Discrete Time Fourier Transform

Discrete Time Fourier Transform:

$$\boldsymbol{F\{x[n]\} = X\{e^{j\omega}\} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}}$$

Properties of DTFT:

1. **Periodicity**
   The discrete time Fourier Transform $\boldsymbol{X\{e^{j\omega}\}}$ is periodic in ω with period 2π.

   If $F\{\,x[n]\} = X(\boldsymbol{e^{j\omega}})$, then

   $X(\boldsymbol{e^{j\omega}}) = X(\boldsymbol{e^{j(\omega+2\pi)}})$
   $X(\boldsymbol{e^{j\omega}}) = X(\boldsymbol{e^{j(\omega+2k\pi)}})$, where k is any integer.

2. **Linearity**

It satisfies the superposition principle.

If,

$F\{\,x1[n]\} = X_1(\boldsymbol{e^{j\omega}})$

$F\{\,x2[n]\} = X_2(\boldsymbol{e^{j\omega}})$, then

$F\{a_1x1[n] + a_2x2[n]\} \leftrightarrow a_1X_1(\boldsymbol{e^{j\omega}}) + a_2X_2(\boldsymbol{e^{j\omega}})$

3. **Time Shifting**

If $F\{\,x[n]\} = X(\boldsymbol{e^{j\omega}})$, then

$F\{\,x[n-k]\} = e^{-j\omega k}X(\boldsymbol{e^{j\omega}})$

4. **Frequency Shifting**

If $F\{\,x[n]\} = X(\boldsymbol{e^{j\omega}})$, then

$F\{\,x[n]e^{j\omega_o n}\} = X(\boldsymbol{e^{j(\omega-\omega_o)}})$

   Shifting the signal in frequency domain results in multiplying the signal $x[n]$ by $e^{j\omega_o n}$
   This property is the dual of the time shifting property.

5. **Conjugate**

$x^*[n] \leftrightarrow X^*(\boldsymbol{e^{-j\omega}})$

6. Time Reversal

If $F\{\,x[n]\} = X\left(\boldsymbol{e^{j\omega}}\right)$, then

$F\{\,x[-n]\} = X\left(\boldsymbol{e^{-j\omega}}\right)$

7. Differentiation in Frequency

If $F\{\,x[n]\} = X\left(\boldsymbol{e^{j\omega}}\right)$, then

$$F\{\,nx[n]\} = j\frac{d}{d\omega}X(\boldsymbol{e^{j\omega}})$$

8. Symmetry Property

The Fourier transform $X\left(\boldsymbol{e^{j\omega}}\right)$ is a complex function of ω.

$X\left(\boldsymbol{e^{j\omega}}\right) = X_R\left(\boldsymbol{e^{j\omega}}\right) + jX_I\left(\boldsymbol{e^{j\omega}}\right)$

Where,

$X\left(\boldsymbol{e^{j\omega}}\right) = \sum_{\boldsymbol{n}=-\infty}^{\infty} \boldsymbol{x[n]e^{-j\omega n}}$

$$X_R\left(\boldsymbol{e^{j\omega}}\right) = \sum_{n=-\infty}^{\infty} x[n]\cos\omega n$$

$$X_I\left(\boldsymbol{e^{j\omega}}\right) = -\sum_{n=-\infty}^{\infty} x[n]\sin\omega n$$

Magnitude of the DTFT,

$$\left|X\left(\boldsymbol{e^{j\omega}}\right)\right| = \sqrt{{X_R}^2(\boldsymbol{e^{j\omega}}) + {X_I}^2(\boldsymbol{e^{j\omega}})}$$

The phase of DTFT,

$$\theta(\omega) = tan^{-1}\left|\frac{X_I\left(\boldsymbol{e^{j\omega}}\right)}{X_R\left(\boldsymbol{e^{j\omega}}\right)}\right|$$

$$X\left(\boldsymbol{e^{j\omega}}\right) = \left|X\left(\boldsymbol{e^{j\omega}}\right)\right|\left|X\left(\boldsymbol{e^{j\omega}}\right)\right|$$

9. Even Symmetry

$X_R\left(\boldsymbol{e^{-j\omega}}\right) = X_R\left(\boldsymbol{e^{j\omega}}\right)$

10. Odd Symmetry

$X_I\left(\boldsymbol{e^{-j\omega}}\right) = -X_I\left(\boldsymbol{e^{j\omega}}\right)$

11. Convolution in Time Domain

$F\{\,x1[n]\} = X_1\left(\boldsymbol{e^{j\omega}}\right)$

$F\{\,x2[n]\} = X_2\left(\boldsymbol{e^{j\omega}}\right)$, then

$F\{\,x1[n] *\; x2[n]\} = X_1\left(\boldsymbol{e^{j\omega}}\right)X_2\left(\boldsymbol{e^{j\omega}}\right)$

Convolution in time domain is multiplication in frequency domain.

12. Multiplication or Convolution in Frequency Domain

$F\{\, x1[n]\} = X_1\left(\boldsymbol{e^{j\omega}}\right)$

$F\{\, x2[n]\} = X_2\left(\boldsymbol{e^{j\omega}}\right)$, then

$F\{\, x1[n]\; x2[n]\} = X_1\left(\boldsymbol{e^{j\omega}}\right) * X_2\left(\boldsymbol{e^{j\omega}}\right)$

Multiplication in time domain is Convolution in frequency domain.

13. Modulation Theorem

If $F\{\, x[n]\} = X\left(\boldsymbol{e^{j\omega}}\right)$, then

$F\{\, x[n]\cos\omega_0 n\} = \frac{1}{2}X\left(\boldsymbol{e^{j(\omega+\omega_o)}}\right) + X\left(\boldsymbol{e^{j(\omega-\omega_o)}}\right)$

14. Correlation in Time

$F\{\, x1[n]\} = X_1\left(\boldsymbol{e^{j\omega}}\right)$

$F\{\, x2[n]\} = X_2\left(\boldsymbol{e^{j\omega}}\right)$, then

$F\{\, Corr(x1[n], x2[n])\} = X_1\left(\boldsymbol{e^{j\omega}}\right)X_2\left(\boldsymbol{e^{-j\omega}}\right)$

15. Parseval's Relation

Represents the energy density function of Discrete time signal.

If $F\{\, x[n]\} = X\left(\boldsymbol{e^{j\omega}}\right)$, then

E=$\sum_{n=-\infty}^{\infty}|x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X\left(e^{j\omega}\right)\right|^2 d\omega$

Condition for the existence of DTFT:

$\sum_{n=-\infty}^{\infty}|x[n]| < \infty$ -> Finite or Absolutely Summable

Q. Find the sequence of given transform,

$\boldsymbol{x[n]} = \{\ldots\}$ **for**

$\boldsymbol{X\left(e^{j\omega}\right) = 3e^{j2\omega} + 2e^{j\omega} - 1 + e^{-j\omega} + 2e^{-3j\omega}}$

Solution:

n=[-2,-1,0,1,2,3]

$x(n) = \{3,2,\boldsymbol{-1},1,0,2\}$

Q. Find the transform of x[n]

$x(n) = \{3, 2, -1, 1, 0, 0, -1\}$

Answer:

Q1. Find FT $x(n) = \delta(n)$

Solution:

$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \delta(n)e^{-j\omega n}$

$\delta(n) = 1$ when n=0,

$X(e^{j\omega}) = 1$

Q2. Find FT $x(n) = u(n)$

Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty} u(n)e^{-j\omega n}$$

$u(n) = 1$ when $n \geq 0$,

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} e^{-j\omega n} = 1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + \cdots$$

$$Use\ AP\ formula: 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

$$X(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}}$$