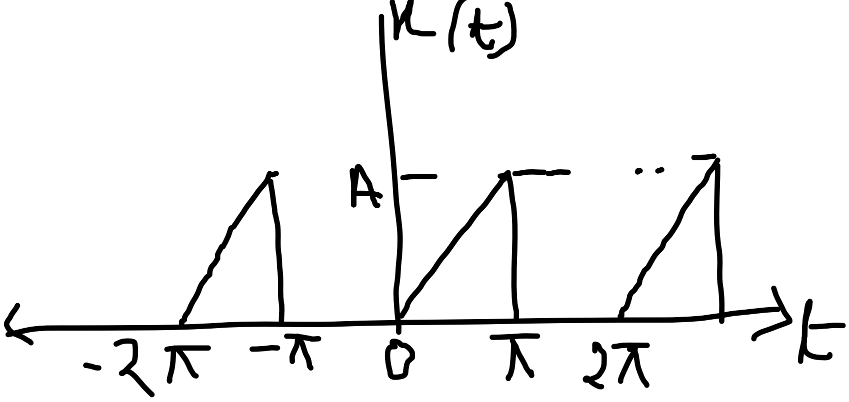
Trigonometric Fourier Series

Q. Obtain the trigonometric Fourier series expansion of the figure shown.



Solution:

Time period of the waveform: 2π

Find the straight line equation from point (0,0) to (π, A).

$$Slope = \frac{A-0}{\pi-0} = \frac{A}{\pi}$$

$$x(t) - 0 = \frac{A}{\pi}(t-0)$$

$$x(t) = \begin{cases} (A/\pi)\ t; & 0 < t < \pi \\ 0; & \pi < t < 2\pi \end{cases}$$

The trigonometric Fourier series expansion of x(t) is,

- $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\ ,\ t_0 < t < t_0 + \frac{2\pi}{\omega_0}$
- $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$
- $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$, 0< $t < 2\pi$
- $a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt = \frac{1}{2\pi}\int_0^{2\pi} x(t)\, dt = \frac{A}{2\pi^2}\int_0^{\pi} t\, dt = \frac{A}{2\pi^2} \times \frac{\pi^2}{2} = \frac{A}{4}$
- $a_n = \frac{2}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} x(t) \cos n\omega_0 t\, dt = \frac{A}{\pi^2}\int_0^{\pi} t \cos nt\, dt = \frac{A}{\pi^2}\left[\frac{t \sin nt}{n} + \frac{\cos nt}{n^2}\right]_0^{\pi}$
- $a_n = \frac{A}{\pi^2}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$
- $if\ n\ is\ even$
- $a_n = 0$;   n = 2, 4, 6, …

- $if\ n\ is\ odd$
- $a_n = \frac{-2A}{n^2\pi^2}$; n= 1, 3, 5, …
- $b_n = \frac{2}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} x(t)\sin n\omega_0 t\,dt = \frac{A}{\pi^2}\int_0^{\pi} t\sin nt\ dt = \frac{A}{\pi^2}\{\frac{-t\cos nt}{n} + \frac{\sin nt}{n^2}\}_0^{\pi}$
- $b_n = \frac{A}{\pi^2}[\frac{-\pi\cos n\pi}{n}] = -\frac{A\cos n\pi}{n\pi}$
- $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt\,;\, 0 < t < 2\pi$
- By substituting the values, we will get
- $x(t) = \frac{A}{4} + \sum_{n=odd}^{\infty} \frac{-2A}{n^2\pi^2}\cos nt + \sum_{n=1}^{\infty} \frac{-A\cos n\pi}{n\pi}\sin nt$
- $x(t) = \frac{A}{4} - \frac{2A}{\pi^2}[\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 3t + \cdots +] + \frac{A}{\pi}\{\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \cdots\}$; 0<t<2$\pi$

Cosine Fourier Series representation

The trigonometric Fourier series expansion of a periodic signal is given by,

- $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\ ,\ t_0 < t < t_0 + \frac{2\pi}{\omega_0}$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{{a_n}^2 + {b_n}^2}\left(\frac{a_n}{\sqrt{{a_n}^2 + {b_n}^2}}\cos n\omega_0 t + \frac{b_n}{\sqrt{{a_n}^2 + {b_n}^2}}\sin n\omega_0 t\right)$$

Let $\cos\emptyset = \frac{a_n}{\sqrt{{a_n}^2+{b_n}^2}}$ and $\sin\emptyset = \frac{-\,b_n}{\sqrt{{a_n}^2+{b_n}^2}}$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{{a_n}^2 + {b_n}^2}\,(\cos n\omega_0 t\cos\emptyset - \sin n\omega_0 t\sin\emptyset)$$

$$= a_0 + \sum_{n=1}^{\infty} \sqrt{{a_n}^2 + {b_n}^2}\cos(\ n\omega_0 t + \phi)$$

$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi)$, which is called cosine Fourier series expansion.

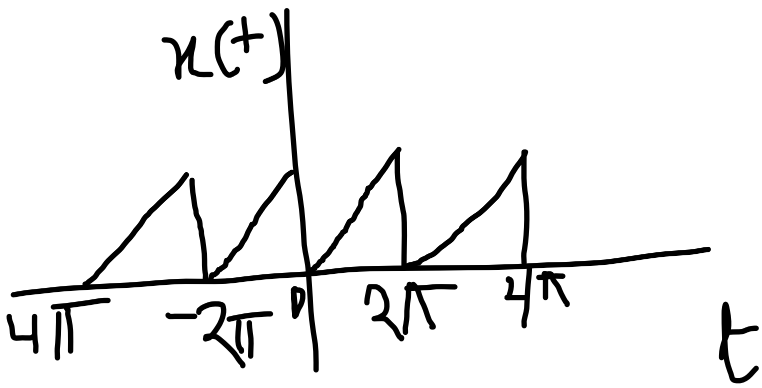
Where, $A_0 = a_0$

$$A_n = \sqrt{{a_n}^2 + {b_n}^2}$$

$$\tan\phi = -\frac{b_n}{a_n}$$

$$\phi = \tan^{-1}\frac{b_n}{a_n}$$

Q. Find the cosine Fourier series of the waveform shown in the figure below,



Solution:

T = 2π

Find the straight line equation from (0,0) and $(2\pi, A)$.

$$slope = \frac{A-0}{2\pi - 0} = \frac{A}{2\pi}$$

$$x(t) - 0 = \frac{A}{2\pi}(t-0)$$

$x(t) = \frac{A}{2\pi} t$; $0<t<2\pi$