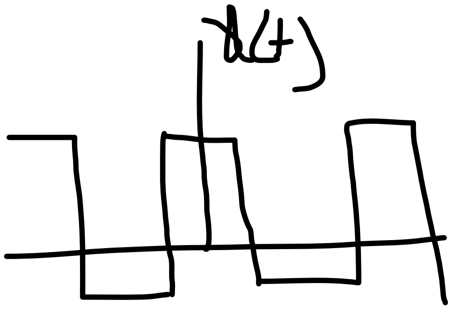
Symmetric condition:

A signal $x(t)$ is said to have even symmetry or mirror symmetry, if

$$x(-t) = x(t)$$

Ex:



$$\int_{-T}^{T} x(t)\,dt = \int_{-T}^{0} x(t)\,dt + \int_{0}^{T} x(t)\,dt$$

Put $t = -t$

$dt = -dt$

$\int_{-T}^{T} x(t)\,dt = -\int_{T}^{0} x(t)\,dt + \int_{0}^{T} x(t)\,dt$, if $x(t)$ has even symmetry

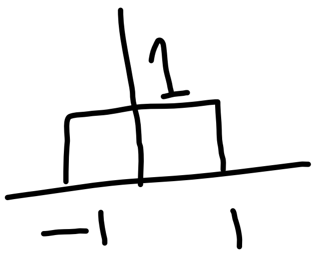
$= -\int_{T}^{0} x(t)dt + \int_{0}^{T} x(t)\,dt$

$= \int_{0}^{T} x(t)dt + \int_{0}^{T} x(t)\,dt$=2 $\int_{0}^{T} x(t)$

if $x(t)$ has even symmetry

$\int_{-T}^{T} x(t)\,dt$ =2 $\int_{0}^{T} x(t)$

Ex:



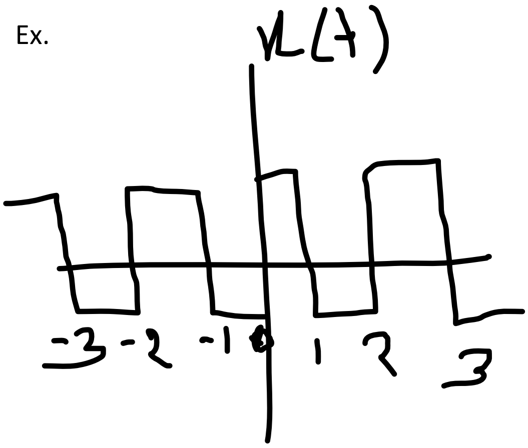
$\int_{-1}^{1} 1\,dt = 2$ => 2 $\int_{0}^{1} 1dt = 2$

Odd Symmetry:

A signal $x(t)$ is said to have odd symmetry or rotational symmetry, if

$x(-t) = -x(t)$ for all ‘t’

Ex.



$$\int_{-T}^{T} x(t)\,dt = \int_{-T}^{0} x(t)\,dt + \int_{0}^{T} x(t)\,dt$$

Put $t = \ -t$

$dt = \ -dt$

$\int_{-T}^{T} x(t)\,dt = -\int_{T}^{0} x(-t)\,dt + \int_{0}^{T} x(t)\,dt$, if $x(t)$ has odd symmetry

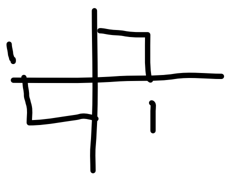
$= \int_{T}^{0} x(t)dt + \int_{0}^{T} x(t)\,dt$

$= -\int_{0}^{T} x(t)dt + \int_{0}^{T} x(t)\,dt$ =0

If the waveform has odd symmetry,

$$\int_{-T}^{T} x(t)\,dt = 0$$

Ex:



$\int_{-1}^{1} 1\,dt = 2$ => $-\int_{0}^{1} 1dt + \int_{0}^{1} 1dt = 0$

Trigonometric Fourier series with Symmetric Conditions:

The trigonometric Fourier series of a periodic signal $x(t)$ is given by,

- $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\ ,\ t_0 < t < t_0 + \frac{2\pi}{\omega_0}$
- $a_0 = \frac{1}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} x(t)dt$, is called average or dc value of the signal $x(t)$
- $a_n = \frac{2}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} x(t) \cos n\omega_0 t\, dt$
- $b_n = \frac{2}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} x(t) \sin n\omega_0 t\, dt$
- Let $t_0 = -\frac{T}{2}$
- $a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt$
- $a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos n\omega_0 t\, dt$
- $b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin n\omega_0 t\, dt$

Substitute the symmetric condition

Case 1: if $x(t)$ has even or mirror symmetry,

$x(-t) = x(t)$

- $a_0 = \frac{2}{T}\int_{0}^{\frac{T}{2}} x(t)dt$
- $a_n = \frac{2}{T}\int_{0}^{\frac{T}{2}} 2 * x(t) \cos n\omega_0 t\, dt = \frac{4}{T}\int_{0}^{\frac{T}{2}} x(t) \cos n\omega_0 t\, dt$ (Even *even=even)
- $b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin n\omega_0 t\, dt = 0$ (Even*Odd=Odd)

NOTE: The trigonometric Fourier series expansion of even symmetric signal consists of only cosine terms.

Case 2: if $x(t)$ has odd or rotational symmetry,

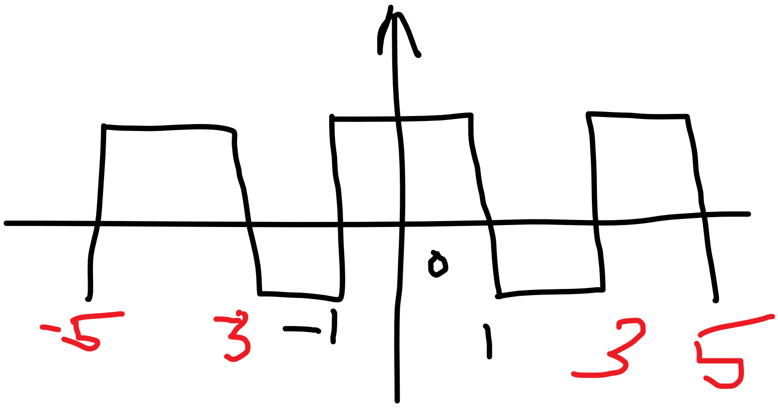
$x(-t) = -x(t)$

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = 0$$

- $a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cos n\omega_0 t\, dt$=0 (odd * even = odd) cosine terms are 0.
- $b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin n\omega_0 t\, dt$ = $\frac{4}{T}\int_{0}^{\frac{T}{2}} x(t)\sin n\omega_0 t\, dt$ (odd * odd=even)

- NOTE: The trigonometric Fourier series expansion of odd symmetric signal consists of only sine terms.

Q1. Find the trigonometric Fourier series of periodic signal $x(t)$ (Even),



Solution:

The trigonometric Fourier series of $x(t)$ is

- $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\ ,\ t_0 < t < t_0 + \frac{2\pi}{\omega_0}$

The given signal $x(t)$ has even symmetry. The trigonometry Fourier series expansion consists of only "Cosine terms".

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$b_n = 0$

$a_0 = \frac{1}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} x(t)dt$ = $\frac{1}{4}\int_{-1}^{3} x(t)dt = \frac{1}{4}\left[\int_{-1}^{1} dt - \int_{1}^{3} dt\right] = \frac{1}{4}[2-2] = 0$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} x(t)\cos n\omega_0 t\, dt = \frac{1}{2}[\int_{-1}^{1} \cos\left(\frac{n\pi t}{2}\right) dt - \int_{1}^{3} \cos\left(\frac{n\pi t}{2}\right) dt]$$

$$= \frac{1}{2}[\frac{2}{n\pi}\left\{\sin\left(\frac{n\pi t}{2}\right)\right\}_{-1}^{1} - \frac{2}{n\pi}\left\{\sin\left(\frac{n\pi t}{2}\right)\right\}_{1}^{3}$$

$$=\frac{1}{2}\left[\frac{2}{n\pi}\left\{\sin\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}\right)\right\}-\frac{2}{n\pi}\left\{\sin\left(\frac{3n\pi}{2}\right)-\sin\left(\frac{n\pi}{2}\right)\right\}\right]$$

$$\sin\left(2n\pi-\frac{n\pi}{2}\right)=-\sin\left(\frac{n\pi}{2}\right)$$

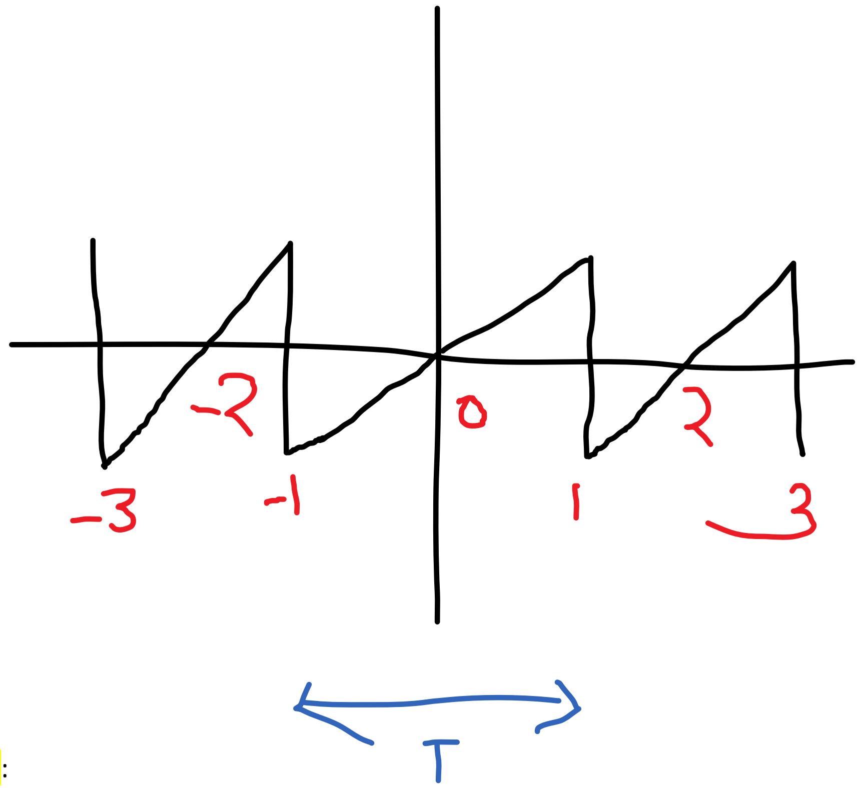
$$=\frac{1}{2}\left[\frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)+\frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)=\frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$b_n=0$$

$$x(t)=a_0+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi t}{2}\right)+b_n\sin\left(\frac{n\pi t}{2}\right);\ -1<t<4$$

$$x(t)=\sum_{n=1}^{\infty}\frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi t}{2}\right),-1<t<4$$

Q2. Find the trigonometric Fourier series of periodic signal $x(t)$ (Odd),



Solution:

(-1, -1) and (1, 2)

$$slope=\frac{2}{2}=1$$

$x(t) + 1 = 1(t + 1)$

$x(t) = t$;   -1<t<1

The trigonometric Fourier series expansion of $x(t)$ is,

- $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\ ,\ t_0 < t < t_0 + \frac{2\pi}{\omega_0}$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

Since the given $x(t)$ has odd symmetry, its trigonometric Fourier series expansion consists of only sine terms.

So, $a_n = 0$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt = \frac{1}{2}\int_{-1}^{1} t\,dt = \frac{1}{2} \times \frac{1}{2}\{t^2\}_{-1}^{\ 1} = 0$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} x(t)\sin n\omega_0 t\,dt = \int_{-1}^{1} t\sin(n\pi t)\,dt = \left\{-\frac{t\cos(n\pi t)}{n\pi} + \frac{\sin n\pi t}{n^2\pi}\right\}_{-1}^{1}$$

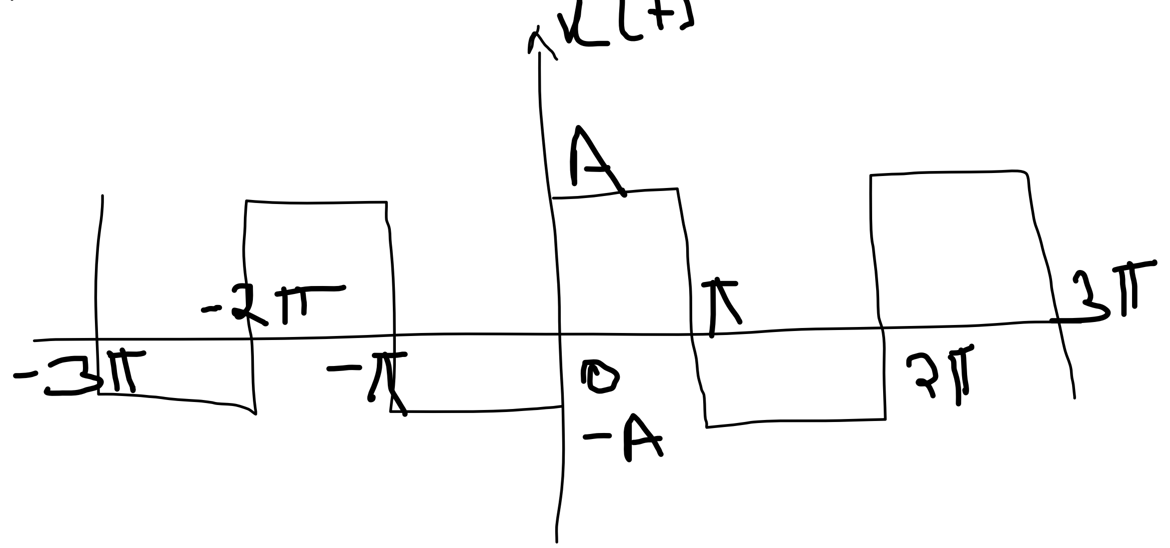
$$b_n = -\frac{\cos n\pi}{n\pi} - \left\{\frac{\cos n\pi}{n\pi}\right\} = \frac{-2\cos n\pi}{n\pi}$$

- $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\ ,\ -1 < t < 1$

$$x(t) = \sum_{n=1}^{\infty} \frac{-2\cos n\pi}{n\pi}\sin(n\pi t)\,; -1 < t < 1$$

Complex Exponential Fourier series:

Q. Find the exponential Fourier series for the waveform shown and also draw the frequency spectrum.



Solution:

The exponential series of periodic signal $x(t)$ is,

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}\,;\; t_0 < t < t_0 + \frac{2\pi}{\omega_0}$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnt}; 0 < t < 2\pi$$

Here,

$$c_n = \frac{1}{T}\int_{t_o}^{t_0+T} x(t)e^{-jn\omega_0 t}dt = \frac{1}{2\pi}\int_0^{2\pi} x(t)e^{-jnt}dt = \frac{1}{2\pi}[A\int_0^{\pi} e^{-jnt}dt - A\int_{\pi}^{2\pi} e^{-jnt}\,dt]$$

$$= \frac{A}{2\pi}\left[\left(\frac{e^{-jnt}}{(-jn)}\right)\begin{matrix}\pi\\0\end{matrix} + \left(\frac{e^{-jnt}}{jn}\right)\begin{matrix}2\pi\\\pi\end{matrix}\right]$$

$$c_n = \frac{A}{2\pi}\left[\left(\frac{e^{-jnt}}{(-jn)}\right)\begin{matrix}\pi\\0\end{matrix} + \left(\frac{e^{-jnt}}{jn}\right)\begin{matrix}2\pi\\\pi\end{matrix}\right]$$

$$= \frac{A}{j2\pi n}\left[-\left(e^{-jn\pi} - 1\right) + \left(e^{-j2\pi n} - e^{-jn\pi}\right)\right]$$

$$e^{-j2\pi n} = \cos 2\pi n - j\sin 2\pi n = 1$$

$e^{-jn\pi} = \cos n\pi - j\sin n\pi = \cos n\pi$

$c_n = \frac{A}{j2\pi n}[(1-\cos n\pi) + (1-\cos n\pi)] = \frac{A}{j\pi n}(1-\cos n\pi)$

If n = even,

$c_n = 0, n = 2, 4, 6, \ldots$

If n = odd,

$$c_n = \frac{2A}{jn\pi}; n = 1, 3, 5, \ldots$$

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{2A}{jn\pi} e^{jnt}; 0 < t < 2\pi$$

$c_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt$ is the average value or DC value.

$$= \frac{1}{2\pi}\int_0^{2\pi} x(t)dt$$

$$= \frac{1}{2\pi}\left[A\int_0^{\pi} dt - A\int_{\pi}^{2\pi} dt\right] = 0$$

<u>Frequency spectrum:</u>

Plotting magnitude of exponential Fourier Series Coefficient vs frequency is called Frequency spectrum.

i.e $|c_n|\ Vs\ n\omega_0$

Q. Plot the frequency spectrum of Exponential Fourier series coefficient

$c_n = \frac{2A}{-jn\pi} = \frac{-j2A}{n\pi}$, for odd value on n=1,3,5,7 …

$$c_1 = \frac{-j2A}{\pi}$$

$$|c_1| = \frac{2A}{\pi}$$

$$c_{-1} = \frac{j2A}{\pi}$$

$$|c_{-1}| = \frac{2A}{\pi}$$

$$c_3 = \frac{-j2A}{3\pi}$$

$|c_3| = \frac{2A}{3\pi}$

$$c_{-3} = \frac{j2A}{3\pi}$$

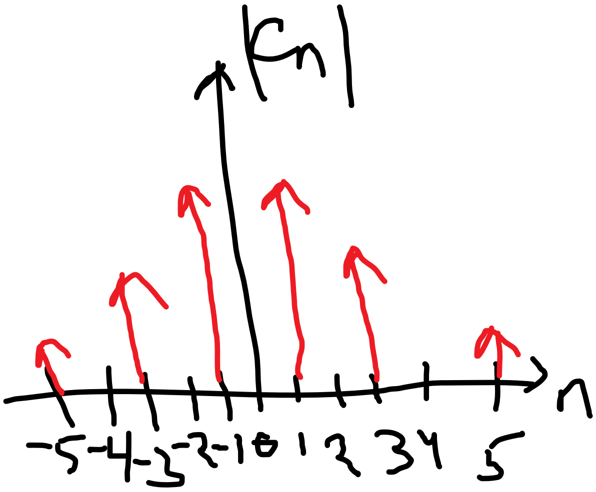
$|c_{-3}| = \frac{2A}{3\pi}$

$c_5 = \frac{-j2A}{5\pi}$

$|c_5| = \frac{2A}{5\pi}$

$$c_{-5} = \frac{j2A}{5\pi}$$

$$|c_{-5}| = \frac{2A}{5\pi}$$



The magnitude spectrum of exponential Fourier series coefficient has even symmetry.