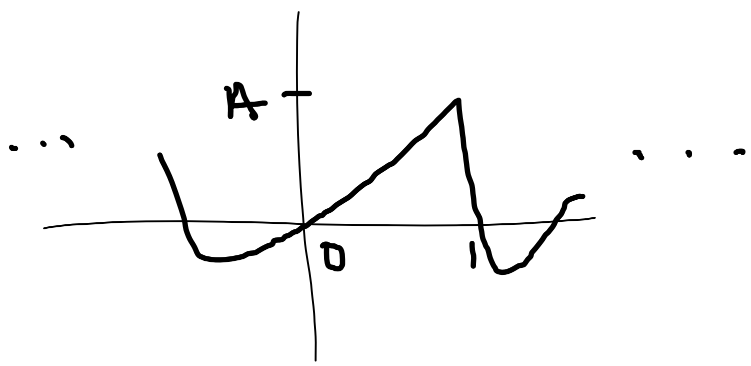
Trigonometric Fourier Series

- It is mainly used for the analysis of periodic signals
- Let us consider a periodic signal $x(t)$ with period T and frequency ω_0
- Where, $\omega_0 = \frac{2\pi}{T}$ => $T = \frac{2\pi}{\omega_0}$
- We want to approximate $x(t)$ in the interval $t_0 < t < t_0 + \frac{2\pi}{\omega_0}$
- To approximate any signal we need orthogonal signals like $\sin n\omega_0 t$ and $\sin m\omega_0 t$ are orthogonal to each other in the interval $t_0 < t < t_0 + \frac{2\pi}{\omega_0}$
- $\sin n\omega_0 t$ and $\cos m\omega_0 t$ are orthogonal to each other in the interval $t_0 < t < t_0 + \frac{2\pi}{\omega_0}$
- So, $x(t) \cong a_0 \boldsymbol{\cos(0 \times \omega_0 t)} + a_1 \cos(1 \times \omega_0 t) + a_2 \cos(2 \times \omega_0 t) + \cdots + a_n \cos(n \times \omega_0 t) + \cdots + b_0 \sin(0 \times \omega_0 t) + b_1 \sin(1 \times \omega_0 t) + \cdots + b_n \sin(n \times \omega_0 t) + \cdots$

  In the interval $t_0 < t < t_0 + \frac{2\pi}{\omega_0}$.
- $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\ ,\ t_0 < t < t_0 + \frac{2\pi}{\omega_0}$
- This is called trigonometric Fourier series
- Where a_0, a_n and b_n are called trigonometric Fourier series coefficient
- $a_0 = \frac{1}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} x(t)dt$, is called average or dc value of the signal $x(t)$
- $a_n = \frac{2}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} x(t) \cos n\omega_0 t\, dt$
- $b_n = \frac{2}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} x(t) \sin n\omega_0 t\, dt$

Q. Find the trigonometric Fourier series for the signal in the interval 0<t<1. (**∫ uv dx = u ∫ v dx - ∫ (u' ∫ v dx) dx**)



- A straight line between (0, 0) and (1, A). So $Slop = \frac{A-0}{1-0} = A$
- Straight line equation = $y_2 - y_1 = x_2 - x_1$
- => $x(t) - 0 = A(t - 0)$
- $x(t) = At; 0 < t < 1$
- Since it is periodic signal, the trigonometric Fourier series of $x(t)$ is,
- $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\ ,\ t_0 < t < t_0 + \frac{2\pi}{\omega_0}$
- $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1}$
- $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n2\pi t + b_n \sin n2\pi t, t_0 < t < 1$
- $a_0 = \frac{1}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} x(t)dt$ = $\int_{t0}^{to+T} x(t)dt = A\int_0^1 tdt = \frac{A}{2}$
- $a_n = \frac{2}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} x(t)\cos n\omega_0 t\, dt = 2A\int_0^1 t\cos n2\pi t\, dt = 2A\left\{t\frac{sin\ 2\pi nt}{2\pi n} - \frac{\cos 2\pi nt}{4n^2\pi^2}\right\}\begin{matrix}1\\0\end{matrix} = 2A\{\frac{1}{4n^2\pi^2} - \frac{1}{4n^2\pi^2}\}$=0
- $b_n = \frac{2}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} x(t)\sin n\omega_0 t\, dt$ =$2A\int_0^1 t\sin n2\pi t\, dt = 2A\left\{-t\frac{\cos 2\pi nt}{2\pi n} + \frac{sin 2\pi nt}{4n^2\pi^2}\right\}\begin{matrix}1\\0\end{matrix} = 2A\left\{-\frac{1}{2\pi n}\right\} = -\frac{A}{\pi n}$
- $x(t) = \frac{A}{2} + \sum_{n=1}^{\infty}\frac{-A}{n\pi}\sin 2\pi nt\ ; 0 < t < 1$
- $x(t) = \frac{A}{2} - \frac{A}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2\pi nt}{n}; 0 < t < 1$

Q2. Find the trigonometric Fourier series expansion of the half wave rectifier sine wave shown below.

Solution:

Time period: 2π

$$x(t) = \begin{cases} A\, sin\omega_0 t; \ 0 < t < \pi \\ 0; \quad \pi < t < 2\pi \end{cases}$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$x(t) = \begin{cases} A \sin t\,; \ 0 < t < \pi \\ 0; \quad \pi < t < 2\pi \end{cases}$$

The formula for trigonometric Fourier series:

- $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n2\pi t + b_n \sin n2\pi t; t_0 < t < t_0 + \frac{2\pi}{\omega_0}$
- $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n2\pi t + b_n \sin n2\pi t; t_0 < t < t_0 + 2\pi$
- $a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt = \frac{1}{2\pi}\int_0^{2\pi} x(t)\, dt = \frac{A}{2\pi}\int_0^{\pi} \sin t\, dt = \frac{-A}{2\pi}\{cos\}_0^{\pi} = \frac{A}{\pi}$
- $a_n = \frac{2}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} x(t) \cos n\omega_0 t\, dt = \frac{A}{2\pi}\left[\frac{-\cos(1+n)t}{1+n} - \frac{\cos(1-n)t}{1-n}\right]_0^{\pi}$

  $= \frac{A}{2\pi}[\frac{\cos n\pi}{1+n} + \frac{\cos n\pi}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}]$ [Use formula $\cos(\pi + n\pi) = -\cos n\pi$, $\cos(\pi - n\pi) = -\cos n\pi$]

  For n is even => $a_n = \frac{A}{2\pi}\left[\frac{2}{1+n} + \frac{2}{1-n}\right] = \frac{A}{\pi}[\frac{1-n+1+n}{1-n^2}]$

  For n is odd => $a_n = 0$; n= 1, 3, 5, 7, . . .

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} x(t) \sin n\omega_0 t\, dt = \frac{A}{2\pi}\int_0^{\pi} 2 \sin t \sin nt\, dt$$

$2 \sin A\ \sin B = \cos(A - B) - \cos(A + B)$

$$b_n = \frac{A}{2\pi}[\int_0^{\pi} [\cos(1-n)t - \cos(1+n)t]\, dt$$

$$b_n = \frac{A}{2\pi}[\frac{\sin(1-n)\, t}{1-n} - \frac{\sin(1+n)t}{1+n}]_0^{\pi}$$

$\sin(\pi - n\pi) = \sin n\pi$

$\sin(\ \pi + n\pi) = -\sin n\pi$

If n=1; $b_n = \infty$, for remaining values, $b_n = 0$

$$b_n = \frac{A}{2\pi}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \sin t \sin t \, dt$$

$$=\frac{A}{2\pi}\int_0^{\pi}(1-\cos 2t)\,dt = \frac{A}{2\pi}\left[\pi - \frac{1}{2}[\sin 2t]_0^{\pi}\right] = \frac{A}{2}$$

$$x(t) = \frac{A}{\pi} + \frac{A}{2}\sin t + \sum_{n=even}^{\infty}\frac{2A}{\pi(1-n^2)}\cos nt;\ 0<t<2\pi$$