UNIVERSITY OF TECHNOLOGY
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CHEMNITZ

Deep Reinforcement Learning

Temporal Difference learning

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1 - Temporal Difference Learning

Temporal-Difference (TD) learning

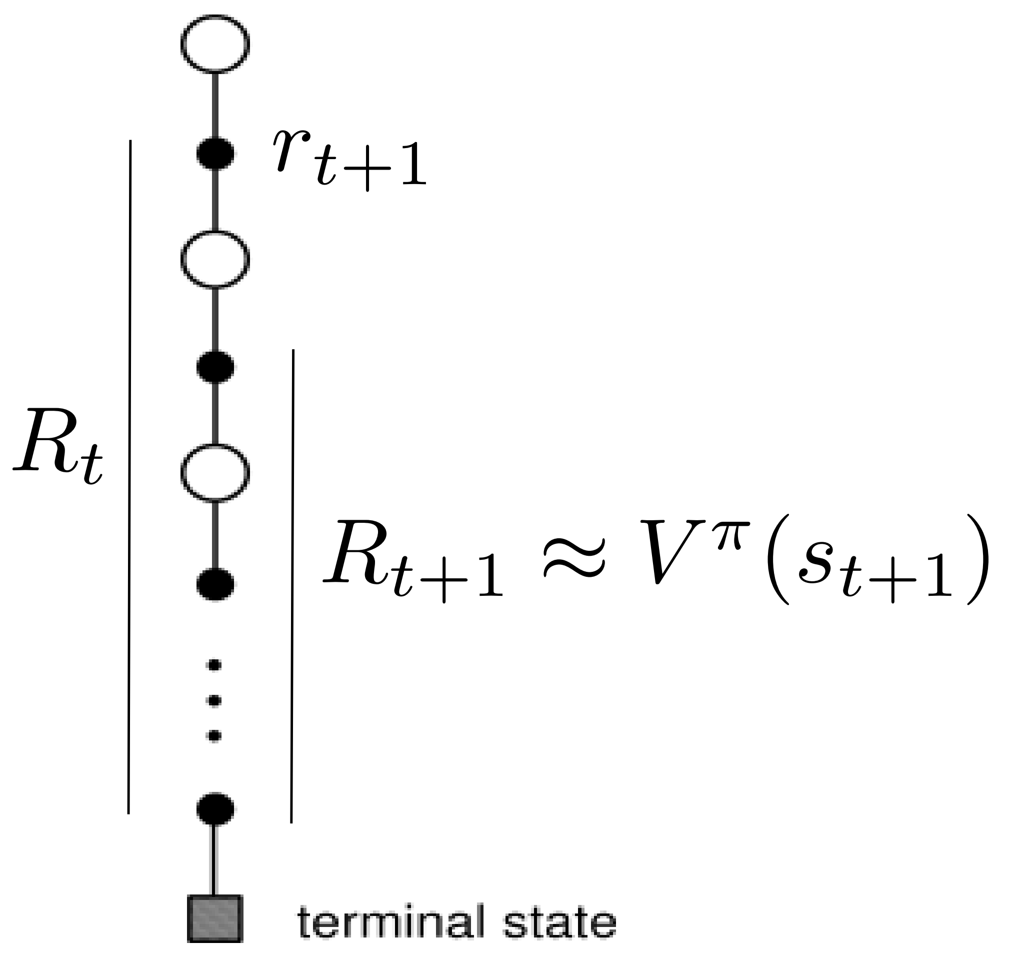
- MC methods wait until the end of the episode to compute the obtained return:

$$V(s_t) = V(s_t) + \alpha (R_t - V(s_t))$$

- If the episode is very long, learning might be very slow. If the task is continuing, it is impossible.

- Considering that the return at time t is the immediate reward plus the return in the next step:

$$R_t = r_{t+1} + \gamma \, R_{t+1}$$



we could replace R_{t+1} by an estimate, which is the value of the next state $V^\pi(s_{t+1}) = \mathbb{E}_\pi [R_{t+1} | s_{t+1} = s]$:

$$R_t \approx r_{t+1} + \gamma \, V^\pi(s_{t+1})$$

- **Temporal-Difference (TD)** methods simply replace the actual return by an estimation in the update rule:

$$V(s_t) = V(s_t) + \alpha \, (r_{t+1} + \gamma \, V(s_{t+1}) - V(s_t))$$

where $r_{t+1} + \gamma \, V(s_{t+1})$ is a sampled estimate of the return.

Temporal-Difference (TD) learning

- The quantity

$$\delta_t = r_{t+1} + \gamma \, V(s_{t+1}) - V(s_t)$$

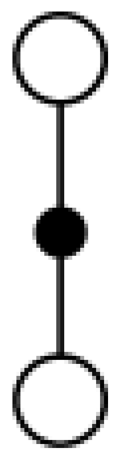
is called equivalently the **reward prediction error** (RPE), the **TD error** or the **advantage** of the action a_t.

- It is the difference between:
  - the estimated return in state s_t: $V(s_t)$.
  - the actual return $r_{t+1} + \gamma \, V(s_{t+1})$, computed with an estimation.
- If $\delta_t > 0$, it means that:
  - we received more reward r_{t+1} than expected, or:
  - we arrive in a state s_{t+1} that is better than expected.
  - we should increase the value of s_t as we **underestimate** it.
- If $\delta_t < 0$, we should decrease the value of s_t as we **overestimate** it.

TD policy evaluation TD(0)

- The learning procedure in TD is then possible after each transition: the backup diagram is limited to only one state and its follower.

Backup diagram of TD(0)



- **while** True:
    - Start from an initial state s_0.
    - **foreach** step t of the episode:
        - Select a_t using the current policy π in state s_t.
        - Apply a_t, observe r_{t+1} and s_{t+1}.
        - Compute the TD error:

$$\delta_t = r_{t+1} + \gamma \, V(s_{t+1}) - V(s_t)$$

        - Update the state-value function of s_t:

$$V(s_t) = V(s_t) + \alpha \, \delta_t$$

        - **if** s_{t+1} is terminal: **break**

- TD learns from experience in a fully incremental manner. It does not need to wait until the end of an episode. It is therefore possible to learn continuing tasks. TD converges to V^π if the step-size parameter α is small enough.

Bias-variance trade-off

- The **TD error** is used to evaluate the policy:

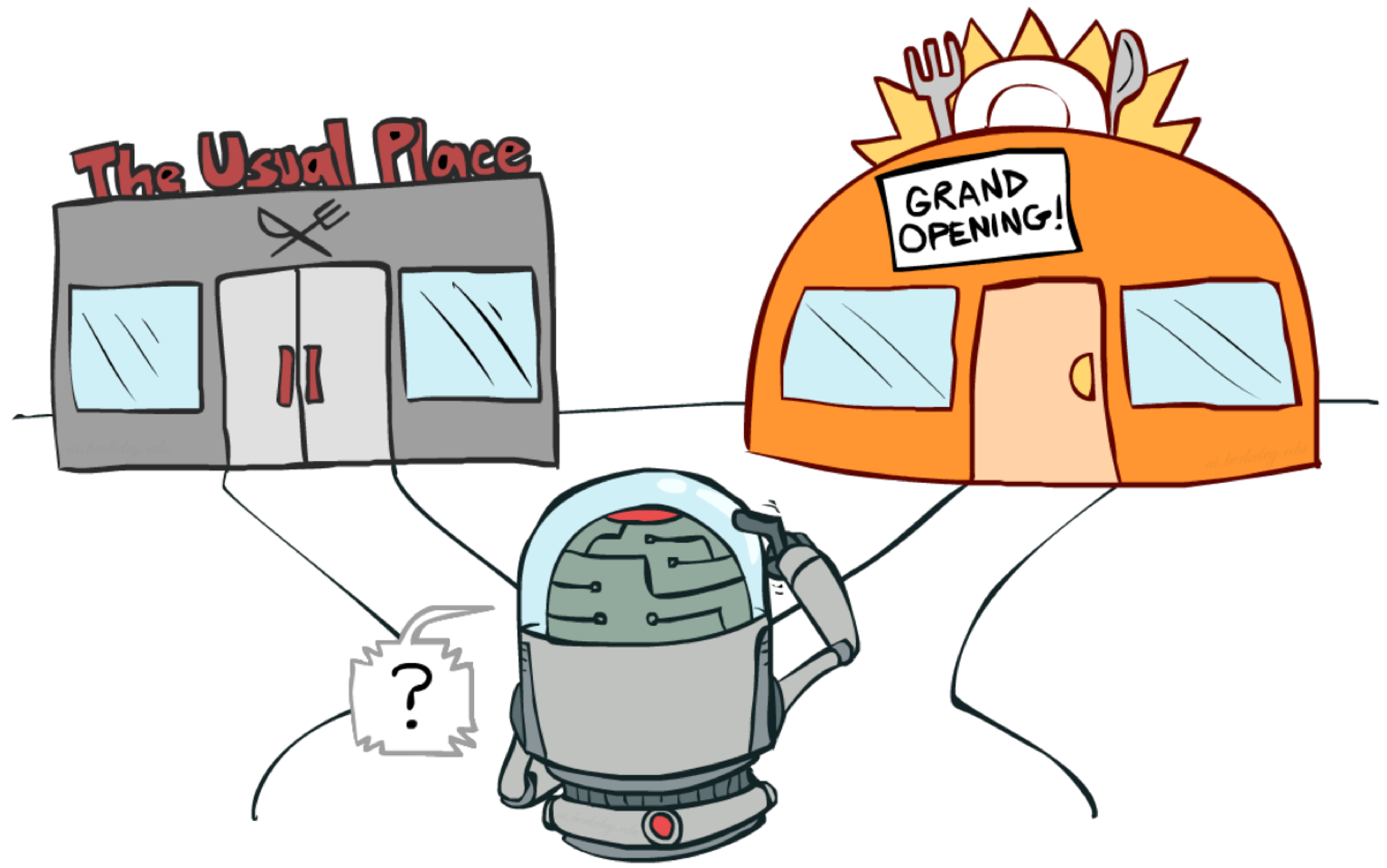
$$V(s_t) = V(s_t) + \alpha \, (r_{t+1} + \gamma \, V(s_{t+1}) - V(s_t)) = V(s_t) + \alpha \, \delta_t$$

- The estimates converge to:

$$V^\pi(s) = \mathbb{E}_\pi[r(s, a, s') + \gamma \, V^\pi(s')]$$

- By using an **estimate of the return** R_t instead of directly the return as in MC,
  - we **increase the bias** (estimates are always wrong, especially at the beginning of learning)
  - but we **reduce the variance**: only $r(s, a, s')$ is stochastic, not the value function V^π.
- We can therefore expect **less optimal solutions**, but we will also need **less samples**.
  - better **sample efficiency** than MC.
  - worse **convergence** (suboptimal).

Exploration-exploitation problem



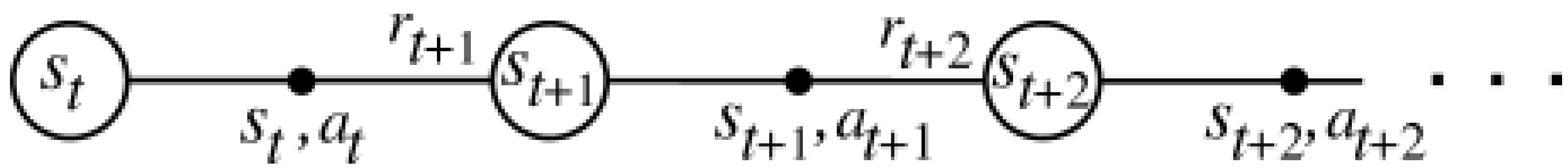
- Q-values can be estimated in the same way:

$$Q(s_t, a_t) = Q(s_t, a_t) + \alpha \, (r_{t+1} + \gamma \, Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$$

- Like for MC, the exploration/exploitation trade-off has to be managed: what is the next action a_{t+1}?
- There are therefore two classes of TD control algorithms:
  - **on-policy** (SARSA)
  - **off-policy** (Q-learning).

SARSA: On-policy TD control

- **SARSA** (state-action-reward-state-action) updates the value of a state-action pair by using the predicted value of the next state-action pair according to the current policy.



- When arriving in s_{t+1} from (s_t, a_t), we already sample the next action:

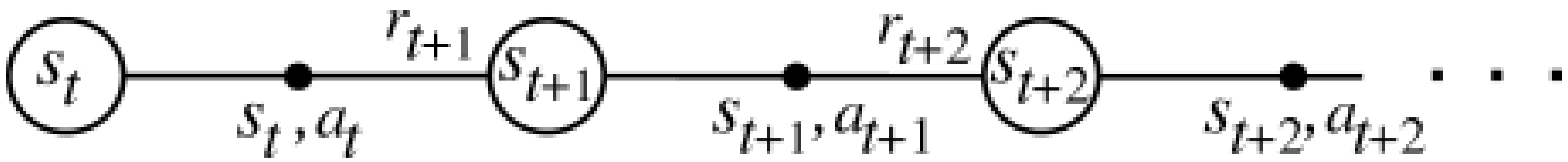
$$a_{t+1} \sim \pi(s_{t+1}, a)$$

- We can now update the value of (s_t, a_t):

$$Q(s_t, a_t) = Q(s_t, a_t) + \alpha \, (r_{t+1} + \gamma \, Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$$

- The next action a_{t+1} will **have to** be executed next: SARSA is **on-policy**. You cannot change your mind and execute another a_{t+1}.
- The learned policy must be ϵ-soft (stochastic) to ensure exploration.
- SARSA converges to the optimal policy if α is small enough and if ϵ (or τ) slowly decreases to 0.

SARSA: On-policy TD control



- **while** True:
  - Start from an initial state s_0 and select a_0 using the current policy π.
  - **foreach** step t of the episode:
    - Apply a_t, observe r_{t+1} and s_{t+1}.
    - Select a_{t+1} using the current **stochastic** policy π.
    - Update the action-value function of (s_t, a_t):

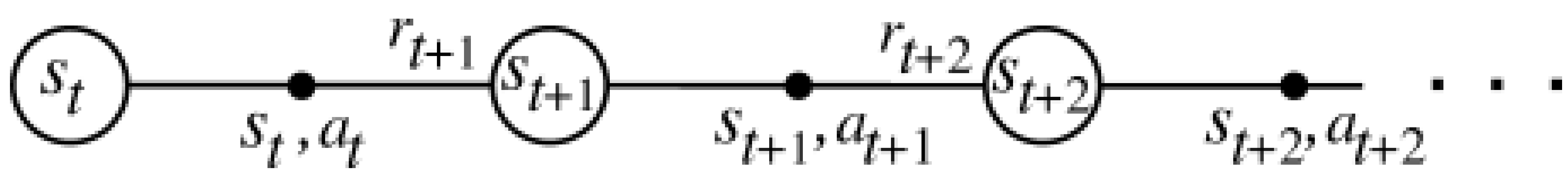
$$Q(s_t, a_t) = Q(s_t, a_t) + \alpha \, (r_{t+1} + \gamma \, Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$$

    - Improve the stochastic policy, e.g:

$$\pi(s_t, a) = \begin{cases} 1 - \epsilon \; \text{if} \; a = \text{argmax} \, Q(s_t, a) \\ \frac{\epsilon}{|\mathcal{A}(s_t) - 1|} \; \text{otherwise.} \end{cases}$$

    - **if** s_{t+1} is terminal: **break**

Q-learning: Off-policy TD control



- **Q-learning** directly approximates the optimal action-value function Q^* independently of the current policy, using the greedy action in the next state.

$$Q(s_t, a_t) = Q(s_t, a_t) + \alpha \, (r_{t+1} + \gamma \, \max_a Q(s_{t+1}, a) - Q(s_t, a_t))$$

- The next action a_{t+1} can be generated by a behavior policy: Q-learning is **off-policy**.
- The learned policy can be deterministic.
- The behavior policy can be an ϵ-soft policy derived from Q or expert knowledge.
- The behavior policy only needs to visit all state-action pairs during learning to ensure optimality.

Q-learning: Off-policy TD control

- **while** True:
  - Start from an initial state s_0.
  - **foreach** step t of the episode:
    - Select a_t using the behavior policy b (e.g. derived from π).
    - Apply a_t, observe r_{t+1} and s_{t+1}.
    - Update the action-value function of (s_t, a_t):

$$Q(s_t, a_t) = Q(s_t, a_t) + \alpha \left(r_{t+1} + \gamma \, \max_a Q(s_{t+1}, a) - Q(s_t, a_t)\right)$$

    - Improve greedily the learned policy:

$$\pi(s_t, a) = \begin{cases} 1 \text{ if } a = \text{argmax} \, Q(s_t, a) \\ 0 \text{ otherwise.} \end{cases}$$

    - **if** s_{t+1} is terminal: **break**

No need for importance sampling in Q-learning

- In off-policy Monte-Carlo, Q-values are estimated using the return of the rest of the episode on average:

$$Q^\pi(s, a) = \mathbb{E}_{\tau \sim \rho_b}[\rho_{0:T-1} \, R(\tau) | s_0 = s, a_0 = a]$$

- As the rest of the episode is generated by b, we need to correct the returns using the importance sampling weight.
- In Q-learning, Q-values are estimated using other estimates:

$$Q^\pi(s, a) = \mathbb{E}_{s_t \sim \rho_b, a_t \sim b}[r_{t+1} + \gamma \, \max_a Q^\pi(s_{t+1}, a) | s_t = s, a_t = a]$$

- As we only sample **transitions** using b and not episodes, there is no need to correct the returns:
  - The returns use estimates Q^π, which depend on π and not b.
  - The immediate reward r_{t+1} is stochastic, but is the same whether you sample a_t from π or from b.

Temporal Difference learning

- **Temporal Difference** allow to learn Q-values from single transitions instead of complete episodes.
- MC methods can only be applied to episodic problems, while TD works for continuing tasks.
- MC and TD methods are **model-free**: you do not need to know anything about the environment ($p(s'|s,a)$ and $r(s,a,s')$) to learn.
- The **exploration-exploitation** dilemma must be dealt with:
  - **On-policy** TD (SARSA) follows the learned stochastic policy.

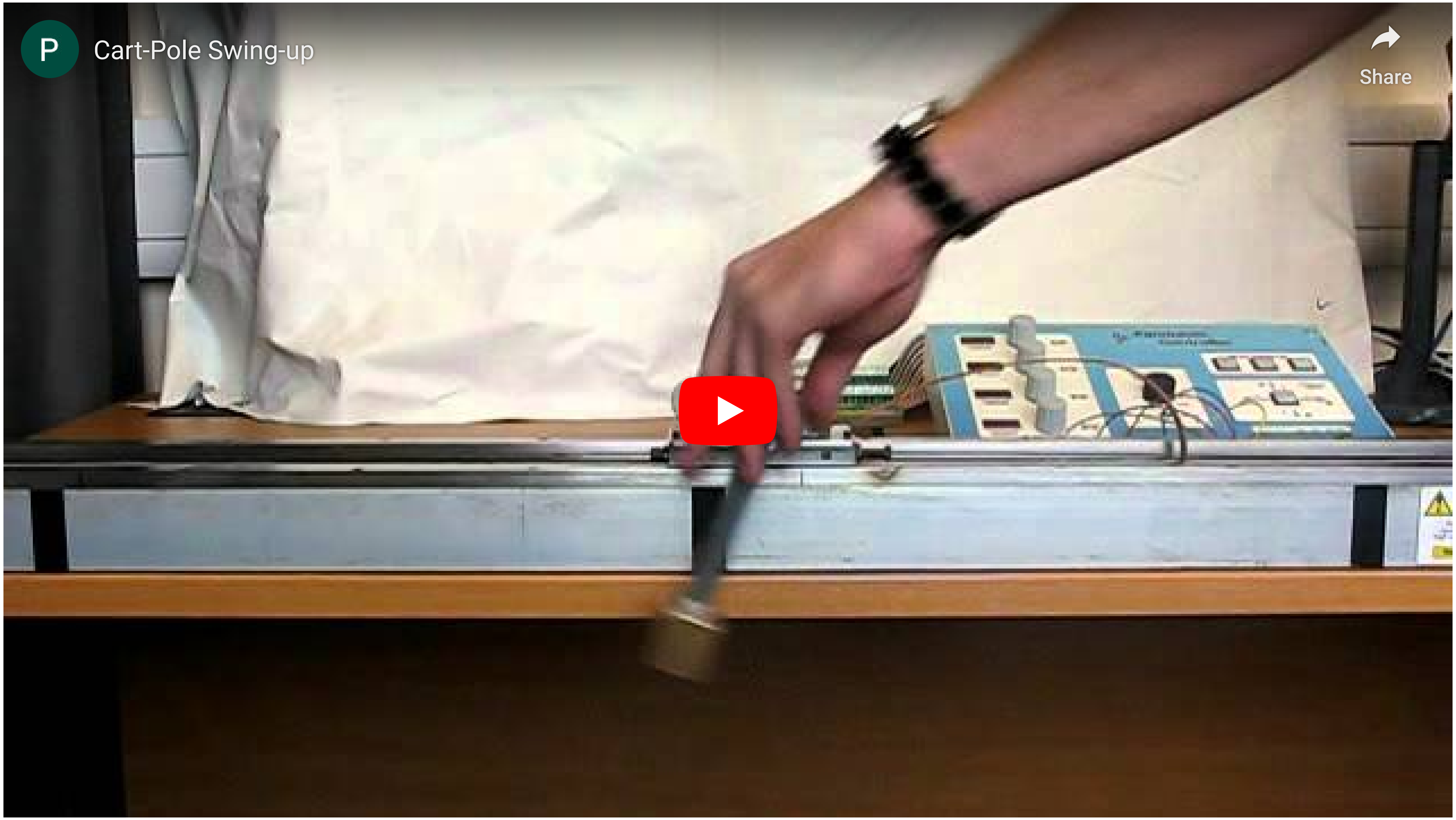
$$Q(s,a) = Q(s,a) + \alpha\,(r(s,a,s') + \gamma\,Q(s',a') - Q(s,a))$$

  - **Off-policy** TD (Q-learning) follows a behavior policy and learns a deterministic policy.

$$Q(s,a) = Q(s,a) + \alpha\,(r(s,a,s') + \gamma\,\max_a Q(s',a) - Q(s,a))$$

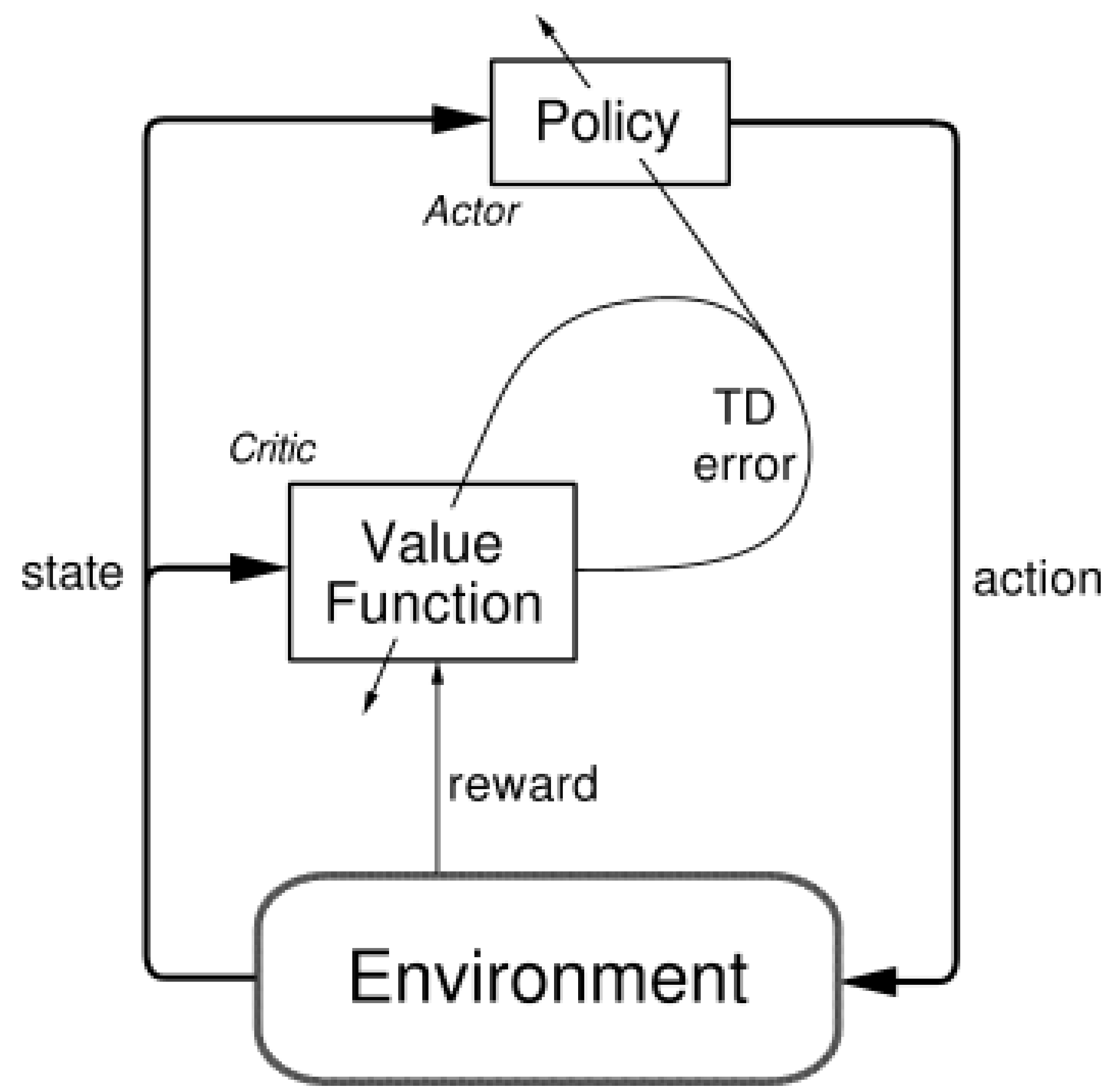
- TD uses **bootstrapping** like DP: it uses other estimates to update one estimate.
- Q-learning is the go-to method in tabular RL.

Optimal control with Q-learning



2 - Actor-critic methods

Actor-critic methods



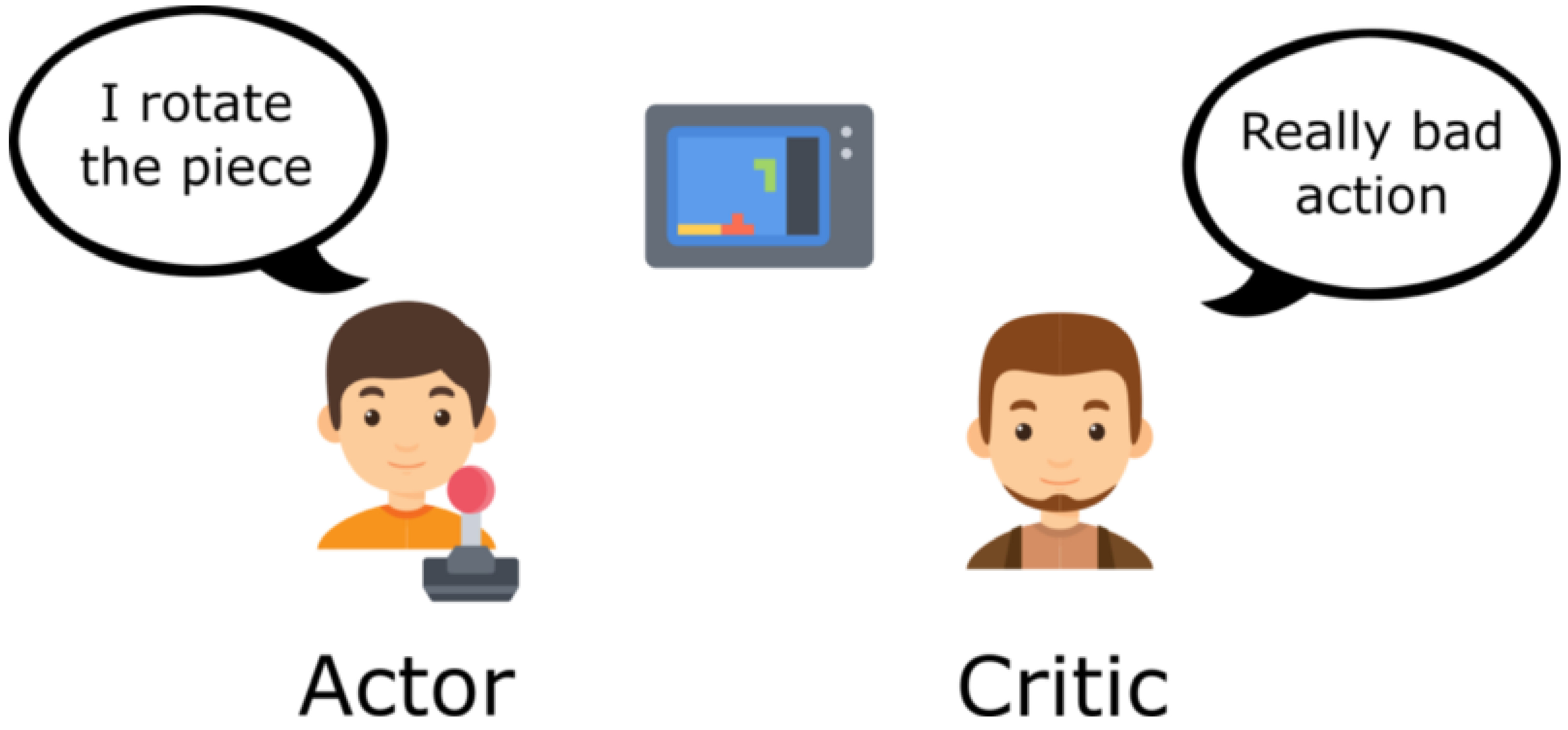
- Actor-critic methods are TD methods that have a separate memory structure to explicitly represent the policy independent of the value function.
- The policy π is implemented by the **actor**, because it is used to select actions.
- The estimated values $V(s)$ are implemented by the **critic**, because it criticizes the actions made by the actor.
- Learning is always **on-policy**: the critic must learn about and critique whatever policy is currently being followed by the actor.

- The critic computes the **TD error** or **1-step advantage**:

$$\delta_t = r_{t+1} + \gamma\, V(s_{t+1}) - V(s_t)$$

- This scalar signal is the output of the critic and drives learning in both the actor and the critic.

Actor-critic methods



Source: https://www.freecodecamp.org/news/an-intro-to-advantage-actor-critic-methods-lets-play-sonic-the-hedgehog-86d6240171d/

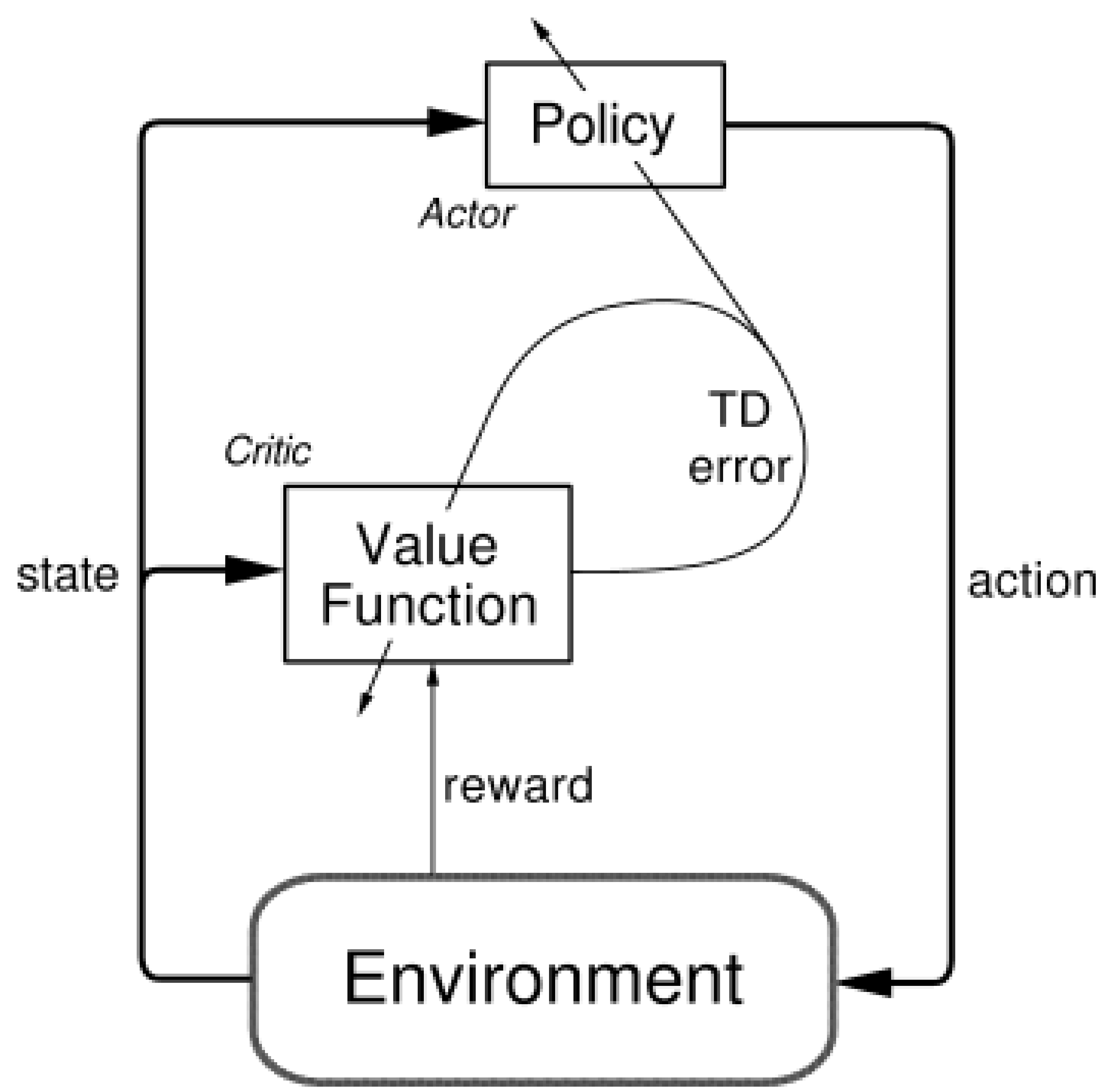
- The TD error after each transition $(s_t, a_t, r_{t+1}, s_{t+1})$:

$$\delta_t = r_{t+1} + \gamma V(s_{t+1}) - V(s_t)$$

tells us how good the action a_t was compared to our expectation $V(s_t)$.

- When the advantage $\delta_t > 0$, this means that the action lead to a better reward or a better state than what was expected by $V(s_t)$, which is a **good surprise**, so the action should be reinforced (selected again) and the value of that state increased.
- When $\delta_t < 0$, this means that the previous estimation of (s_t, a_t) was too high (**bad surprise**), so the action should be avoided in the future and the value of the state reduced.

Actor-critic methods



- TD error after each transition:

$$\delta_t = r_{t+1} + \gamma V(s_{t+1}) - V(s_t)$$

- The critic is updated using this scalar signal:

$$V(s_t) \leftarrow V(s_t) + \alpha\, \delta_t$$

- The actor is updated according to this TD error signal. For example a softmax actor over preferences:

$$p(s_t, a_t) \leftarrow p(s_t, a_t) + \beta\, \delta_t$$

$$\pi(s, a) = \frac{\exp p(s, a)}{\sum_b \exp p(s, b)}$$

- When $\delta_t > 0$, the preference is increased, so the probability of selecting it again increases.
- When $\delta_t < 0$, the preference is decreased, so the probability of selecting it again decreases.
- This is the equivalent of **reinforcement comparison** for bandits.

Actor-critic algorithm with preferences

- Start in s_0. Initialize the preferences $p(s, a)$ for each state action pair and the critic $V(s)$ for each state.
- **foreach** step t:
    - Select a_t using the **actor** π in state s_t:

$$\pi(s_t, a) = \frac{\exp p(s, a)}{\sum_b \exp p(s, b)}$$

    - Apply a_t, observe r_{t+1} and s_{t+1}.
    - Compute the TD error in s_t using the **critic**:

$$\delta_t = r_{t+1} + \gamma \, V(s_{t+1}) - V(s_t)$$

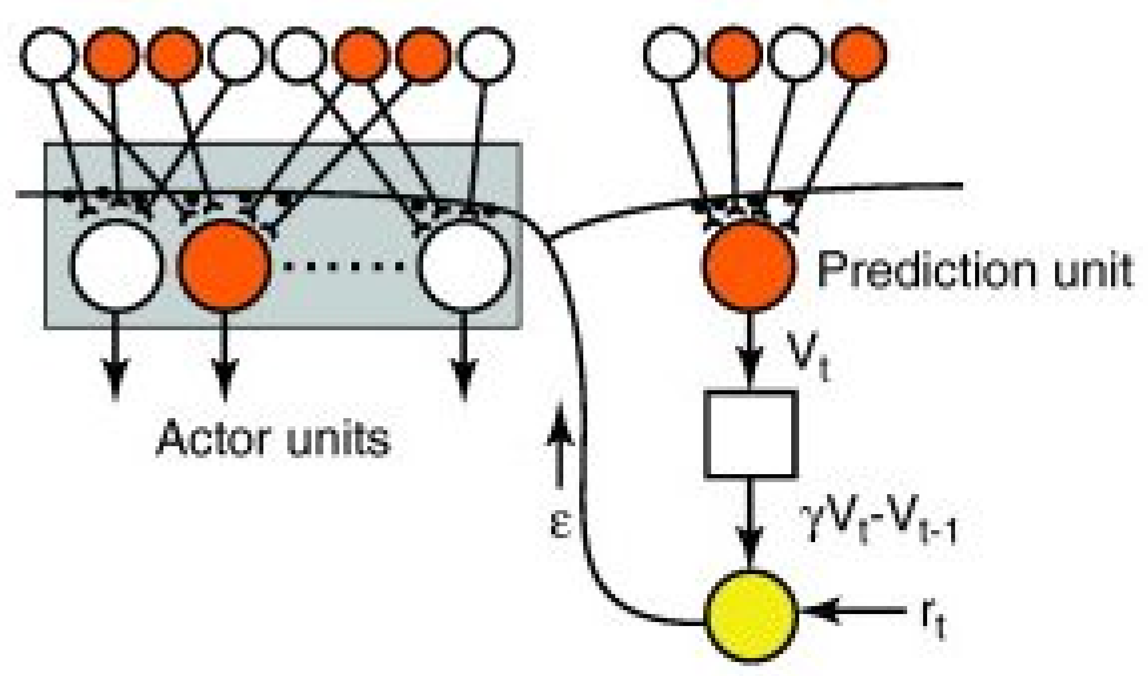
    - Update the **actor**:

$$p(s_t, a_t) \leftarrow p(s_t, a_t) + \beta \, \delta_t$$

    - Update the **critic**:

$$V(s_t) \leftarrow V(s_t) + \alpha \, \delta_t$$

Actor-critic methods



- The advantage of the separation between the actor and the critic is that now the actor can take any form (preferences, linear approximation, deep networks).
- It requires minimal computation in order to select the actions, in particular when the action space is huge or even continuous.
- It can learn stochastic policies, which is particularly useful in non-Markov problems.

- **It is obligatory to learn on-policy:**
  - the critic must evaluate the actions taken by the current actor.
  - the actor must learn from the current critic, not "old" V-values.

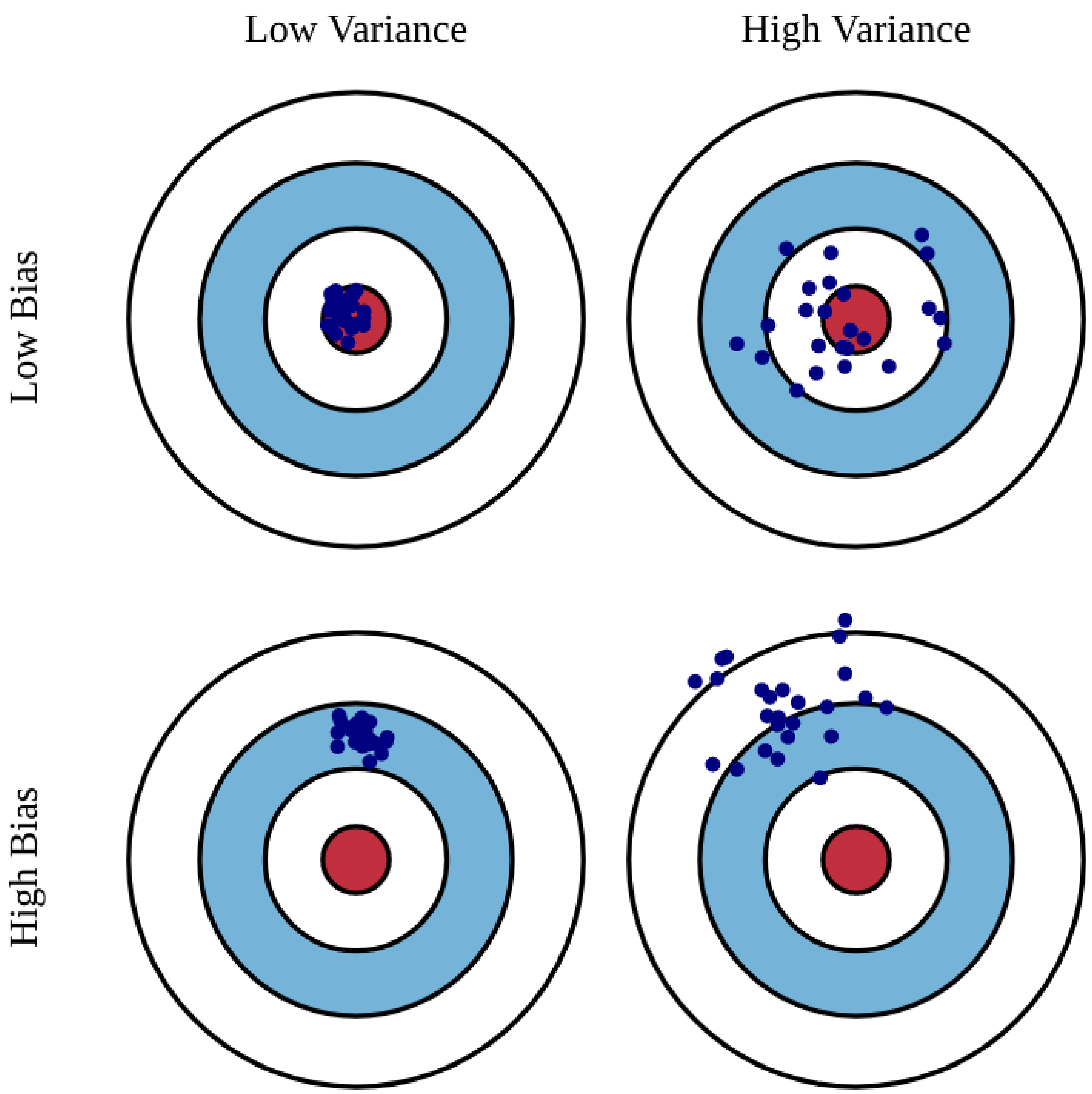
Actor-critic methods

- **Value-based** methods use value estimates $Q(s, a)$ to infer a policy:
  - **On-policy** methods learn and use a stochastic policy to explore.
  - **Off-policy** methods learn a deterministic policy but use a (stochastic) behavior policy to explore.
- **Policy-based** methods directly learn the policy $\pi(s, a)$ (**actor**) using preferences or function approximators.
  - A **critic** learning values is used to improve the policy w.r.t a performance baseline.
  - Actor-critic architectures are strictly **on-policy**.

	Bandits	**MDP**
Value-based		
On-policy	ϵ-greedy, softmax	SARSA
Off-policy	greedy	Q-learning
Policy-based		
On-policy	Reinforcement comparison	Actor-critic

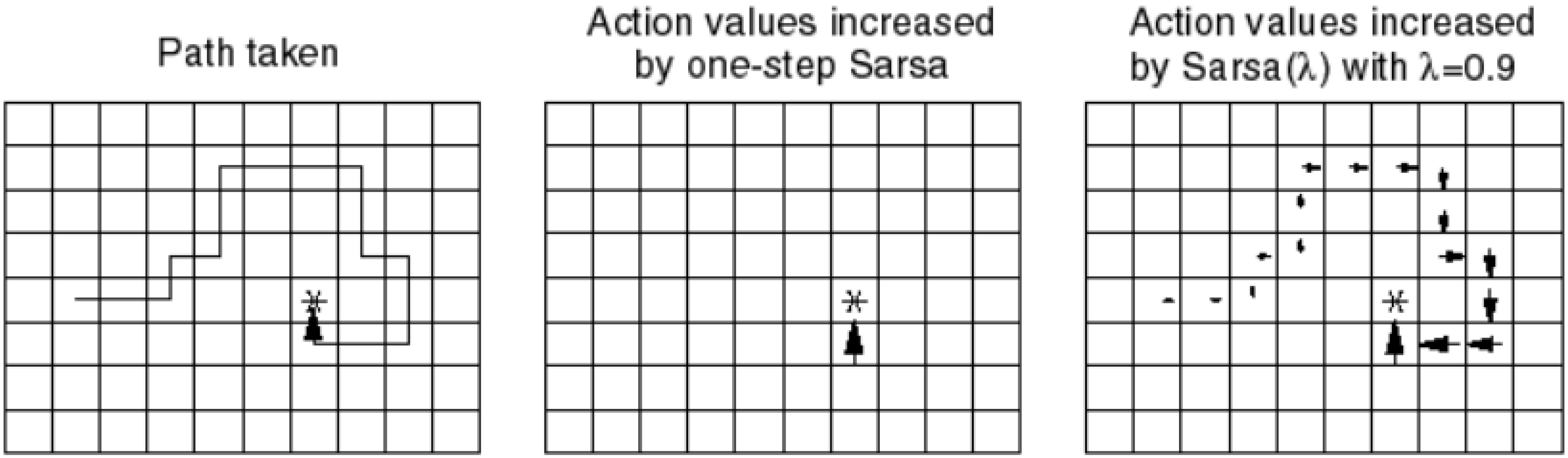
3 - Eligibility traces and advantage estimation

Bias-variance trade-off



- MC has **high variance, zero bias**:
  - Good convergence properties. We are more likely to find the optimal policy.
  - Not very sensitive to initial estimates.
  - Very simple to understand and use.
- TD has **low variance, some bias**:
  - Usually more **sample efficient** than MC.
  - TD(0) converges to $V^\pi(s)$ (but not always with function approximation). The policy might be suboptimal.
  - More sensitive to initial values (bootstrapping).

Drawback of learning from single transitions

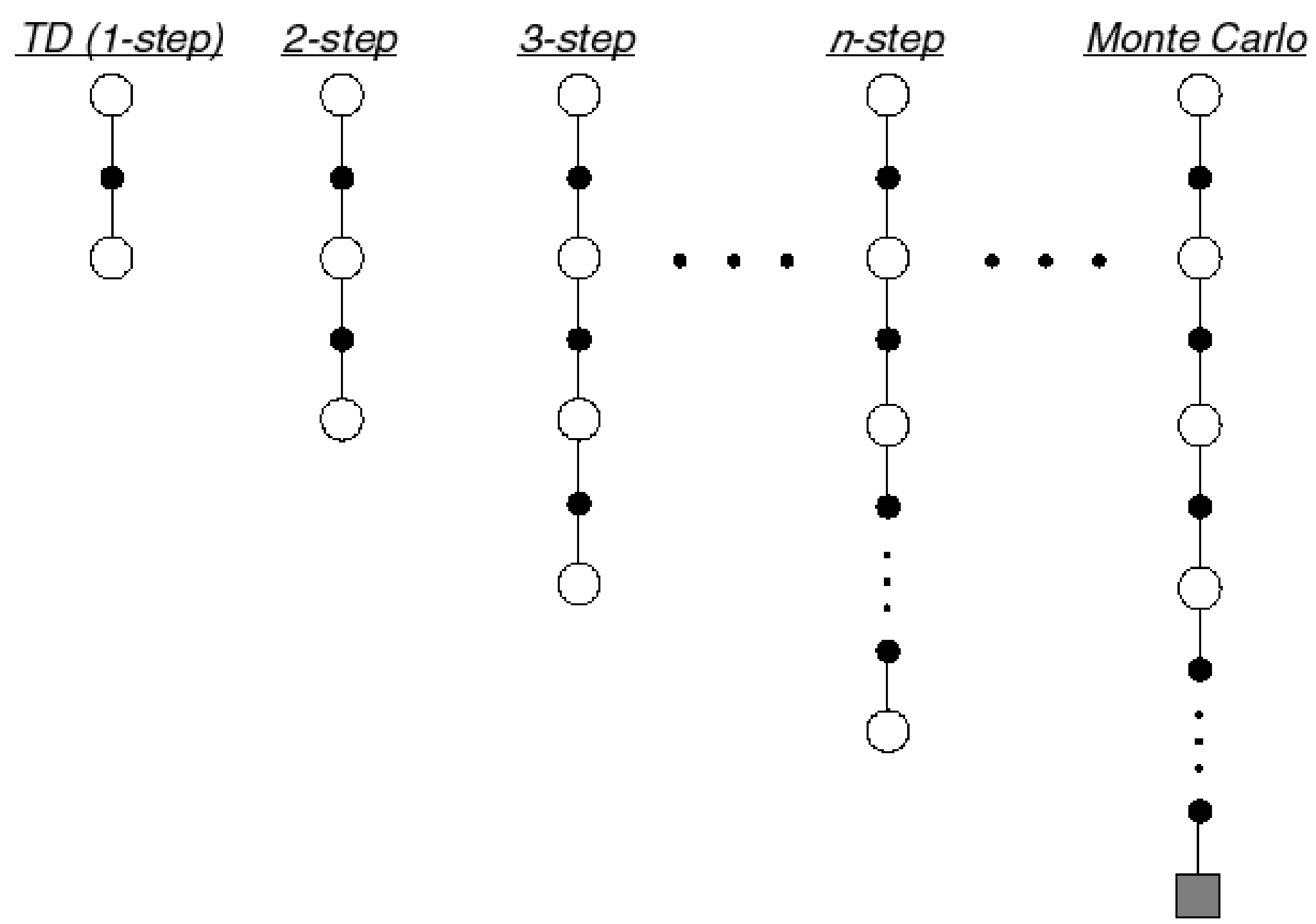


- When the reward function is sparse (e.g. only at the end of a game), only the last action, leading to that reward, will be updated the first time in TD.

$$Q(s, a) = Q(s, a) + \alpha \, (r(s, a, s') + \gamma \, \max_a Q(s', a) - Q(s, a))$$

- The previous actions, which were equally important in obtaining the reward, will only be updated the next time they are visited.
- This makes learning very slow: if the path to the reward has n steps, you will need to repeat the same episode at least n times to learn the Q-value of the first action.

n-step advantage



- Optimally, we would like a trade-off between:
  - TD (only one state/action is updated each time, small variance but significant bias)
  - Monte-Carlo (all states/actions in an episode are updated, no bias but huge variance).
- In **n-step TD prediction**, the next n rewards are used to estimate the return, the rest is approximated.

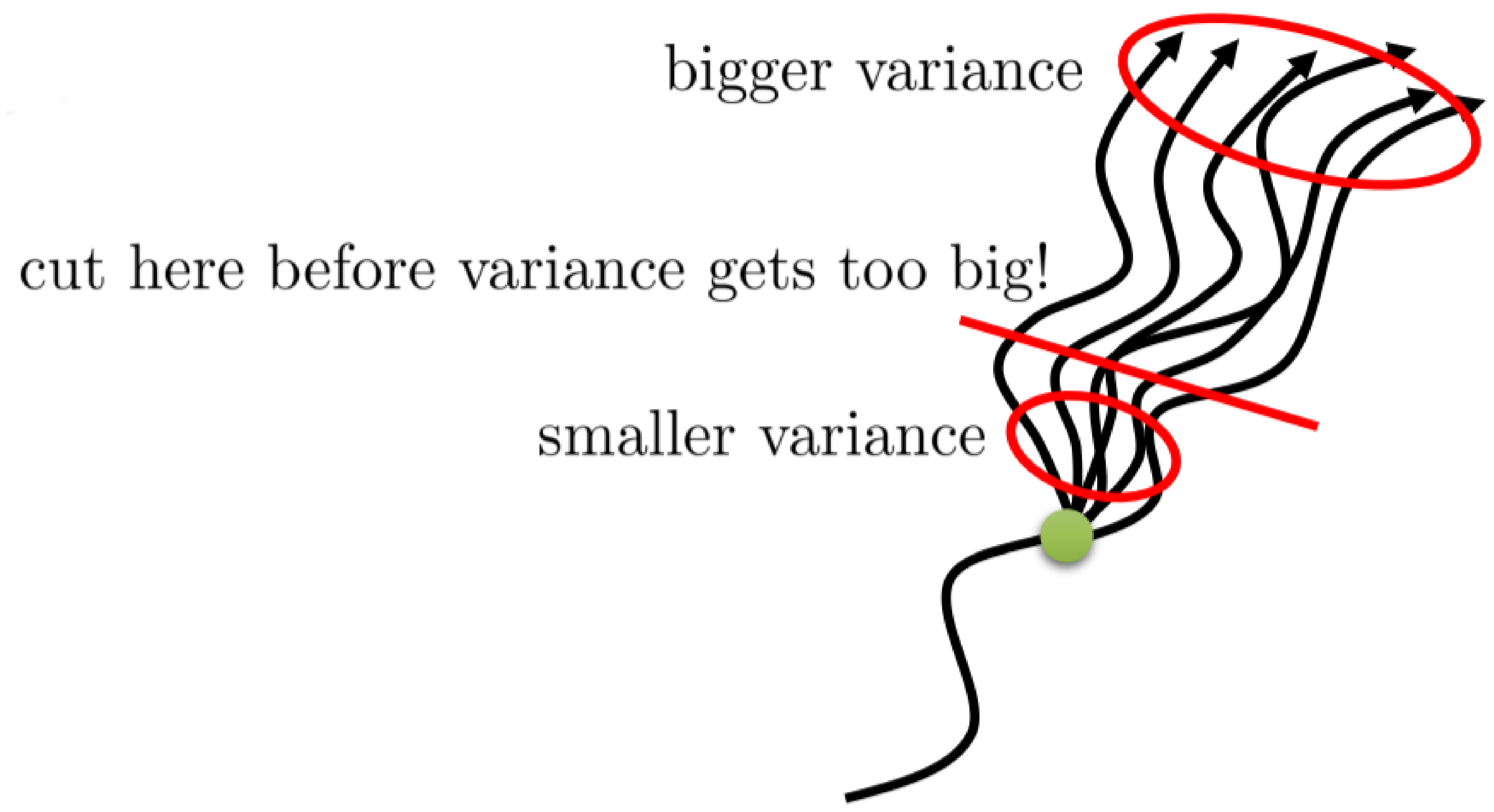
- The **n-step return** is the discounted sum of the n next rewards is computed as in MC plus the predicted value at step $t + n$ which replaces the rest as in TD.

$$R_t^n = \sum_{k=0}^{n-1} \gamma^k \, r_{t+k+1} + \gamma^n \, V(s_{t+n})$$

- We can update the value of the state with this n-step return:

$$V(s_t) = V(s_t) + \alpha \, (R_t^n - V(s_t))$$

n-step advantage



Credit: S. Levine

- The **n-step advantage** at time t is:

$$A_t^n = \sum_{k=0}^{n-1} \gamma^k \, r_{t+k+1} + \gamma^n \, V(s_{t+n}) - V(s_t)$$

- It is easy to check that the **TD error** is the 1-step advantage:

$$\delta_t = A_t^1 = r_{t+1} + \gamma \, V(s_{t+1}) - V(s_t)$$

- As you use more “real” rewards, you **reduce the bias** of Q-learning.
- As you use estimates for the rest of the episode, you **reduce the variance** of MC methods.
- But how to choose n?

Eligibility traces : forward view

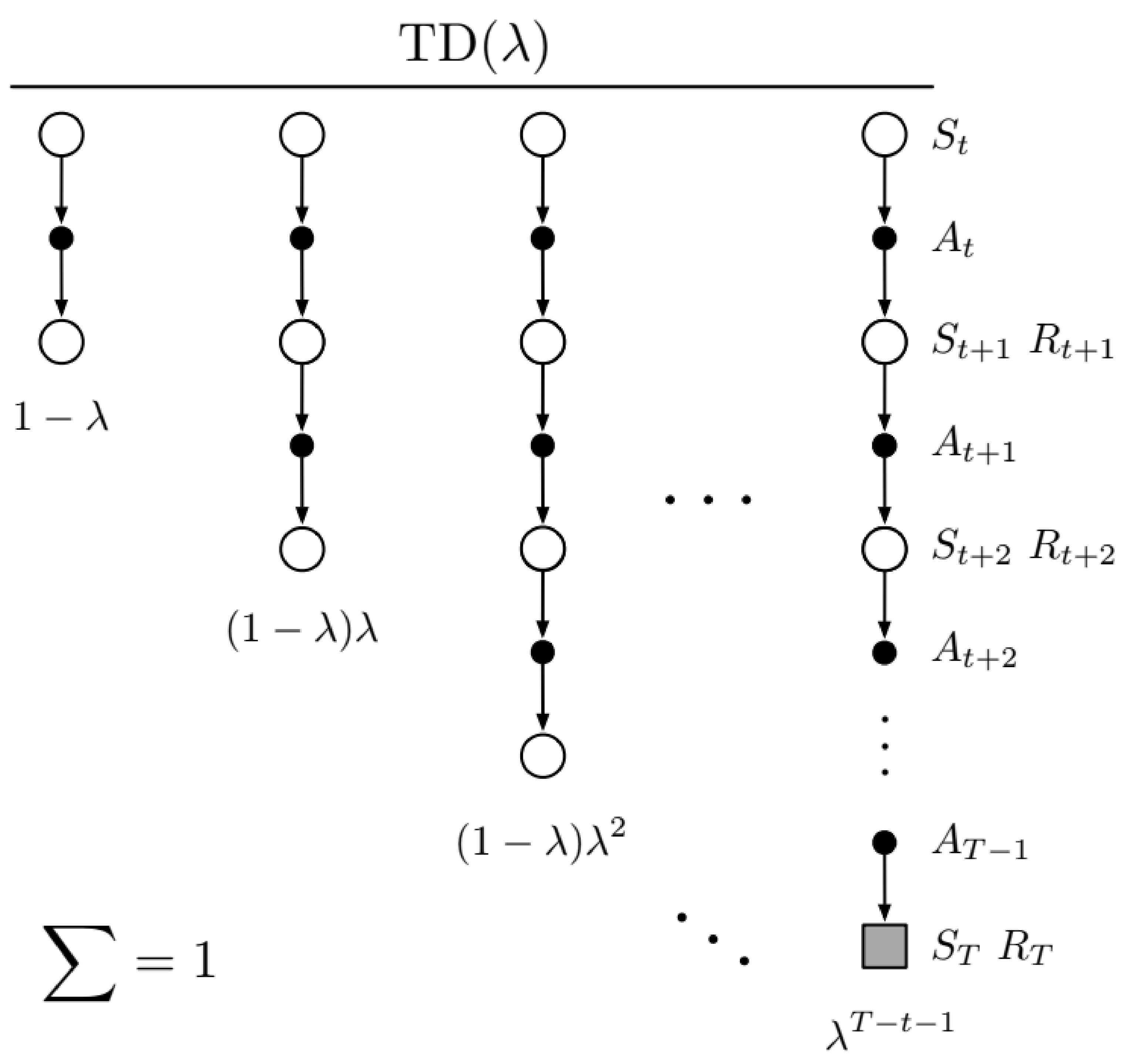
- One solution is to **average** the n-step returns, using a discount factor λ :

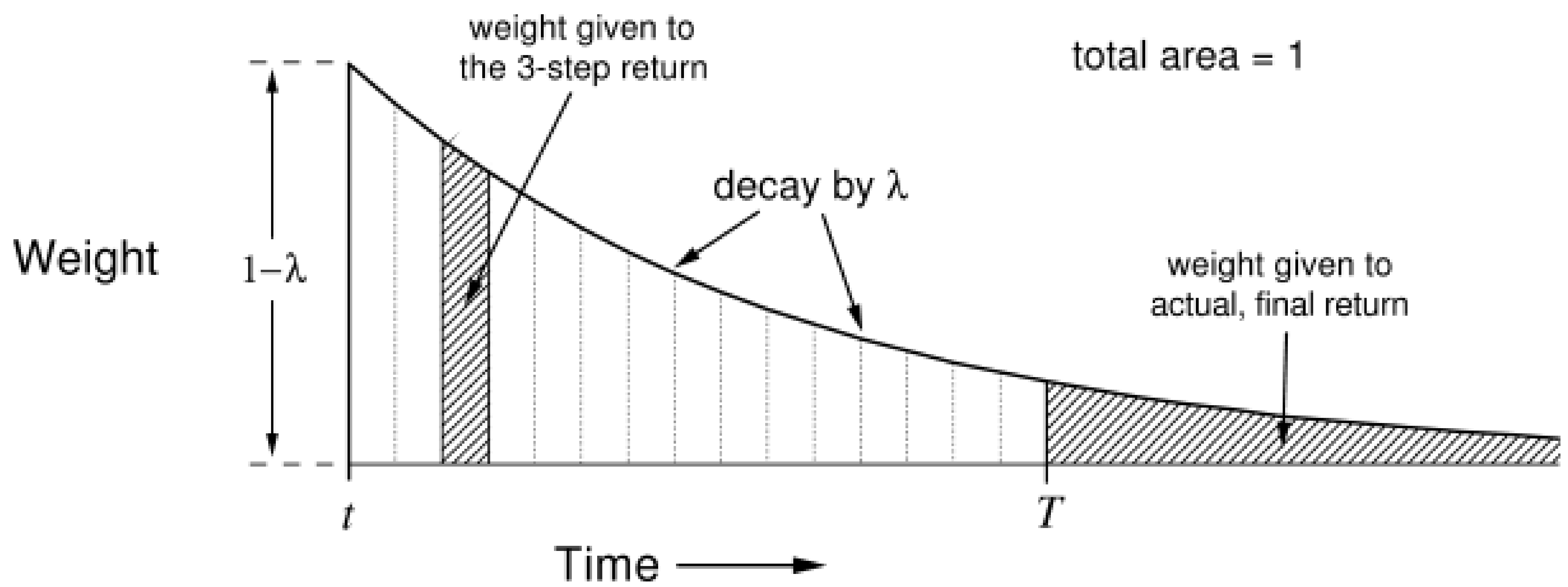
$$R_t^\lambda = (1 - \lambda) \sum_{n=1}^{\infty} \lambda^{n-1} R_t^n$$

- The term $1 - \lambda$ is there to ensure that the coefficients λ^{n-1} sum to one.

$$\sum_{n=1}^{\infty} \lambda^{n-1} = \frac{1}{1 - \lambda}$$

- Each reward r_{t+k+1} will count multiple times in the λ-return. Distant rewards are discounted by λ^k in addition to γ^k.
- Large n-step returns (MC) should not have as much importance as small ones (TD), as they have a high variance.



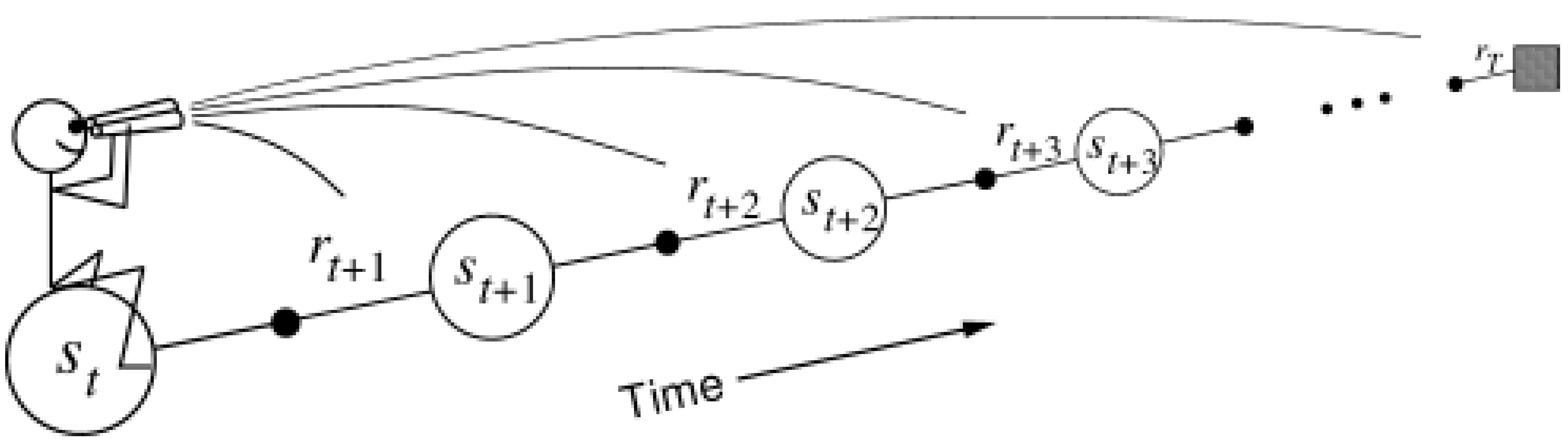


Eligibility traces : forward view

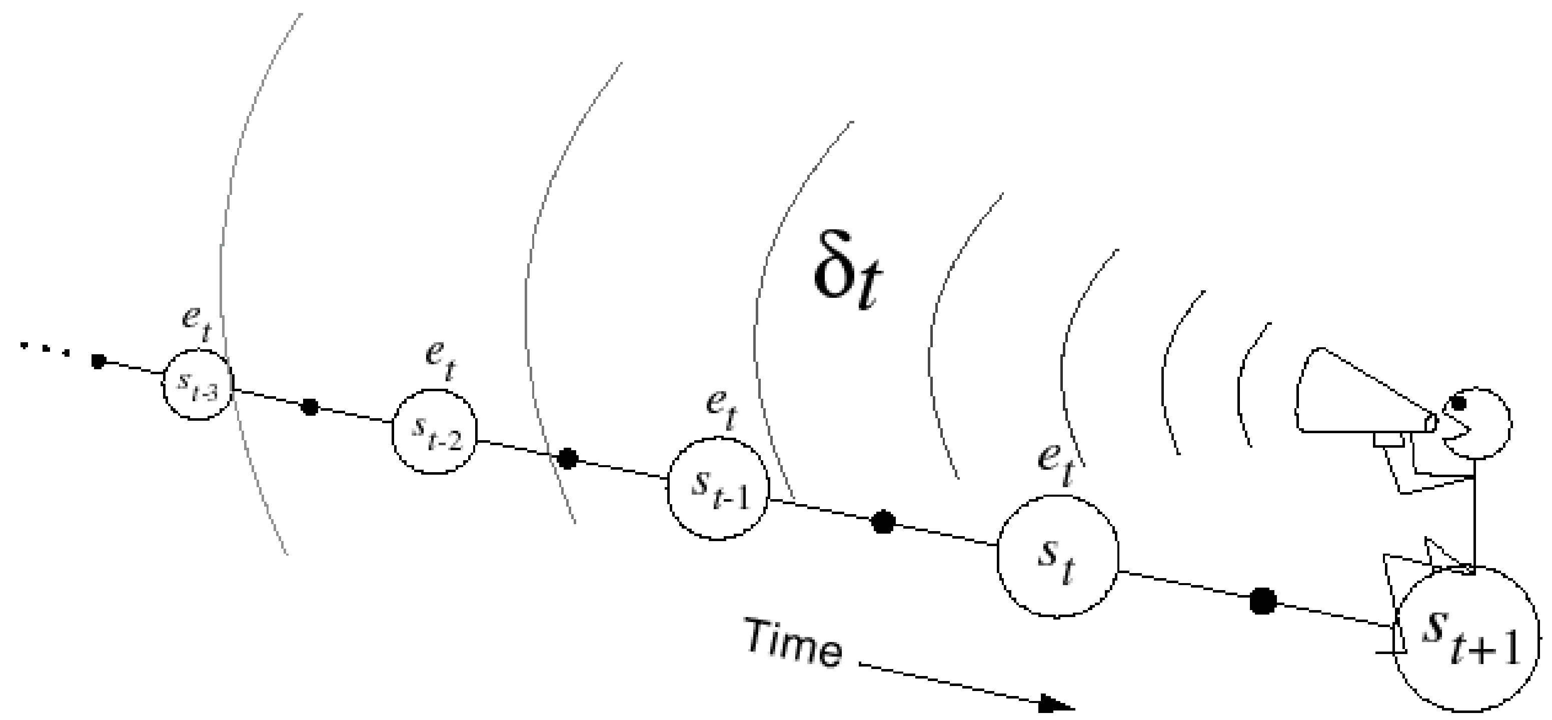
- To understand the role of λ, let's split the infinite sum w.r.t the end of the episode at time T. n-step returns with $n \geq T$ all have a MC return of R_t:

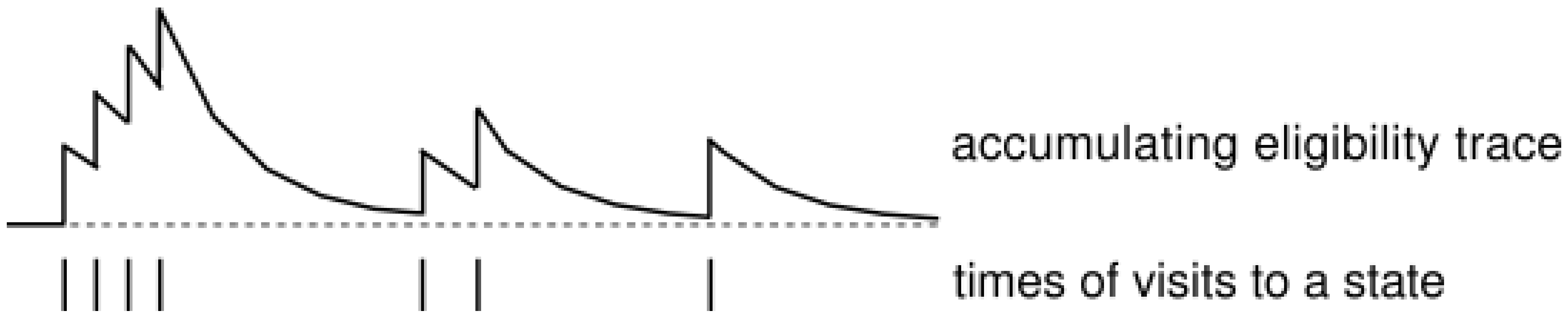
$$R_t^\lambda = (1 - \lambda) \sum_{n=1}^{T-t-1} \lambda^{n-1} \, R_t^n + \lambda^{T-t-1} \, R_t$$

- λ controls the bias-variance trade-off:
  - If $\lambda = 0$, the λ-return is equal to $R_t^1 = r_{t+1} + \gamma \, V(s_{t+1})$, i.e. TD: high bias, low variance.
  - If $\lambda = 1$, the λ-return is equal to $R_t = \sum_{k=0}^\infty \gamma^k \, r_{t+k+1}$, i.e. MC: low bias, high variance.
- This **forward view** of eligibility traces implies to look at all future rewards until the end of the episode to perform a value update. This prevents online learning using single transitions.



Eligibility traces : backward view





- Another view on eligibility traces is that the **TD reward prediction error** at time t is sent backwards in time:

$$\delta_t = r_{t+1} + \gamma V(s_{t+1}) - V(s_t)$$

- Every state s previously visited during the episode will be updated proportionally to the current TD error and an **eligibility trace** $e_t(s)$:

$$V(s) \leftarrow V(s) + \alpha \, \delta_t \, e_t(s)$$

- The eligibility trace defines since how long the state was visited:

$$e_t(s) = \begin{cases} \gamma \, \lambda \, e_{t-1}(s) \qquad \text{if} \quad s \neq s_t \\ e_{t-1}(s) + 1 \qquad \text{if} \quad s = s_t \end{cases}$$

- λ defines how important is a future TD error for the current state.

TD(λ) algorithm: policy evaluation

- **foreach** step t of the episode:
    - Select a_t using the current policy π in state s_t, observe r_{t+1} and s_{t+1}.
    - Compute the TD error in s_t:

    $$\delta_t = r_{t+1} + \gamma \, V_k(s_{t+1}) - V_k(s_t)$$

    - Increment the trace of s_t:

    $$e_{t+1}(s_t) = e_t(s_t) + 1$$

    - **foreach** state $s \in [s_o, \dots, s_t]$ in the episode:
        - Update the state value function:

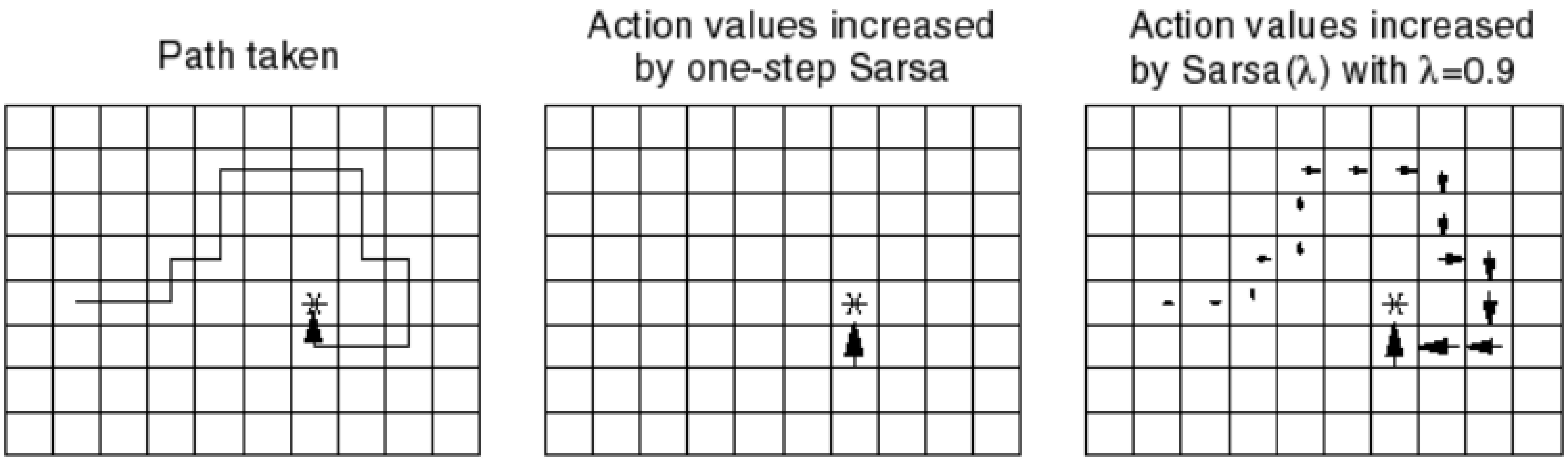
        $$V_{k+1}(s) = V_k(s) + \alpha \, \delta_t \, e_t(s)$$

        - Decay the eligibility trace:

        $$e_{t+1}(s) = \lambda \, \gamma \, e_t(s)$$

    - **if** s_{t+1} is terminal: **break**

Eligibility traces



- The backward view of eligibility traces can be applied on single transitions, given we know the history of visited states and maintain a trace for each of them.
- Eligibility traces are a very useful way to speed learning up in TD methods and control the bias/variance trade-off.
- This modification can be applied to all TD methods: TD(λ) for states, SARSA(λ) and Q(λ) for actions.
- The main drawback is that we need to keep a trace for ALL possible state-action pairs: memory consumption. Clever programming can limit this issue.
- The value of λ has to be carefully chosen for the problem: perhaps initial actions are random and should not be reinforced.
- If your problem is not strictly Markov (POMDP), eligibility traces can help as they update the history!

Generalized advantage estimation (GAE)

- The **n-step advantage** at time t:

$$A_t^n = \sum_{k=0}^{n-1} \gamma^k \, r_{t+k+1} + \gamma^n \, V(s_{t+n}) - V(s_t)$$

can be written as function of the TD error of the next n transitions:

$$A_t^n = \sum_{l=0}^{n-1} \gamma^l \, \delta_{t+l}$$

Proof with $n=2$:

$$A_t^2 = r_{t+1} + \gamma \, r_{t+2} + \gamma^2 \, V(s_{t+2}) - V(s_t)$$

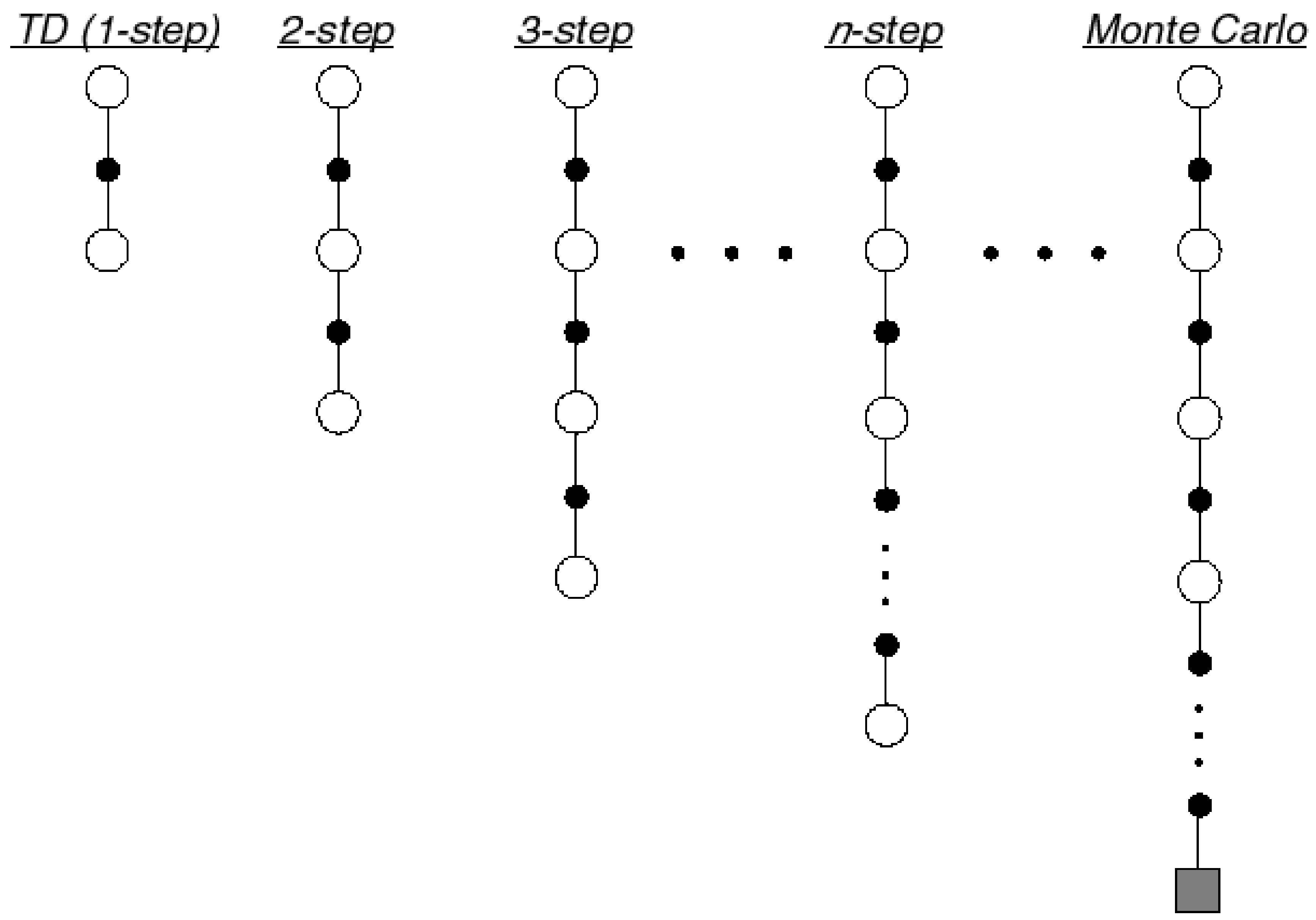
$$= (r_{t+1} - V(s_t)) + \gamma \, (r_{t+2} + \gamma \, V(s_{t+2}))$$

$$= (r_{t+1} + \gamma \, V(s_{t+1}) - V(s_t)) + \gamma \, (r_{t+2} + \gamma \, V(s_{t+2}) - V(s_{t+1}))$$

$$= \delta_t + \gamma \, \delta_{t+1}$$

Schulman, J., Moritz, P., Levine, S., Jordan, M., and Abbeel, P. (2015). High-Dimensional Continuous Control Using Generalized Advantage Estimation. arXiv:1506.02438.

Generalized advantage estimation (GAE)



- The **n-step advantage** realizes a bias/variance trade-off, but which value of n should we choose?

$$A_t^n = \sum_{k=0}^{n-1} \gamma^k \, r_{t+k+1} + \gamma^n \, V(s_{t+n}) - V(s_t)$$

- Schulman et al. (2015) proposed a **generalized advantage estimate** (GAE) $A_t^{\text{GAE}(\gamma, \lambda)}$ summing all possible n-step advantages with a discount parameter λ:

$$A_t^{\text{GAE}(\gamma, \lambda)} = (1 - \lambda) \sum_{n=1}^\infty \lambda^n \, A_t^n$$

- This is just a forward eligibility trace over distant n-step advantages: the 1-step advantage is more important the the 1000-step advantage (too much variance).
- We can show that the GAE can be expressed as a function of the future 1-step TD errors:

$$A_t^{\text{GAE}(\gamma, \lambda)} = \sum_{k=0}^\infty (\gamma \, \lambda)^k \, \delta_{t+k}$$

Schulman, J., Moritz, P., Levine, S., Jordan, M., and Abbeel, P. (2015). High-Dimensional Continuous Control Using Generalized Advantage Estimation. arXiv:1506.02438.

Generalized advantage estimation (GAE)

- **Generalized advantage estimate** (GAE) :

$$A_t^{\text{GAE}(\gamma, \lambda)} = (1 - \lambda) \sum_{n=1}^\infty \lambda^n \, A^n_t = \sum_{k=0}^\infty (\gamma \, \lambda)^k \, \delta_{t+k}$$

- The parameter λ controls the **bias-variance** trade-off.
- When $\lambda = 0$, the generalized advantage is the TD error:

$$A_t^{\text{GAE}(\gamma, 0)} = r_{t+1} + \gamma \, V(s_{t+1}) - V(s_t) = \delta_{t}$$

- When $\lambda = 1$, the generalized advantage is the MC advantage:

$$A_t^{\text{GAE}(\gamma, 1)} = \sum_{k=0}^\infty \gamma^k \, r_{t+k+1} - V(s_t) = R_t - V(s_t)$$

- Any value in between controls the bias-variance trade-off: from the high bias / low variance of TD to the small bias / high variance of MC.
- In practice, it leads to a better estimation than n-step advantages, but is more computationally expensive.

Schulman, J., Moritz, P., Levine, S., Jordan, M., and Abbeel, P. (2015). High-Dimensional Continuous Control Using Generalized Advantage Estimation. arXiv:1506.02438.