Differential Equations	Name
MAA121/MAG131 Sheet 3	Number
Hand in by 11/03/2010	Year
	Mark: /10
	date marked: / /2010

Please attach your working, with this sheet at the front.

These handouts can be downloaded on the Internet at:
`http://www-maths.swan.ac.uk/staff/vm/ODE/ODE.html`

The mark for this assignment does not count towards your final mark.

1. Find general solution for the following ODEs using the method of separation of variables:

   (*a*) $\dot{x} = -x^2$;

   (*b*) $\dot{x} = t^2 x$;

   (*c*) $\dot{x} = e^{-t^2} x^2$ (give the solution in terms of an integral).

2. Find solution for the following initial value problems using the method of separation of variables:

   (*a*) $\dot{x} = t^3(1-x), \quad x(0) = 5$;

   (*b*) $\dot{x} + px = q, \quad x(0) = 1$ (here p and q are nonzero constants).

3. Consider the initial value problem

$$\dot{x} = x^\alpha, \quad x(0) = x_0.$$

Show that:

a) for $\alpha > 1$ show that solutions with $x_0 > 0$ blow up in a finite time (compare with Example 4.1 in the Lecture Notes);

b) for $0 < \alpha < 1$ show that solutions with $x_0 = 0$ are not unique (compare with Example 5.2 in the Lecture Notes).

Turn over.

4. The *Existence and Uniqueness Theorem* for 1st order ODEs states that if

$$f(x,t) \quad \text{and} \quad \frac{\partial f}{\partial x}(x,t)$$

are continuous for $a < x < b$ and for $c < t < d$, then for any $x_0 \in (a,b)$ and $t_0 \in (c,d)$ the initial value problem

$$\dot{x} = f(x,t), \qquad x(t_0) = x_0,$$

has a unique solution $x(t)$, defined on some existence interval $[t_0, T)$.

Determine, which of the following initial value problems have unique solution ?

(a) $\quad \dot{x} = x(1 - x^4), \quad x(0) = x_0.$

(b) $\quad \dot{x} = t^2 x^{1/3}(1 - x)^2, \quad x(1) = 0;$

(c) $\quad t^2 \dot{x} = e^{-t^2} x^2, \quad x(0) = 1.$

Differential Equations
MAA121/MAG131 Sheet 3 (Solutions)

1. (*a*) This is a 1st order nonlinear *separable* equation. The unknown function is $x(t)$ and the independent variable is t. Separating the variables in the equation we obtain

$$\frac{dx}{x^2} = -dt.$$

Integration both sides

$$\int \frac{dx}{x^2} = -\int dt,$$

we obtain

$$-\frac{1}{x} + C_1 = -t + C_2.$$

Multiplying by -1 and absorbing the constants C_1 and C_2 into $C = C_2 - C_1$ we arrive to a slightly simpler expression

$$\frac{1}{x} = t - C.$$

Resolving with respect to x we finally obtain the required general solution

$$x(t) = \frac{1}{t - C},$$

where $C \in \mathbb{R}$ is an arbitrary constant.

(*b*) This is a 1st order linear *separable* equation. The unknown function is $x(t)$ and the independent variable is t. Separating the variables in the equation we obtain

$$\frac{dx}{x} = t^2 dt.$$

Integration both sides

$$\int \frac{dx}{x} = \int t^2 dt,$$

and absorbing the constants we obtain

$$\log|x| = \frac{t^3}{3} + C.$$

Exponentiating both sides we get

$$|x| = e^{\frac{t^3}{3} + C},$$

or

$$|x(t)| = Ae^{\frac{t^3}{3}},$$

where $A = e^C$ is positive. Taking $|x(t)| = x(t)$ gives a positive solution, while taking $|x(t)| = -x(t)$ gives a negative solution. Notice that $x(t) = 0$ is also a solution of the equation. Thus the general solution is

$$x(t) = Ae^{\frac{t^3}{3}},$$

allowing any $A \in \mathbb{R}$.

(c) This is a 1st order nonlinear *separable* equation. The unknown function is $x(t)$ and the independent variable is t. Separating the variables in the equation we obtain

$$\frac{dx}{x^2} = e^{-t^2} dt$$

Integration both sides

$$\int \frac{dx}{x^2} = \int e^{-t^2} dt$$

and absorbing the constants we obtain

$$-\frac{1}{x} = \int e^{-t^2} dt + C.$$

Resolving with respect to x we finally obtain the required general solution

$$x(t) = -\frac{1}{\int e^{-t^2} dt + C},$$

where $C \in \mathbb{R}$ is an arbitrary constant.

2. (a) This is an initial value problem for a 1st order linear *separable* equation. The unknown function is $x(t)$ and the independent variable is t.

First, we obtain the general solution of the equation. Separating the variables, we get

$$\frac{dx}{1-x} = t^3 dt.$$

Integrating both sides we obtain the general solution

$$x(t) = 1 + Ce^{-\frac{t^4}{4}},$$

where $C \in \mathbb{R}$ is an arbitrary constant to be determined from the initial condition $x(0) = 5$.

Substituting the initial condition into the general solution, we obtain

$$\underbrace{x(0)}_{=5} = 1 + Ce^0 = 1 + C,$$

so $C = 4$ and hence the required solution of the initial value problem is

$$x(t) = 1 + 4e^{-\frac{t^4}{4}}.$$

(b) This is an initial value problem for a 1st order linear *separable* equation. The unknown function is $x(t)$ and the independent variable is t.

First, we obtain the general solution of the equation. Separating the variables, we get

$$\frac{dx}{q - px} = dt.$$

Integrating both sides we obtain

$$-\frac{1}{p}\log|q-px| = t + C.$$

Multiplying by $-p$ and exponentiating both sides gives

$$|q-px| = Ae^{-pt}$$

where $A = e^{pC}$. Taking either sign for the modulus gives positive or negative values of A, so we have

$$q - px = Ae^{-pt},$$

and finally, the general solution of the equation is written as

$$x(t) = Be^{-pt} + \frac{q}{p},$$

where $B = -A/p$ is a constant, to be determined from the initial condition $x(0) = 1$.

Substituting the initial condition into the general solution, we obtain

$$\underbrace{x(0)}_{=1} = Be^0 + \frac{q}{p} = B + \frac{q}{p},$$

so

$$B = 1 - \frac{q}{p}$$

and hence the required solution of the initial value problem is

$$x(t) = \left(1 - \frac{q}{p}\right)e^{-pt} + \frac{q}{p}.$$

3. *a*) Assume that $\alpha > 1$. Then a direct computation (similar to (*a*) above) shows that for any $C > 0$ the function

$$x(t) = (\alpha - 1)^{\frac{1}{1-\alpha}}(C - t)^{\frac{1}{1-\alpha}}$$

is a solution of the ODE for $0 \le t < C$ with $x_0 = (C(\alpha-1))^{\frac{1}{1-\alpha}} > 0$ and $x(t) \to +\infty$ as $t \to C$, that is $x(t)$ is a solution which blows up when $t \to C$.

b) Assume that $0 < \alpha < 1$. Then a direct computation (e.g. using Mathematica's `DSolve` to guess a form of the solution) shows that the function

$$x(t) = \begin{cases} (1-\alpha)^{\frac{1}{1-\alpha}}(t-C)^{\frac{1}{1-\alpha}}, & t \ge C \\ 0, & 0 \le t < C. \end{cases}$$

is a solution of the ODE with $x_0 = 0$ for any $C \ge 0$, that is the initial value problem has infinitely many solutions!

4. (a) The function $f(t, x)$ in this case is

$$f(x, t) = x(1 - x^4).$$

This is a continuous function for all $x \in \mathbb{R}$ (and it does not depend on t). Its derivative in x is computed as

$$\frac{\partial f}{\partial x}(x, t) = 1 - 5x^4,$$

which is again a continuous function for all $x \in \mathbb{R}$. Hence the ODE has unique solution for any initial condition $x(0) = x_0$.

(b) The function $f(t, x)$ in this case is

$$f(x, t) = t^2 x^{1/3}(1 - x)^2.$$

It is convenient to rewrite it as

$$f(x, t) = t^2(x^{1/3} - 2x^{4/3} + x^{7/3}).$$

This is a continuous function for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$. Its derivative in x is computed as

$$\frac{\partial f}{\partial x}(x, t) = t^2 \left(\frac{1}{3}x^{-2/3} - \frac{8}{3}x^{1/3} + \frac{7}{3}x^{4/3} \right).$$

This is a function which has a discontinuity if $x = 0$, or we may say that $x_0 = 0$ is a "singular" initial condition. Therefore the ODE may not have a unique solution for the initial condition $x(1) = 0$.

(c) We first rewrite the equation in the standard form

$$\dot{x} = \frac{e^{-t^2}}{t^2} x^2.$$

Thus the function $f(t, x)$ is

$$f(x, t) = \frac{e^{-t^2}}{t^2} x^2.$$

This is a function which has a discontinuity if $t = 0$, or we may say that $t_0 = 0$ is a "singular" initial time. Therefore, for the initial condition $x(0) = 1$ the ODE may not have unique solution.