Tutorial: Similarity and Numerical Solutions for Third Order, Partial Differential Equations

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Similarity Solutions for the Third Order Linear PDE

Third Order linear PDE

$$u_t = u_{xxx}, \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+$$

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Similarity Solutions

$$u(x,t) = t^{-a}F(y), \quad y = xt^{-b}.$$

Reduces to an ODE

$$-at^{-a-1}F - bt^{-a-1}F'y = t^{-a-3b}F'''.$$

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Equating powers of t

$$a+1 = a+3b.$$

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Equating powers of t

$$b = \tfrac{1}{3}.$$

Equating coefficients

$$a = \tfrac{1}{3}.$$

F solves the ODE

$$F''' = -\frac{1}{3}\,F'y - \frac{1}{3}\,F.$$

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Yields the Airy equation.

Asymptotics

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Asymptotic To

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Order Less Than

$$f(x) = o(g(x)) \quad \text{as} \quad x \rightarrow x_0 \quad \Longleftrightarrow \quad \lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = 0.$$

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Assume solution, $F(y)$, is exponential as $y \to +\infty$

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Assume that $s(y)$ is some polynomial, such that $s(y) \sim \alpha y^{\beta}$.
Then $s' \sim \alpha\beta y^{\beta-1}$ and $s'' \sim \alpha\beta(\beta-1)\, y^{\beta-2}$.

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Second order linear ODE

$$F'' = -\tfrac{1}{3} F y.$$

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Substituting into the ODE

$$\alpha\beta(\beta-1)y^{\beta-2} + \alpha^2\beta^2 y^{2\beta-2} \sim -\tfrac{1}{3}y$$

Two cases: $\beta \leq 0$ and $\beta > 0$.

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$$\alpha\beta(\beta - 1)y^{\beta-2} + \alpha^2\beta^2 y^{2\beta-2} \sim -\frac{1}{3}y$$

For $\beta \leq 0$

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small term + small term

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For $\beta \le 0$

$$y^{\beta-2} = o(y) \quad \text{and} \quad y^{2\beta-2} = o(y) \quad \Rightarrow \text{no balance.}$$

small term + small term $\nsim$ larger term

$$\alpha\beta(\beta-1)y^{\beta-2}+\alpha^2\beta^2y^{2\beta-2}\sim-\frac{1}{3}y$$

For $\beta > 0$

$$y^{\beta-2}=o(y^{2\beta-2}) \quad \text{as} \quad y\to\infty$$

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small term larger term

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For $\beta > 0$

$$y^{\beta-2} = o(y^{2\beta-2}) \quad \text{as} \quad y \to \infty \qquad \Rightarrow \text{Terms can balance}$$

small term + 2nd larger term $\sim$ larger term

Balancing Leading Order Terms

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$$\alpha^2\beta^2 y^{2\beta-2} \sim -\tfrac{1}{3}y$$

Equating Powers of y we have

$$\beta = \tfrac{3}{2} \quad \text{and} \quad \alpha = \tfrac{2i}{3\sqrt{3}}.$$

$$s(y) = -\tfrac{2i}{3\sqrt{3}}y^{3/2} + c(y), \quad c(y) = o(y^{3/2})$$

Insert $s(y)$ into the ODE again. Higher order terms cancel due to previous balance.

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Second term in the expansion.

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$$c(y) \sim -\tfrac{1}{4}\ln y$$

$$s(y) \sim -\tfrac{2\mathrm{i}}{3\sqrt{3}}\,y^{3/2} - \tfrac{1}{4}\ln y$$

Can keep expanding with as many terms as you like for greater accuracy.

Asymptotic Behaviour as $y \to +\infty$

$$F(y) \sim y^{-1/4} \exp\left(-\frac{2\mathrm{i}}{3\sqrt{3}}\, y^{3/2}\right) \quad \text{as} \quad y \to +\infty.$$

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Numerical Solutions using the Boundary Value Problem Solver

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Maximum attained at some point $\hat{a}$.

Can solve for solutions left and right of this point. Match the two to find the full solution.

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Need to move the point $\hat{a}$, until correct profile is found.

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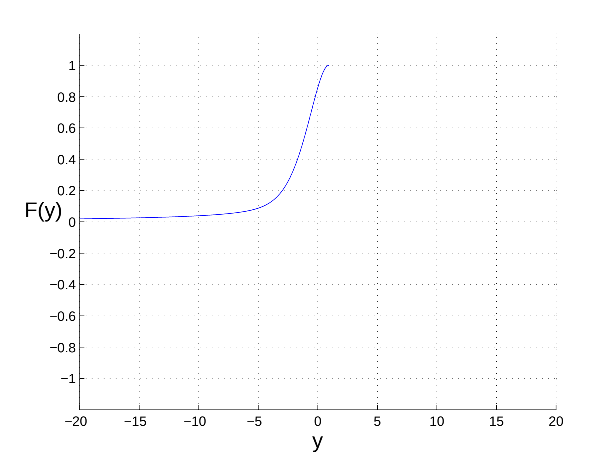
Profile is correct if the second derivative, for both left and right solutions, are the same at the point $\hat{a}$. This ensures continuity at the point $\hat{a}$.

Rescaled Fundamental Solution to $u_t = u_{xxx}$



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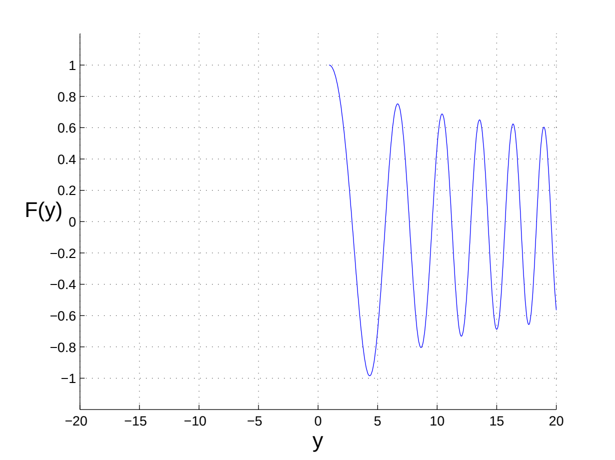

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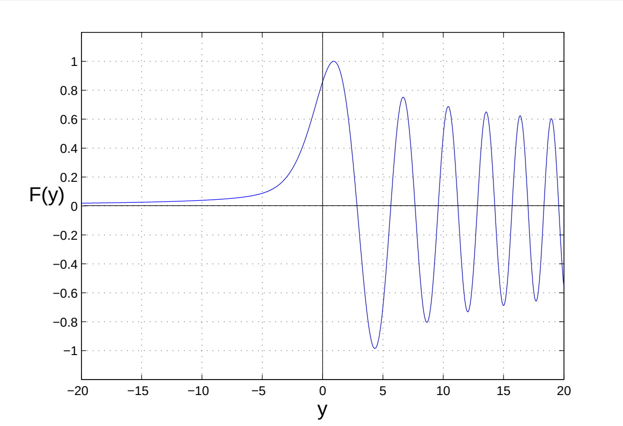

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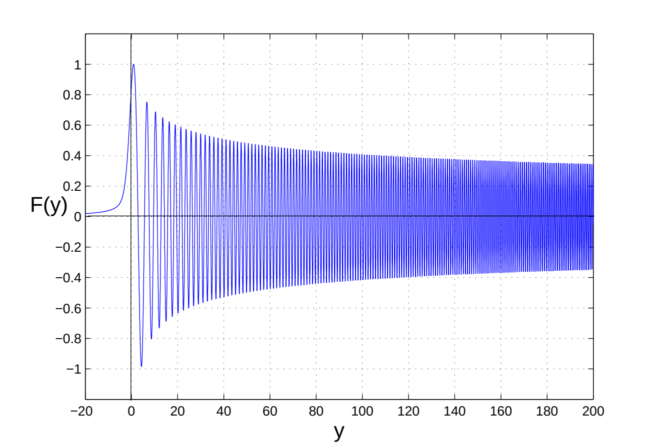

Third Order Semilinear Model

Semilinear PDE

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$$u_*(x,t) = t^{-1/(p-1)} f(y), \quad y = xt^{-1/3}.$$

f solves the ODE

$$f''' + \tfrac{1}{3} f' y + \tfrac{1}{p-1} f - |f|^{p-1} f = 0 \quad \text{in} \quad \mathbb{R}.$$

Asymptotic methods like that used for the linear case, cannot be applied here.

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Can use knowledge of linear results, since semilinear case is the same, with a perturbation of $|f|^{p-1}f$.

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Use 2 boundary conditions at the initial point: $f = 0$ and $f' = 0$.
Use 1 boundary condition at the end point: $f = 0$.

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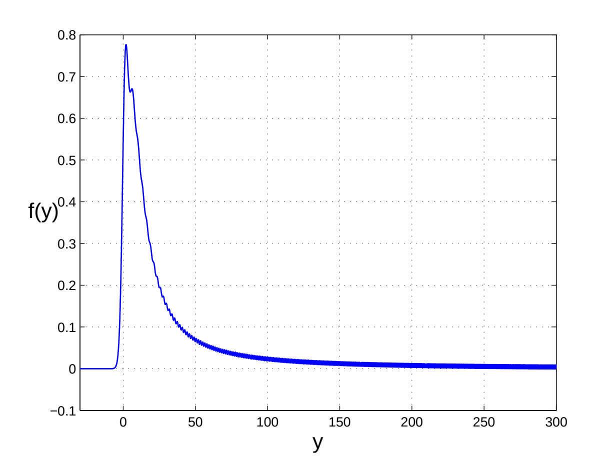
Move end point until best profile is found.

Wrong profile to $u_t = u_{xxx} - u^p$ with $p = 2.9$



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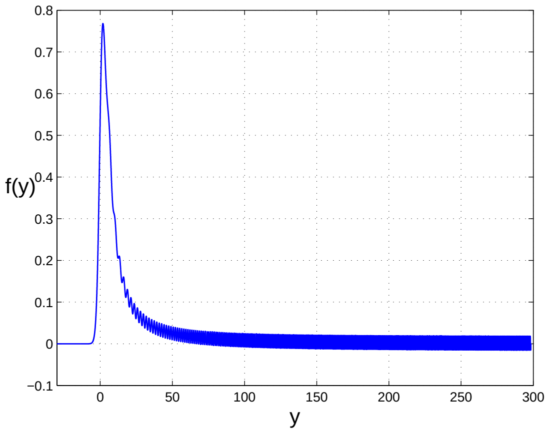

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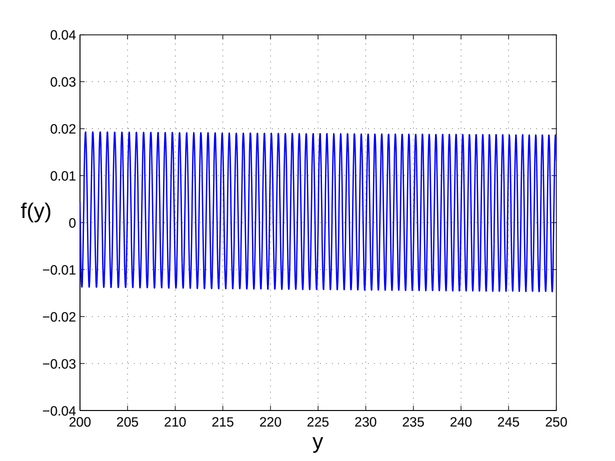

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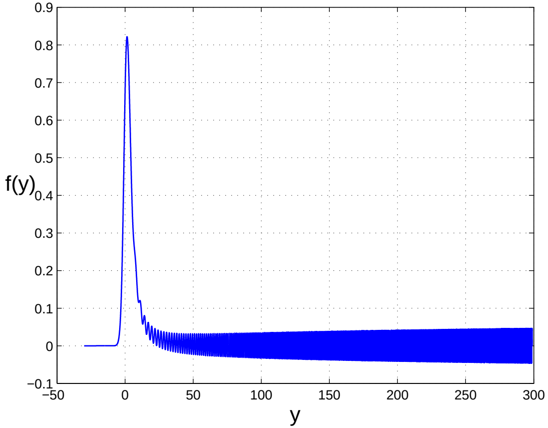

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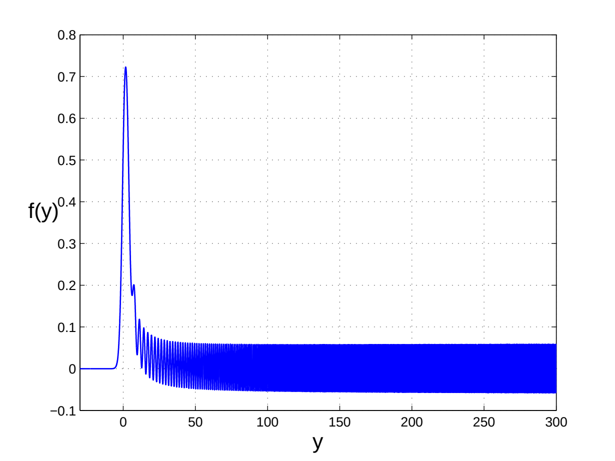

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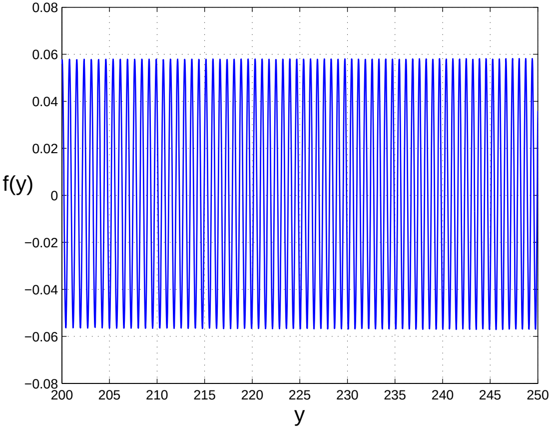

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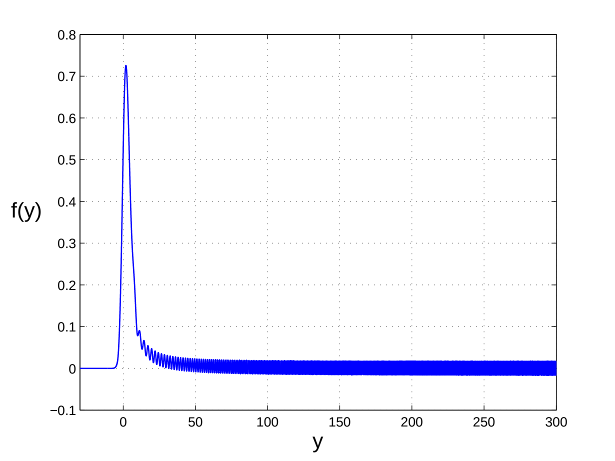

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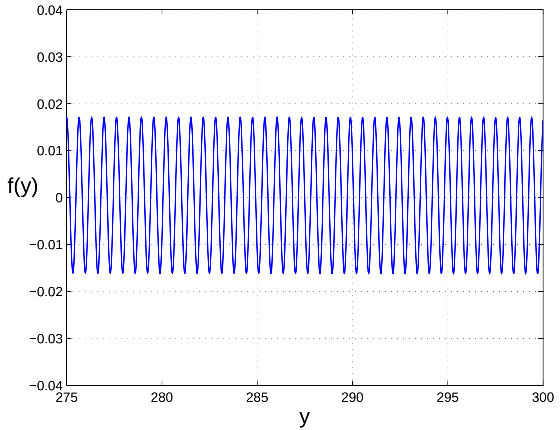
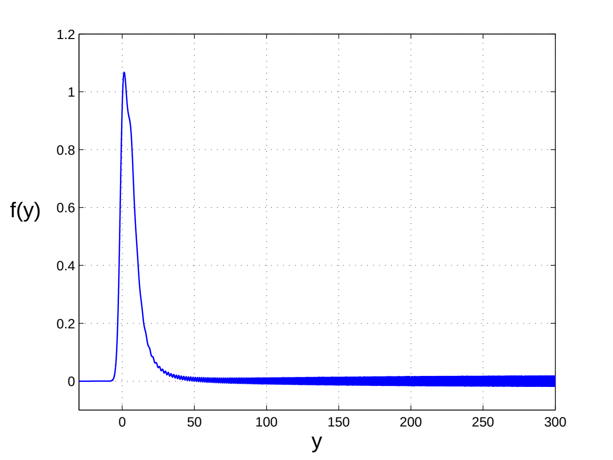


Fundamental solution to $u_t = u_{xxx} - u^p$ with $p = 2$



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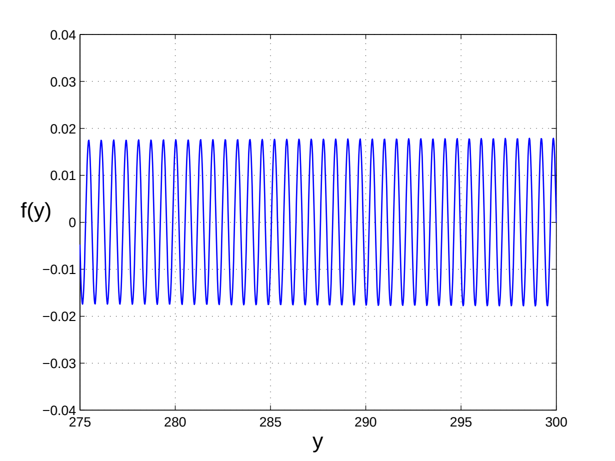

Conditions we look at to ensure the best possible profiles:

- Symmetry of tail for $y \gg 1$.
- Symmetry of tail occurs as close to 0 as possible.
- Minimisation of tail.

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