Fourier Transform

Fourier Transform from exponential Fourier series:

$x_T(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, t_0 < t < t_0 + \frac{2\pi}{\omega_0}$

$$c_k = \frac{1}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} x_T(t)e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-jk\omega_0 t}\,dt$$

$$Tc_k = \int_{-T/2}^{T/2} x_T(t)e^{-jk\omega_0 t}\,dt$$

If T-> ∞

$x_T(t) = x(t), -\infty < t < \infty$

$k\omega_0 \rightarrow \omega$

$Tc_n = x(\omega)$

$x(\omega)$= $\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$

This is called Fourier transform of aperiodic signal $x(t)$.

$x(t) \overset{FT}{\leftrightarrow} x(\omega)$

Inverse Fourier Transform

$x_T(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, Tc_n = x(\omega), c_n = \frac{x(\omega)}{T}$

$$= \sum_{n=-\infty}^{\infty} \frac{X(\omega)}{T} e^{jn\omega_0 t}$$

If $\omega_0 = \frac{2\pi}{T} => T = \frac{2\pi}{\omega_0}$

$$x_T(t) = \sum_{n=-\infty}^{\infty} \frac{X(\omega)}{2\pi} e^{jn\omega_0 t}\omega_0$$

If $T \rightarrow \infty; X_T(t) \rightarrow x(t), \omega_0 = \frac{2\pi}{T}, n\omega_0 \rightarrow \omega, \omega_0 \rightarrow d\omega$

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$ is called Inverse Fourier Transform.

Problem:

Find the Fourier transform of $x(t) = e^{-at}\, u(t)$. Find its magnitude and phase spectrum.

Solution:

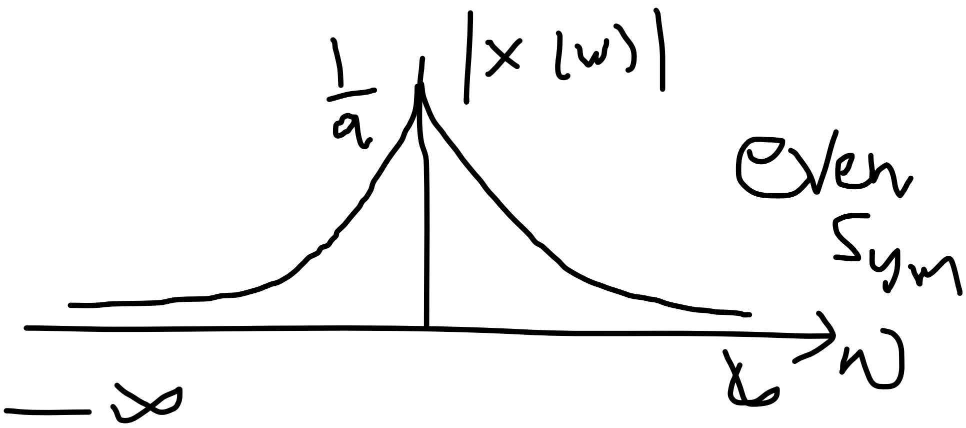
$x(\omega)= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$

$x(\omega)= \int_{-\infty}^{\infty} e^{-at}\, u(t)e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(a+j\omega)t} dt = \frac{-1}{(a+j\omega)} \left\{e^{-(a+j\omega)t}\right\}_{0}^{\infty}$

$$x(\omega) = \frac{-1}{(a+j\omega)}(e^{-\infty} - 1) = \frac{1}{(a+j\omega)}$$

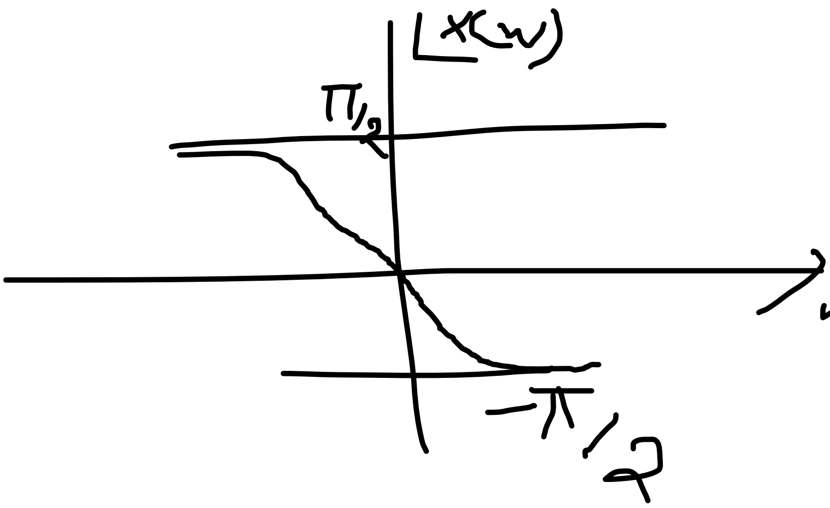
$$|x(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

ω	$\lvert x(\omega)\rvert$
0	1/a
$-\infty$	0
∞	0



angle of $x(\omega) = -tan^{-1}\left(\frac{\omega}{a}\right)$

ω	$\lvert x(\omega)\rvert$
0	0
$-\infty$	$\frac{\pi}{2}$
∞	$-\frac{\pi}{2}$



odd sym

__Properties of Continuous time Fourier transform__

1. Linearity

$$a_1x1(t) + a_2x2(t) = a_1X_1(\omega) + a_2X_2(\omega)$$

2. Time Shifting

$$x(t - t_0) \leftrightarrow e^{-j\omega t_0}X(\omega)$$

3. Frequency Shifting

$$e^{j\omega_0 t}x(t) \leftrightarrow X(\omega - \omega_0)$$

4. Time scaling

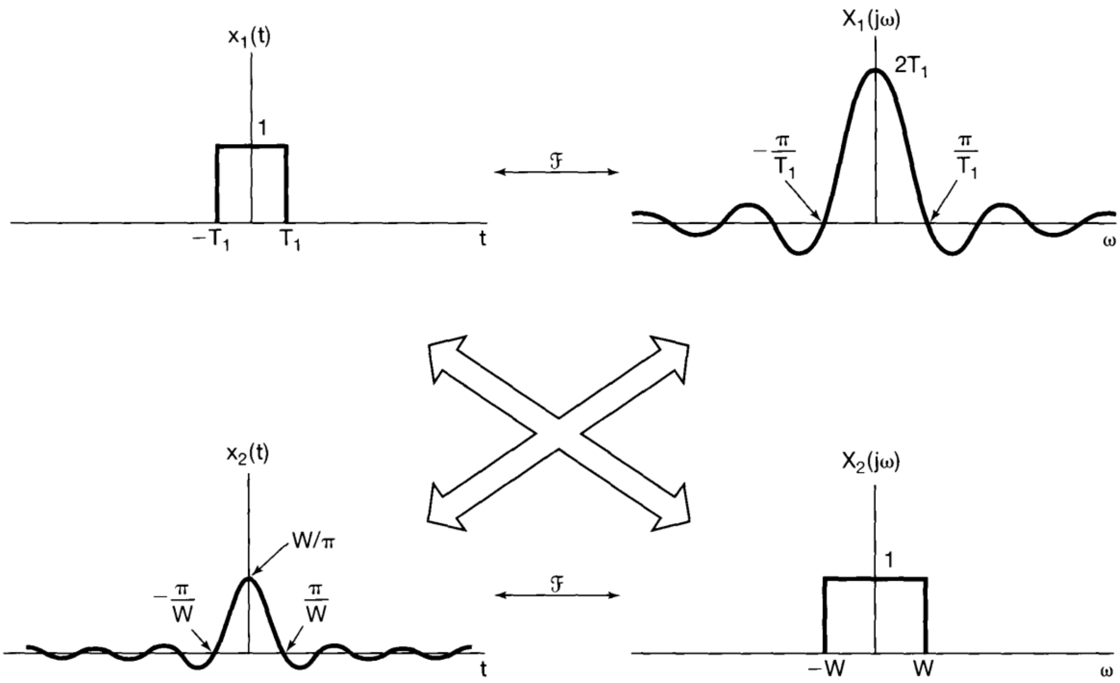
$$x(at) \leftrightarrow \frac{1}{|a|}X(\frac{\omega}{a})$$

5. Time Reversal

$$x(-t) = X(-\omega)$$

6. Duality (or symmetry)

$$X(t) \leftrightarrow 2\pi x(-\omega)$$



7. Differentiation in time domain

$$\frac{d}{dt}(x(t)) \leftrightarrow j\omega X(\omega)$$

8. Differentiation in the frequency domain

$$(-jt)x(t) \leftrightarrow \frac{dX(\omega)}{d\omega}$$

9. Integration in time domain

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \pi X(0)\delta(\omega) + \frac{1}{j\omega}X(\omega)$$

10. Convolution

$$x_1(t) * x_2(t) \leftrightarrow X_1(\omega)X_2(\omega)$$

11. Multiplication

$$x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi}X_1(\omega) * X_2(\omega)$$

12. Additional properties

$$x(t) = x_e(t) + x_0(t)$$

Where, $x_e(t)$ and $x_0(t)$ are even and odd components of $x(t)$.

$$X(t) \leftrightarrow X(\omega) = A(\omega) + jB(\omega)$$

Then, $X(-\omega) = X^*(\omega)$

Then $x_e(t) \leftrightarrow Re\{X(\omega)\} = A(\omega)$

$x_o(t) \leftrightarrow jIm\{X(\omega)\} = jB(\omega)$

13. Parseval's Relations

$$\int_{-\infty}^{\infty} x_1(t)x_2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)X_2(-\omega)d\omega$$

$$if\, x_1(t) = x_2(t)$$

$\int_{-\infty}^{\infty}|x_1(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X_1(\omega)|^2 d\omega$, is known as energy theorem.

Problem: Using duality property, find out the Fourier transform $G(j\omega)$ of the signal $g(t) = \frac{2}{1+t^2}$.

Solution:

The FT of g(t) is the x(t) for which X(ω) $= \frac{2}{1+\omega^2}$

When $x(t) = e^{-|t|} \leftrightarrow \frac{2}{1+\omega^2}$

So as per duality property:

FT of $g(t) = 2\pi e^{-|\omega|}$.

Some common Fourier Transform pairs

$x(t)$	$x(\omega)$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\omega t_0}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin\omega_0 t$	$-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$u(-t)$	$\pi\delta(\omega)-\frac{1}{j\omega}$
$e^{-at}, u(t)$ a>0	$\frac{1}{j\omega+a}$
$t\,e^{-at}, u(t)$ a>0	$\frac{1}{(j\omega+a)^2}$
$e^{-a\|t\|}$ a>0	$\frac{2a}{a^2+\omega^2}$
$\frac{1}{a^2+t^2}$	$e^{-a\|\omega\|}$
e^{-at^2} a>0	$\sqrt{\frac{\pi}{a}}e^{-\frac{\omega^2}{4a}}$
$P_a(t)=\begin{cases}1 & \|t\|<a\\ 0 & \|t\|>a\end{cases}$	$2a\frac{\sin\omega a}{\omega a}$
$sgn\ t$	$\frac{2}{j\omega}$
$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0), \omega_0=\frac{2\pi}{T}$
$\frac{sgn\ at}{\pi t}$	$P_a(\omega)=\begin{cases}1 & \|\omega\|<a\\ 0 & \|\omega\|>a\end{cases}$