Adventures in de Sitter space

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This is my contribution to the Festschrift honoring Stephen Hawking on his 60th birthday. Twenty-five years ago, Gibbons and Hawking laid out the semi-classical properties of de Sitter space. After a summary of their main results, I discuss some further quantum aspects that have since been understood. The largest de Sitter black hole displays an intriguing pattern of instabilities, which can render the boundary structure arbitrarily complicated. I review entropy bounds specific to de Sitter space and outline a few of the strategies and problems in the search for a full quantum theory of the spacetime.

Contents

I. INTRODUCTION

It is a pleasure to help celebrate Stephen Hawking's 60th birthday (not least because Stephen knows how to party). I am grateful for the good fortune I had in working with him, and for the physics I learned from him as his student from 1997 to 2000. But what I am most thankful for is the homework. Stephen's discoveries amount to a formidable problem set. I'm afraid we're late turning it in. Without question, it will keep us happily occupied for years to come.

Stephen and I have published five journal articles together (Bousso and Hawking, 1995, 1996b, 1997c, 1998b, 1999a), as well as a number of proceedings (Bousso and Hawking, 1996a,c, 1997a,b, 1999b). Most of these papers, and much of my subsequent work, are concerned in one way or another with aspects of quantum gravity in de Sitter space. In my contribution to this Festschrift I should therefore like to survey of some of the results, problems, and speculations surrounding this topic.

Stephen's contributions to black hole physics and cosmology find a synthesis in his semi-classical treatment of de Sitter space. Gibbons and Hawking (1977) demonstrated that the de Sitter horizon, like a black hole, is endowed with entropy and temperature. Thus, the quantum properties of black holes will extend to the universe as a whole, if the vacuum energy is positive.

More than ever, the implications of this work are under active investigation. The Bekenstein-Hawking entropy of the spacetime, in particular, has been taken either as a starting point, or alternatively, as a crucial test, of various approaches to a full quantum gravity theory for de Sitter space.

A brief description of the de Sitter geometry is given in Sec. II. In Sec. III.A, I summarize the concepts of black hole entropy and the generalized second law of thermodynamics (Bekenstein, 1972, 1973, 1974), as well as Hawking's (1974, 1975) results on black hole temperature and radiation. This will provide a context for a review of the main conclusions of Gibbons and Hawking (1977), in Sec. III.B.

The generalized second law has been used to infer universal bounds on the entropy of matter systems. In Sec. IV.A, I describe how the Bekenstein (1981) bound, $S \leq 2\pi ER$, is obtained from a gedankenexperiment involving a black hole. When this kind of argument is extended to the de Sitter horizon, one obtains analogous bounds in various limits (Schiffer, 1992; Bousso, 2001). In Sec. IV.B, I discuss one of these bounds, the D-bound, whose good agreement with the Bekenstein bound is nontrivial.

Roughly speaking, the Bekenstein-Hawking entropy of empty de Sitter space is the largest entropy attainable in any asymptotically de Sitter spacetime (Fischler, 2000a,b; Banks, 2000). In Sec. V, I make this absolute entropy bound more precise. One must distinguish be-

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tween spacetimes that are de Sitter in the past, in the future, or both. Moreover, it is natural to include at least some spacetimes with positive cosmological constant but no asymptotic de Sitter region. The more carefully one characterizes the spacetime *portions* whose entropy need be considered, the broader the *class* of spacetimes obeying the bound. It is argued that only the entropy contained in causal diamonds is observable (Bousso, 2000). However, even with this restriction, one can find some $\Lambda > 0$ universes with unbounded entropy (Bousso, DeWolfe, and Myers, 2002).

The formulation of a quantum gravity theory describing de Sitter space remains an open problem. In Sec. VI, I outline some of the strategies that can be adopted and the difficulties they face. In particular, asymptotic regions, which are conventionally used to define observables, are not globally accessible to any de Sitter observer; quantum mechanical obstructions may be even more severe. I also discuss the possibility that a new class of theories, with a manifestly finite number of states, may play a role in the description of de Sitter space.

Sec. VII is most closely related to my joint work with Stephen. It discusses the largest possible black hole in de Sitter space. This spacetime, the Nariai solution, exhibits a remarkable set of instabilities. Small perturbations can lead to an infinite variety of global structures, including the fragmentation of the spatial geometry into disconnected de Sitter universes (Bousso, 1998). I place these results in the context of present approaches to de Sitter space.

This article is by no means an attempt to review the subject. It touches upon a small portion of the literature, whose selection is biased by my current interests and by some of my own adventures in de Sitter space. Many of these were shared with Stephen and with other collaborators: Andrew Chamblin, Oliver DeWolfe, Andrei Linde, Alex Maloney, Rob Myers, Jens Niemeyer, Joe Polchinski, and Andy Strominger. I would like to thank them, and I emphasize that the shortcomings of this article, for which I apologize, are mine.

Planck units are used throughout. For definiteness, the number of spacetime dimensions is taken to be $D = 4$ unless noted otherwise. The discussion generalizes trivially to higher dimensions except where special cases are pointed out. For a review of de Sitter space, see, e.g., Spradlin, Strominger, and Volovich (2001). Extensive lists of references are also given by Balasubramanian, de Boer, and Minic (2001); Spradlin and Volovich (2001).

II. DE SITTER SPACE

This section summarizes a number of the classical properties of de Sitter space that are used below. A more extensive discussion is found in Hawking and Ellis (1973).

de Sitter space is the maximally symmetric solution of the vacuum Einstein equations with a positive cosmological constant, Λ. It is positively curved with characteristic

Spherical

$\Lambda > 0$

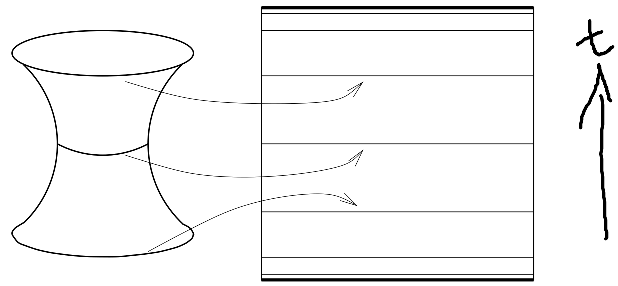

FIG. 1 de Sitter space as a hyperboloid. Time goes up.— Right: Penrose diagram. Horizontal lines represent three-spheres.

length

length

$$\ell = \sqrt{\frac{3}{\Lambda}}. \tag{2.1}$$

Friedmann - Lemaître - Robertson - Walker metric

Globally, de Sitter space can be written as a closed FRW universe:

$$\frac{ds^2}{\ell^2} = -d\tau^2 + \cosh^2 \tau \, d\Omega_3^2 \tag{2.2}$$

The spacelike slices are three-spheres. The space-time can be visualized as a hyperboloid (Fig. 1). The smallest S^3 is at the throat of the hyperboloid, at $\tau = 0$. For $\tau > 0$, the three-spheres expand exponentially without bound. The time evolution is symmetric about $\tau = 0$, so three-spheres in the past are arbitrarily large and contracting.

The Penrose diagram of de Sitter space is a square (Fig. 1). The spatial three-spheres are horizontal lines. As usual, every point represents a two-sphere, except for the points on the left and right edge of the square, which represent the poles of the three-sphere. The top and bottom edge are future and past infinity, $\mathcal{I}^+$ and $\mathcal{I}^-$, where all spheres become arbitrarily large.

In the static coordinate system,

$$\frac{ds^2}{\ell^2} = -V(r)\, dt^2 + \frac{1}{V(r)} dr^2 + r^2 d\Omega_2^2, \tag{2.3}$$

where

$$V(r) = 1 - r^2, \tag{2.4}$$

it becomes manifest that an observer at $r = 0$ is surrounded by a cosmological horizon at $r = 1$. These coordinates cover only part of the space-time, namely the interior of a cavity bounded by $r = 1$ (Fig. 2). This is precisely the operationally meaningful portion of de Sitter space, i.e., the region that can be probed by a single observer. The upper and lower triangles contain exponentially large regions that cannot be observed; in particular, they contain the conformal boundaries $\mathcal{I}^+$ and $\mathcal{I}^-$.

An object held at a fixed distance from the observer is redshifted. The red-shift, $V(r)^{1/2}$, diverges near the heat energy cannot be created nor destroyed

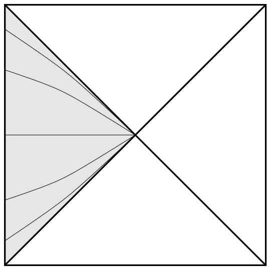

FIG. 2 Past and future event horizon (diagonal lines). The static slicing covers the interior of the cosmological horizon (shaded).

horizon. If released, the object will accelerate towards the horizon. Once it crosses the horizon, it can no longer be retrieved. In short, the cosmological horizon acts like a black hole "surrounding" the observer. Note that the symmetry of the space-time implies that the location of the cosmological horizon is observer-dependent. The black hole analogy carries over to the semi-classical level; this is discussed further in Sec. III.B.

III. ENTROPY AND TEMPERATURE OF EVENT HORIZONS

A. Black holes

entropy never decreases in any classical process

The entropy of an ordinary object is lost to an outside observer when the object falls into a black hole. However, the black hole's horizon area increases in this process. (Indeed, Hawking, 1971, showed that it never decreases in any classical process.) In order to salvage the second law of thermodynamics, Bekenstein (1972, 1973, 1974) proposed that a black hole carries an entropy on the order of its horizon area, A, in Planck units. Moreover, he conjectured a generalized second law of thermodynamics: the sum of ordinary entropy and horizon entropy never decreases.

The analogy between the laws of thermodynamics and classical properties of black hole spacetimes—with the surface gravity, κ, playing the role of temperature, and the horizon area, A, mimicking entropy—was soon understood in great detail. Still, Bekenstein's proposal met with scepticism (Bardeen, Carter, and Hawking, 1973), because it appeared to lead to a contradiction. If A represented an actual entropy, the temperature ($\sim \kappa$) had to be a physical effect as well. But how could black holes radiate?

Using Bogolubov transformation techniques, Hawking (1974, 1975) demonstrated that black holes emit radiation by a quantum process, at a temperature

$$T_{\rm hor} = \frac{\kappa}{2\pi}. \tag{3.1}$$

For a Schwarzschild black hole of mass M, $\kappa = (4M)^{-1}$. Via the first law of thermodynamics,

$$\frac{1}{T_{\rm hor}} = \frac{\partial S_{\rm hor}}{\partial M}, \tag{3.2}$$

Hawking's calculation confirmed Bekenstein's entropy formula (up to an additive constant which can be argued to vanish) and determined its numerical coefficient:

$$S_{\rm hor} = \frac{A}{4}. \tag{3.3}$$

B. de Sitter space

In spacetimes which are asymptotically de Sitter in the future, any observer is surrounded by an event horizon. At late times, its area is given by

surface area equation of sphere

$$A_0 = 4\pi\ell^2, \tag{3.4}$$

where

$$\ell = \sqrt{\frac{3}{\Lambda}} \tag{3.5}$$

is the curvature radius, and Λ is the cosmological constant. Gibbons and Hawking (1977) noted that this horizon possesses surface gravity $\kappa = 1/\ell$ and satisfies analogues to the classical laws of black hole mechanics (Bardeen, Carter, and Hawking, 1973).

This suggests that the horizons of black holes and of de Sitter space share quantum properties as well. In analogy to Hawking's (1974) result, one would expect the horizon to be at a non-zero temperature according to Eq. (3.1). Moreover, ordinary matter entropy is lost when systems cross the de Sitter horizon. In analogy to Bekenstein's (1972) argument, one would expect that the de Sitter horizon must have non-zero entropy according to Eq. (3.3).

Using Euclidean techniques, Gibbons and Hawking (1977) demonstrated that an observer in de Sitter space does detect thermal radiation at a temperature

$$T_{\rm dS} = \frac{1}{2\pi\ell}, \tag{3.6}$$

in agreement with expectation. The presence of thermal Green functions had been noticed previously in 1+1 dimensional de Sitter space (Figari, Hoegh-Krohn, and Nappi, 1975).

One would like to infer the cosmological horizon entropy from the Gibbons-Hawking temperature by the first law of thermodynamics, Eq. (3.2). The mass of a cosmological horizon is not defined *a priori*. However, only a mass differential is needed. Hence, let us express the first law in an alternate form, which refers to the change in matter energy rather than the change in "horizon energy".[1]

[1] Gibbons and Hawking (1977) formally assign a negative mass to

Leaving de Sitter space aside for a minute, consider a closed system consisting of a black hole initially well separated from ordinary matter. The total energy is conserved in any process whereby energy is exchanged between the components. Hence, $dM = -dE$, where M is the black hole mass and E is the energy of matter, and Eq. (3.2) becomes

$$\frac{1}{T_{\rm hor}} = -\frac{\partial S_{\rm hor}}{\partial E}. \tag{3.7}$$

In this form, the first law makes no reference to the "energy of the horizon". It can be adapted to the de Sitter case, where the matter energy M is perturbatively defined in terms of the timelike Killing vector field in the interior of the cosmological horizon. By studying black hole solutions in de Sitter space,[2] one finds that

$$\left.\frac{\partial A_{\rm hor}}{\partial E}\right|_{A_{\rm hor}=A_0} = 8\pi\ell. \tag{3.8}$$

for the derivative of the cosmological horizon area [see also Eq. (4.5) below]. With the Bekenstein ansatz, $S_{\rm hor} = \eta A_{\rm hor}$, for the entropy of the cosmological horizon, the usual coefficient, $\eta = 1/4$, follows from Eqs. (3.6)–(3.8). Up to an additive constant, which is taken to vanish, Eq. (3.3) thus applies both to cosmological and to black hole horizons. In particular, the total entropy of empty de Sitter space is given by its horizon entropy, which is

hc = hbar

$$S_0 = \frac{A_0}{4} = \frac{3\pi}{\Lambda}. \tag{3.9}$$

Thus, Gibbons and Hawking (1977) showed that the de Sitter horizon is endowed with the same quantum properties as a black hole horizon: a temperature and an entropy. They noted that the de Sitter horizon, unlike a black hole horizon, is observer dependent. They interpreted their results as an indication that quantum gravity may not admit a single, objective and complete description of the universe. Rather, its laws may have to be formulated with reference to an observer—no more than one at a time. These insights foreshadow more radical assertions of the need for complementary descriptions (Susskind, Thorlacius, and Uglum, 1993), which eventually arose from considerations of unitarity in the presence of black holes.

the cosmological horizon, but this is a mere convenience. The differential treatment of energy-momentum flux across the horizon makes reference only to the matter stress tensor.

[2] By finding the zeros of $V(r)$ in Eq. (7.2) as a function of E, one estimates the horizon area as a function of enclosed energy. Unlike Gibbons and Hawking (1977), this argument is quick and dirty. It assumes that a small Schwarzschild-de Sitter black hole with mass parameter E in Eq. (7.2), and a perturbative Killing mass $E = \int d^3x\sqrt{h}T_{\mu\nu}\chi^\mu n^\nu$, have equal cosmological horizon area after back-reaction is taken into account. (Here $\chi = \partial/\partial t$ and $n = \chi/|\chi|$.)

IV. ENTROPY BOUNDS FROM HORIZONS

A. Black holes and the Bekenstein bound

When a matter system falls into a black hole, the matter entropy disappears. At the same time, the horizon area (and hence, the black hole entropy) increases. Bekenstein's generalized second law *may* hold, but only if the horizon entropy increases by enough, i.e., if

entropy

$$\frac{\Delta A_{\rm hor}}{4} \geq S_{\rm matter}. \tag{4.1}$$

Bekenstein (1981) estimated a lower bound on $\Delta A_{\rm hor}$ based on the "Geroch process", by which a the system of mass E is added to the black hole only after first extracting a maximum amount of work. This minimizes the increase in the black hole mass, and hence, in its area.

One finds that $\Delta A_{\rm hor} \leq 8\pi ER$, where R is the largest[3] dimension of the system. Hence,

$$S_{\rm matter} \leq 2\pi ER. \tag{4.2}$$

Remarkably, this conclusion does not depend on microscopic properties of matter and thus betrays a fundamental aspect of nature. However, the bound applies only to matter systems that can actually be added to black holes. In particular, one must assume that gravity is not the dominant force in the system.

B. de Sitter space and the D-bound

Analogous arguments can be made for matter systems crossing the de Sitter horizon. One possibility is to study matter systems which are very small compared to the cosmological horizon (Schiffer, 1992). In this case the cosmological constant is exploited only to provide a horizon; its effect on spatial curvature is negligible over the scale of the system. One finds agreement with the Bekenstein bound. The Unruh-Wald analysis can also be generalized in this limit (Davies, 1984).

A different possibility is to consider systems whose size is comparable to the horizon radius. The resulting entropy bound is called the D-bound. The cosmological curvature is significant in large systems, and it

[3] This is an empirical choice. Classical analysis of the Geroch process would suggest that R can be the smallest dimension, which would lead to contradictions. However, Unruh (1976) radiation must be taken into account for very flat systems. More generally, the proper quantum treatment of the Geroch process is under debate (see, for example, Bekenstein, 1999; Wald, 2001; Marolf and Sorkin, 2002). Independently of its logical status, there is empirical evidence that Bekenstein's bound holds for all weakly gravitating matter systems that can actually be constructed (Schiffer and Bekenstein, 1989; Wald, 2001). See Bekenstein (2001) and Wald (2001) for reviews; further references are also given in Bousso (2002).

is not obvious that the D-bound will agree with Bekenstein's bound, which applies to systems which perturb flat space weakly. With reasonable definitions of mass and "largest dimension" of the system, however, one finds precise agreement. This section follows Bousso (2001).

Consider a matter system in an asymptotically de Sitter spacetime, i.e., a spacetime in which an observer's causal domain agrees well with empty de Sitter space at late times. The total initial entropy is the sum of the matter entropy, $S_{\rm matter}$, and the horizon entropy, Eq. (3.3). The total final entropy is $A_0/4$, the entropy of empty de Sitter space. By the generalized second law,

$$S_{\rm matter} \le \frac{A_0 - A_{\rm hor}(\text{initial})}{4}. \tag{4.3}$$

This is the D-bound. It holds for any matter system that can be contained in a causal domain of an asymptotically de Sitter universe; no assumptions about weak gravity are necessary. In particular, the D-bound predicts that the cosmological horizon will have area $A_{\rm hor} < A_0$ as long as matter is present. This can be verified explicitly for many solutions, e.g., for the Schwarzschild-de Sitter spacetimes. For light matter systems within the cosmological horizon, the D-bound is more stringent than the holographic bound, $S_{\rm matter} \le A_{\rm hor}/4$. This can be seen readily in the limit of empty de Sitter space, for which $A_{\rm hor} \to A_0$ and the D-bound vanishes.

Equation (4.3) is not always the most useful form of the D-bound. One would like to evaluate the right hand side in terms of intrinsic characteristics of the matter system. Let us assume that the matter system is light, i.e., it does not affect the de Sitter horizon much:

$$A_0 - A_{\rm hor}(\text{initial}) \ll A_0. \tag{4.4}$$

Next, suitable quantities must be defined. We seek a definition of "mass" in de Sitter space that makes reference only to the causally accessible region (which prevents us from using the definition of Abbott and Deser, 1982), but does not assume a fixed background (which precludes the use of a timelike Killing vector). A convenient definition makes use of the cosmological horizon area.

The mass E of a system in asymptotically flat space could be alternatively written as a "gravitational radius" $R_{\rm g}$, the radius[4] of the Schwarzschild black hole with the same mass. In this vein, let us define the mass of a matter system in de Sitter space to be the radius of the particular Schwarzschild-de Sitter black hole that leads to the same value of the cosmological horizon area.

By substituting $\partial A_{\rm hor} \to 4S_{\rm matter}$ and $\partial E \to R_{\rm g}/2$ in Eq. (3.8), the D-bound can be expressed in the form

$$S_{\rm matter} \le \pi R_{\rm g} R_{\rm c}, \tag{4.5}$$

That is, the entropy of a spherical system in de Sitter space cannot be larger than π times the product of its gravitational radius and the radius of the cosmological horizon, $R_{\rm c}$.

In order to compare this result with the Bekenstein bound, it is useful to express Eq. (4.2) in terms of the gravitational radius $R_{\rm g} = 2E$:

$$S_{\rm matter} \le \pi R_{\rm g} R. \tag{4.6}$$

The "largest dimension", R, plays a role comparable to $R_{\rm c}$, at least in the sense that the horizon size in de Sitter space places an upper bound on the extent of the system. Hence, the two bounds agree for large dilute systems in de Sitter space. This is not trivial as the spacetime background differs significantly. For smaller systems, the Bekenstein bound is more stringent. Of course, in the limit of very small systems, one expects the Bekenstein bound to hold, since the deviations from flat space will be negligible.

Both the Bekenstein bound and the D-bound can be extended to $D > 4$ spatial dimensions. They continue to agree exactly in the limit of large dilute systems. However, surprisingly, one finds that black holes saturate the Bekenstein bound only for $D = 4$.

V. ABSOLUTE ENTROPY BOUNDS IN SPACETIMES WITH $\Lambda > 0$

The generalized second law states, loosely speaking, that the final entropy is the largest entropy. In any spacetime which asymptotes to de Sitter space this implies that the maximal entropy is given by Eq. (3.9). This statement was used to derive the D-bound in the previous section. However, de Sitter space has two conformal boundaries, one in the past, and one in the future. At fixed value of Λ, one can demand the presence of both boundaries, only one, or neither.

Interestingly, there are in fact several different statements of the type $S \le S_0$ for $\Lambda > 0$ universes, depending on asymptotic conditions. With liberal conditions on the asymptotic structure, S must be rather restrictively defined for $S \le S_0$ to obtain. With stringent boundary conditions, $S \le S_0$ holds more broadly.

All of these bounds are *absolute* in the sense that they refer to the maximal entropy that can be contained (or probed) in a spacetime. They are a consequence of (but not as general as) the holographic bound ('t Hooft, 1993; Susskind, 1995; Bousso, 1999a,b), which limits entropy in the neighborhood of arbitrarily chosen codimension two spatial surfaces, relative to the surface area.

A. $\mathbf{dS}^+$ and the second law

Let us begin with the class of universes familiar from the previous section. The set $\mathbf{dS}^+(\Lambda)$ is defined to contain all spacetimes which possess an asymptotic de Sitter

[4] This refers to the area radius of a horizon, defined as the root of the proper horizon area divided by the area of a unit sphere. $R_{\rm g}$ and $R_{\rm c}$ are physical lengths, unlike the coordinate r used in Eqs. (2.3) and (7.1).

region in the future, with cosmological constant Λ. I will not make specific assumptions about matter content either here or below, except to demand that reasonable energy conditions are satisfied, e.g., the dominant energy condition (Wald, 1984).

At least for those observers who reach the asymptotic region, the second law argument can be completed, and one may conclude that the D-bound holds, and moreover, that the entropy of empty de Sitter space provides an upper bound on the total matter and horizon entropy at any prior time.

The reference to an observer is crucial, however. The $\mathbf{dS}^+$ class contains, for example, a spatially flat Friedmann-Robertson-Walker universe that starts with a big bang singularity, is radiation or matter dominated initially, and dominated by a cosmological constant at late times. (This may be a good approximation to our own universe.) The spatial extent of this universe is infinite at all times, and the entropy on a global time slice is clearly unbounded. However, the cosmological horizon shields all but a finite portion of the universe from any single observer.

The statement $S \leq S_0$ refers only to the entropy of matter inside the observer's horizon (that is, matter in the observer's causal past), plus the horizon entropy. Otherwise, it would obviously be violated. This restriction is implicit in the generalized second law argument. The growth of the de Sitter horizon to its asymptotic value at late times can only be a response to the matter that actually that crosses that horizon. The horizon has no knowledge of the entropy of matter that was already beyond it to start with.

In a dS^+ universe, the generalized second law thus implies the following result. *Consider an observer whose worldline approaches $\mathcal{I}^+$, the future de Sitter infinity, and let P be the causal past of the point where the worldline meets $\mathcal{I}^+$. Let Σ be any (suitably smooth and complete) timelike hypersurface. The spatial region $\Sigma' \equiv \Sigma \cap P$ corresponds to a particular instant of time in the observer's causal past, and the boundary of Σ' in Σ, $\partial\Sigma'$, is the event horizon at that time. Let S_{matter} be the entropy of matter in the region Σ', and let A_{hor} be the area of $\partial\Sigma'$. Then*

$$S_{\text{matter}} + \frac{A_{\text{hor}}}{4} \leq S_0. \tag{5.1}$$

Note that this argument applies to any observer who reaches $\mathcal{I}^+$, but not to observers who fail to do so, for example because they fall into a black hole. With suitable energy conditions preventing the formation of large regions in the interior of a black hole, one still expects that the entropy in the causal past of such observers will be bounded as in Eq. (5.1). However, I will not give a detailed argument.

B. $\mathbf{dS}^\pm$ and global entropy

Next, consider a subset of $\mathbf{dS}^+$. Rather than demanding an asymptotic de Sitter region only in the future ($\mathcal{I}^+$), let us insist on a similar region in the past as well ($\mathcal{I}^-$). The spacetimes that possess both infinities define the set $\mathbf{dS}^\pm$.

The Penrose diagram of empty de Sitter space, Fig. 2, is exactly a square, because a light-ray starting in the infinite past will barely reach the opposite end of the universe in an infinite time. If de Sitter space is not completely empty, the Penrose diagram will be deformed. An example is the big bang universe described earlier; it corresponds to a diagram whose width exceeds its height. Following Leblonde, Marolf, and Myers (2002), I will call such diagrams "short". This shape reflects the property that no complete Cauchy surface is contained in any observer's past. Examples of this type preclude a bound on global entropy in the $\mathbf{dS}^+$ class of universes.

However, if a universe is in $\mathbf{dS}^\pm$, i.e., if it also has a past asymptotic de Sitter region, one can use a theorem of Gao and Wald (2000) to argue that its Penrose diagram is necessarily "tall", except for the exact vacuum de Sitter solution. Hence, the observer's event horizon will cross the entire diagram. Its area at late times will be A_0; at some finite earlier time t_{form}, its area will vanish. Hence, there will be early time ($t \leq t_{\text{form}}$) Cauchy surfaces which are completely contained ($\Sigma = \Sigma'$) in an observer's causal past P. Simply put, an observer in a $\mathrm{dS}^\pm$ universe can see what the *whole* universe looked like in its early days.

Cauchy surfaces that precede the formation of the event horizon do not intersect the horizon and hence have $A_{\text{hor}} = 0$. Then Eq. (5.1) reduces to $S_{\text{matter}} \leq S_0$, where S_{matter} refers to the global entropy on a sufficiently early Cauchy surface (Bousso, 2002). The existence of such surfaces is guaranteed by the Gao-Wald result for all $\mathrm{dS}^\pm$ spacetimes (except for the exact de Sitter solution, in which the matter entropy classically vanishes at all times).[5]

To summmarize, Eq. (5.1) and the theorem of Gao and Wald (2000) imply the following statement: *Consider a spacetime in $\mathbf{dS}^\pm$, and let S_{matter} be the total matter entropy on a sufficiently early Cauchy surface. Then*

$$S_{\text{matter}} \leq S_0. \tag{5.2}$$

The boundedness of the global entropy may seem a surprising result, since the early-time Cauchy surfaces have divergent volume. Hence, an arbitrary amount of

[5] The result of Gao and Wald (2000), and hence, our conclusion, relies on a number of technical requirements. In particular, the spacetime must be geodesically complete. The presence of both infinities is not sufficient to guarantee geodesic completeness even with reasonable smoothness requirements, because black holes can form. However, one would not expect the geodesic incompleteness due to black hole singularities to invalidate the above conclusions (Susskind, Thorlacius, and Uglum, 1993).

entropy can be placed on them without significant local back-reaction. However, with this choice of coordinates, the spacetime will be collapsing initially. The characteristic scale it will reach before it can re-expand is set by the value of the cosmological constant. However, if the matter density becomes larger than the energy density of the cosmological constant during the collapsing phase, it begins to dominate the evolution, and the universe will collapse to a big crunch.[6] Then there will be no future infinity, in contradiction to our assumption.

C. General $\Lambda > 0$ universes and the covariant entropy bound

Beginning with the set $\mathbf{dS}^+(\Lambda)$ in Sec. V.A, an absolute bound was derived on the entropy in an observer's causal past. Specializing to the $\mathbf{dS}^\pm(\Lambda)$ subset in Sec. V.B, this result was see to imply a bound on global entropy. Let us now explore the opposite direction and try to generalize to larger set of spacetimes. Can the $\mathbf{dS}^+(\Lambda)$ set be augmented non-trivially while retaining an absolute entropy bound of the type $S \leq S_0$?

Let us define $\mathbf{all}(\Lambda)$ to be the set of all spacetimes with positive cosmological constant Λ. Let's further assume that matter satisfies the dominant energy condition and that the number of species is not exponentially large. Note that Λ must be taken to be the lowest accessible vacuum energy. (For example, if there are scalar fields, Λ will refer to the cosmological constant at the minimum of their potential.)

$\mathbf{dS}^+(\Lambda)$ is a proper subset of $\mathbf{all}(\Lambda)$, that is, there are spacetimes with $\Lambda > 0$ that do not asymptote to de Sitter space in the future. Consider, for example, a closed universe which begins with an initial singularity (a big bang). If the density of ordinary matter is sufficiently high, the universe will cease its expansion before the cosmological constant can begin to dominate the evolution. The universe will then recollapse to a big crunch. Another example is the time-reversal of our own universe (supposing that it started with a big bang and is about to be dominated by a cosmological constant, as some observations suggest).

It is more difficult to obtain an absolute entropy bound in $\Lambda > 0$ spacetimes without a future de Sitter region, because the second law is no longer of any use. Indeed, it turns out that $\mathbf{all}(\Lambda)$ contains spacetimes with unbounded observable entropy (Bousso, DeWolfe, and Myers, 2002). The significance of these examples will be discussed later.

[6] Weakly perturbed de Sitter space is protected from singularity theorems (Hawking and Ellis, 1973; Wald, 1984) only because the strong energy condition is violated by a positive cosmological constant. If matter satisfying the strong energy condition dominates while the universe contracts, one expects this protection to break down.

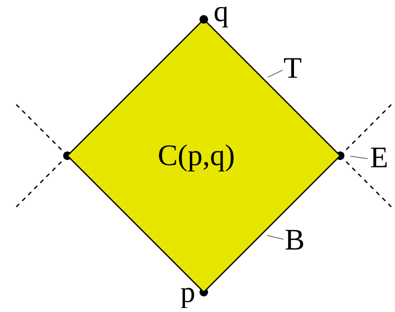

FIG. 3 Causal diamond $C(p,q)$, bounded by top (T) and bottom (B) cone, which intersect on the edge (E). A spatial region that fails to fit into *any* causal diamond cannot be probed in any experiment.

Nevertheless, it is possible to prove an entropy bound for an interesting subset of $\mathbf{all}(\Lambda)$ which includes certain spacetimes without $\mathcal{I}^+$. The place of the generalized second law is taken by the covariant entropy bound (Bousso, 1999a; Fischler and Susskind, 1998), which states that the entropy on any *light-sheet* (a contracting null hypersurface) will not exceed a quarter of its largest area (see Bousso, 2002, for a review). We will now summarize this argument.

In order to prove an entropy bound in this context, one must take great care to determine which parts of the spacetime are accessible to an observer. In order to show Eq. (5.1) for $\mathbf{dS}^+$, it sufficed to restrict to the portion of the spacetime in the observer's past. However, it is not really enough for information to be present in our past; for it to have operational meaning, it has to get to us—or, at the very least, it has to enter a spacetime region that we can actively probe. Such a region has the shape of a *causal diamond* (Bousso, 2000).

Given an observer's worldline between two points p and q, the causal diamond $C(p,q)$ is defined by the intersection of the past of q with the future of p (Fig. 3):

$$C(p,q) \equiv J^-(q) \cap J^+(p). \tag{5.3}$$

In order to determine the largest amount of information available to any observer in a spacetime $\mathcal{M}$, it suffices to find an upper bound on the entropy in the causal diamond $C(p,q)$, for all p and q in $\mathcal{M}$.[7] To demonstrate an absolute entropy bound for a whole set of spacetimes, one has to repeat this argument for each spacetime in the set (or, more efficiently, show that it applies to all).

The set considered in Bousso (2000) is a subset of $\mathbf{all}(\Lambda)$, namely the spherically symmetric spacetimes

[7] Causal diamonds may well play a rather general role in quantum gravity. Restriction to regions of the form $C(p,q)$ may alleviate various problems. For example, classically, no more than one point of any spacelike singularity can be contained in a causal diamond. Thus it is not clear that a quantum treatment of the vicinity of a spacelike singularity will need to address the singularity globally.

with cosmological constant Λ. Moreover, not all observers will be considered (i.e., not all p and q), but only the observers which are central (i.e., respect the spherical symmetry). The entropy seen by such observers, including horizon entropy if horizons are present near $C(p,q)$, will not exceed S_0.

This is shown as follows. At the top (the "future end"), the causal diamond $C(p,q)$ is bounded by a kind of past light-cone (the boundary of the past of q). Let us call this the top cone, $T(p,q)$. By the second law, the entropy captured on $T(p,q)$ is an upper bound on the entropy on any other time slice of the causal diamond. Hence it suffices to consider $T(p,q)$.

The cone $T(p,q)$ can be thought of as consisting of one light-like direction and $D-2$ spatial directions. The latter span cross-sectional spatial surfaces whose area depends on the position along the light-like direction. The focussing theorem implies that this cross-sectional area takes on precisely one maximal value, $A_{\max}$. This can be a local maximum along $T(p,q)$ (an apparent horizon), or it can be reached on the edge, E, of the cone, where T meets a similar cone, B, extending from p to the future (Fig. 3).

In either case, one can show that the surface of maximal area is either *normal* (that is, it is neither properly trapped nor properly anti-trapped), or that its area is smaller than the area of some normal surface. Hence, $A_{\max}$ must be smaller than the largest possible normal sphere in any spherically symmetric $\Lambda > 0$ universe. By Birkhoff's theorem, it must be possible to match such a normal sphere to a portion of a Schwarzschild-de Sitter solution. One easily verifies that no Schwarzschild-de Sitter solution contains any normal spherically symmetric surfaces with area greater than $4S_0$. It follows that

$$A_{\max} \leq 4S_0. \tag{5.4}$$

Then a rough argument shows immediately that the maximal entropy will be on the order of S_0. Namely, the surface $A_{\max}$ divides the top cone into one or two portions. On each portion, the cross-sectional area decreases in the direction away from $A_{\max}$. Hence, each portion is a light-sheet of $A_{\max}$ and contains entropy no greater than $A_{\max}/4 = S_0$. Therefore, $S_{\text{matter}} \leq 2S_0$. One also has to take into account the potential presence of a quasistatic event horizon. Its area, however, will not exceed $A_{\max}$. Hence, the total observable entropy, $S_{\text{matter}} + (A_{\text{hor}}/4)$, will not exceed $3S_0$.

A more detailed analysis (Bousso, 2000) shows that the D-bound, Eq. (4.3), applies to portions of the top cone in some cases, yielding tighter inequalities. Moreover, the presence of a quasistatic event horizon can be excluded in many cases. Overall, these considerations allow a tightening of the above inequality by a factor of 3. This leads to the following conclusion.

Observable matter entropy must be contained in a causal diamond. Let $C(p,q)$ be an arbitrary spherically symmetric causal diamond in a spacetime with positive cosmological constant Λ. Let $S_{\text{matter}}[C(p,q)]$ be the total matter entropy contained in $C(p,q)$. Define the total observable entropy $S[C(p,q)]$ to be $S_{\text{matter}}[C(p,q)]$ plus the horizon entropy $A_{\text{hor}}/4$ of any quasistatic horizons in the vicinity of the boundary of $C(p,q)$. Then

$$S[C(p,q)] \leq S_0. \tag{5.5}$$

The assumption of spherical symmetry was necessary in order to apply Birkhoff's theorem. This assumption is most relevant for the spacetimes in $\mathbf{all}(\Lambda) - \mathbf{dS}^+(\Lambda)$, where other arguments for an absolute entropy bound are lacking. For observers which reach $\mathcal{I}^+$ in a dS^+ spacetime, the bound 5.5 holds generally. We expect that Eq. (5.5) holds for all observers in $\mathbf{dS}^+$, but this has not been proven.

Yet, the fact that Eq. (5.1) is independent of spherical symmetry in $\mathbf{dS}^+$ suggests that its validity in $\mathbf{all}(\Lambda)$ is more generic than the preceding proof. It would be surprising if Eq. (5.5) could be violated by small deviations from spherical symmetry. Hence, it was conjectured in Bousso (2000) that Eq. (5.5), the "N-bound", holds generally in $\mathbf{all}(\Lambda)$.

However, the N-bound is in fact violated for some spacetimes in $\mathbf{all}(\Lambda)$ (Bousso, DeWolfe, and Myers, 2002). The known counterexamples are topologically different from spherically symmetric solutions. Thus, the possibility remains that a set of spacetimes larger than $\mathbf{dS}^+(\Lambda)$ but smaller than $\mathbf{all}(\Lambda)$ can be identified for which the N-bound holds generally. The existence of such a set may be of some significance for the prospects of quantum gravity theories with a finite number of states. This will be discussed at the end of the next section.

VI. QUANTUM GRAVITY IN DE SITTER SPACE

There is evidence that we possess a quantum theory of gravity for certain asymptotically Anti-de Sitter (AdS) spacetimes, which are negatively curved. String theory is known to admit spacetimes of the form $\text{AdS}_m \times M_n$ where M_n is a suitable compact Euclidean manifold, for some dimensions (m,n). In these backgrounds, a nonperturbative definition of the theory (and thus, of quantum gravity) has been found in terms of a conformal field theory (Maldacena, 1998; Gubser, Klebanov, and Polyakov, 1998; Witten, 1998). String theory (and nonperturbatively, M(atrix)-theory; Banks *et al.*, 1997) also defines S-matrix amplitudes in some asymptotically flat geometries and in other backgrounds.

Rather than attacking the problem of quantum gravity in complete generality, these successes suggest that progress can be made by restricting to suitable classes of spacetimes, characterized by asymptotic boundaries which are protected from quantum fluctuations. After addressing flat and negatively curved geometries, a natural step is to consider a positive cosmological constant next. This case is of particular interest because it might include the universe we inhabit (Riess *et al.*, 1998; Perlmutter *et al.*, 1999).

However, twenty-five years after the semi-classical results of Gibbons and Hawking (1977), a full quantum gravity theory for asymptotically de Sitter spacetimes is still lacking. In particular, it has turned out very difficult to realize de Sitter space in string theory (for recent approaches, see, e.g., Hull, 2001; Silverstein, 2001; Gutperle and Strominger, 2002; and the contribution by Maloney, Silverstein, and Strominger to this volume).

A. To drop infinity or to keep infinity

It is not clear how broad a set of spacetimes would be described by a "quantum theory of de Sitter space". From the experience with string theory, one expects that a particular matter content, presumably compatible with reasonable energy conditions, will arise from the theory. But what about asymptotic conditions? Should both asymptotic regions be demanded ($\mathbf{dS}^{\pm}$)? Will the theory describe the broader class $\mathbf{dS}^{+}(\Lambda)$, requiring only that the spacetime asymptote to de Sitter space in the future? Or should we abandon asymptotic conditions entirely and seek a theory of $\mathbf{all}(\Lambda)$, spacetimes characterized merely by a particular positive value of the cosmological constant?

No such confusions arise for asymptotically AdS and flat universes, because their asymptotic boundaries have only one connected component. Moreover, in the AdS and flat cases, the presence of the asymptotic region is not affected by continuous changes to Cauchy data. For spacetimes with positive cosmological constant, however, small changes in the stress tensor at one time may affect the presence of asymptotic de Sitter regions in the past or future (Bousso, 2000; Bousso, DeWolfe, and Myers, 2002). Of course, it is conceivable that a non-perturbative quantum theory will preclude variations of Cauchy data that would classically change the asymptotic structure. However, at least from a low-energy perspective, this would seem unnatural.

Classically, one must keep in mind that even in a dS^+ universe, not all observers reach future infinity; some fall into black holes. Moreover, an observer's causal diamond contains at most one point each of $\mathcal{I}^+$ and $\mathcal{I}^-$, so that physical observables cannot be defined in the asymptotic regions. At the quantum level, one expects all structures to be thermalized by Gibbons-Hawking radiation within finite time, which prevents observers from reaching $\mathcal{I}^+$ altogether. Together with the difficulty mentioned in the previous paragraph, this makes it desirable to seek a theory which ultimately does not require or make use of asymptotic de Sitter regions (for example a theory of $\mathbf{all}(\Lambda)$, with suitable matter restrictions).

However, one has no control over metric fluctuations in the spacetime interior, and no symmetries can be assumed. This impedes concrete progress without some reference to asymptotic regions. Moreover, whether or not it will ultimately survive in the formulation, the structure of the de Sitter infinities may well provide clues to properties of a quantum gravity theory. This has been the subject of numerous studies, especially in the context of the correspondence between de Sitter space and a Euclidean conformal field theory recently conjectured by Strominger (2001a).[8]

B. A theory with finite-dimensional Hilbert space?

Reproduction of the entropy S_0 of de Sitter space provides a key test for any formulation of quantum gravity. For $\mathrm{dS}^{\pm}$ spacetimes, Witten (2001) has argued that a Hilbert space of dimension e^{S_0} might arise from a larger space of states via a non-standard inner product. In the context of the dS/CFT correspondence, the entropy of de Sitter and Kerr-de Sitter spacetimes has been numerically reproduced (e.g., Balasubramanian, de Boer, and Minic, 2001; Bousso, Maloney, and Strominger, 2001). However, this was done by methods whose justification from the CFT point of view is still incomplete.

Fischler (2000a,b) and Banks (2000) have proposed that the finiteness of the de Sitter entropy should be elevated to a defining principle for the theory. The bound on observable entropy in all dS^+ universes, Eq. (5.1), implies that a finite number of states suffices to completely describe all of physics in such universes. It would be most economical, therefore, to seek a quantum gravity theory with Hilbert space of finite dimension e^{S_0}. Conversely, perhaps a positive cosmological constant should be regarded as nature's way of ensuring that entropies greater than S_0 simply cannot occur—an essential cutoff if our Hilbert space is really finite.

The Banks-Fischler proposal suggests that a positive cosmological constant should not be regarded as a consequence of complicated quantum corrections and cancellations. Rather, $\Lambda > 0$ constitutes a direct and fundamental reflection of the size of the Hilbert space of quantum gravity. A correpondence between the value of the cosmological constant and the number $\mathcal{N}$ of states in the Hilbert space is thus implied. Thus, the proposal offers a fresh perspective on the cosmological constant problem. (This is called the "Λ-N correspondence", where $N = \log \mathcal{N}$, in Bousso, 2000.)

It must be stressed that this proposal goes beyond what is necessary to explain de Sitter entropy. It exploits the fact that a cosmological constant can be regarded as a fixed property of a theory, rather than a variable parameter associated with a solution. The finite entropy of a black hole, by contrast, reflects only those states (of

[8] I will not attempt to survey the literature on this approach. Extensive lists of references may be found in Balasubramanian, de Boer, and Minic (2001), and in Spradlin and Volovich (2001). In viewing cosmological evolution as inverse RG flow, Strominger (2001b) has also outlined a possible approach to understanding the apparent increase in the number of available degrees of freedom with time.

a larger or infinite Hilbert space) which actually correspond to the black hole. As the mass of the black hole is usually considered a variable parameter, it cannot possibly constrain the dimension of the full Hilbert space, which ought to be infinite for asymptotically flat or AdS spacetimes.

The Banks-Fischler proposal asserts that the de Sitter entropy will not arise from a subset of states, but represents the complete Hilbert space of a theory. In particular, there would be no possibility of describing de Sitter by a theory with infinitely many states. This would be a remarkable constraint. For example, a single harmonic oscillator has an infinite-dimensional Hilbert space, not to speak of quantum field theory or string theory.

The Banks-Fischler proposal assigns a crucial role to theories with Hilbert space of finite dimension $\mathcal{N}$ for the description of certain cosmological spacetimes. The physics of asymptotically flat or AdS universes (e.g., string theory) would be recovered only in the limit $\mathcal{N} \to \infty$. Conversely, this would explain why de Sitter space has not been found in string theory.

For $D > 4$, the Λ-N correspondence suffers from the shortcoming that the specification of a positive cosmological constant alone does not guarantee the entropy bound (5.5). As discussed at the end of the previous section, explicit counterexamples are known (Bousso, DeWolfe, and Myers, 2002). In other words, $\mathbf{all}(\Lambda)$ contains spacetimes with observable entropy greater than S_0. Such spacetimes cannot possibly be described by a theory with only $\mathcal{N} = e^{S_0}$ states. Some $\Lambda > 0$ spacetimes must be excluded from the "gravity dual" of any finite-$\mathcal{N}$ theory.

Thus, a simple relation between the size of Hilbert space and the cosmological constant cannot hold unless additional conditions are specified. Clearly, the demand of a future asymptotic de Sitter region is a sufficient condition. However, as discussed in Sec. VI.A, it is both artificial and operationally questionable to distinguish spacetimes in $\mathbf{dS}^+$ from at least some of the closely related spacetimes in $\mathbf{all}(\Lambda) - \mathbf{dS}^+(\Lambda)$.

The search for a suitable completion of the set $\mathbf{dS}^+(\Lambda)$, in which Eq. (5.5) would hold, has not yet succeeded. Such a completion would give support to the Banks-Fischler proposal. It would provide a concrete candidate set of spacetimes that might be described by quantum gravity theories with finite $\mathcal{N}$, if such theories exist.

C. Other questions

There are many other open questions about de Sitter space. If the region near $\mathcal{I}^+$ cannot be observed, then what are the observables? Is the evolution of matter fields and horizon in de Sitter space unitary or is information lost?

Hawking (1976) claimed that black holes convert pure states to mixed states. The debate continues despite recent results in support of unitarity (Strominger and Vafa, 1996; Maldacena, 1998). For a de Sitter horizon, it is not clear whether the question is even well-posed. One can attempt to extend black hole complementarity (Susskind, Thorlacius, and Uglum, 1993) to de Sitter space (Dyson, Lindesay, and Susskind, 2002) and restrict to a causal diamond region (Bousso, 2000). However, asymptotic states cannot be defined, and it is not clear how unitary evolution would be verified in any experiment.

In particular, a black hole can evaporate completely; in principle, it can return information in correlations of the Hawking radiation. On the other hand, the cosmological horizon never disappears completely except in the catastrophic collapse of the entire spacetime. No more than a third of the degrees of freedom are available in matter form (this limit arises from the largest black hole in de Sitter space). Thus, if the whole system, consisting of matter and the cosmological horizon, were in a pure state, the matter subsystem would be unlikely to contain any information at all (Page, 1993). Finite observer lifetimes further complicate this problem.

Yet, whether or not we live in de Sitter space, many of the above conceptual problems arise in any attempt at a quantum treatment of cosmology. de Sitter space offers a relatively simple arena for their investigation.

VII. INSTABILITIES OF THE NARIAI SOLUTION

A. Schwarzschild-de Sitter and Nariai

Black holes in de Sitter space cannot be larger than the de Sitter horizon. Small Schwarzschild-de Sitter black holes are much hotter than the cosmological horizon, and the geometry in their neighborhood is a good approximation of a Schwarzschild black hole in flat space. Their evolution will not differ much from their flat space cousins.

In this section I will discuss quantum aspects of black holes which are of a size comparable to the cosmological horizon. These "large" Schwarzschild-de Sitter black holes have no flat space analogue. They constitute interesting physical systems in their own right. The interplay between the two horizons leads to novel effects. Instabilities arise which can complicate the global structure of asymptotically de Sitter spacetimes.

The Schwarzschild-de Sitter solution is given by the metric

$$\frac{ds^2}{\ell^2} = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_2^2, \qquad (7.1)$$

where

$$V(r) = 1 - \frac{2E}{r} - r^2. \qquad (7.2)$$

This metric is static and covers only a portions of the maximally extended spacetime, as seen in the Penrose diagram of Fig. 4.

The mass parameter E grows monotonically with the size of the black hole. For $E = 0$ one recovers empty

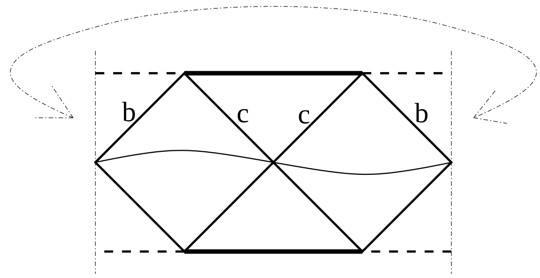

FIG. 4 Penrose diagram of a Schwarzschild-de Sitter spacetime. The curved line is a slice of equal time in the static coordinates; its geometry is a warped product of S^1 and S^2. The S^2 directions are suppressed in this diagram; the S^1 arises because the left and right ends are identified. The black hole (b) and cosmological (c) horizons are indicated. The static coordinates, Eq. (7.1), cover one of the diamond-shaped regions. The black hole singularity and the de Sitter infinity are shown as dashed and bold lines, respectively. The Nariai solution has the same Penrose diagram except for the nature of the boundaries. The Penrose diagram for a multiple Schwarzschild-de Sitter solution is obtained by joining several copies of this diagram before identifying the ends.

de Sitter space. If $0 < E < E_{\max} \equiv 3^{-3/2}$, $V(r)$ has two positive zeros, corresponding to the black hole and the cosmological horizon. The fully extended spatial geometry at $t = 0$ has topology $S^1 \times S^2$. The size of the S^2 varies as a function of the coordinate on the S^1; it is minimal on the black hole horizon and maximal on the cosmological horizon.

The cosmological horizon decreases as E is increased. For $E = E_{\max}$, both horizons are at the same value of r, at $r = 3^{-1/2}$. In the metric (7.1) it would appear that there is no space left in the geometry. In fact, however, only the coordinate r becomes degenerate and ceases to be useful. A proper limiting procedure (Ginsparg and Perry, 1983) shows that the geometry of the $t = 0$ slice remains perfectly regular as $E \to E_{\max}$ and becomes the geometry of the Nariai solution. This space is the direct product of an S^1 and an S^2, both with radius $r = 3^{-1/2}$.

B. First-mode instability

Although the Nariai and Schwarzschild-de Sitter geometries nearly agree at an instant of time, they differ markedly in their temporal evolution. The S^1 factor of the Nariai solution expands exponentially, forming a 1+1 dimensional de Sitter spacetime. The S^2 factor remains constant. In global coordinates, the Nariai metric is given by

$$\frac{ds^2}{\ell^2} = \frac{1}{3}\left(-dT^2 + \cosh^2 T\, dx^2 + d\Omega_2^2\right). \qquad (7.3)$$

Unlike all of the Schwarzschild-de Sitter solutions, the Nariai spacetime is homogeneous. It does not possess any singularity, nor does it possess four-dimensional asymptotic de Sitter regions.[9]

These properties make a classical instability of the Nariai solution manifest. Consider a small perturbation of the $T = 0$ slice, such that the two-sphere area is not constant but instead is given by $(\ell^2/3)(1 + \epsilon \cos x)$. That is, the two-sphere size oscillates once as a function of the angular variable on the S^1. To leading order, this will revert the geometry to a nearly maximal Schwarzschild-de Sitter metric (Ginsparg and Perry, 1983). The two-spheres that are smaller than the Nariai value will collapse to form the black hole interior. The larger two-spheres expand exponentially to generate an asymptotic de Sitter region.

A nearly maximal Schwarzschild-de Sitter black hole is classically stable. Quantum mechanically one expects Hawking radiation to be emitted both by the black hole, and by the cosmological horizon that surrounds it. In the Nariai solution, the two horizons would be in equilibrium at a temperature $T = 3^{1/2}/(2\pi\ell)$ (Bousso and Hawking, 1996b). The black hole and the cosmological horizon emit and receive equal amounts of radiation.

One expects perturbations of the Nariai geometry to disturb this equilibrium. This question was studied by Bousso and Hawking (1998b). The quantum radiation was included in the s-wave approximation at the level of a one-loop effective action. (Different actions were employed by Nojiri and Odintsov, 1999a,b, 2001). We found that large Schwarzschild-de Sitter black holes are unstable to radiation.

Because of the interplay between the two nearly equal horizons, the dynamics of the evaporation can be more involved than it is for small black holes. For some perturbations, the black hole horizon grows towards the Nariai value at early times. However, the shrinking mode is expected to dominate at late times.

Our analysis was carried out perturbatively about the Nariai solution. This did not amount to a conclusive argument that the evaporation of black holes will continue well into the small black hole regime, where the effect of the cosmological horizon can be neglected and one can be sure that the evaporation process will in fact complete. Later this was demonstrated by a non-linear numerical analysis (Niemeyer and Bousso, 2000). Schwarzschild-de Sitter black holes evaporate, be they large or small.

[9] The singularities as well as the asymptotic boundaries of the Schwarzschild-de Sitter spacetimes lie in the far future or past. They are not places in space. Hence the different boundary structure of the Nariai solution does not contradict the similarity of the spatial metrics at one instant of time [the $T = 0$ slice of (7.3) and the $E \to E_{\max}$ limit of (the full extension of) the $t = 0$ slice of (7.1)].

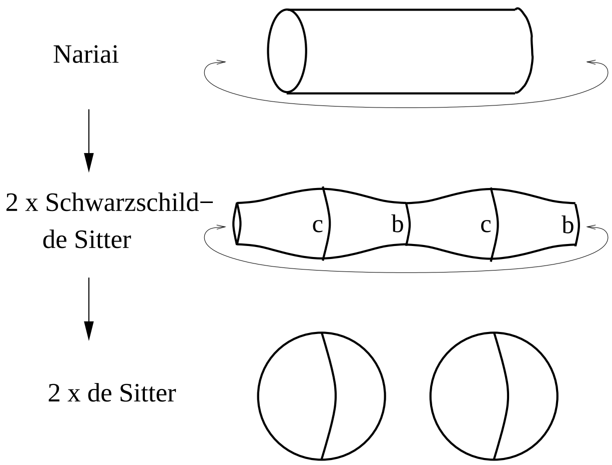

FIG. 5 Fragmentation. Nariai space is a product $S^1 \times S^2$. The S^2 is represented as an S^1 here; the time direction is suppressed in the drawings and indicated only through the arrows evolving the snapshots of the spatial geometry. Upon a higher-mode perturbation (here, $n = 2$), Nariai space can evolve into a sequence of Schwarzschild-de Sitter universes. Black hole (b) and cosmological (c) horizons are indicated. When the black holes evaporate, the geometry pinches in several places, and only disconnected de Sitter portions remain.

C. Higher modes and fragmentation

So far, I have discussed only a first-mode perturbation. The Nariai solution has more in store. A constant mode perturbation[10] makes the two-sphere area everywhere smaller, or everywhere larger, than the value $\ell^2/3$. The former case is like trying to make a black hole that doesn't fit into de Sitter space; the spacetime collapses globally in a big crunch. The latter case will lead to a solution in which all regions are locally de Sitter but the global topology is non-trivial.

Higher-mode perturbations of the Nariai solution give rise to rather drastic global effects (Bousso, 1998). Consider the n-th mode ($n > 1$), which perturbs the two-sphere radius as $(\ell^2/3)(1 + \epsilon \cos nx)$. If this fluctuation dominates, one finds the following classical evolution. The mode is oscillatory at first, increasing in proper wavelength as the S^1 expands. When the S^1 has expanded by a factor n, the wavelength becomes larger than the horizon scale, and the mode grows exponentially. This marks the beginning of the formation of n black hole interiors and n asymptotically de Sitter regions (Fig. 5).

The geometry then resembles a sequence of Schwarzschild-de Sitter solutions. In the ordinary Schwarzschild-de Sitter solution ($n = 1$), the black hole connects opposite ends of a single asymptotic de Sitter region (Fig. 4). There are two black hole horizons, but only one black hole interior. In the solutions obtained for $n > 1$, however, each black hole connects two different de Sitter regions; after traversing[11] n such regions and n black holes, one is back at the first region.

The higher-mode instabilities of the Nariai solution become even more interesting when Hawking radiation is taken into account (Bousso, 1998). Perturbatively, each of the black holes is found to evaporate, much like a large Schwarzschild-de Sitter black hole in a single asymptotic region. At the non-linear level, the evaporation was again found to continue until the black holes are much smaller than the cosmological horizons (Niemeyer and Bousso, 2000). Then nothing can stabilize them, and one expects that they will disappear altogether.

If $n = 1$, the complete evaporation of a Schwarzschild-de Sitter black hole can be visualized as a deformation and eventual topological transition of the $S^1 \times S^2$ spatial sections. The direct product metric (Nariai) becomes a warped product (Schwarzschild-de Sitter), in which the two-sphere size varies along the S^1. Eventually the S^2 vanishes at one point; the $S^1 \times S^2$ geometry pinches off and reverts to an S^3 topology (empty de Sitter space). If $n > 1$, however, the S^2 size decreases at n points on the S^1, as all n black holes evaporate. Hence, the spatial geometry is pinched in several places, leaving behind n disconnected spatial manifolds, each of topology S^3.

Thus, the spacetime fragments. Notice that this is not an off-shell Planckian quantum fluctuation of the metric by which some kind of baby universe is created. The process is slow and under good semi-classical control. Planckian curvatures enter only at the endpoint of the evaporation, which is usually assumed to correspond to the disappearance of the black hole. The fragments can be arbitrarily large.

D. Global structure, black hole creation, and proliferation

The instability of the Nariai solution is remarkable in that arbitrarily small variations in the spatial geometry can lead to an unlimited variety of causal structures. The mode number of the dominant perturbation determines the number of copies of $\mathcal{I}^+$; upon inclusion of Hawking radiation, it determines the number of disconnected components of space at late times. This can be regarded as a generalization of an effect discussed in Sec. VI above. There it was noted that small changes in Cauchy data can determine whether or not $\mathcal{I}^+$ is present at all.

In this section, I have described the decay of the Nariai solution, not the decay of de Sitter space. However,

[10] Classically, the perturbations in vacuum must satisfy certain constraint equations. In the one-loop model studied by Bousso and Hawking (1998b) this becomes a constraint on the initial distribution of the radiation, which can be satisfied for all metric perturbations at leading order.

[11] This is a description of spacelike surfaces, not of causal paths through the spacetime.

it is possible for black holes to nucleate spontaneously in de Sitter space. A number of arguments lead to this expectation and yield mutually compatible estimates of the nucleation rate. For example, black holes can be assumed to lie in the exponential tail of the thermal radiation emitted by the cosmological horizon. However, the gravitational instanton approach has been most commonly used (Ginsparg and Perry, 1983; Bousso and Hawking, 1995, 1996b, 1999a; Mann and Ross, 1995; Chao, 1997; Bousso and Chamblin, 1999). In this picture, the formation of black holes is regarded as a tunneling event, much like the Schwinger pair creation of charged particles in a strong electric field. One finds at leading order that the rate of black hole formation is suppressed by the difference between the de Sitter entropy and the Schwarzschild-de Sitter entropy:

$$\Gamma \sim \exp(S_{\mathrm{SdS}} - S_0). \tag{7.4}$$

In particular, a Nariai black hole might nucleate, with a rate of $\exp(-\pi/\Lambda)$. Thus, the possibility arises that de Sitter space proliferates (Bousso, 1998) by the iterated production, evaporation, and fragmentation of Nariai geometries. However, it is not clear whether a global description of the nucleation process, suggested by the instanton approach, is adequate (Sec. V.C). In fact, the repeated nucleation of black holes at different times, which occurs naturally in a single causal region from a statistical mechanics point of view, faces global obstructions in the instanton picture, as the nucleation surfaces would mutually intersect (Jacobson, unpublished). For the purpose of generating an unlimited number of components of $\mathcal{I}^+$, at least, those obstructions can be circumvented by considering black holes of sufficient charge (Bousso, 1999c).

E. Discussion

The work reported in this section precedes the current surge of interest in de Sitter space and accelerating universes. Many of the present approaches are guided by the desire to apply string theory, or at least some of the lessons learned from recent developments in string theory, to the problem of de Sitter quantum gravity. The study of the Nariai solution might add some useful perspectives to these endeavors.

It is possible to explore the vacuum structure of string theory by identifying certain unstable configurations (e.g., the vacuum of bosonic string theory, or a D-brane/anti-D-brane pair). The idea is to study their evolution and try to describe the structure of the decay product (see, e.g., Sen, 1999). In this sense, the Nariai solution may offer a way of circumventing the difficulty of incorporating de Sitter space in string theory. If a Nariai solution could be constructed, its decay would naturally lead to de Sitter space, and perhaps to the more complicated configurations obtained by the fragmentation process. Though there is no guarantee that a Nariai solution can be implemented in string theory, the possibility of this approach should be noted.

In some of the present approaches to de Sitter quantum gravity (Sec. VI), asymptotic boundaries play a central role, and the presence of a single copy each of $\mathcal{I}^+$ and $\mathcal{I}^-$ is often assumed (e.g., Strominger, 2001a; Witten 2001). We noted earlier that the presence of these boundaries is by no means guaranteed and is sensitive to small variations of Cauchy data. The present section has shown that the conformal boundary can become arbitrarily complicated in de Sitter-like spacetimes. There can be an unlimited number of disconnected components of $\mathcal{I}^+$. Moreover, spacetimes with a large variety of different asymptotic structures can be obtained from small variations of topologically identical initial conditions. It is conceivable that this additional structure might play a role in the formulation of a quantum gravity theory in de Sitter space.

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