

POSITIVITY PRINCIPLES AND DECAY OF SOLUTIONS IN SEMILINEAR ELLIPTIC PROBLEMS

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ABSTRACT. This is an outline of the lectures given at Roma Tor Vergata in May-June 2024. Please, email me if you spot any typos or mistakes.

CONTENTS

1. The AAP positivity principle	2
2. Hardy type inequalities	3
3. The energy space	5
4. A Phragmén–Lindelöf principle	9
5. Half–space ansatz	12
6. A nonlinear Liouville theorem	15
7. Appendix: A Brezis–Browder type result	18
References	19

Consider the linear equation

$$(*) \quad -\Delta u + Vu = f \quad \text{in } \Omega,$$

where Ω is a domain (open connected set) in \mathbb{R}^N with $N \geq 2$ and V is a potential and $0 \leq f \in L_{loc}^1(\Omega)$ is a nonnegative right hand side. We assume that $V = V^+ - V^-$ and

$$V^+ \in L_{loc}^\infty(\Omega), \quad V^- \in L_{loc}^1(\Omega).$$

The natural quadratic form associated to the Schrödinger operator $-\Delta + V$ is given by

$$\mathcal{E}_V(\varphi) := \int_\Omega |\nabla \varphi|^2 dx + \int_\Omega V \varphi^2 dx \quad (\varphi \in H_c^1(\Omega) \cap L_c^\infty(\Omega)),$$

where the subscript c denotes the *compact support*. We say that \mathcal{E}_V is non-negative if

$$\mathcal{E}_V(\varphi) \geq 0, \quad \forall \varphi \in H_c^1(\Omega) \cap L_c^\infty(\Omega).$$

We are going to study the relationship between the existence and some properties of positive supersolutions to $(*)$ and non-negativity of the quadratic form \mathcal{E}_V . Such a relationship is commonly referred to as Agmon–Allegretto–Piepenbrink’s (AAP) Principle or “Ground

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State transformation". Our exposition is inspired and largely based on [4–7, 15] but is adapted with applications to semilinear equations in mind, as developed in [12–14].

1. THE AAP POSITIVITY PRINCIPLE

A *weak* supersolution to $(*)$ is a function $u \in H_{loc}^1(\Omega) \cap L_{loc}^1(\Omega, V dx)$ such that

$$(1.1) \quad \int_{\Omega} \nabla u \cdot \nabla \varphi dx + \int_{\Omega} V u \varphi dx \geq \int_{\Omega} f \varphi dx, \quad \forall 0 \leq \varphi \in H_c^1(\Omega) \cap L_c^{\infty}(\Omega).$$

The notions of a weak sub-solution and weak solution are defined similarly by replacing " \geq " with " \leq " and " $=$ " respectively. Note that if $u \geq 0$ is a weak supersolution to $(*)$ with $f \geq 0$ then $u > 0$ in Ω , in the sense that $u^{-1} \in L_{loc}^{\infty}(\Omega)$. This follows from the weak Harnack inequality which is ensured by the assumption $V^+ \in L_{loc}^{\infty}(\Omega)$.

In what follows we often omit the word *weak* and use simply solution, super and sub-solution. We say that u is a solution to $-\Delta + V$ if u is a solution for $(*)$ with $f = 0$. Similarly for sub and supersolutions.

When $V \geq 0$, the quadratic form \mathcal{E}_V is non-negative, and any positive constant is a supersolution. Thus, the interesting case to consider is when V is negative or changes sign. The fundamental relation between the existence of positive supersolutions to $(*)$ and non-negativity of the quadratic form \mathcal{E}_V is described in the following result, which was originally proved by W. Allegretto [2] and J. Piepenbrink [3] in 1974, and later became the foundation of S. Agmon's Criticality Theory [4].

Theorem 1.1 (AAP Positivity Principle). *Assume that $(*)$ admits a weak positive (super)solution $u_* > 0$. Then*

$$(1.2) \quad \mathcal{E}_V(\varphi) (\geq) = \int_{\Omega} \left| \nabla \left(\frac{\varphi}{u_*} \right) \right|^2 u_*^2 dx + \int_{\Omega} \varphi^2 \frac{f}{u_*} dx, \quad \forall 0 \leq \varphi \in H_c^1(\Omega) \cap L_c^{\infty}(\Omega).$$

Proof. Let $0 \leq \varphi \in H_c^1(\Omega) \cap L_c^{\infty}(\Omega)$. Then $\psi := \frac{\varphi^2}{u_*} \in H_c^1(\Omega) \cap L_c^{\infty}(\Omega)$. Testing $(*)$ against ψ we arrive at

$$2 \int_{\Omega} \varphi \nabla \varphi \frac{\nabla u_*}{u_*} dx (\geq) = \int_{\Omega} \varphi^2 \frac{|\nabla u_*|^2}{u_*^2} dx - \int_{\Omega} V \varphi^2 dx + \int_{\Omega} \frac{f}{u_*} \varphi^2 dx.$$

Direct computation yields

$$\begin{aligned} & \int_{\Omega} (|\nabla \varphi|^2 + V \varphi^2) dx - \int_{\Omega} \left| \nabla \left(\frac{\varphi}{u_*} \right) \right|^2 u_*^2 dx \\ &= \int_{\Omega} (|\nabla \varphi|^2 + V \varphi^2) dx - \int_{\Omega} \left(\frac{|\nabla \varphi|^2}{u_*^2} - 2\varphi \nabla \varphi \frac{\nabla u_*}{u_*^3} + \varphi^2 \frac{|\nabla u_*|^2}{u_*^4} \right) u_*^2 dx \\ &= 2 \int_{\Omega} \varphi \nabla \varphi \frac{\nabla u_*}{u_*} dx - \int_{\Omega} \varphi^2 \frac{|\nabla u_*|^2}{u_*^2} dx + \int_{\Omega} V \varphi^2 dx (\geq) = \int_{\Omega} \varphi^2 \frac{f}{u_*} dx. \end{aligned}$$

This proves (1.2) on $H_c^1(\Omega) \cap L_c^{\infty}(\Omega)$. \square

The following straightforward corollary of Theorem 1.1 is crucial in the analysis of nonexistence of positive solutions to semilinear equations.

Corollary 1.2 (Nonexistence principle). *Assume there exists $\varphi \in C_c^\infty(\Omega)$ such that $\mathcal{E}_V(\varphi) < 0$. Then $-\Delta + V$ has no positive weak supersolutions in Ω .*

Remark 1.3. The computation in the proof of the AAP is often known as the *Picone's identity*:

$$|\nabla\varphi|^2 - \nabla\left(\frac{\varphi^2}{u}\right) \cdot \nabla u = \left|\nabla\left(\frac{\varphi}{u}\right)\right|^2 u^2 \geq 0,$$

for any $\varphi, v \in H_{loc}^1(\Omega)$ such that $\varphi \geq 0$, $u > 0$ in Ω .

Remark 1.4. It is simple to see that the " u_* -ground state transformed" quadratic form

$$\mathcal{E}_{u_*}(\varphi) = \int_\Omega \left| \nabla \left(\frac{\varphi}{u_*} \right) \right|^2 u_*^2 dx$$

is generated by the nonnegative selfadjoint operator

$$-\Delta + 2\nabla \log(u_*) \nabla$$

in $L^2(\Omega, u_*^2 dx)$.

Remark 1.5. (a) We do not assume any boundary conditions on the reference supersolution u_* in Lemma 1.1. We also do not require any boundary regularity of u_* . All we need is that $u_* \in H_{loc}^1(\Omega)$ is a positive weak supersolution of $-\Delta + V$. In many examples u_* is positive or even singular on parts of the boundary (Example 2.1), but u_* also may be zero on the boundary (Examples 2.4 and 2.6).

(b) We do not assume any boundary regularity of the potential V , which could be 'very' singular on $\partial\Omega$ (Examples 2.1, 2.4).

(c) Domain Ω need not be smooth (e.g. $\mathbb{R}^N \setminus \{0\}$ in Example 2.1).

2. HARDY TYPE INEQUALITIES

In this section we will use the AAP positivity principle to prove several Hardy type inequalities. We start with the hardy inequality in \mathbb{R}^N .

Example 2.1 (Hardy inequality in \mathbb{R}^N). Consider

$$(2.1) \quad -\Delta u - \frac{C_H}{|x|^2} u = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

where $N \geq 3$ and $C_H := (N-2)^2/4$. Clearly,

$$u_*(|x|) = |x|^{\frac{2-N}{2}}$$

is a positive weak solution of (2.1). Hence, by Lemma 1.1 we obtain Hardy's inequality

$$(2.2) \quad \int_{\mathbb{R}^N} |\nabla\varphi|^2 \geq C_H \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}).$$

Remark 2.2. We had to remove the origin in (2.1) because $|x|^{\frac{2-N}{2}} \notin H_{loc}^1(\mathbb{R}^N)$ – the singularity at the origin is too strong. To prove (2.1) in \mathbb{R}^N , first prove

$$(2.3) \quad \int_{\mathbb{R}^N} |\nabla \varphi|^2 \geq c \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N).$$

for any $c \in (0, C_H)$. To do this, use $u_* = |x|^{\alpha_+}$, where α_+ is the biggest root of the scalar equation $-\alpha(\alpha + N - 2) = c$. Since $\alpha_+ > -\frac{N-2}{2}$, we have $|x|^{\alpha_+} \in H_{loc}^1(\mathbb{R}^N)$ and the AAP principle is valid. Since (2.3) is valid for any $c < C_H$, it is also valid for $c = C_H$.

Exercise 2.3. Show that $C_H = \frac{(N-2)^2}{4}$ ($N \geq 3$) is the *best constant* in the Hardy inequality (2.2) i.e. the inequality fails for any $c > C_H$.

Hint. Take a family of trial functions $\varphi_R = u_* \eta_R$ where $u_* = |x|^{\frac{2-N}{2}}$ and $\eta_R(r)$ is a non-negative C^1 -cut-off function such that $\eta_R = 1$ for $r \in [0, R]$, $\eta_R = 0$ for $r > 2R$ and $|\eta'(r)| \leq 2/R$. Then use representation (1.2) to show that for any $\varepsilon > 0$,

$$\int_{\mathbb{R}^N} |\nabla \varphi_R|^2 - C_H \int_{\mathbb{R}^N} \frac{\varphi_R^2}{|x|^2} - \varepsilon \int_{\mathbb{R}^N} \frac{\varphi_R^2}{|x|^2} \rightarrow -\infty$$

as $R \rightarrow \infty$.

Example 2.4 (Hardy inequality in \mathbb{R}^2). Consider

$$(2.4) \quad -\Delta u - \frac{1/4}{|x|^2 \left(\log \frac{|x|}{\rho} \right)^2} u = 0 \quad \text{in } \mathbb{R}^2 \setminus B_\rho(0).$$

where $\rho > 0$. Clearly,

$$u_*(|x|) = \left(\log \frac{|x|}{\rho} \right)^{1/2}$$

is a positive weak solution of (2.4). Hence, by Lemma 1.1 we obtain a Hardy type inequality

$$\int_{\mathbb{R}^2} |\nabla \varphi|^2 \geq \frac{1}{4} \int_{\mathbb{R}^2} \frac{\varphi^2}{|x|^2 \left(\log \frac{|x|}{\rho} \right)^2}, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^2 \setminus B_\rho(0)).$$

Exercise 2.5 (Improved Hardy inequality). For $N \geq 2$ prove the following inequality

$$\int_{\mathbb{R}^N} |\nabla \varphi|^2 \geq C_H \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} + \frac{1}{4} \int_{\mathbb{R}^2} \frac{\varphi^2}{|x|^2 (\log |x|)^2}, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus B_1).$$

Hint: Take $u_*(|x|) = |x|^{-\frac{N-2}{2}} (\log |x|)^{1/2}$

Example 2.6 (Principal Dirichlet eigenvalue). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\lambda_1 > 0$ be the principal Dirichlet eigenvalue of $-\Delta$ on Ω and $u_* := \phi_1 > 0$ be the corresponding principal eigenfunction. By Lemma 1.1 we obtain the inequality

$$(2.5) \quad \int_{\Omega} |\nabla \varphi|^2 \geq \lambda_1 \int_{\Omega} \varphi^2, \quad \forall \varphi \in C_c^\infty(\Omega).$$

Example 2.7 (Torsional Hardy inequality). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\psi_\Omega > 0$ be a torsion function of Ω , that is the unique solution of

$$-\Delta\psi = 1 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Then we obtain “torsional Hardy inequality”

$$\int_{\Omega} |\nabla\varphi|^2 \geq \int_{\Omega} \frac{\varphi^2}{\psi_\Omega} \quad \forall \varphi \in C_c^\infty(\Omega).$$

Now take $\varphi = \varphi_1$, the principal Dirichlet eigenvalue of Ω such that $\|\varphi_1\|_2 = 1$. Then

$$(2.6) \quad \lambda_1 = \int_{\Omega} |\nabla\varphi|^2 \geq \int_{\Omega} \frac{\varphi_1^2}{\psi_\Omega} \geq \frac{\|\varphi_1\|_2^2}{\|\psi_\Omega\|_\infty} = \frac{1}{\|\psi_\Omega\|_\infty}.$$

Hence we deduce a lower bound

$$\|\psi_\Omega\|_\infty \geq \frac{1}{\lambda_1}.$$

Exercise 2.8 (Barta’s inequality). Prove the following inequality

$$(2.7) \quad \inf_{x \in \Omega} \frac{(-\Delta - V)\phi}{\phi} \leq \inf_{0 \neq u \in C_c^\infty(\Omega)} \frac{\mathcal{E}_V(u, u)}{\|u\|_{L^2}^2}.$$

Hint: Similar to (2.6).

Exercise 2.9. For $R > 0$ and $B_R \subset \mathbb{R}^N$, prove the inequality

$$\int_{B_R} |\nabla\varphi|^2 \geq 2N \int_{B_R} \frac{\varphi^2}{R^2 - |x|^2} \quad \forall \varphi \in C_c^\infty(B_R).$$

3. THE ENERGY SPACE

3.1. The λ -property and the energy space. Following [5], we say that the form \mathcal{E}_V satisfies the λ -property if there exists a function $0 < \lambda \in L_{loc}^1(\Omega)$ such that $\lambda^{-1} \in L_{loc}^\infty(\Omega)$ and

$$(3.1) \quad \mathcal{E}_V(u) \geq \int_{\Omega} u^2 \lambda(x) dx, \quad \forall u \in C_c^\infty(\Omega).$$

It is elementary to see that if \mathcal{E}_V satisfies the λ -property then $\|\cdot\|_V := \sqrt{\mathcal{E}_V(\cdot)}$ is a norm on $C_c^\infty(\Omega)$. Denote by $D_V^1(\Omega)$ the family of measurable a.e. finite functions $u : \Omega \rightarrow \mathbb{R}$ such that there exists an $\|\cdot\|_V$ -Cauchy sequence $(u_n) \subset C_c^\infty(\Omega)$ that converges to u a.e. in Ω . This sequence (u_n) is called an *approximating sequence* for $u \in D_V^1(\Omega)$. Then the limit

$$\mathcal{E}_V(u) := \lim_{n \rightarrow \infty} \mathcal{E}_V(u_n)$$

exists and is independent of the choice of the approximating sequence. Thus \mathcal{E}_V is extended uniquely to a nonnegative definite quadratic form on $D_V^1(\Omega)$. The family $D_V^1(\Omega)$ is called the *energy space* of \mathcal{E}_V .

Theorem 3.1. *The energy space $D_V^1(\Omega)$ is a Hilbert space with the inner product*

$$\langle u, v \rangle_V := \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Omega} V u v \, dx$$

Moreover, $D_V^1(\Omega)$ is continuously embedded into $L^2(\Omega, \lambda \, dx)$. Additionally, if u_* is locally Lipschitz in Ω then $D_V^1(\Omega)$ is continuously embedded in $H_{loc}^1(\Omega)$.

Proof. Clearly, (3.1) implies that (u_n) converges to u in $L^2(\Omega, \lambda \, dx)$ and hence $D_V^1(\Omega)$ is continuously embedded into $L^2(\Omega, \lambda \, dx)$.

To prove the embedding $D_V^1(\Omega) \subset H_{loc}^1(\Omega)$, following [10] we first observe that

$$\begin{aligned} \int_{\Omega} \left| \nabla \left(\frac{\varphi}{u_*} \right) \right|^2 u_*^2 \, dx &= \int_{\Omega} (|\nabla \varphi|^2 - 2\nabla \varphi \cdot \frac{\varphi}{u} \nabla u + \frac{\varphi^2}{u^2} |\nabla u|^2) \, dx \\ &\geq \frac{1}{2} \int_{\Omega} (|\nabla \varphi|^2 - |\nabla \log(u)|^2 \varphi^2) \, dx, \end{aligned}$$

where we used the Cauchy–Schwartz to estimate

$$\nabla \varphi \cdot \frac{\varphi}{u} \nabla u \leq \frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{\varepsilon^{-1}}{2} \frac{\varphi^2}{u^2} |\nabla u|^2$$

and set $\varepsilon = \frac{1}{2}$. Then using (1.2) we obtain

$$\begin{aligned} \frac{1}{2} \int_{\Omega'} |\nabla \varphi|^2 &\leq \int_{\Omega} \left| \nabla \left(\frac{\varphi}{u_*} \right) \right|^2 u_*^2 \, dx + \frac{1}{2} \int_{\Omega'} |\nabla \log(u)|^2 \varphi^2 \, dx \\ &\leq \mathcal{E}_V(\varphi) + c' \int_{\Omega'} \varphi^2 \lambda(x) \, dx \leq (1 + c') \mathcal{E}_V(\varphi), \end{aligned}$$

where the constant $c' > 0$ depends on the choice of subdomian Ω' . \square

Remark 3.2. If \mathcal{E}_V satisfies the λ -property in Ω and Ω' is a bounded subdomain such that $\bar{\Omega}' \subset \Omega$, then \mathcal{E}_V satisfies the λ -property in Ω' with $\lambda^{-1} \in L^\infty(\Omega')$. Hence $D_V^1(\Omega') = H_0^1(\Omega')$.

Denote by $D_V^{-1}(\Omega)$ the dual space of all linear continuous functionals on $D_V^1(\Omega)$. The following lemma is a standard consequence of the Riesz Representation Theorem.

Lemma 3.3. *Assume that \mathcal{E}_V satisfies the λ -property. Let $l \in D_V^{-1}(\Omega)$. Then there exists a unique $u_* \in D_V^1(\Omega)$ such that*

$$(3.2) \quad \langle u_*, \varphi \rangle = l(\varphi), \quad \forall \varphi \in D_V^1(\Omega).$$

Since $D_V^1(\Omega) \subset L^2(\Omega, \lambda \, dx)$, by duality we conclude that $L^2(\Omega, \lambda^{-1} \, dx) \subset D_V^{-1}(\Omega)$, in the sense that if $f \in L^2(\Omega, \lambda^{-1} \, dx)$ then

$$l_f(\varphi) := \int_{\Omega} f \varphi \, dx \in D_V^{-1}(\Omega).$$

Thus Lemma 3.3 implies that for any $f \in L^2(\Omega, \lambda^{-1} \, dx)$ the equation

$$(3.3) \quad -\Delta u + Vu = f \quad \text{in } D_V^1(\Omega)$$

has a unique solution $u_f \in D_V^1(\Omega)$.

Exercise 3.4 (The space $H^1(\mathbb{R}^N)$). Show that $-\Delta + 1$ satisfies the λ -property in \mathbb{R}^N and that the corresponding energy space $D_1^1(\mathbb{R}^N)$ coincides with the standard Sobolev space $H^1(\mathbb{R}^N)$.

Example 3.5 (The space $D_0^1(\mathbb{R}^N)$ for $N \geq 3$). Consider $-\Delta$ on \mathbb{R}^N with $N \geq 3$. Take

$$u_*(|x|) := (1 + |x|^2)^{-\frac{N-2}{2}},$$

which is known as a Talenti function¹. Then

$$-\Delta u^* = N(N-2)(1 + |x|^2)^{-\frac{N+2}{2}}$$

and

$$\lambda(x) = \frac{-\Delta u_*(x)}{u_*(x)} = N(N-2)(1 + |x|^2)^{-2}.$$

Hence the space $D_0^1(\mathbb{R}^N)$, a completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the homogenous gradient norm $\|\nabla u\|_{L^2(\mathbb{R}^N)}$, is a well-defined Hilbert space and moreover,

$$D_0^1(\mathbb{R}^N) \subset L^2(\mathbb{R}^N, (1 + |x|^2)^{-2} dx).$$

It is easy to see that $D_0^1(\mathbb{R}^N) \not\subset L^2(\mathbb{R}^N)$ and hence $D_0^1(\mathbb{R}^N) \neq H^1(\mathbb{R}^N)$. To see this when $N = 3, 4$, check that $\|\nabla u_*\|_{L^2(\mathbb{R}^N)} < \infty$ but $u_* \notin L^2(\mathbb{R}^N)$ when $N = 3, 4$. In fact, $D_0^1(\mathbb{R}^N) \subset L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ – this follows from the Sobolev inequality.²

Example 3.6 (The space $D^1(\mathbb{R}^2)$ is not well-defined). According to the Liouville's theorem, every positive super solution on \mathbb{R}^2 is constant. This means that the Laplacian $-\Delta$ does not satisfy λ -property on \mathbb{R}^2 . Hence we can not apply Theorem 3.1 to construct the energy space $D_0^1(\mathbb{R}^2)$. In fact, such a space is not well defined as the space of functions, see [15] for details.

Example 3.7 (The space $D^1(\mathbb{R}^2 \setminus \bar{B}_\rho)$ in \mathbb{R}^2). Consider $-\Delta$ on $\Omega = \mathbb{R}^2 \setminus \bar{B}_\rho$, for a $\rho > 0$. Take

$$u_*(|x|) = \left(\log \frac{|x|}{\rho} \right)^{1/2}.$$

As in the Example 2.4, we conclude that $-\Delta$ in Ω satisfies the λ -property with

$$\lambda(x) = \frac{1/4}{|x|^2 \left(\log \frac{|x|}{\rho} \right)^2}.$$

Hence the space $D_0^1(\mathbb{R}^2 \setminus \bar{B}_\rho)$, the completion of $C_c^\infty(\mathbb{R}^2 \setminus \bar{B}_\rho)$ with respect to the homogenous gradient norm $\|\nabla u\|_{L^2(\mathbb{R}^2)}$ is a well-defined Hilbert space.

Exercise 3.8. Use the improved Hardy Inequality 2.5 to show that for $N \geq 3$ the critical Hardy operator

$$-\Delta - \frac{C_H}{|x|^2}, \quad C_H := \left(\frac{N-2}{2} \right)^2,$$

¹Up to a rescaling, it is a minimizer of the Sobolev inequality

²The space $D_0^1(\mathbb{R}^N)$ from this example is also often denoted as $D^1(\mathbb{R}^N)$ or $\dot{H}^1(\mathbb{R}^N)$.

satisfies the λ -property in $\mathbb{R}^N \setminus \bar{B}_1$ and define the energy space $D_{-\frac{C_H}{|x|^2}}^1(\mathbb{R}^N \setminus \bar{B}_1)$. Then use the arguments in Exercise 2.3 to show that

$$u_0 = |x|^{\frac{2-N}{2}} \in D_{-\frac{C_H}{|x|^2}}^1(\mathbb{R}^N \setminus \bar{B}_1).$$

Note that $u_0 \notin H_{loc}^1(\mathbb{R}^N)!$

3.2. Maximum and comparison principles. If \mathcal{E}_V satisfies the λ -property then $-\Delta + V$ satisfies weak maximum and comparison principles. In order to prove that we need to know that $D_V^1(\Omega)$ is invariant under the standard truncations.

Lemma 3.9. *If $u \in D_V^1(\Omega)$ then $u^+, u^- \in D_V^1(\Omega)$ and*

$$(3.4) \quad \mathcal{E}_V(u^\pm) \leq \mathcal{E}(u), \quad \forall u \in D_V^1(\Omega).$$

If $u, v \in D_V^1(\Omega)$ then $u \vee v, u \wedge v \in D_V^1(\Omega)$.

Proof. See [14, Lemma A.1]. □

Remark 3.10. We do not claim that $u \in D_V^1(\Omega)$ implies $u \wedge 1 \in D_V^1(\Omega)$.

Exercise 3.11. Let $(\varphi_n) \subset C_c^\infty(\Omega)$ be an approximating sequence for $0 \leq w \in D_V^1(\Omega)$. Show that $(\varphi_n^+) \subset D_V^1(\Omega)$ is also an approximating sequence for w , in the sense that $\mathcal{E}_V(\varphi_n^+) \rightarrow \mathcal{E}_V(w)$.

Hint. Show that (φ_n^-) is a Cauchy sequence and hence $\mathcal{E}_V(\varphi_n^-) \rightarrow 0$.

Lemma 3.12. *Assume that \mathcal{E}_V satisfies the λ -property. Let $w \in H_{loc}^1(\Omega) \cap L^1(\Omega, V dx)$ be a super-solution to $(*)$ such that $w^- \in D_V^1(\Omega)$. Then $w \geq 0$ in Ω .*

Proof. Let $(\varphi_n) \subset C_c^\infty(\Omega)$ be an approximating sequence for $w^- \in D_V^1(\Omega)$. Set

$$w_n := \varphi_n^+ \wedge w^-.$$

Hence $0 \leq w_n \in D_V^1(\Omega)$, by Lemma 3.9. Note that $w_n = w^- - (\varphi_n^+ - w^-)^-$. Therefore

$$\mathcal{E}_V(w^- - w_n) = \mathcal{E}_V((\varphi_n^+ - w^-)^-) \leq \mathcal{E}_V(\varphi_n^+ - w^-) \rightarrow 0.$$

Thus (w_n) is a nonnegative approximating sequence for w^- . Since $0 \leq w_n \leq w^-$ we have $w^+ \wedge w_n = 0$, and then we obtain

$$0 \leq \langle w, w_n \rangle_V = -\langle w^-, w_n \rangle_V \rightarrow -\mathcal{E}_V(w^-) \leq 0.$$

We conclude that $w^- = 0$. □

The following comparison principle is a straightforward consequence of Lemma 3.12.

Corollary 3.13. *Assume that \mathcal{E}_V satisfies the λ -property. Let $w \in H_{loc}^1(\Omega) \cap L^1(\Omega, V dx)$ be a super-solution to $(*)$ and $v \in H_{loc}^1(\Omega) \cap L^1(\Omega, V dx)$ be a sub-solution to $(*)$ such that $(w - v)^- \in D_V^1(\Omega)$. Then $w \geq v$ in Ω .*

Remark 3.14. In particular, Lemmas 3.3 and 3.12 imply that if \mathcal{E}_V satisfies the λ -property then equation $(*)$ has a “rich” cone of positive super-solutions. Indeed, if $0 \leq f \in L^2(\Omega, \lambda^{-1}dx)$ and $u_f \in D_V^1(\Omega)$ is the solution to (3.3) constructed in Lemma 3.3 then $u_f > 0$ in Ω by Lemma 3.12 and hence u_f is a weak positive supersolution for $-\Delta + V$. The situation is different if \mathcal{E}_V is nonnegative but *does not* satisfy the λ -property.

Example 3.15 (Liouville’s Theorem on the plane). Consider $-\Delta$ on \mathbb{R}^2 . Obviously, in this case the form \mathcal{E}_0 is nonnegative. Yet, according to the classical Liouville’s theorem the only positive superharmonic for $-\Delta$ on \mathbb{R}^2 is constant.

Example 3.16 (Principal Dirichlet eigenvalue). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\lambda_1 > 0$ be the principal Dirichlet eigenvalue of $-\Delta$ on Ω as in Example 2.6. By (2.5), the form \mathcal{E}_{λ_1} is nonnegative. Yet, the corresponding principal eigenfunction $\phi_1 > 0$ is (up to a constant) the only positive supersolution to $-\Delta - \lambda_1$ in Ω (see Corollary 1.2).

The above examples clarify the following classification:

- $-\Delta + V$ is *subcritical* if \mathcal{E}_V satisfies the λ -property. In this case $-\Delta + V$ has a “rich” cone of positive supersolutions (see Remark 3.14).
- $-\Delta + V$ is *critical* if \mathcal{E}_V is nonnegative but does not satisfy the λ -property. In this case $-\Delta + V$ has exactly one (up to a scalar) positive supersolution, which is actually a solution (see [4, Theorem 5.2]).
- $-\Delta + V$ is *supercritical* if \mathcal{E}_V is not nonnegative. In this case $-\Delta + V$ has no positive supersolutions (see Corollary 1.2).

Further study of $-\Delta + V$ from the point of view of this classification is known as *criticality theory*, see [4, 6, 7]. In this lectures we are interested in one particular aspect only – we want to characterise the “size” of the cone of positive super-solutions of subcritical operators in terms of the admissible decay (or growth) of supersolutions “at infinity”.

4. A PHRAGMÉN-LINDELÖF PRINCIPLE

In this section we assume that Ω is an exterior domain in \mathbb{R}^N such that $\mathbb{R}^N \setminus \bar{B}_1 \subset \Omega$ and $\partial\Omega \neq \emptyset$. We also always assume that $-\Delta + V$ satisfies the λ -property in Ω . Note that then $-\Delta + V$ also satisfies the λ -property in any subdomain $\Omega' \subset \Omega$, see Remark 3.2. We are going to study admissible decay at infinity, i.e. as $|x| \rightarrow \infty$, of all positive supersolutions to $-\Delta + V$ in Ω .

Definition 4.1. We say $u > 0$ is a *small* (sub)solution at infinity for $-\Delta + V$ if u is a (sub)solution for $-\Delta + V$ in B_R^c for some $R \geq 1$ and there exists a supersolution $w_* > 0$ for $-\Delta + V$ in Ω , such that

$$(4.1) \quad \lim_{|x| \rightarrow \infty} \frac{w_*}{u} = +\infty.$$

We say $U > 0$ is a *large* (sub)solution at infinity for $-\Delta + V$ in Ω if U is (sub)solution for $-\Delta + V$ in B_R^c for some $R \geq 1$ and U is *not* a small (sub)solution, i.e. for any supersolution

$w > 0$ for $-\Delta + V$ in Ω ,

$$(4.2) \quad \liminf_{|x| \rightarrow \infty} \frac{w}{U} < +\infty.$$

Exercise 4.2. Prove that if $u > 0$ is a small and $U > 0$ a large solution at infinity for $-\Delta + V$ in Ω then

$$(4.3) \quad \limsup_{|x| \rightarrow \infty} \frac{U}{u} = +\infty.$$

Intuitively, a small solution at infinity is “smaller” than one of the supersolutions. Then a *large* solution at infinity is *not* “smaller” than any of the supersolutions, i.e. in some sense it dominates at infinity all positive supersolutions.³ An essential observation is that if a solution is smaller than *one* of the positive supersolutions in the sense of (4.1), then up to a constant it is dominated by *every* positive supersolution.

Lemma 4.3 (Small solution lemma). *If $u > 0$ is a small solution at infinity for $-\Delta + V$ in Ω then for any supersolution $w > 0$ for $-\Delta + V$ in Ω there exists $c > 0$ such that*

$$(4.4) \quad u(x) \leq cw(x) \quad (|x| \geq 1).$$

Proof. Choose $c > 0$ such that $cu \leq w$ as $|x| = 1$. For small $\varepsilon > 0$ consider the barrier functions

$$u_\varepsilon := cu - \varepsilon w_*.$$

In view of (4.1), there exists $\rho_\varepsilon > 1$ such that

$$u_\varepsilon \leq 0 \quad (|x| \geq \rho_\varepsilon)$$

and moreover, $\rho_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

Set $\Omega_\varepsilon := \{x \in \mathbb{R}^N : 1 < |x| < \rho_\varepsilon\}$. Since $w > 0$, we conclude that

$$u_\varepsilon \leq w \quad \text{on } \partial\Omega_\varepsilon$$

and hence $(w - u_\varepsilon)^- \in H_0^1(\Omega_\varepsilon)$, see Remark 3.2. By the comparison principle of Corollary 3.13 we conclude that

$$u_\varepsilon \leq w \quad \text{in } \Omega_\varepsilon.$$

Then the assertion follows, since $\varepsilon > 0$ could be taken arbitrary small and taking into account that $\rho_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$. \square

Proposition 4.4 (Phragmén-Lindelöf principle for supersolutions). *Let $u > 0$ be a small and $U > 0$ a large solution at infinity for $-\Delta + V$ in Ω . Then for any supersolution $w > 0$ for $-\Delta + V$ in Ω :*

$$\liminf_{|x| \rightarrow \infty} \frac{w}{u} > 0, \quad \liminf_{|x| \rightarrow \infty} \frac{w}{U} < \infty.$$

Proof. The first \liminf is simply the reformulation of (4.4), the second \liminf is the inversion of (4.2). \square

³Similarly, we could define small and large *subsolutions* but we will omit this for simplicity.

Lemma 4.5 (Large solution lemma). *Assume that $U > 0$ satisfies*

$$(4.5) \quad -\Delta U + VU = 0 \quad (|x| > 1) \quad \text{and} \quad U = 0 \quad (|x| = 1).$$

Then U is a large solution for $-\Delta + V$.

Proof. Assume that U is not a large solution for $-\Delta + V$, i.e. U is a small solution. Then there exists a supersolution $w_* > 0$ for $-\Delta + V$ such that (4.1) holds.

For small $\varepsilon > 0$ consider the barrier functions

$$U_\varepsilon := U - \varepsilon w_*.$$

In view of (4.1), there exists $\rho_\varepsilon > 1$ such that

$$U_\varepsilon \leq 0 \quad (|x| \geq \rho_\varepsilon)$$

and moreover, $\rho_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

Set $\Omega_\varepsilon := \{x \in \mathbb{R}^N : 1 < |x| < \rho_\varepsilon\}$ and choose a small $\varepsilon > 0$ such that $U_\varepsilon^+ > \delta > 0$ on an open set $G \subset \Omega_\varepsilon$. Since $w_* > 0$ and $U = 0$ as $|x| = 1$ and $|x| = \rho_\varepsilon$, we conclude that for any $n \in \mathbb{N}$,

$$nU_\varepsilon \leq w_* \quad \text{on } \partial\Omega_\varepsilon$$

and hence $(w_* - nU_\varepsilon)^- \in H_0^1(\Omega_\varepsilon)$, see Remark 3.2. By the comparison principle of Corollary 3.13 we conclude that

$$nU_\varepsilon \leq w_* \quad \text{in } \Omega_\varepsilon.$$

Then $w_* \geq n\delta$ on G and since $n \in \mathbb{N}$ can be taken arbitrary large, $w_* = +\infty$ on G , which is a contradiction. \square

Remark 4.6. If U satisfies (4.5) then $U \notin D_V^1(B_1^c)$. Indeed, assume $U \in D_V^1(B_1^c)$. Then $\mathcal{E}_V(U) = 0$ by (4.2). But $\mathcal{E}_V(u) > 0$ for any $0 \neq u \in D_V^1(B_1^c)$ since \mathcal{E}_V satisfies the λ -property, a contradiction.

Remark 4.7. Assume that $U > 0$ is a (sub)solution in $|x| > 1$ and there exists a supersolution $w_* > 0$ such that

$$\lim_{|x| \rightarrow \infty} \frac{U}{w} = +\infty.$$

Then U is a large sub-solution at infinity.

Hint: Consider $U_* = U - cw_*$ for a sufficiently large $c > 0$.

Example 4.8 ($-\Delta$ in \mathbb{R}^N with $N \geq 3$). If $-\Delta w \geq 0$ in $\mathbb{R}^N \setminus \bar{B}_1$ then

$$\liminf_{|x| \rightarrow \infty} \frac{w(x)}{|x|^{-(N-2)}} > 0, \quad \liminf_{|x| \rightarrow \infty} w(x) < \infty.$$

To see this, take $u(x) = |x|^{-(N-2)}$ as a small solution, $U(x) = 1 - |x|^{-(N-2)}$ as a large solution for $-\Delta$ in $\mathbb{R}^N \setminus \bar{B}_1$ and use Phragmén-Lindelöf principle for supersolutions. To check that $|x|^{-(N-2)}$ is a small solution, take $w_* = 1$ as a reference supersolution in (4.1).

Example 4.9 ($-\Delta$ in \mathbb{R}^2). If $-\Delta w \geq 0$ in $\mathbb{R}^N \setminus \bar{B}_1$ then

$$\liminf_{|x| \rightarrow \infty} w(x) > 0, \quad \liminf_{|x| \rightarrow \infty} \frac{w(x)}{\log |x|} < \infty.$$

To see this, take $u(x) = 1$ as a small solution, $U(x) = \log |x|$ as a large solution for $-\Delta$ in $\mathbb{R}^2 \setminus \bar{B}_1$ and use Phragmén-Lindelöf principle for supersolutions. To check that 1 is a small solution, take $w_* = \log |x|$ as a reference supersolution in (4.1).

Example 4.10 ($-\Delta + 1$ in \mathbb{R}^3). If $-\Delta w + w \geq 0$ in $\mathbb{R}^3 \setminus \bar{B}_1$ then

$$\liminf_{|x| \rightarrow \infty} \frac{w(x)}{\frac{e^{-|x|}}{|x|}} > 0, \quad \liminf_{|x| \rightarrow \infty} \frac{w(x)}{\frac{e^{|x|}}{|x|}} < \infty.$$

To see this, take

$$u(x) = \frac{e^{-|x|}}{|x|}, \quad U(x) = \frac{e^{|x|-1} - e^{1-|x|}}{|x|},$$

as a small and large solution for $-\Delta + 1$ in $\mathbb{R}^3 \setminus \bar{B}_1$, respectively. Then use Phragmén-Lindelöf principle for supersolutions. To check that u is a small solution, take $w_* = 1$ as a reference supersolution in (4.1).

Exercise 4.11. Show that if

$$-\Delta w + \frac{c}{|x|^2} w \geq 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}_1,$$

and $c > -C_H := -\frac{(N-2)^2}{4}$, then

$$\liminf_{|x| \rightarrow \infty} \frac{w(x)}{|x|^{\alpha_-}} > 0, \quad \liminf_{|x| \rightarrow \infty} \frac{w(x)}{|x|^{\alpha_+}} < \infty,$$

where $\alpha_- < \alpha_+$ are the two roots of the quadratic equation $\alpha(\alpha + N - 2) + c = 0$.

Note that if $c > 0$ then $\alpha_+ > 0$ and U is growing at infinity, while if $c < 0$ then $\alpha_+ < 0$ and U is decaying at infinity.

5. HALF-SPACE ANSATZ

Consider the following singularly perturbed linear problem model in the half-space

$$(A_{\mu,\varepsilon}) \quad \begin{cases} -\Delta h = 0 & \text{in } \mathbb{R}_+^N, \\ h = 0 & \text{on } \partial\mathbb{R}_+^N \setminus \{0\}, \end{cases}$$

where $\mathbb{R}_+^N = \{x \in \mathbb{R}^N \mid x_1 > 0\}$. We are going to construct explicit solutions of $(A_{\mu,\varepsilon})$.

Let

$$h(x) = x_1^\alpha r^\beta.$$

We compute

$$\begin{aligned} -\Delta h &= -\alpha(\alpha-1)x_1^{\alpha-2}r^\beta - \beta(\beta+N-2+2\alpha)x_1^\alpha r^{\beta-2} \\ &= \left(\frac{-\alpha(\alpha-1)}{x_1^2} + \frac{-\beta(\beta+N-2+2\alpha)}{r^2} \right) h. \end{aligned}$$

If $\alpha_- \leq \alpha_+$ are as before the root of $\alpha(\alpha - 1) = \mu$ and

$$\beta_+ := -(N - 2) - 2\alpha_+, \quad \beta_- := -(N - 2) - 2\alpha_-,$$

then the functions

$$h_+(x) = x_1^{\alpha_+} r^{\beta_+}, \quad h_-(x) = x_1^{\alpha_-} r^{\beta_-},$$

are explicit solutions of the equation $(A_{\mu,0})$,

$$-\Delta h - \frac{\mu}{x_1^2} h = 0 \quad \text{in } \mathbb{R}_+^N.$$

Observe that these functions are distinct if and only if $\alpha_+ \neq \alpha_-$ or equivalently $\mu < \frac{1}{4}$.

We shall also need solutions of $(A_{\mu,\varepsilon})$ with the same properties as h_\pm on the boundary, especially when we are dealing with the case $\mu = 1/4$. Let $\beta_{\pm,-} \leq \beta_{\pm,+}$ be the roots of

$$\beta(\beta + N - 2 + 2\alpha_\pm) = \varepsilon,$$

or equivalently, of

$$(5.1) \quad \beta(\beta + N - 1 \pm \sqrt{1 - 4\mu}) = \varepsilon.$$

Then any of the *four* functions

$$\begin{aligned} h_+(x) &= x_1^{\alpha_+} r^{\beta_{+,+}}, & h_-(x) &= x_1^{\alpha_+} r^{\beta_{+,-}}, \\ H_+(x) &= x_1^{\alpha_-} r^{\beta_{-,+}}, & H_-(x) &= x_1^{\alpha_-} r^{\beta_{-,-}}, \end{aligned}$$

give a solution to $(A_{\mu,\varepsilon})$.

Remark 5.1. Although we are interested only in $\varepsilon \geq 0$, note that for a given $\mu \leq 1/4$ solutions h_+ and h_- can be constructed for any

$$\varepsilon \geq -\left(\frac{N-1}{2} + \sqrt{\frac{1}{4} - \mu}\right)^2.$$

Via the AAP principle this leads to the family of Hardy type inequalities

$$\int_{\mathbb{R}_+^N} |\nabla \phi|^2 \geq \mu \int_{\mathbb{R}_+^N} \frac{\phi^2}{x_1^2} + \left(\frac{N-1}{2} + \sqrt{\frac{1}{4} - \mu}\right)^2 \int_{\mathbb{R}_+^N} \frac{\phi^2}{|x|^2}.$$

Interestingly, solutions H_+ and H_- are defined only for smaller range

$$\varepsilon \geq -\left(\frac{N-1}{2} - \sqrt{\frac{1}{4} - \mu}\right)^2,$$

and the latter constant vanishes if $\mu = -\frac{1}{4}N(N - 2)$. Y. Pinchover pointed out that the constant is optimal. He refers to the arguments in the recent paper [21].

Lemma 5.2. (MINIMAL POSITIVE SOLUTION LEMMA) *Let $\mu < 1/4$. Then the problem*

$$(5.2) \quad L_\mu h_R = 0 \quad \text{in } \mathbb{R}_+^N \cap B_R(0), \quad h_R = x_1^{\alpha_-} \quad \text{on } \mathbb{R}_+^N \cap S_R(0)$$

admits a positive solution such that for $x \in \mathbb{R}_+^N \cap B_{R/2}(0)$ it holds

$$(5.3) \quad \limsup_{x_1 \rightarrow 0} \frac{h_R(x)}{x_1^{\alpha_+}} < \infty.$$

Proof. Let $\psi : [0, R] \rightarrow \mathbb{R}$ be a smooth function such that $\psi(R) = 1$, $0 \leq \psi(r) \leq 1$ and $\psi(r) = 0$ for $r \leq R/2$. Set

$$f(x) := L_\mu(x_1^{\alpha_-} \psi) = x_1^{\alpha_-} \left(-\psi'' - \frac{N-1+2\alpha_-}{r} \psi' \right).$$

Clearly $f \in L^1(\mathbb{R}_+^N \cap B_R(0), x_1^{\alpha_-} dx)$ and $f(x) = 0$ in $\mathbb{R}_+^N \cap B_{R/2}(0)$. Then the problem

$$L_\mu \eta = -f \quad \text{in } \mathbb{R}_+^N \cap B_R(0), \quad \eta = 0 \quad \text{on } \partial(\mathbb{R}_+^N \cap S_R(0))$$

admits a solution η such that for $x \in \mathbb{R}_+^N \cap B_{R/2}(0)$ it holds

$$\limsup_{x_1 \rightarrow 0} \frac{\eta(x)}{x_1^{\alpha_+}} < \infty.$$

Then clearly

$$h_R := x_1^{\alpha_-} \psi + \eta$$

is the required solution of (5.2). \square

We establish a version of the Phragmen–Lindelöf type comparison principle for sub-harmonics of L_μ , which shows that sub-harmonics either have a “strong” singularity at the origin or have a “minimal” decay at the origin. See [1, pp. 93–106] for a classical reference to the Phragmen–Lindelöf principle.

Lemma 5.3. (PHRAGMEN–LINDELÖF TYPE ESTIMATE) *Let $\mu < 1/4$. Let h be an L_μ –sub-harmonic in $\mathbb{R}_+^N \cap B_R(0)$, for some $R > 0$. Assume that $x \in \mathbb{R}_+^N \cap B_R(0)$ and*

$$(5.4) \quad \lim_{x_1 \rightarrow 0} \frac{h(x)}{x_1^{\alpha_-} + x_1^{\alpha_+} |x|^{-(N-2+2\alpha_+)}} = 0.$$

Then for $x \in \mathbb{R}_+^N \cap B_{R/2}(0)$ it holds

$$(5.5) \quad \limsup_{x_1 \rightarrow 0} \frac{h(x)}{x_1^{\alpha_+}} < \infty.$$

Proof. Without loss of generality we may assume that h is continuous on $\mathbb{R}_+^N \cap \bar{B}_R(0)$. Let $h_R > 0$ be the minimal positive solution of (5.2), as constructed in Lemma 5.2. Note that

$$h_R(x) \leq cx_1^{\alpha_+} \quad \text{in } \mathbb{R}_+^N \cap B_{R/2}(0).$$

For $\tau > 0$, define a comparison function

$$h_\tau := h - h_R - \tau(x_1^{\alpha_-} + x_1^{\alpha_+} |x|^{-(N-2+2\alpha_+)}).$$

Clearly, h_τ is L_μ –sub-harmonic in $\mathbb{R}_+^N \cap B_R(0)$.

For every $\tau > 0$, (5.4) combined with the construction of h_R implies that $h_\tau \leq 0$ on a neighbourhood of $\partial(\mathbb{R}_+^N \cap B_R(0))$. Hence we can apply the classical comparison principle for L_μ in a proper subdomain of $\mathbb{R}_+^N \cap B_R(0)$ and deduce that $h_\tau \leq 0$ everywhere in $\mathbb{R}_+^N \cap B_R(0)$. By considering arbitrary small $\tau > 0$, we conclude that $h \leq h_R$ in $\mathbb{R}_+^N \cap B_R(0)$. \square

Further reading. For the development of the criticality theory beyond Agmon's ideas in [4, 5] see e.g. [6–8]. Some Phragmén–Lindelöf type results in the context of linear elliptic equations can be found in [1]. Powerful applications to the analysis of Hardy type inequalities involving distance to the boundary are given in [9]. More recent developments of the criticality theory presented e.g. in [15, 21]. For an extension to p -Laplacian see [17]; to local and nonlocal Dirichlet forms see [19, 22]. This list is very far from being complete.

6. A NONLINEAR LIOUVILLE THEOREM

We are going to use the AAP and Phragmén–Lindelöf principles developed in the previous section to study positive supersolutions of the semilinear elliptic equation

$$(6.1) \quad -\Delta u = u^p \quad \text{in } \Omega,$$

where Ω is an exterior domain in \mathbb{R}^N such that $\mathbb{R}^N \setminus \bar{B}_1 \subset \Omega$ and $\partial\Omega \neq \emptyset$, and $p \in \mathbb{R}$ is the nonlinear *exponent*, which could take both positive and negative values.

A *weak* (super) solution to (6.1) is a function $u \in H_{loc}^1(\Omega)$ such that $u^p \in L_{loc}^1(\Omega)$ and

$$(6.2) \quad \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx (\geq) = \int_{\Omega} u^p \varphi \, dx, \quad \forall 0 \leq \varphi \in H_c^1(\Omega) \cap L_c^{\infty}(\Omega).$$

Note that if $u \geq 0$ is a weak supersolution to (6.1) then $-\Delta u \geq 0$ in Ω , and hence $u > 0$ in Ω . In what follows we often omit the *weak* and use simply (super) solution.

Theorem 6.1. *Let $N \geq 3$. Equation (6.1) admits a positive weak supersolution if and only if $p > p_S := \frac{N}{N-2}$.*

This theorem was proved by J. Serrin in the 70s for the radial functions, see the introduction to [11] for the references and a general overview of nonlinear Liouville's theorem. The exponent p_S is often known as the Serrin's critical exponent. The idea to use the AAP and Phragmén–Lindelöf principles in the context of nonlinear Liouville's theorems goes back to [12] and was developed in [13, 14, 16, 18, 20].

Proof. Our proof of Theorem 6.1 will be split into nonexistence and existence part. In the proof of nonexistence we will distinguish four separate cases: $1 < p < p_S$, $p = p_S$, $p = 1$, $p < 1$. The nonexistence in the superlinear case $1 < p < p_S$ relies on the *lower bound* in the Phragmén–Lindelöf principle while the nonexistence in the sublinear case uses the *upper bound* in the Phragmén–Lindelöf principle.

Before we start the proof, we present two technical lemmas. The first one is a particular case of the nonexistence counterpart of the AAP principle (Corollary 1.2).

Lemma 6.2. *Assume that $u > 0$ satisfies*

$$(6.3) \quad -\Delta u - c|x|^{-2+\varepsilon} u \geq 0 \quad (|x| > 1),$$

for some $\varepsilon > 0$ and $c > 0$. Then $u \equiv 0$.

Proof. Consider the quadratic form

$$\mathcal{E}(u) := \int_{\Omega} |\nabla u|^2 dx - c \int_{\Omega} \frac{\varphi^2}{|x|^{2+\varepsilon}} dx \quad (\varphi \in C_c^\infty(\Omega)).$$

Take $\varphi \in C_c^\infty(\Omega)$ such that $\varphi \geq 0$, $0 \leq \varphi \leq 1$ and $\varphi = 0$ for $|x| > 2$ and $|x| < 1$. Then for $R > 1$ the rescaling

$$\varphi_R(x) := \varphi\left(\frac{x}{R}\right) \in C_c^\infty(\Omega)$$

and by the change of variables we compute

$$\mathcal{E}(\varphi_R) = R^{N-2} \int_{\Omega} |\nabla \varphi|^2 dx - c R^{N-2+\varepsilon} \int_{\Omega} \frac{\varphi^2}{|x|^{2-\varepsilon}} dx \rightarrow -\infty,$$

as $R \rightarrow \infty$. By the AAP principle (Corollary 1.2) we conclude that $u \equiv 0$. \square

The second lemma is an an apriori lower bound on positive solutions of (6.1) in the sublinear case. Note that the bound (6.3) depends on the value of $p < 1$.

Lemma 6.3. *Let $p < 1$, $s \in \mathbb{R}$ and assume that $u > 0$ satisfies*

$$-\Delta u \geq |x|^s u^p \quad \text{in } \Omega.$$

Then

$$(6.4) \quad u \geq c|x|^{\frac{2+s}{1-p}} \quad (|x| > 1).$$

Proof. The proof uses the AAP principle and weak Harnack's inequality, see [14, Lemma 6.1]. \square

Now we are in a position to prove Theorem 6.1. We will proceed case by case.

Nonexistence in the superlinear subcritical case $1 < p < p_S$. Assume that $u > 0$ is a super-solution to (6.1). Then $-\Delta u \geq 0$ in Ω and hence, as in Example 4.8 we conclude that for some $c > 0$,

$$(6.5) \quad u \geq c|x|^{-(N-2)} \quad (|x| > 1)$$

Consider the *linearisation* of (6.1),

$$(6.6) \quad -\Delta u + V(x)u \geq 0 \quad \text{in } \Omega,$$

where $V(x) := u^{p-1}$. Since $p > 1$, using (6.5) we can estimate

$$(6.7) \quad V(x) \geq c_1|x|^{-(N-2)(p-1)} \quad (|x| > 1),$$

where $c_1 = c^{p-1} > 0$. Note that if $1 < p < p_S$ then

$$(6.8) \quad -(N-2)(p-1) > -2.$$

Hence $u > 0$ satisfies

$$(6.9) \quad -\Delta u + c_1|x|^{-2+\varepsilon}u \geq 0 \quad (|x| > 1),$$

for some $\varepsilon > 0$. By Lemma 6.2, we conclude that $u \equiv 0$. \square

Nonexistence in the critical case $p = p_S$. In this case we have $\varepsilon = 0$ in (6.9) and the previous argument fails.⁴ Instead, we will iterate the previous to improve the lower bound (6.7).

Indeed, we may assume that $0 < c_1 < C_H$ and $u > 0$ satisfies

$$(6.10) \quad -\Delta u - c_1|x|^{-2}u \geq 0 \quad (|x| > 1).$$

As in the Exercise 4.11 we conclude that for some $c_2 > 0$,

$$(6.11) \quad u \geq c_2|x|^{\alpha_-} \quad (|x| > 1),$$

where $\alpha_- > -(N - 2)$ is the smallest root of $-\alpha(\alpha + N - 2) = c_1$. Then $u > 0$ satisfies the linearisation equation (6.6) and we can estimate

$$(6.12) \quad V(x) \geq c_3|x|^{\alpha_-(p-1)} \quad (|x| > 1),$$

where $c_3 = c_2^{p-1}$. Since $p = p_S$ and $\alpha_- > -(N - 2)$,

$$(6.13) \quad \alpha_-(p-1) > -2.$$

Hence $u > 0$ satisfies (6.9) with some $\varepsilon > 0$ and as before, by Lemma 6.2, we conclude that $u \equiv 0$. \square

Nonexistence in the linear case $p = 1$. In this case the equation (6.1) is linear. We simply note that there exists $\varphi \in C_c^\infty(\Omega)$ such that the corresponding quadratic form

$$\mathcal{E}_{-1}(\varphi) = \int_{\Omega} |\nabla \varphi|^2 dx - \int_{\Omega} \varphi^2 dx < 0.$$

(This can be seen using the same family of test functions φ_R as in the proof of Lemma 6.2.) By the AAP principle (Corollary 1.2) we conclude that $u \equiv 0$. \square

Nonexistence in the sublinear case $p < 1$. Assume that $u > 0$ is a supersolution to (6.1). Then $-\Delta u \geq 0$ in Ω and using the upper bound in Example 4.8 we conclude that

$$(6.14) \quad \liminf_{|x| \rightarrow \infty} u(x) < \infty.$$

But according to Lemma 6.3,

$$(6.15) \quad u \geq c|x|^{\frac{2}{1-p}} \quad (|x| > 1).$$

Since $p < 1$ these two bounds are incompatible with each other and we conclude that $u \equiv 0$. \square

Existence in the case $p > p_S$. A direct computation shows that for every $p > p_S$

$$u = c_p|x|^{-\frac{2}{p-1}}, \quad c_p^{p-1} = \frac{2}{(p-1)^2}((N-2)p-N)$$

is a solution of (6.1).⁵ \square

⁴If $c_1 > C_H$, the critical Hardy constant, we would conclude again by the AAP principle. However we do not control the size of $c_1 > 0$ and in general, c_1 could be small.

⁵Note that $c_p < 0$ for $p < p_S$ and $c_p = 0$ if $p = p_S$.

Exercise 6.4. Modify the previous arguments to show that if $N = 2$ then equation (6.1) has no positive weak supersolutions for any $p \in \mathbb{R}$.

Exercise 6.5. Let $N \geq 2$, $c > 0$ and $p \in \mathbb{R}$. Show that the equation

$$-\Delta u + \frac{c}{|x|^2} u = u^p \quad \text{in } |x| > 1$$

admits a positive weak supersolution if and only if $p \notin [1 - \frac{2}{\alpha_+}, 1 - \frac{2}{\alpha_-}]$, where α_- and α_+ are defined in Exercise 4.11.

Hint: Use small and large solutions constructed in Exercise 4.11. The nonexistence in the lower critical case $p = 1 - \frac{2}{\alpha_+} < 0$ is difficult, see [14, Lemma 6.6]. All other regimes could be studied similarly to the proof of Theorem 6.1.

Exercise 6.6. Let $N \geq 3$, $s > -2$ and $p \in \mathbb{R}$. Show that the equation

$$-\Delta u = |x|^s u^p \quad \text{in } |x| > 1$$

admits a positive weak supersolution if and only if $p > \frac{N+s}{N-2}$.

Hint: In the case $p < 1$ use (6.4).

Further reading. Similar methods based on the AAP and Phragmén–Lindelöf principles were used to prove Liouville’s theorems for divergence type semilinear equations in conical domains [13], equations with Hardy type potentials in conical domains [14], quasilinear equations involving p -Laplacian [16], equations with Hardy type potentials involving distance to the boundary of a bounded domain [18] and nonlocal Choquard’s equations [20].

7. APPENDIX: A BREZIS–BROWDER TYPE RESULT

By modifying the proof of Lemma 3.12, we can establish the following approximation lemma, cf. [23, Theorem 3.4.1].

Lemma 7.1. *Assume that \mathcal{E}_V satisfies the λ -property and $0 \leq w \in D_V^1(\Omega)$. Then there exists a sequence $(w_n) \in D_V^1(\Omega) \cap L^\infty(\Omega)$ such that:*

- (a) $\text{supp}(w_n)$ is compact in Ω ;
- (b) $0 \leq w_n \leq w$;
- (c) $\mathcal{E}_V(w - w_n) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Let $(\varphi_n) \subset C_c^\infty(\Omega)$ be an approximating sequence for $w \in D_V^1(\Omega)$. Set

$$w_n := \varphi_n^+ \wedge w \wedge n.$$

Then (w_n) has all the required properties and (c) follows as in the proof of Lemma 3.12. \square

An important consequence of Lemma 7.1 is a Brezis–Browder type result, cf. [23, Theorem 3.4.1] and a historical discussion in [23, p.82].

Theorem 7.2. Let $T \in D_V^{-1}(\Omega) \cap L_{loc}^1(\Omega)$ and $w \in D_V^1(\Omega)$. Assume

$$(7.1) \quad T(x)w(x) \geq 0 \quad \text{a.e. in } \Omega.$$

Then $Tw \in L^1(\Omega)$ and

$$(7.2) \quad \langle T, w \rangle = \int_{\Omega} T(x)w(x)dx,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $D_V^1(\Omega)$ and $D_V^{-1}(\Omega)$.

Proof. Let (w_n) be an approximating sequence for w , defined in Lemma 7.1. Since $T \in L_{loc}^1(\Omega)$ and $w_n \in L_c^\infty(\Omega)$, we know that

$$\langle T, w_n \rangle = \int_{\Omega} T(x)w_n(x)dx.$$

By Lemma 7.1 (c), $\langle T, w_n \rangle \rightarrow \langle T, w \rangle$ as $n \rightarrow \infty$. On the other hand, $T(x)w_n(x) \geq 0$ a.e. in Ω and then by Fatou's lemma $Tw \in L^1(\Omega)$. But we also know that $T(x)w_n(x) \leq T(x)w(x)$ a.e. in Ω . Then

$$\lim_{n \rightarrow \infty} \int_{\Omega} T(x)w_n(x)dx = \int_{\Omega} T(x)w(x)dx,$$

by dominated convergence. \square

Exercise 7.3. Show that (7.1) can be replaced by one of the weaker assumptions:

$$(7.3) \quad T(x) \geq 0 \quad \text{a.e. in } \Omega,$$

or

$$(7.4) \quad T(x)w(x) \geq -|f(x)| \quad \text{a.e. in } \Omega,$$

for some $f \in L^1(\Omega)$.

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