Hilbert Transform:

The Hilbert transform of $f(t)$ is represented by $f_h(t)$.

$$f_h(t) = f(t) * \frac{1}{\pi t}$$

i.e. $f_h(t)$ is the convolution of $f(t)$ and $\frac{1}{\pi t}$.

Using convolution formula, we will get

$$f_h(t) = \int_{-\infty}^{\infty} f(\tau) \frac{1}{\pi(t-\tau)} d\tau = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau} d\tau$$

$$f_h(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau} d\tau$$

Properties:

- A signal $f(t)$ and its Hilbert Transform $f_h(t)$ have same autocorrelation function.
- A signal $f(t)$ and its Hilbert transform $f_h(t)$ have same Energy density Spectrum.
- A signal $f(t)$ and its Hilbert transform $f_h(t)$ are orthogonal to each other.
  $$\int_{-\infty}^{\infty} f(t)\, f_h(t) dt = 0$$
- If Hilbert Transform of $f(t)$ is $f_h(t)$, then the Hilbert Transform of $f_h(t)$ is - $f(t)$.
- It is used to find pre-envelope of a signal in Analog Communication.
  $f_p(t) = f(t) + jf_h(t)$, where $f_p(t)$ is the pre-envelope of a signal $f(t)$ and $f_h(t)$ is the Hilbert Transform of $f(t)$.

Applications:

- It is used in the generation of SSB (single side band) signal in Analog Communication.
- It is used in the design of minimum phase shift filters.
- It is used to represent Band Pass signals in Analog Communication.

Inverse Hilbert Transform:

The inverse Hilbert Transform is defined as,

$$f(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f_h(\tau)}{t-\tau} d\tau$$

Proof:

$$f_h(t) = f(t) * \frac{1}{\pi t}$$

Take Fourier Transform of two side,

$$F_h(\omega) = F(\omega) FT\{\frac{1}{\pi t}\}$$

$sgn(t) \overset{FT}{\leftrightarrow} \frac{2}{j\omega}$ (Use Duality property)

$$\frac{2}{jt} \overset{FT}{\leftrightarrow} 2\pi\, sgn(-\omega)$$

$$\frac{1}{\pi t} \overset{FT}{\leftrightarrow} j\, sgn(-\omega)$$

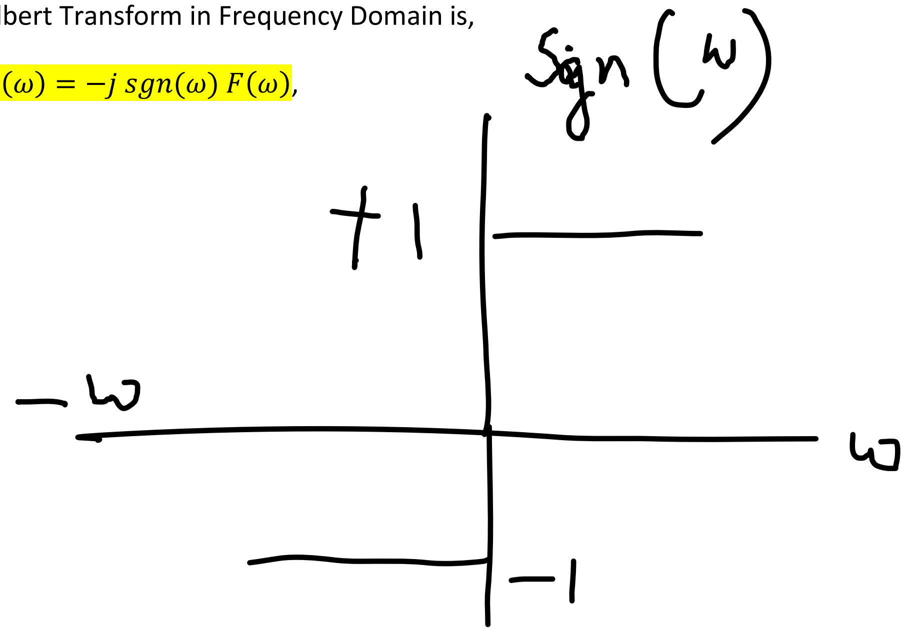
$sgn(-\omega)$ an odd function,

$$\frac{1}{\pi t} \overset{FT}{\leftrightarrow} -j\, sgn(\omega)$$

$F_h(\omega) = -j\, sgn(\omega)\, F(\omega)$,

Hilbert Transform in Frequency Domain is,

$F_h(\omega) = -j\, sgn(\omega)\, F(\omega)$,



Case 1: When $\omega > 0$;

$$F_h(\omega) = -jF(\omega)$$

All the frequencies will undergo a phase shift of $-90°$.

Case 2: When $\omega < 0$;

$$F_h(\omega) = jF(\omega),$$

These frequencies will undergo a phase shift of $90°$.

Practice Problems:

Q1. Find the Hilbert Transform of $\boldsymbol{\delta(t)}$

Solution:

$$f_h(t) = f(t) * \frac{1}{\pi t}$$

$$f_h(t) = \boldsymbol{\delta(t)} * \frac{1}{\pi t}$$

$f_h(t) = \int_{-\infty}^{\infty} \delta(\tau) \frac{1}{\pi(t-\tau)} d\tau$ Impulse exist at $\tau = 1$

So,

$$f_h(t) = \frac{1}{\pi t}$$

Q2. Find the Hilbert Transform of $\boldsymbol{\sin \omega_0 t}$

Solution:

$$f_h(t) = f(t) * \frac{1}{\pi t}$$

So,

$F_h(\omega) = -j\, sgn(\omega)\, F(\omega)$

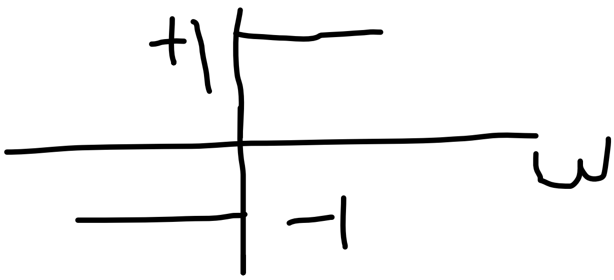
$\mathrm{f(t) = \sin \omega_0 t}$

FT of $\sin \omega_0 t$ is,

$$F(\omega) = \pi j[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

$$F_h(\omega) = -j\, sgn(\omega)\, \pi j[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

$$F_h(\omega) = \pi sgn(\omega)[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$



$F_h(\omega) = -\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$

Taking Inverse FT on both sides,

$F_h(t) = -\cos \omega_o t$

Q3. Find the Hilbert Transform of $\boldsymbol{cos\ \omega_0 t}$

Solution:

Q4. Find the Hilbert Transform of $\boldsymbol{cos\ \omega_1 t\ + sin\ \omega_2 t}$

Solution: