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Quantum Harmonic Black Holes

Roberto Casadio, Alessio Orlandi

Abstract Inspired by the recent conjecture that black holes are condensates of gravitons, we investigate a simple model for the black hole degrees of freedom that is consistent both from the point of view of Quantum mechanics and of General Relativity. Since the two perspectives should “converge” into a unified picture for small, Planck size, objects, we expect our construction is a useful step for understanding the physics of microscopic, quantum black holes. In particular, we show that a harmonically trapped condensate gives rise to two horizons, whereas the extremal case (corresponding to a remnant with vanishing Hawking temperature) is not contained in the spectrum.

1 Introduction

One of the major mysteries in modern theoretical physics is to understand what are the internal degrees of freedom of black holes (BHs). Our best starting point is the classical description of BHs provided by General Relativity [6], along with well established semiclassical results, such as the predicted Hawking radiation [15, 16].

It was recently proposed by Dvali and Gomez that BHs are Bose-Einstein Condensates (BECs) of gravitons at a critical point, with Bogoliubov modes that become degenerate and nearly gapless representing the holographic quantum degrees of freedom responsible for the BH entropy and the information storage [11, 12, 13, 14]. In order to support this view, they consider a collection of objects (gravitons) interacting via Newtonian gravitational potential $V_N \sim -\frac{G_N \mu}{r}$ and whose effective

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mass μ is related to their characteristic quantum mechanical size via the Compton/de Broglie wavelength $\ell \simeq \frac{\hbar}{\mu} = \ell_p \frac{m_p}{\mu}$.

These bosons can superpose and form a “ball” of radius ℓ , and total energy $M = N\mu$, where N is the total number of constituents. Within the Newtonian approximation, there is then a value of N for which the whole system becomes a BH. In details, given the coupling constant $\alpha = \frac{\ell_p^2}{\ell^2} = \frac{\mu^2}{m_p^2}$ there exists an integer N such that no constituent can escape the gravitational well it contributed to create, and which can be approximately described by the potential

$$U(r) \simeq V_N(\ell) \simeq -N\alpha \frac{\hbar}{\ell} \Theta(\ell - r), \quad (1)$$

where Θ is the Heaviside step function. This implies that components in the depleting region are “marginally bound” when the kinetic energy given by $E_K \simeq \mu$ equals the potential energy

$$E_K + U \simeq 0 \quad \Longleftrightarrow \quad N\alpha = 1. \quad (2)$$

Consequently, the effective boson mass and total mass of the BH scale according to

$$\mu \simeq \frac{m_p}{\sqrt{N}} \quad \text{and} \quad M = N\mu \simeq \sqrt{N}m_p. \quad (3)$$

Note that one has here assumed the ball is of size ℓ (since bosons superpose) and, therefore, the constituents will interact at a maximum distance of order $r \sim \ell$, with fixed ℓ . The Hawking radiation and the negative specific heat spontaneously result from quantum depletion of the condensate for the states satisfying Eq. (2). This description is partly Quantum Mechanics and partly classical Newtonian physics, but no General Relativity is involved, in that geometry does not appear in the argument.

2 Quantum Mechanical Model

We can improve on the former model by employing the Quantum Mechanical theory of the harmonic oscillator as a (better) mean field approximation for the Newtonian gravitational interaction acting on each boson inside the BEC¹. The potential U in Eq. (1) is therefore replaced by²

$$V = \frac{1}{2}\mu\omega^2(r^2 - d^2)\Theta(d - r) \equiv V_0(r)\Theta(d - r) \quad (4)$$

¹ We shall use units with $c = 1$, $\hbar = \ell_p m_p$ and the Newton constant $G_N = \ell_p/m_p$.

² This is nothing but Newton oscillator, which would correspond to a homogenous BEC distribution in the Newtonian approximation.

and we further set $V(0) = U(0)$, so that $\frac{1}{2}\mu\omega^2 d^2 = N\alpha\frac{\hbar}{\ell}$. We also assume that the effective mass, length and frequency of a single graviton mode are related by $\mu = \hbar\omega = \hbar/\ell$, which leads to $d = \sqrt{2N\alpha}\ell = \sqrt{2N}\ell_p$.

If we neglect the finite size of the well, the Schrödinger equation in polar coordinates yields the well-known eigenfunctions

$$\psi_{nlm}(r, \theta, \phi) = \mathcal{N} r^l e^{-\frac{r^2}{2\ell^2}} {}_1F_1(-n, l+3/2, r^2/\ell^2) Y_{lm}(\theta, \phi), \quad (5)$$

where \mathcal{N} is a normalization constant, ${}_1F_1$ the Kummer confluent hypergeometric function of the first kind and $Y_{lm}(\theta, \phi)$ are the usual spherical harmonics. The corresponding energy eigenvalues are given by $E_{nl} = \hbar\omega [2n + l + \frac{3}{2} - V(0)] = \hbar\omega [2n + l + \frac{1}{2} (3 - \frac{d^2}{\ell^2})]$, where n is the radial quantum number and l the angular momentum (not to be confused with ℓ). Following the idea in Ref. [11, 12, 13, 14], we view the above spectrum as representing the effective Quantum Mechanical dynamics of depleting modes, which can be described by the first (non-rotating) excited state³ $\psi_{100}(r) = \sqrt{\frac{2}{3\ell^2\sqrt{\pi}}} e^{-\frac{r^2}{2\ell^2}} (2r^2 - 3\ell^2)$. The marginally binding condition (2), that is $E_{10} \simeq 0$, then leads to the desired scaling laws $\ell = \sqrt{\frac{2N}{7}}\ell_p$ and $\mu = \sqrt{\frac{7}{2N}}m_p$. We can now estimate the effect of the finite width of the potential well (4) by simply applying first order perturbation theory and obtain $\Delta E_{10} = -\int_d^\infty r^2 dr \psi_{100}^2(r) V_0(r) \simeq -\frac{0.1}{\sqrt{N}}m_p$. This can now be compared, for example, with the ground state energy $E_{00} = -\sqrt{14/N}m_p \simeq -3.7m_p/\sqrt{N}$. Since $|\Delta E_{10}| \ll |E_{00}|$, our approximation appears reasonable. We however remark that the ground state energy in this model has no physical meaning. Indeed, the Schrödinger equation must be viewed as describing the effective dynamics of BH constituents, and the total energy of the “harmonic black hole” is still given by the sum of the individual boson effective masses,

$$M = N\mu \simeq \sqrt{\frac{7N}{2}}m_p, \quad (6)$$

in agreement with the “maximal packing” of Eq. (3) and the expected mass spectrum of quantum BHs (see, for example, Refs. [2, 10]).

3 Regular geometry

It is now reasonable to assume that the actual density profile of the BEC gravitational source is related to the ground state wave function in Eq. (5) according to

³ Note we have already integrated out the angular coordinates.

$$\rho(r) \simeq M \psi_{000}^2 \simeq \frac{7^2 m_p e^{-\frac{7r^2}{2N\ell_p^2}}}{\sqrt{\pi} N \ell_p^3}. \quad (7)$$

Similar Gaussians profiles have been extensively studied in Refs. [17, 18], where it was proven that such densities satisfy the Einstein field equations with a “de Sitter vacuum” equation of state, $\rho = -p$, where p is the pressure. Curiously, BECs can display this particular equation of state [7, 8, 20]. This feature provides a connection between Quantum Mechanics and the geometrical description.

Let us indeed take the static and normalised, energy density profile of Ref. [18]⁴,

$$\rho(r) = \frac{M e^{-\frac{r^2}{4\theta}}}{\sqrt{4\pi} \theta^{3/2}}, \quad (8)$$

where $\sqrt{\theta}$ is viewed as a fundamental length related to space-time noncommutativity, and r is the radial coordinate such that the integral inside a sphere of area $4\pi r^2$ gives the total Arnowitt-Deser-Misner (ADM) mass M of the object for $r \rightarrow \infty$, i.e.: $M(r) = \int_0^r \rho(\bar{r}) \bar{r}^2 d\bar{r} = M \frac{\gamma(3/2, r^2/4\theta)}{\Gamma(3/2)}$. Here, $\Gamma(3/2)$ and $\gamma(3/2, r^2/4\theta)$ are the complete and upper incomplete Euler Gamma functions, respectively. This energy distribution then satisfies Einstein field equations together with the Schwarzschild-like metric $ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 d\Omega^2$, where $f(r) = 1 - \frac{2G_N M(r)}{r}$. According to Ref. [18], one has a BH only if the *mass-to-characteristic length* ratio is sufficiently large, namely for

$$M \gtrsim 1.9 \frac{\sqrt{\theta}}{G_N} = 1.9 m_p \frac{\sqrt{\theta}}{\ell_p} \equiv M_*. \quad (9)$$

If the above inequality is satisfied, the metric function $f = f(r)$ has two zeros and there are two distinct horizons. For $M = M_*$, $f = f(r)$ has only one zero which corresponds to an “extremal” BH, with two coinciding horizons (and vanishing Hawking temperature). The latter represents the minimum mass BH, and a candidate BH remnant of the Hawking decay [3]. Further, the classical Schwarzschild case is precisely recovered in the limit $G_N M / \sqrt{\theta} \rightarrow \infty$, so that departures from the standard geometry become quickly negligible for very massive BHs.

Going back to the BEC model, whose total ADM mass is given in Eq. (6), and comparing the Gaussian profile (7) with Eq. (8), that is setting $\theta = N \ell_p^2 / 14$, one finds that the condition in Eq. (9) reads $1.8 \sqrt{N} \gtrsim 0.5 \sqrt{N}$, and is always satisfied (for $N \geq 1$). We can therefore conclude that harmonic black holes always have two horizons, and the degenerate case is not realised in their spectrum. Although this mismatch might appear as a shortcoming of our construction, it is actually consistent with the idea that the extremal case should have vanishing Hawking temperature and therefore no depleting modes. It also implies that the final evaporation phase, if it

⁴ The squared length θ should not be confused with one of the angular coordinates of the previous expressions. Also, note ρ has already been integrated over the angles.

ends in the extremal case, must be realised by a transition that most likely drives the BEC out of the critical point. The precise nature of such a “quantum black hole” state remains, however, unclear.

4 Conclusions and outlook

We have shown that the scenario of Ref. [11, 12, 13, 14], in which BH inner degrees of freedom (as well as the Hawking radiation) correspond to depleting states in a BEC, can be understood and recovered in the context of General Relativity by viewing a BH as made of the superposition of N constituents, with a Gaussian density profile, whose characteristic length is given by the constituents’ effective Compton wavelength. From the point of view of Quantum Mechanics, such states straightforwardly arise from a binding harmonic oscillator potential. Moreover, requiring the existence of (at least) a horizon showed that the extremal case, corresponding to a remnant with vanishing Hawking temperature, is not realised in the harmonic spectrum (6). Such states will therefore have to be described by a different model.

At the threshold of BH formation (see, for example, Ref. [5]), for a total ADM mass $M \simeq m_p$ (thus $N \simeq 1$), the above description should allow us to describe Quantum Mechanical processes involving BH intermediate (or metastable) states. However, we can already anticipate that quantum BHs with spin should be relatively easy to accommodate in our description, by simply considering states in Eq. (5) with $l > 0$. This should allow us to consider more realistic quantum BH formation from particle collisions.

Many questions are still left open. First of all, the discretisation of the mass has an important consequence in the classical limit. For example, let us look again at Eq. (6), and consider two non-rotating BHs with mass $M_1 = \sqrt{\frac{7}{2}N_1}m_p$ and $M_2 = \sqrt{\frac{7}{2}N_2}m_p$, where N_1 and N_2 are positive integers, which slowly merge in a head-on collision (with zero impact parameter). The resulting BH should have a mass M which is also given by Eq. (6). However, there is in general no integer N_3 such that $\sqrt{N_3} = \sqrt{N_1} + \sqrt{N_2}$. It therefore appears that either the mass should not be conserved, $M \neq M_1 + M_2$, or the mass spectrum described by Eq. (6) is not complete. This problem, which is manifestly more significant for small BH masses (or, equivalently, integers N), is shared by all those models in which the BH mass does not scale exactly like an integer. If we wish to keep Eq. (6), or any equivalent mass spectrum, we might then argue that a suitable amount of energy (of order $M_1 + M_2 - M_3$) should be expelled during the merging, in order to accommodate the overall mass into an allowed part of the spectrum. In this case, one may also wonder if this emission can be thought of as some sort of Hawking radiation⁵, or if it is completely different in nature.

⁵ Note that for vanishing impact parameter, one does not expect any emission of classical gravitational waves.

Another issue regards the assumption in Eq. (7), i.e. the idea that the classical density profile corresponds to the square modulus of the (normalised) wavefunction. At the semiclassical level, this seems reasonable and intuitive, but necessarily removes the concept of “point-like test particle” from General Relativity, thus forcing us to reconsider the idea of geodesics only in terms of propagation of extended wave packets, which might show unexpected features or remove others from the classical theory. Also, elementary particles would not differ from extended massive objects and therefore should have an equation of state (see, for instance, the old shell model in Refs. [4]).

Last but not least, there is the question of describing the formation of a BEC during a stellar collapse. Condensation is usually achieved at extremely low temperature, when the thermal de Broglie wavelength becomes comparable to the inter-particle spacing. Whereas one has no doubt that particles inside a BH are extremely packed, it is not clear how such a dramatic drop of temperature could occur.

Acknowledgements

This work is supported in part by the European Cooperation in Science and Technology (COST) action MP0905 “Black Holes in a Violent Universe”.

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