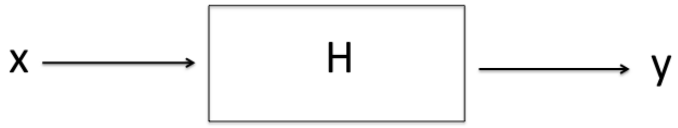
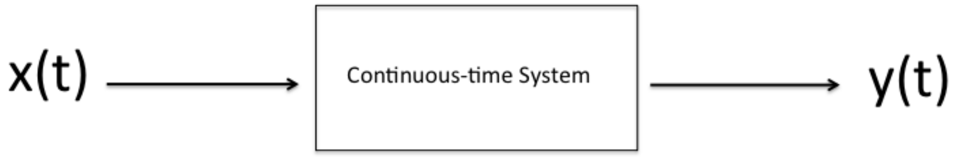
Interconnection of Systems & Convolution

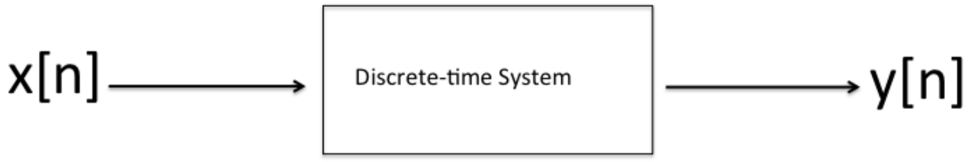
For us a **system** is a box that takes in a signal x as an input and outputs a signal y as illustrated below:



CONTINUOUS-TIME systems process *inputs* and *outputs* of **continuous-time** signals.



DISCRETE-TIME systems process *inputs* and *outputs* of **discrete-time** signals.



We can represent the system by an operator H:

$$y = H \cdot x$$

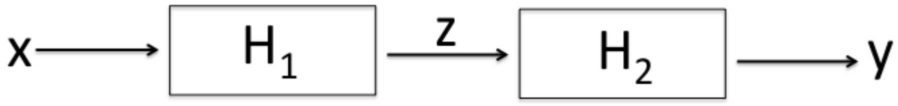
The notation indicates that y changes with x. For example the nature of the operator specifies the type of system.

Interconnection of Systems

Systems can be combined to form more complex systems otherwise known as the **interconnection of systems**.

Series/Cascade Interconnection

A series or cascade interconnection is the results of an input x into system H1 which results in an output z that is in turn the input for system H2 which results in an output y as illustrated below.



A cascade system can mathematically be represented as:

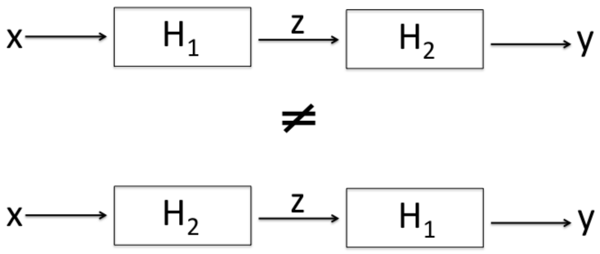
$$z=H1\cdot x$$

$$y=H2\cdot z$$

$$y=H2\cdot H1\cdot x$$

Does the order of systems H1 and H2 matter? It does indeed matter.

As illustrated below the first cascade system is not necessarily equal to the second cascade system.



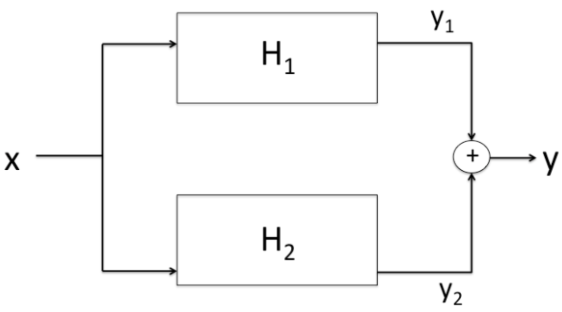
Case 1: $y=H2\cdot H1\cdot x$

Case2: $y=H1\cdot H2\cdot x$

In general
$H1\cdot H2\neq H2\cdot H1$ exept for some special cases

Parallel Interconnection

In a parallel interconnection the input x goes into both systems H1 and H2 simulatenously. The output from both systems are added together to produce the resulting output y :



Mathematically:

$$y1=H1\cdot x \text{ and } y2=H2\cdot x$$

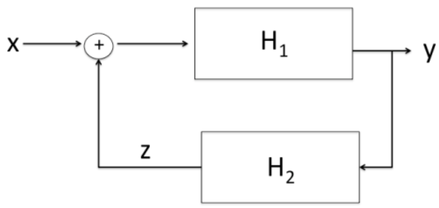
$$y=y1+y2$$

$$y=H1\cdot x+H2\cdot x$$

$$y=(H1+H2)\cdot x$$

Feedback Interconnection

In a feedback interconnection the output itself affects the output.



$$y=(x+z)\cdot H1$$

$$z=H2\cdot y$$

$$y=H1\cdot(x+H2\cdot y)$$

Simple System Examples

Identity Transform

The output is exactly the input. This is a "do-nothing" system

$$y=H\cdot x=x \text{ Notation: } H=I$$

$$x=I\cdot x$$

Delay

The delay system introduces a delay in the signal.

$$y=H\cdot x=x(t-\tau)(CT)$$

$$y=H\cdot x=x[n-N](DT)$$

System Properties

Memoryless System

A system is said to be **memoryless** if its output at any time depends only on its input at the **same** instant with no reference to input values at other times. Mathematically this can be illustrated in CT and DT.

$$x(t_o)\rightarrow y(t_o) \text{ (CT)}$$

$$x[n_o]\rightarrow y[n_o] \text{ (DT)}$$

<u>EXAMPLES</u>

Memoryless

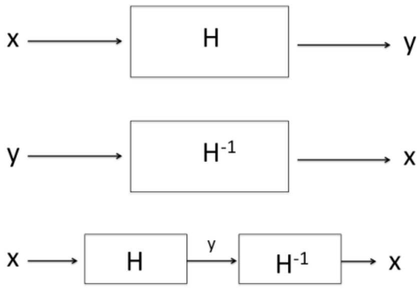
1. y(t)=4∗x(t)y(t)
2. y[n]=x2[n]y[n]

Non-memoryless Systems

1. $y[n] = \sum_{-\infty}^{n} x[n]$
2. $y[n] = x[n-k]$ for $k \neq 0$

Invertibility

A system H is said to be **invertible** if for every $y=H\cdot x$. There exist another system H^{-1} such that x can be recovered from y as $x=H^{-1}\cdot y$



$$x=H^{-1}\cdot H\cdot x$$

where
$H^{-1}\cdot H=I$ (identity)

<u>EXAMPLES</u>

Invertible Systems

1. y(t)=4x(t)
   - x(t)=$\frac{1}{4}$y(t) is the *inverse*

Non-invertible Systems

1. y(t)=x2(t)
   - Cannot determine sign of input from knowledge of output so the system in *not invertible*

Impulse Response and Convolution

Let us now consider how LTI systems respond to inputs

Any signal x[n] can be written as:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Also recall the definition of an LSI system

<u>CT</u>
x1(t)→y1(t)

x2(t)→y2(t)

ax1(t−τ)+bx2(t−τ)→ay1(t−τ)+by2(t−τ)

<u>DT</u>
x1[n]→y1[n]

$$x2[n] \rightarrow y2[n]$$

$$ax1[n-k]+bx2[n-k] \rightarrow ay1[n-k]+by2[n-k]$$

Consider an LSI system:

$x[n] \rightarrow y[n]$

Specifically consider how the system responds to an impulse input:

$\delta[n] \rightarrow h[n]$

We call h[n] the **impulse response** of the system. It is the output it generates in response to an input impulse. We can define an impulse response to CT systems as well

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$=x[0]\delta[n] \rightarrow x[0]h[n]$$

$$=x[1]\delta[n-1] \rightarrow x[1]h[n-1]$$

$$=x[-1]\delta[n+1] \rightarrow x[-1]h[n+1]$$

$$=x[2]\delta[n-2] \rightarrow x[2]h[n-2]$$

$$=x[-2]\delta[n+2] \rightarrow x[-2]h[n+2]$$

....

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Since $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

$$x[n] \rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Implications:

- In an LSI system the response of the system to any input is completely determined from its impulse response.
- In other words, the system is completely specified by its impulse response h[n].
- The equation relating x to y is :

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

is knowns as the CONVOLUTION of x[n] & h[n].

It is usually represented a $y[n]=H[n]*x[n]$

<u>A similar relation can be found for CT LTI systems</u>

If $\delta(t)\rightarrow h(t)$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

Understanding the Convolution

So to visualize how to compute $y[n]$, we flip $h[n]$ and shift it forward one step at a time. At each shift n we multiple the $x[k]$ & $h[n-k]$ sample by sample and add the samples up to get $y[n]$.

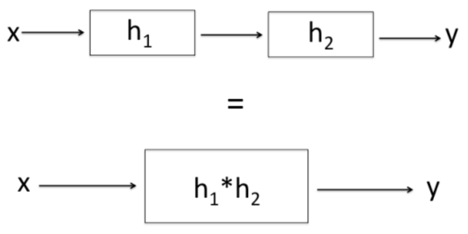
Properties of Convolution

Commutativity

$x*h=h*x$

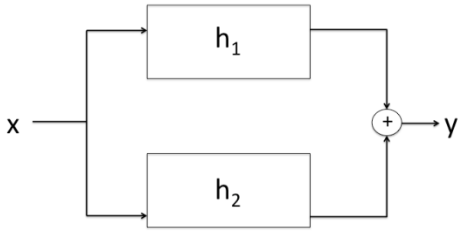
Associativity

$$x*h1*h2=(x*h1)*h2=x*(h1*h2)$$



Distributive

$$y=x*h1+x*h2=x*(h1+h2)$$



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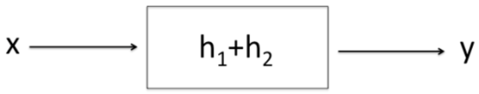

LSI systems are:

- Commutative
- Associative
- Distributive