

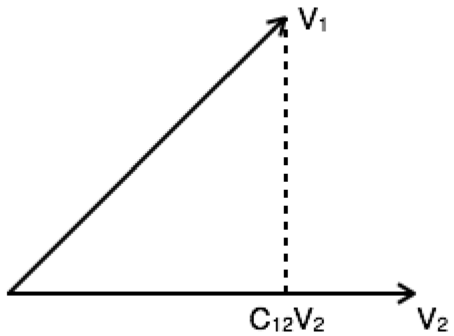
Analogy Between Vectors and Signals

There is a perfect analogy between vectors and signals.

Vector

A vector contains magnitude and direction. The name of the vector is denoted by bold face type and their magnitude is denoted by light face type.

Example: V is a vector with magnitude V. Consider two vectors V_1 and V_2 as shown in the following diagram. Let the component of V_1 along with V_2 is given by $C_{12}V_2$. The component of a vector V_1 along with the vector V_2 can obtained by taking a perpendicular from the end of V_1 to the vector V_2 as shown in diagram:



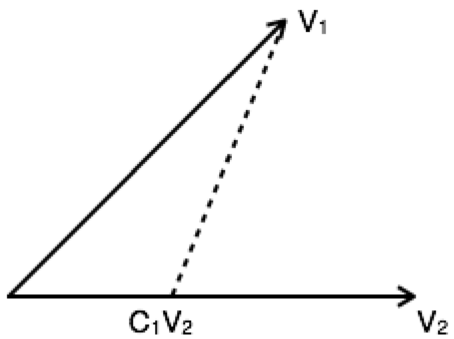
The vector V_1 can be expressed in terms of vector V_2

$V_1 = C_{12}V_2 + V_e$

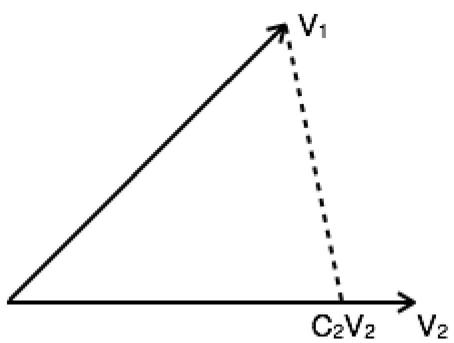
Where Ve is the error vector.

But this is not the only way of expressing vector V_1 in terms of V_2. The alternate possibilities are:

$V_1=C_1V_2+V_{e1}$



$V_2=C_2V_2+V_{e2}$



The error signal is minimum for large component value. If C_{12}=0, then two signals are said to be orthogonal.

Dot Product of Two Vectors

$V_1 . V_2 = V_1.V_2 \cos\theta$

θ = Angle between V1 and V2

$V_1 . V_2 = V_2.V_1$

The components of V_1 alog$_n$ V_2 = V_1 Cos θ = $\frac{V1.V2}{V2}$

From the diagram, components of V_1 alog$_n$ V_2 = C $_{12}$ V_2

$$\frac{V_1.V_2}{V_2 = C_1 2\, V_2}$$

$$\Rightarrow C_{12} = \frac{V_1.V_2}{V_2}$$

Signal

The concept of orthogonality can be applied to signals. Let us consider two signals $f_1 t$ and $f_2 t$. Similar to vectors, you can approximate $f_1 t$ in terms of $f_2 t$ as

$f_1 t = C_{12}\, f_2 t + f_e t$ for ($t_1 < t < t_2$)

$\Rightarrow f_e t = f_1 t - C_{12}\, f_2 t$

One possible way of minimizing the error is integrating over the interval t_1 to t_2.

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_e(t)] dt$$

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_1(t) - C_{12} f_2(t)] dt$$

However, this step also does not reduce the error to appreciable extent. This can be corrected by taking the square of error function.

$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_e(t)]^2 dt$

$\Rightarrow \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_e(t) - C_{12} f_2]^2 dt$

Where ε is the mean square value of error signal. The value of C_{12} which minimizes the error, you need to calculate $\frac{d\varepsilon}{dC_{12}} = 0$

$\Rightarrow \frac{d}{dC_{12}} [\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_1(t) - C_{12} f_2(t)]^2 dt] = 0$

$\Rightarrow \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [\frac{d}{dC_{12}} f_1^2(t) - \frac{d}{dC_{12}} 2 f_1(t) C_{12} f_2(t) + \frac{d}{dC_{12}} f_2^2(t) C_{12}^2] dt = 0$

Derivative of the terms which do not have C12 term are zero.

$\Rightarrow \int_{t_1}^{t_2} -2 f_1(t) f_2(t) dt + 2 C_{12} \int_{t_1}^{t_2} [f_2^2(t)] dt = 0$

If $C_{12} = \frac{\int_{t_1}^{t_2} f_1(t)f_2(t)dt}{\int_{t_1}^{t_2} f_2^2(t)dt}$ component is zero, then two signals are said to be orthogonal.

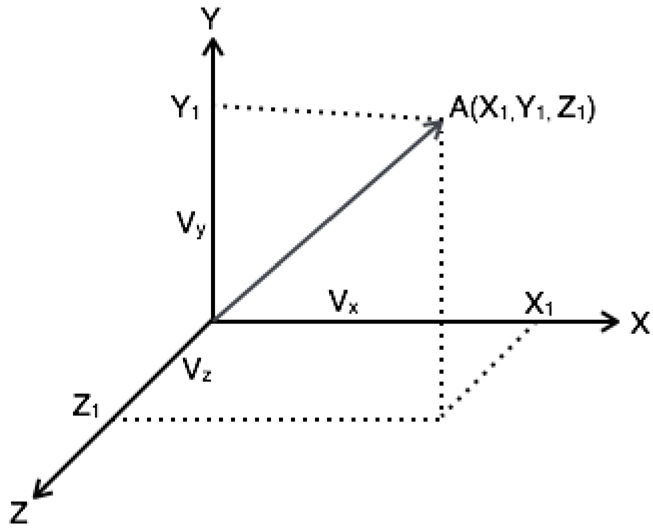
Put $C_{12} = 0$ to get condition for orthogonality.

$0 = \frac{\int_{t_1}^{t_2} f_1(t)f_2(t)dt}{\int_{t_1}^{t_2} f_2^2(t)dt}$

$$\int_{t_1}^{t_2} f_1(t)f_2(t)dt = 0$$

Orthogonal Vector Space

A complete set of orthogonal vectors is referred to as orthogonal vector space. Consider a three dimensional vector space as shown below:



Consider a vector A at a point (X_1, Y_1, Z_1). Consider three unit vectors (V_X, V_Y, V_Z) in the direction of X, Y, Z axis respectively. Since these unit vectors are mutually orthogonal, it satisfies that

$$V_X.V_X = V_Y.V_Y = V_Z.V_Z = 1$$
$$V_X.V_Y = V_Y.V_Z = V_Z.V_X = 0$$

You can write above conditions as

$$V_a.V_b = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$

The vector A can be represented in terms of its components and unit vectors as

$$A = X_1V_X + Y_1V_Y + Z_1V_Z \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Any vectors in this three dimensional space can be represented in terms of these three unit vectors only.

If you consider n dimensional space, then any vector A in that space can be represented as

$$A = X_1V_X + Y_1V_Y + Z_1V_Z + \ldots + N_1V_N \ldots\ldots(2)$$

As the magnitude of unit vectors is unity for any vector A

The component of A along x axis = $A.V_X$

The component of A along Y axis = $A.V_Y$

The component of A along Z axis = A.V_Z

Similarly, for n dimensional space, the component of A along some G axis

$= A.VG...............(3)$

Substitute equation 2 in equation 3.

$\Rightarrow CG = (X_1 V_X + Y_1 V_Y + Z_1 V_Z + ... + G_1 V_G ... + N_1 V_N) V_G$

$= X_1 V_X V_G + Y_1 V_Y V_G + Z_1 V_Z V_G + ... + G_1 V_G V_G ... + N_1 V_N V_G$

$= G_1 \quad \text{since } V_G V_G = 1$

$If V_G V_G \neq 1 \text{ i.e.} V_G V_G = k$

$AV_G = G_1 V_G V_G = G_1 K$

$G_1 = \frac{(AV_G)}{K}$

Orthogonal Signal Space

Let us consider a set of n mutually orthogonal functions $x_1 t$, $x_2 t$... $x_n t$ over the interval t_1 to t_2. As these functions are orthogonal to each other, any two signals $x_j t$, $x_k t$ have to satisfy the orthogonality condition. i.e.

$$\int_{t_1}^{t_2} x_j(t) x_k(t) dt = 0 \text{ where } j \neq k$$

$$\text{Let} \int_{t_1}^{t_2} x_k^2(t) dt = k_k$$

Let a function ft, it can be approximated with this orthogonal signal space by adding the components along mutually orthogonal signals i.e.

$$f(t) = C_1 x_1(t) + C_2 x_2(t) + ... + C_n x_n(t) + f_e(t)$$

$$= \Sigma_{r=1}^{n} C_r x_r(t)$$

$$f(t) = f(t) - \Sigma_{r=1}^{n} C_r x_r(t)$$

Mean sqaure error $\varepsilon = \frac{1}{t_2 - t_2} \int_{t_1}^{t_2} [f_e(t)]^2 dt$

$$= \frac{1}{t_2 - t_2} \int_{t_1}^{t_2} [f[t] - \sum_{r=1}^{n} C_r x_r(t)]^2 dt$$

The component which minimizes the mean square error can be found by

$$\frac{d\varepsilon}{dC_1} = \frac{d\varepsilon}{dC_2} = ... = \frac{d\varepsilon}{dC_k} = 0$$

Let us consider $\frac{d\varepsilon}{dC_k} = 0$

$$\frac{d}{dC_k}[\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f(t) - \Sigma_{r=1}^{n} C_r x_r(t)]^2 dt] = 0$$

All terms that do not contain C_k is zero. i.e. in summation, r=k term remains and all other terms are zero.

$$\int_{t_1}^{t_2} -2f(t)x_k(t)dt + 2C_k \int_{t_1}^{t_2} [x_k^2(t)]dt = 0$$

$$\Rightarrow C_k = \frac{\int_{t_1}^{t_2} f(t)x_k(t)dt}{int_{t_1}^{t_2} x_k^2(t)dt}$$

$$\Rightarrow \int_{t_1}^{t_2} f(t)x_k(t)dt = C_k K_k$$

Mean Square Error

The average of square of error function f$_e t$ is called as mean square error. It is denoted by ε $epsilon$.

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$\varepsilon = \frac{1}{t_2-t_1} \int_{t_1}^{t_2} [f_e(t)]^2 dt$

$= \frac{1}{t_2-t_1} \int_{t_1}^{t_2} [f_e(t) - \Sigma_{r=1}^{n} C_r x_r(t)]^2 dt$

$= \frac{1}{t_2-t_1} [\int_{t_1}^{t_2} [f_e^2(t)]dt + \Sigma_{r=1}^{n} C_r^2 \int_{t_1}^{t_2} x_r^2(t)dt - 2\Sigma_{r=1}^{n} C_r \int_{t_1}^{t_2} x_r(t)f(t)dt$

You know that $C_r^2 \int_{t_1}^{t_2} x_r^2(t)dt = C_r \int_{t_1}^{t_2} x_r(t)f(d)dt = C_r^2 K_r$

$\varepsilon = \frac{1}{t_2-t_1} [\int_{t_1}^{t_2} [f^2(t)]dt + \Sigma_{r=1}^{n} C_r^2 K_r - 2\Sigma_{r=1}^{n} C_r^2 K_r]$

$= \frac{1}{t_2-t_1} [\int_{t_1}^{t_2} [f^2(t)]dt - \Sigma_{r=1}^{n} C_r^2 K_r]$

$\therefore \varepsilon = \frac{1}{t_2-t_1} [\int_{t_1}^{t_2} [f^2(t)]dt + (C_1^2 K_1 + C_2^2 K_2 + ... + C_n^2 K_n)]$

The above equation is used to evaluate the mean square error.

Closed and Complete Set of Orthogonal Functions

Let us consider a set of n mutually orthogonal functions x$_1 t$, x$_2 t$...x$_n t$ over the interval t_1 to t_2. This is called as closed and complete set when there exist no function ft satisfying the condition $\int_{t_1}^{t_2} f(t)x_k(t)dt = 0$

If this function is satisfying the equation $\int_{t_1}^{t_2} f(t)x_k(t)dt = 0 \text{ for} k = 1, 2, ..$ then ft is said to be orthogonal to each and every function of orthogonal set. This set is incomplete without ft. It becomes closed and complete set when ft is included.

ft can be approximated with this orthogonal set by adding the components along mutually orthogonal signals i.e.

$$f(t) = C_1 x_1(t) + C_2 x_2(t) + ... + C_n x_n(t) + f_e(t)$$

If the infinite series $C_1 x_1(t) + C_2 x_2(t) + ... + C_n x_n(t)$ converges to ft then mean square error is zero.

Orthogonality in Complex Functions

If f$_1 t$ and f$_2 t$ are two complex functions, then f$_1 t$ can be expressed in terms of f$_2 t$ as

$f_1(t) = C_{12} f_2(t)$ ..with negligible error

Where $C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) f_2^*(t) dt}{\int_{t_1}^{t_2} |f_2(t)|^2 dt}$

Where $f_2^*(t)$ = complex conjugate of $f_2 t$.

If $f_1 t$ and $f_2 t$ are orthogonal then $C_{12} = 0$

$$\frac{\int_{t_1}^{t_2} f_1(t) f_2^*(t) dt}{\int_{t_1}^{t_2} |f_2(t)|^2 dt} = 0$$

$$\Rightarrow \int_{t_1}^{t_2} f_1(t) f_2^*(dt) = 0$$

The above equation represents orthogonality condition in complex functions.