arXiv:gr-qc/9411070v2 6 Dec 1994

DFTT 64/94
November 26, 1994

HAMILTONIAN FORMALISM FOR BLACK HOLES AND QUANTIZATION

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ABSTRACT

Starting from the Lagrangian formulation of the Einstein equations for the vacuum static spherically symmetric metric, we develop a canonical formalism in the radial variable r that is time–like inside the Schwarzschild horizon. The Schwarzschild mass turns out to be represented by a canonical function that commutes with the r–Hamiltonian. We investigate the Wheeler–DeWitt quantization and give the general representation for the solution as superposition of eigenfunctions of the mass operator.

PACS: 04.20.Fy, 04.60.Ds, 04.70.-s.

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1. Introduction.

Recently the dynamics of primordial Schwarzschild black holes has been cast in canonical formalism and the quantization procedure has been discussed [1]. A complete bibliography that covers the history of the subject is also contained there. Indeed, the Hamiltonian formalism is a fundamental key to obtain a quantum description of a gravitational system and a great deal of work has been devoted to the construction of a canonical formalism for the classical black hole solutions (see also [2,3]).

In the present paper we derive the canonical formalism for the vacuum static spherically symmetric metric in a simple direct way by foliation in the coordinate r. Classically the general vacuum spherically symmetric solution of the Einstein equations is locally isometric to the Schwarzschild metric. In order to obtain a Hamiltonian description of the Schwarzschild metric we start from the general static spherically symmetric line element [4]

$$ds^2 = -a(r)dt^2 + N(r)dr^2 + 2B(r)dtdr + b(r)^2 d\Omega^2, \tag{1.1}$$

where a, B, N and b are real functions of r and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the metric of the two–sphere.

Usually, redefining the coordinate time and fixing $b = r$, (1.1) is cast in the form

$$ds^2 = -A(r)dt^2 + C(r)dr^2 + r^2 d\Omega^2, \tag{1.2}$$

where r is now the "area coordinate" since the area of the two-sphere of radius r is $4\pi r^2$. One is then left with two functions $A(r)$ and $C(r)$ that can be determined by the Einstein equations. The line element (1.2) is the "standard form" of the general static isotropic metric (1.1) [4].

The line element (1.1) will be our starting point for a canonical treatment, formulated in the coordinate r. We are of course aware that the line element (1.1) does not cover the complete spacetime since it describes only a half of the Kruskal–Szekeres plane and pure r–coordinate transformations do not lead to a complete covering of the Kruskal–Szekeres manifold starting from the metric (1.1). In spite of this, the analysis of reparametrizations from this point of view may lead to interesting consequences. Indeed, later we will consider a formal r–quantization scheme and investigate the ensuing Wheeler – DeWitt (WDW) equation [5,6].

Since the metric tensor in (1.1,2) does not depend on t, no t–differentiation appears in the expression of the minisuperspace action derived from (1.1); starting from the Lagrangian we may develop a formal Hamiltonian scheme in the variable r and obtain the corresponding r–super Hamiltonian $H(a, p_a, b, p_b)$ after having introduced the r–conjugate momenta p_a and p_b.

Note that there is a range where r is a timelike variable. The signs of N and a are the key. In fact, inside the Schwarzschild horizon of the black hole the area coordinate r in (1.2) is a time variable while t is spacelike, and our formalism is a true canonical motion in time. In the range where r is timelike, H generates the dynamics and plays the role of the usual ADM Hamiltonian; in general the r–super Hamiltonian is related to the reparametrizations of the variable r. In the metric (1.1) $\sqrt{|N|}$ plays essentially the role of the ADM lapse function with respect to the r–slicing [7]. Since we must allow for negative values of $N(r)$, we need a slight modification of the ADM formalism, similar to what has been done, for instance continuing from a Lorentzian to an Euclidean signature (see e.g. [7]).

A single Lagrange multiplier imposes the constraint of vanishing of the r–super Hamiltonian

$$H(a, p_a, b, p_b) = 0, \tag{1.3}$$

so of course the Hamiltonian "r–dynamics" is generated by a constraint that is quadratic in the momenta, as predicted by the ADM canonical formalism. It is easy to check that this formalism is equivalent to the Einstein equations for the static solution.

The canonical formalism allows for an interesting algebraic structure of constants of the motion: in particular we will see that the Schwarzschild mass is expressed by a constant canonical quantity, of course gauge invariant.

The constraint equation $H = 0$ is independent of r, indeed there has been no gauge fixing and r is not determined. The identification of r should be obtained by connecting it to the canonical coordinates of the problem (gauge fixing). This procedure can be carried on by the method proposed in [8] for quantum cosmological models. We defer to further study the analysis of gauge fixing and quantization in the reduced space.

We will investigate the quantization of the system by the method of enforcing the condition $H = 0$ as an operator condition over wave functions (WDW equation). We find the form of the general solution of the equation diagonalizing the Schwarzschild mass operator and a commuting operator. The solutions have an oscillatory behaviour in the classically allowed regions and an exponential behaviour in the classically forbidden ones.

Thus in this approach the mass plays the role of the quantum number determining the wave function; in this respect our result is in agreement with the conclusions obtained in [1].

The outline of the paper is as follows. In the next section we discuss the classical r–Lagrangian and r–Hamiltonian formalisms for the metric (1.1). In section 3 we integrate the infinitesimal gauge transformations and obtain the entire group. We identify the gauge invariant quantities and discuss their algebra. Section 4 is devoted to the study of the WDW equation.

2. Lagrangian formulation.

Our starting point is the line element (1.1) where the Lagrangian coordinates a, b, B, N are functions of r. As mentioned in the introduction, changes of sign in the metric coefficients a and N are allowed (note that the signature is Minkowskian over the whole manifold: for instance, if $B = 0$, $aN > 0$). r can be a timelike coordinate and t spacelike over part of the manifold, so it is a matter of preference to define *a priori* t or r as the timelike variable. Hence, we develop a formal canonical structure in r in which the r–super Hamiltonian H is a generator of gauge canonical transformations that correspond to reparametrizations of the r coordinate in the Lagrangian formulation (and thus in the region where r is timelike it generates the dynamics). Hence it seems worthwile to study in detail this r–canonical structure.

Let us consider the line element (1.1) that corresponds essentially to use a Gaussian normal system of coordinates with respect to the three–surface (t, θ, ϕ), i.e. to perform the 3+1 slicing with respect to the r coordinate. As remarked in the introduction, looking at (1.1) one realizes that the variable $\sqrt{|N(r)|}$ plays the role of the r–lapse function in our foliation [7]. The Einstein–Hilbert action

$$S = \frac{1}{16\pi G}\int_{V_4} d^4x\sqrt{-g}R - \frac{1}{8\pi G}\int_{\partial V_4} d^3x\ \sqrt{h}\ \mathbf{K} \tag{2.1}$$

can be cast in the form

$$S = \int_{t_1}^{t_2} dt \int_{r_1}^{r_2} dr L(a, b, \Delta), \tag{2.2}$$

where

$$L = 2\sqrt{\Delta}\left(\frac{a'bb'}{\Delta} + \frac{ab'^2}{\Delta} + 1\right). \tag{2.3}$$

(primes denote differentiation with respect to r). In (2.3) we have set $4G = 1$ and Δ is given by

$$\Delta(r) = aN + B^2. \tag{2.4}$$

Eq. (2.3) requires that $\Delta > 0$ and from (2.4) the signature of (1.1) is Minkowskian for any value of r. From (2.3) the Einstein equations of motion can be recovered considering formally $a(r)$, $N(r)$, $B(r)$ and $b(r)$ as Lagrangian coordinates evolving in r. Of course, $\sqrt{\Delta}$ acts as a Lagrange multiplier (and we still have the freedom of choosing $B(r)$ or $N(r)$). From the vacuum Einstein equations derived from (2.1), or directly from (2.3), one obtains

$$\Delta = k^2 b'^2, \tag{2.5a}$$

$$a = k^2\left(1 - \frac{2M}{b}\right), \tag{2.5b}$$

where k and M are two integration constants. Since the metric is t–independent, we can arbitrarily rescale t in (1.1). This corresponds essentially to fix k in (2.5), so we can set $k = 1$; then the metric coincides with the standard Schwarzschild form. M is the Schwarzschild mass. Eqs. (2.5) will be useful for comparison with the Hamiltonian formalism that will be developed below. Note that the Lagrange multiplier $\sqrt{\Delta}$ can be arbitrarily fixed; furthermore, since Δ is related to N and B by eq. (2.4), also N, or B, can be arbitrarily chosen; these two choices correspond to the freedom in the definition of t and r in the line element (1.1). For instance, the choice $\Delta = 1$ corresponds to the area gauge since from (2.5) we obtain $r = b$:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + N(r) dr^2 \pm 2\left[1 - \left(1 - \frac{2M}{r}\right) N(r)\right]^{1/2} dt\ dr + r^2 d\Omega^2. \tag{2.6}$$

The line element (2.6) corresponds to the standard form of the Schwarzschild solution for $N(r) = (1 - 2M/r)^{-1}$, to the Eddington–Finkelstein metric for $N = 1 + 2M/r$ and to the line element of ref. [2] choosing $N = 1$.

Let us now set up the Hamiltonian formalism in r. We introduce the r–conjugate momenta as

$$p_a = \frac{2bb'}{\sqrt{\Delta}}, \tag{2.7a}$$

$$p_b = \frac{2}{\sqrt{\Delta}}(a'b + 2ab'), \tag{2.7b}$$

and by the usual Legendre transformation we obtain the density of the action (with respect to the coordinate t)

$$\mathcal{S} = \int_{r_1}^{r_2} dr \left\{ \frac{1}{2}(a'p_a + b'p_b - ap'_a - bp'_b) - lH \right\}. \tag{2.8}$$

H is the Schwarzschild r–super Hamiltonian

$$H = p_a(bp_b - ap_a) - 4b^2, \tag{2.9}$$

and

$$l = \frac{\sqrt{\Delta}}{2b^2} \tag{2.10}$$

has been chosen as Lagrange multiplier. Note that the Legendre transformation used to write (2.8) is singular for $b = 0$, but not for $a = 0$. As a consequence of (2.8) we have the constraint

$$H = 0. \tag{2.11}$$

This constraint expresses the invariance under r–reparametrization and inside the region where r is timelike it generates the dynamics.

Eqs. (2.1) – (2.11) can be easily extended to the Reissner – Nordström (RN) case, i.e. to a static electrically charged black hole. Let us consider a radial electric field whose potential 1-form is

$$A = A(r)dt \tag{2.12}$$

(this Ansatz was used in [9,10] for the discussion of Euclidean electromagnetic black holes). Adding to (2.3) the electromagnetic Lagrangian and using (2.12) the Hamiltonian becomes

$$H_{\rm RN} = p_a(bp_b - ap_a) - 4b^2 + P_A^2 = H + P_A^2, \tag{2.13}$$

where P_A is the conjugate momentum to A. Since (2.13) is separable we can solve the equation of motion for the electromagnetic field and have $P_A = Q$ where Q is the charge of the black hole. Eq. (2.11) becomes

$$H_{RN} = H + Q^2 = 0. \tag{2.14}$$

The RN case is equivalent to the Schwarzschild case with the constraint (2.14) in place of (2.11).

3. Algebra and Gauge Transformations.

The gauge transformations of the system are generated by H $(i = a, b)$:

$$\delta q_i = \alpha(r)\frac{\partial H}{\partial p_i} = \alpha(r)\big[q_i, H\big]_P, \tag{3.1a}$$

$$\delta p_i = -\alpha(r)\frac{\partial H}{\partial q_i} = \alpha(r)\big[p_i, H\big]_P, \tag{3.1b}$$

$$\delta l = \frac{d\alpha}{dr}. \tag{3.1c}$$

The action (2.8) is invariant under (3.1) apart from a boundary term that does not change the classical equations of motion:

$$\mathcal{S} = \int_{r_1}^{r_2} dr \frac{d}{dr}\left[\alpha\big(p_i\frac{\partial H}{\partial p_i} + q_i\frac{\partial H}{\partial q_i} - 2H\big)\right]. \tag{3.2}$$

With $\alpha \to l(r)dr$ eqs. (3.1a,b) are the equations of motion.

The system described by the action (2.8) has remarkable algebraic properties. Consider the following canonical quantities:

$$J = 8b - p_a p_b, \tag{3.3a}$$

$$I = b/p_a. \tag{3.3b}$$

J and I are canonically conjugate gauge invariant quantities (and also obviously integrals of the motion):

$$\big[J, I\big]_P = 1, \qquad \big[J, H\big]_P = 0, \qquad \big[I, H\big]_P = 0. \tag{3.4}$$

It is also interesting to consider the canonical quantity

$$N = bp_b - 2ap_a. \tag{3.5}$$

We have

$$N = IJ + 2H/p_a, \tag{3.6}$$

and the relations

$$\big[N, H\big]_P = -2H, \quad \big[N, I\big]_P = -I, \quad \big[N, J\big]_P = J. \tag{3.7}$$

N is not gauge invariant, however, in the case $Q^2 = 0$, i.e. for a Schwarzschild metric, it is constant on the constraint $H = 0$. We shall see that N plays an interesting role in the frame of the WDW equation.

The gauge transformations (3.1) from a gauge l_1 to l_2 can be integrated explicitly. We have

$$a = -H\alpha^2 + IJ\alpha + 4I^2, \tag{3.8a}$$

$$b = -\frac{I}{\alpha}, \tag{3.8b}$$

$$p_a = -\frac{1}{\alpha}, \tag{3.8c}$$

$$p_b = J\alpha + 8I, \tag{3.8d}$$

$$\alpha = \int dr\big(l_2 - l_1\big). \tag{3.8e}$$

From eqs. (3.8) the gauge independent relation follows

$$a = \frac{I^2}{b^2}\big(4b^2 - Jb - H\big). \tag{3.9}$$

On the constraint $H = 0$ (Schwarzschild metric)

$$a = 4I^2\left(1 - \frac{J}{4b}\right). \tag{3.10}$$

Therefore from (2.5)

$$J = 8M, \tag{3.11}$$

where M is the Schwarzschild mass. On the constraint $H = -Q^2$ the two roots of $a = 0$ in (3.9) correspond to the two horizons of the RN metric.

It follows that in the case of the Schwarzschild metric I is the momentum conjugate to the Schwarzschild mass. This suggests to perform a canonical transformation to new pair of canonical variables, $(J, I; q_a, p_a)$ where $q_a = -H/p_a^2$. This motivates our choice of the eigenfunctions in the discussion of the WDW equation.

4. Quantization.

The quantization of this apparently simple system exhibits ambiguities that are characteristic of the canonical quantization of systems described by general relativity [11].

A main problem in general is that, in order to set up canonical quantization rules, we must know a priori the causal structure of the model representing a physical system. To be more specific, we must know which coordinate plays the role of time and consequently write down equal time canonical commutation relations.

This is usually an ambiguous procedure. In the classical treatment, the identification – if any – of the time variable results from the solution of the classical equations of motion and it is not determined a priori. Of course, in some cases, as for instance the Friedmann – Robertson – Walker (FRW) model, one assumes the signature of the metric (see e.g. [12]). This is because the outcome of the equations of motion is anticipated, and a limitation in the signature of the metric is consequently assumed. However, strictly speaking, these limitations are not always known at the start. This becomes evident whenever the classical equations allow for a change in the signature of the metric (see e.g. [7]) or when, as in the present case, the presence of a horizon induces a double change of signature in the metric.

In our present case we know from classical solutions that the signature of the metric (and the gauge fixing of the coordinate) implies for r a timelike range. It is then tempting to explore the implications of a canonical quantization of this system imposing equal r commutation relations. This will be carried out in the present section.

We shall impose the constraint (2.11) as an operator condition on the wave function. This is the WDW equation. It expresses a necessary condition for the wave function, although it does not in general contain all the information relevant to the quantum form of the theory. Indeed as it is well known the time is not identified, the solution contains both positive and negative frequencies, it is a hyperbolic differential operator and thus it does not lead to a well defined boundary value problem. It is also plagued by ambiguities since the metric in the Hilbert space is not defined.

We believe that the correct procedure [8] requires identification of the parameter r (our internal time) through a gauge fixing condition that defines r in terms of the canonical variables and leads to a unitary Hamiltonian in the reduced canonical space. When this is possible the quantization of the system is non ambiguous and the solutions contain also the information from the constraint. The problem of the gauge fixing in the present case will be treated elsewhere; here we shall limit ourselves to explore the properties of the solutions of the WDW equation.

The fundamental commutation relations are:

$$\big[a, p_a\big] = i, \tag{4.1a}$$

$$\big[b, p_b\big] = i. \tag{4.1b}$$

The operators a, p_a, b, p_b have the Schrödinger representation, there being the usual ambiguities about the measure to be used.

We introduce also the mass operator J and its conjugate I according to eqs. (3.4). We have

$$\big[I, J\big] = i. \tag{4.2}$$

We remark in particular that J commutes with p_a.

The expression of the WDW Hamiltonian operator is (we consider for simplicity the case of the Schwarzschild metric, $Q = 0$)

$$H_{WDW} = -ap_a^2 - bJ + 4b^2 + i\lambda p_a. \tag{4.3}$$

The term λ depends on the ordering and on the representation of p_a and p_b. The choice of the covariant Laplace – Beltrami operator [13] leads to $\lambda = 1$ while the symmetric ordering of the operators a, p_a and b, p_b leads to $\lambda = 1/2$. In what follows we shall keep λ undetermined.

First of all we determine the eigenfunctions of the commuting operators J and p_a. We choose the simplest representation,

$$p_q \to -i\frac{\partial}{\partial q}, \tag{4.4}$$

$(q = a, b)$. Then the eigenvalue equation for J is

$$\big(8b + \partial_a\partial_b\big)\ \psi_M = 8M\ \psi_M, \tag{4.5}$$

and the eigenfunctions of J and p_a are given by

$$\psi_{pM}(a, b) = \sqrt{\frac{8}{p}}\frac{1}{2\pi}\exp\ i\left(pa + p^{-1}\beta_M\right), \tag{4.6}$$

where p is the eigenvalue of p_a and

$$\beta_M = 4b(b - 2M). \tag{4.7}$$

The set (4.6) is orthonormal in $-\infty < a < +\infty$, $-\infty < b < +\infty$ with unit measure. Let us remark that this approach can be easily adapted to a different interval in a, b and to different representations for p_a, p_b; the wave function will change correspondingly, however the important properties to exploit remain the role of J and of the commutation relation $[J, p_a] = 0$.

The form of β_M is related to the existence of the horizon at $b = 2M$ for positive M. Expressing the solution of the WDW equation

$$H_{WDW}\Psi \ = 0 \tag{4.8}$$

as superposition of ψ_{pM}, the general representation of the WDW wave function for the Schwarzschild black hole is given by

$$\Psi(a, b) \ = \int dp \ p^{\lambda-3/2} \int dm \ C(m) \ \psi_{pm}(a, b). \tag{4.9}$$

$C(m)$ is arbitrary. It is interesting to remark that there is a priori no limitation on the sign of the mass m.

Using a well known representation for the solutions of the Bessel equation [14], the representation (4.9) can be cast in the form

$$\Psi(a, b) \ = \int dm \ C(m) \left(-\frac{\beta_m}{a}\right)^{(\lambda-1)/2} \ K_{1-\lambda}(2\sqrt{-a\beta_m}) \tag{4.10}$$

and the solution with fixed mass M is

$$\Psi_M \ = C_M \left(-\frac{\beta_M}{a}\right)^{(\lambda-1)/2} K_{1-\lambda}(2\sqrt{-a\beta_M}). \tag{4.11}$$

It is natural to assume the form (4.11) of the solution in the regions where $a\beta_M < 0$, namely in the classically forbidden regions $a < 0, \ b > 2M$ and $a > 0, \ b < 2M$, where (4.11) is damped exponentially for large b. In the two classically allowed regions for the black hole, namely $b > 2M$, $a > 0$ and $b < 2M$, $a < 0$, the behaviour is oscillatory and one should write the appropriate oscillating solutions with outgoing or incoming asymptotic conditions. Note that for large b in these regions the phase approaches the value of the action evaluated on the classical solution for the asymptotically flat spacetime. We are not discussing the joining of the wave functions between the different regions as this depends on the choice of the ordering and also on the representation assumed for the momenta.

Suitable superpositions of the kind (4.10) may give wave functions that are regular also for $b \to 0$ [10,15]. We note also that the general solution for the Kantowski-Sachs Euclidean wormhole found in [10] corresponds to the solutions of the present WDW equation obtained by diagonalizing the operator N (see (3.5)) in place of J. Indeed, using the same choice for ordering ($\lambda = 1$) and measure in superspace as in [10], the differential representation of N is

$$N = -i(b\partial_b - 2a\partial_a), \tag{4.12}$$

and the solutions of the WDW equation that are eigenfunctions of N with eigenvalue ν are:

$$\Psi_\nu(a,b) = \frac{8}{\pi}\left(\frac{2\sinh \pi\nu}{\nu}\right)^{1/2}(-a)^{i\nu/2}K_{i\nu}\left(4b\sqrt{-a}\right). \tag{4.13}$$

These solutions are real in the region $a < 0$ and orthonormal in $0 \le b \le \infty$, $-\infty \le a \le 0$ with measure $b\ da\ db$:

$$(\Psi_\nu, \Psi_{\nu'}) \equiv \int_{-\infty}^{0} da \int_{0}^{\infty} db\ b\ \Psi_\nu^*\ \Psi_{\nu'}\ = \delta(\nu - \nu'). \tag{4.14}$$

Again the phase factor coincides asymptotically with the classical phase factor as for (4.11). It is interesting to note that when $\nu = 0$ the solution (4.13), namely

$$\Psi_{\nu=0}\ = \frac{8\sqrt{2}}{\sqrt{\pi}}K_0\left(4b\sqrt{-a}\right), \tag{4.15}$$

coincides with (4.11) for $M = 0$ (and $\lambda = 1$), as expected since $J\Psi_{M=0}(a,b) = N\Psi_{M=0}(a,b) = 0$ on the constraint shell $H = 0$. This wave function describes a vacuum wormhole in the classically forbidden region. This equivalence supports the conjecture [16] that the ultimate remnant in the evaporation process of a black hole is a vacuum wormhole.

5. Conclusions.

The classical Einstein equations for a static spherically symmetric metric can be cast in Hamiltonian form. The starting point is the ADM foliation performed along the coordinate r. This is of course a constrained canonical formalism, the constraint being that the Hamiltonian vanishes. The Hamiltonian generates gauge transformations of the canonical variables that correspond to the reparametrization of the coordinate r in the customary formalism of General Relativity.

By a suitable, self – suggesting choice of the Lagrangian multiplier (analogously to what done in [17,8] for the FRW universe) the Hamiltonian assumes a beautiful polynomial form. The infinitesimal gauge transformations can be integrated, thanks essentially to Einstein and Schwarzschild. This is an interesting integrable non linear system. Integrability is due to its simple algebraic structure. Indeed, one identifies a pair of conjugate gauge invariant quantities: one of them is the Schwarzschild mass.

Then, the temptation to explore the quantization of this system is big and we have carried on the investigation of the WDW equation. In doing this, one is comforted by the fact that inside the horizon of a black hole r is a timelike variable.

Note that if we do not fix the coordinate gauge by expressing r in terms of the canonical coordinates, this statement is vague: for instance the trivially different fixings $b = r$ (area gauge) and $b = e^r$ lead to obviously different values for the horizon in terms of r. However, this does not matter much: there is a region where r is timelike.

Thus we have studied the WDW equation and give the general representation of the solution in terms of superpositions of eigenfunctions of the mass operator. It is interesting to observe that there is no reason why the sum should be limited to positive eigenvalues of the mass only.

There is nothing in the form of the WDW equation that reminds us of the region in a, b where it is valid, as the WDW equation does not contain r. So we may determine the solution in the four regions $a \gtrless 0$, $b \gtrless 2M$ (for positive mass). We have not discussed the joining of the solutions between these regions, as the result may be affected by the ambiguities in the ordering of the operators and in the choice of the measure.

The solution in the classically forbidden regions can also be cast in a form identical to the solution representing a Euclidean wormhole in the Kantowski – Sachs spacetime [10]. These solutions are eigenfunctions of a different operator N that commutes weakly with the Hamiltonian. In particular the state with eigenvalue 0 of N is also eigenstate of the mass with eigenvalue 0. This equivalence may support the conjecture [16] that the ultimate remnant in the process of evaporation of a black hole is a vacuum wormhole.

The WDW equation is plagued by the so well known problems. A more natural way to investigate the quantum properties of the system seems to be the introduction of a gauge fixing of the parameter in the canonical treatment [8] that connects r to the canonical variables and leads to a unitary Hamiltonian in the reduced canonical space. We defer to a next paper the investigation of this method as well as of the connection between the WDW equation and the gauge fixed quantization for integrable systems.

Acknowledgments

It is a pleasure to thank Orfeu Bertolami, Fernando de Felice and Luis J. Garay for interesting discussions on the subject of this paper and related topics. One of the authors (A.T.F.) acknowledges a partial support from the Russian Science Foundation (grant 93 - 02 - 3827) and from the International Science Foundation (grant RF 000).

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