Laplace Transform

- **We use the Laplace transformation to transform the circuit from the time domain to the frequency domain, obtain the solution, and apply the inverse Laplace transform to the result to transform it back to the time domain**

Advantages of Laplace transform in circuits solving

- **First, it can be applied to a wider variety of inputs than phasor analysis.**

- **Second, it provides an easy way to solve circuit problems involving initial conditions, because it allows us to work with algebraic equations instead of differential equations.**

- **Third, the Laplace transform is capable of providing us, in one single operation, the total response of the circuit comprising both the natural and forced responses**

The **Laplace transform** is an integral transformation of a function $f(t)$ from the time domain into the complex frequency domain, giving $F(s)$.

Given a function $f(t)$, its Laplace transform, denoted by $F(s)$ or $\mathcal{L}[f(t)]$, is defined by

$$\boxed{\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt}$$

where s is a complex variable given by

$$s = \sigma + j\omega$$

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}\,dt$$

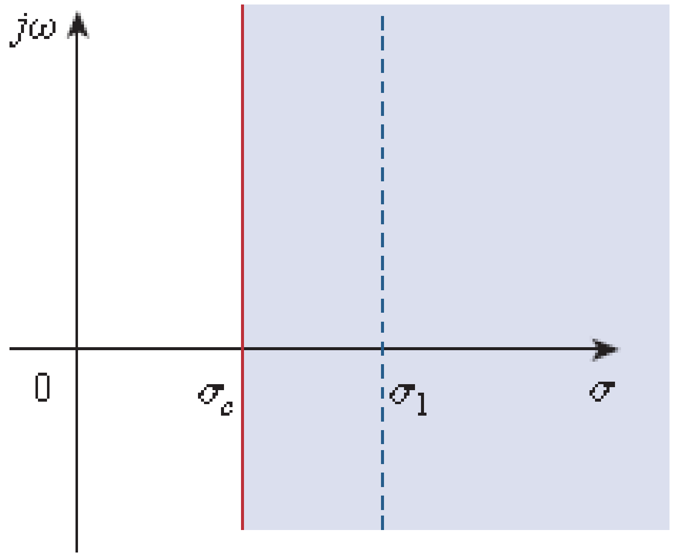
Region of Convergence

- All functions, f(t) may not have LTs, such that integral converges to a finite value,

$$\int_{0^-}^{\infty} e^{-\sigma t}|f(t)|\,dt < \infty$$

for some real value $\sigma = \sigma_c$

- Thus the region of convergence for LT is
  - Re(s)= $\sigma > \sigma_c$
- In this region $|F(s)| < \infty$ and $F(s)$ exists.



- $F(s)$ is undefined outside the region of convergence.

Inverse LT

- A companion to the direct LT is its inverse LT

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}\, ds$$

Where the integration is performed along a straight line

$(\sigma_1 + j\omega, -\infty < \omega < \infty)$ in the region of convergence $\sigma_1 \gtrsim \sigma_c$.

- However, for simplicity, look up tables are used for finding inverse LT
- $f(t) \quad \Leftrightarrow \quad F(s)$ is regarded as Laplace transform pair.

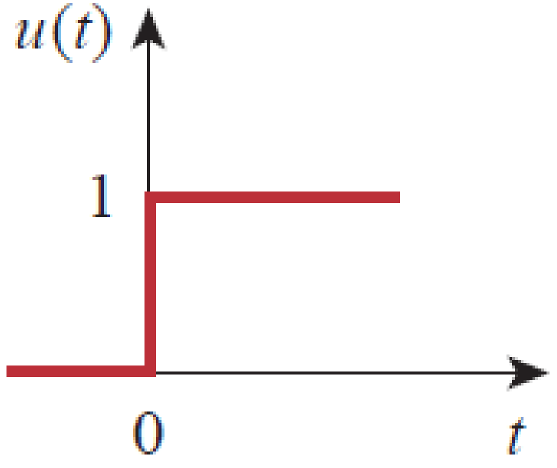
$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}\,ds$$

$$f(t) \quad \Leftrightarrow \quad F(s)$$

(a) For the unit step function $u(t)$, shown in Fig. the Laplace transform is

$$\mathcal{L}[u(t)] = \int_{0^-}^{\infty} 1e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^{\infty}$$

$$= -\frac{1}{s}(0) + \frac{1}{s}(1) = \frac{1}{s}$$

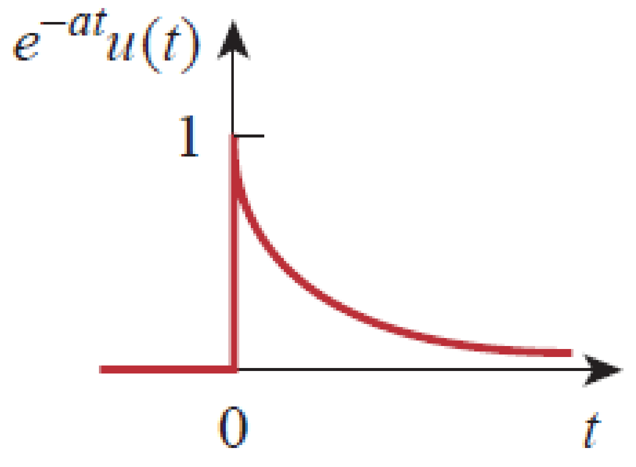


(a)

(b) For the exponential function, shown in Fig. the Laplace transform is

$$\mathcal{L}[e^{-at}u(t)] = \int_{0^-}^{\infty} e^{-at}e^{-st}\,dt$$

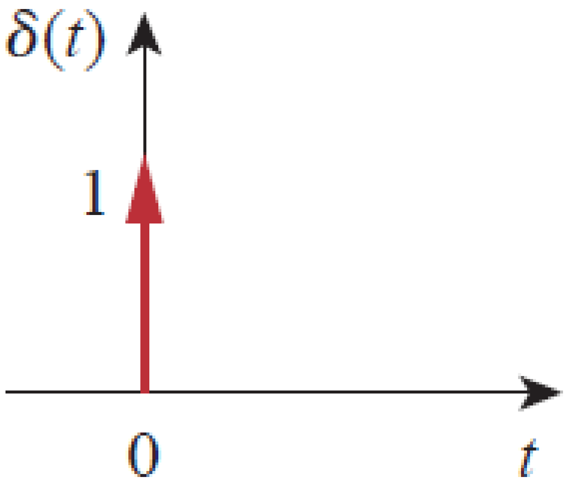
$$= -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^{\infty} = \frac{1}{s+a}$$



(c) For the unit impulse function, shown in Fig. 15.2(c),

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt = e^{-0} = 1$$

since the impulse function $\delta(t)$ is zero everywhere except at $t = 0$.



Determine the Laplace transform of $f(t) = \sin\omega t\, u(t)$.

Solution:

Using Eq. (B.27) in addition to Eq. (15.1), we obtain the Laplace transform of the sine function as

$$\begin{aligned} F(s) = \mathcal{L}[\sin\omega t] &= \int_0^\infty (\sin\omega t)e^{-st}\,dt = \int_0^\infty \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)e^{-st}\,dt \\ &= \frac{1}{2j}\int_0^\infty \left(e^{-(s-j\omega)t} - e^{-(s+j\omega)t}\right)dt \\ &= \frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) = \frac{\omega}{s^2+\omega^2} \end{aligned}$$

Properties of the Laplace transform.

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as} F(s)$
Frequency shift	$e^{-at} f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2 f}{dt^2}$	$s^2 F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3 f}{dt^3}$	$s^3 F(s) - s^2 f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - f^{(n-1)}(0^-)$

Properties of the Laplace transform.

Property	$f(t)$	$F(s)$
Time integration	$\int_0^t f(t)\,dt$	$\frac{1}{s}F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds}F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)\,ds$
Time periodicity	$f(t) = f(t + nT)$	$\frac{F_1(s)}{1 - e^{-sT}}$
Initial value	$f(0)$	$\lim_{s \to \infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s \to 0} sF(s)$
Convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$

Obtain the Laplace transform of $f(t) = \delta(t) + 2u(t) - 3e^{-2t}u(t)$.

Solution:

By the linearity property,

$$F(s) = \mathcal{L}[\delta(t)] + 2\mathcal{L}[u(t)] - 3\mathcal{L}[e^{-2t}u(t)]$$
$$= 1 + 2\frac{1}{s} - 3\frac{1}{s+2} = \frac{s^2 + s + 4}{s(s+2)}$$

Find the Laplace transform of $f(t) = (\cos(3t) + e^{-5t})u(t)$.

Answer: $\dfrac{2s^2 + 5s + 9}{(s+5)(s^2+9)}$.

Example 15.5

Find the Laplace transform of the gate function in Fig. 15.5.

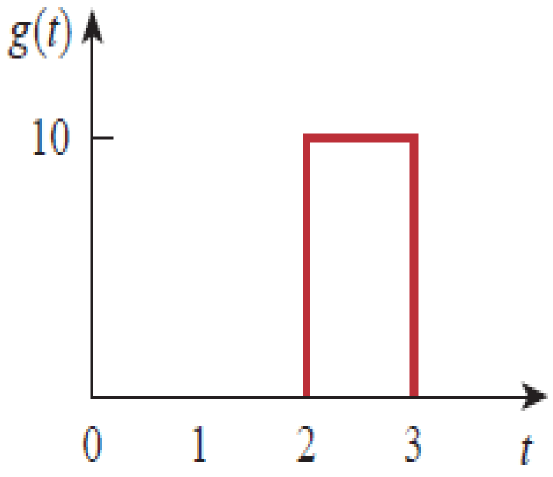


Figure 15.5
The gate function; for Example 15.5.

Solution:

We can express the gate function in Fig. 15.5 as

$$g(t) = 10[u(t-2) - u(t-3)]$$

Since we know the Laplace transform of $u(t)$, we apply the time-shift property and obtain

$$G(s) = 10\left(\frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}\right) = \frac{10}{s}(e^{-2s} - e^{-3s})$$

Calculate the Laplace transform of the periodic function in Fig. 15.7.

Example 15.6

Solution:

The period of the function is $T = 2$. To apply Eq. (15.40), we first obtain the transform of the first period of the function.

$$
\begin{aligned}
f_1(t) &= 2t[u(t) - u(t-1)] = 2tu(t) - 2tu(t-1) \\
&= 2tu(t) - 2(t - 1 + 1)u(t-1) \\
&= 2tu(t) - 2(t-1)u(t-1) - 2u(t-1)
\end{aligned}
$$

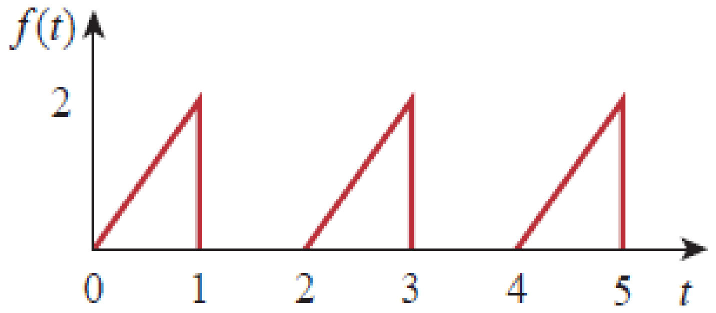


Figure 15.7
For Example 15.6.

Using the time-shift property,

$$F_1(s) = \frac{2}{s^2} - 2\frac{e^{-s}}{s^2} - \frac{2}{s}e^{-s} = \frac{2}{s^2}(1 - e^{-s} - se^{-s})$$

Thus, the transform of the periodic function in Fig. 15.7 is

$$F(s) = \frac{F_1(s)}{1 - e^{-Ts}} = \frac{2}{s^2(1 - e^{-2s})}(1 - e^{-s} - se^{-s})$$

Practice Problem 15.6

Determine the Laplace transform of the periodic function in Fig. 15.8.

Answer: $\dfrac{1 - e^{-2s}}{s(1 - e^{-5s})}$.

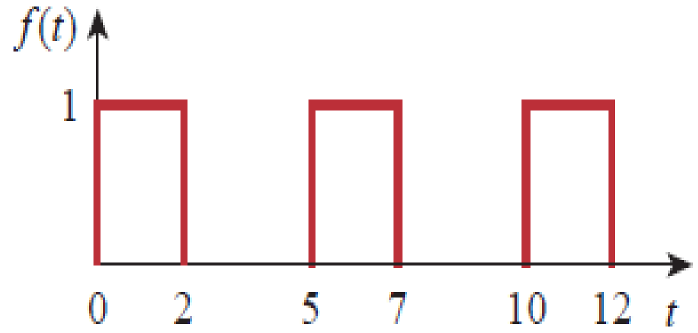


Figure 15.8
For Practice Prob. 15.6.

Example 15.8

Find the inverse Laplace transform of

$$F(s) = \frac{3}{s} - \frac{5}{s+1} + \frac{6}{s^2+4}$$

Solution:

The inverse transform is given by

$$\begin{aligned} f(t) = \mathcal{L}^{-1}[F(s)] &= \mathcal{L}^{-1}\left(\frac{3}{s}\right) - \mathcal{L}^{-1}\left(\frac{5}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{6}{s^2+4}\right) \\ &= (3 - 5e^{-t} + 3\sin 2t)u(t), \qquad t \geq 0 \end{aligned}$$

where Table 15.2 has been consulted for the inverse of each term.

Practice Problem 15.8

Determine the inverse Laplace transform of

$$F(s) = 1 + \frac{3}{s+4} - \frac{5s}{s^2+25}$$

Answer: $\delta(t) + (4e^{-4t} - 5\cos(5t))u(t)$.

Example 15.9

Find $f(t)$ given that

$$F(s) = \frac{s^2 + 12}{s(s + 2)(s + 3)}$$

Solution:

Unlike in the previous example where the partial fractions have been provided, we first need to determine the partial fractions. Since there are three poles, we let

$$\frac{s^2 + 12}{s(s + 2)(s + 3)} = \frac{A}{s} + \frac{B}{s + 2} + \frac{C}{s + 3} \qquad \textbf{(15.9.1)}$$

where A, B, and C are the constants to be determined. We can find the constants using two approaches.

METHOD 1 **Residue method:**

$$A = sF(s)\,|_{s=0} = \frac{s^2 + 12}{(s + 2)(s + 3)}\bigg|_{s=0} = \frac{12}{(2)(3)} = 2$$

$$B = (s + 2)F(s)\,|_{s=-2} = \frac{s^2 + 12}{s(s + 3)}\bigg|_{s=-2} = \frac{4 + 12}{(-2)(1)} = -8$$

$$C = (s + 3)F(s)\,|_{s=-3} = \frac{s^2 + 12}{s(s + 2)}\bigg|_{s=-3} = \frac{9 + 12}{(-3)(-1)} = 7$$

METHOD 2 **Algebraic method:** Multiplying both sides of Eq. (15.9.1) by $s(s + 2)(s + 3)$ gives

$$s^2 + 12 = A(s + 2)(s + 3) + Bs(s + 3) + Cs(s + 2)$$

or

$$s^2 + 12 = A(s^2 + 5s + 6) + B(s^2 + 3s) + C(s^2 + 2s)$$

Equating the coefficients of like powers of s gives

Constant:	$12 = 6A$	$\Rightarrow$	$A = 2$
s:	$0 = 5A + 3B + 2C$	$\Rightarrow$	$3B + 2C = -10$
s^2:	$1 = A + B + C$	$\Rightarrow$	$B + C = -1$

Thus, $A = 2$, $B = -8$, $C = 7$, and Eq. (15.9.1) becomes

$$F(s) = \frac{2}{s} - \frac{8}{s + 2} + \frac{7}{s + 3}$$

By finding the inverse transform of each term, we obtain

$$f(t) = (2 - 8e^{-2t} + 7e^{-3t})u(t)$$

Practice Problem 15.9

Find $f(t)$ if

$$F(s) = \frac{6(s+2)}{(s+1)(s+3)(s+4)}$$

Answer: $f(t) = (e^{-t} + 3e^{-3t} - 4e^{-4t})u(t)$.

Example 15.10

Calculate $v(t)$ given that

$$V(s) = \frac{10s^2 + 4}{s(s + 1)(s + 2)^2}$$

Solution:

While the previous example is on simple roots, this example is on repeated roots. Let

$$\begin{aligned} V(s) &= \frac{10s^2 + 4}{s(s + 1)(s + 2)^2} \\ &= \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{(s + 2)^2} + \frac{D}{s + 2} \end{aligned} \tag{15.10.1}$$

METHOD 1 **Residue method:**

$$A = sV(s)\,|_{s=0} = \left.\frac{10s^2 + 4}{(s+1)(s+2)^2}\right|_{s=0} = \frac{4}{(1)(2)^2} = 1$$

$$B = (s+1)V(s)\,|_{s=-1} = \left.\frac{10s^2 + 4}{s(s+2)^2}\right|_{s=-1} = \frac{14}{(-1)(1)^2} = -14$$

$$C = (s+2)^2V(s)\,|_{s=-2} = \left.\frac{10s^2 + 4}{s(s+1)}\right|_{s=-2} = \frac{44}{(-2)(-1)} = 22$$

$$D = \frac{d}{ds}[(s+2)^2V(s)]\Big|_{s=-2} = \frac{d}{ds}\left(\frac{10s^2 + 4}{s^2 + s}\right)\Big|_{s=-2}$$

$$= \left.\frac{(s^2 + s)(20s) - (10s^2 + 4)(2s + 1)}{(s^2 + s)^2}\right|_{s=-2} = \frac{52}{4} = 13$$

METHOD 2 **Algebraic method:** Multiplying Eq. (15.10.1) by $s(s+1)(s+2)^2$, we obtain

$$10s^2 + 4 = A(s+1)(s+2)^2 + Bs(s+2)^2 + Cs(s+1) + Ds(s+1)(s+2)$$

or

$$10s^2 + 4 = A(s^3 + 5s^2 + 8s + 4) + B(s^3 + 4s^2 + 4s) + C(s^2 + s) + D(s^3 + 3s^2 + 2s)$$

Equating coefficients,

$$\begin{array}{llll}
\text{Constant:} & 4 = 4A & \Rightarrow & A = 1 \\
s: & 0 = 8A + 4B + C + 2D & \Rightarrow & 4B + C + 2D = -8 \\
s^2: & 10 = 5A + 4B + C + 3D & \Rightarrow & 4B + C + 3D = 5 \\
s^3: & 0 = A + B + D & \Rightarrow & B + D = -1
\end{array}$$

Solving these simultaneous equations gives $A = 1$, $B = -14$, $C = 22$, $D = 13$, so that

$$V(s) = \frac{1}{s} - \frac{14}{s+1} + \frac{13}{s+2} + \frac{22}{(s+2)^2}$$

Taking the inverse transform of each term, we get

$$v(t) = (1 - 14e^{-t} + 13e^{-2t} + 22te^{-2t})u(t)$$

Laplace transform pairs.*

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\dfrac{1}{s}$
e^{-at}	$\dfrac{1}{s+a}$
t	$\dfrac{1}{s^2}$
t^n	$\dfrac{n!}{s^{n+1}}$

te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$

$\sin(\omega t + \theta)$	$\dfrac{s\sin\theta + \omega\cos\theta}{s^2 + \omega^2}$
$\cos(\omega t + \theta)$	$\dfrac{s\cos\theta - \omega\sin\theta}{s^2 + \omega^2}$
$e^{-at}\sin\omega t$	$\dfrac{\omega}{(s + a)^2 + \omega^2}$
$e^{-at}\cos\omega t$	$\dfrac{s + a}{(s + a)^2 + \omega^2}$

*Defined for $t \geq 0$; $f(t) = 0$, for $t < 0$.

Applications of the Laplace Transform

Steps in Applying the Laplace Transform:

1. Transform the circuit from the time domain to the s-domain.
2. Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique with which we are familiar.
3. Take the inverse transform of the solution and thus obtain the solution in the time domain.

Resistor

For a resistor, the voltage-current relationship in the time domain is

$$v(t) = Ri(t)$$

Taking the Laplace transform, we get

$$\boxed{V(s) = RI(s)}$$

Inductor

For an inductor,

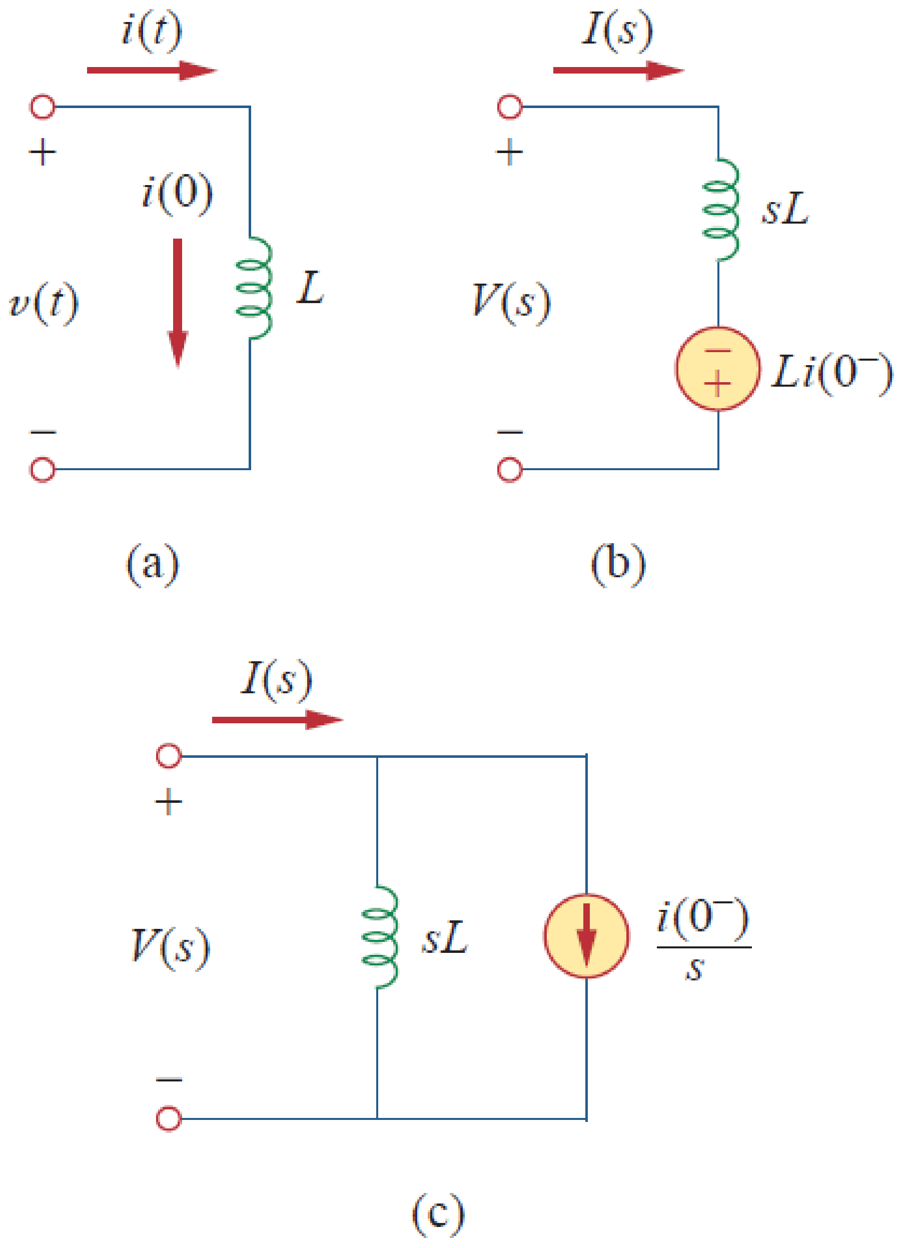
$$v(t) = L\frac{di(t)}{dt}$$

Taking the Laplace transform of both sides gives

$$V(s) = L[sI(s) - i(0^-)] = sLI(s) - Li(0^-)$$

or

$$\boxed{I(s) = \frac{1}{sL}V(s) + \frac{i(0^-)}{s}}$$



(a) (b) (c)

Capacitor

For a capacitor,

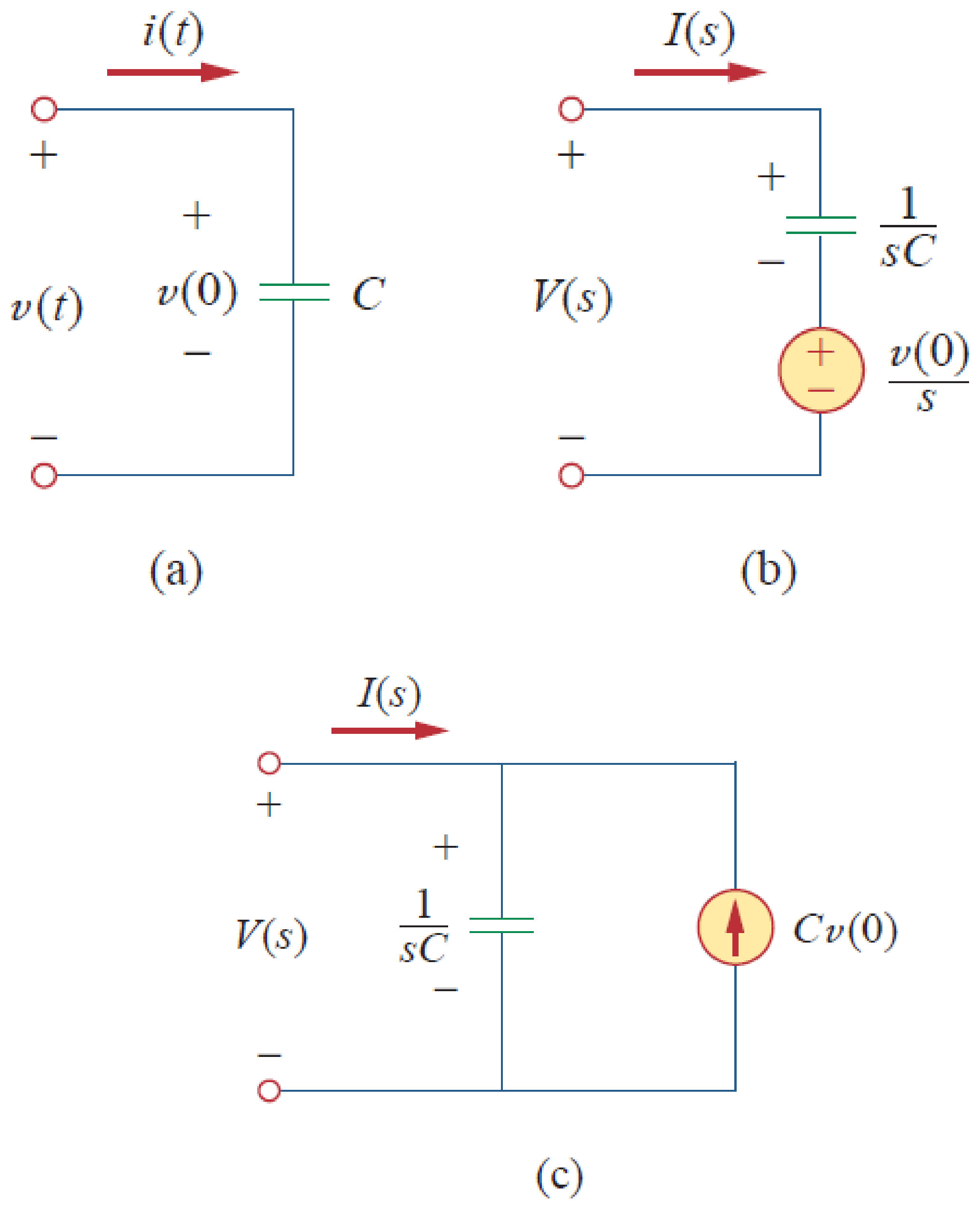
$$i(t) = C\frac{dv(t)}{dt}$$

which transforms into the s-domain as

$$I(s) = C[sV(s) - v(0^-)] = sCV(s) - Cv(0^-)$$

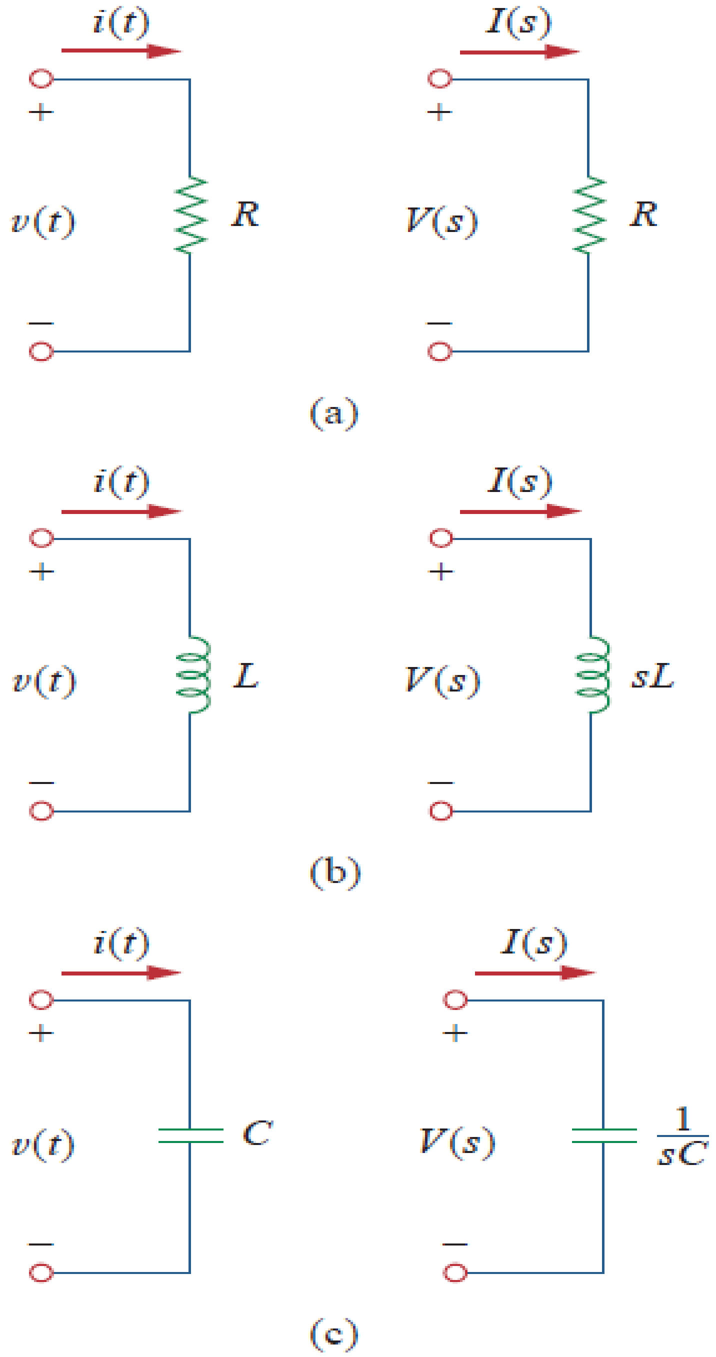
or

$$\boxed{V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}}$$



The elegance of using the Laplace transform in circuit analysis lies in the automatic inclusion of the initial conditions in the transformation process, thus providing a complete (transient and steady-state) solution.

Time-domain and *s*-domain representations of passive elements under zero initial conditions.



If we assume zero initial conditions for the inductor and the capacitor, the above equations reduce to:

$$\text{Resistor:} \quad V(s) = RI(s)$$

$$\text{Inductor:} \quad V(s) = sLI(s)$$

$$\text{Capacitor:} \quad V(s) = \frac{1}{sC} I(s)$$

We define the impedance in the s-domain as the ratio of the voltage transform to the current transform under zero initial conditions; that is,

$$Z(s) = \frac{V(s)}{I(s)}$$

Thus, the impedances of the three circuit elements are

$$\begin{aligned} &\text{Resistor:} && Z(s) = R \\ &\text{Inductor:} && Z(s) = sL \\ &\text{Capacitor:} && Z(s) = \frac{1}{sC} \end{aligned}$$

The admittance in the s-domain is the reciprocal of the impedance, or

$$Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)}$$

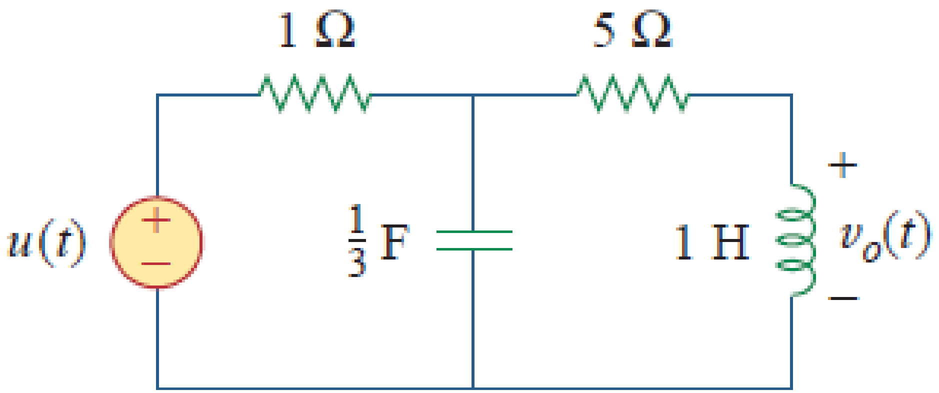
Impedance of an element in the *s*-domain.*

Element	$Z(s) = V(s)/I(s)$
Resistor	R
Inductor	sL
Capacitor	$1/sC$

* Assuming zero initial conditions

Find $v_o(t)$ in the circuit of Fig. assuming zero initial conditions.

Example



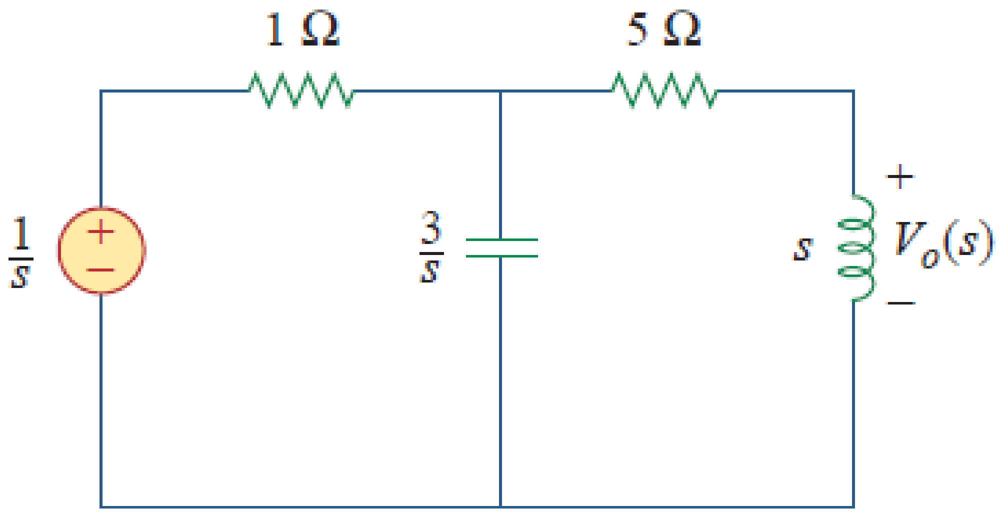
Solution:

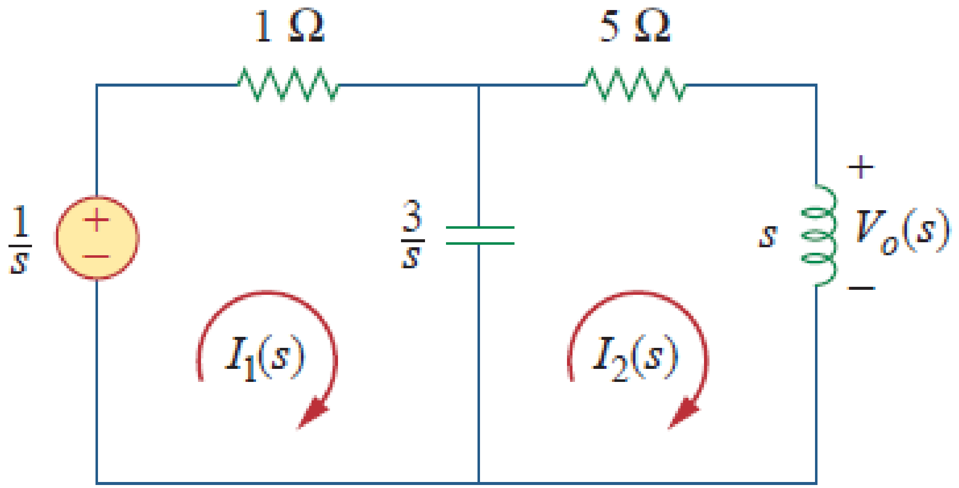
We first transform the circuit from the time domain to the s-domain.

$$u(t) \quad \Rightarrow \quad \frac{1}{s}$$

$$1\text{ H} \quad \Rightarrow \quad sL = s$$

$$\frac{1}{3}\text{F} \quad \Rightarrow \quad \frac{1}{sC} = \frac{3}{s}$$





Mesh analysis of the frequency-domain equivalent of the same circuit.

For mesh 1,

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right)I_1 - \frac{3}{s}I_2$$

For mesh 2,

$$0 = -\frac{3}{s}I_1 + \left(s + 5 + \frac{3}{s}\right)I_2$$

$$I_1 = \frac{1}{3}(s^2 + 5s + 3)I_2$$

Substituting this into Eq. (16.1.1),

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right)\frac{1}{3}(s^2 + 5s + 3)I_2 - \frac{3}{s}I_2$$

Multiplying through by $3s$ gives

$$3 = (s^3 + 8s^2 + 18s)I_2 \qquad \Rightarrow \qquad I_2 = \frac{3}{s^3 + 8s^2 + 18s}$$

$$V_o(s) = sI_2 = \frac{3}{s^2 + 8s + 18} = \frac{3}{\sqrt{2}}\frac{\sqrt{2}}{(s+4)^2 + (\sqrt{2})^2}$$

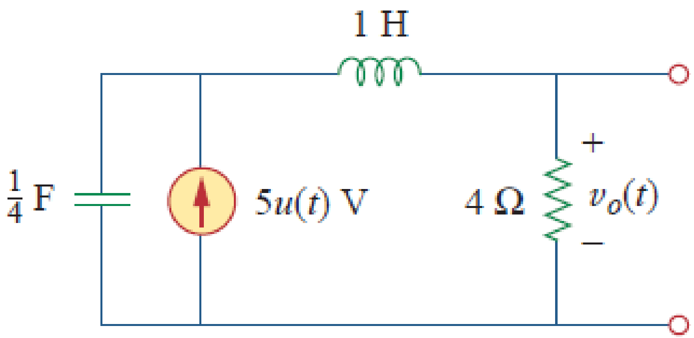
Taking the inverse transform yields

$$v_o(t) = \frac{3}{\sqrt{2}}e^{-4t}\sin\sqrt{2}t \text{ V}, \qquad t \geq 0$$

Practice Problem 16.1

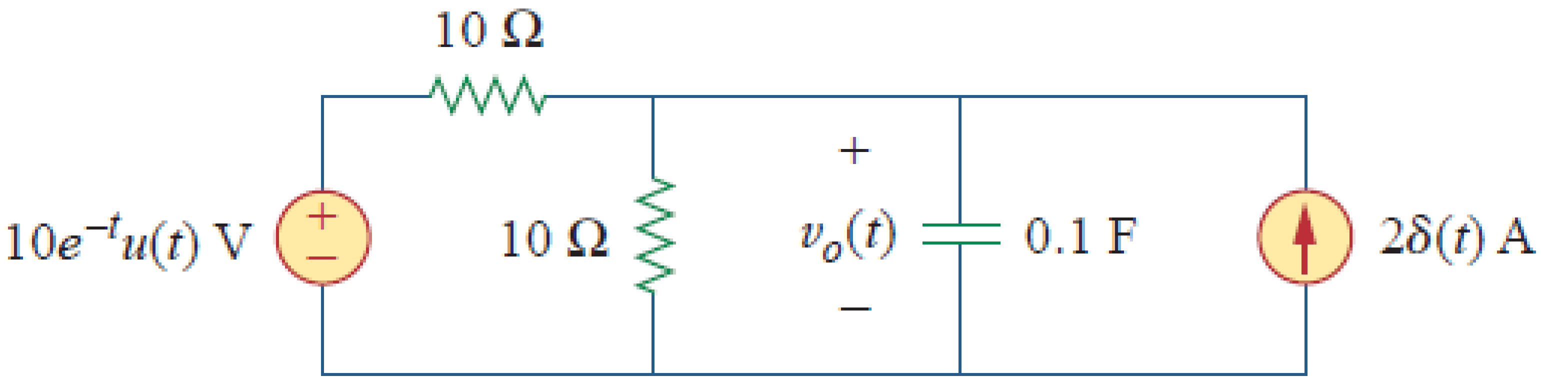
Determine $v_o(t)$ in the circuit of Fig. 16.6, assuming zero initial conditions.

Answer: $20(1 - e^{-2t} - 2te^{-2t})u(t)$ V.

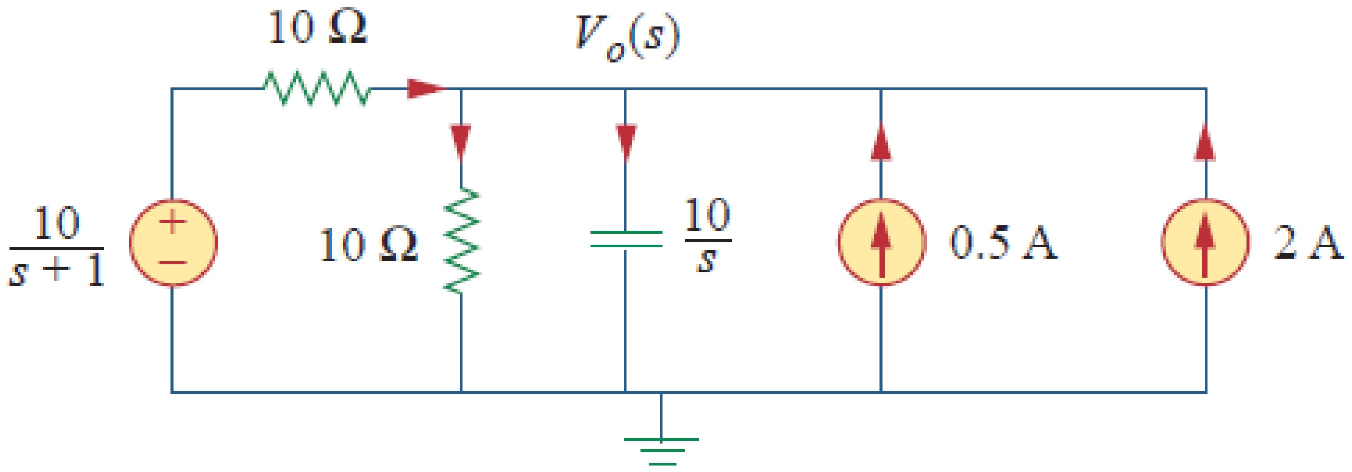


Example 16.2

Find $v_o(t)$ in the circuit of Fig. 16.7. Assume $v_o(0) = 5$ V.

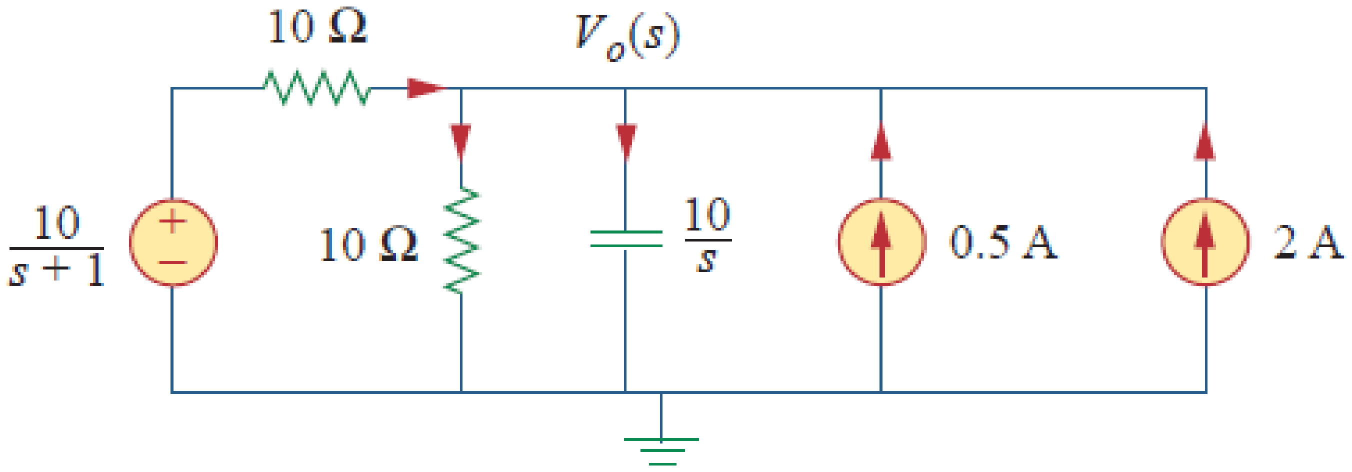


We transform the circuit to the s-domain as shown in Fig.



The initial condition is included in the form of the current source

$$Cv_o(0) = 0.1(5) = 0.5 \text{ A}.$$



We transform the circuit to the s-domain as shown in Fig.

$$\frac{10/(s+1) - V_o}{10} + 2 + 0.5 = \frac{V_o}{10} + \frac{V_o}{10/s}$$

$$\frac{1}{s+1} + 2.5 = \frac{2V_o}{10} + \frac{sV_o}{10} = \frac{1}{10}V_o(s+2)$$

Multiplying through by 10,

$$\frac{10}{s+1} + 25 = V_o(s+2)$$

or

$$V_o = \frac{25s+35}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

where

$$A = (s+1)V_o(s)\,\Big|_{s=-1} = \left.\frac{25s+35}{(s+2)}\right|_{s=-1} = \frac{10}{1} = 10$$

$$B = (s+2)V_o(s)\,\Big|_{s=-2} = \left.\frac{25s+35}{(s+1)}\right|_{s=-2} = \frac{-15}{-1} = 15$$

Thus,

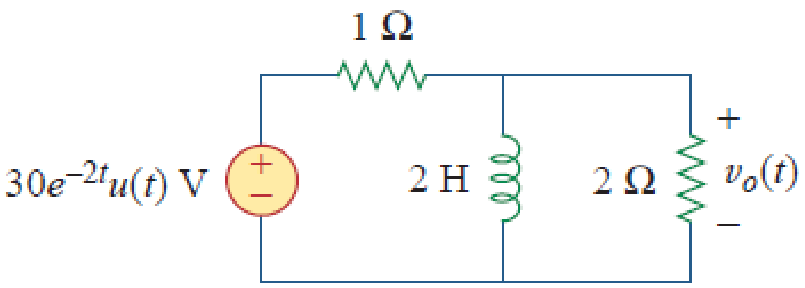
$$V_o(s) = \frac{10}{s + 1} + \frac{15}{s + 2}$$

Taking the inverse Laplace transform, we obtain

$$v_o(t) = (10e^{-t} + 15e^{-2t})u(t) \text{ V}$$

Practice Problem 16.2

Find $v_o(t)$ in the circuit shown in Fig. 16.9. Note that, since the voltage input is multiplied by $u(t)$, the voltage source is a short for all $t < 0$ and $i_L(0) = 0$.



Answer: $(24e^{-2t} - 4e^{-t/3})u(t)$ V.

Home work

- **Example problem 16.3 (page 721-Sadiku)**
- **Practice problem 16.3 (page 722-Sadiku)**

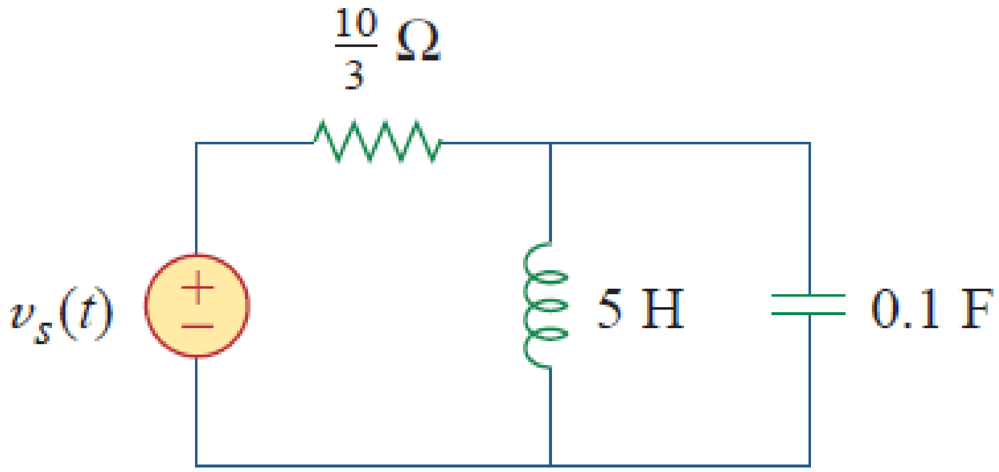
Circuit Analysis

- Need to transform a complicated set of mathematical relationships in the time domain into the *s*-domain where we convert operators (derivatives and integrals) into simple multipliers of *s* and $1/s$
- Laplace transform helps to use algebra to set up and solve our circuit equations
- All of the circuit theorems and relationships for dc circuits are perfectly valid in the *s*-domain.

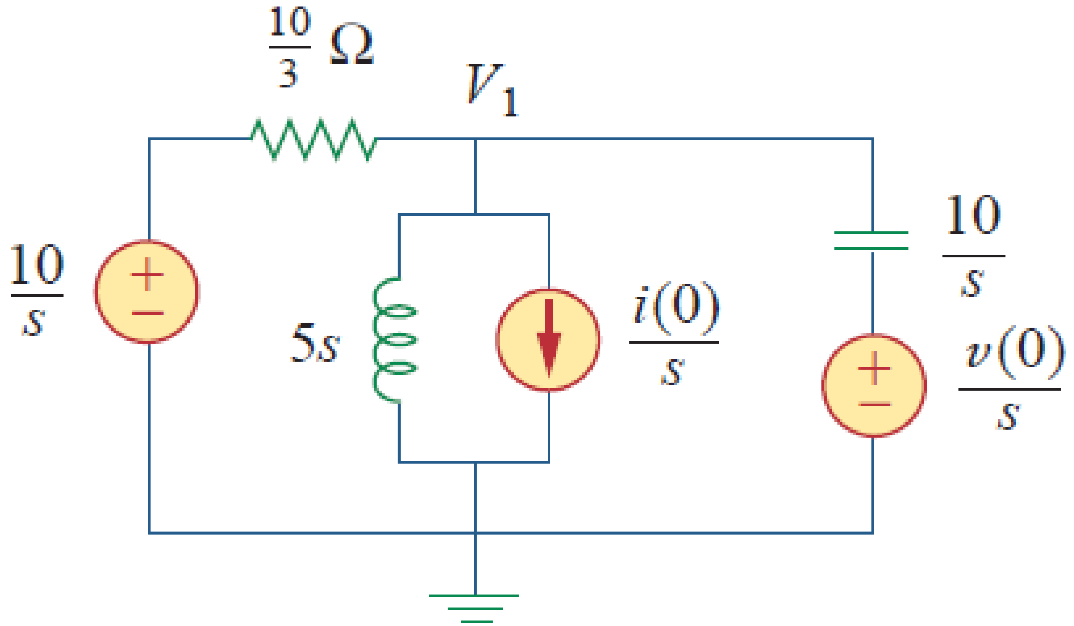
> Remember, equivalent circuits, with capacitors and inductors, only exist in the *s*-domain; they cannot be transformed back into the time domain.

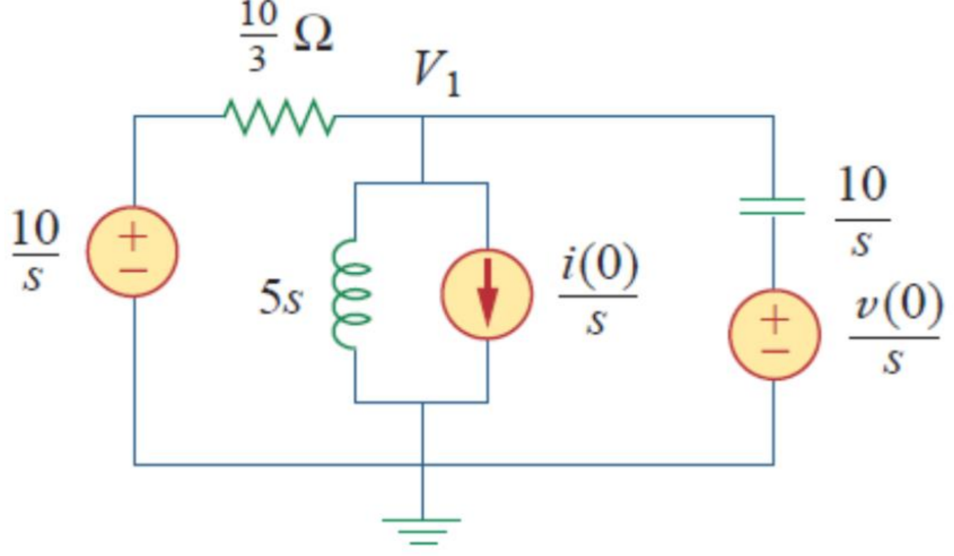
Example

Consider the circuit in Fig. 16.12(a). Find the value of the voltage across the capacitor assuming that the value of $v_s(t) = 10u(t)$ V and assume that at $t = 0$, -1 A flows through the inductor and $+5$ V is across the capacitor.



(a)





$$\frac{V_1 - 10/s}{10/3} + \frac{V_1 - 0}{5s} + \frac{i(0)}{s} + \frac{V_1 - [v(0)/s]}{1/(0.1s)} = 0$$

or

$$0.1\left(s + 3 + \frac{2}{s}\right)V_1 = \frac{3}{s} + \frac{1}{s} + 0.5$$

where $v(0) = 5$ V and $i(0) = -1$ A. Simplifying we get

$$(s^2 + 3s + 2)\,V_1 = 40 + 5s$$

or

$$V_1 = \frac{40 + 5s}{(s+1)(s+2)} = \frac{35}{s+1} - \frac{30}{s+2}$$

$$V_1 = \frac{40 + 5s}{(s + 1)(s + 2)} = \frac{35}{s + 1} - \frac{30}{s + 2}$$

Taking the inverse Laplace transform yields

$$v_1(t) = (35e^{-t} - 30e^{-2t})u(t)\ \text{V}$$

Practice Problem 16.4

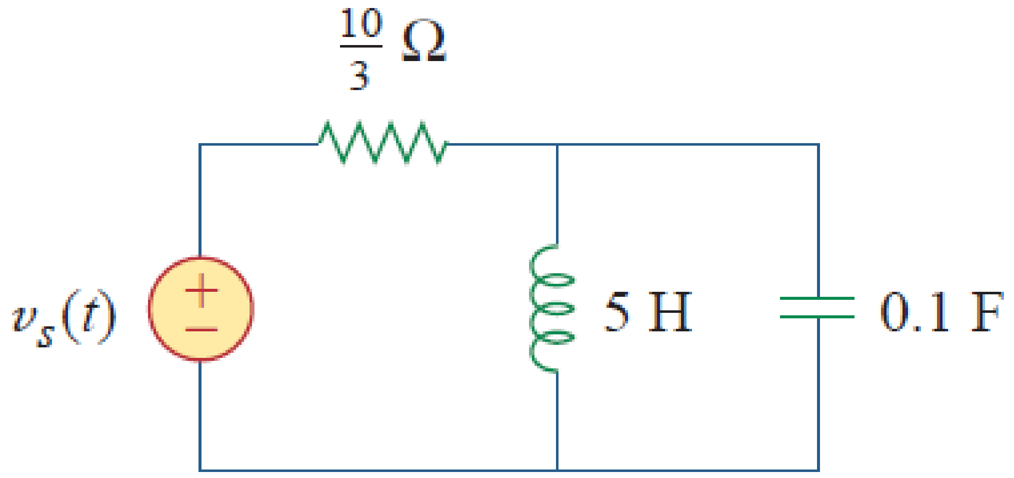
For the circuit shown in Fig. 16.12 with the same initial conditions, find the current through the inductor for all time $t > 0$.

Answer: $i(t) = (3 - 7e^{-t} + 3e^{-2t})u(t)$ A.

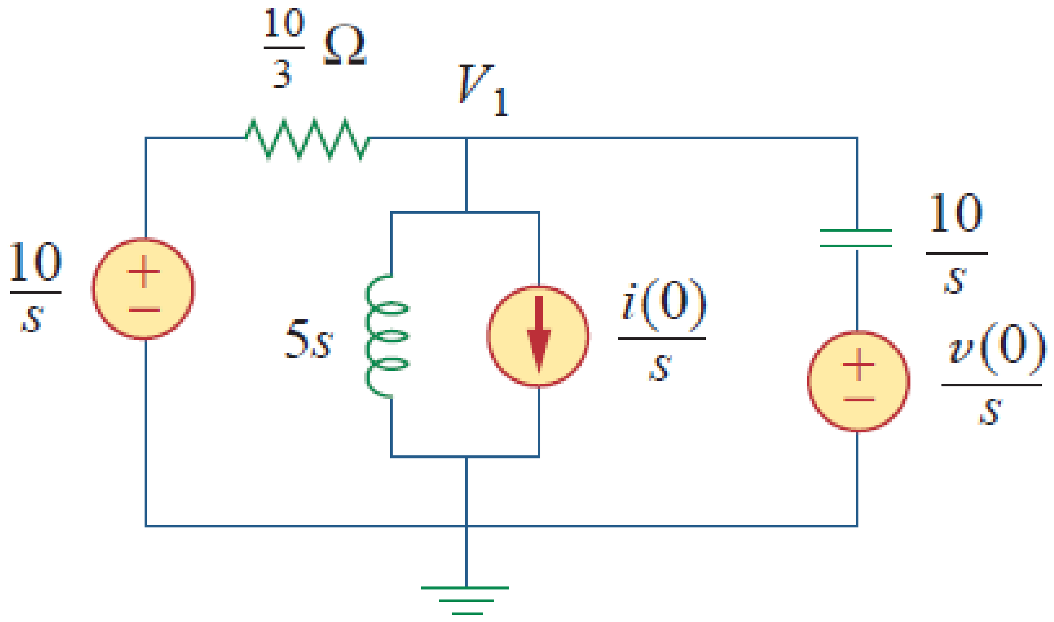
Example

For the following circuit shown in Fig a. and t use superposition to find the value of the capacitor voltage

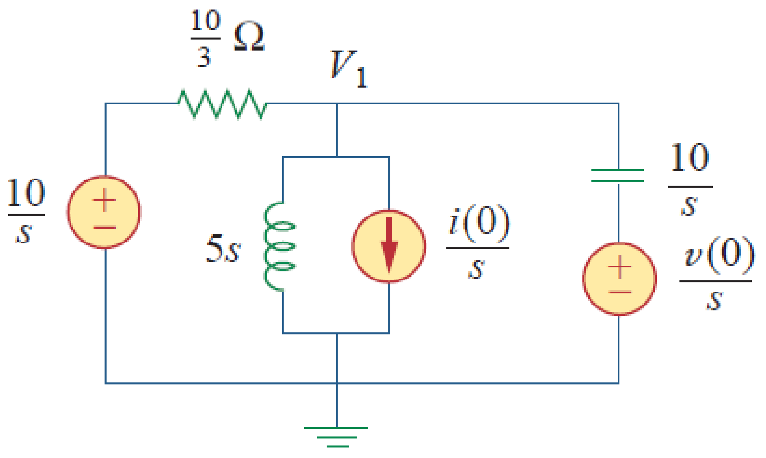
assume that at $t = 0$, -1 A flows through the inductor and $+5$ V is across the capacitor.

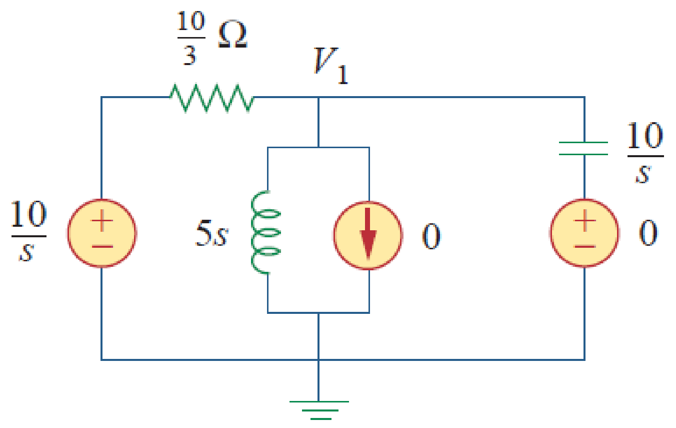


(a)

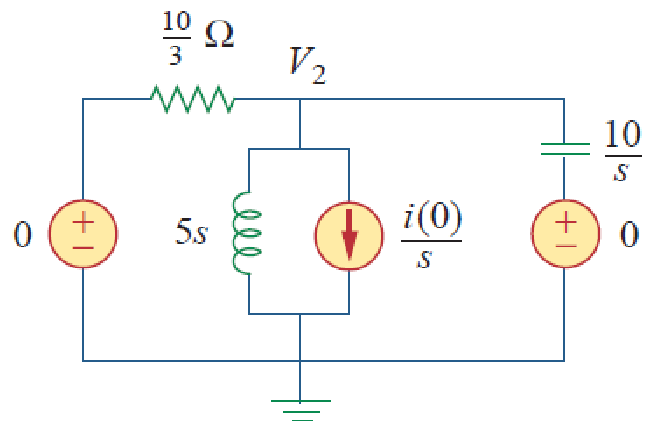


(b)

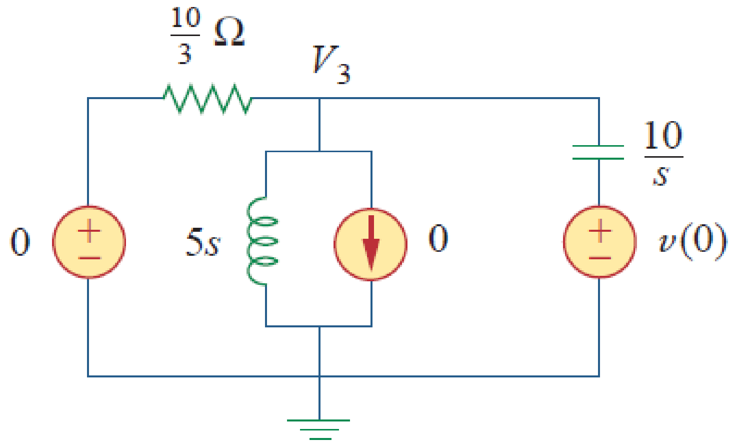

10/3 Ω
V1
10/s
5s
i(0)/s
10/s
v(0)/s


10/3 Ω
V1
10/s
5s
0
10/s
0

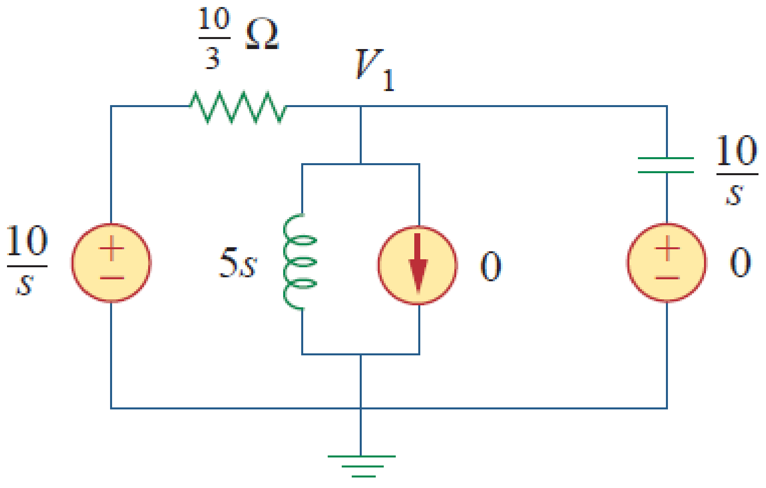
(a)


10/3 Ω
V2
0
5s
i(0)/s
10/s
0

(b)


10/3 Ω
V3
0
5s
0
10/s
v(0)

(c)



(a)

$$\frac{V_1 - 10/s}{10/3} + \frac{V_1 - 0}{5s} + 0 + \frac{V_1 - 0}{1/(0.1s)} = 0$$

or

$$0.1\left(s + 3 + \frac{2}{s}\right)V_1 = \frac{3}{s}$$

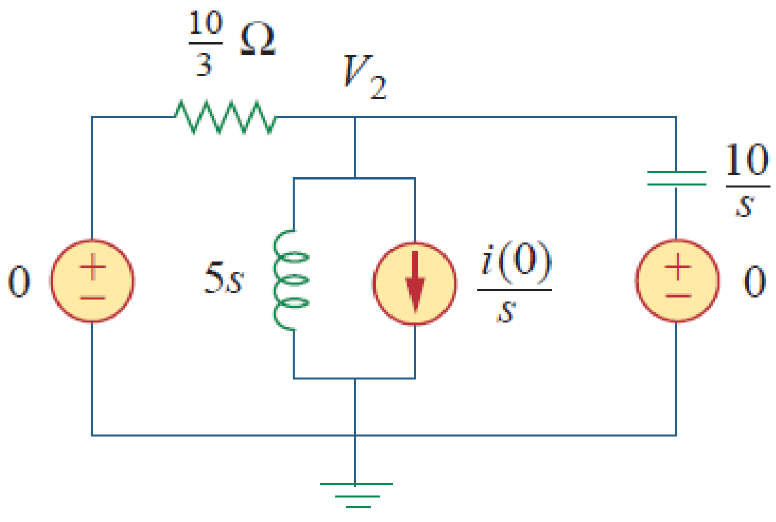
Simplifying we get

$$(s^2 + 3s + 2)V_1 = 30$$

$$V_1 = \frac{30}{(s+1)(s+2)} = \frac{30}{s+1} - \frac{30}{s+2}$$

or

$$v_1(t) = (30e^{-t} - 30e^{-2t})u(t)\text{ V}$$



(b)

For Fig. 16.13(b) we get,

$$\frac{V_2 - 0}{10/3} + \frac{V_2 - 0}{5s} - \frac{1}{s} + \frac{V_2 - 0}{1/(0.1s)} = 0$$

or

$$0.1\left(s + 3 + \frac{2}{s}\right)V_2 = \frac{1}{s}$$

This leads to

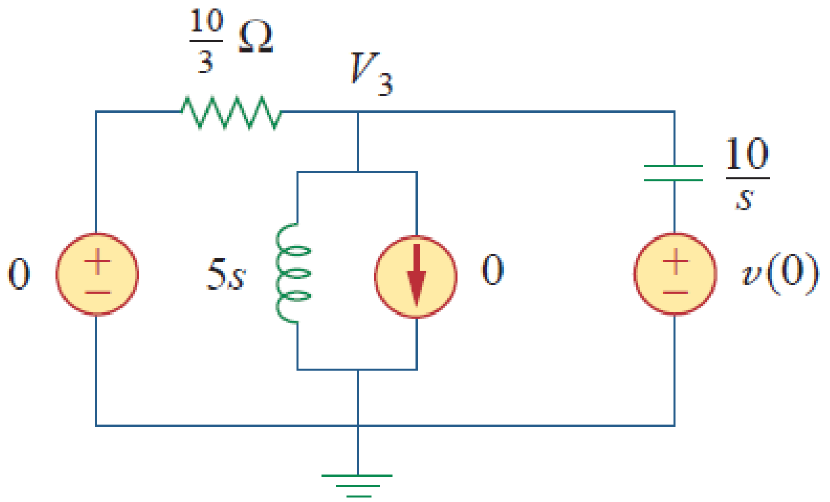
$$V_2 = \frac{10}{(s+1)(s+2)} = \frac{10}{s+1} - \frac{10}{s+2}$$

Taking the inverse Laplace transform, we get

$$v_2(t) = (10e^{-t} - 10e^{-2t})u(t)\ \text{V}$$

For Fig. 16.13(c),

$$\frac{V_3 - 0}{10/3} + \frac{V_3 - 0}{5s} - 0 + \frac{V_3 - 5/s}{1/(0.1s)} = 0$$



(c)

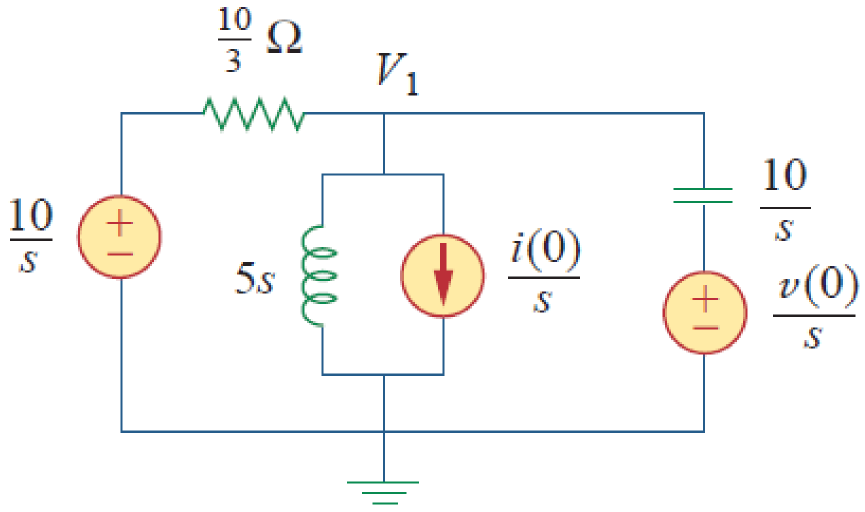
or

$$0.1\left(s + 3 + \frac{2}{s}\right)V_3 = 0.5$$

$$V_3 = \frac{5s}{(s+1)(s+2)} = \frac{-5}{s+1} + \frac{10}{s+2}$$

This leads to

$$v_3(t) = (-5e^{-t} + 10e^{-2t})u(t)\ \text{V}$$



Now all we need to do is to add Eqs. (16.5.1), (16.5.2), and (16.5.3):

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) + v_3(t) \\ &= \{(30 + 10 - 5)e^{-t} + (-30 + 10 - 10)e^{-2t}\}u(t) \text{ V} \end{aligned}$$

or

$$v(t) = (35e^{-t} - 30e^{-2t})u(t) \text{ V}$$

which agrees with our answer in Example 16.4.

Practice Problem 16.5

For the circuit shown in Fig. 16.12, and the same initial conditions in Example 16.4, find the current through the inductor for all time $t > 0$ using superposition.

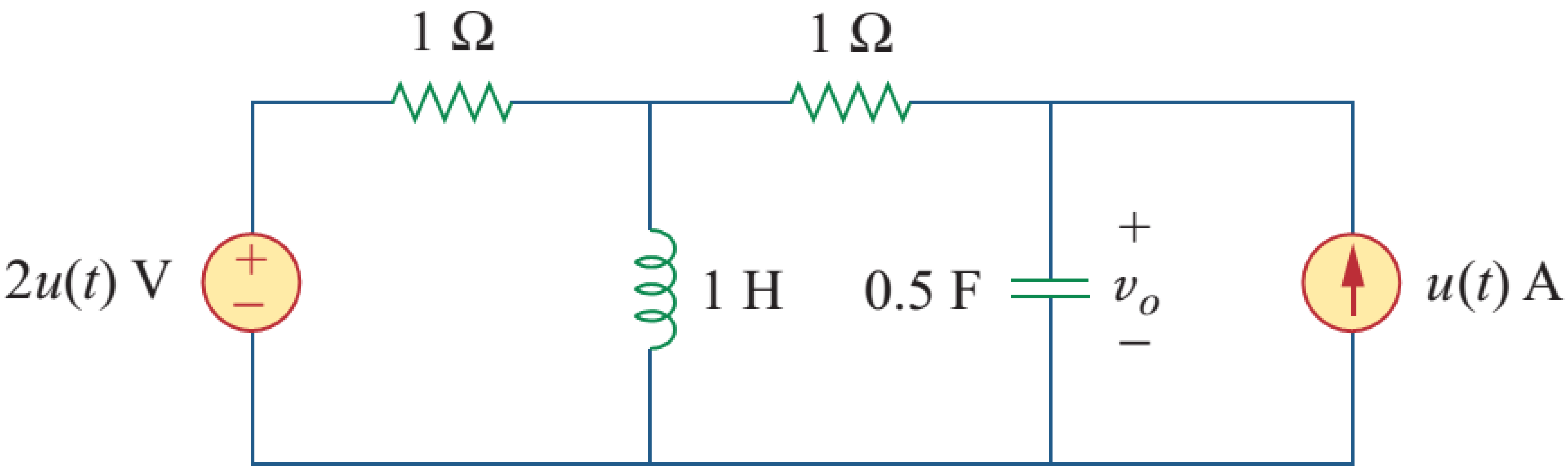
Answer: $i(t) = (3 - 7e^{-t} + 3e^{-2t})u(t)$ A.

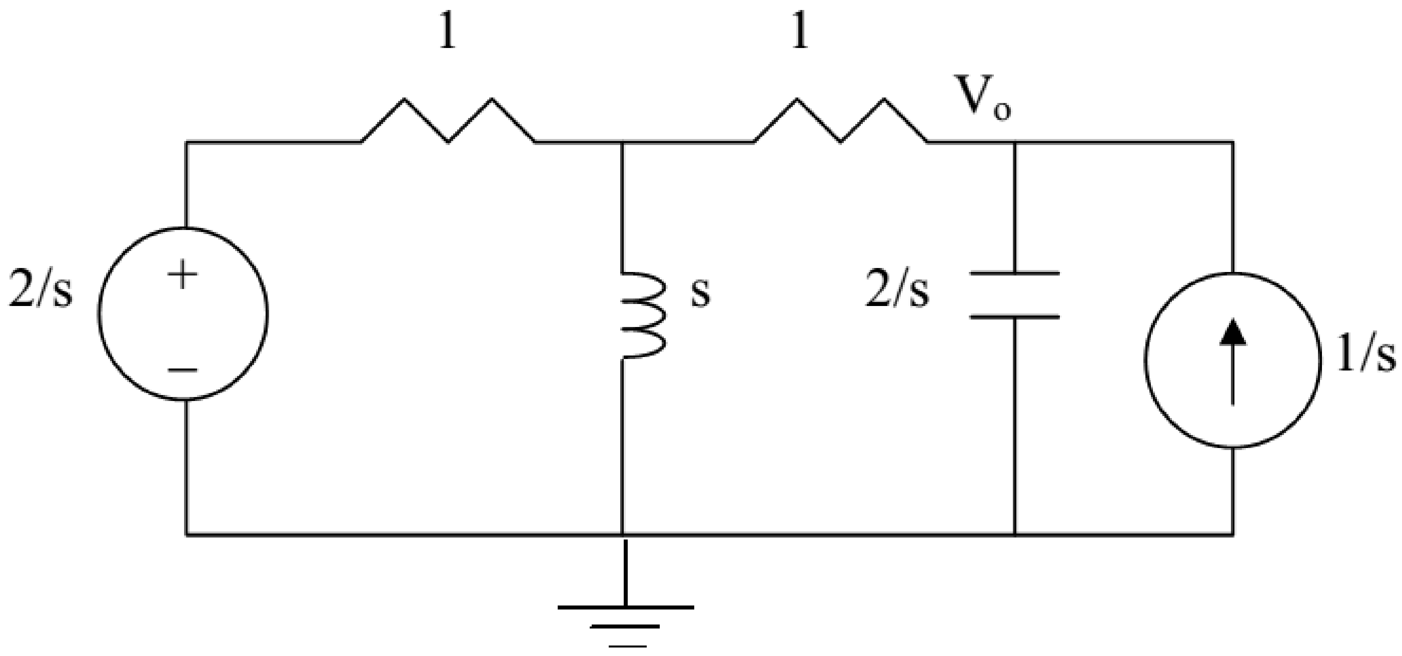
Home work

- Example problem 16.3 (page 721-Sadiku)
- Practice problem 16.3 (page 722-Sadiku)
- Example problem 16.6 (page 724-Sadiku)
- Practice problem 16.6 (page 725-Sadiku

Example

16.7 Find $v_o(t)$, for all $t > 0$, in the circuit of Fig. 16.41.




1
1
Vo
2/s
+
−
s
2/s
1/s

$$\frac{2/s - V_1}{1} = \frac{V_1}{s} + \frac{V_1 - V_o}{1} \longrightarrow \frac{2}{s} = V_1(2 + 1/s) - V_o \qquad (1)$$

$$\frac{V_1 - V_o}{1} + \frac{1}{s} = \frac{V_o}{2/s} = \frac{s}{2}V_o \longrightarrow V_1 = (1 + s/2)V_o - 1/s \qquad (2)$$

$$V_o = \frac{(4s+1)}{s(s^2 + 1.5s + 1)}$$

Partial Fraction Rules

- For a linear term $ax + b$ we get a contribution of $\dfrac{A}{ax+b}$.
- For a repeated linear term, such as $(ax+b)^3$, we get a contribution of

$$\frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3}.$$

  We have *three* terms which matches that $(ax+b)$ occurs to the *third* power.

- For a quadratic term $ax^2 + bx + c$ we get a contribution of $\dfrac{Ax+B}{ax^2+bx+c}$.
- For a repeated quadratic term such as $(ax^2+bx+c)^2$ we get a contribution of

$$\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2}.$$

These rules can be mixed together in any way.

$$V_o = \frac{(4s+1)}{s(s^2+1.5s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1.5s+1}$$

$$4s+1 = A(s^2+1.5s+1) + Bs^2 + Cs$$

We equate coefficients.

s^2 : $0 = A + B$ or $B = -A$

s: $4 = 1.5A + C$

constant: $1 = A$, $B = -1$, $C = 4 - 1.5A = 2.5$

$$V_o = \frac{1}{s} + \frac{-s+2.5}{s^2+1.5s+1}$$

$$V_o = \frac{1}{s} + \frac{-s + 2.5}{s^2 + 1.5s + 1}$$

$e^{-at}\cos(\omega_d t)$	$\frac{s + a}{(s + a)^2 + \omega_d^2}$

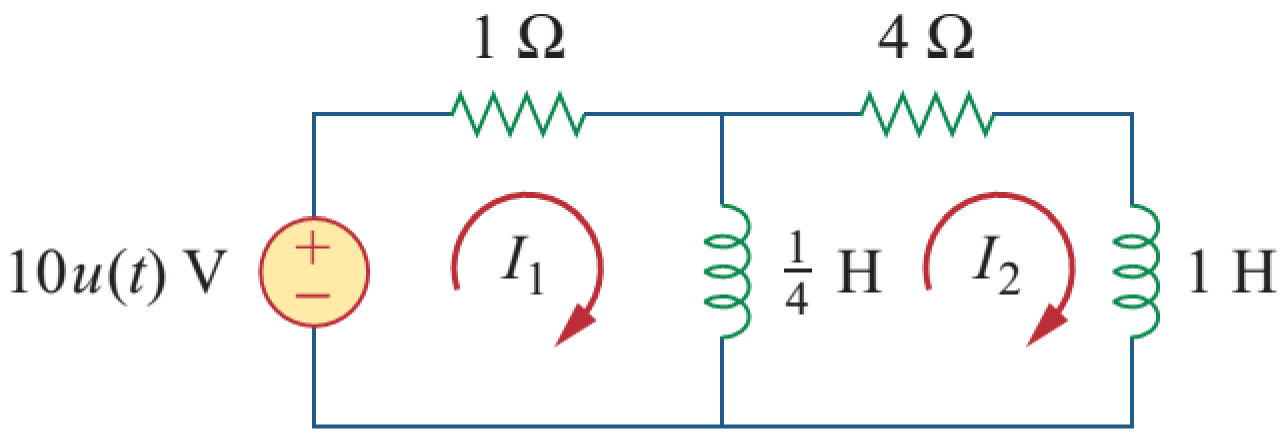
$$V_o = \frac{1}{s} + \frac{-s+2.5}{s^2+1.5s+1}$$

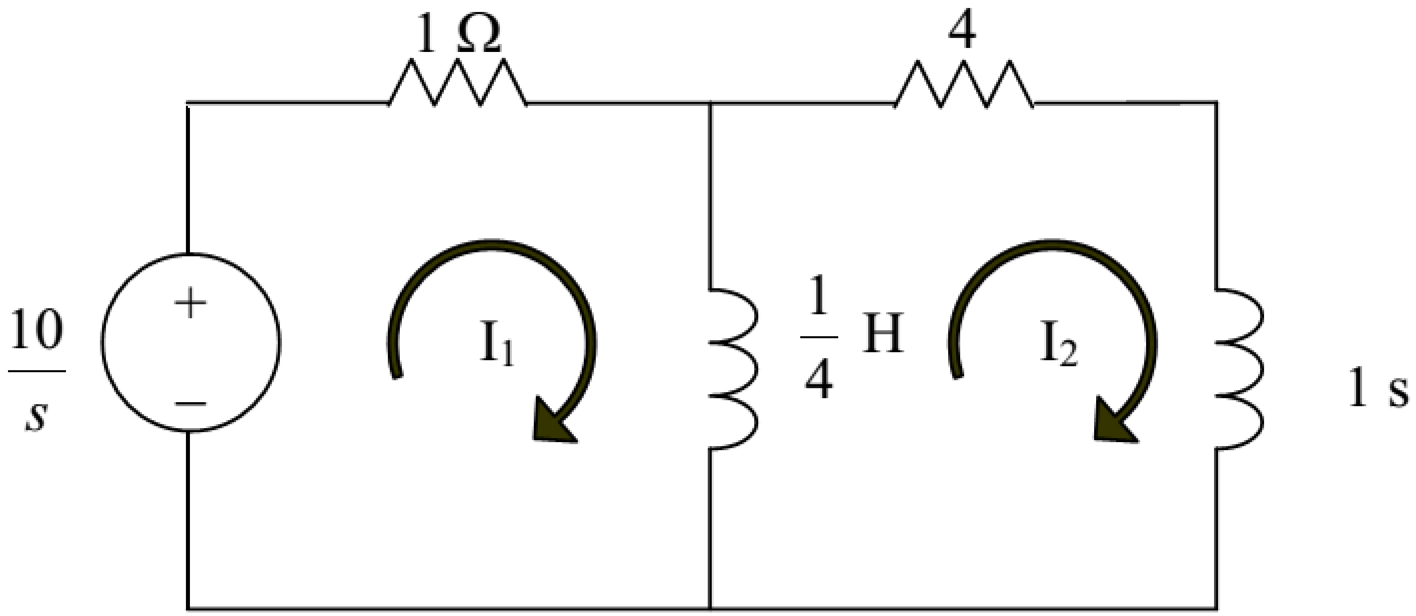
$$\frac{1}{s} + \frac{-s+2.5}{s^2+1.5s+1} = \frac{1}{s} - \frac{s+3/4}{(s+3/4)^2+\left(\frac{\sqrt{7}}{4}\right)^2} + \frac{\frac{\frac{3.25}{\sqrt{7}}}{4} x \frac{\sqrt{7}}{4}}{(s+3/4)^2+\left(\frac{\sqrt{7}}{4}\right)^2}$$

$$v(t) = \underline{u(t) - e^{-3t/4}\cos\frac{\sqrt{7}}{4}t + 4.9135e^{-3t/4}\sin\frac{\sqrt{7}}{4}t}$$

Example

Solve for the mesh currents in the circuit of Fig.





$$\frac{10}{s} = (1+\frac{s}{4})I_1 - \frac{1}{4}sI_2$$

$$-\frac{1}{4}sI_1 + I_2(4+\frac{5}{4}s) = 0$$

$$\frac{10}{s} = (1+\frac{s}{4})I_1 - \frac{1}{4}sI_2$$

$$-\frac{1}{4}sI_1 + I_2(4+\frac{5}{4}s) = 0$$

$$I_1 = \underline{\frac{50s+160}{s(s^2+9s+16)}}$$

$$I_2 = \underline{\frac{10}{s^2+9s+16}}$$

Transfer Functions

- The transfer function is a key concept in signal processing because it indicates how a signal is processed as it passes through a network.
- It is a fitting tool for finding the network response, determining (or designing for) network stability, and network synthesis.
- The transfer function of a network describes how the output behaves with respect to the input.
- It specifies the transfer from the input to the output in the s-domain, assuming no initial energy.

The **transfer function** $H(s)$ is the ratio of the output response $Y(s)$ to the input excitation $X(s)$, assuming all initial conditions are zero.

$$\boxed{H(s) = \frac{Y(s)}{X(s)}}$$

$$H(s) = \text{Voltage gain} = \frac{V_o(s)}{V_i(s)}$$

$$H(s) = \text{Current gain} = \frac{I_o(s)}{I_i(s)}$$

$$H(s) = \text{Impedance} = \frac{V(s)}{I(s)}$$

$$H(s) = \text{Admittance} = \frac{I(s)}{V(s)}$$

- The transfer function depends on what we define as input and output.
- Based on the input and output, either current or voltage at any place in the circuit, there are four possible transfer functions
- Thus, a circuit can have many transfer functions
- **H(s)** is dimensionless

$$H(s) = \frac{Y(s)}{X(s)}$$

$$Y(s) = H(s)X(s)$$

The inverse transform of Y(s) gives $y(t)$.

A special case is when the input $x(t) = \delta(t)$, so that $X(s) = 1$.
is the unit impulse function,

so that For this case, $Y(s) = H(s)$ or $y(t) = h(t)$

$$h(t) = \mathcal{L}^{-1}[H(s)]$$

The term h(t) represents the unit impulse response—it is the time-domain response of the network to a unit impulse.

Example

The output of a linear system is $y(t) = 10e^{-t}\cos 4t\, u(t)$ when the input is $x(t) = e^{-t}u(t)$. Find the transfer function of the system and its impulse response.

Solution:

If $x(t) = e^{-t}u(t)$ and $y(t) = 10e^{-t}\cos 4t\, u(t)$, then

$$X(s) = \frac{1}{s+1} \qquad \text{and} \qquad Y(s) = \frac{10(s+1)}{(s+1)^2 + 4^2}$$

Hence,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{10(s+1)^2}{(s+1)^2 + 16} = \frac{10(s^2 + 2s + 1)}{s^2 + 2s + 17}$$

To find $h(t)$, we write $H(s)$ as

$$H(s) = 10 - 40\frac{4}{(s+1)^2 + 4^2}$$

From Table 15.2, we obtain

$$h(t) = 10\delta(t) - 40e^{-t}\sin 4t\, u(t)$$

Practice Problem 16.7

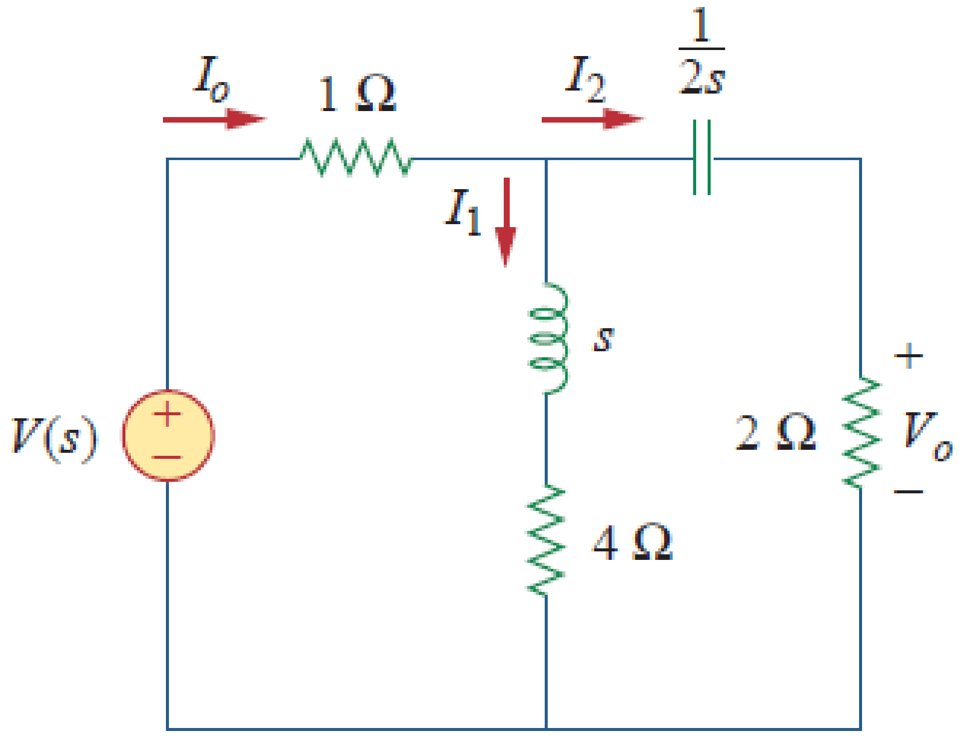
The transfer function of a linear system is

$$H(s) = \frac{2s}{s + 6}$$

Find the output $y(t)$ due to the input $5e^{-3t}u(t)$ and its impulse response.

Answer: $-10e^{-3t} + 20e^{-6t}$, $t \geq 0$, $2\delta(t) - 12e^{-6t}u(t)$.

Determine the transfer function $H(s) = V_o(s)/I_o(s)$ of the circuit



METHOD 1 By current division,

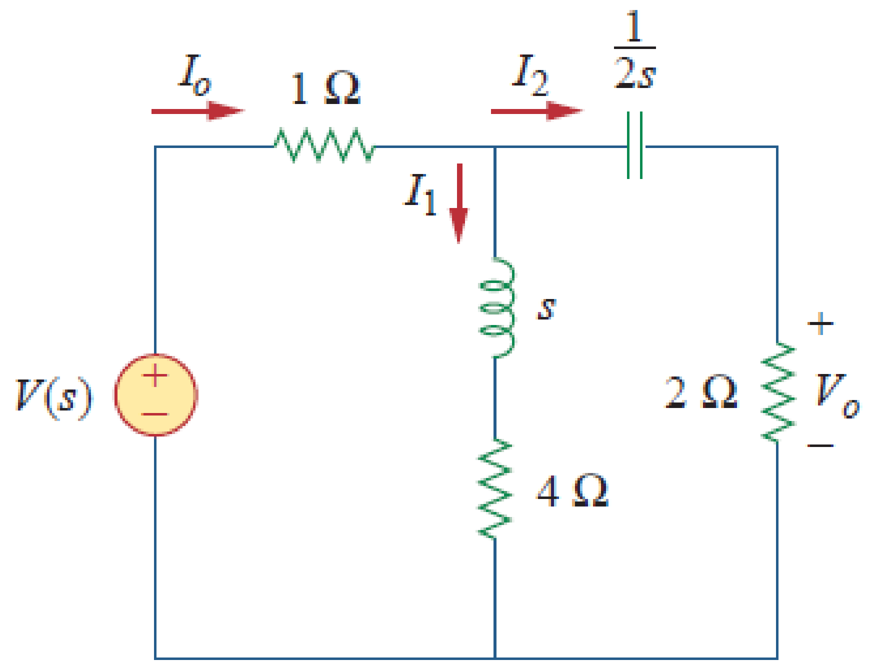
$$I_2 = \frac{(s+4)I_o}{s+4+2+1/2s}$$

But

$$V_o = 2I_2 = \frac{2(s+4)I_o}{s+6+1/2s}$$

Hence,

$$H(s) = \frac{V_o(s)}{I_o(s)} = \frac{4s(s+4)}{2s^2+12s+1}$$



METHOD 2 We can apply the ladder method. We let $V_o = 1$ V. By Ohm's law, $I_2 = V_o/2 = 1/2$ A. The voltage across the $(2 + 1/2s)$ impedance is

$$V_1 = I_2\left(2 + \frac{1}{2s}\right) = 1 + \frac{1}{4s} = \frac{4s + 1}{4s}$$

This is the same as the voltage across the $(s + 4)$ impedance. Hence

$$I_1 = \frac{V_1}{s + 4} = \frac{4s + 1}{4s(s + 4)}$$

Applying KCL at the top node yields

$$I_o = I_1 + I_2 = \frac{4s + 1}{4s(s + 4)} + \frac{1}{2} = \frac{2s^2 + 12s + 1}{4s(s + 4)}$$

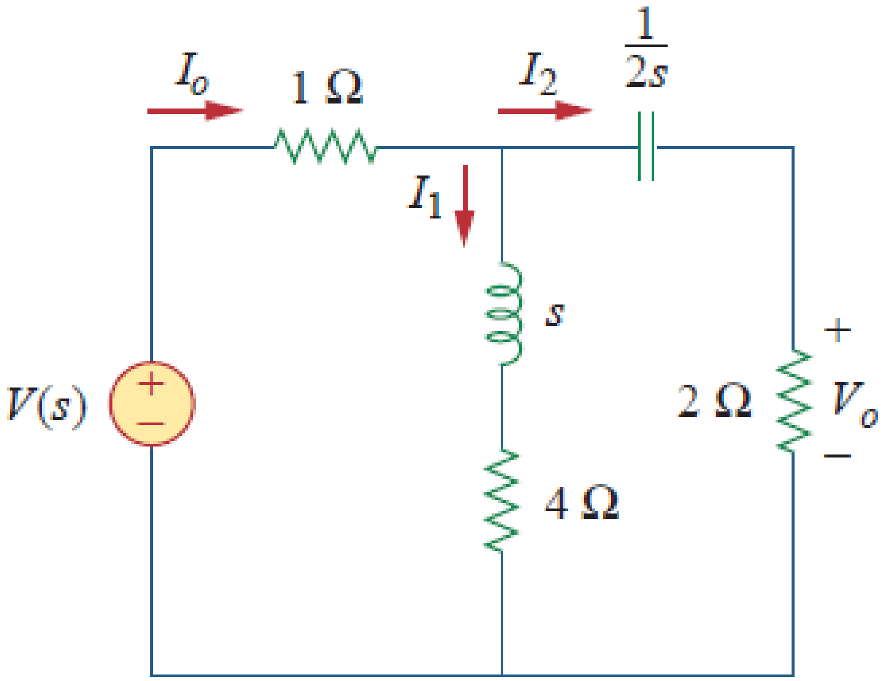
Hence,

$$H(s) = \frac{V_o}{I_o} = \frac{1}{I_o} = \frac{4s(s + 4)}{2s^2 + 12s + 1}$$

as before.

Practice Problem 16.8

Find the transfer function $H(s) = I_1(s)/I_o(s)$ in the circuit of Fig. 16.18.



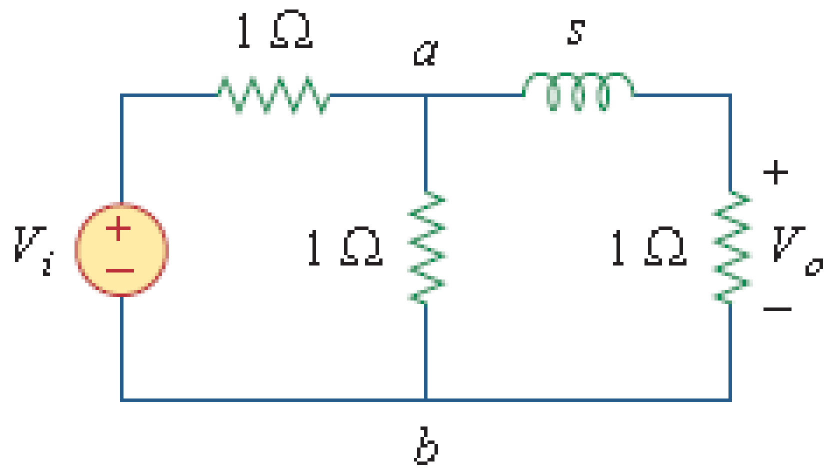
Answer: $\dfrac{4s + 1}{2s^2 + 12s + 1}$.

Home work

- Example problem 16.9 (page 729-Sadiku)
- Practice problem 16.9 (page 730-Sadiku)

Example

For the s-domain circuit in Fig. 16.19, find: (a) the transfer function $H(s) = V_o/V_i$, (b) the impulse response, (c) the response when $v_i(t) = u(t)$ V, (d) the response when $v_i(t) = 8\cos 2t$ V.



$$H(s) = \frac{V_o}{V_i} = \frac{1}{2s + 3}$$

$$h(t) = \frac{1}{2}e^{-3t/2}u(t)$$

$$v_o(t) = \frac{1}{3}(1 - e^{-3t/2})u(t) \text{ V}$$

$$v_o(t) = \frac{24}{25}\left(-e^{-3t/2} + \cos 2t + \frac{4}{3}\sin 2t\right)u(t) \text{ V}$$

Example

When the input to a system is a unit step function, the response is 10 cos 2*t*. Obtain the transfer function of the system

$$x(t) = u(t) \longrightarrow X(s) = \frac{1}{s}$$

$$y(t) = 10\cos(2t) \longrightarrow Y(s) = \frac{10s}{s^2+4}$$

$$H(s) = \frac{Y(s)}{X(s)} = \underline{\frac{\mathbf{10s^2}}{\mathbf{s^2+4}}}$$

Example

The transfer function of a certain circuit is

$$H(s) = \frac{5}{s+1} - \frac{3}{s+2} + \frac{6}{s+4}$$

Find the impulse response of the circuit.

$$h(t) = \underline{\left(5e^{-t} - 3e^{-2t} + 6e^{-4t}\right)u(t)}$$

Example

When a unit step is applied to a system at $t = 0$ its response is

$$y(t) = \left[4 + \frac{1}{2}e^{-3t} - e^{-2t}(2\cos 4t + 3\sin 4t)\right]u(t)$$

What is the transfer function of the system?

$$H(s) = \frac{Y(s)}{X(s)}, \qquad X(s) = \frac{1}{s}$$

$$Y(s) = \frac{4}{s} + \frac{1}{2(s+3)} - \frac{2s}{(s+2)^2 + 16} - \frac{(3)(4)}{(s+2)^2 + 16}$$

$$H(s) = s\,Y(s) = \underline{4 + \frac{s}{2(s+3)} - \frac{2s(s+2)}{s^2 + 4s + 20} - \frac{12s}{s^2 + 4s + 20}}$$

A circuit is known to have its transfer function as

$$H(s) = \frac{s+3}{s^2+4s+5}$$

Find its output when:

(a) the input is a unit step function

(b) the input is $6te^{-2t}\,u(t)$.

(a) $$Y(s) = H(s)\,X(s)$$

$$= \frac{s+3}{s^2+4s+5} \cdot \frac{1}{s}$$

$$= \frac{s+3}{s\,(s^2+4s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+5}$$

$$s+3 = A\,(s^2+4s+5) + Bs^2 + Cs$$

Equating coefficients :

$$s^0: \quad 3 = 5A \longrightarrow A = 3/5$$

$$s^1: \quad 1 = 4A + C \longrightarrow C = 1 - 4A = -7/5$$

$$s^2: \quad 0 = A + B \longrightarrow B = -A = -3/5$$

$$Y(s) = \frac{3/5}{s} - \frac{1}{5} \cdot \frac{3s+7}{s^2+4s+5}$$

$$Y(s) = \frac{0.6}{s} - \frac{1}{5} \cdot \frac{3(s+2)+1}{(s+2)^2+1}$$

$$y(t) = \underline{\left[\mathbf{0.6 - 0.6\,e^{-2t}\cos(t) - 0.2\,e^{-2t}\sin(t)}\right]\mathbf{u(t)}}$$

$$x(t) = 6\,t\,e^{-2t} \quad \longrightarrow \quad X(s) = \frac{6}{(s+2)^2}$$

$$Y(s) = H(s)\,X(s) = \frac{s+3}{s^2+4s+5} \cdot \frac{6}{(s+2)^2}$$

$$Y(s) = \frac{6(s+3)}{(s+2)^2(s^2+4s+5)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{Cs+D}{s^2+4s+5}$$

Equating coefficients :

$$s^3: \quad 0 = A + C \quad \longrightarrow \quad C = -A \qquad (1)$$

$$s^2: \quad 0 = 6A + B + 4C + D = 2A + B + D \qquad (2)$$

$$s^1: \quad 6 = 13A + 4B + 4C + 4D = 9A + 4B + 4D \qquad (3)$$

$$s^0: \quad 18 = 10A + 5B + 4D = 2A + B \qquad (4)$$

Solving (1), (2), (3), and (4) gives

$A = 6$, $B = 6$, $C = -6$, $D = -18$

$$Y(s) = \frac{6}{s+2} + \frac{6}{(s+2)^2} - \frac{6s+18}{(s+2)^2+1}$$

$$Y(s) = \frac{6}{s+2} + \frac{6}{(s+2)^2} - \frac{6(s+2)}{(s+2)^2+1} - \frac{6}{(s+2)^2+1}$$

$$y(t) = \underline{\left[\mathbf{6e^{-2t} + 6te^{-2t} - 6e^{-2t}\cos(t) - 6e^{-2t}\sin(t)}\right]\mathbf{u(t)}}$$

Example

A circuit has a transfer function

$$H(s) = \frac{s+4}{(s+1)(s+2)^2}$$

Find the impulse response.

$$H(s)=\frac{s+4}{(s+1)(s+2)^2}=\frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{(s+2)^2}$$

$$s+4=A(s+2)^2+B(s+1)(s+2)+C(s+1)=A(s^2+2s+4)+B(s^2+3s+2)+C(s+1)$$

We equate coefficients.

s^2 : 0=A+B or B=-A

s: 1=4A+3B+C=B+C

constant: 4=4A+2B+C =2A+C

Solving these gives A=3, B=-3, C=-2

$$H(s)=\frac{3}{s+1}-\frac{3}{s+2}-\frac{2}{(s+2)^2}$$

$$h(t)=\underline{(3e^{-t}-3e^{-2t}-2te^{-2t})u(t)}$$