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DUST TIME IN QUANTUM COSMOLOGY

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We give a formulation of quantum cosmology with a pressureless dust and arbitrary additional matter fields. The dust provides a natural time gauge corresponding to a cosmic time, yielding a physical time independent Hamiltonian. The approach simplifies the analysis of both Wheeler-deWitt and loop quantum cosmology models, broadening the applicability of the latter.

Keywords: quantum cosmology, canonical formalism, time

Introduction: In the process of constructing a consistent non-perturbative theory of quantum gravity the principal obstacle is the problem of time: due to the time-reparametrization invariance of general relativity, the physical Hamiltonian is replaced by a Hamiltonian constraint in the canonical formulation. Explicit time gauge fixing leads in general to non-unique and gauge-dependent theory.

A possible solution is deparametrization, where a time variable is provided by suitable matter fields. This method however usually produces a Hamiltonian of the form of a square root – which is often too complicated to apply to physical scenarios.

One completion of the gravity quantization procedure[1] avoids these problems: one couples gravity (and other matter) with a single timelike irrotational dust field.[2] For this system an application of a natural time gauge distinguished by the dust field, and the diffeomorphism invariant formalism of loop quantum gravity,[3] results in a theory with a true Hamiltonian which is not a square root. Quantum evolution is described by a Schrödinger equation with time independent Hamiltonian.

Here we discuss an application[4] of this development in both the Wheeler-DeWitt (WDW) and loop quantum cosmology (LQC) quantization. To illustrate the useful properties of the approach we present an example of the flat isotropic universe with cosmological constant.

The model: The theory is given by the Einstein-Hilbert action with timelike dust field T and an arbitrary matter Lagrangian $\mathcal{L}_m$

$$S = \frac{1}{4G}\int \mathrm{d}^4x\sqrt{-g}R - \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_m - \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}M(g^{ab}\partial_a T\partial_b T + 1). \quad (1)$$

where M is a Lagrange multiplier.

In the canonical formalism this action leads to a Hamiltonian constraint linear in the canonical momentum p_T of T and the usual spatial diffeomorphism constraint. The dust field admits a natural time gauge $T = t$ in which the true Hamiltonian is

$$\boldsymbol{H} \equiv -p_T = \boldsymbol{H}_G + \boldsymbol{H}_m. \tag{2}$$

Here $\boldsymbol{H}_G$ and $\boldsymbol{H}_m$ are respectively the gravitational and the matter part of the Hamiltonian constraint resulting from Eq. (1).

In the flat isotropic setting considered here the spacetime metric is of the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)(\mathrm{dx}^2 + \mathrm{dy}^2 + \mathrm{dz}^2), \tag{3}$$

where the unity of the lapse follows directly from equations of motion derived from Eq. (2). Thus t reproduces exactly the cosmic time.

To present the formalism we further restrict our attention to the case $\boldsymbol{H}_m = 0$ and $\Lambda = 0$. Now, the complete information about the system is captured in the canonical pair (v, b), where $v = \alpha^{-1}a^3$ (with $\alpha \approx 1.35\ell_{\mathrm{PL}}^3$) and $\{v, b\} = 2$.

Quantization: The classical system may be quantized using two distinct techniques. These are the geometrodynamic (or Wheeler-DeWitt approach) and and loop techniques (Loop Quantum Cosmology).[5]

• *WDW (Schrödinger) quantization:* Here one uses the standard Schrödinger representation. The canonical variables are promoted to operators acting on the dense domain in $L^2_s(\mathbb{R}, \mathrm{d}v)$ (where s denotes symmetric functions). The (symmetrically ordered) Hamiltonian (2) takes the form

$$\hat{\boldsymbol{H}}_G = \frac{3\pi G}{2\alpha}\sqrt{|\hat{v}}\hat{B}^2\sqrt{|\hat{v}|}. \tag{4}$$

so Eq. (2) leads to a time-dependent Schrödinger equation

$$i\partial_t\Psi = \hat{\boldsymbol{H}}_G\Psi \tag{5}$$

The operator $\hat{\boldsymbol{H}}_G$ *admits a* 1*-parameter family of self-adjoint extensions.* For each of them there exists an invertible map $\underline{P}_\beta$ to auxiliary Hilbert spaces. In a new variable $x = -1/b$, proportional to inverse Hubble parameter, the time evolution of the physical state is represented by a simple formula

$$[\underline{P}_\beta\Psi](x,t) = \int_{\mathbb{R}^+} \mathrm{d}k\, \tilde{\Psi}(k)\, [\theta(x)e^{ikx} + \theta(-x)e^{i\beta}e^{ikx}]e^{i\omega_k t}, \quad \omega_k = 3\pi\ell_{\mathrm{Pl}}^2\alpha^{-1}k, \tag{6}$$

This formula describes a coherent plain wave packet propagating freely through configuration space with exception of the point $x = 0$ (corresponding to a classical singularity), where it is rotated by an extension-dependent angle. The interesting physical observables, like the volume $\hat{V} = |\hat{a}^3|$ when mapped to auxiliary spaces take simple analytic form, allowing to evaluate the trajectory. In particular

$$\langle\hat{V}\rangle(t) = V(t) = 2\alpha^{-1}\langle\hat{k}\rangle\ [3\pi\ell_{\mathrm{Pl}}^2(t - t_s)]^2 + 2\alpha\tilde{\sigma}_x^2, \quad \hat{k} = k\mathbb{I}, \tag{7}$$

where t_s has an interpretation as the big crunch/bang time. and $\tilde{\sigma}_x$ is related to the dispersion of $\hat{x}$ at t_s.

• <u>*LQC quantization:*</u> The physical Hilbert space is $\mathcal{H} = L^2(\bar{\mathbb{R}}, \mathrm{d}\mu_B)$, where $\bar{\mathbb{R}}$ is the Bohr compactification of the real line and $\mathrm{d}\mu_B$ is the Haar measure on it. The Hamiltonian $\boldsymbol{H}_G$ is now a difference operator in v

$$\hat{\mathcal{H}}_G \;=\; -\frac{3\pi G}{8\alpha}\sqrt{|\hat{v}|}(\hat{\mathcal{N}} - \hat{\mathcal{N}}^{-1})^2\sqrt{|\hat{v}|}\ , \quad \hat{\mathcal{N}}|v\rangle \;=\; |v+1\rangle\ , \tag{8}$$

and is *non-negative definite and essentially self-adjoint.* By introducing a variable $x = -\cot(b)$ and repeating the procedure applied in WDW quantization we arrive to the analog of the Schrödinger eq. (5) generating the quantum evolution

$$[P\Psi](x,t) \;=\; \int_{\mathbb{R}^+} \mathrm{d}k\ \tilde{\Psi}(k) e^{-ikx} e^{i\omega_k t}\ , \quad \tilde{\Psi} \in L^2(\mathbb{R}^+, k\mathrm{d}k)\ , \tag{9}$$

where ω_k is given by (6) and P is a mapping analogous to $\underline{P}_\beta$. This in turn leads to a quantum trajectory (where the meaning of t_s and $\tilde{\sigma}_x^2$ is the same as in Eq. (7))

$$V(t) \;=\; 2\alpha^{-1}\langle\hat{k}\rangle\ [3\pi\ell_{\mathrm{Pl}}^2(t-t_s)]^2 + 2\alpha\tilde{\sigma}_x^2 + 2\alpha\langle\hat{k}\rangle. \tag{10}$$

The above description easily generalizes to the case $\Lambda \neq 0$, where the Hamiltonian remains self-adjoint and the state evolution is still given by Eq. (9) with the form of $x(b)$ distinct from $\Lambda = 0$ case but still explicitly known.

Comparison: The above examples are an explicit illustration of singularity resolution in LQC and the lack of it in WDW, in the following sense.

• In WDW, evolving the state across the singularity requires specification of additional boundary data (choice of an extension). Furthermore the quantum trajectory $V(t)$ approaches $V = 0$ up to quantum variation.

• In LQC the quantum evolution is unique and the minimal volume of the universe is well isolated from zero. This illustrates the dynamical resolution of a singularity through a big bounce.[5]

Application - modified Friedmann equation: In the LQC quantization the Hubble and gravitational energy density operators

$$\hat{H} = \pi\ell_{\mathrm{Pl}}^2\alpha^{-1}\sin(2\hat{b}), \quad \hat{\rho}_G \;=\; -\rho_c\sin^2(\hat{b}), \quad \rho_c \approx 0.41\rho_{\mathrm{Pl}}. \tag{11}$$

are physical observables. The precise relation between their expectation values takes the form (where σ_H, σ_ρ are the dispersions of $\hat{H}$ and $\hat{\rho}_G$ respectively)

$$\langle\hat{H}\rangle^2 \;=\; \frac{8\pi G}{3}\langle-\hat{\rho}_G\rangle\left(1 - \frac{\langle-\hat{\rho}_G\rangle}{\rho_c}\right) \;-\; \left[\frac{8\pi G}{3}\frac{\sigma_\rho^2}{\rho_c} + \sigma_H^2\right]\ , \tag{12}$$

which is the *exact* realization of the so called *modified Friedmann equation* valid for all physical states and the arbitrary matter content.

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