparison principles.** If  $\mathcal{E}_V$  satisfies the  $\lambda$ -property then  $-\Delta + V$  satisfies weak maximum and comparison principles. In order to prove that we need to know that  $D^1_V(\Omega)$  is invariant under the standard truncations.

**Lemma 3.9.** *If  $u \in D^1_V(\Omega)$  then  $u^+, u^- \in D^1_V(\Omega)$  and*

$$(3.4) \quad \mathcal{E}_V(u^\pm) \leq \mathcal{E}(u), \quad \forall u \in D^1_V(\Omega).$$

*If  $u, v \in D^1_V(\Omega)$  then  $u \vee v, u \wedge v \in D^1_V(\Omega)$ .*

*Proof.* See [14, Lemma A.1]. □

**Remark 3.10.** We do not claim that  $u \in D^1_V(\Omega)$  implies  $u \wedge 1 \in D^1_V(\Omega)$ .

**Exercise 3.11.** Let  $(\varphi_n) \subset C_c^\infty(\Omega)$  be an approximating sequence for  $0 \leq w \in D^1_V(\Omega)$ . Show that  $(\varphi_n^+) \subset D^1_V(\Omega)$  is also an approximating sequence for  $w$ , in the sense that  $\mathcal{E}_V(\varphi_n^+) \rightarrow \mathcal{E}_V(w)$ .

*Hint.* Show that  $(\varphi_n^-)$  is a Cauchy sequence and hence  $\mathcal{E}_V(\varphi_n^-) \rightarrow 0$ .

**Lemma 3.12.** *Assume that  $\mathcal{E}_V$  satisfies the  $\lambda$ -property. Let  $w \in H^1_{loc}(\Omega) \cap L^1(\Omega, Vdx)$  be a super-solution to  $(*)$  such that  $w^- \in D^1_V(\Omega)$ . Then  $w \geq 0$  in  $\Omega$ .*

*Proof.* Let  $(\varphi_n) \subset C_c^\infty(\Omega)$  be an approximating sequence for  $w^- \in D^1_V(\Omega)$ . Set

$$w_n := \varphi_n^+ \wedge w^-.$$

Hence  $0 \leq w_n \in D^1_V(\Omega)$ , by Lemma 3.9. Note that  $w_n = w^- - (\varphi_n^+ - w^-)^-$ . Therefore

$$\mathcal{E}_V(w^- - w_n) = \mathcal{E}_V((\varphi_n^+ - w^-)^-) \leq \mathcal{E}_V(\varphi_n^+ - w^-) \rightarrow 0.$$

Thus  $(w_n)$  is a nonnegative approximating sequence for  $w^-$ . Since  $0 \leq w_n \leq w^-$  we have  $w^+ \wedge w_n = 0$ , and then we obtain

$$0 \leq \langle w, w_n \rangle_V = -\langle w^-, w_n \rangle_V \rightarrow -\mathcal{E}_V(w^-) \leq 0.$$

We conclude that  $w^- = 0$ . □

The following comparison principle is a straightforward consequence of Lemma 3.12.

**Corollary 3.13.** *Assume that  $\mathcal{E}_V$  satisfies the  $\lambda$ -property. Let  $w \in H^1_{loc}(\Omega) \cap L^1(\Omega, Vdx)$  be a super-solution to  $(*)$  and  $v \in H^1_{loc}(\Omega) \cap L^1(\Omega, Vdx)$  be a sub-solution to  $(*)$  such that  $(w - v)^- \in D^1_V(\Omega)$ . Then  $w \geq v$  in  $\Omega$ .*

**Remark 3.14.** In particular, Lemmas 3.3 and 3.12 imply that if  $\mathcal{E}_V$  satisfies the  $\lambda$ -property then equation  $(*)$  has a “rich” cone of positive super-solutions. Indeed, if  $0 \leq f \in L^2(\Omega, \lambda^{-1}dx)$  and  $u_f \in D_V^1(\Omega)$  is the solution to (3.3) constructed in Lemma 3.3 then  $u_f > 0$  in  $\Omega$  by Lemma 3.12 and hence  $u_f$  is a weak positive supersolution for  $-\Delta + V$ . The situation is different if  $\mathcal{E}_V$  is nonnegative but *does not* satisfy the  $\lambda$ -property.

**Example 3.15** (Liouville’s Theorem on the plane). Consider  $-\Delta$  on  $\mathbb{R}^2$ . Obviously, in this case the form  $\mathcal{E}_0$  is nonnegative. Yet, according to the classical Liouville’s theorem the only positive superharmonic for  $-\Delta$  on  $\mathbb{R}^2$  is constant.

**Example 3.16** (Principal Dirichlet eigenvalue). Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain,  $\lambda_1 > 0$  be the principal Dirichlet eigenvalue of  $-\Delta$  on  $\Omega$  as in Example 2.6. By (2.5), the form  $\mathcal{E}_{\lambda_1}$  is nonnegative. Yet, the corresponding principal eigenfunction  $\phi_1 > 0$  is (up to a constant) the only positive supersolution to  $-\Delta - \lambda_1$  in  $\Omega$  (see Corollary 1.2).

The above examples clarify the following classification:

- $-\Delta + V$  is *subcritical* if  $\mathcal{E}_V$  satisfies the  $\lambda$ -property. In this case  $-\Delta + V$  has a “rich” cone of positive supersolutions (see Remark 3.14).
- $-\Delta + V$  is *critical* if  $\mathcal{E}_V$  is nonnegative but does not satisfy the  $\lambda$ -property. In this case  $-\Delta + V$  has exactly one (up to a scalar) positive supersolution, which is actually a solution (see [4, Theorem 5.2]).
- $-\Delta + V$  is *supercritical* if  $\mathcal{E}_V$  is not nonnegative. In this case  $-\Delta + V$  has no positive supersolutions (see Corollary 1.2).

Further study of  $-\Delta + V$  from the point of view of this classification is known as *criticality theory*, see [4, 6, 7]. In this lectures we are interested in one particular aspect only – we want to characterise the “size” of the cone of positive super-solutions of subcritical operators in terms of the admissible decay (or growth) of supersolutions “at infinity”.

#### 4. A PHRAGMÉN-LINDELÖF PRINCIPLE

In this section we assume that  $\Omega$  is an exterior domain in  $\mathbb{R}^N$  such that  $\mathbb{R}^N \setminus \bar{B}_1 \subset \Omega$  and  $\partial\Omega \neq \emptyset$ . We also always assume that  $-\Delta + V$  satisfies the  $\lambda$ -property in  $\Omega$ . Note that then  $-\Delta + V$  also satisfies the  $\lambda$ -property in any subdomain  $\Omega' \subset \Omega$ , see Remark 3.2. We are going to study admissible decay at infinity, i.e. as  $|x| \rightarrow \infty$ , of all positive supersolutions to  $-\Delta + V$  in  $\Omega$ .

**Definition 4.1.** We say  $u > 0$  is a *small* (sub)solution at infinity for  $-\Delta + V$  if  $u$  is a (sub)solution for  $-\Delta + V$  in  $B_R^c$  for some  $R \geq 1$  and there exists a supersolution  $w_* > 0$  for  $-\Delta + V$  in  $\Omega$ , such that

$$(4.1) \quad \lim_{|x| \rightarrow \infty} \frac{w_*}{u} = +\infty.$$

We say  $U > 0$  is a *large* (sub)solution at infinity for  $-\Delta + V$  in  $\Omega$  if  $U$  is (sub)solution for  $-\Delta + V$  in  $B_R^c$  for some  $R \geq 1$  and  $U$  is *not* a small (sub)solution, i.e. for any supersolution

$w > 0$  for  $-\Delta + V$  in  $\Omega$ ,

$$(4.2) \quad \liminf_{|x| \rightarrow \infty} \frac{w}{U} < +\infty.$$

**Exercise 4.2.** Prove that if  $u > 0$  is a small and  $U > 0$  a large solution at infinity for  $-\Delta + V$  in  $\Omega$  then

$$(4.3) \quad \limsup_{|x| \rightarrow \infty} \frac{U}{u} = +\infty.$$

Intuitively, a small solution at infinity is “smaller” than one of the supersolutions. Then a *large* solution at infinity is *not* “smaller” than any of the supersolutions, i.e. in some sense it dominates at infinity all positive supersolutions.<sup>3</sup> An essential observation is that if a solution is smaller than *one* of the positive supersolutions in the sense of (4.1), then up to a constant it is dominated by *every* positive supersolution.

**Lemma 4.3** (Small solution lemma). *If  $u > 0$  is a small solution at infinity for  $-\Delta + V$  in  $\Omega$  then for any supersolution  $w > 0$  for  $-\Delta + V$  in  $\Omega$  there exists  $c > 0$  such that*

$$(4.4) \quad u(x) \leq cw(x) \quad (|x| \geq 1).$$

*Proof.* Choose  $c > 0$  such that  $cu \leq w$  as  $|x| = 1$ . For small  $\varepsilon > 0$  consider the barrier functions

$$u_\varepsilon := cu - \varepsilon w_*.$$

In view of (4.1), there exists  $\rho_\varepsilon > 1$  such that

$$u_\varepsilon \leq 0 \quad (|x| \geq \rho_\varepsilon)$$

and moreover,  $\rho_\varepsilon \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ .

Set  $\Omega_\varepsilon := \{x \in \mathbb{R}^N : 1 < |x| < \rho_\varepsilon\}$ . Since  $w > 0$ , we conclude that

$$u_\varepsilon \leq w \quad \text{on } \partial\Omega_\varepsilon$$

and hence  $(w - u_\varepsilon)^- \in H_0^1(\Omega_\varepsilon)$ , see Remark 3.2. By the comparison principle of Corollary 3.13 we conclude that

$$u_\varepsilon \leq w \quad \text{in } \Omega_\varepsilon.$$

Then the assertion follows, since  $\varepsilon > 0$  could be taken arbitrary small and taking into account that  $\rho_\varepsilon \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ .  $\square$

**Proposition 4.4** (Phragmén-Lindelöf principle for supersolutions). *Let  $u > 0$  be a small and  $U > 0$  a large solution at infinity for  $-\Delta + V$  in  $\Omega$ . Then for any supersolution  $w > 0$  for  $-\Delta + V$  in  $\Omega$ :*

$$\liminf_{|x| \rightarrow \infty} \frac{w}{u} > 0, \quad \liminf_{|x| \rightarrow \infty} \frac{w}{U} < \infty.$$

*Proof.* The first lim inf is simply the reformulation of (4.4), the second lim inf is the inversion of (4.2).  $\square$

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<sup>3</sup>Similarly, we could define small and large *subsolutions* but we will omit this for simplicity.

**Lemma 4.5** (Large solution lemma). *Assume that  $U > 0$  satisfies*

$$(4.5) \quad -\Delta U + VU = 0 \quad (|x| > 1) \quad \text{and} \quad U = 0 \quad (|x| = 1).$$

*Then  $U$  is a large solution for  $-\Delta + V$ .*

*Proof.* Assume that  $U$  is not a large solution for  $-\Delta + V$ , i.e.  $U$  is a small solution. Then there exists a supersolution  $w_* > 0$  for  $-\Delta + V$  such that (4.1) holds.

For small  $\varepsilon > 0$  consider the barrier functions

$$U_\varepsilon := U - \varepsilon w_*.$$

In view of (4.1), there exists  $\rho_\varepsilon > 1$  such that

$$U_\varepsilon \leq 0 \quad (|x| \geq \rho_\varepsilon)$$

and moreover,  $\rho_\varepsilon \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ .

Set  $\Omega_\varepsilon := \{x \in \mathbb{R}^N : 1 < |x| < \rho_\varepsilon\}$  and choose a small  $\varepsilon > 0$  such that  $U_\varepsilon^+ > \delta > 0$  on an open set  $G \subset \Omega_\varepsilon$ . Since  $w_* > 0$  and  $U = 0$  as  $|x| = 1$  and  $|x| = \rho_\varepsilon$ , we conclude that for any  $n \in \mathbb{N}$ ,

$$nU_\varepsilon \leq w_* \quad \text{on } \partial\Omega_\varepsilon$$

and hence  $(w_* - nU_\varepsilon)^- \in H_0^1(\Omega_\varepsilon)$ , see Remark 3.2. By the comparison principle of Corollary 3.13 we conclude that

$$nU_\varepsilon \leq w_* \quad \text{in } \Omega_\varepsilon.$$

Then  $w_* \geq n\delta$  on  $G$  and since  $n \in \mathbb{N}$  can be taken arbitrary large,  $w_* = +\infty$  on  $G$ , which is a contradiction.  $\square$

**Remark 4.6.** If  $U$  satisfies (4.5) then  $U \notin D_V^1(B_1^c)$ . Indeed, assume  $U \in D_V^1(B_1^c)$ . Then  $\mathcal{E}_V(U) = 0$  by (4.2). But  $\mathcal{E}_V(u) > 0$  for any  $0 \neq u \in D_V^1(B_1^c)$  since  $\mathcal{E}_V$  satisfies the  $\lambda$ -property, a contradiction.

**Remark 4.7.** Assume that  $U > 0$  is a (sub)solution in  $|x| > 1$  and there exists a supersolution  $w_* > 0$  such that

$$\lim_{|x| \rightarrow \infty} \frac{U}{w} = +\infty.$$

Then  $U$  is a large sub-solution at infinity.

*Hint:* Consider  $U_* = U - cw_*$  for a sufficiently large  $c > 0$ .

**Example 4.8** ( $-\Delta$  in  $\mathbb{R}^N$  with  $N \geq 3$ ). If  $-\Delta w \geq 0$  in  $\mathbb{R}^N \setminus \bar{B}_1$  then

$$\liminf_{|x| \rightarrow \infty} \frac{w(x)}{|x|^{-(N-2)}} > 0, \quad \liminf_{|x| \rightarrow \infty} w(x) < \infty.$$

To see this, take  $u(x) = |x|^{-(N-2)}$  as a small solution,  $U(x) = 1 - |x|^{-(N-2)}$  as a large solution for  $-\Delta$  in  $\mathbb{R}^N \setminus \bar{B}_1$  and use Phragmén-Lindelöf principle for supersolutions. To check that  $|x|^{-(N-2)}$  is a small solution, take  $w_* = 1$  as a reference supersolution in (4.1).

**Example 4.9** ( $-\Delta$  in  $\mathbb{R}^2$ ). If  $-\Delta w \geq 0$  in  $\mathbb{R}^N \setminus \bar{B}_1$  then

$$\liminf_{|x| \rightarrow \infty} w(x) > 0, \quad \liminf_{|x| \rightarrow \infty} \frac{w(x)}{\log |x|} < \infty.$$

To see this, take  $u(x) = 1$  as a small solution,  $U(x) = \log |x|$  as a large solution for  $-\Delta$  in  $\mathbb{R}^2 \setminus \bar{B}_1$  and use Phragmén-Lindelöf principle for supersolutions. To check that 1 is a small solution, take  $w_* = \log |x|$  as a reference supersolution in (4.1).

**Example 4.10** ( $-\Delta + 1$  in  $\mathbb{R}^3$ ). If  $-\Delta w + w \geq 0$  in  $\mathbb{R}^3 \setminus \bar{B}_1$  then

$$\liminf_{|x| \rightarrow \infty} \frac{w(x)}{\frac{e^{-|x|}}{|x|}} > 0, \quad \liminf_{|x| \rightarrow \infty} \frac{w(x)}{\frac{e^{|x|}}{|x|}} < \infty.$$

To see this, take

$$u(x) = \frac{e^{-|x|}}{|x|}, \quad U(x) = \frac{e^{|x|-1} - e^{1-|x|}}{|x|},$$

as a small and large solution for  $-\Delta + 1$  in  $\mathbb{R}^3 \setminus \bar{B}_1$ , respectively. Then use Phragmén-Lindelöf principle for supersolutions. To check that  $u$  is a small solution, take  $w_* = 1$  as a reference supersolution in (4.1).

**Exercise 4.11.** Show that if

$$-\Delta w + \frac{c}{|x|^2} w \geq 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}_1,$$

and  $c > -C_H := -\frac{(N-2)^2}{4}$ , then

$$\liminf_{|x| \rightarrow \infty} \frac{w(x)}{|x|^{\alpha_-}} > 0, \quad \liminf_{|x| \rightarrow \infty} \frac{w(x)}{|x|^{\alpha_+}} < \infty,$$

where  $\alpha_- < \alpha_+$  are the two roots of the quadratic equation  $\alpha(\alpha + N - 2) + c = 0$ .

Note that if  $c > 0$  then  $\alpha_+ > 0$  and  $U$  is growing at infinity, while if  $c < 0$  then  $\alpha_+ < 0$  and  $U$  is decaying at infinity.

## 5. HALF-SPACE ANSATZ

Consider the following singularly perturbed linear problem model in the half-space

$$(A_{\mu,\varepsilon}) \quad \begin{cases} -\Delta h = 0 & \text{in } \mathbb{R}_+^N, \\ h = 0 & \text{on } \partial\mathbb{R}_+^N \setminus \{0\}, \end{cases}$$

where  $\mathbb{R}_+^N = \{x \in \mathbb{R}^N \mid x_1 > 0\}$ . We are going to construct explicit solutions of  $(A_{\mu,\varepsilon})$ .

Let

$$h(x) = x_1^\alpha r^\beta.$$

We compute

$$\begin{aligned} -\Delta h &= -\alpha(\alpha-1)x_1^{\alpha-2}r^\beta - \beta(\beta+N-2+2\alpha)x_1^\alpha r^{\beta-2} \\ &= \left( \frac{-\alpha(\alpha-1)}{x_1^2} + \frac{-\beta(\beta+N-2+2\alpha)}{r^2} \right) h. \end{aligned}$$

If  $\alpha_- \leq \alpha_+$  are as before the root of  $\alpha(\alpha - 1) = \mu$  and

$$\beta_+ := -(N - 2) - 2\alpha_+, \quad \beta_- := -(N - 2) - 2\alpha_-,$$

then the functions

$$h_+(x) = x_1^{\alpha_+} r^{\beta_+}, \quad h_-(x) = x_1^{\alpha_-} r^{\beta_-},$$

are explicit solutions of the equation  $(A_{\mu,0})$ ,

$$-\Delta h - \frac{\mu}{x_1^2} h = 0 \quad \text{in } \mathbb{R}_+^N.$$

Observe that these functions are distinct if and only if  $\alpha_+ \neq \alpha_-$  or equivalently  $\mu < \frac{1}{4}$ .

We shall also need solutions of  $(A_{\mu,\epsilon})$  with the same properties as  $h_{\pm}$  on the boundary, especially when we are dealing with the case  $\mu = 1/4$ . Let  $\beta_{\pm,-} \leq \beta_{\pm,+}$  be the roots of

$$\beta(\beta + N - 2 + 2\alpha_{\pm}) = \epsilon,$$

or equivalently, of

$$(5.1) \quad \beta(\beta + N - 1 \pm \sqrt{1 - 4\mu}) = \epsilon.$$

Then any of the *four* functions

$$\begin{aligned} h_+(x) &= x_1^{\alpha_+} r^{\beta_{+,+}}, & h_-(x) &= x_1^{\alpha_+} r^{\beta_{+,-}}, \\ H_+(x) &= x_1^{\alpha_-} r^{\beta_{-,+}}, & H_-(x) &= x_1^{\alpha_-} r^{\beta_{-,-}}, \end{aligned}$$

give a solution to  $(A_{\mu,\epsilon})$ .

**Remark 5.1.** Although we are interested only in  $\epsilon \geq 0$ , note that for a given  $\mu \leq 1/4$  solutions  $h_+$  and  $h_-$  can be constructed for any

$$\epsilon \geq -\left(\frac{N-1}{2} + \sqrt{\frac{1}{4} - \mu}\right)^2.$$

Via the AAP principle this leads to the family of Hardy type inequalities

$$\int_{\mathbb{R}_+^N} |\nabla \phi|^2 \geq \mu \int_{\mathbb{R}_+^N} \frac{\phi^2}{x_1^2} + \left(\frac{N-1}{2} + \sqrt{\frac{1}{4} - \mu}\right)^2 \int_{\mathbb{R}_+^N} \frac{\phi^2}{|x|^2}.$$

Interestingly, solutions  $H_+$  and  $H_-$  are defined only for smaller range

$$\epsilon \geq -\left(\frac{N-1}{2} - \sqrt{\frac{1}{4} - \mu}\right)^2,$$

and the latter constant vanishes if  $\mu = -\frac{1}{4}N(N-2)$ . Y. Pinchover pointed out that the constant is optimal. He refers to the arguments in the recent paper [21].

**Lemma 5.2.** (MINIMAL POSITIVE SOLUTION LEMMA) *Let  $\mu < 1/4$ . Then the problem*

$$(5.2) \quad L_{\mu} h_R = 0 \quad \text{in } \mathbb{R}_+^N \cap B_R(0), \quad h_R = x_1^{\alpha_-} \quad \text{on } \mathbb{R}_+^N \cap S_R(0)$$

*admits a positive solution such that for  $x \in \mathbb{R}_+^N \cap B_{R/2}(0)$  it holds*

$$(5.3) \quad \limsup_{x_1 \rightarrow 0} \frac{h_R(x)}{x_1^{\alpha_+}} < \infty.$$

*Proof.* Let  $\psi : [0, R] \rightarrow \mathbb{R}$  be a smooth function such that  $\psi(R) = 1$ ,  $0 \leq \psi(r) \leq 1$  and  $\psi(r) = 0$  for  $r \leq R/2$ . Set

$$f(x) := L_\mu(x_1^{\alpha_-} \psi) = x_1^{\alpha_-} \left( -\psi'' - \frac{N-1+2\alpha_-}{r} \psi' \right).$$

Clearly  $f \in L^1(\mathbb{R}_+^N \cap B_R(0), x_1^{\alpha_-} dx)$  and  $f(x) = 0$  in  $\mathbb{R}_+^N \cap B_{R/2}(0)$ . Then the problem

$$L_\mu \eta = -f \quad \text{in } \mathbb{R}_+^N \cap B_R(0), \quad \eta = 0 \quad \text{on } \partial(\mathbb{R}_+^N \cap B_R(0))$$

admits a solution  $\eta$  such that for  $x \in \mathbb{R}_+^N \cap B_{R/2}(0)$  it holds

$$\limsup_{x_1 \rightarrow 0} \frac{\eta(x)}{x_1^{\alpha_+}} < \infty.$$

Then clearly

$$h_R := x_1^{\alpha_-} \psi + \eta$$

is the required solution of (5.2).  $\square$

We establish a version of the Phragmen–Lindelöf type comparison principle for sub-harmonics of  $L_\mu$ , which shows that sub-harmonics either have a “strong” singularity at the origin or have a “minimal” decay at the origin. See [1, pp. 93-106] for a classical reference to the Phragmen–Lindelöf principle.

**Lemma 5.3.** (PHRAGMEN–LINDELÖF TYPE ESTIMATE) *Let  $\mu < 1/4$ . Let  $h$  be an  $L_\mu$ -sub-harmonic in  $\mathbb{R}_+^N \cap B_R(0)$ , for some  $R > 0$ . Assume that  $x \in \mathbb{R}_+^N \cap B_R(0)$  and*

$$(5.4) \quad \lim_{x_1 \rightarrow 0} \frac{h(x)}{x_1^{\alpha_-} + x_1^{\alpha_+} |x|^{-(N-2+2\alpha_+)}} = 0.$$

*Then for  $x \in \mathbb{R}_+^N \cap B_{R/2}(0)$  it holds*

$$(5.5) \quad \limsup_{x_1 \rightarrow 0} \frac{h(x)}{x_1^{\alpha_+}} < \infty.$$

*Proof.* Without loss of generality we may assume that  $h$  is continuous on  $\mathbb{R}_+^N \cap \bar{B}_R(0)$ . Let  $h_R > 0$  be the minimal positive solution of (5.2), as constructed in Lemma 5.2. Note that

$$h_R(x) \leq c x_1^{\alpha_+} \quad \text{in } \mathbb{R}_+^N \cap B_{R/2}(0).$$

For  $\tau > 0$ , define a comparison function

$$h_\tau := h - h_R - \tau(x_1^{\alpha_-} + x_1^{\alpha_+} |x|^{-(N-2+2\alpha_+)}).$$

Clearly,  $h_\tau$  is  $L_\mu$ -sub-harmonic in  $\mathbb{R}_+^N \cap B_R(0)$ .

For every  $\tau > 0$ , (5.4) combined with the construction of  $h_R$  implies that  $h_\tau \leq 0$  on a neighbourhood of  $\partial(\mathbb{R}_+^N \cap B_R(0))$ . Hence we can apply the classical comparison principle for  $L_\mu$  in a proper subdomain of  $\mathbb{R}_+^N \cap B_R(0)$  and deduce that  $h_\tau \leq 0$  everywhere in  $\mathbb{R}_+^N \cap B_R(0)$ . By considering arbitrary small  $\tau > 0$ , we conclude that  $h \leq h_R$  in  $\mathbb{R}_+^N \cap B_R(0)$ .  $\square$

**Further reading.** For the development of the criticality theory beyond Agmon's ideas in [4, 5] see e.g. [6–8]. Some Phragmén–Lindelöf type results in the context of linear elliptic equations can be found in [1]. Powerful applications to the analysis of Hardy type inequalities involving distance to the boundary are given in [9]. More recent developments of the criticality theory presented e.g. in [15, 21]. For an extension to  $p$ -Laplacian see [17]; to local and nonlocal Dirichlet forms see [19, 22]. This list is very far from being complete.

## 6. A NONLINEAR LIOUVILLE THEOREM

We are going to use the AAP and Phragmén–Lindelöf principles developed in the previous section to study positive supersolutions of the semilinear elliptic equation

$$(6.1) \quad -\Delta u = u^p \quad \text{in } \Omega,$$

where  $\Omega$  is an exterior domain in  $\mathbb{R}^N$  such that  $\mathbb{R}^N \setminus \bar{B}_1 \subset \Omega$  and  $\partial\Omega \neq \emptyset$ , and  $p \in \mathbb{R}$  is the nonlinear *exponent*, which could take both positive and negative values.

A *weak* (super) solution to (6.1) is a function  $u \in H_{loc}^1(\Omega)$  such that  $u^p \in L_{loc}^1(\Omega)$  and

$$(6.2) \quad \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx (\geq) = \int_{\Omega} u^p \varphi \, dx, \quad \forall 0 \leq \varphi \in H_c^1(\Omega) \cap L_c^\infty(\Omega).$$

Note that if  $u \not\geq 0$  is a weak supersolution to (6.1) then  $-\Delta u \geq 0$  in  $\Omega$ , and hence  $u > 0$  in  $\Omega$ . In what follows we often omit the *weak* and use simply (super) solution.

**Theorem 6.1.** *Let  $N \geq 3$ . Equation (6.1) admits a positive weak supersolution if and only if  $p > p_S := \frac{N}{N-2}$ .*

This theorem was proved by J. Serrin in the 70s for the radial functions, see the introduction to [11] for the references and a general overview of nonlinear Liouville's theorem. The exponent  $p_S$  is often known as the Serrin's critical exponent. The idea to use the AAP and Phragmén–Lindelöf principles in the context of nonlinear Liouville's theorems goes back to [12] and was developed in [13, 14, 16, 18, 20].

*Proof.* Our proof of Theorem 6.1 will be split into nonexistence and existence part. In the proof of nonexistence we will distinguish four separate cases:  $1 < p < p_S$ ,  $p = p_S$ ,  $p = 1$ ,  $p < 1$ . The nonexistence in the superlinear case  $1 < p < p_S$  relies on the *lower* bound in the Phragmén–Lindelöf principle while the nonexistence in the sublinear case uses the *upper* bound in the Phragmén–Lindelöf principle.

Before we start the proof, we present two technical lemmas. The first one is a particular case of the nonexistence counterpart of the AAP principle (Corollary 1.2).

**Lemma 6.2.** *Assume that  $u > 0$  satisfies*

$$(6.3) \quad -\Delta u - c|x|^{-2+\varepsilon}u \geq 0 \quad (|x| > 1),$$

*for some  $\varepsilon > 0$  and  $c > 0$ . Then  $u \equiv 0$ .*

*Proof.* Consider the quadratic form

$$\mathcal{E}(u) := \int_{\Omega} |\nabla \varphi|^2 dx - c \int_{\Omega} \frac{\varphi^2}{|x|^{2+\varepsilon}} dx \quad (\varphi \in C_c^\infty(\Omega)).$$

Take  $\varphi \in C_c^\infty(\Omega)$  such that  $\varphi \geq 0$ ,  $0 \leq \varphi \leq 1$  and  $\varphi = 0$  for  $|x| > 2$  and  $|x| < 1$ . Then for  $R > 1$  the rescaling

$$\varphi_R(x) := \varphi\left(\frac{x}{R}\right) \in C_c^\infty(\Omega)$$

and by the change of variables we compute

$$\mathcal{E}(\varphi_R) = R^{N-2} \int_{\Omega} |\nabla \varphi|^2 dx - c R^{N-2+\varepsilon} \int_{\Omega} \frac{\varphi^2}{|x|^{2-\varepsilon}} dx \rightarrow -\infty,$$

as  $R \rightarrow \infty$ . By the AAP principle (Corollary 1.2) we conclude that  $u \equiv 0$ .  $\square$

The second lemma is an an a priori lower bound on positive solutions of (6.1) in the sublinear case. Note that the bound (6.3) depends on the value of  $p < 1$ .

**Lemma 6.3.** *Let  $p < 1$ ,  $s \in \mathbb{R}$  and assume that  $u > 0$  satisfies*

$$-\Delta u \geq |x|^s u^p \quad \text{in } \Omega.$$

*Then*

$$(6.4) \quad u \geq c|x|^{\frac{2+s}{1-p}} \quad (|x| > 1).$$

*Proof.* The proof uses the AAP principle and weak Harnack's inequality, see [14, Lemma 6.1].  $\square$

Now we are in a position to prove Theorem 6.1. We will proceed case by case.

*Nonexistence in the superlinear subcritical case  $1 < p < p_S$ .* Assume that  $u > 0$  is a supersolution to (6.1). Then  $-\Delta u \geq 0$  in  $\Omega$  and hence, as in Example 4.8 we conclude that for some  $c > 0$ ,

$$(6.5) \quad u \geq c|x|^{-(N-2)} \quad (|x| > 1)$$

Consider the *linearisation* of (6.1),

$$(6.6) \quad -\Delta u + V(x)u \geq 0 \quad \text{in } \Omega,$$

where  $V(x) := u^{p-1}$ . Since  $p > 1$ , using (6.5) we can estimate

$$(6.7) \quad V(x) \geq c_1|x|^{-(N-2)(p-1)} \quad (|x| > 1),$$

where  $c_1 = c^{p-1} > 0$ . Note that if  $1 < p < p_S$  then

$$(6.8) \quad -(N-2)(p-1) > -2.$$

Hence  $u > 0$  satisfies

$$(6.9) \quad -\Delta u + c_1|x|^{-2+\varepsilon}u \geq 0 \quad (|x| > 1),$$

for some  $\varepsilon > 0$ . By Lemma 6.2, we conclude that  $u \equiv 0$ .  $\square$

*Nonexistence in the critical case  $p = p_S$ .* In this case we have  $\varepsilon = 0$  in (6.9) and the previous argument fails.<sup>4</sup> Instead, we will iterate the previous to improve the lower bound (6.7).

Indeed, we may assume that  $0 < c_1 < C_H$  and  $u > 0$  satisfies

$$(6.10) \quad -\Delta u - c_1|x|^{-2}u \geq 0 \quad (|x| > 1).$$

As in the Exercise 4.11 we conclude that for some  $c_2 > 0$ ,

$$(6.11) \quad u \geq c_2|x|^{\alpha_-} \quad (|x| > 1),$$

where  $\alpha_- > -(N-2)$  is the smallest root of  $-\alpha(\alpha + N - 2) = c_1$ . Then  $u > 0$  satisfies the linearisation equation (6.6) and we can estimate

$$(6.12) \quad V(x) \geq c_3|x|^{\alpha_-(p-1)} \quad (|x| > 1),$$

where  $c_3 = c_2^{p-1}$ . Since  $p = p_S$  and  $\alpha_- > -(N-2)$ ,

$$(6.13) \quad \alpha_-(p-1) > -2.$$

Hence  $u > 0$  satisfies (6.9) with some  $\varepsilon > 0$  and as before, by Lemma 6.2, we conclude that  $u \equiv 0$ .  $\square$

*Nonexistence in the linear case  $p = 1$ .* In this case the equation (6.1) is linear. We simply note that there exists  $\varphi \in C_c^\infty(\Omega)$  such that the corresponding quadratic form

$$\mathcal{E}_{-1}(\varphi) = \int_{\Omega} |\nabla \varphi|^2 dx - \int_{\Omega} \varphi^2 dx < 0.$$

(This can be seen using the same family of test functions  $\varphi_R$  as in the proof of Lemma 6.2.) By the AAP principle (Corollary 1.2) we conclude that  $u \equiv 0$ .  $\square$

*Nonexistence in the sublinear case  $p < 1$ .* Assume that  $u > 0$  is a supersolution to (6.1). Then  $-\Delta u \geq 0$  in  $\Omega$  and using the upper bound in Example 4.8 we conclude that

$$(6.14) \quad \liminf_{|x| \rightarrow \infty} u(x) < \infty.$$

But according to Lemma 6.3,

$$(6.15) \quad u \geq c|x|^{\frac{2}{1-p}} \quad (|x| > 1).$$

Since  $p < 1$  these two bounds are incompatible with each other and we conclude that  $u \equiv 0$ .  $\square$

*Existence in the case  $p > p_S$ .* A direct computation shows that for every  $p > p_S$

$$u = c_p|x|^{-\frac{2}{p-1}}, \quad c_p^{p-1} = \frac{2}{(p-1)^2}((N-2)p - N)$$

is a solution of (6.1).<sup>5</sup>  $\square$

<sup>4</sup>If  $c_1 > C_H$ , the critical Hardy constant, we would conclude again by the AAP principle. However we do not control the size of  $c_1 > 0$  and in general,  $c_1$  could be small.

<sup>5</sup>Note that  $c_p < 0$  for  $p < p_S$  and  $c_p = 0$  if  $p = p_S$ .

**Exercise 6.4.** Modify the previous arguments to show that if  $N = 2$  then equation (6.1) has no positive weak supersolutions for any  $p \in \mathbb{R}$ .

**Exercise 6.5.** Let  $N \geq 2$ ,  $c > 0$  and  $p \in \mathbb{R}$ . Show that the equation

$$-\Delta u + \frac{c}{|x|^2} u = u^p \quad \text{in } |x| > 1$$

admits a positive weak supersolution if and only if  $p \notin [1 - \frac{2}{\alpha_+}, 1 - \frac{2}{\alpha_-}]$ , where  $\alpha_-$  and  $\alpha_+$  are defined in Exercise 4.11.

*Hint:* Use small and large solutions constructed in Exercise 4.11. The nonexistence in the lower critical case  $p = 1 - \frac{2}{\alpha_+} < 0$  is difficult, see [14, Lemma 6.6]. All other regimes could be studied similarly to the proof of Theorem 6.1.

**Exercise 6.6.** Let  $N \geq 3$ ,  $s > -2$  and  $p \in \mathbb{R}$ . Show that the equation

$$-\Delta u = |x|^s u^p \quad \text{in } |x| > 1$$

admits a positive weak supersolution if and only if  $p > \frac{N+s}{N-2}$ .

*Hint:* In the case  $p < 1$  use (6.4).

**Further reading.** Similar methods based on the AAP and Phragmén–Lindelöf principles were used to prove Liouville’s theorems for divergence type semilinear equations in conical domains [13], equations with Hardy type potentials in conical domains [14], quasilinear equations involving  $p$ -Laplacian [16], equations with Hardy type potentials involving distance to the boundary of a bounded domain [18] and nonlocal Choquard’s equations [20].

## 7. APPENDIX: A BREZIS–BROWDER TYPE RESULT

By modifying the proof of Lemma 3.12, we can establish the following approximation lemma, cf. [23, Theorem 3.4.1].

**Lemma 7.1.** *Assume that  $\mathcal{E}_V$  satisfies the  $\lambda$ -property and  $0 \leq w \in D_V^1(\Omega)$ . Then there exists a sequence  $(w_n) \in D_V^1(\Omega) \cap L^\infty(\Omega)$  such that:*

- (a)  $\text{supp}(w_n)$  is compact in  $\Omega$ ;
- (b)  $0 \leq w_n \leq w$ ;
- (c)  $\mathcal{E}_V(w - w_n) \rightarrow 0$  as  $n \rightarrow \infty$ .

*Proof.* Let  $(\varphi_n) \subset C_c^\infty(\Omega)$  be an approximating sequence for  $w \in D_V^1(\Omega)$ . Set

$$w_n := \varphi_n^+ \wedge w \wedge n.$$

Then  $(w_n)$  has all the required properties and (c) follows as in the proof of Lemma 3.12.  $\square$

An important consequence of Lemma 7.1 is a Brezis–Browder type result, cf. [23, Theorem 3.4.1] and a historical discussion in [23, p.82].

**Theorem 7.2.** Let  $T \in D_V^{-1}(\Omega) \cap L_{loc}^1(\Omega)$  and  $w \in D_V^1(\Omega)$ . Assume

$$(7.1) \quad T(x)w(x) \geq 0 \quad \text{a.e. in } \Omega.$$

Then  $Tw \in L^1(\Omega)$  and

$$(7.2) \quad \langle T, w \rangle = \int_{\Omega} T(x)w(x)dx,$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality between  $D_V^1(\Omega)$  and  $D_V^{-1}(\Omega)$ .

*Proof.* Let  $(w_n)$  be an approximating sequence for  $w$ , defined in Lemma 7.1. Since  $T \in L_{loc}^1(\Omega)$  and  $w_n \in L_c^\infty(\Omega)$ , we know that

$$\langle T, w_n \rangle = \int_{\Omega} T(x)w_n(x)dx.$$

By Lemma 7.1 (c),  $\langle T, w_n \rangle \rightarrow \langle T, w \rangle$  as  $n \rightarrow \infty$ . On the other hand,  $T(x)w_n(x) \geq 0$  a.e. in  $\Omega$  and then by Fatou's lemma  $Tw \in L^1(\Omega)$ . But we also know that  $T(x)w_n(x) \leq T(x)w(x)$  a.e. in  $\Omega$ . Then

$$\lim_{n \rightarrow \infty} \int_{\Omega} T(x)w_n(x)dx = \int_{\Omega} T(x)w(x)dx,$$

by dominated convergence. □

**Exercise 7.3.** Show that (7.1) can be replaced by one of the weaker assumptions:

$$(7.3) \quad T(x) \geq 0 \quad \text{a.e. in } \Omega,$$

or

$$(7.4) \quad T(x)w(x) \geq -|f(x)| \quad \text{a.e. in } \Omega,$$

for some  $f \in L^1(\Omega)$ .

**Acknowledgement.** VM is grateful to Dr Roberto Ognibene for his careful reading of the manuscript which helped to eliminate a number of typos.

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