Swansea Summer School in Nonlinear PDEs

Minicourses abstracts

Martino Bardi

Università degli Studi di Padova

Fully nonlinear degenerate elliptic equations: qualitative properties of viscosity solutions

Abstract. The course is an introduction to the theory of viscosity solutions to fully nonlinear degenerate elliptic equations. After presenting some model problems and motivating the definitions, I will illustrate the basic properties of viscosity solutions, in particular stability and the comparison principles.
In the second part I'll discuss two qualitative properties of subsolutions: the Strong Maximum Principle, i.e., if a subsolution in an open connected set attains an interior maximum then it is constant, and the Liouville property, i.e., if a subsolution in the whole space is bounded form above then it is constant. They are standard results for classical solutions of linear elliptic PDEs, and many extensions are known. I will show how the viscosity methods allow to turn around the difficulties of non-smooth solutions, fully nonlinear equations, and their possible degeneracies

Serena Dipierro

University of Western Australia

Nonlocal equations and applications

Abstract. We present some equation of nonlocal type arising in material sciences (atom dislocations in crystals) and biology (optimal foraging). Topics will include:

- Averaging procedures and the Malmheden Theorem
- All functions are locally s-harmonic
- Harmonic extensions
- Atom dislocations in crystals

Louis Jeanjean

Université de Franche-Comté

On nonlinear quantum graphs

Abstract. The nonlinear Schrodinger equation is a ubiquitous model in physics, with many applications in fields as diverse as Bose-Einstein condensation or non-linear optics. In many physical situations, the underlying space (i.e. the domain on which the solutions of the equation are defined) is essentially one-dimensional and can be modeled as a metric graph, also called a quantum graph, i.e. a locally one-dimensional domain obtained by gluing together one–dimensional, possibly unbounded intervals, the edges, through the identification of some of their endpoints, the vertices. The mathematical study of this type of model is very recent and is growing incredibly. As when the non-linear Schrodinger equation is placed on the entire space the study of stationary solutions is the subject of special attention. One can then focus either on solutions with a given frequency or on solutions having a prescribed mass, namely a prescribed L2 norm. This second type of solution is particularly interesting from a physical point of view, as this quantity is preserved over time.
In this course, we will concentrate on this second type of solution. First, we will look at the mass sub-critical case, whose study naturally leads us to consider minimization problems. Secondly, we will study the mass-supercritical case, where a minimization approach is no longer possible. In particular, we will present the various tools, such as variational methods, blow-up techniques or Morse index estimates, that are needed to deal with this case.

Enrico Valdinoci

University of Western Australia

Nonlocal minimal surfaces and phase transitions

Abstract. We present some results about nonlocal perimeters and their applications to models of capillarity and long-range phase transitions. Topics will include:

- Landau theory of phase transitions and long-range interactions
- Gamma-convergence and convergence of level sets
- Inequalities from differential and integral geometry
- Regularity of (non)local minimal surfaces
- A conjecture of De Giorgi

Jean Van Schaftingen

UCLouvain

Sobolev mappings into manifolds

Abstract. Sobolev mappings arise naturally in many partial differential equations and calculus of variations problems arising in physical models, geometry and computer graphics. Although they can be naturally defined as Sobolev functions satisfying almost everywhere a nonlinear manifold constraint, not all the properties of classical Sobolev spaces pass to the corresponding spaces of mappings. In particular the approximation by smooth maps and the surjectivity of the traces on fractional Sobolev spaces fail when the integrability exponent is small. The obstacles to these properties are intimately connected to obstruction theory and the finiteness or triviality of some homotopy groups of the target manifold. In these lectures, I will describe these pathologies and the nonlinear constructions leading to positive results, and relate them to the lifting and homotopy problems.

Guest lectures abstracts

John Ball

Heriot-Watt University and Maxwell Institute for Mathematical Sciences

Understanding material microstructure

Abstract. Under temperature changes or loading, alloys can form beautiful patterns of microstructure that largely determine their macroscopic behaviour. These patterns result from phase transformations involving a change of shape of the underlying crystal lattice, together with the requirement that such changes in different parts of the crystal fit together geometrically. Similar considerations apply to plastic slip. The lecture will explain both successes in explaining such microstructure mathematically, and how resolving deep open questions of the calculus of variations could lead to a better understanding.

Nicolas Dirr

Cardiff University

Fluctuations of SPDES and interacting particle systems

Abstract. Stochastic Partial Differential Equations and interacting particle systems are both meant to model the same physical reality, i.e. a deterministic evolution subject to stochastic fluctuations. Here we focus on the Simple Symmetric Exclusion Process and its deterministic limit under diffusive scaling, the heat equation. If we take into account fluctuations, then the large deviations rate functionals and fluctuation fields of the particles coincide formally with those of a stochastic heat equation with nonlinear divergence form noise. This equation has no solution for white noise, i.e. for randomness on all scales. The particles however, have a natural cut off on small scales: The particle size. This indicates

that the SPDE is a useful approximation of the particle system at intermediate scales much larger than the particle size but smaller than the macroscopic scale. On these scales, regularized noise is a natural choice and for such noise a solution theory has been developed by B. Fehrman and B. Gess. We will sketch this concept of solution and explain the Large Deviations and Central Limit Theorem for them. Then we will present numerical evidence to show that the SPDE nonlinear noise term is a good approximation for the particle systems on intermediate scales. This is joint work with Benjamin Fehrman and Benjamin Gess.

Juan Davila

University of Bath

Overhanging solitary waves with vorticity

Abstract. We find solutions for an overdetermined elliptic problem, resulting in solitary waves that are non-graphical, within a fluid with constant vorticity. Although numerical evidence supports their existence, constructing these nearly singular solutions remains challenging using complex variables or bifurcation theory. Our method resembles the desingularisation process used for constant mean curvature surfaces. This work is a collaboration with Manuel del Pino, Monica Musso, and Miles Wheeler from the University of Bath.

Xue-Mei Li

Imperial College London

Non-linear wave equation with rough random Gaussian noise

Abstract. We consider a stochastic wave equations on the 2 dinensional torus with non-linear potential, with distributional valued noise. I will discuss an attempt with X. Ren, attempting to make sense of a solution theory by a renormalization procedure.

Florian Theil

University of Warwick

Convergence of particle systems to kinetic equations

Abstract. We consider the asymptotic behaviour of solutions of particle systems in the limit where the number of particles tends to infinity. In the case of short-range interactions and small densities it is expected that the solutions approximate solutions of the Boltzmann equation. Since the convergence is only known for short time, we consider regularisations of the particle evolution where the expected number of collisions per particle is bounded. Here we can establish convergence for large times and study finite-size corrections.

Swansea Summer School in Nonlinear PDEs

Poster sessions - order of short talks

Monday Session, 15.30-16.30	**Wednesday Session, 14.00-15.00**
• Marco Gallo • Raffaele Grande • Karol Hajduk • Jacopo Schino • Ye Zhang	• Damiano Greco • Sho Katayama • Jordan Marajh • Prachi Sahjwani

Posters abstracts

Matias Gomez Aedo

Imperial College London

A approach to estimate solutions of Hamilton-Jacobi equations based on data-driven methods using second-order information.

Abstract. From approximation theory, the problem of reconstructing a function from a dataset of images of points in an efficient way has been studied by Adcock and Sui. On the other hand, in the context of differential equations, it is sought to find solutions that cannot be explicitly determined using numerical analysis tools. We will present an approach to estimating solutions of Hamilton-Jacobi equations related to optimal control problems based on the work of Azmi, Kalise and Kunisch, where they reconstruct the function using polynomials in a way that avoids very high computational costs as the dimension grows. We will also compare, in a practical case, how the use of second-order information, which can be obtained from the structure of the optimal control problem, helps to achieve a good approximation with a smaller dataset.

Asma Benhamida

University Paris Est Creteil Val De Marne

Critical Sobolev Problem with p-Laplacian operator and with weight

Abstract. We consider the problem $-div(\alpha(x)|\nabla u|^{p-2}\nabla u) = \lambda|u|^{q-2}u + |u|^{p^\star-2}u, u > 0$ in Ω, $u = 0$ on $\partial\Omega,$ where Ω is a bounded domain in R^N, $N > p$, α a positive continuous potential on $\bar{\Omega}$, p^* the critical Sobolev exponent and $2 \leq p \leq q < p^*$ and λ is a real constant. We prove the existence of some positive solutions which depends, among others, on the behavior of the potential $\alpha(.)$ near its minima, the position of p^2 with respect to the dimension of the space and on the position of q with respect to some precise values.

James Coe

University of Edinburgh

Sharp quasi-invariance threshold for the cubic Szegő equation

Abstract. We consider the flow of Gaussian fields under the cubic Szegő equation, a toy model for dispersionless Hamiltonian dynamics. We show that for a class of Gaussian fields below a critical regularity, the induced measure at almost all times is singular with respect to the initial distribution, but above the critical regularity, the distribution is quasi-invariant under the flow. We introduce a new method to show singularity, first by exhibiting an instantaneous growth of Sobolev norm of the solution (at high frequencies), and then employing an abstract argument to show that such a property cannot hold with positive probability for uncountably many times.

Abdalaziz Elhaj Bakhit Elkhwad

Vilnius University

Discrete Sturm-Liouville Problem for Two-Dimensional Elliptic Equation with the Multiple Integral

Abstract. We will consider finite difference approximation for solving the nonlocal boundary value problem for two-dimensional Poisson equation in a square domain with Dirichlet and double integral boundary conditions. Motivated by some theoretical results we will study the spectrum structure of the corresponding difference eigenvalue problem.

Habib Fourti

King Faisal University

Multispike Solutions for a slightly subcritical elliptic problem with non-power nonlinearity

Abstract. "In this work, we are concerned with the following elliptic equation

$$\begin{cases} -\Delta u &= |u|^{4/(n-2)}u/[\ln(e+|u|)]^{\varepsilon} \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded open domain in R^n; $n \geq 4$ and $\varepsilon > 0$. By using a Ljapunov-Schmidt reduction method, Clapp, Pardo, Pistoia and Saldana proved in [1] that there exists a single-peak positive solution for small ε. This solution blows up at a non-degenerate critical point of the Robin function as ε goes to 0. Here we construct positive as well as changing sign solutions concentrated at several points inside the domain Ω at the same time. More precisely, we build solutions which blow up (positively or negatively) at distinct points which form a non-degenerate critical point of a function defined explicitly in terms of the Green function and its regular part. Our proof follows the finite reduction method introduced by Bahri, Li and Rey in [2]. (Joint work with M. Ben Ayed and R. Ghoudi)

References:

[1] M. Clapp, R. Pardo, A. Pistoia, A. Saldana, A solution to a slightly subcritical elliptic problem with non-power nonlinearity, Journal of Diff. Eq. Vol 275, (2021).

[2] A. Bahri, YY. Li and O. Rey, On a variational problem with lack of compactness: The topological effect of the critical points at infinity, Calculus of Variation and Partial Diff. Equa. 3 (1995), 67-94.

Marco Gallo
Università Cattolica del Sacro Cuore, Brescia

A doubly nonlocal interaction in the asymptotic decay of solutions

Abstract. In this poster session we will discuss the asymptotic decay of solutions for equations of the type

$$(-\Delta)^s u + u = \left(\frac{1}{|x|^{N-\alpha}} * u^{p+1}\right) u^p \quad \text{in } \mathbb{R}^N$$

where $(-\Delta)^s$ denotes the fractional Laplacian, $s \in (0,1)$, and $\frac{1}{|x|^{N-\alpha}} *$ denotes the convolution with the Riesz potential, $\alpha \in (0,N)$. Both the operator and the nonlinearity are thus nonlocal. Here the power p varies in the range $[\frac{\alpha}{N}, \frac{\alpha+2s}{N-2s}]$. In the local case $s=1$ it has been shown that the asymptotic decay of $u(x)$ as $|x| \to +\infty$ hardly changes when p is smaller or greater than 1: in this doubly nonlocal case, instead, we will see that the interaction of the two nonlocalities generates a different threshold, $p_{s,\alpha} := \frac{\alpha+2s}{N+2s}$, which verifies $p_{1,\alpha} \neq 1$.

Raffaele Grande
Czech Academy of Sciences

Horizontal mean curvature flow and asymptotic rescaling in the Heisenberg group

Abstract. We derive the geometric evolution by (horizontal) mean curvature flow of a hypersurface embedded in the 1-dimensional Heisenberg group from a formal asymptotic expansion of a nonlocal mean-field equation. This result is obtained by using the anisotropic rescaling induced by the Carnot group structure of the Heisenberg group. This is motivated by the aim of connecting mechanisms at a microscopic (i.e. cellular) level to macroscopic models of image processing through a multi-scale approach. This is a joint work with G.Citti (Università di Bologna), N.Dirr (Cardiff University) and F.Dragoni (Cardiff University)

Damiano Greco
Swansea University

On some Thomas–Fermi type variational problems: an attractive and a repulsive case

Abstract. We consider two Thomas-Fermi type variational problems. The first one concerns studying existence and qualitative properties of the minimizers for a Thomas–Fermi type energy functional with non local repulsion involving a convolution with the Riesz kernel and interaction with an external potential. Under mild assumptions on the latter, we establish uniqueness and qualitative properties such as positivity, regularity and decay at infinity of the global minimizer. The second problem concerns the study of optimisers for a Gagliardo–Nirenberg type inequality. Such problem is well understood in connection with Keller-Segel models and appears in the study of Thomas-Fermi limit regimes for the Choquard equations with local repulsion. We establish optimal ranges of parameters for the validity of the inequality, discuss the existence and qualitative properties of the optimisers.

Matthias Grutzner

Humboldt-Universität Berlin

Using complex analysis in 2D Stokes flow

Abstract. Researching the dynamics of microswimmers in Stokes flow has a variety of applications throughout multiple fields. In this work an approach to the low Reynolds number case of the Navier Stokes equation in two dimensions using complex analysis is studied. Specifically, a circular microswimmer modelled by the first two terms in the Blake squirmer model is considered in wedge boundary conditions and equivalent singularities describing the microswimmer are obtained. By transforming the boundary conditions, implicit analytical solutions for the flow field for an arbitrary wedge angle are obtained.

Karol Hajduk

Institute of Mathematics, Polish Academy of Sciences

Incompressible convective Brinkman-Forchheimer equations in a thin domain

Abstract. The flow of a fluid through a porous medium is clasically described by Darcy's law. However, it typically applies for sufficiently slow viscous flows, e.g. for flows with small Reynolds number (laminar flows). When the flow is non-Darcian (e.g. turbulent flows), various modifications of Darcy's law are used to describe it. In the poster we will present one of such models, namely the convective Brinkman-Forchheimer equations (CBF). From the mathematical perspective, this model can be seen also as the Navier-Stokes equations with damping term $|u|^{r-1}u$, called the absorption term (or the Forchheimer term). We will give an overwiev of some available results for this model. We will also discuss stationary CBF flow through a 2D thin channel. We are interested in existence of solutions of such flows and asymptotic analysis of the model. This is an ongoing project in collaboration with Marko Radulović (University of Zagreb).

Kazuya Hirose

Hokkaido University, Japan

Lower gradient estimates for viscosity solutions of Hamilton-Jacobi equation depending on the unknown function

Abstract. In this poster, we derive the lower gradient estimate for viscosity solutions of the Hamilton--Jacobi equation with the convex Hamiltonian depending on the unknown function. We obtain gradient estimates in two different ways.

Sho Katayama

University of Tokyo

Supercritical Lane-Emden equation with a forcing term

Abstract. This study concerns the structure of positive solutions to the elliptic problem for the Lane--Emden equation on the whole space with a positive forcing term. Under a suitable assumption on the forcing term, we give a complete classification of the unique existence/multiple existence/nonexistence of positive solutions with respect to the size of the forcing term and the exponent of nonlinearity.

Jordan Marajh
Queen Mary University London

Controlled regularity at future null infinity from past asymptotic initial data: wave equation

Abstract. In this work, we present the results obtained from a problem inspired by conformal scattering in a neighbourhood of spatial and null infinity for the conformal wave equation on the Minkowski spacetime. Moreover, we show how one can prescribe characteristic initial data on past null infinity to guarantee a certain regularity on future null infinity. Some key techniques used in this analysis to construct the estimates are Friedrich's cylinder at spatial infinity, a Gr¨onwall argument which is non-degenerate at the critical sets and some general theory of symmetric hyperbolic systems. This is work to appear soon on arXiv in collaboration with Grigalius Taujanskas and Juan A. Valiente Kroon (Marajh et al. 2024).

Prachi Sahjwani
Cardiff University

Stability of Alexandrov-Fenchel inequalities in hyperbolic space

Abstract. In this talk, I will discuss the stability of Alexandrov-Fenchel inequalities in hyperbolic space. I will give a brief overview of the inequalities and their stability problems. To understand what I mean by stability, I will first discuss it for Isoperimetric inequality, which is a special case of Alexandrov-Fenchel inequalities. This is joint work with Prof. Dr. Julian Scheuer.

Jacopo Schino
University of Warsaw

About solutions with prescribed norm to a Schrödinger equation in a mixed regime

Abstract. Schrödinger-type equations model a lot of natural phenomena and their solutions have interesting and important properties. This gives rise to the search for normalised solutions, i.e., when the L^2-norm is prescribed. Here, I will present results in a mixed regime, that is, when the non-linear term behaves differently at the origin and at infinity.

Billy Sumners
Heriot-Watt University

Triplets and Compatibility in Martensitic Phase Transformations

Abstract. Recently, a nickel-titanium-based shape-memory alloy which displays good reversibility under thermal cycling was developed by the group of Tomonari Inamura. We attempt to continue the work started by themselves and Francesco Della Porta of modelling the microstructure that appears in the martensitic phase. The key factor distinguishing this alloy from ones with poorer reversibility are that its variants are highly compatible, represented by the satisfaction of a property of matrices known as a triplet condition. Also of interest are the lack of identifiable remnants in the martensite phase of the phase transition from the austenite phase. In light of this, we study the rigidity properties of Sobolev mappings whose gradient lives in a set of matrices satisfying a triplet condition. Furthermore, in order to model the phase transition, we discuss when such mappings are also allowed to have their gradient live in an additional set representing the austenite phase, and determine in particular at least some of the interfaces between the austenite and martensite phases.

Ye Zhang

Okinawa Institute of Science and Technology Graduate University

Horizontal semiconcavity for the square of Carnot-Carath\'eodory distance on Carnot groups and applications to Hamilton-Jacobi equations

Abstract. In our recent work, we establish the horizontal semiconcavity (h-semiconcavity) of the square of Carnot-Caratheodory distance from the origin, in a class of Carnot groups. Via Hopf-Lax formula, we apply this property to show h-semiconcavity for the solutions of a class of non-coercive evolutive Hamilton-Jacobi equations.

Erbol Zhanpeisov

Okinawa Institute of Science and Technology

Liouville-type theorem for fully nonlinear elliptic and parabolic equations with boundary degeneracy

Abstract. We study a general class of fully nonlinear boundary-degenerate elliptic or parabolic equations that admit a trivial solution. Although no boundary conditions are posed together with the equations, we show that the operator degeneracy actually generates an implicit boundary condition. Under appropriate assumptions on the degeneracy rate and regularity of the operator, we then prove that there exist no bounded solutions other than the trivial one. Our method is based on the uniqueness arguments for viscosity solutions of state constraint problems for Hamilton-Jacobi equations.

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