The Fourier Series

- The sinusoid is the most simple and useful periodic function. This chapter is concerned with analysing circuits with periodic, non-sinusoidal excitations.

- The Fourier series, a technique for expressing a periodic function in terms of sinusoids. Once the source function is expressed in terms of sinusoids, we can apply the phasor method to analyse circuits

- Trigonometric Fourier series. Later we consider the exponential Fourier series
- Apply Fourier series in circuit analysis.
- Practical applications of Fourier series in spectrum analysers and filters

Trigonometric Fourier Series

Periodic function is one that repeats every *T* seconds.

$$f(t) = f(t + nT)$$

where *n* is an integer and *T* is the period of the function

According to the *Fourier theorem*, any practical periodic function of frequency ω_0 can be expressed as an infinite sum of sine or cosine functions that are integral multiples of ω_0 Thus, $f(t)$ can be expressed as

$$f(t) = a_0 + a_1 \cos \omega_0 t + b_1 \sin \omega_0 t + a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t + a_3 \cos 3\omega_0 t + b_3 \sin 3\omega_0 t + \cdots$$

$$f(t) = \underbrace{a_0}_{\text{dc}} + \underbrace{\sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{\text{ac}}$$

$$f(t) = \underbrace{a_0}_{\text{dc}} + \underbrace{\sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{\text{ac}}$$

Trigonometric Fourier series of $f(t)$;

- where ω_0 is called the *fundamental frequency* in radians per second.
- The sinusoid $\sin n\omega_0 t$ *or* $\cos n\omega_0 t$ is called the *n*th harmonic of $f(t)$;
- it is an odd harmonic if *n* is odd and an even harmonic if *n* is even.
- The constants a_n and b_n are the Fourier coefficients
- The coefficient a_0 is the dc component or the average value of $f(t)$. (Recall that sinusoids have zero average values.)
- The coefficients a_n and b_n (for $n \neq 0$) are the amplitudes of the sinusoids in the ac component.

The **Fourier series** of a periodic function $f(t)$ is a representation that resolves $f(t)$ into a dc component and an ac component comprising an infinite series of harmonic sinusoids.

The harmonic frequency ω_n is an integral multiple of the fundamental frequency ω_0, i.e., $\omega_n = n\omega_0$.

Dirichlet conditions

1. $f(t)$ is single-valued everywhere.
2. $f(t)$ has a finite number of finite discontinuities in any one period.
3. $f(t)$ has a finite number of maxima and minima in any one period.
4. The integral $\int_{t_0}^{t_0+T} |f(t)|\,dt < \infty$ for any t_0.

The process of determining the coefficients is called Fourier analysis

$$\int_0^T \sin n\omega_0 t \, dt = 0$$

$$\int_0^T \cos n\omega_0 t \, dt = 0$$

$$\int_0^T \sin n\omega_0 t \cos m\omega_0 t \, dt = 0$$

$$\int_0^T \sin n\omega_0 t \sin m\omega_0 t \, dt = 0, \qquad (m \neq n)$$

$$\int_0^T \cos n\omega_0 t \cos m\omega_0 t \, dt = 0, \qquad (m \neq n)$$

$$\int_0^T \sin^2 n\omega_0 t \, dt = \frac{T}{2}$$

$$\int_0^T \cos^2 n\omega_0 t \, dt = \frac{T}{2}$$

$$\int_0^T f(t)\,dt = \int_0^T \left[a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\right] dt$$

$$= \int_0^T a_0\,dt + \sum_{n=1}^{\infty}\left[\int_0^T a_n \cos n\omega_0 t\,dt + \int_0^T b_n \sin n\omega_0 t\,dt\right] dt$$

$$\int_0^T f(t)\,dt = \int_0^T a_0\,dt = a_0 T$$

$$\boxed{a_0 = \frac{1}{T}\int_0^T f(t)\,dt}$$

Showing that a_0 is the average value of $f(t)$.

$$\int_0^T f(t)\cos m\omega_0 t\,dt$$

$$= \int_0^T \left[a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\right]\cos m\omega_0 t\,dt$$

$$= \int_0^T a_0 \cos m\omega_0 t\,dt + \sum_{n=1}^{\infty}\left[\int_0^T a_n \cos n\omega_0 t \cos m\omega_0 t\,dt\right.$$

Type equation here.

$$\left. + \int_0^T b_n \sin n\omega_0 t \cos m\omega_0 t\,dt\right] dt$$

$$\boxed{a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt}$$

$$\int_0^T f(t)\cos m\omega_0 t\,dt = a_n\frac{T}{2}, \qquad \text{for } m = n$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt$$

$$f(t) = \underbrace{a_0}_{\text{dc}} + \underbrace{\sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{\text{ac}}$$

An alternative form of above is the amplitude-phase form

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$$

$$\boxed{f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)}$$

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

$$a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n) = a_0 + \sum_{n=1}^{\infty} (A_n \cos\phi_n) \cos n\omega_0 t - (A_n \sin\phi_n) \sin n\omega_0 t$$

$$a_n = A_n \cos\phi_n, \qquad b_n = -A_n \sin\phi_n$$

$$\boxed{A_n = \sqrt{a_n^2 + b_n^2}, \qquad \phi_n = -\tan^{-1}\frac{b_n}{a_n}}$$

$$\boxed{A_n = \sqrt{a_n^2 + b_n^2}, \qquad \phi_n = -\tan^{-1} \frac{b_n}{a_n}}$$

$$A_n \underline{/\phi_n} = a_n - jb_n$$

The plot of the amplitude A_n of the harmonics versus $n\omega_0$ is called the *amplitude spectrum* of $f(t)$; the plot of the phase ϕ_n versus $n\omega_0$ is the *phase spectrum* of $f(t)$. Both the amplitude and phase spectra form the *frequency spectrum* of $f(t)$.

The frequency spectrum of a signal consists of the plots of the amplitudes and phases of the harmonics versus frequency.

To evaluate the Fourier coefficients a_0, a_n, and b_n, we often need to apply the following integrals:

$$\int \cos at\, dt = \frac{1}{a}\sin at$$

$$\int \sin at\, dt = -\frac{1}{a}\cos at$$

$$\int t\cos at\, dt = \frac{1}{a^2}\cos at + \frac{1}{a}t\sin at$$

$$\int t\sin at\, dt = \frac{1}{a^2}\sin at - \frac{1}{a}t\cos at$$

It is also useful to know the values of the cosine, sine, and exponential functions for integral multiples of π. where n is an integer.

TABLE

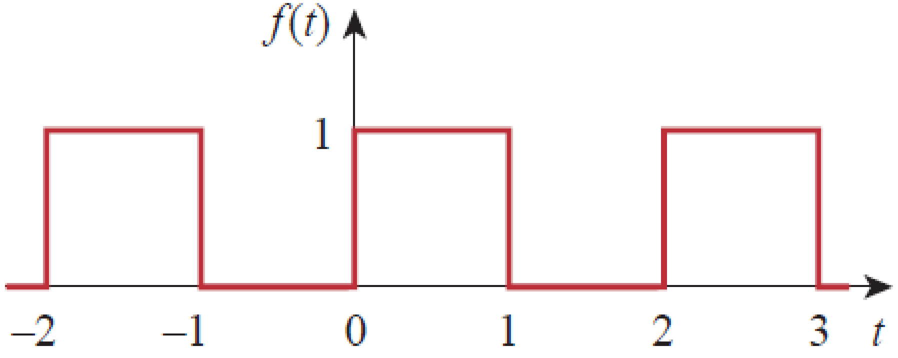
Values of cosine, sine, and exponential functions for integral multiples of π.

Function	Value
$\cos 2n\pi$	1
$\sin 2n\pi$	0
$\cos n\pi$	$(-1)^n$
$\sin n\pi$	0
$\cos\dfrac{n\pi}{2}$	$\begin{cases}(-1)^{n/2}, & n = \text{even}\\ 0, & n = \text{odd}\end{cases}$

Function	Value
$\sin\dfrac{n\pi}{2}$	$\begin{cases}(-1)^{(n-1)/2}, & n = \text{odd}\\ 0, & n = \text{even}\end{cases}$
$e^{j2n\pi}$	1
$e^{jn\pi}$	$(-1)^n$
$e^{jn\pi/2}$	$\begin{cases}(-1)^{n/2}, & n = \text{even}\\ j(-1)^{(n-1)/2}, & n = \text{odd}\end{cases}$

Example

Determine the Fourier series of the waveform shown in Fig. Obtain the amplitude and phase spectra.



a square wave.

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

Solution:

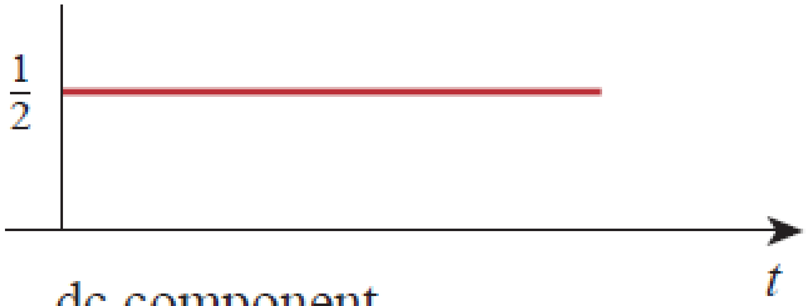
The Fourier series is given by Eq.

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

obtain the Fourier coefficients a_0, a_n, and b_n

and $f(t) = f(t + T)$. Since $T = 2$, $\omega_0 = 2\pi/T = \pi$. Thus,

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\left[\int_0^1 1\,dt + \int_1^2 0\,dt\right] = \frac{1}{2}t\Big|_0^1 = \frac{1}{2}$$



dc component

$$\int \cos at\, dt = \frac{1}{a}\sin at$$

$$\begin{aligned}
a_n &= \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\, dt \\
&= \frac{2}{2}\left[\int_0^1 1\cos n\pi t\, dt + \int_1^2 0\cos n\pi t\, dt\right. \\
&= \frac{1}{n\pi}\sin n\pi t\,\Big|_0^1 = \frac{1}{n\pi}[\sin n\pi - \sin(0)] = 0
\end{aligned}$$

$$\int \sin at\, dt = -\frac{1}{a}\cos at$$

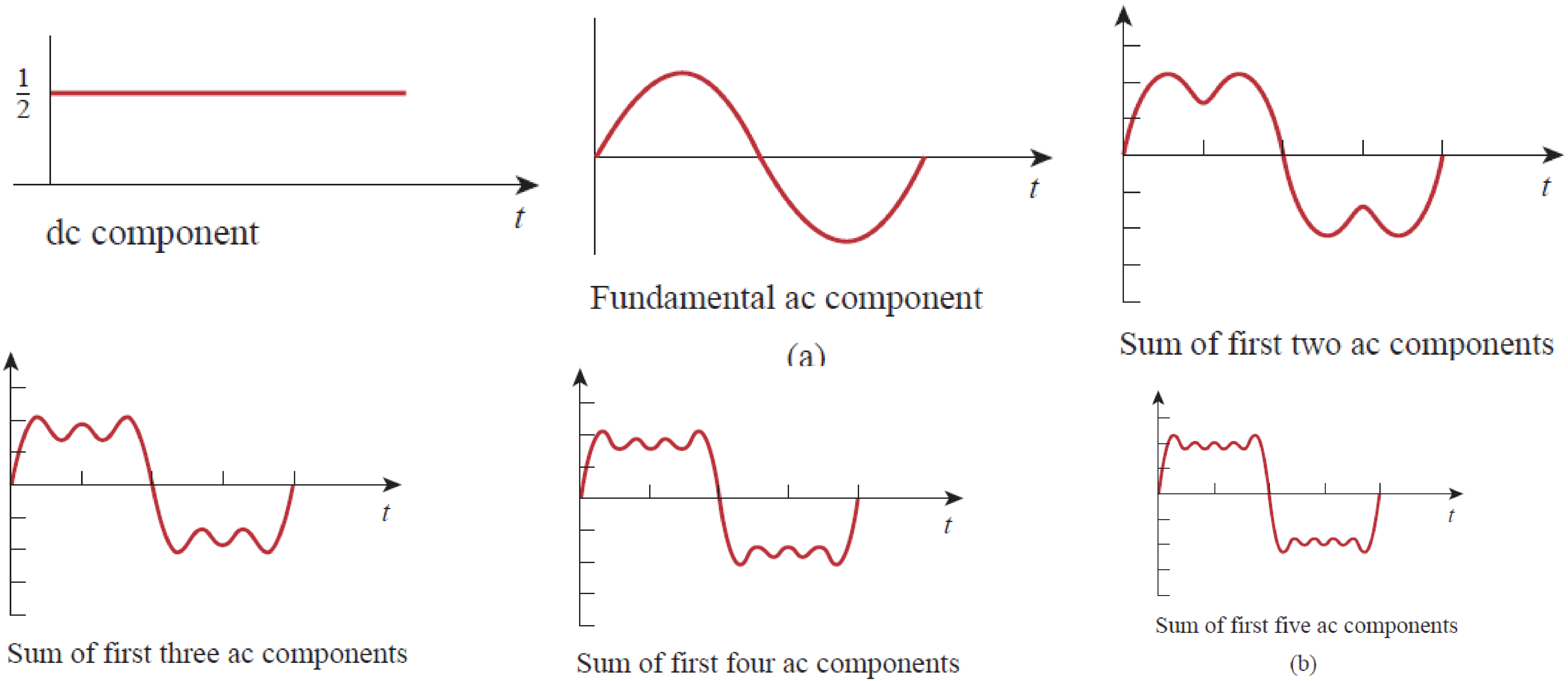
$$\begin{aligned}
b_n &= \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\, dt \\
&= \frac{2}{2}\left[\int_0^1 1 \sin n\pi t\, dt + \int_1^2 0 \sin n\pi t\, dt\right] \\
&= -\frac{1}{n\pi}\cos n\pi t \,\Big|_0^1 \\
&= -\frac{1}{n\pi}(\cos n\pi - 1), \qquad \cos n\pi = (-1)^n \\
&= \frac{1}{n\pi}[1 - (-1)^n] = \begin{cases} \dfrac{2}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}
\end{aligned}$$

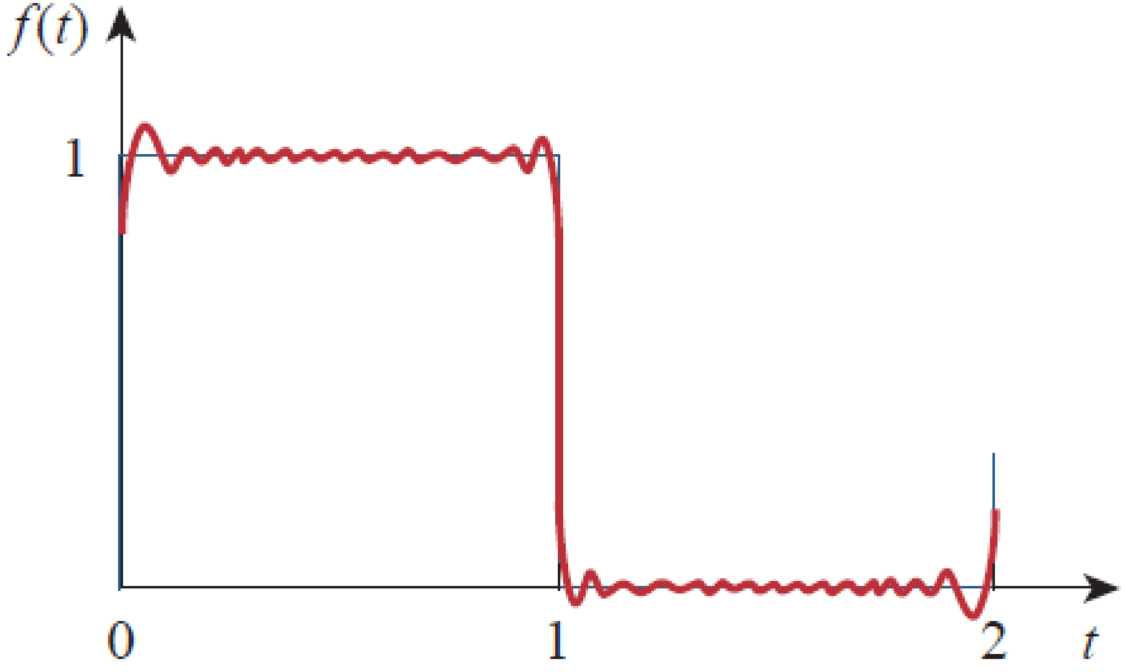
$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t + \cdots$$

Since $f(t)$ contains only the dc component and the sine terms with the fundamental component and odd harmonics, it may be written as

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{1}{n}\sin n\pi t, \qquad n = 2k - 1$$

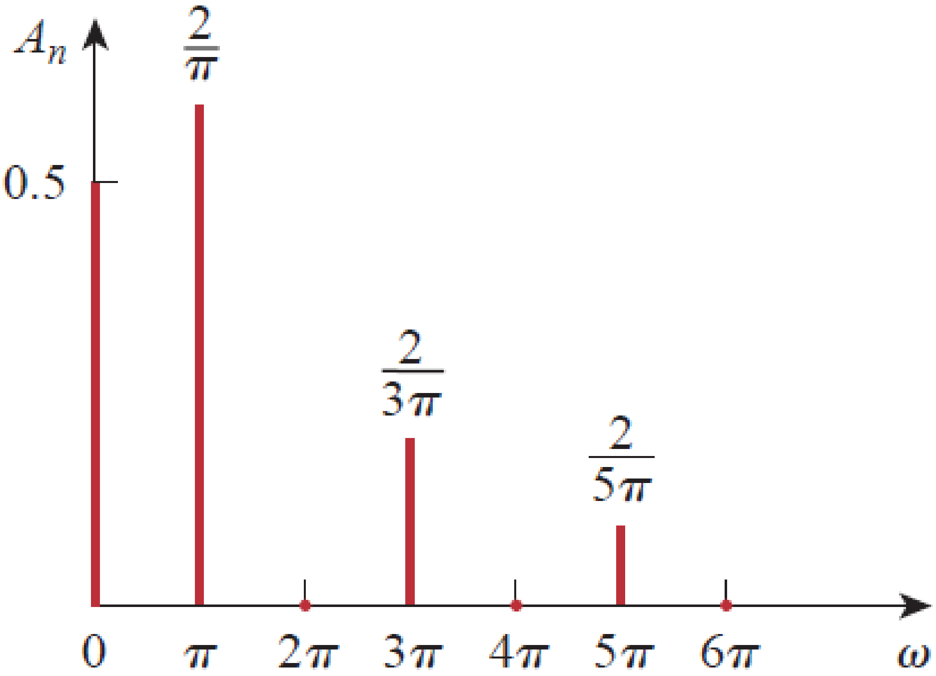
$$f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{n} \sin n\pi t, \qquad n = 2k - 1$$



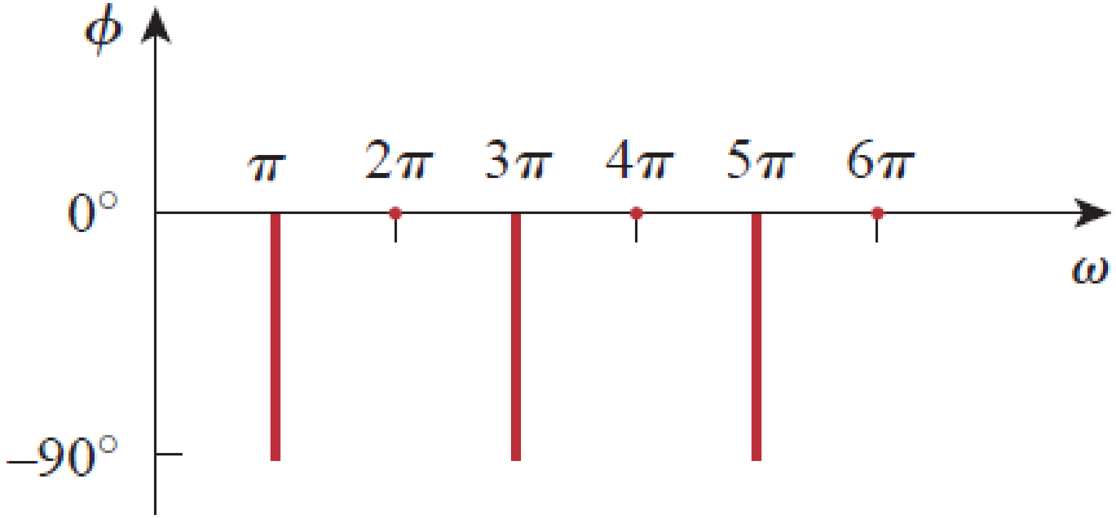


Truncating the Fourier series at N=11 Gibbs phenomenon.

we notice that the partial sum oscillates above and below the actual value of $f(t)$. At the neighborhood of the points of discontinuity $(x = 0, 1, 2, \ldots)$, there is overshoot and damped oscillation. In fact, an overshoot of about 9 percent of the peak value is always present, regardless of the number of terms used to approximate $f(t)$. This is called the *Gibbs phenomenon*.



(a)



(b)

$$A_n = \sqrt{a_n^2 + b_n^2} = |b_n| = \begin{cases} \dfrac{2}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

$$\phi_n = -\tan^{-1}\frac{b_n}{a_n} = \begin{cases} -90^\circ, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

The plots of A_n and ϕ_n for different values of $n\omega_0 = n\pi$ provide the amplitude and phase spectra in Fig. Notice that the amplitudes of the harmonics decay very fast with frequency.

TABLE 17.2

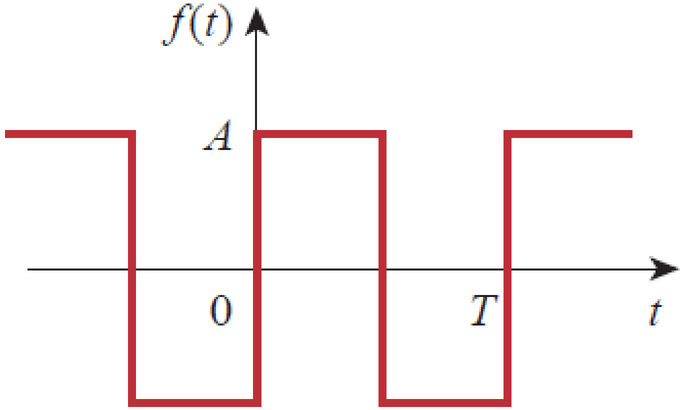
Effects of symmetry on Fourier coefficients.

Symmetry	a_0	a_n	b_n	Remarks
Even	$a_0 \neq 0$	$a_n \neq 0$	$b_n = 0$	Integrate over $T/2$ and multiply by 2 to get the coefficients.
Odd	$a_0 = 0$	$a_n = 0$	$b_n \neq 0$	Integrate over $T/2$ and multiply by 2 to get the coefficients.
Half-wave	$a_0 = 0$	$a_{2n} = 0$ $a_{2n+1} \neq 0$	$b_{2n} = 0$ $b_{2n+1} \neq 0$	Integrate over $T/2$ and multiply by 2 to get the coefficients.

The Fourier series of common functions.

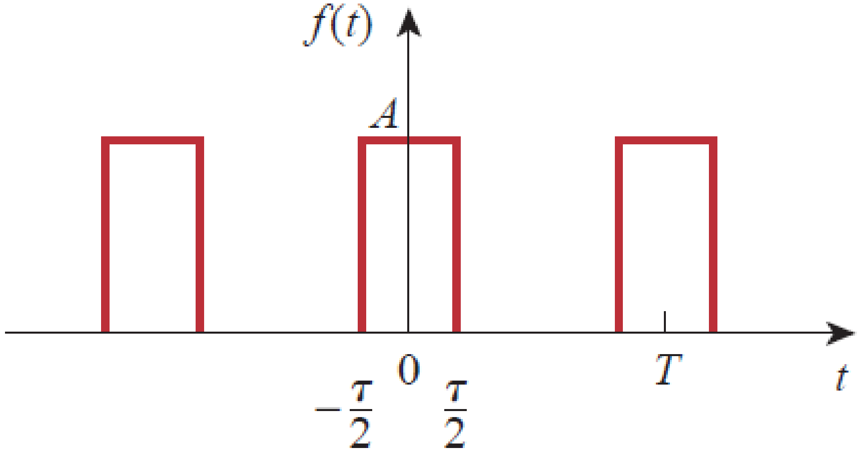
Function	Fourier series

1. Square wave



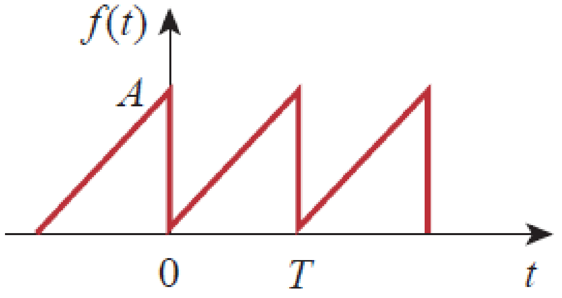
$$f(t) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n - 1} \sin(2n - 1)\omega_0 t$$

2. Rectangular pulse train



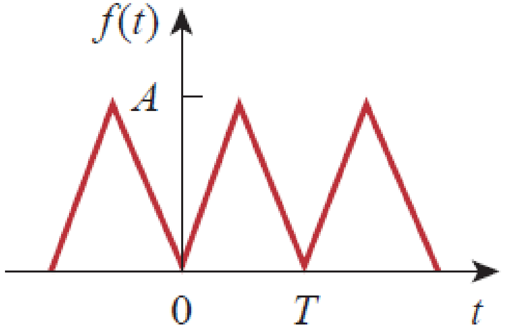
$$f(t) = \frac{A\tau}{T} + \frac{2A}{T} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi\tau}{T} \cos n\omega_0 t$$

3. Sawtooth wave



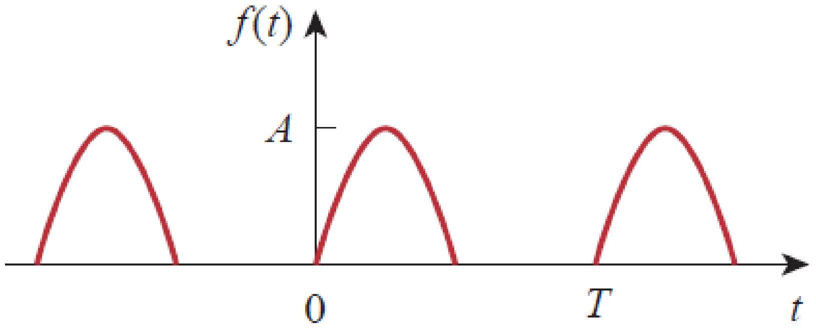
$$f(t) = \frac{A}{2} - \frac{A}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\omega_0 t}{n}$$

4. Triangular wave



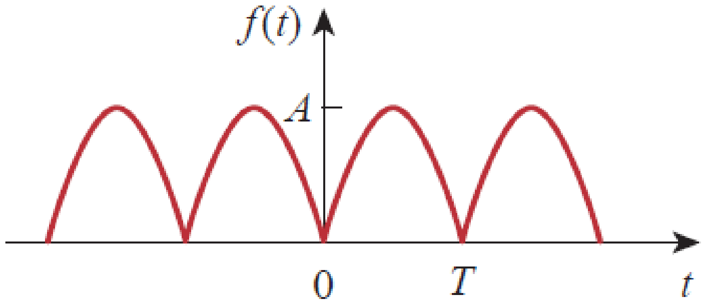
$$f(t) = \frac{A}{2} - \frac{4A}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)\omega_0 t$$

5. Half-wave rectified sine



$$f(t) = \frac{A}{\pi} + \frac{A}{2}\sin\omega_0 t - \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2n\omega_0 t$$

6. Full-wave rectified sine



$$f(t) = \frac{2A}{\pi} - \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos n\omega_0 t$$

Home work

- **Practice problem 17.2 (page 768-Sadiku)**
- **Review symmetry considerations (Even symmetry, odd symmetry, half wave symmetry (page 768-Sadiku)**
- **Example problem 17.3 (page 775-Sadiku)**
- **Practice problem 17.3 (page 775-Sadiku)**
- **Example problem 17.4 (page 776-Sadiku)**
- **Practice problem 17.4 (page 777-Sadiku)**
- **Example problem 17.5 (page 777-Sadiku)**
- **Practice problem 17.5 (page 778-Sadiku)**

Circuit Applications

To find the steady-state response of a circuit to a non-sinusoidal periodic excitation requires the application of a Fourier series, ac phasor analysis, and the superposition principle

Steps for Applying Fourier Series:

1. Express the excitation as a Fourier series.
2. Transform the circuit from the time domain to the frequency domain.
3. Find the response of the dc and ac components in the Fourier series.
4. Add the individual dc and ac responses using the superposition principle.

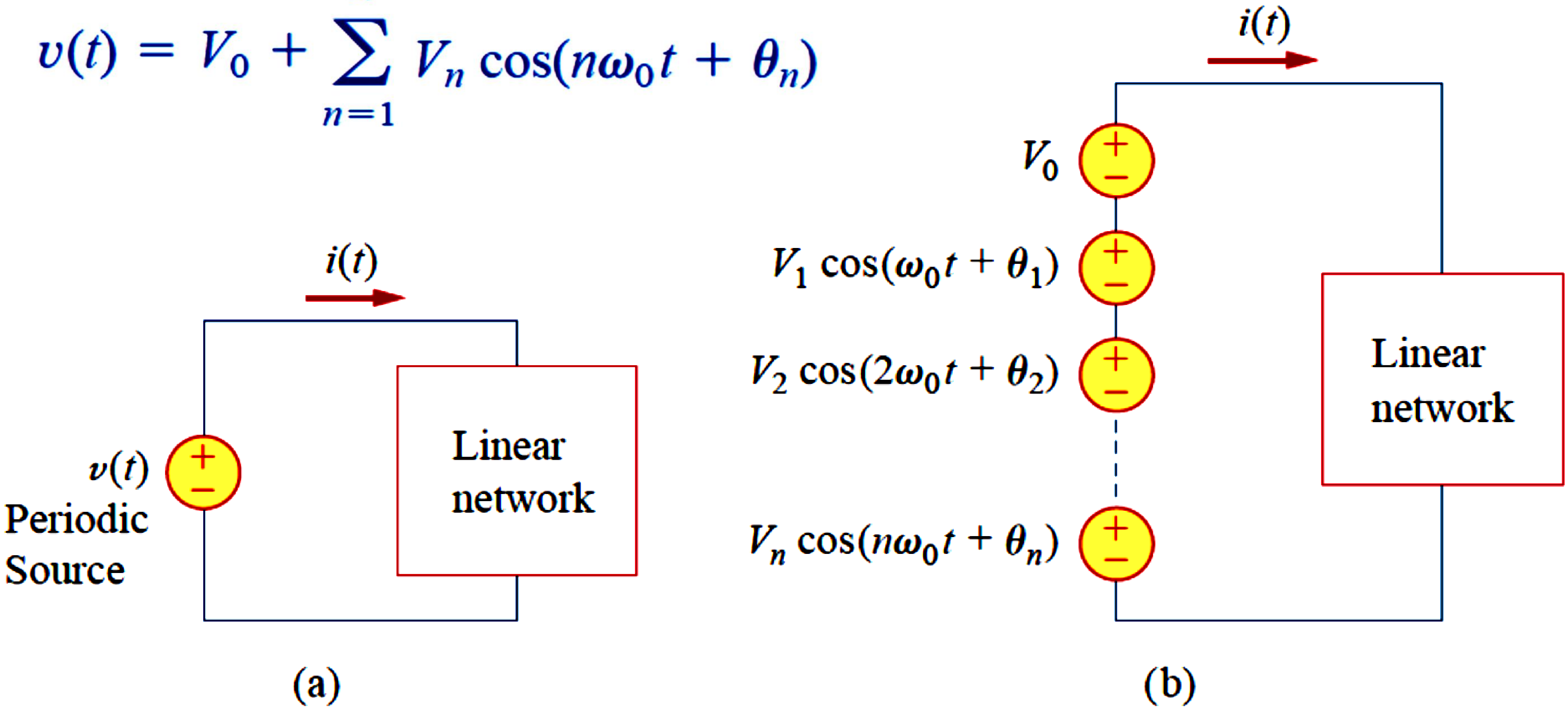
1. Express the excitation as a Fourier series.

$$v(t) = V_0 + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t + \theta_n)$$

(The same could be done for a periodic current source.)

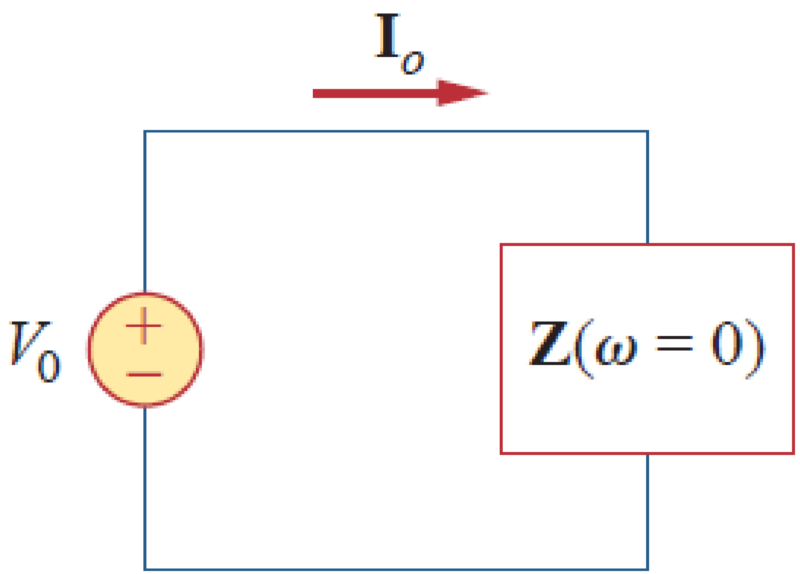
$v(t)$ consists of two parts: the dc component V_0 and the ac component $\mathbf{V}_n = V_n \underline{/\theta_n}$ with several harmonics. This Fourier series representation may be regarded as a set of series-connected sinusoidal sources, with each source having its own amplitude and frequency

$$v(t) = V_0 + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t + \theta_n)$$

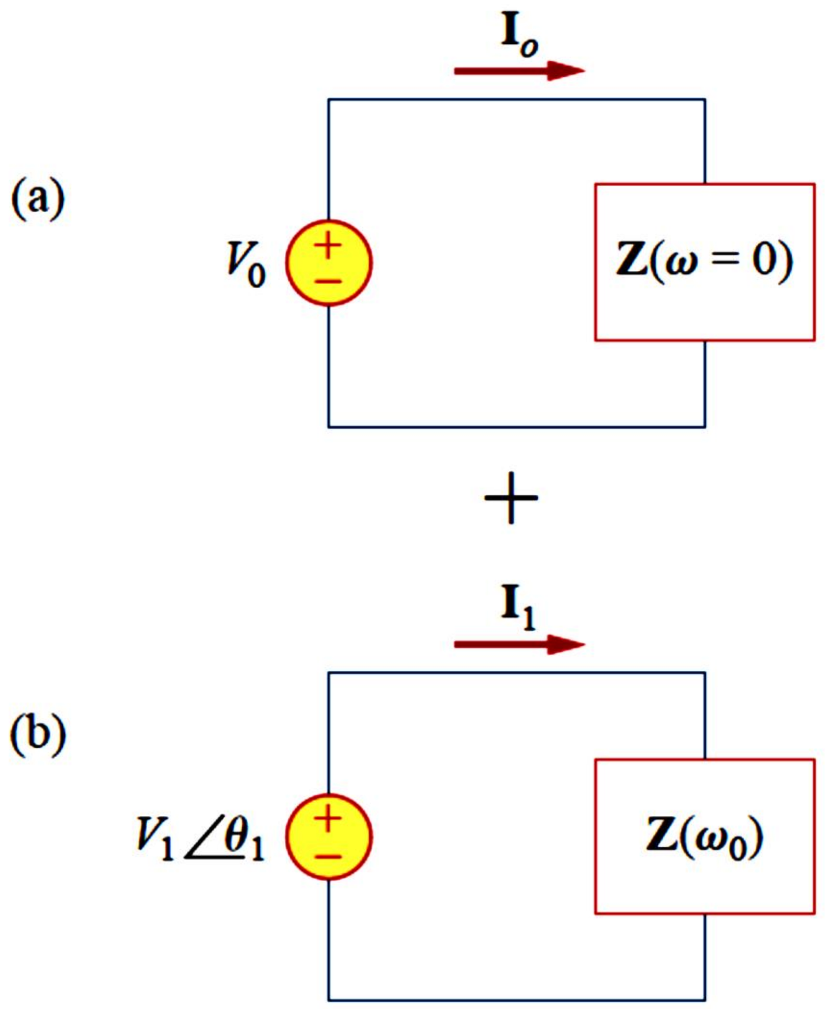


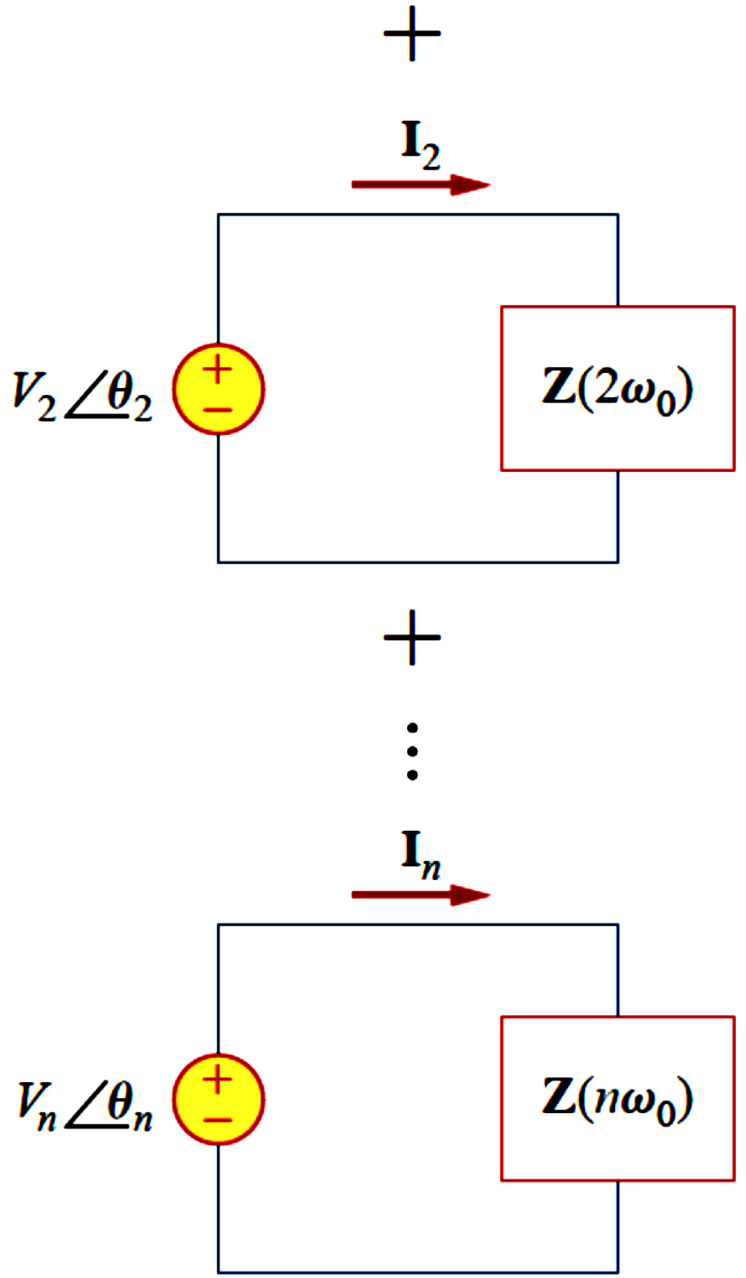
(a) Linear network excited by a periodic voltage source
(b) Fourier series representation (time-domain).

(a)



The response to the dc component can be determined in the frequency domain by setting $n = 0$ or $\omega = 0$as in Fig. or in the time domain by replacing all inductors with short circuits and all capacitors with open circuits.





Steady-state responses: (a) dc component, (b) ac component (frequency domain).

Finally, following the principle of superposition, we add all the individual responses. For the case shown in Fig.

$$
\begin{aligned}
i(t) &= i_0(t) + i_1(t) + i_2(t) + \cdots \\
&= \mathbf{I}_0 + \sum_{n=1}^{\infty} |\mathbf{I}_n| \cos(n\omega_0 t + \psi_n)
\end{aligned}
$$

where each component $\mathbf{I}_n$ with frequency $n\omega_0$ has been transformed to the time domain to get $i_n(t)$, and ψ_n is the argument of $\mathbf{I}_n$.

Example 17.6

Let the function $f(t)$ in Example 17.1 be the voltage source $v_s(t)$ in the circuit of Fig. 17.20. Find the response $v_o(t)$ of the circuit.

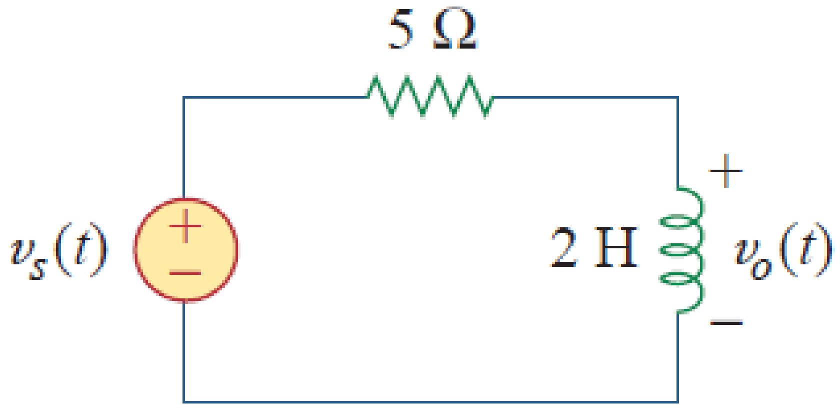


Figure 17.20

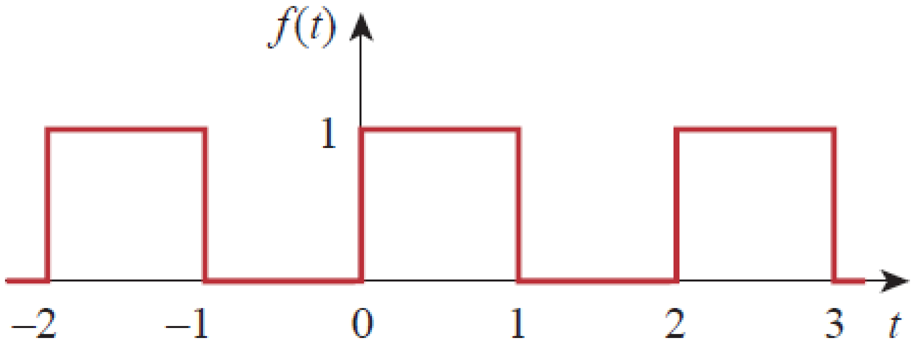


Figure 17.1
For Example 17.1; a square wave.

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

and $f(t) = f(t + T)$. Since $T = 2$, $\omega_0 = 2\pi/T = \pi$.

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

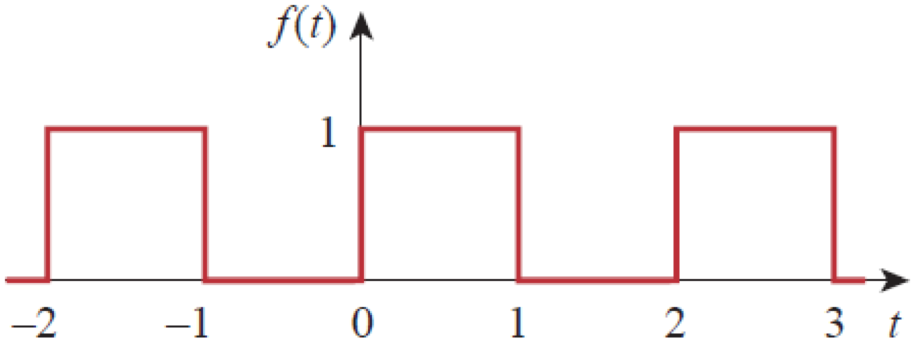


Figure 17.1
For Example 17.1; a square wave.

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t + \cdots$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin n\pi t, \qquad n = 2k - 1$$

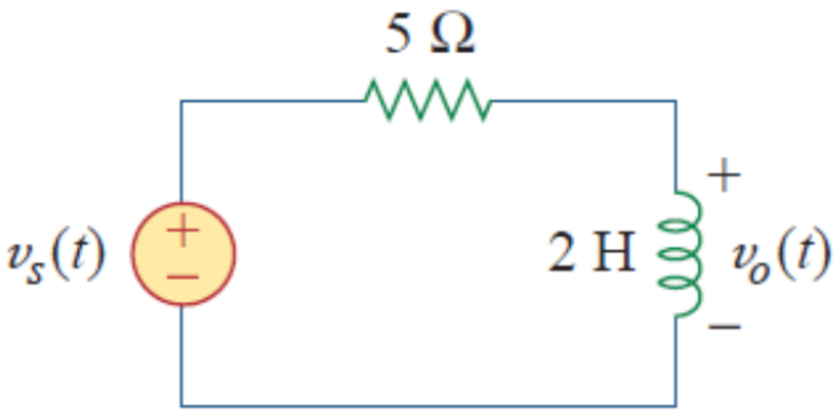


Figure 17.20

where $\omega_n = n\omega_0 = n\pi$ rad/s. Using phasors, we obtain the response $\mathbf{V}_o$ in the circuit of Fig. 17.20 by voltage division:

$$\mathbf{V}_o = \frac{j\omega_n L}{R + j\omega_n L}\mathbf{V}_s = \frac{j2n\pi}{5 + j2n\pi}\mathbf{V}_s$$

For the dc component ($\omega_n = 0$ or $n = 0$)

$$\mathbf{V}_s = \frac{1}{2} \quad \Rightarrow \quad \mathbf{V}_o = 0$$

the inductor is a short circuit to dc.

$$\boxed{f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)}$$

$$A_n = \sqrt{a_n^2 + b_n^2} = |b_n| = \begin{cases} \dfrac{2}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

$$\phi_n = -\tan^{-1}\frac{b_n}{a_n} = \begin{cases} -90^\circ, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

$$A_n \angle \phi_n = a_n - jb_n$$

For the nth harmonic, $\mathbf{V}_s = \dfrac{2}{n\pi} \underline{/-90^\circ}$

We have $$\mathbf{V}_o = \frac{j2n\pi}{5 + j2n\pi}\mathbf{V}_s$$

Therefore the corresponding response is,

$$\mathbf{V}_o = \frac{2n\pi \underline{/90^\circ}}{\sqrt{25 + 4n^2\pi^2}\underline{/\tan^{-1} 2n\pi/5}}\left(\frac{2}{n\pi}\underline{/-90^\circ}\right)$$

$$= \frac{4\underline{/-\tan^{-1} 2n\pi/5}}{\sqrt{25 + 4n^2\pi^2}}$$

$$\mathbf{V}_o = \frac{4\angle -\tan^{-1} 2n\pi/5}{\sqrt{25 + 4n^2\pi^2}}$$

In the time domain,

$$v_o(t) = \sum_{k=1}^{\infty} \frac{4}{\sqrt{25 + 4n^2\pi^2}} \cos\left(n\pi t - \tan^{-1}\frac{2n\pi}{5}\right), \qquad n = 2k - 1$$

The first three terms ($k = 1, 2, 3$ or $n = 1, 3, 5$) of the odd harmonics in the summation give us

$$v_o(t) = 0.4981\cos(\pi t - 51.49^\circ) + 0.2051\cos(3\pi t - 75.14^\circ) + 0.1257\cos(5\pi t - 80.96^\circ) + \cdots \text{ V}$$

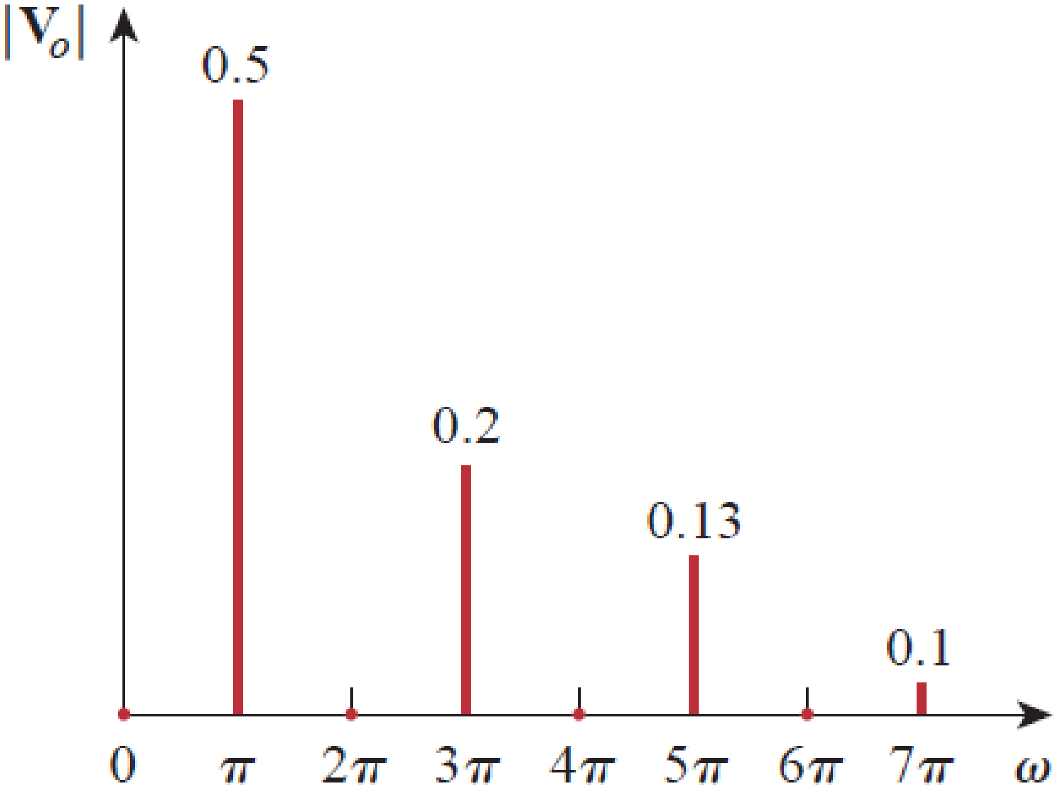


Figure 17.21
For Example 17.6: Amplitude spectrum of the output voltage.

Practice Problem 17.6

If the sawtooth waveform in Fig. 17.9 (see Practice Prob. 17.2) is the voltage source $v_s(t)$ in the circuit of Fig. 17.22, find the response $v_o(t)$.

Answer: $v_o(t) = \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi n t - \tan^{-1} 4n\pi)}{n\sqrt{1 + 16n^2\pi^2}}$ V.

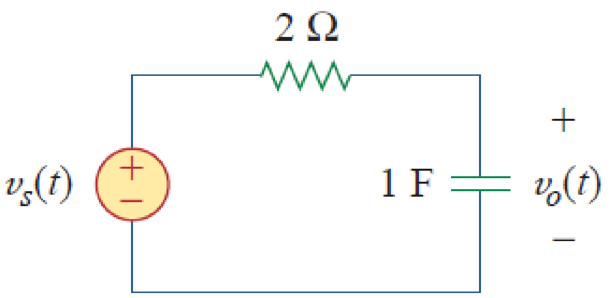


Figure 17.22
For Practice Prob. 17.6.

Home work

- **Example problem 17.7 (page 780-Sadiku)**
- **Practice problem 17.7 (page 781-Sadiku)**