POSITIVITY PRINCIPLES AND DECAY OF SOLUTIONS IN SEMILINEAR ELLIPTIC PROBLEMS

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Abstract. This is an outline of the lectures given at *Online Mini-courses in Mathematical Analysis 2020*, organised by the University of Padova 14–17 September 2020. Please, email me if you spot any typos or mistakes.

Contents

Consider the linear equation

$$-\Delta u + Vu = f \quad \text{in } \Omega, \tag{$*$}$$

where Ω is a domain (open connected set) in $\mathbb{R}^N$ with $N \geq 2$ and V is a potential and $0 \leq f \in L^1_{loc}(\Omega)$ is a nonnegative right hand side. We assume that $V = V^+ - V^-$ and

$$V^+ \in L^\infty_{loc}(\Omega), \quad V^- \in L^1_{loc}(\Omega).$$

The natural quadratic form associated to the Schrödinger operator $-\Delta + V$ is given by

$$\mathcal{E}_V(\varphi) := \int_\Omega |\nabla\varphi|^2\,dx + \int_\Omega V\varphi^2\,dx \qquad (\varphi \in H^1_c(\Omega) \cap L^\infty_c(\Omega)),$$

where the subscript c denotes the *compact support*. We say that $\mathcal{E}_V$ is non-negative if

$$\mathcal{E}_V(\varphi) \geq 0, \qquad \forall\varphi \in H^1_c(\Omega) \cap L^\infty_c(\Omega)).$$

We are going to study the relationship between the existence and some properties of positive supersolutions to $(*)$ and non-negativity of the quadratic form $\mathcal{E}_V$. Such a relationship is commonly referred to as Agmon–Allegretto–Piepenbrink's (AAP) Principle or "Ground State transformation". Our exposition is inspired and largely based on [3, 4, 6, 7, 14] but is adapted with applications to semilinear equations in mind, as developed in [11–13].

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1. The AAP positivity principle

A *weak* supersolution to $(*)$ is a function $u \in H^1_{loc}(\Omega) \cap L^1_{loc}(\Omega, V dx)$ such that

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx + \int_\Omega V u \varphi \, dx \geq \int_\Omega f \varphi \, dx, \qquad \forall \, 0 \leq \varphi \in H^1_c(\Omega) \cap L^\infty_c(\Omega). \tag{1.1}$$

The notions of a weak sub-solution and weak solution are defined similarly by replacing "$\geq$" with "$\leq$" and "$=$" respectively. Note that if $u \gneq 0$ is a weak supersolution to $(*)$ with $f \geq 0$ then $u > 0$ in Ω, in the sense that $u^{-1} \in L^\infty_{loc}(\Omega)$. This follows from the weak Harnack inequality which is ensured by the assumption $V^+ \in L^\infty_{loc}(\Omega)$.

In what follows we often omit the word *weak* and use simply solution, super and subsolution. We say that u is a solution to $-\Delta + V$ if u is a supersolution for $(*)$ with $f = 0$. Similarly for sub and supersolutions.

When $V \geq 0$, the quadratic form $\mathcal{E}_V$ is non-negative, and any positive constant is a supersolution. Thus, the interesting case to consider is when V is negative or changes sign. The fundamental relation between the existence of positive supersolutions to $(*)$ and non-negativity of the quadratic form $\mathcal{E}_V$ is described in the following result, which was originally proved by W. Allegretto [1] and J. Piepenbrink [2] in 1974, and later became the foundation of S. Agmon's Criticality Theory [3].

Theorem 1.1 (AAP Positivity Principle)**.** *Assume that $(*)$ admits a weak positive (super)solution $u_* > 0$. Then*

$$\mathcal{E}_V(\varphi) \, (\geq) = \int_\Omega \left| \nabla \left(\frac{\varphi}{u_*} \right) \right|^2 u_*^2 dx + \int_\Omega \varphi^2 \, \frac{f}{u_*} \, dx, \qquad \forall \, 0 \leq \varphi \in H^1_c(\Omega) \cap L^\infty_c(\Omega). \tag{1.2}$$

Proof. Let $0 \leq \varphi \in H^1_c(\Omega) \cap L^\infty_c(\Omega)$. Then $\psi := \frac{\varphi^2}{u_*} \in H^1_c(\Omega) \cap L^\infty_c(\Omega)$. Testing $(*)$ against ψ we arrive at

$$2 \int_\Omega \varphi \, \nabla \varphi \, \frac{\nabla u_*}{u_*} \, dx \, (\geq) = \int_\Omega \varphi^2 \frac{|\nabla u_*|^2}{u_*^2} \, dx - \int_\Omega V \varphi^2 \, dx + \int_\Omega \frac{f}{u_*} \varphi^2 \, dx.$$

Direct computation yields

$$\begin{aligned} \int_\Omega \left(|\nabla \varphi|^2 + V \varphi^2 \right) dx - \int_\Omega \left| \nabla \left(\frac{\varphi}{u_*} \right) \right|^2 u_*^2 dx \\ = \int_\Omega \left(|\nabla \varphi|^2 + V \varphi^2 \right) dx - \int_\Omega \left(\frac{|\nabla \varphi|^2}{u_*^2} - 2 \varphi \, \nabla \varphi \, \frac{\nabla u_*}{u_*^3} + \varphi^2 \frac{|\nabla u_*|^2}{u_*^4} \right) u_*^2 dx \\ = 2 \int_\Omega \varphi \, \nabla \varphi \, \frac{\nabla u_*}{u_*} \, dx - \int_\Omega \varphi^2 \frac{|\nabla u_*|^2}{u_*^2} \, dx + \int_\Omega V \varphi^2 \, dx \, (\geq) = \int_\Omega \varphi^2 \, \frac{f}{u_*} \, dx. \end{aligned}$$

This proves (1.2) on $H^1_c(\Omega) \cap L^\infty_c(\Omega)$. □

Remark 1.2. It is simple to see that the "u_*-ground state transformed" quadratic form

$$\mathcal{E}_{u_*}(\varphi) = \int_\Omega \left| \nabla \left(\frac{\varphi}{u_*} \right) \right|^2 u_*^2 dx$$

is generated by the nonnegative selfadjoint operator

$$-\Delta + 2\nabla \log(u_*)\nabla$$

in $L^2(\Omega, u_*^2\, dx)$.

Remark 1.3. (a) We do not assume any boundary conditions on the reference supersolution u_* in Lemma 1.1. We also do not require any boundary regularity of u_*. All we need is that $u_* \in H^1_{loc}(\Omega)$ is a positive weak supersolution of $-\Delta + V$. In many examples u_* is positive or even singular on parts of the boundary (Example 1.3), but u_* also may be zero on the boundary (Examples 1.6 and 1.8).

(b) We do not assume any boundary regularity of the potential V, which could be 'very' singular on $\partial\Omega$ (Examples 1.3, 1.6).

(c) Domain Ω need not be smooth (e.g. $\mathbb{R}^N \setminus \{0\}$ in Example 1.3).

Example 1.4 (Hardy inequality in $\mathbb{R}^N$)**.** Consider

$$-\Delta u - \frac{C_H}{|x|^2} u = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{1.3}$$

where $N \geq 3$ and $C_H := (N-2)^2/4$. Clearly,

$$u_*(|x|) = |x|^{\frac{2-N}{2}}$$

is a positive weak solution of (1.3). Hence, by Lemma 1.1 we obtain Hardy's inequality

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2 \geq C_H \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}, \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}). \tag{1.4}$$

Remark 1.5. We had to remove the origin in (1.3) because $|x|^{\frac{2-N}{2}} \notin H^1_{loc}(\mathbb{R}^N)$ – the singularity at the origin is too strong. To prove (1.3) in $\mathbb{R}^N$, first prove

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2 \geq c \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}, \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^N). \tag{1.5}$$

for any $c \in (0, C_H)$. To do this, use $u_* = |x|^{\alpha_+}$, where α_+ is the biggest root of the scalar equation $-\alpha(\alpha + N - 2) = c$. Since $\alpha_+ > -\frac{N-2}{2}$, we have $|x|^{\alpha_+} \in H^1_{loc}(\mathbb{R}^N)$ and the AAP principle is valid. Since (1.5) is valid for any $c < C_H$, it is also valid for $c = C_H$.

Example 1.6 (Hardy inequality in $\mathbb{R}^2$)**.** Consider

$$-\Delta u - \frac{1/4}{|x|^2 \Big(\log \frac{|x|}{\rho}\Big)^2} u = 0 \quad \text{in } \mathbb{R}^2 \setminus B_\rho(0). \tag{1.6}$$

where $\rho > 0$. Clearly,

$$u_*(|x|) = \Big(\log \frac{|x|}{\rho}\Big)^{1/2}$$

is a positive weak solution of (1.6). Hence, by Lemma 1.1 we obtain a Hardy type inequality

$$\int_{\mathbb{R}^2}|\nabla\varphi|^2 \geq \frac{1}{4}\int_{\mathbb{R}^2}\frac{\varphi^2}{|x|^2\Big(\log\frac{|x|}{\rho}\Big)^2}, \qquad \forall\varphi\in C_c^\infty(\mathbb{R}^2\setminus B_\rho(0)).$$

Exercise 1.7 (Improved Hardy inequality)**.** For $N\geq 2$ prove the following inequality

$$\int_{\mathbb{R}^N}|\nabla\varphi|^2 \geq C_H\int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^2}+\frac{1}{4}\int_{\mathbb{R}^2}\frac{\varphi^2}{|x|^2(\log|x|)^2}, \qquad \forall\varphi\in C_c^\infty(\mathbb{R}^N\setminus B_1).$$

Hint: Take $u_*(|x|)=|x|^{-\frac{N-2}{2}}(\log|x|)^{1/2}$

Example 1.8 (Principal Dirichlet eigenvalue)**.** Let $\Omega\subset\mathbb{R}^N$ be a bounded domain, $\lambda_1>0$ be the principal Dirichlet eigenvalue of $-\Delta$ on Ω and $u_*:=\phi_1>0$ be the corresponding principal eigenfunction. By Lemma 1.1 we obtain the inequality

$$\int_\Omega|\nabla\varphi|^2\geq\lambda_1\int_\Omega\varphi^2, \qquad \forall\varphi\in C_c^\infty(\Omega). \tag{1.7}$$

Example 1.9 (Torsional Hardy inequality)**.** Let $\Omega\subset\mathbb{R}^N$ be a bounded domain and $\psi_\Omega>0$ be a torsion function of Ω, that is the unique solution of

$$-\Delta\psi=1\quad\text{in }\Omega,\qquad u=0\quad\text{on }\partial\Omega.$$

Then we obtain "torsional Hardy inequality"

$$\int_\Omega|\nabla\varphi|^2\geq\int_\Omega\frac{\varphi^2}{\psi_\Omega}\qquad\forall\varphi\in C_c^\infty(\Omega).$$

Now take $\varphi=\varphi_1$, the principal Dirichlet eigenvalue of Ω such that $\|\varphi_1\|_2=1$. Then

$$\lambda_1=\int_\Omega|\nabla\varphi|^2\geq\int_\Omega\frac{\varphi_1^2}{\psi_\Omega}\geq\frac{\|\varphi_1\|_2^2}{\|\psi_\Omega\|_\infty}=\frac{1}{\|\psi_\Omega\|_\infty}. \tag{1.8}$$

Hence we deduce a lower bound

$$\|\psi_\Omega\|_\infty\geq\frac{1}{\lambda_1}.$$

Exercise 1.10 (Barta's inequality)**.** Prove the following inequality

$$\inf_{x\in\Omega}\frac{(-\Delta-V)\phi}{\phi}\leq\inf_{0\neq u\in C_c^\infty(\Omega)}\frac{\mathcal{E}_V(u,u)}{\|u\|_{L^2}^2}. \tag{1.9}$$

Hint: Similar to (1.8).

The following straightforward corollary of Lemma 1.1 is crucial in the analysis of nonexistence of positive solutions to semilinear equations.

Corollary 1.11 (Nonexistence principle)**.** *Assume there exists $\varphi\in C_c^\infty(\Omega)$ such that $\mathcal{E}_V(\varphi)<0$. Then $-\Delta+V$ has no positive weak supersolutions in Ω.*

2. The energy space

2.1. **The λ–property and the energy space.** Following [4], we say that the form $\mathcal{E}_V$ satisfies the *λ–property* if there exists a function $0 < \lambda \in L^1_{loc}(G)$ such that $\lambda^{-1} \in L^\infty_{loc}(G)$ and

$$\mathcal{E}_V(u) \geq \int_\Omega u^2\, \lambda(x)\, dx, \qquad \forall\, u \in C_c^\infty(\Omega). \tag{2.1}$$

It is elementary to see that if $\mathcal{E}_V$ satisfies the *λ–property* then $\|\cdot\|_V := \sqrt{\mathcal{E}_V(\cdot)}$ is a norm on $C_c^\infty(\Omega)$. Denote by $D^1_V(\Omega)$ the family of measurable a.e. finite functions $u : \Omega \to \mathbb{R}$ such that there exists an $\|\cdot\|_V$–Cauchy sequence $(u_n) \subset C_c^\infty(\Omega)$ that converges to u a.e. in Ω. This sequence (u_n) is called an *approximating sequence* for $u \in D^1_V(\Omega)$. Then the limit

$$\mathcal{E}_V(u) := \lim_{n\to\infty} \mathcal{E}_V(u_n)$$

exists and is independent of the choice of the approximating sequence. Thus $\mathcal{E}_V$ is extended uniquely to a nonnegative definite quadratic form on $D^1_V(\Omega)$. The family $D^1_V(\Omega)$ is called the *energy space* of $\mathcal{E}_V$. Moreover, (2.1) implies that (u_n) converges to u in $L^2(\Omega, \lambda\, dx)$ and hence $D^1_V(\Omega)$ is continuously embedded into $L^2(\Omega, \lambda\, dx)$. This is summarized in the following theorem.

Theorem 2.1. *The energy space $D^1_V(\Omega)$ is a Hilbert space with the inner product*

$$\langle u, v\rangle_V := \int_\Omega \nabla u \cdot \nabla v\, dx + \int_\Omega V uv\, dx$$

Moreover, $D^1_V(\Omega)$ is continuously embedded into $L^2(\Omega, \lambda\, dx)$.

Remark 2.2. If $\mathcal{E}_V$ satisfies the λ–property in Ω and Ω' is a bounded subdomain such that $\bar{\Omega}' \subset \Omega$, then $\mathcal{E}_V$ satisfies the λ–property in Ω' with $\lambda^{-1} \in L^\infty(\Omega')$. Hence $D^1_V(\Omega') = H^1_0(\Omega')$.

Denote by $D^{-1}_V(\Omega)$ the dual space of all linear continuous functionals on $D^1_V(\Omega)$. The following lemma is a standard consequence of the Riesz Representation Theorem.

Lemma 2.3. *Assume that $\mathcal{E}_V$ satisfies the λ–property. Let $l \in D^{-1}_V(\Omega)$. Then there exists a unique $u_* \in D^1_V(\Omega)$ such that*

$$\langle u_*, \varphi\rangle = l(\varphi), \qquad \forall\, \varphi \in D^1_V(\Omega). \tag{2.2}$$

Since $D^1_V(\Omega) \subset L^2(\Omega, \lambda\, dx)$, by duality we conclude that $L^2(\Omega, \lambda^{-1} dx) \subset D^{-1}_V(\Omega)$, in the sense that if $f \in L^2(\Omega, \lambda^{-1} dx)$ then

$$l_f(u) := \int_\Omega f\varphi\, dx \in D^{-1}_V(\Omega).$$

Thus Lemma 2.3 implies that for any $f \in L^2(\Omega, \lambda^{-1} dx)$ the equation

$$-\Delta u + Vu = f \quad \text{in } D^1_V(\Omega) \tag{2.3}$$

has a unique solution $u_f \in D^1_V(\Omega)$.

Exercise 2.4 (The space $H^1(\mathbb{R}^N)$)**.** Show that $-\Delta + 1$ satisfies the λ–property in $\mathbb{R}^N$ and that the corresponding energy space $D^1_1(\mathbb{R}^N)$ coincides with the standard Sobolev space $H^1(\mathbb{R}^N)$.

Example 2.5 (The space $D^1_0(\mathbb{R}^N)$ for $N \geq 3$)**.** Consider $-\Delta$ on $\mathbb{R}^N$ with $N \geq 3$. Take

$$u_*(|x|) := (1 + |x|^2)^{-\frac{N-2}{2}},$$

which is known as a Talenti function[1]. Then

$$-\Delta u^* = N(N-2)(1 + |x|^2)^{-\frac{N+2}{2}}$$

and

$$\lambda(x) = \frac{-\Delta u_*(x)}{u_*(x)} = N(N-2)(1 + |x|^2)^{-4}.$$

Hence the space $D^1_0(\mathbb{R}^N)$, a completion of $C^\infty_c(\mathbb{R}^N)$ with respect to the homogenous gradient norm $\|\nabla u\|_{L^2(\mathbb{R}^N)}$, is a well–defined Hilbert space and moreover,

$$D^1_0(\mathbb{R}^N) \subset L^2(\mathbb{R}^N, (1 + |x|^2)^{-4} dx).$$

It is easy to see that $D^1_0(\mathbb{R}^N) \not\subset L^2(\mathbb{R}^N)$ and hence $D^1_0(\mathbb{R}^N) \neq H^1(\mathbb{R}^N)$. To see this when $N = 3, 4$, check that $u_* \in D^1(\mathbb{R}^N)$ but $u_* \not\in L^2(\mathbb{R}^N)$ when $N = 3, 4$. In fact, $D^1_0(\mathbb{R}^N) \subset L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ – this follows from the Sobolev inequality.[2]

Example 2.6 (The space $D^1(\mathbb{R}^2)$ is not well-defined)**.** According to the Liouville's theorem, every positive super solution on $\mathbb{R}^2$ is constant. This means that the Laplacian $-\Delta$ does not satisfy λ–property on $\mathbb{R}^2$. Hence we can not apply Theorem 2.1 to construct the energy space $D^1_0(\mathbb{R}^2)$. In fact, such a space is not well defined as the space of functions, see [14] for details.

Example 2.7 (The space $D^1(\mathbb{R}^2 \setminus \bar{B}_\rho)$ in $\mathbb{R}^2$)**.** Consider $-\Delta$ on $\Omega = \mathbb{R}^2 \setminus \bar{B}_\rho$, for a $\rho > 0$. Take

$$u_*(|x|) = \Big(\log \frac{|x|}{\rho}\Big)^{1/2}.$$

As in the Example 1.6, we conclude that $-\Delta$ in Ω satisfies the λ–property with

$$\lambda(x) = \frac{1/4}{|x|^2 \Big(\log \frac{|x|}{\rho}\Big)^2}.$$

Hence the space $D^1_0(\mathbb{R}^2 \setminus \bar{B}_\rho)$, the completion of $C^\infty_c(\mathbb{R}^N \setminus \bar{B}_\rho)$ with respect to the homogenous gradient norm $\|\nabla u\|_{L^2(\mathbb{R}^2)}$ is a well–defined Hilbert space.

Exercise 2.8. Use the improved Hardy Inequality 1.7 to show that for $N \geq 3$ the critical Hardy operator

$$-\Delta - \frac{C_H}{|x|^2}, \qquad C_H := \big(\tfrac{N-2}{2}\big)^2,$$

[1] Up to a rescaling, it is a minimizer of the Sobolev inequality

[2] The space $D^1_0(\mathbb{R}^N)$ from this example is usually denoted as $D^1(\mathbb{R}^N)$ or $\dot{H}^1(\mathbb{R}^N)$.

satisfies the λ–property in $\mathbb{R}^N \setminus \bar{B}_1$ and define the energy space $D^1_{-\frac{C_H}{|x|^2}}(\mathbb{R}^N \setminus \bar{B}_1)$.

2.2. **Maximum and comparison principles.** If $\mathcal{E}_V$ satisfies the λ–property then $-\Delta+V$ satisfies weak maximum and comparison principles. In order to prove that we need to know that $D^1_V(\Omega)$ is invariant under the standard truncations.

Lemma 2.9. *If $u \in D^1_V(\Omega)$ then $u^+, u^- \in D^1_V(\Omega)$ and*

$$\mathcal{E}_V(u^{\pm}) \le \mathcal{E}(u), \qquad \forall u \in D^1_V(\Omega). \tag{2.4}$$

If $u, v \in D^1_V(\Omega)$ then $u \vee v, u \wedge v \in D^1_V(\Omega)$.

Proof. See [13, Lemma A.1]. □

Remark 2.10. We do not claim that $u \in D^1_V(\Omega)$ implies $u \wedge 1 \in D^1_V(\Omega)$.

Lemma 2.11. *Assume that $\mathcal{E}_V$ satisfies the λ–property. Let $w \in H^1_{loc}(\Omega) \cap L^1(\Omega, V dx)$ be a super-solution to $(*)$ such that $w^- \in D^1_V(\Omega)$. Then $w \ge 0$ in Ω.*

Proof. Let $(\varphi_n) \subset C^\infty_c(\Omega)$ be an approximating sequence for $w^- \in D^1_V(\Omega)$. Set

$$w_n := \varphi_n^+ \wedge w^-.$$

Hence $0 \le w_n \in D^1_V(\Omega)$, by Lemma 2.9. Note that $w_n = w^- + (\varphi_n^+ - w^-)^-$. Therefore

$$\mathcal{E}_V(w^- - w_n) = \mathcal{E}_V((\varphi_n^+ - w^-)^-) \le \mathcal{E}_V(\varphi_n - w^-) \to 0.$$

Thus (w_n) is a nonnegative approximating sequence for w^-. Since $w^+ \wedge w_n = 0$, we obtain

$$0 \le \langle w, w_n \rangle_V = -\langle w^-, w_n \rangle_V \to -\mathcal{E}_V(w^-) \le 0.$$

We conclude that $w^- = 0$. □

The following comparison principle is a straightforward consequence of Lemma 2.11.

Corollary 2.12. *Assume that $\mathcal{E}_V$ satisfies the λ–property. Let $w \in H^1_{loc}(\Omega) \cap L^1(\Omega, V dx)$ be a super-solution to $(*)$ and $v \in H^1_{loc}(\Omega) \cap L^1(\Omega, V dx)$ be a sub-solution to $(*)$ such that $(w - v)^- \in D^1_V(\Omega)$. Then $w \ge v$ in Ω.*

Remark 2.13. In particular, Lemmas 2.3 and 2.11 imply that if $\mathcal{E}_V$ satisfies the λ–property then equation $(*)$ has a "rich" cone of positive super-solutions. Indeed, if $0 \lneqq f \in L^2(\Omega, \lambda^{-1} dx)$ and $u_f \in D^1_V(\Omega)$ is the solution to (2.3) constructed in Lemma 2.3 then $u_f > 0$ in Ω by Lemma 2.11 and hence u_f is a weak positive supersolution for $-\Delta + V$. The situation is different if $\mathcal{E}_V$ is nonnegative but *does not* satisfy the λ–property.

Example 2.14 (Liouville's Theorem on the plane)**.** Consider $-\Delta$ on $\mathbb{R}^2$. Obviously, in this case the form $\mathcal{E}_0$ is nonnegative. Yet, according to the classical Liouville's theorem the only positive superharmonic for $-\Delta$ on $\mathbb{R}^2$ is constant.

Example 2.15 (Principal Dirichlet eigenvalue)**.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\lambda_1 > 0$ be the principal Dirichlet eigenvalue of $-\Delta$ on Ω as in Example 1.8. By (1.7), the form $\mathcal{E}_{\lambda_1}$ is nonnegative. Yet, the the corresponding principal eigenfunction $\phi_1 > 0$ is (up to a constant) the only positive supersolution to $-\Delta - \lambda_1$ in Ω (see Corollary 1.11).

The above examples clarify the following classification:

- $-\Delta + V$ is *subcritical* if $\mathcal{E}_V$ satisfies the λ–property. In this case $-\Delta + V$ has a "rich" cone of positive supersolutions (see Remark 2.13).
- $-\Delta + V$ is *critical* if $\mathcal{E}_V$ is nonnegative but does not satisfy the λ–property. In this case $-\Delta + V$ has exactly one (up to a scalar) positive supersolution, which is actually a solution (see [3, Theorem 5.2]).
- $-\Delta+V$ is *supercritical* if $\mathcal{E}_V$ is not nonnegative. In this case $-\Delta+V$ has no positive supersolutions (see Corollary 1.11).

Further study of $-\Delta+V$ from the point of view of this classification is known as *criticality theory*, see [3,6,7]. In this lectures we are interested in one particular aspect only – we want to characterise the "size" of the cone of positive super-solutions of subcritical operators in terms of the admissible decay (or growth) of supersolutions "at infinity".

3. A Phragmén-Lindelöf principle

In this section we assume that Ω is an exterior domain in $\mathbb{R}^N$ such that $\mathbb{R}^N \setminus \bar{B}_1 \subset \Omega$ and $\partial\Omega \neq \emptyset$. We also always assume that $-\Delta + V$ satisfies the λ–property in Ω. Note that then $-\Delta + V$ also satisfies the λ–property in any subdomain $\Omega' \subset \Omega$, see Remark 2.2. We are going to study admissible decay at infinity, i.e. as $|x| \to \infty$, of all positive supersolutions to $-\Delta + V$ in Ω.

Definition 3.1. We say $u > 0$ is a *small* (sub) solution at infinity for $-\Delta + V$ if u is a (sub) solution for $-\Delta + V$ in B_R^c for some $R \geq 1$ and there exists a supersolution $w_* > 0$ for $-\Delta + V$ in Ω, such that

$$\lim_{|x|\to\infty} \frac{u}{w_*} = 0. \tag{3.1}$$

We say $U > 0$ is a *large* (sub) solution at infinity for $-\Delta + V$ in Ω if U is (sub) solution for $-\Delta+V$ in B_R^c for some $R \geq 1$ and U is *not* a small (sub) solution, i.e. for any supersolution $w > 0$ for $-\Delta + V$ in Ω,

$$\limsup_{|x|\to\infty} \frac{U}{w} > 0. \tag{3.2}$$

Exercise 3.2. Prove that if $u > 0$ is a small and $U > 0$ a large solution at infinity for $-\Delta + V$ in Ω then

$$\limsup_{|x|\to\infty} \frac{U}{u} = +\infty. \tag{3.3}$$

Intuitively, a small solution at infinity is "smaller" than one of the supersolutions. Then a *large* solution at infinity is *not* "smaller" than any of the supersolutions, i.e. in some sense it dominates at infinity all positive supersolutions. [3] An essential observation is that if a solution is smaller than *one* of the positive supersolutions in the sense of (3.1), then up to a constant it is dominated by *every* positive supersolution.

[3] Similarly, we could define small and large *sub*solutions but we will omit this for simplicity.

Lemma 3.3 (Small solution lemma)**.** *If $u > 0$ is a small solution at infinity for $-\Delta + V$ in Ω then for any supersolution $w > 0$ for $-\Delta + V$ in Ω there exists $c > 0$ such that*

$$u(x) \leq cw(x) \qquad (|x| \geq 1). \tag{3.4}$$

Proof. Choose $c > 0$ such that $cu \leq w$ as $|x| = 1$. For small $\varepsilon > 0$ consider the barrier functions

$$u_\varepsilon := cu - \varepsilon w_*.$$

In view of (3.1), there exists $\rho_\varepsilon > 1$ such that

$$u_\varepsilon \leq 0 \qquad (|x| \geq \rho_\varepsilon)$$

and moreover, $\rho_\varepsilon \to \infty$ as $\varepsilon \to 0$.

Set $\Omega_\varepsilon := \{x \in \mathbb{R}^N : 1 < |x| < \rho_\varepsilon\}$. Since $w > 0$, we conclude that

$$u_\varepsilon \leq w \quad \text{on } \partial\Omega_\varepsilon$$

and hence $(w - u_\varepsilon)^- \in H^1_0(\Omega_\varepsilon)$, see Remark 2.2. By the comparison principle of Corollary 2.12 we conclude that

$$u_\varepsilon \leq w \quad \text{in } \Omega_\varepsilon.$$

Then the assertion follows, since $\varepsilon > 0$ could be taken arbitrary small and taking into account that $\rho_\varepsilon \to \infty$ as $\varepsilon \to 0$. □

Proposition 3.4 (Phragmén-Lindelöf principle for supersolutions)**.** *Let $u > 0$ be a small and $U > 0$ a large solution at infinity for $-\Delta + V$ in Ω. Then for any supersolution $w > 0$ for $-\Delta + V$ in Ω:*

$$\liminf_{|x|\to\infty} \frac{w}{u} > 0, \qquad \liminf_{|x|\to\infty} \frac{w}{U} < \infty.$$

Proof. The first lim inf is simply the reformulation of (3.4), the second lim inf is the inversion of (3.2). □

Lemma 3.5 (Large solution lemma)**.** *Assume that $U > 0$ satisfies*

$$-\Delta U + VU = 0 \quad (|x| > 1) \quad \textit{and} \quad U = 0 \quad (|x| = 1). \tag{3.5}$$

Then U is a large solution for $-\Delta + V$.

Proof. Assume that U is not a large solution for $-\Delta + V$, i.e. U is a small solution. Then there exists a supersolution $w_* > 0$ for $-\Delta + V$ such that (3.1) holds.

For small $\varepsilon > 0$ consider the barrier functions

$$U_\varepsilon := U - \varepsilon w_*.$$

In view of (3.1), there exists $\rho_\varepsilon > 1$ such that

$$U_\varepsilon \leq 0 \qquad (|x| \geq \rho_\varepsilon)$$

and moreover, $\rho_\varepsilon \to \infty$ as $\varepsilon \to 0$.

Set $\Omega_\varepsilon := \{x \in \mathbb{R}^N : 1 < |x| < \rho_\varepsilon\}$ and choose a small $\varepsilon > 0$ such that $U_\varepsilon^+ > \delta > 0$ on an open set $G \subset \Omega_\varepsilon$. Since $w_* > 0$ and $U = 0$ as $|x| = 1$ and $|x| = \rho_\varepsilon$, we conclude that for any $n \in \mathbb{N}$,

$$nU_\varepsilon \le w_* \quad \text{on } \partial\Omega_\varepsilon$$

and hence $(w_* - nU_\varepsilon)^- \in H_0^1(\Omega_\varepsilon)$, see Remark 2.2. By the comparison principle of Corollary 2.12 we conclude that

$$nU_\varepsilon \le w_* \quad \text{in } \Omega_\varepsilon.$$

Then $w_* \ge n\delta$ on G and since $n \in \mathbb{N}$ can be taken arbitrary large, $w_* = +\infty$ on G, which is a contradiction. □

Remark 3.6. If U satisfies (3.5) then $U \notin D_V^1(B_1^c)$. Indeed, assume $U \in D_V^1(B_1^c)$. Then $\mathcal{E}_V(U) = 0$ by (3.2). But $\mathcal{E}_V(u) > 0$ for any $0 \ne u \in D_V^1(B_1^c)$ since $\mathcal{E}_V$ satisfies the λ–property, a contradiction.

Example 3.7 ($-\Delta$ in $\mathbb{R}^N$ with $N \ge 3$)**.** If $-\Delta w \ge 0$ in $\mathbb{R}^N \setminus \bar{B}_1$ then

$$\liminf_{|x|\to\infty} \frac{w(x)}{|x|^{-(N-2)}} > 0, \qquad \liminf_{|x|\to\infty} w(x) < \infty.$$

To see this, take $u(x) = |x|^{-(N-2)}$ as a small solution, $U(x) = 1 - |x|^{-(N-2)}$ as a large solution for $-\Delta$ in $\mathbb{R}^N \setminus \bar{B}_1$ and use Phragmén-Lindelöf principle for supersolutions. To check that $|x|^{-(N-2)}$ is a small solution, take $w_* = 1$ as a reference supersolution in (3.1).

Example 3.8 ($-\Delta$ in $\mathbb{R}^2$)**.** If $-\Delta w \ge 0$ in $\mathbb{R}^N \setminus \bar{B}_1$ then

$$\liminf_{|x|\to\infty} w(x) > 0, \qquad \liminf_{|x|\to\infty} \frac{w(x)}{\log|x|} < \infty.$$

To see this, take $u(x) = 1$ as a small solution, $U(x) = \log|x|$ as a large solution for $-\Delta$ in $\mathbb{R}^2 \setminus \bar{B}_1$ and use Phragmén-Lindelöf principle for supersolutions. To check that 1 is a small solution, take $w_* = \log|x|$ as a reference supersolution in (3.1).

Example 3.9 ($-\Delta + 1$ in $\mathbb{R}^3$)**.** If $-\Delta w + w \ge 0$ in $\mathbb{R}^3 \setminus \bar{B}_1$ then

$$\liminf_{|x|\to\infty} \frac{w(x)}{\frac{e^{-|x|}}{|x|}} > 0, \qquad \liminf_{|x|\to\infty} \frac{w(x)}{\frac{e^{|x|}}{|x|}} < \infty.$$

To see this, take

$$u(x) = \frac{e^{-|x|}}{|x|}, \qquad U(x) = \frac{e^{|x|-1} - e^{1-|x|}}{|x|},$$

as a small and large solution for $-\Delta + 1$ in $\mathbb{R}^3 \setminus \bar{B}_1$, respectively. Then use Phragmén-Lindelöf principle for supersolutions. To check that u is a small solution, take $w_* = 1$ as a reference supersolution in (3.1).

Exercise 3.10. Show that if

$$-\Delta w + \frac{c}{|x|^2} w \ge 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}_1,$$

and $c > -C_H := -\frac{(N-2)^2}{4}$, then

$$\liminf_{|x|\to\infty} \frac{w(x)}{|x|^{\alpha_-}} > 0, \qquad \liminf_{|x|\to\infty} \frac{w(x)}{|x|^{\alpha_+}} < \infty,$$

where $\alpha_- < \alpha_+$ are the two roots of the quadratic equation $\alpha(\alpha + N - 2) + c = 0$.

Note that if $c > 0$ then $\alpha_+ > 0$ and U is growing at infinity, while if $c < 0$ then $\alpha_+ < 0$ and U is decaying at infinity.

Further reading. For the development of the criticality theory beyond Agmon's ideas in [3, 4] see e.g. [6–8]. Some Phragmén-Lindelöf type results in the context of linear elliptic equations can be found in [5]. Powerful applications to the analysis of Hardy type inequalities involving distance to the boundary are given in [9]. More recent developments of the criticality theory presented e.g. in [14, 20]. For an extension to p–Laplacian see [16]; to local and nonlocal Dirichlet forms see [18, 21]. This list is very far from being complete.

4. A nonlinear Liouville theorem

We are going to use the AAP and Phragmén–Lindelöf principles developed in the previous section to study positive supersolutions of the semilinear elliptic equation

$$-\Delta u = u^p \quad \text{in } \Omega, \tag{4.1}$$

where Ω is an exterior domain in $\mathbb{R}^N$ such that $\mathbb{R}^N \setminus \bar{B}_1 \subset \Omega$ and $\partial\Omega \neq \emptyset$, and $p \in \mathbb{R}$ is the nonlinear *exponent*, which could take both positive and negative values.

A *weak* (super) solution to (4.1) is a function $u \in H^1_{loc}(\Omega)$ such that $u^p \in L^1_{loc}(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx \; (\geq) = \int_\Omega u^p \varphi \, dx, \qquad \forall\, 0 \leq \varphi \in H^1_c(\Omega) \cap L^\infty_c(\Omega). \tag{4.2}$$

Note that if $u \gneq 0$ is a weak supersolution to (4.1) then $-\Delta u \geq 0$ in Ω, and hence $u > 0$ in Ω. In what follows we often omit the *weak* and use simply (super) solution.

Theorem 4.1. *Let $N \geq 3$. Equation (4.1) admits a positive weak supersolution if and only if $p \leq p_S := \frac{N}{N-2}$.*

This theorem was proved by J. Serrin in the 70s for the radial functions, see the introduction to [10] for the references and a general overview of nonlinear Liouville's theorem. The exponent p_S is often known as the Serrin's critical exponent. The idea to use the AAP and Phragmén–Lindelöf principles in the context of nonlinear Liouville's theorems goes back to [11] and was developed in [12, 13, 15, 17, 19].

Proof. Our proof of Theorem 4.1 will be split into nonexistence and existence part. In the proof of nonexistence we will distinguish four separate cases: $1 < p < p_S$, $p = p_S$, $p = 1$, $p < 1$. The nonexistence in the superlinear case $1 < p < p_S$ relies on the *lower* bound in the Phragmén–Lindelöf principle while the nonexistence in the sublinear case uses the *upper* bound in the Phragmén–Lindelöf principle.

Before we start the proof, we present two technical lemmas. The first one is a particular case of the nonexistence counterpart of the AAP principle (Corollary 1.11).

Lemma 4.2. *Assume that $u > 0$ satisfies*

$$-\Delta u - c|x|^{-2+\varepsilon}u \geq 0 \quad (|x| > 1), \tag{4.3}$$

for some $\varepsilon > 0$ and $c > 0$. Then $u \equiv 0$.

Proof. Consider the quadratic form

$$\mathcal{E}(u) := \int_\Omega |\nabla\varphi|^2\,dx - c\int_\Omega \frac{\varphi^2}{|x|^{2+\varepsilon}}\,dx \qquad (\varphi \in C_c^\infty(\Omega).$$

Take $\varphi \in C_c^\infty(\Omega)$ such that $\varphi \gneq 0$, $0 \leq \varphi \leq 1$ and $\varphi = 0$ for $|x| > 2$ and $|x| < 1$. Then for $R > 1$ the rescaling

$$\varphi_R(x) := \varphi\big(\tfrac{x}{R}\big) \in C_c^\infty(\Omega)$$

and by the change of variables we compute

$$\mathcal{E}(\varphi_R) = R^{N-2}\int_\Omega |\nabla\varphi|^2\,dx - cR^{N-2+\varepsilon}\int_\Omega \frac{\varphi^2}{|x|^{2-\varepsilon}}\,dx \to -\infty,$$

as $R \to \infty$. By the AAP principle (Corollary 1.11) we conclude that $u \equiv 0$. □

The second lemma is an an apriori lower bound on positive solutions of (4.1) in the sublinear case. Note that the bound (4.3) depends on the value of $p < 1$.

Lemma 4.3. *Let $p < 1$ and assume that $u > 0$ satisfies*

$$-\Delta u \geq u^p \quad \textit{in } \Omega.$$

Then

$$u \geq c|x|^{\frac{2}{1-p}} \quad (|x| > 1). \tag{4.4}$$

Proof. The proof uses the AAP principle and weak Harnack's inequality, see [13, Lemma 6.1]. □

Now we are in a position to prove Theorem 4.1. We will proceed case by case.

Nonexistence in the superlinear subcritical case $1 < p < p_S$. Assume that $u > 0$ is a supersolution to (4.1). Then $-\Delta u \geq 0$ in Ω and hence, as in Example 3.7 we conclude that for some $c > 0$,

$$u \geq c|x|^{-(N-2)} \quad (|x| > 1) \tag{4.5}$$

Consider the *linearisation* of (4.1),

$$-\Delta u + V(x)u \geq 0 \quad \text{in } \Omega, \tag{4.6}$$

where $V(x) := u^{p-1}$. Since $p > 1$, using (4.5) we can estimate

$$V(x) \geq c_1|x|^{-(N-2)(p-1)} \quad (|x| > 1), \tag{4.7}$$

where $c_1 = c^{p-1} > 0$. Note that if $1 < p < p_S$ then

$$-(N-2)(p-1) > -2. \tag{4.8}$$

Hence $u > 0$ satisfies

$$-\Delta u + c_1|x|^{-2+\varepsilon}u \geq 0 \quad (|x| > 1), \tag{4.9}$$

for some $\varepsilon > 0$. By Lemma 4.2, we conclude that $u \equiv 0$. □

Nonexistence in the critical case $p = p_S$. In this case we have $\varepsilon = 0$ in (4.9) and the previous argument fails.[4] Instead, we will iterate the previous to improve the lower bound (4.7).

Indeed, we may assume that $0 < c_1 < C_H$ and $u > 0$ satisfies

$$-\Delta u - c_1|x|^{-2}u \geq 0 \quad (|x| > 1). \tag{4.10}$$

As in the Exercise 3.10 we conclude that for some $c_2 > 0$,

$$u \geq c_2|x|^{\alpha_-} \quad (|x| > 1), \tag{4.11}$$

where $\alpha_- > -(N-2)$ is the smallest root of $-\alpha(\alpha + N - 2) = c_1$. Then $u > 0$ satisfies the linearisation equation (4.6) and we can estimate

$$V(x) \geq c_3|x|^{\alpha_-(p-1)} \quad (|x| > 1), \tag{4.12}$$

where $c_3 = c_2^{p-1}$. Since $p = p_S$ and $\alpha_- > -(N-2)$,

$$\alpha_-(p-1) > -2. \tag{4.13}$$

Hence $u > 0$ satisfies (4.9) with some $\varepsilon > 0$ and as before, by Lemma 4.2, we conclude that $u \equiv 0$. □

Nonexistence in the linear case $p = 1$. In this case the equation (4.1) is linear. We simply note that there exists $\varphi \in C_c^\infty(\Omega)$ such that the corresponding quadratic form

$$\mathcal{E}_{-1}(\varphi) = \int_\Omega |\nabla\varphi|^2\,dx - \int_\Omega \varphi^2\,dx < 0.$$

(This can be seen using the same family of test functions φ_R as in the proof of Lemma 4.2.) By the AAP principle (Corollary 1.11) we conclude that $u \equiv 0$. □

Nonexistence in the sublinear case $p < 1$. Assume that $u > 0$ is a supersolution to (4.1). Then $-\Delta u \geq 0$ in Ω and using the upper bound in Example 3.7 we conclude that

$$\liminf_{|x|\to\infty} u(x) < \infty. \tag{4.14}$$

But according to Lemma 4.3,

$$u \geq c|x|^{\frac{2}{1-p}} \quad (|x| > 1). \tag{4.15}$$

[4] If $c_1 > C_H$, the critical Hardy constant, we would conclude again by the AAP principle. However we do not control the size of $c_1 > 0$ and in general, c_1 could be small.

Since $p < 1$ these two bounds are incompatible with each other and we conclude that $u \equiv 0$. □

Existence in the case $p > p_S$. A direct computation shows that for every $p > p_S$

$$u = c_p|x|^{-\frac{2}{p-1}}, \qquad c_p^{p-1} = \tfrac{2}{(p-1)^2}((N-2)p - N)$$

is a solution of (4.1).[5] □

Exercise 4.4. Modify the previous arguments to show that if $N = 2$ then equation (4.1) has no positive weak supersolutions for any $p \in \mathbb{R}$.

Exercise 4.5. Let $N \geq 2$ and $c > 0$. Show that the equation

$$-\Delta u + \frac{c}{|x|^2}u = u^p \quad \text{in } \Omega$$

admits a positive weak supersolution if and only if $p \notin [1 - \frac{2}{\alpha_+}, 1 - \frac{2}{\alpha_-}]$, where α_- and α_+ are defined in Exercise 3.10.

Hint: Use small and large solutions constructed in Exercise 3.10. The nonexistence in the lower critical case $p = 1 - \frac{2}{\alpha_+} < 0$ is difficult, see [13, Lemma 6.6]. All other regimes could be studied similarly to the proof of Theorem 4.1.

Further reading. Similar methods based on the AAP and Phragmén–Lindelöf principles were used to prove Liouville's theorems for divergence type semilinear equations in conical domains [12], equations with Hardy type potentials in conical domains [13], quasilinear equations involving p-Laplacian [15], equations with Hardy type potentials involving distance to the boundary of a bounded domain [17] and nonlocal Choquard's equations [19].

[5] Note that $c_p < 0$ for $p < p_S$ and $c_p = 0$ if $p = p_S$.

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