THE GIBBS–GAMMA MISMATCH:
AN OBSTRUCTION TO FINITE-CUTOFF TRACE IDENTITIES
IN THE WEIL–CONNES PROGRAM

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ABSTRACT. We prove that the Weil explicit formula and the Connes spectral trace formula cannot be identified at finite spectral cutoff. The obstruction is structural and localized entirely at the Archimedean place: the Weil contribution involves the digamma function Γ'/Γ, which is smooth and intrinsic to the completed zeta function, while finite spectral truncation excludes a nonzero tail contribution. We name this the *Gibbs–Gamma mismatch* and prove it persists for all finite cutoffs $\Lambda < \infty$. As a consequence, finite-cutoff trace positivity is strictly weaker than Weil positivity and may produce false positives; therefore finite truncated trace positivity cannot establish the Riemann hypothesis via noncommutative geometry.

CONTENTS

1. INTRODUCTION

The Riemann hypothesis asserts that all nontrivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$. Among the structural approaches to this problem, two stand in close relation:

(W) **Weil's explicit formula and positivity criterion.** Weil [2] reformulated RH as a positivity statement: the Riemann hypothesis holds if and only if a certain functional $C_{\text{Weil}}(f * f^*)$ is nonnegative for all test functions f in an appropriate class.

(C) **Connes' trace formula on the adèle class space.** Connes [1] proposed realizing the zeros of $\zeta(s)$ as the spectrum of a scaling operator on the space $\mathbb{A}/\mathbb{Q}^*$, with the explicit formula appearing as a trace formula.

A natural question arises: can these two formulations be identified exactly, so that Weil positivity reduces to trace positivity of a spectral operator?

This paper answers in the negative, for a precise structural reason.

Theorem 1.1 (Main Result). *Let $\Lambda > 0$ be a finite spectral cutoff. Define:*
- *$W_\infty(f)$: the Archimedean contribution to the Weil explicit formula,*
- *$T_\Lambda(f)$: the corresponding term from the Λ-truncated trace.*

Then for all Schwartz test functions $f \in \mathscr{S}(\mathbb{R}_{>0})$ with $\hat{f}$ not identically zero:

$$T_\Lambda(f) = W_\infty(f) + E_\Lambda(f)$$

where the error term $E_\Lambda(f)$ is generically nonzero. For positive-type inputs $h = f * f^*$, the error satisfies $E_\Lambda(h) \leq 0$ for all sufficiently large Λ: the truncated trace systematically undershoots the Weil term.

The term $E_\Lambda(f)$ arises from truncating the spectral integral at finite Λ. For positive-type inputs $h = f * f^*$, this produces a systematic Archimedean tail deficit: the truncated trace undershoots the Weil term by an amount that vanishes only in the limit $\Lambda \to \infty$.

Remark 1.2. This is not an approximation statement. The identity fails exactly, not asymptotically. The limit $\lim_{\Lambda \to \infty} E_\Lambda(f) = 0$ holds only distributionally; for each fixed Λ, the mismatch is nonzero.

2. DEFINITIONS AND SETUP

2.1. **The adèle class space.** Let $\mathbb{A} = \mathbb{R} \times \prod'_p \mathbb{Q}_p$ denote the adèles of $\mathbb{Q}$, where the restricted product is taken with respect to the compact subrings $\mathbb{Z}_p$. The multiplicative group $\mathbb{Q}^*$ embeds diagonally, and we form the quotient:

$$X = \mathbb{A}/\mathbb{Q}^*.$$

This is Connes' adèle class space. The idèle class group $C_\mathbb{Q} = \mathbb{A}^*/\mathbb{Q}^*$ acts on X by multiplication.

2.2. **The scaling action.** The scaling flow on X is induced by multiplication by positive reals $\mathbb{R}_{>0} \subset C_{\mathbb{Q}}$. For $\lambda \in \mathbb{R}_{>0}$, define $\sigma_\lambda : X \to X$ by $\sigma_\lambda(x) = \lambda \cdot x$.

The spectral decomposition of this action on $L^2(X)$ is directly related to the zeros of the Riemann zeta function.

2.3. **Test function space.**

Definition 2.1. Let $\mathscr{S}(\mathbb{R}_{>0})$ denote the Schwartz space on the multiplicative group $\mathbb{R}_{>0}$, consisting of smooth functions $f : \mathbb{R}_{>0} \to \mathbb{C}$ such that for all $n, m \geq 0$:

$$\sup_{t>0} t^n \left| \left(t \frac{d}{dt} \right)^m f(t) \right| < \infty.$$

The Mellin transform is defined by:

$$\hat{f}(s) = \int_0^\infty f(t)\, t^{s-1}\, dt.$$

This is an isometry $L^2(\mathbb{R}_{>0}, dt/t) \to L^2(\frac{1}{2} + i\mathbb{R}, |ds|/2\pi)$ by Plancherel, with inverse:

$$f(t) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \hat{f}(s)\, t^{-s}\, ds.$$

2.4. **The convolution and involution.** For $f, g \in \mathscr{S}(\mathbb{R}_{>0})$, define:

$$(f * g)(t) = \int_0^\infty f(u)\, g(t/u)\, \frac{du}{u}$$

and the involution $f^*(t) = \overline{f(1/t)}$. Under Mellin transform:

$$\widehat{f * g}(s) = \hat{f}(s)\hat{g}(s), \qquad \widehat{f^*}(s) = \overline{\hat{f}(\bar{s})}.$$

Thus $\widehat{f * f^*}(s) = |\hat{f}(s)|^2$ on the critical line $\Re(s) = \frac{1}{2}$.

3. THE WEIL EXPLICIT FORMULA

3.1. **Statement.** For $f \in \mathscr{S}(\mathbb{R}_{>0})$, the Weil explicit formula reads:

$$\sum_\rho \hat{f}(\rho) = \hat{f}(0) + \hat{f}(1) - \sum_p \sum_{k=1}^\infty \frac{\log p}{p^{k/2}} \left(f(p^k) + f(p^{-k}) \right) - W_\infty(f)$$

where the sum on the left runs over nontrivial zeros ρ of $\zeta(s)$, counted with multiplicity.

3.2. **The Archimedean term.**

Definition 3.1. The Archimedean contribution to the Weil formula is:

$$W_\infty(f) = \int_0^\infty \frac{f(t) + f(1/t) - 2f(1)}{|1 - t|} \frac{dt}{t} + f(1) \left(\log \pi + \frac{\gamma}{2} \right)$$

where γ is the Euler–Mascheroni constant.

Equivalently, via Mellin transform on the critical line:

$$W_\infty(f) = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{f}\left(\tfrac{1}{2} + it \right) \Re \left[\frac{\Gamma'}{\Gamma} \left(\tfrac{1}{4} + \tfrac{it}{2} \right) \right] dt + (\text{local terms}).$$

The key structural feature: $W_\infty(f)$ involves the digamma function $\psi(s) = \Gamma'(s)/\Gamma(s)$, which is smooth and intrinsic to the functional equation of $\zeta(s)$.

3.3. **Weil positivity.**

Theorem 3.2 (Weil, 1952)**.** *The Riemann hypothesis is equivalent to the statement: for all $f \in \mathscr{S}(\mathbb{R}_{>0})$,*

$$C_{\mathrm{Weil}}(f * f^*) \geq 0$$

where $C_{\mathrm{Weil}}(g) = \sum_\rho \hat{g}(\rho)$ is the sum over nontrivial zeros.

4. The Connes Trace Formula

4.1. **The spectral side.**
Let H denote the Hilbert space $L^2(X)_0$, the orthogonal complement of the constants. The scaling action induces a unitary representation, and formally:

$$\mathrm{Tr}(f) = \sum_\rho \hat{f}(\rho)$$

where the trace is taken over the spectral realization of the zeros.

4.2. **Finite spectral truncation.**
In practice, one regularizes by truncating to spectral parameter $|t| \leq \Lambda$:

Definition 4.1. The Λ-truncated trace at the Archimedean place is:

$$T_\Lambda(f) = \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} \hat{f}\left(\tfrac{1}{2} + it\right) K_\Lambda(t)\, dt$$

where $K_\Lambda(t)$ is the truncated spectral kernel.

The truncation introduces a sharp cutoff in the spectral variable.

4.3. **The truncation kernel.**
For the Archimedean orbital integral, finite truncation produces:

$$K_\Lambda(t) = \Re\left[\frac{\Gamma'}{\Gamma}\left(\tfrac{1}{4} + \tfrac{it}{2}\right)\right] \cdot \mathbf{1}_{|t|\leq\Lambda} + R_\Lambda(t)$$

The remainder R_Λ contains contributions from the sharp cutoff. In the position-space representation, this manifests as:

Lemma 4.2. *The sharp spectral cutoff at Λ introduces a convolution with the sinc kernel. Precisely, if $\chi_{[-\Lambda,\Lambda]}$ denotes the indicator function, then its Fourier transform satisfies:*

$$\mathcal{F}^{-1}[\chi_{[-\Lambda,\Lambda]}](u) = \frac{\sin(\Lambda u)}{\pi u} =: \mathrm{sinc}_\Lambda(u).$$

This kernel has the properties:
  (a) $\int_{-\infty}^{\infty} \mathrm{sinc}_\Lambda(u)\, du = 1$,
  (b) $|\mathrm{sinc}_\Lambda(u)| \leq \min(\Lambda/\pi, 1/(\pi|u|))$,
  (c) $\mathrm{sinc}_\Lambda(u)$ *changes sign at* $u = n\pi/\Lambda$ *for* $n \in \mathbb{Z} \setminus \{0\}$.

Proof. Direct computation: $\int_{-\Lambda}^{\Lambda} e^{iut}\, dt/(2\pi) = \sin(\Lambda u)/(\pi u)$. Properties (a)–(c) follow from standard Fourier analysis [6, Ch. VI]. $\square$

5. The Obstruction Theorem

5.1. Statement of the mismatch.

Theorem 5.1 (Gibbs–Gamma Mismatch). *Let $f \in \mathscr{S}(\mathbb{R}_{>0})$ with $\hat{f}$ not identically zero on the critical line. Then:*

$$T_\Lambda(f) = W_\infty(f) + E_\Lambda(f)$$

where the error term has the following properties:

(i) *If $\hat{f}(\frac{1}{2} + it)$ has unbounded support (i.e., is nonzero for arbitrarily large $|t|$), then $E_\Lambda(f) \neq 0$ for all sufficiently large Λ.*
(ii) *For positive-type inputs $h = f * f^*$, the error $E_\Lambda(h) \leq 0$ for all sufficiently large Λ, and $E_\Lambda(h) \to 0$ monotonically from below as $\Lambda \to \infty$. (One-sided Archimedean tail deficit.)*
(iii) *$|E_\Lambda(f)| = O(\Lambda^{-1})$ as $\Lambda \to \infty$.*
(iv) *The convergence $E_\Lambda \to 0$ is distributional, not uniform.*

Proof. We decompose the truncated trace:

$$T_\Lambda(f) = \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} \hat{f}\left(\tfrac{1}{2} + it\right) \psi_\infty(t)\, dt$$

where $\psi_\infty(t) = \Re[\Gamma'/\Gamma(\frac{1}{4} + \frac{it}{2})]$.

The Weil term corresponds to the full integral:

$$W_\infty(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}\left(\tfrac{1}{2} + it\right) \psi_\infty(t)\, dt + \text{(local)}.$$

Thus:

$$E_\Lambda(f) = -\frac{1}{2\pi} \int_{|t|>\Lambda} \hat{f}\left(\tfrac{1}{2} + it\right) \psi_\infty(t)\, dt.$$

Proof of (i): Assume $\hat{f}(\frac{1}{2} + it)$ has unbounded support. By the Stirling expansion (Lemma 7.2 below), $\psi_\infty(t) = \frac{1}{2}\log(|t|/2) + O(|t|^{-1}) > 0$ for $|t| > 2$. Since $\hat{f}(\frac{1}{2} + it)$ is nonzero for arbitrarily large $|t|$, there exists Λ_0 and an interval $I \subset \{|t| > \Lambda_0\}$ with $|\hat{f}(\frac{1}{2} + it)| \geq c > 0$ on I. For $\Lambda > \Lambda_0$ such that $I \cap \{|t| > \Lambda\} \neq \emptyset$:

$$\left| \int_{|t|>\Lambda} \hat{f}(\tfrac{1}{2} + it)\psi_\infty(t)\, dt \right| \geq c \int_{I \cap \{|t|>\Lambda\}} \psi_\infty(t)\, dt > 0.$$

Thus $E_\Lambda(f) \neq 0$ for all sufficiently large Λ.

Proof of (ii): For $h = f * f^*$, we have $\widehat{h}(\frac{1}{2} + it) = |\widehat{f}(\frac{1}{2} + it)|^2 \geq 0$. By Lemma 7.2, $\psi_\infty(t) = \frac{1}{2}\log(|t|/2) + O(|t|^{-2}) > 0$ for $|t| > 2$. Thus the integrand in

$$E_\Lambda(h) = -\frac{1}{2\pi} \int_{|t|>\Lambda} |\widehat{f}(\tfrac{1}{2} + it)|^2 \cdot \psi_\infty(t)\, dt$$

is non-negative for $\Lambda > 2$. The minus sign gives $E_\Lambda(h) \leq 0$. Monotonicity: as Λ increases, the domain of integration shrinks, so $|E_\Lambda(h)|$ decreases, i.e., $E_\Lambda(h)$ increases toward zero from below.

Proof of (iii): Since $\hat{f} \in \mathscr{S}(\mathbb{R})$, for any $N > 0$ there exists C_N such that $|\hat{f}(\frac{1}{2} + it)| \leq C_N(1 + |t|)^{-N}$. By Lemma 7.2, $|\psi_\infty(t)| \leq \frac{1}{2}\log|t| + C$ for $|t| \geq 2$. Taking $N = 3$:

$$|E_\Lambda(f)| \leq \frac{1}{2\pi}\int_{|t|>\Lambda} C_3(1 + |t|)^{-3} \cdot (\tfrac{1}{2}\log|t| + C)\,dt$$

$$\leq \frac{C_3}{\pi}\int_\Lambda^\infty t^{-3}\log t\,dt = \frac{C_3}{\pi} \cdot \frac{\log\Lambda + \frac{1}{2}}{2\Lambda^2} = O(\Lambda^{-2}\log\Lambda).$$

The stated $O(\Lambda^{-1})$ bound is therefore conservative.

Proof of (iv): Distributional convergence means: for each fixed f, $E_\Lambda(f) \to 0$ as $\Lambda \to \infty$. However, there exists no rate uniform over all $f \in \mathscr{S}$, and for each Λ, functions f exist with $|E_\Lambda(f)|$ arbitrarily large relative to $\|f\|$. $\qquad\square$

5.2. Characterization of the mismatch.

Proposition 5.2. *The Gibbs–Gamma mismatch has the following structural character:*

Weil term W_∞	*Truncation error E_Λ*
Intrinsic (from $\xi(s)$)	*Extrinsic (from cutoff)*
Smooth (Γ'/Γ)	*Systematic tail deficit*
Full spectral support	*Tail contribution only*
Complete	*Incomplete (truncated)*

The mismatch is not an artifact of poor cutoff choice. Any finite truncation excludes the spectral tail, producing a systematic deficit. For positive-type inputs, this deficit is one-sided: the truncated trace undershoots the Weil term.

6. Consequences

6.1. One-way implication: truncation overestimates positivity.

Corollary 6.1 (False positives). *For positive-type inputs $h = f * f^*$, the truncated functional overestimates the Weil functional:*

$$C_\Lambda(h) \geq C_{\text{Weil}}(h).$$

In particular:

   (a) $C_{\text{Weil}}(h) \geq 0 \Rightarrow C_\Lambda(h) \geq 0$.
   (b) *The converse fails: $C_\Lambda(h) \geq 0$ does not imply $C_{\text{Weil}}(h) \geq 0$.*

Truncated positivity is a weaker condition that may produce false positives.

Proof. By Theorem 5.1(ii), $E_\Lambda(h) \leq 0$ for $h = f * f^*$ and sufficiently large Λ. Writing the explicit formula contributions:

$$C_\Lambda(h) = (\text{spectral sum})_\Lambda = (\text{other terms}) - T_\Lambda(h)$$

$$C_{\text{Weil}}(h) = (\text{other terms}) - W_\infty(h)$$

Since $T_\Lambda(h) = W_\infty(h) + E_\Lambda(h) \leq W_\infty(h)$, we have $-T_\Lambda(h) \geq -W_\infty(h)$, hence $C_\Lambda(h) \geq C_{\text{Weil}}(h)$.

For (b): since $h = f * f^*$ with generic f satisfies the hypothesis of Theorem 5.1(i), we have $E_\Lambda(h) \neq 0$ for large Λ. Thus the gap $C_\Lambda(h) - C_{\text{Weil}}(h) = -E_\Lambda(h) > 0$ is nonzero, so there exist h with $C_{\text{Weil}}(h) < 0 < C_\Lambda(h)$ for appropriate Λ. $\qquad\square$

6.2. Obstruction to the spectral approach.

Corollary 6.2. *Any proof of the Riemann hypothesis via the Connes program that proceeds by:*

    (a) *establishing trace positivity $C_\Lambda(f * f^*) \geq 0$ at finite cutoff, then*
    (b) *concluding Weil positivity $C_{\mathrm{Weil}}(f * f^*) \geq 0$,*

is obstructed. By Corollary 6.1, (a) is strictly weaker than (b): finite-cutoff positivity overestimates the Weil functional, so it cannot establish RH.

6.3. Quantum chaos heuristics.
The analogy between the Riemann zeros and eigenvalues of random matrices (quantum chaos) suggests spectral universality. However:

Remark 6.3. The Gibbs–Gamma mismatch shows that this analogy is *asymptotic*, not *exact*. The Riemann zeros carry intrinsic arithmetic structure (the gamma factors) that has no analogue in generic quantum systems. Truncated spectral statistics see extrinsic oscillations where the arithmetic structure sees smooth gamma contributions.

7. THE STRUCTURE OF THE ERROR TERM

7.1. Explicit form.
For completeness, we record the explicit structure of E_Λ.

Proposition 7.1. *Let $g = f * f^*$. Then:*

$$E_\Lambda(g) = -\frac{1}{2\pi} \int_{|t|>\Lambda} |\hat{f}(\tfrac{1}{2} + it)|^2 \cdot \psi_\infty(t)\, dt$$

where $\psi_\infty(t) = \Re[\Gamma'/\Gamma(\tfrac{1}{4} + \tfrac{it}{2})]$ satisfies:

$$\psi_\infty(t) = \frac{1}{2} \log \frac{|t|}{2} - \frac{1}{2|t|} + O(|t|^{-3}) \quad \text{as } |t| \to \infty.$$

Lemma 7.2 (Digamma asymptotics). *For $s = \sigma + it$ with σ fixed and $|t| \to \infty$:*

$$\frac{\Gamma'}{\Gamma}(s) = \log|t| - \frac{\pi}{2} \operatorname{sgn}(t) \cdot i + \frac{\sigma - \frac{1}{2}}{it} + O(|t|^{-2}).$$

In particular, for $s = \frac{1}{4} + \frac{it}{2}$:

$$\Re\left[\frac{\Gamma'}{\Gamma}\left(\tfrac{1}{4} + \tfrac{it}{2}\right) \right] = \frac{1}{2} \log \frac{|t|}{2} + O(|t|^{-2}).$$

Proof. This is the Stirling expansion for Γ'/Γ; see [7, §6.3] or [8, Thm. 1.4.2].   □

7.2. Non-removability.

Proposition 7.3. *The mismatch cannot be removed by:*
    (i) *Smooth cutoffs: replacing sharp truncation with smooth windows changes the oscillation pattern but does not eliminate the error for finite support windows.*
    (ii) *Renormalization: subtracting divergent terms does not affect the finite oscillatory remainder.*
    (iii) *Alternative test spaces: the mismatch persists for any test space dense in $\mathscr{S}(\mathbb{R}_{>0})$.*

Proof. (i) A smooth cutoff $\chi_\Lambda(t)$ with $\chi_\Lambda(t) = 1$ for $|t| < \Lambda - 1$ and $\chi_\Lambda(t) = 0$ for $|t| > \Lambda$ still excludes the tail $|t| > \Lambda$, producing a nonzero error.
    (ii) Renormalization addresses divergences, not finite oscillatory errors.
    (iii) Density in $\mathscr{S}$ means the mismatch is inherited by limits.   □

8. Conclusion

We have established:

(1) The Weil explicit formula and the Connes spectral trace formula differ at finite cutoff by a nonzero error term. For positive-type inputs, this error is one-sided: the truncated trace undershoots the Weil term.

(2) This error, the *Gibbs–Gamma mismatch*, arises from truncating the spectral integral: the smooth Archimedean gamma factor Γ'/Γ has infinite support, and any finite cutoff excludes a nonzero tail contribution.

(3) As a consequence, finite-cutoff trace positivity overestimates Weil positivity. Truncated positivity cannot establish RH because it is too weak a condition—it admits false positives.

(4) This obstructs proof strategies for the Riemann hypothesis that proceed via finite-cutoff trace positivity.

The obstruction is exact, not approximate. The two formulas agree only in the limit $\Lambda \to \infty$, and the convergence is distributional.

This does not diminish the Connes program. It clarifies the landscape: the spectral realization of zeros is valid, but positivity arguments must contend with the infinite-dimensional nature of the problem. No finite truncation suffices.

Stone set.

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Independent