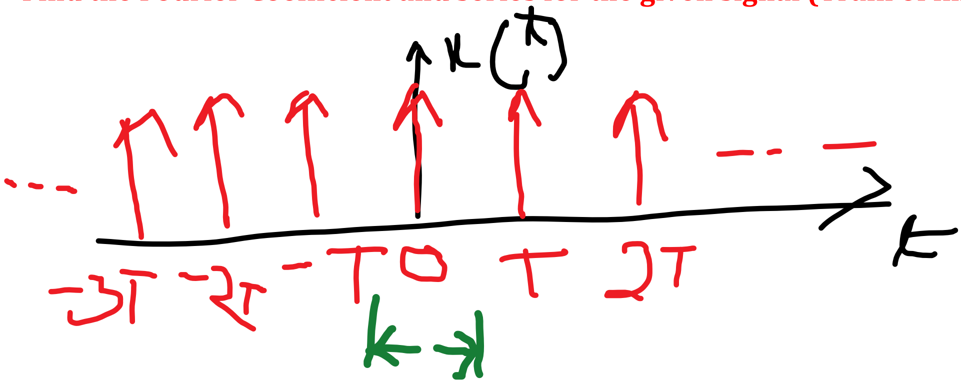
Find the Fourier Coefficient and series for the given signal (Train of impulse).



Solution:

$T = \frac{2\pi}{\omega_0}, \omega_0 = \frac{2\pi}{T}$

$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jn\omega_0 t}dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jn\omega_0 t}dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{0}dt$ signal exist at t = 0.

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, dt = \frac{1}{T}$$

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn(\frac{2\pi}{T})t}$$

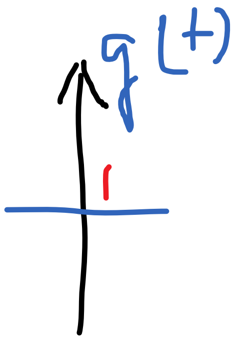
$$C_n = \frac{a_n - jb_n}{2}$$

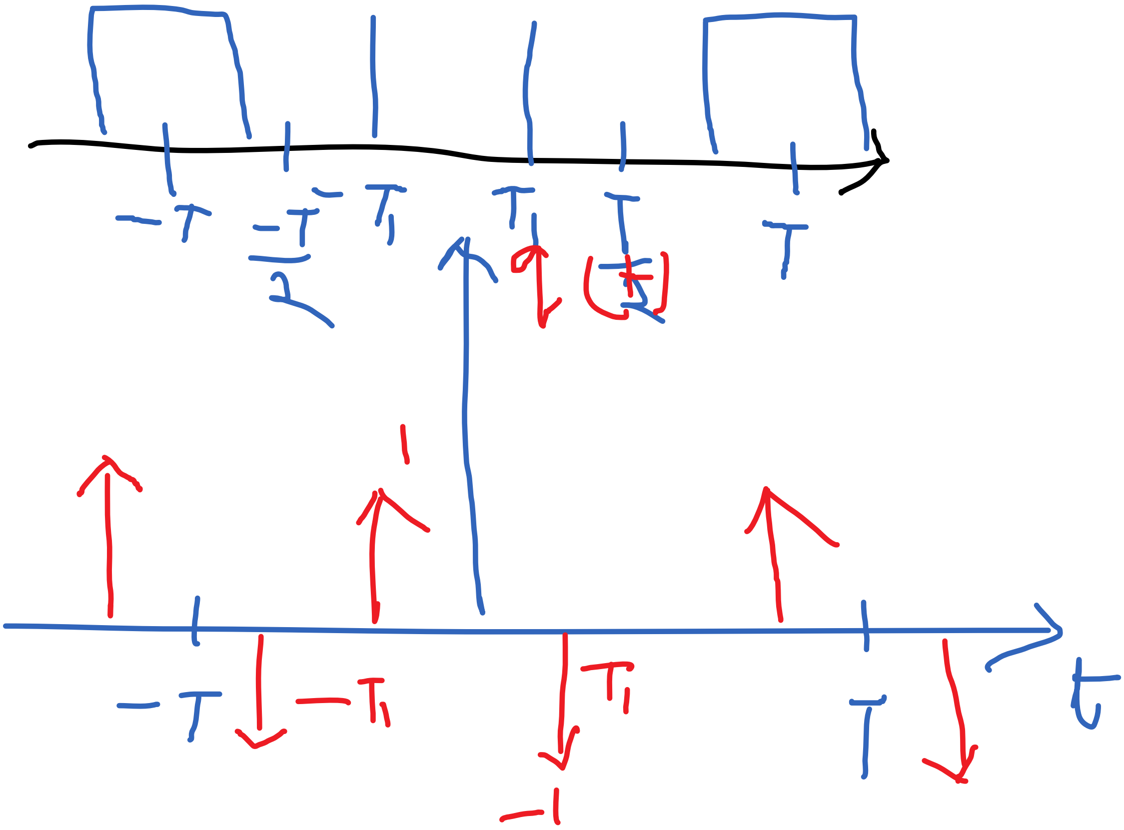
$\frac{1}{T} = \frac{a_n - jb_n}{2}$, b_n is zero as signal is even.

$$a_n = \frac{2}{T}$$

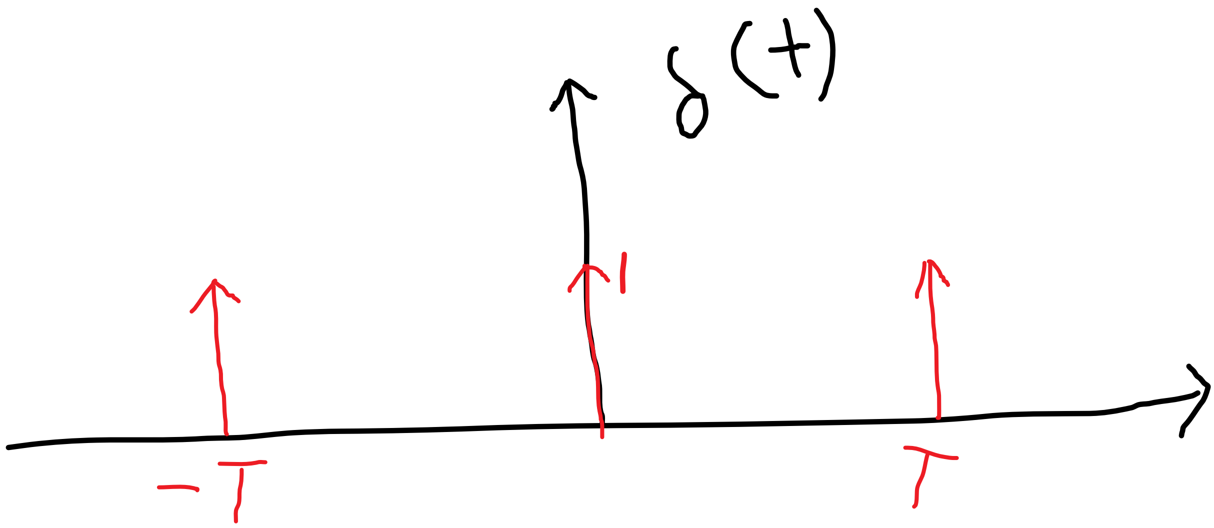
$$a_0 = c_0 = \frac{1}{T}$$

<u>Impulse train and square wave signal.</u>





$g(t)$ is a square wave signal and $q(t)$ is the difference of two shifted version of an impulse signal given below.



$$q(t) = \delta(t + T_1) - \delta(t - T_1)$$

Using properties of Fourier Series, we can get

i. Time shifting and linearity

$b_n = e^{jn\omega_o T_1} a_n - e^{-jk\omega_o T_1} a_n$

Where, $\omega_0 = \frac{2\pi}{T}$

$b_n = \frac{1}{T}\left[e^{jn\omega_o T_1} - e^{-jn\omega_0 T_1}\right] = \frac{2j\sin(n\omega_0 T_1)}{T}$

ii. As $q(t)$ is the derivative of $g(t)$, we can give the differentiation property

$b_n = jn\omega_0 c_n$

Where, c_n is the Fourier Series coefficients of $g(t)$.

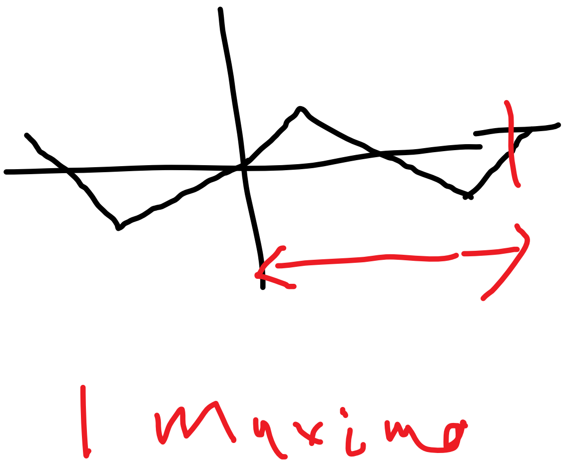
$C_n = \frac{b_n}{jn\omega_0} = \frac{2j\sin(k\omega_0 T_1)}{jk\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, k \neq 0$

$C_0 = \frac{2T_1}{T}$, average value of $g(t)$ over one period. As at k=0, we cannot find the value of C_0.

Convergence of Fourier Series:

- Fourier Series is used to analyse the periodic signals
- All Periodic signals are not eligible for **Fourier Series.** This is proved by PL Dirichlet. So the conditions are known as Dirichlet conditions.
- **Condition 1:**

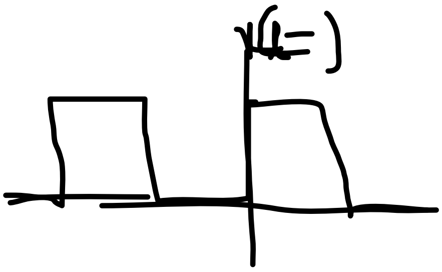
  Signal should have finite number of maxima and minima over the range of time period.

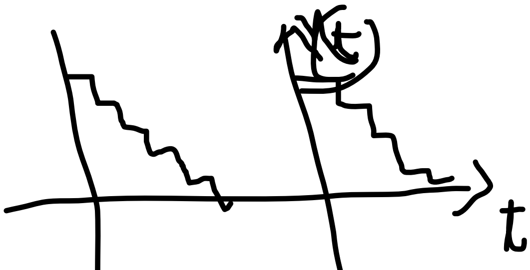


So for signal 2, FS will not exist.

- **Condition 2:**

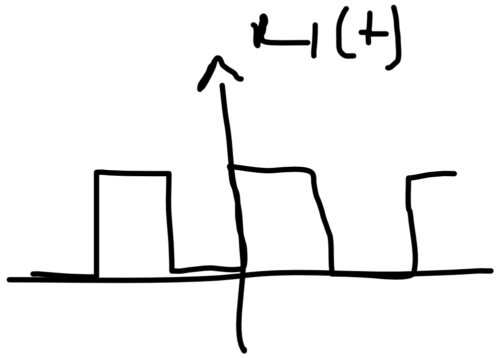
  Signal should have finite number of discontinuities over the range of time period.

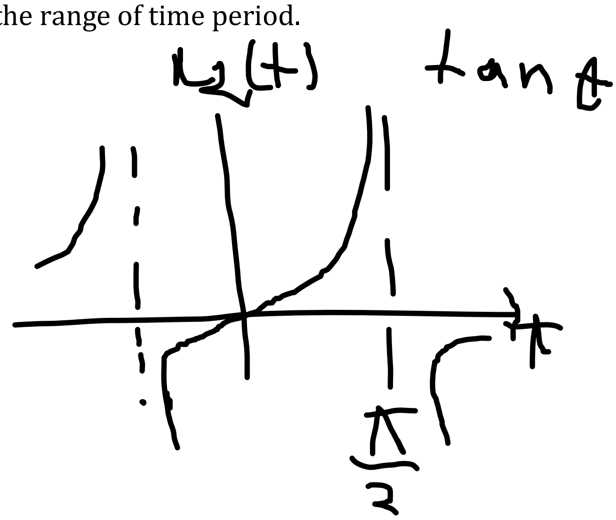




- **Condition 3:**

  Signal should be absolutely integrable over the range of time period.



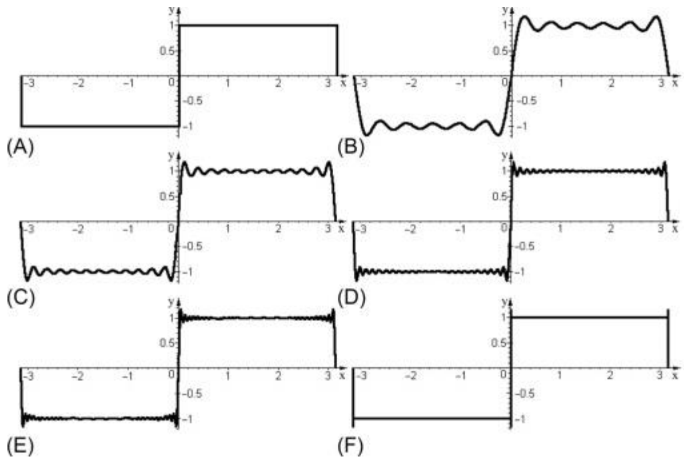


- Periodic signals over the range of $(-\infty, \infty)$ are not energy signals as the area will always be infinite. But, the periodic signals over a period may or may not be infinite.
- So if $\int_0^{T_0} |x_1(t)| dt < \infty$ then the signal is absolute integrable and FS is possible.
- So if $\int_0^{T_0} |x_1(t)| dt = \infty$ then the signal is not absolute integrable and FS is not possible

<u>Gibb's Phenomenon</u>

For a periodic signal with discontinuities, if the signal is constructed by adding the Fourier series, then overshoot appears around the edges. These overshoots decay outwards in a damped oscillatory manner away from the edges. This is known as GIBBS phenomenon and is shown as,

$x(t) \cong a_0 \, \mathbf{cos(0 \times \boldsymbol{\omega_0 t})} + a_1 \cos(1 \times \omega_0 t) + a_2 \cos(2 \times \omega_0 t) + \cdots + a_n \cos(n \times \omega_0 t) + \cdots +$
$b_0 \sin(0 \times \omega_0 t) + b_1 \sin(1 \times \omega_0 t) + \cdots + b_n \sin(n \times \omega_0 t) + \cdots$



Convergence of Fourier Transform:

- Similar to Fourier Series, Fourier transform exist only when Dirichlets Condition is satisfied.
- **Condition 1:**
- Signal should have finite number of maxima and minima over within any finite interval.
- **Condition 2:**
- Signal should have finite number of discontinuities over any finite interval and each of these discontinuities is finite.
- **Condition 3:**
- The signal, $x(t)$, is absolutely integrable,
- $\int_{-\infty}^{\infty} |x(t)| dt < \infty$

Q. Is Fourier transform exist for $(t) = e^{-at}u(t)$ $a > 0$? If yes, then find the transform.

Solution:

Check Dirichlets Condition.

Condition 1: The function is a decaying exponential and continuous without any discontinuities so condition 1 satisfied.

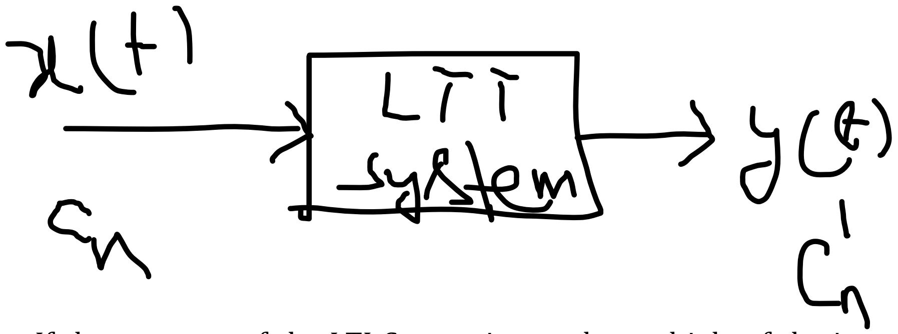
Condition 2: The function is a smooth function so the number of maxima and minima are fixed.

Condition 3: The integration is finite for a finite a, so its absolutely integrable. (Find the integration to check this)

So the Fourier transform exist.

$$F[x(t)] = X(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)\, e^{-j\omega t} dt = \frac{1}{a + j\omega}$$

Fourier Series for LTI System



- If the response of the LTI System is a scaler multiple of the input,, then the input is known as the Eigen function of the LTI system and the scaler quantity is referred as system's Eigen value.
- $If\ y(t) = N\, x(t)$, the N is system's Eigen value.
- If the Impulse response of the LTI System is $N\delta(t - t_0)$, then all periodic input signal with period t_0 will form Eigen function.
- $y(t) = h(t) * x(t)$
- The complex exponential FS is,
- $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$
- $y(t) = \sum_{n=-\infty}^{\infty} C_n{}' e^{jn\omega_0 t}$
- The complex exponential FS coefficients of the O/P of the LTI system is equal to the multiplication of the frequency response of the LTI system with the complex exponential FS coefficients of the I/P.
- $C_n{}' = H(jn\omega_0). C_n$

For continuous signal,

$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau} d\tau$, where $h(t)$ is the impulse response of the LTI system.

For discrete signal,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-jn\omega}$$

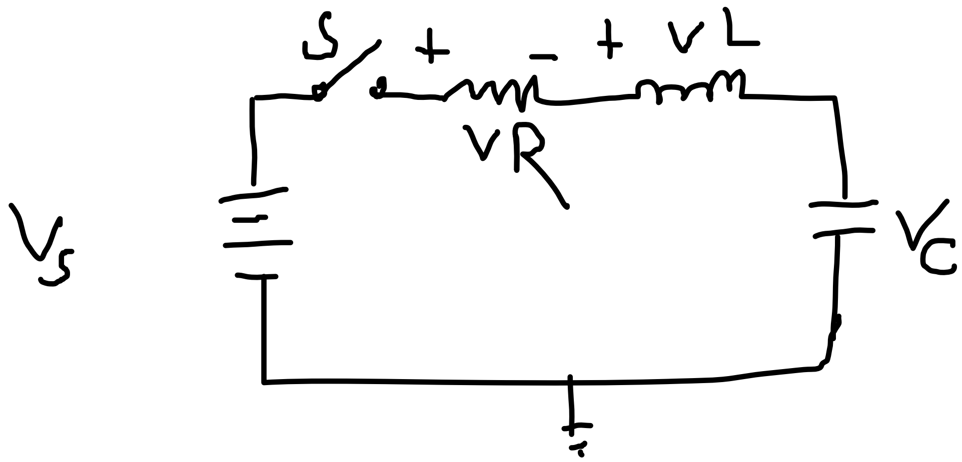
If x(t) is a periodic signal and the FS representation of the signal is,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

And

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

<u>Frequency Response of a RLC circuit</u>



Steps:

- **Transformation of circuit analysis from time to frequency domain**
- **General format :**

$$\boldsymbol{V_{out}(\omega) = V_{in}(\omega) \times H(\omega)}$$

- $\boldsymbol{V_{in}(\omega)}$: ***Input function in terms of frequency***
- $\boldsymbol{V_{out}(\omega)}$: ***Output function in terms of Frequecny***
- $\boldsymbol{H(\omega)}$: ***Transfer Function*** **(Multiplier in frequency domain and convolution in time domain)**
- **Step 1: Convert input into frequency domain (including $\boldsymbol{V_{in}}$ and any circuit components like R, L and C)**
- **Step 2: Solve for transfer function by using voltage divider for the component you would like to analyse.**
- **Step 3: Use $\boldsymbol{V_{in}}$ and $\boldsymbol{H(\omega)}$ functions into general format of Frequency domain shown in above equation.**

- **Plug in circuit component values into the final equation**

Solution:

1. **The impedance across different components are,**

$$Z_R = R = 50\Omega$$

$$Z_L = Lj\omega = 0.3j\omega\Omega$$

$$Z_c = \frac{1}{cj\omega}\Omega$$

2. **Transfer function by using voltage divider for the component (here its across capacitor)**

$$H(\omega) = \frac{Z_c}{Z_R + Z_L + Z_c}$$

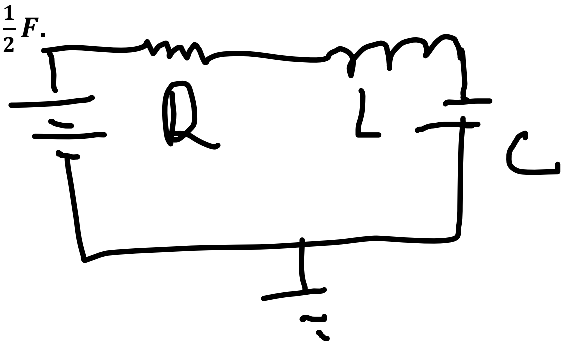
In frequency domain,

$$H(\omega) = \frac{\frac{1}{cj\omega}}{R+Lj\omega+\frac{1}{cj\omega}} = \frac{\frac{1}{cj\omega}}{R+Lj\omega+\frac{1}{cj\omega}} \times \frac{cj\omega}{cj\omega} = \frac{1}{1+RCj\omega-LC\omega^2}$$

3. **Find the Fourier series of $V_{in}(\omega)$ and $V_{out}(\omega)$**

4. $V_{out}(\omega) = V_{in}(\omega) \times H(\omega)$

***Example*: For the following circuit find the output across capacitor if input is $v_s(t)$ and L = 2H, R = 2Ω and C = $\frac{1}{2}F$.**



$$V_s(t) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi t)$$

Solution:

Step 1: Find the Fourier series of $V_s(t)$.

We can rewrite the equation as:

$$V_s(t) = \frac{1}{2} + \sum_{n=1}^{\infty} -\frac{1}{n\pi}\, sin\,(n\pi t)$$

We can relate the equation with

$$V_s(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$a_0 = \frac{1}{2}$, $b_n = -\frac{1}{n\pi}$ **and** $\omega_0 = \pi$

Step 2: Transfer function by using voltage divider for the component (here its across capacitor)

$$H(\omega) = \frac{Z_c}{Z_R + Z_L + Z_c}$$

In frequency domain,

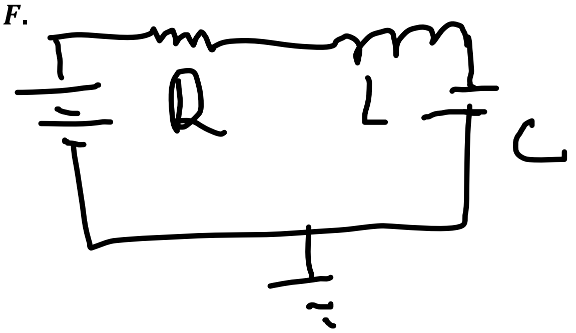
$$H(\omega) = \frac{\frac{1}{cj\omega}}{R+Lj\omega+\frac{1}{cj\omega}} = \frac{\frac{1}{cj\omega}}{R+Lj\omega+\frac{1}{cj\omega}} \times \frac{cj\omega}{cj\omega} = \frac{1}{1+RCj\omega-LC\omega^2}$$

Step 3: $V_{out}(\omega) = V_{in}(\omega) \times H(\omega)$

$V_{out}(\omega) = V_{in}(\omega) \times \frac{1}{1+j\omega-2\omega^2}$, **replace** $\omega = \pi$

$$V_{out}(\omega) = \left(\frac{1}{2} + \sum_{n=1}^{\infty} -\frac{1}{n\pi}\, sin\,(n\pi t)\right) \times \frac{1}{1 + j\pi - 2\pi^2}$$

***Example*: For the following circuit find the output across capacitor if input is sin t and L = 2H, R = 2Ω and C =** $\frac{1}{2}F$.



Solution:

Step 1: Find the Fourier series of Sin t

$$\boldsymbol{\sin t} = \frac{1}{2j}e^{j\omega_0 n_1 t} - \frac{1}{2j}e^{-j\omega_0 n_2 t}$$

$$\boldsymbol{H(\omega)} = \frac{\frac{1}{cj\omega}}{R+Lj\omega+\frac{1}{cj\omega}} = \frac{\frac{1}{cj\omega}}{R+Lj\omega+\frac{1}{cj\omega}} \times \frac{cj\omega}{cj\omega} = \frac{1}{1+RCj\omega-LC\omega^2}, \textbf{ replace } \boldsymbol{\omega} = \mathbf{1} \textbf{ from sin t}$$

Output across Capacitor is:

$$\boldsymbol{\sin t} \times \boldsymbol{H(\omega)} = [\frac{1}{2j}e^{j\omega_0 n_1 t} - \frac{1}{2j}e^{-j\omega_0 n_2 t}] \times \frac{\mathbf{1}}{\mathbf{1}+\boldsymbol{j}-\mathbf{2}}$$

***Example*: Consider a Causal LTI System whose i/p and O/P are related by following differential equation.**

$\frac{dy(t)}{dt} + 4y(t) = x(t)$ **and** $H(jn\omega_0) = \frac{1}{4+jn\omega_0}$

Find the o/p coefficient $C_n{}'$ if i/p is $\cos 2\pi t + \sin 4\pi t$

Solution:

$x(t) = \cos 2\pi t + \sin 4\pi t$

$\omega_0 = HCF(2\pi, 4\pi) = 2\pi$

Given $H(jn\omega_0) = \frac{1}{4+jn\omega_0}$

As discussed earlier,

$$H(jn\omega_0) = \frac{C_n{}'}{C_n}$$

$$x(t) = \cos 2\pi t + \sin 4\pi t = \cos \omega_0 t + \sin 2\omega_0 t = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} + \frac{1}{2j}e^{j2\omega_0 t} - \frac{1}{2j}e^{-j2\omega_0 t}$$

$C_1 = \frac{1}{2}, C_{-1} = \frac{1}{2}$

$C_2 = \frac{1}{2j}, C_{-2} = -\frac{1}{2j}$

All other coefficients are zero.

$$C_1{}' = H(j1\omega_0)C_1 = \frac{1}{4+j2\pi}C_1$$

$$C_{-1}{}' = H(j(-1)\omega_0)C_{-1} = \frac{1}{4-j2\pi}C_{-1}$$

$$C_2{}' = H(j2\omega_0)C_2 = \frac{1}{4+j4\pi}C_2$$

$$C_{-2}{}' = H(j(-2)\omega_0)C_2 = \frac{1}{4-j4\pi}C_{-2}$$

Example: **If x(t)= $1+\frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$ is a periodic signal given as input to the system with impulse response $h(t) = e^{t}u(t)$. Find the Output of the system and $\omega_0 = 2\pi$.**

Solution:

$$H(j\omega) = \int_0^\infty e^{-\tau} e^{-j\omega\tau} d\tau = -\frac{1}{1+j\omega} e^{-\tau}e^{-j\omega\tau}\Big|_0^\infty = \frac{1}{1+j\omega}$$

From x(t) we can find the Fourier series coefficients as

$$a_0 = 1$$

$$a_1 = a_{-1} = \frac{1}{4}$$

$$a_2 = a_{-2} = \frac{1}{2}$$

$$a_3 = a_{-3} = \frac{1}{3}$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)e^{jk\omega_o t} = \sum_{k=-3}^{3} b_k e^{jk2\pi t}$$

$b_k = a_k H(jk2\pi)$, So

$$b_0 = 1$$

$$b_1 = \frac{1}{4}\left(\frac{1}{1+j2\pi}\right)$$

$$b_{-1} = \frac{1}{4}\left(\frac{1}{1 - j2\pi}\right)$$

$$b_2 = \frac{1}{2}\left(\frac{1}{1 + j4\pi}\right)$$

$$b_{-2} = \frac{1}{2}\left(\frac{1}{1 - j4\pi}\right)$$

$$b_3 = \frac{1}{3}(\frac{1}{1 + j6\pi})$$

$$b_{-3} = \frac{1}{3}(\frac{1}{1 - j6\pi})$$

As b_1 and b_{-1} are complex and even conjugate of each other, $y(t)$ is real.