

Deep Reinforcement Learning

Successor representations

Julien Vitay

Professur für Künstliche Intelligenz - Fakultät für Informatik

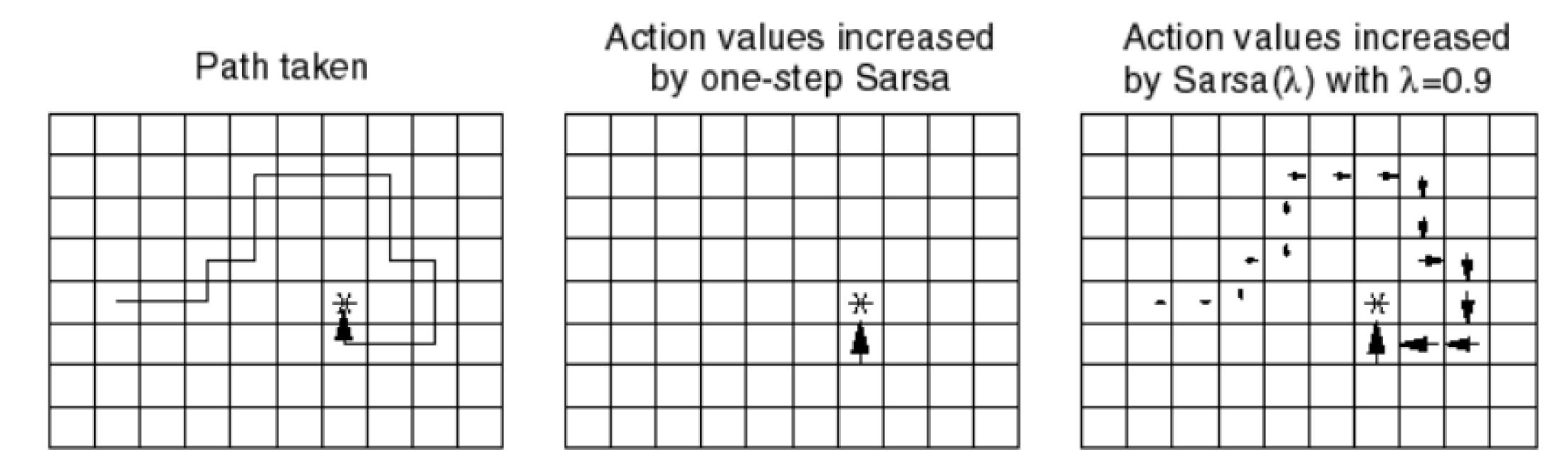
1 - Model-based vs. Model-free

Model-based vs. Model-free

Model-free methods use the reward prediction error (RPE) to update values:

$$egin{aligned} \delta_t &= r_{t+1} + \gamma \, V^\pi(s_{t+1}) - V^\pi(s_t) \ & \Delta V^\pi(s_t) = lpha \, \delta_t \end{aligned}$$

Encountered rewards propagate very slowly to all states and actions.



- If the environment changes (transition probabilities, rewards), they have to relearn everything.
- After training, selecting an action is very fast.

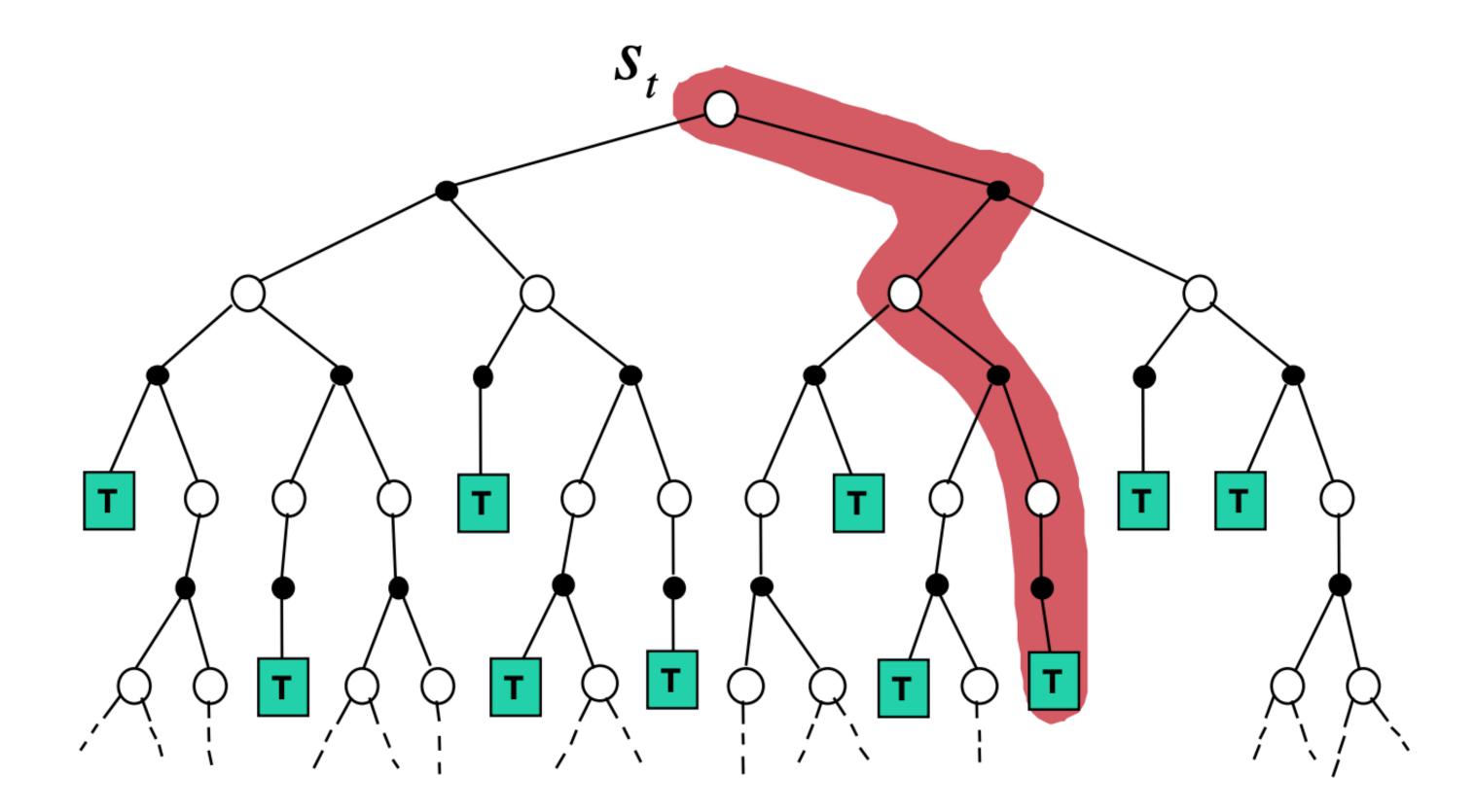
Model-based vs. Model-free

Model-based RL can learn very fast changes in the transition or reward distributions:

$$\Delta r(s_t, a_t, s_{t+1}) = lpha \left(r_{t+1} - r(s_t, a_t, s_{t+1})
ight)$$

$$\Delta p(s'|s_t,a_t) = lpha\left(\mathbb{I}(s_{t+1}=s') - p(s'|s_t,a_t)
ight)$$

• But selecting an action requires planning in the tree of possibilities (slow).



Model-based vs. Model-free

• Relative advantages of MF and MB methods:

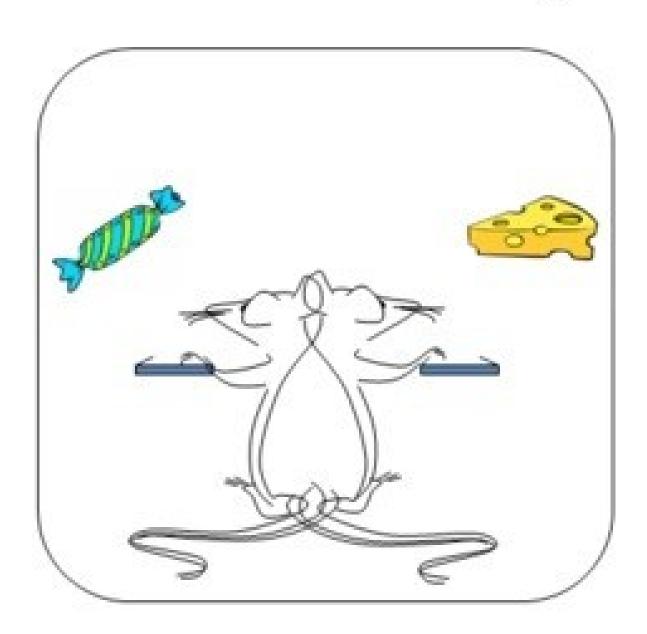
	Inference speed	Sample complexity	Optimality	Flexibility
Model-free	fast	high	yes	no
Model-based	slow	low	as good as the model	yes

• A trade-off would be nice... Most MB models in the deep RL literature are hybrid MB/MF models anyway.

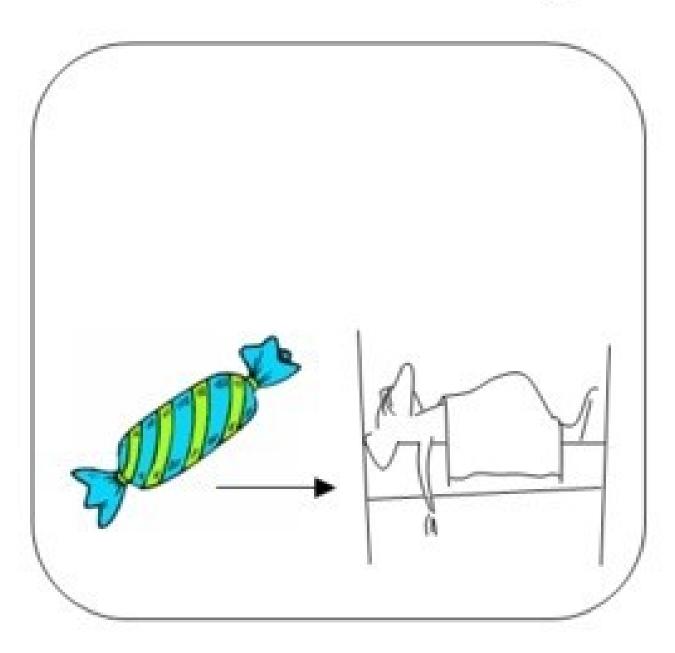
Outcome devaluation

- Two forms of behavior are observed in the animal psychology literature:
- 1. **Goal-directed** behavior learns Stimulus \rightarrow Response \rightarrow Outcome associations.
- 2. **Habits** are developed by overtraining Stimulus \rightarrow Response associations.
- The main difference is that habits are not influenced by **outcome devaluation**, i.e. when rewards lose their value.

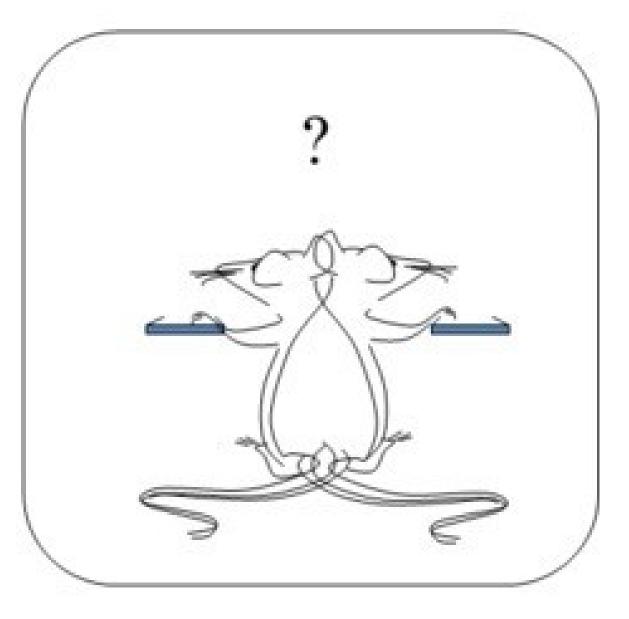
1. Instrumental Learning



2. Taste aversion learning



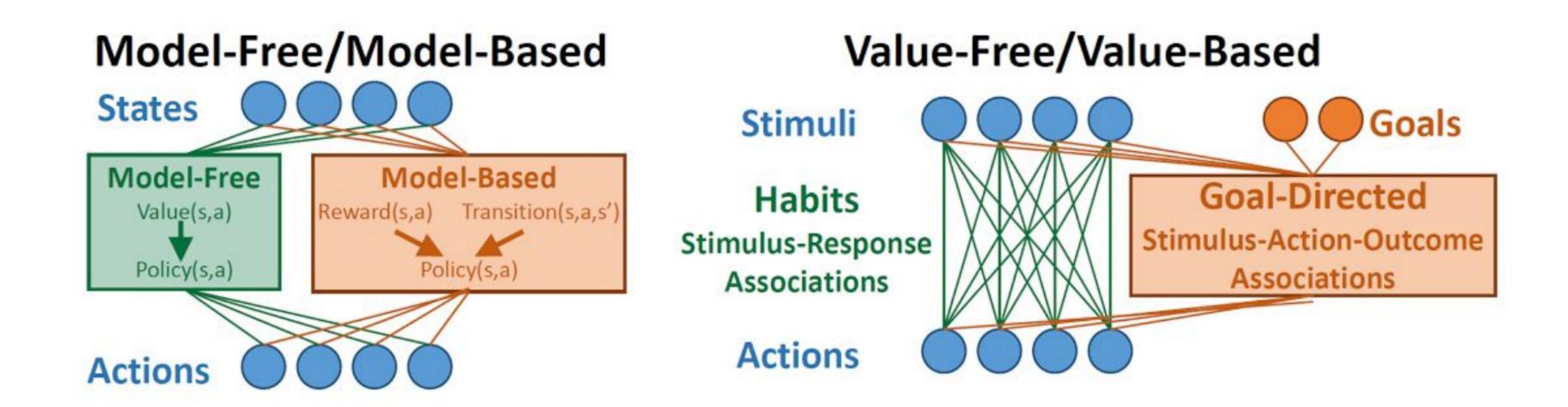
3. Test



Source: Bernard W. Balleine

Goal-directed / habits = MB / MF?

 The classical theory assigns MF to habits and MB to goal-directed, mostly because their sensitivity to outcome devaluation.



- The open question is the arbitration mechanism between these two segregated process: who takes control?
- Recent work suggests both systems are largely overlapping.

References

Doll, B. B., Simon, D. A., and Daw, N. D. (2012). The ubiquity of model-based reinforcement learning. Current Opinion in Neurobiology 22, 1075–1081. doi:10.1016/j.conb.2012.08.003.

Miller, K., Ludvig, E. A., Pezzulo, G., and Shenhav, A. (2018). "Re-aligning models of habitual and goal-directed decision-making," in Goal-Directed Decision Making: Computations and Neural Circuits, eds. A. Bornstein, R. W. Morris, and A. Shenhav (Academic Press)

2 - Successor representations

Successor representations (SR) have been introduced to combine MF and MB properties. Let's split the
definition of the value of a state:

$$V^\pi(s) = \mathbb{E}_\pi[\sum_{k=0}^\infty \gamma^k \, r_{t+k+1} | s_t = s]$$

(2)

$$egin{aligned} &= \mathbb{E}_{\pi} [egin{array}{c} 1 \ \gamma \ \gamma^2 \ \cdots \ \gamma^{\infty} \end{bmatrix} imes egin{bmatrix} \mathbb{I}(s_{t+1}) \ \mathbb{I}(s_{t+2}) \ \cdots \ \mathbb{I}(s_{\infty}) \end{bmatrix} imes egin{bmatrix} r_{t+1} \ r_{t+2} \ r_{t+3} \ \cdots \ r_{t+\infty} \end{bmatrix} |s_t = s] \end{aligned} \end{aligned}$$

where $\mathbb{I}(s_t)$ is 1 when the agent is in s_t at time t, 0 otherwise.

- The left part corresponds to the **transition dynamics**: which states will be visited by the policy, discounted by γ .
- The right part corresponds to the **immediate reward** in each visited state.
- Couldn't we learn the transition dynamics and the reward distribution separately in a model-free manner?

__

• SR rewrites the value of a state into an expected discounted future state occupancy $M^{\pi}(s,s')$ and an expected immediate reward r(s') by summing over all possible states s' of the MDP:

$$V^\pi(s) = \mathbb{E}_\pi[\sum_{k=0}^\infty \gamma^k \, r_{t+k+1} | s_t = s]$$

$$=\sum \mathbb{E}_{\pi}[\sum^{\infty}\gamma^{k}\,\mathbb{I}(s_{t+k}=s') imes r_{t+k+1}|s_{t}=s]$$
 (6)

(7)

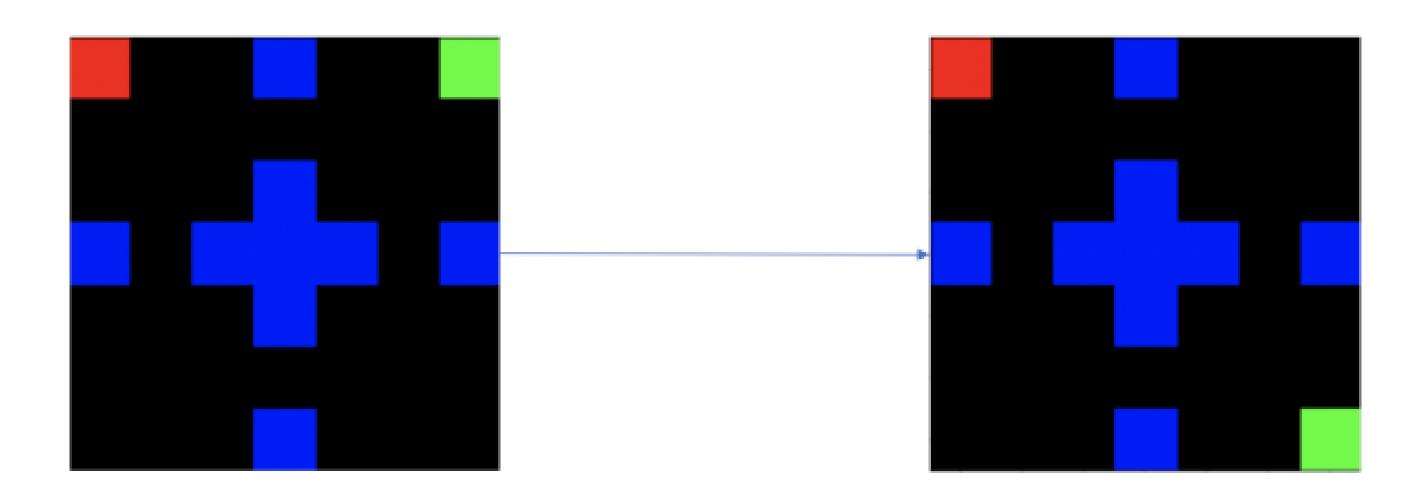
$$pprox \sum_{s' \in \mathcal{S}} \mathbb{E}_{\pi}[\sum_{k=0}^{\infty} \gamma^k \, \mathbb{I}(s_{t+k} = s') | s_t = s] imes \mathbb{E}[r_{t+1} | s_t = s']$$

(9)

$$pprox \sum_{s' \in \mathcal{S}} M^{\pi}(s,s') imes r(s')$$
 (10)

(5)

- The underlying assumption is that the world dynamics are independent from the reward function (which does not depend on the policy).
- This allows to re-use knowledge about world dynamics in other contexts (e.g. a new reward function in the same environment): **transfer learning**.



Source: https://awjuliani.medium.com/the-present-in-terms-of-the-future-successor-representations-in-reinforcement-learning-316b78c5fa3

- What matters is the states that you will visit and how interesting they are, not the order in which you visit them.
- Knowing that being in the mensa will eventually get you some food is enough to know that being in the mensa is a good state: you do not need to remember which exact sequence of transitions will put food in your mouth.

- SR algorithms must estimate two quantities:
 - 1. The **expected immediate reward** received after each state:

$$r(s) = \mathbb{E}[r_{t+1}|s_t=s]$$

2. The expected discounted future state occupancy (the SR itself):

$$M^\pi(s,s') = \mathbb{E}_\pi[\sum_{k=0}^\infty \gamma^k\, \mathbb{I}(s_{t+k}=s')|s_t=s]$$

• The value of a state *s* is then computed with:

$$V^{\pi}(s) = \sum_{s' \in \mathcal{S}} M(s,s') imes r(s')$$

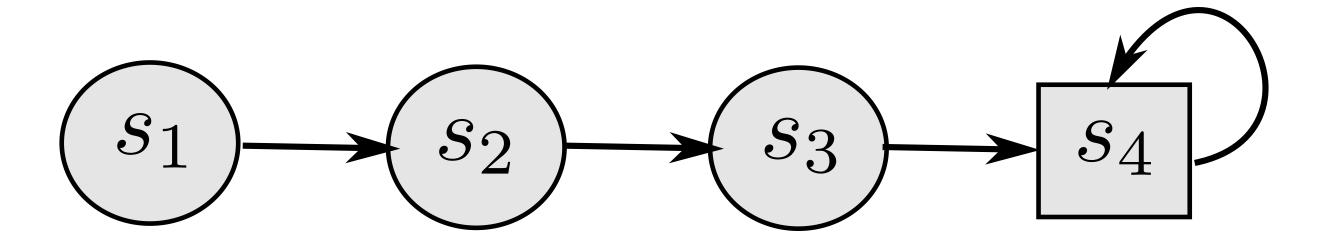
what allows to infer the policy (e.g. using an actor-critic architecture).

• The immediate reward for a state can be estimated very quickly and flexibly after receiving each reward:

$$\Delta \, r(s_t) = lpha \left(r_{t+1} - r(s_t)
ight)$$

SR and transition matrix

• Imagine a very simple MDP with 4 states and a single deterministic action:



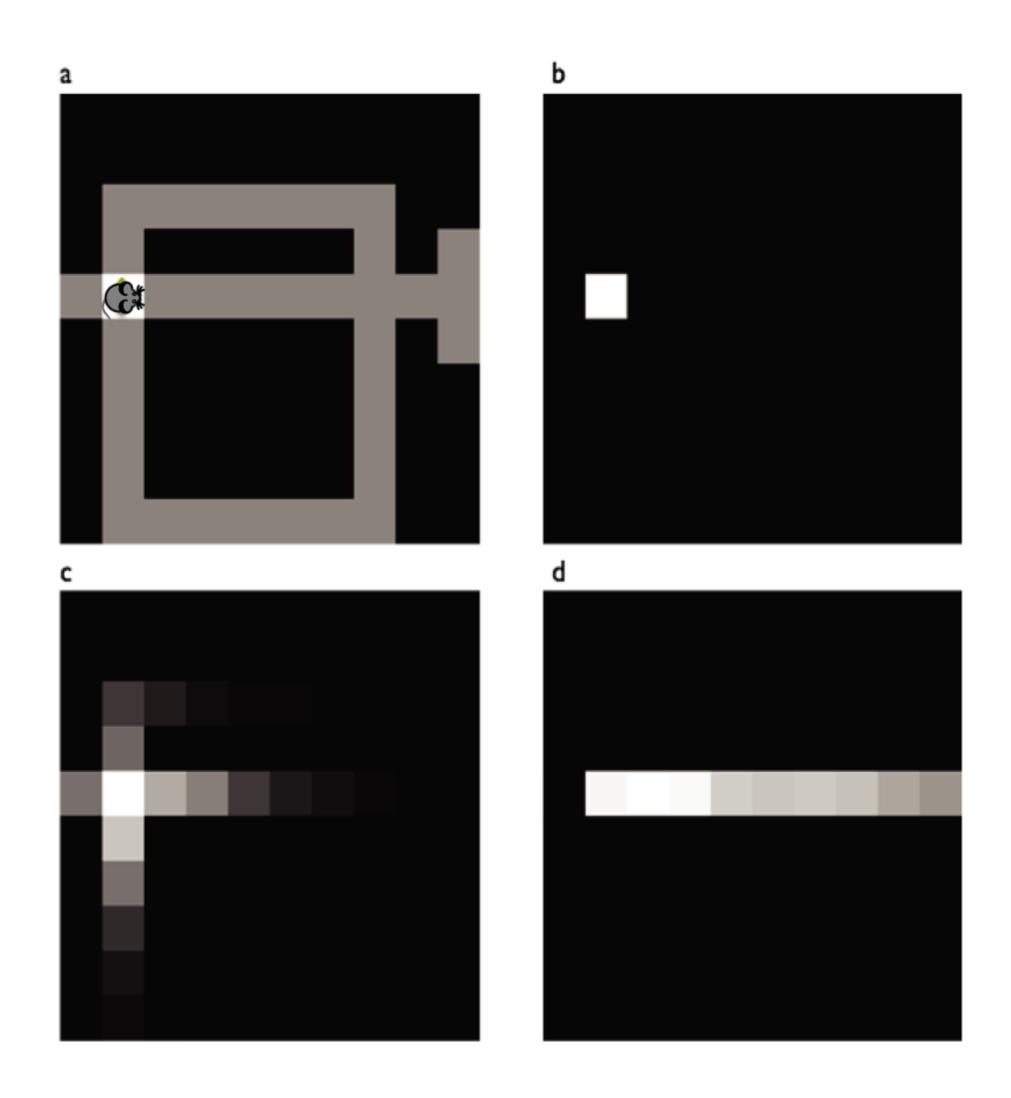
ullet The transition matrix \mathcal{P}^π depicts the possible (s,s') transitions:

$$\mathcal{P}^{\pi} = egin{bmatrix} 0 & 1 & 0 & 0 \ 0 & 0 & 1 & 0 \ 0 & 0 & 0 & 1 \ 0 & 0 & 0 & 1 \end{bmatrix}$$

ullet The SR matrix M also represents the future transitions discounted by γ :

$$M = egin{bmatrix} 1 & \gamma & \gamma^2 & \gamma^3 \ 0 & 1 & \gamma & \gamma^2 \ 0 & 0 & 1 & \gamma \ 0 & 0 & 0 & 1 \end{bmatrix}$$

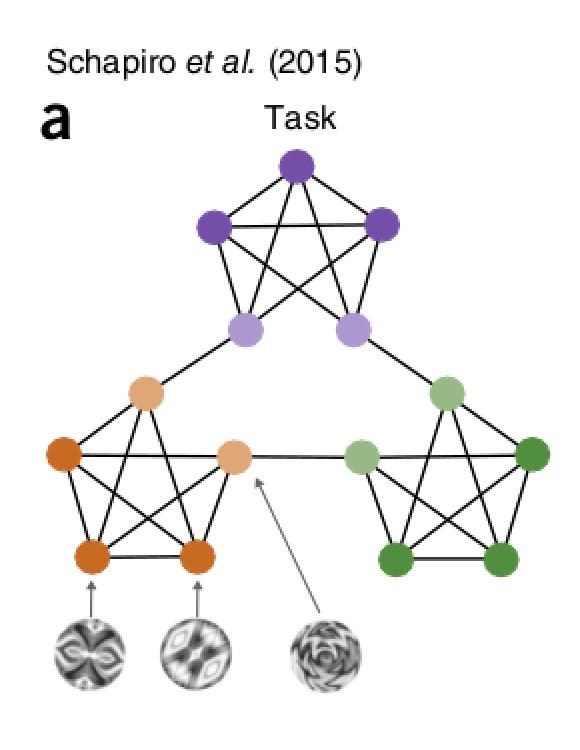
SR matrix in a Tolman's maze

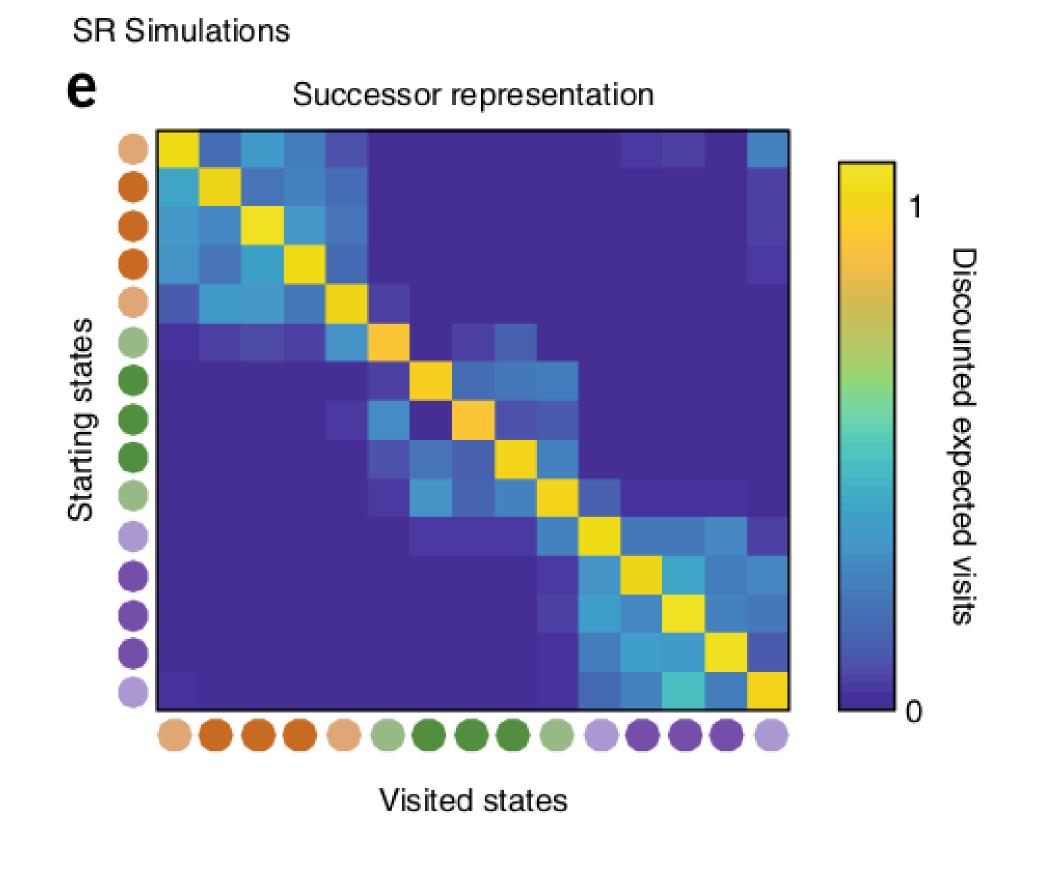


- The SR represents whether a state can be reached soon from the current state (b) using the current policy.
- The SR depends on the policy:
 - A random agent will map the local neighborhood (c).
 - A goal-directed agent will have SR representations that follow the optimal path (d).
- It is therefore different from the transition matrix, as it depends on behavior and rewards.
- The exact dynamics are lost compared to MB: it only represents whether a state is reachable, not how.

Example of a SR matrix

• The SR matrix reflects the proximity between states depending on the transitions and the policy. it does not have to be a spatial relationship.





Learning the SR

How can we learn the SR matrix for all pairs of states?

$$M^\pi(s,s') = \mathbb{E}_\pi[\sum_{k=0}^\infty \gamma^k\, \mathbb{I}(s_{t+k}=s')|s_t=s]$$

We first notice that the SR obeys a recursive Bellman-like equation:

$$egin{aligned} M^{\pi}(s,s') &= \mathbb{I}(s_t = s') + \mathbb{E}_{\pi}[\sum_{k=1}^{\infty} \gamma^k \, \mathbb{I}(s_{t+k} = s') | s_t = s] \ &= \mathbb{I}(s_t = s') + \gamma \, \mathbb{E}_{\pi}[\sum_{k=0}^{\infty} \gamma^k \, \mathbb{I}(s_{t+k+1} = s') | s_t = s] \ &= \mathbb{I}(s_t = s') + \gamma \, \mathbb{E}_{s_{t+1} \sim \mathcal{P}^{\pi}(s'|s)}[\mathbb{E}_{\pi}[\sum_{k=0}^{\infty} \gamma^k \, \mathbb{I}(s_{t+k} = s') | s_{t+1} = s]] \ &= \mathbb{I}(s_t = s') + \gamma \, \mathbb{E}_{s_{t+1} \sim \mathcal{P}^{\pi}(s'|s)}[M^{\pi}(s_{t+1}, s')] \end{aligned}$$

• This is reminiscent of TDM: the remaining distance to the goal is 0 if I am already at the goal, or gamma the distance from the next state to the goal.

Model-based SR

Bellman-like SR:

$$M^\pi(s,s') = \mathbb{I}(s_t=s') + \gamma \, \mathbb{E}_{s_{t+1} \sim \mathcal{P}^\pi(s'|s)}[M^\pi(s_{t+1},s')]$$

• If we know the transition matrix for a fixed policy π :

$$\mathcal{P}^{\pi}(s,s') = \sum_a \pi(s,a) \, p(s'|s,a)$$

we can obtain the SR directly with matrix inversion as we did in dynamic programming:

$$M^\pi = I + \gamma \, \mathcal{P}^\pi imes M^\pi$$

so that:

$$M^\pi = (I - \gamma \, \mathcal{P}^\pi)^{-1}$$

• This DP approach is called **model-based SR** (MB-SR) as it necessitates to know the environment dynamics.

Model-free SR

ullet If we do not know the transition probabilities, we simply sample a single s_t, s_{t+1} transition:

$$M^\pi(s_t,s')pprox \mathbb{I}(s_t=s')+\gamma\,M^\pi(s_{t+1},s')$$

• We can define a **sensory prediction error** (SPE):

$$\delta_t^{ ext{SR}} = \mathbb{I}(s_t = s') + \gamma\,M^\pi(s_{t+1}, s') - M(s_t, s')$$

that is used to update an estimate of the SR:

$$\Delta M^{\pi}(s_t,s') = lpha\,\delta_t^{
m SR}$$

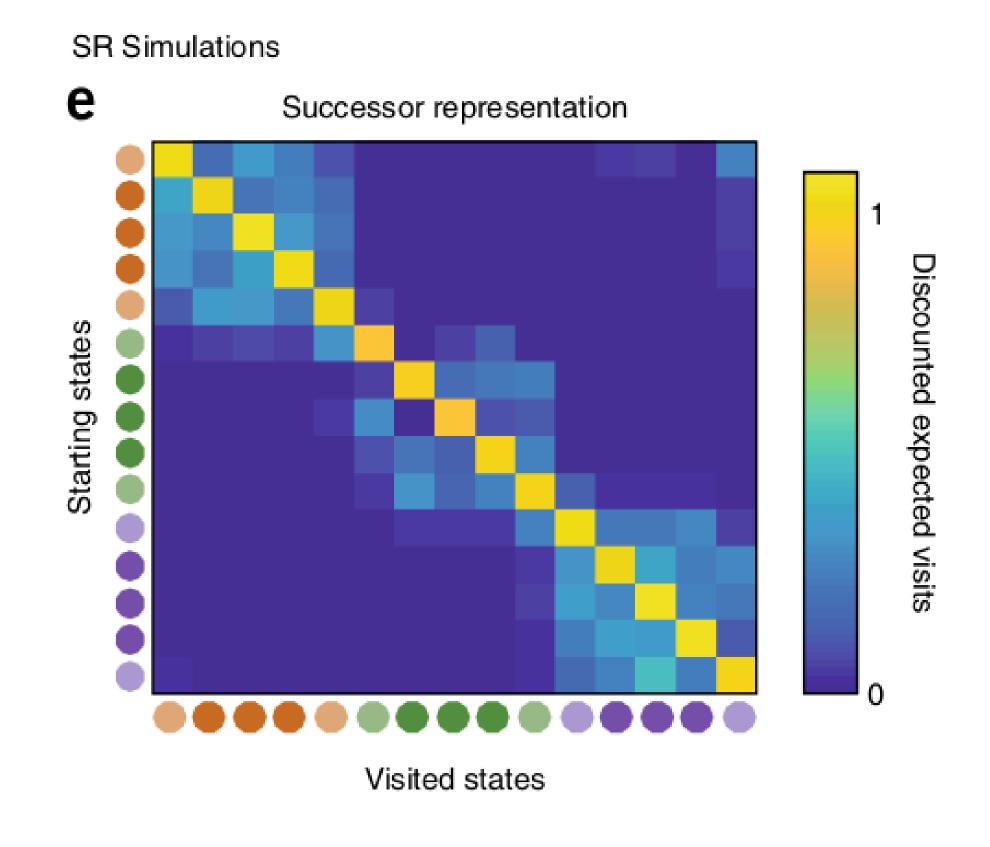
• This is SR-TD, using a SPE instead of RPE, which learns only from transitions but ignores rewards.

The sensory prediction error - SPE

• The SPE has to be applied on ALL successor states s' after a transition (s_t, s_{t+1}) :

$$M^{\pi}(s_t,\mathbf{s'}) = M^{\pi}(s_t,\mathbf{s'}) + lpha\left(\mathbb{I}(s_t=\mathbf{s'}) + \gamma\,M^{\pi}(s_{t+1},\mathbf{s'}) - M(s_t,\mathbf{s'})
ight)$$

- Contrary to the RPE, the SPE is a **vector** of prediction errors, used to update one row of the SR matrix.
- ullet The SPE tells how **surprising** a transition $s_t o s_{t+1}$ is for the SR.



Successor representations

The SR matrix represents the expected discounted future state occupancy:

$$M^\pi(s,s') = \mathbb{E}_\pi[\sum_{k=0}^\infty \gamma^k\, \mathbb{I}(s_{t+k}=s')|s_t=s]$$

It can be learned using a TD-like SPE from single transitions:

$$M^{\pi}(s_t,\mathbf{s'}) = M^{\pi}(s_t,\mathbf{s'}) + \alpha\left(\mathbb{I}(s_t=\mathbf{s'}) + \gamma\,M^{\pi}(s_{t+1},\mathbf{s'}) - M(s_t,\mathbf{s'})
ight)$$

The immediate reward in each state can be learned independently from the policy:

$$\Delta \, r(s_t) = lpha \left(r_{t+1} - r(s_t)
ight)$$

ullet The value $V^\pi(s)$ of a state is obtained by summing of all successor states:

$$V^{\pi}(s) = \sum_{s' \in \mathcal{S}} M(s,s') imes r(s')$$

• This critic can be used to train an **actor** π_{θ} using regular TD learning (e.g. A3C).

Successor representation of actions

Note that it is straightforward to extend the idea of SR to state-action pairs:

$$M^\pi(s,a,s') = \mathbb{E}_\pi[\sum_{k=0}^\infty \gamma^k\, \mathbb{I}(s_{t+k}=s')|s_t=s,a_t=a]$$

allowing to estimate Q-values:

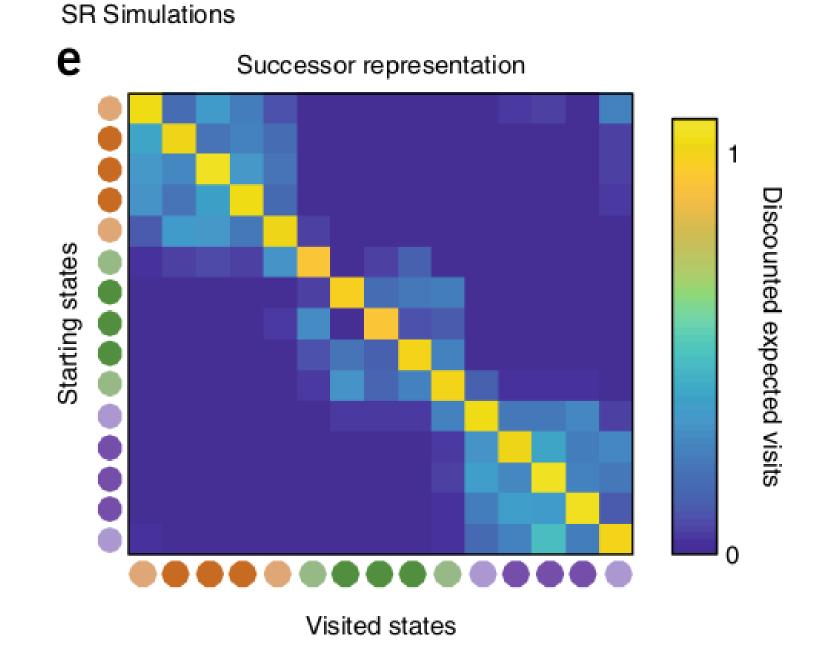
$$Q^{\pi}(s,a) = \sum_{s' \in \mathcal{S}} M(s,a,s') imes r(s')$$

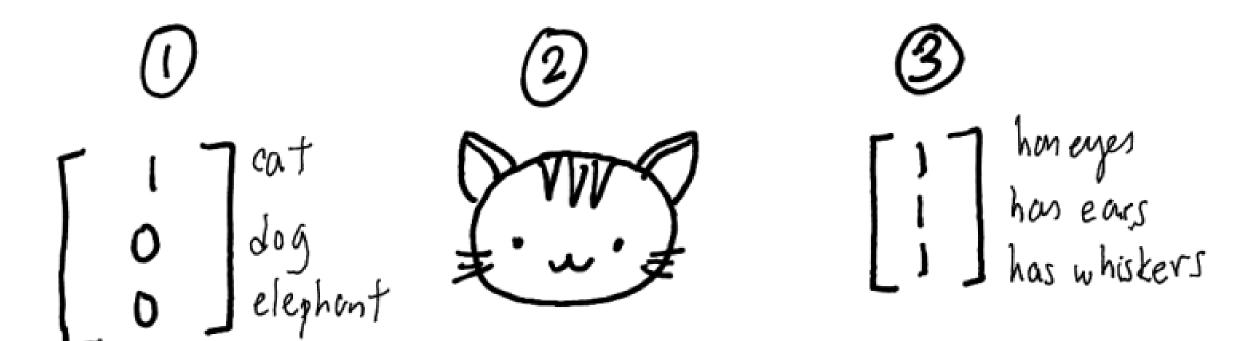
using SARSA or Q-learning-like SPEs:

$$\delta_t^{ ext{SR}} = \mathbb{I}(s_t = s') + \gamma \, M^\pi(s_{t+1}, a_{t+1}, s') - M(s_t, a_t, s')$$

depending on the choice of the next action a_{t+1} (on- or off-policy).

- The SR matrix associates each state to all others (N imes N matrix):
 - curse of dimensionality.
 - only possible for discrete state spaces.
- A better idea is to describe each state s by a feature vector $\phi(s)=[\phi_i(s)]_{i=1}^d$ with less dimensions than the number of states.
- This feature vector can be constructed (see the lecture on function approximation) or learned by an autoencoder (latent representation).





Source: http://www.jessicayung.com/the-successor-representation-1-generalising-between-states/

• The successor feature representation (SFR) represents the discounted probability of observing a feature ϕ_i after being in s.

Source: http://www.jessicayung.com/the-successor-representation-1-generalising-between-states/

• Instead of predicting when the agent will see a cat after being in the current state s, the SFR predicts when it will see eyes, ears or whiskers independently:

$$M_j^\pi(s) = M^\pi(s,\phi_j) = \mathbb{E}_\pi[\sum_{k=0}^\infty \gamma^k \, \mathbb{I}(\phi_j(s_{t+k})) | s_t = s, a_t = a]$$

• Linear SFR (Gehring, 2015) supposes that it can be linearly approximated from the features of the current state:

$$M_j^\pi(s) = M^\pi(s,\phi_j) = \sum_{i=1}^d m_{i,j}\,\phi_i(s)$$

 The value of a state is now defined as the sum over successor features of their immediate reward discounted by the SFR:

$$V^{\pi}(s) = \sum_{j=1}^d M_j^{\pi}(s) \, r(\phi_j) = \sum_{j=1}^d r(\phi_j) \, \sum_{i=1}^d m_{i,j} \, \phi_i(s)$$

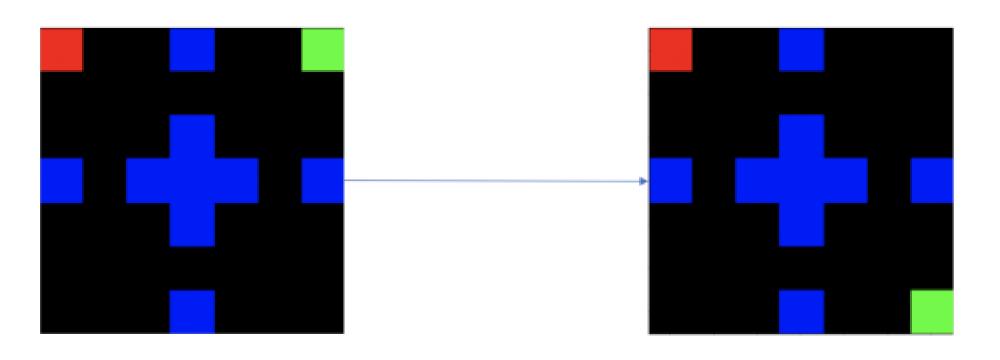
- The SFR matrix $M^\pi=[m_{i,j}]_{i,j}$ associates each feature ϕ_i of the current state to all successor features ϕ_j .
 - Knowing that I see a kitchen door in the current state, how likely will I see a food outcome in the near future?
- ullet Each successor feature ϕ_j is associated to an expected immediate reward $r(\phi_j)$.
 - lacksquare A good state is a state where food features (high $r(\phi_j)$) are likely to happen soon (high $m_{i,j}$).
- In matrix-vector form:

$$V^{\pi}(s) = \mathbf{r}^T imes M^{\pi} imes \phi(s)$$

Value of a state:

$$V^\pi(s) = \mathbf{r}^T imes M^\pi imes \phi(s)$$

- ullet The reward vector ${f r}$ only depends on the features and can be learned independently from the policy, but can be made context-dependent:
 - Food features can be made more important when the agent is hungry, less when thirsty.
- Transfer learning becomes possible in the same environment:
 - Different goals (searching for food or water, going to place A or B) only require different reward vectors.
 - The dynamics of the environment are stored in the SFR.



Source: https://awjuliani.medium.com/the-present-in-terms-of-the-future-successor-representations-in-reinforcement-learning-316b78c5fa3

• How can we learn the SFR matrix M^{π} ?

$$V^\pi(s) = \mathbf{r}^T imes M^\pi imes \phi(s)$$

• We only need to use the sensory prediction error for a transition between the feature vectors $\phi(s_t)$ and $\phi(s_{t+1})$:

$$\delta_t^{
m SFR} = \phi(s_t) + \gamma \, M^\pi imes \phi(s_{t+1}) - M^\pi imes \phi(s_t)$$

and use it to update the whole matrix:

$$\Delta M^\pi = \delta_t^{
m SFR} imes \phi(s_t)^T$$

• However, this linear approximation scheme only works for **fixed** feature representation $\phi(s)$. We need to go deeper...

Deep Successor Reinforcement Learning

Tejas D. Kulkarni* BCS, MIT tejask@mit.edu Ardavan Saeedi*
CSAIL, MIT
ardavans@mit.edu

Simanta Gautam CSAIL, MIT simanta@mit.edu

Samuel J. Gershman
Department of Psychology
Harvard University
gershman@fas.harvard.edu

Kulkarni, T. D., Saeedi, A., Gautam, S., and Gershman, S. J. (2016). Deep Successor Reinforcement Learning. arXiv:1606.02396

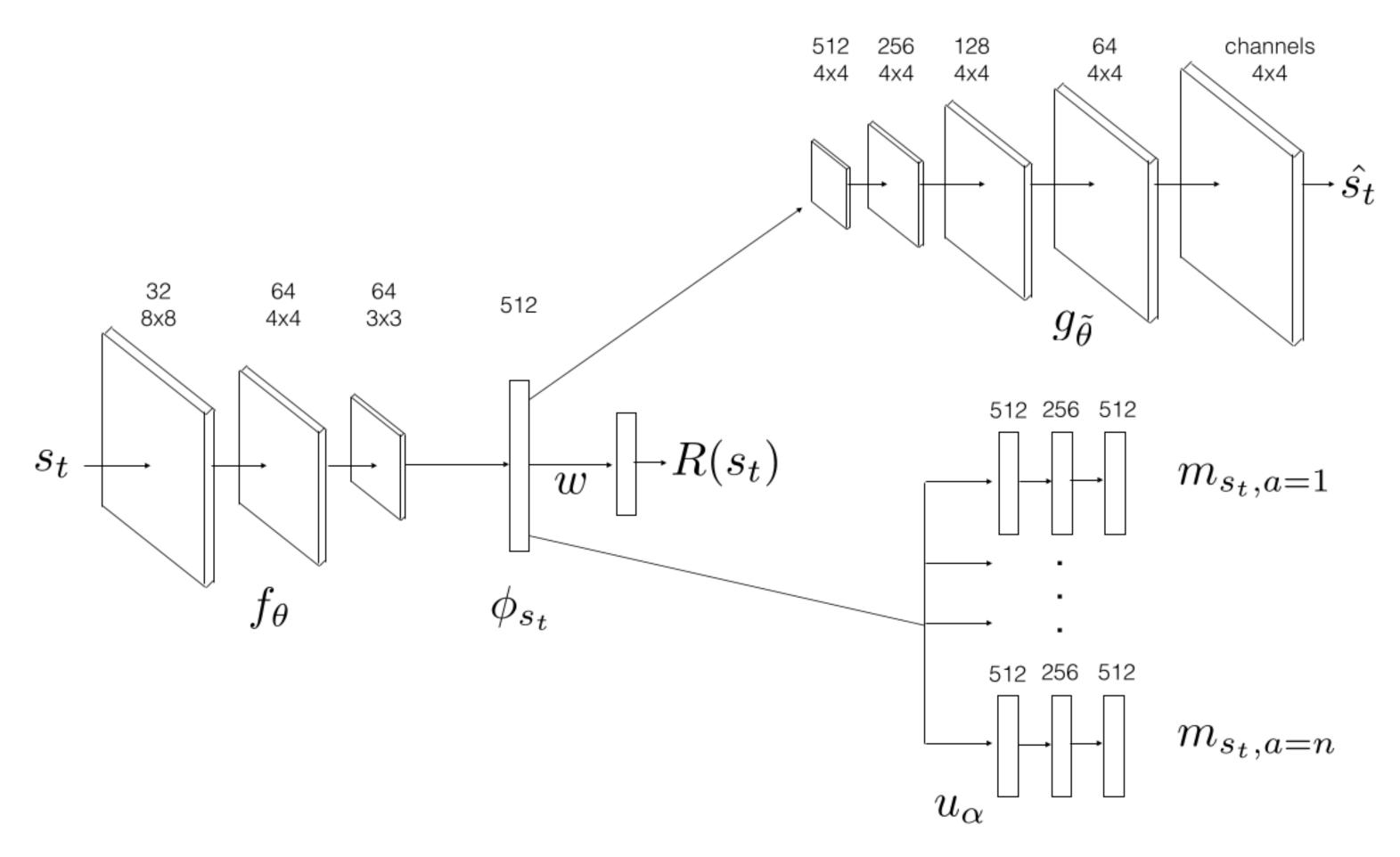


Figure 1: Model Architecture: DSR consists of: (1) feature branch f_{θ} (CNN) which takes in raw images and computes the features ϕ_{s_t} , (2) successor branch u_{α} which computes the SR $m_{s_t,a}$ for each possible action $a \in \mathcal{A}$, (3) a deep convolutional decoder which produces the input reconstruction $\hat{s_t}$ and (4) a linear regressor to predict instantaneous rewards at s_t . The Q-value function can be estimated by taking the inner-product of the SR with reward weights: $Q^{\pi}(s,a) \approx m_{sa} \cdot \mathbf{w}$.

- Each state s_t is represented by a D-dimensional (D=512) vector $\phi(s_t) = f_{\theta}(s_t)$ which is the output of an encoder.
- A decoder $g_{\hat{ heta}}$ is used to provide a reconstruction loss, so $\phi(s_t)$ is a latent representation of an autoencoder:

$$\mathcal{L}_{ ext{reconstruction}}(heta, \hat{ heta}) = \mathbb{E}[(g_{\hat{ heta}}(\phi(s_t)) - s_t)^2]$$

• The immediate reward $R(s_t)$ is linearly predicted from the feature vector $\phi(s_t)$ using a reward vector ${f w}$.

$$R(s_t) = \phi(s_t)^T imes \mathbf{w}$$
 $\mathcal{L}_{ ext{reward}}(\mathbf{w}, heta) = \mathbb{E}[(r_{t+1} - \phi(s_t)^T imes \mathbf{w})^2]$

- The reconstruction loss is important, otherwise the latent representation $\phi(s_t)$ would be too reward-oriented and would not generalize.
- The reward function is learned on a single task, but it can fine-tuned on another task, with all other weights frozen.

ullet For each action a, a NN u_lpha predicts the future feature occupancy M(s,s',a) for the current state:

$$m_{s_t a} = u_lpha(s_t, a)$$

• The Q-value of an action is simply the dot product between the SR of an action and the reward vector w:

$$Q(s_t,a) = \mathbf{w}^T imes m_{s_ta}$$

• The selected action is ϵ -greedily selected around the greedy action:

$$a_t = rg \max_a Q(s_t, a)$$

ullet The SR of each action is learned using the Q-learning-like SPE (with fixed heta and a target network $u_{lpha'}$):

$$\mathcal{L}^{ ext{SPE}}(lpha) = \mathbb{E}[\sum_{a}(\phi(s_t) + \gamma \, \max_{a'} u_{lpha'}(s_{t+1}, a') - u_lpha(s_t, a))^2]$$

• The compound loss is used to train the complete network end-to-end **off-policy** using a replay buffer (DQN-like).

$$\mathcal{L}(\theta, \hat{ heta}, \mathbf{w}, lpha) = \mathcal{L}_{ ext{reconstruction}}(\theta, \hat{ heta}) + \mathcal{L}_{ ext{reward}}(\mathbf{w}, heta) + \mathcal{L}^{ ext{SPE}}(lpha)$$

Algorithm 1 Learning algorithm for DSR

```
1: Initialize experience replay memory \mathcal{D}, parameters \{\theta, \alpha, \mathbf{w}, \theta\} and exploration probability
     \epsilon = 1.
 2: for i = 1 : \#episodes do
          Initialize game and get start state description s
 3:
          while not terminal do
 4:
              \phi_s = f_{\theta}(s)
 5:
               With probability \epsilon, sample a random action a, otherwise choose \operatorname{argmax}_a u_{\alpha}(\phi_s, a) \cdot \mathbf{w}
 6:
               Execute a and obtain next state s' and reward R(s') from environment
               Store transition (s, a, R(s'), s') in \mathcal{D}
 8:
               Randomly sample mini-batches from \mathcal{D}
 9:
               Perform gradient descent on the loss L^r(\mathbf{w}, \theta) + L^a(\tilde{\theta}, \theta) with respect to \mathbf{w}, \theta and \tilde{\theta}.
10:
               Fix (\theta, \tilde{\theta}, \mathbf{w}) and perform gradient descent on L^m(\alpha, \theta) with respect to \alpha.
12:
               s \leftarrow s'
          end while
13:
          Anneal exploration variable \epsilon
14:
15: end for
```

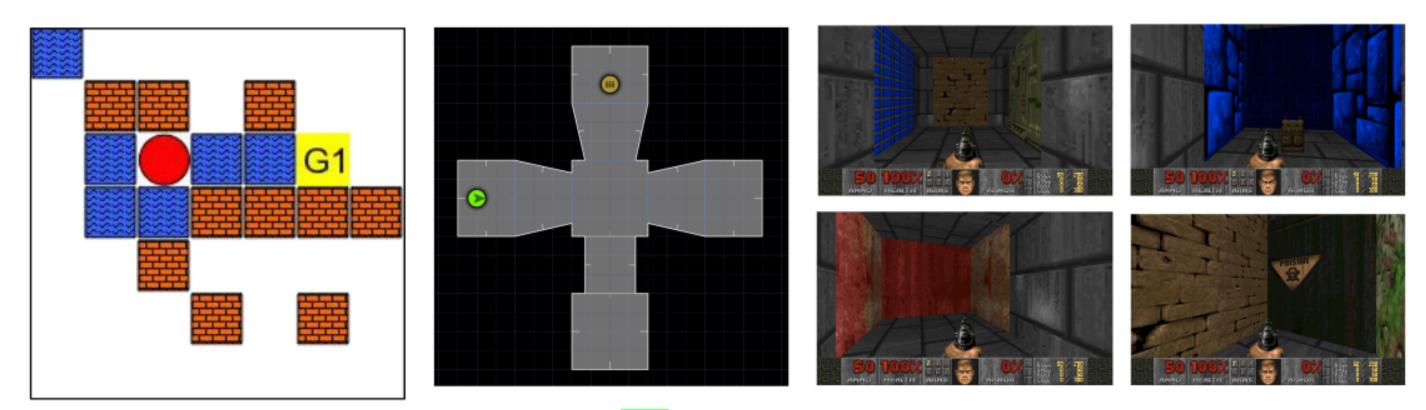


Figure 2: **Environments**: (**left**) MazeBase [37] map where the agent starts at an arbitrary location and needs to get to the goal state. The agent gets a penalty of -0.5 per-step, -1 to step on the water-block (blue) and +1 for reaching the goal state. The model observes raw pixel images during learning. (**center**) A *Doom* map using the VizDoom engine [13] where the agent starts in a room and has to get to another room to collect ammo (per-step penalty = -0.01, reward for reaching goal = +1). (**right**) Sample screen-shots of the agent exploring the 3D maze.

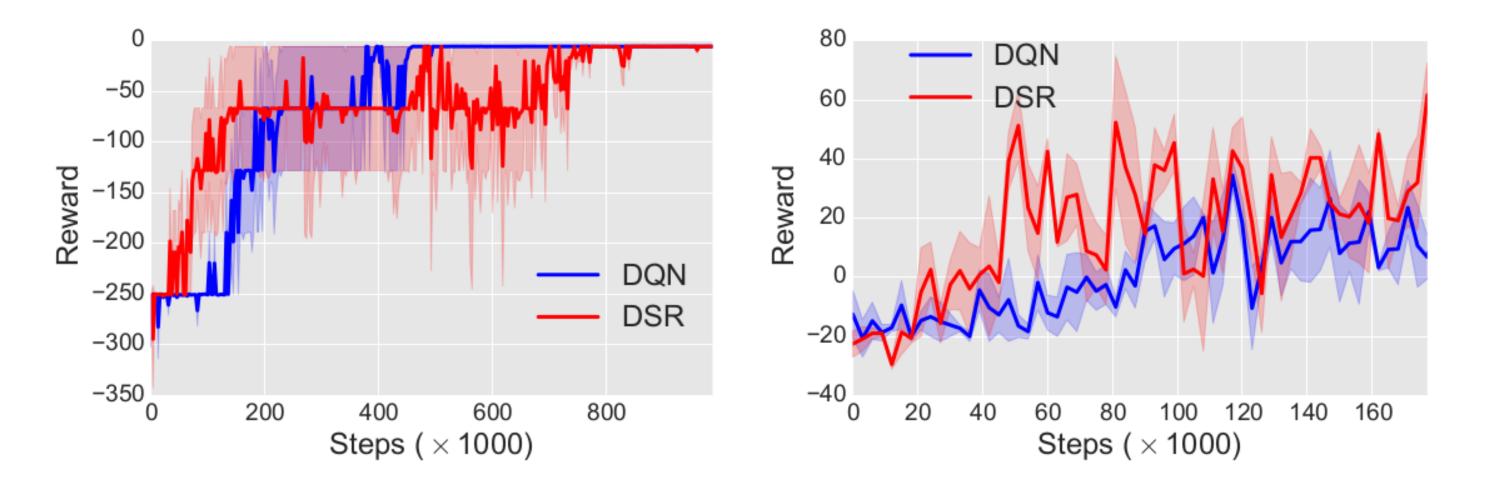
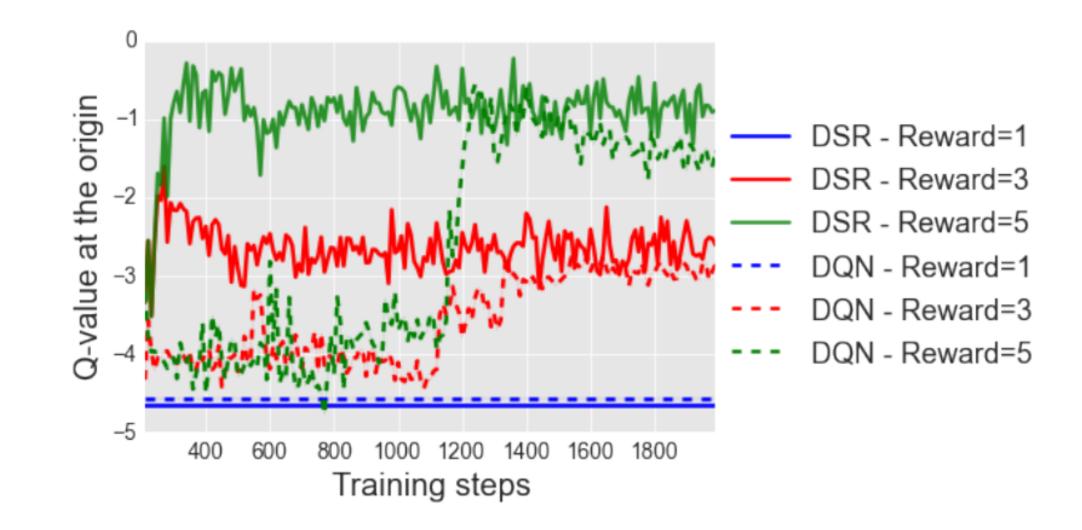


Figure 3: Average trajectory of the reward (left) over 100k steps for the grid-world maze. (right) over 180k steps for the Doom map over multiple runs.

- The interesting property is that you do not need rewards to learn:
 - A random agent can be used to learn the encoder and the SR, but w can be left untouched.
 - When rewards are introduced (or changed),
 only w has to be adapted, while DQN would have to re-learn all Q-values.



- This is the principle of **latent learning** in animal psychology: fooling around in an environment without a goal allows to learn the structure of the world, what can speed up learning when a task is introduced.
- The SR is a **cognitive map** of the environment: learning task-unspecific relationships.

- Note: the same idea was published by three different groups at the same time (preprint in 2016, conference in 2017):
 - Barreto A, Dabney W, Munos R, Hunt JJ, Schaul T, van Hasselt H, Silver D. (2016). Successor Features for Transfer in Reinforcement Learning. arXiv:160605312.
 - Kulkarni, T. D., Saeedi, A., Gautam, S., and Gershman, S. J. (2016). Deep Successor Reinforcement Learning. arXiv:1606.02396.
 - Zhang J, Springenberg JT, Boedecker J, Burgard W. (2016). Deep Reinforcement Learning with Successor Features for Navigation across Similar Environments. arXiv:161205533.
- The (Barreto et al., 2016) is from Deepmind, so it tends to be cited more...

Visual Semantic Planning using Deep Successor Representations

