

Example 7.36. Find the z-transform and the ROC of the signal.

$$x(n) = [3(2^n) - 4(3^n)] \cdot u(n)$$

Solution : The given signal is

$$x(n) = [3(2^n) - 4(3^n)] \cdot u(n) \quad \dots(i)$$

Assuming that

$$x_1(n) = 2^n \cdot u(n)$$

and

$$x_2(n) = 3^n \cdot u(n)$$

Then the signal $x[n]$ may be written as

$$x(n) = 3x_1(n) - 4x_2(n) \quad \dots(ii)$$

According to linearity property, the z-transform of equation (ii) is

$$X(z) = 3X_1(z) - 4X_2(z)$$

Also, we know that

$$\alpha^n \cdot u(n) \xrightarrow{z} \frac{1}{1 - \alpha z^{-1}} \quad \text{ROC : } |z| > |\alpha| \quad \dots(iii)$$

Substituting $\alpha = 2$ and $\alpha = 3$ in equation (iii), we get

$$x_1(n) = 2^n \cdot u(n) \xrightarrow{z} X_1(z)$$

or

$$x_1(n) = \frac{1}{1 - 2z^{-1}}$$

$$\text{ROC : } |z| > 2$$

and

$$x_2(n) = 3^n \cdot u[n] \xrightarrow{z} X_2(z)$$

or

$$x_2(n) = \frac{1}{1 - 3z^{-1}}$$

$$\text{ROC : } |z| > 3$$

It may be observed that the intersection of the ROC of $X_1(z)$ and $X_2(z)$ is $|z| > 3$.

Therefore the overall z-transform $X(z)$ will be

$$X(z) = \frac{3}{1 - 2z^{-1}} - \frac{4}{1 - 3z^{-1}} \quad \text{ROC : } |z| > 3$$

Example 7.37. Evaluate the z-transform of the following signal :

$$x(n) = \left(\frac{1}{2}\right)^n \cdot u(n)$$

Solution : The given signal is

$$x(n) = \left(\frac{1}{2}\right)^n \cdot u(n) \quad \dots(i)$$

We know that z-transform of a signal $x[n]$ is expressed as

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Putting the value of $x[n]$, we get

$$\text{or} \quad X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cdot u(n) z^{-n} \quad [\because u(n) \text{ exists only for positive } n]$$

$$\text{or} \quad X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cdot 1 \cdot z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cdot z^{-n}$$

or
$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n = 1 + \left(\frac{1}{2} z^{-1}\right) + \left(\frac{1}{2} z^{-1}\right)^2 + \left(\frac{1}{2} z^{-1}\right)^3 + \dots \dots \dots (ii)$$

The above expression is a well-known geometric progression of the form

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \quad \text{if } |x| < 1$$

Therefore, the equation (ii) becomes

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} \quad \text{for } \left|\frac{1}{2} z^{-1}\right| < 1 \text{ or } |z| > \frac{1}{2}$$

Hence

$$X(z) = \frac{z}{z - \frac{1}{2}}$$

$$\text{ROC : } |z| > \frac{1}{2}$$

Example 7.38. Find the z -transform of the signal

$$x(n) = -\alpha^n u(-n-1) = \begin{cases} 0 & n \geq 0 \\ -\alpha^n & n \leq -1 \end{cases}$$

Solution : The given signal is

$$x(n) = -\alpha^n u(-n-1) = \begin{cases} 0 & n \geq 0 \\ -\alpha^n & n \leq -1 \end{cases} \quad \dots(i)$$

We know that z -transform is expressed as

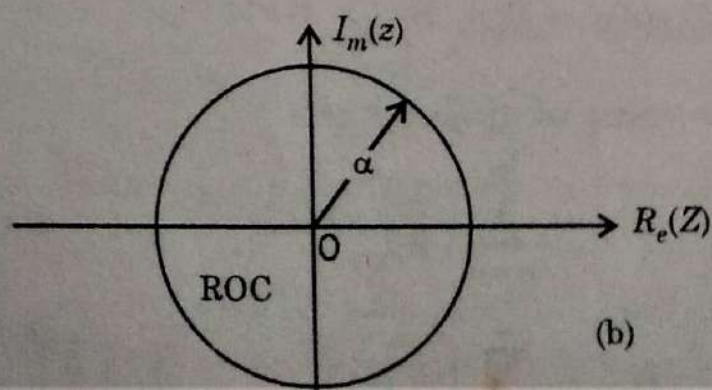
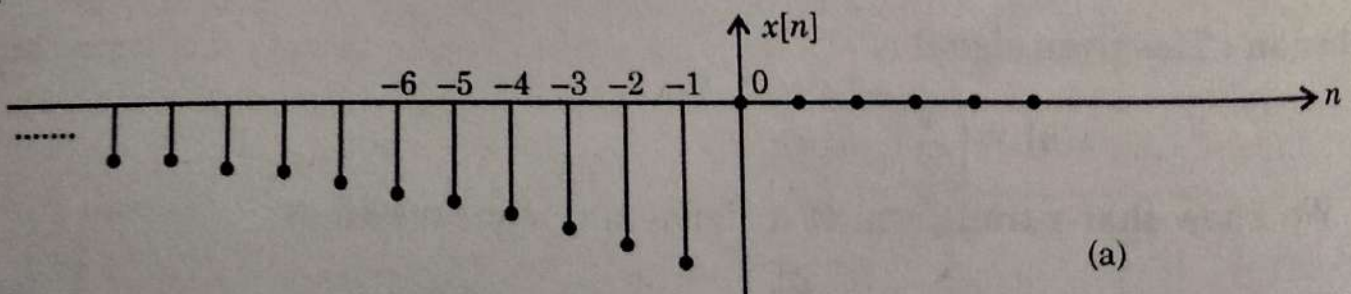
$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

According to equation (i)
$$X(z) = \sum_{n=-\infty}^{\infty} (-\alpha^n) z^{-n}$$

Putting $m = -n$
$$X(z) = -\sum_{m=1}^{\infty} (-\alpha^{-1} z)^m$$

Using the formula $x + x^2 + x^3 + \dots = \frac{x}{1-x}$ for $|x| < 1$

Therefore, We have
$$X(z) = -\frac{\alpha^{-1} z}{1 - \alpha^{-1} z}$$



$$X(z) = -\frac{1}{1 - \alpha z^{-1}} \quad \text{for } |\alpha^{-1}z| < 1 \text{ or } |z| < |\alpha|$$

or

$$X(z) = -\frac{1}{1 - \alpha z^{-1}} \quad \text{ROC : } |z| < |\alpha|$$

Example 7.39. Find the z-transform of the signal

$$x(n) = na^n \cdot u(n)$$

Solution : The given signal is

$$x(n) = na^n \cdot u(n)$$

...(i)

The signal in equation (i) may be expressed as $nx_1(n)$ where

$$x_1(n) = a^n \cdot u(n)$$

But, we know that

$$x_1(n) = a^n \cdot u(n) \xrightarrow{z} X_1(z) = \frac{1}{1 - az^{-1}} \quad \text{ROC : } |z| > |a|$$

Also the differentiation property states that

$$\text{If } x(n) \xrightarrow{z} X(z)$$

$$\text{then } nx(n) \xrightarrow{z} -z \cdot \frac{dX(z)}{dz}$$

$$\text{Therefore, } na^n \cdot u(n) \xrightarrow{z} X(z) = -z \cdot \frac{dX_1(z)}{dz} = \frac{az^{-1}}{(1 - az^{-1})^2} \quad \text{ROC : } |z| > |a|$$

Example 7.40. Find the discrete-time signal $x[n]$ whose z-transform is given as

$$X(z) = \log(1 + az^{-1}) \quad |z| > |a|$$

Solution : The given expression is

$$X(z) = \log(1 + az^{-1}) \quad |z| > |a|$$

Taking the differentiation, we get

$$\frac{dX(z)}{dz} = \frac{-az^{-2}}{1 + az^{-1}}$$

$$\text{Hence } -z \frac{dX(z)}{dz} = az^{-1} \left[\frac{1}{1 - (-a)z^{-1}} \right] \quad |z| > |a| \quad \dots(i)$$

In equation (i), the inverse z-transform of the term in brackets is $(-a)^n$.

The multiplication by z^{-1} means a time-delay of one sample (i.e. time-shifting property) which results in $(-a)^{n-1} \cdot u[n-1]$.

In last, from the differentiation property, we get

$$nx(n) = a(-a)^{n-1} \cdot u(n-1)$$

or

$$x(n) = (-1)^{n+1} \cdot \frac{a^n}{n} \cdot u(n-1) \quad \text{Ans.}$$

Example 7.41. Find the pole-zero plot for the discrete-time causal signal

$$x(n) = a^n \cdot u(n) \quad a > 0$$

Solution : We know that the z-transform is expressed as

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\text{or } X(z) = \sum_{n=0}^{\infty} a^n \cdot u(n) z^{-n} = \sum_{n=0}^{\infty} a^n \cdot 1 \cdot z^{-n} \quad [\because u[n] = 1 \text{ for } n > 0]$$

$$\text{or } X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$\text{or } X(z) = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \dots = \frac{1}{1 - \frac{a}{z}} \quad \left|\frac{a}{z}\right| < 1 \text{ or } |z| > |a|$$

$$\text{or } X(z) = \frac{z}{z-a} \quad \text{ROC : } |z| > |a| \quad \dots(i)$$

From equation (i), it is clear that $X(z)$ has one zero at $z_1 = 0$ and one pole at $p_1 = a$.

Figure 7.12. shows the pole-zero plot.

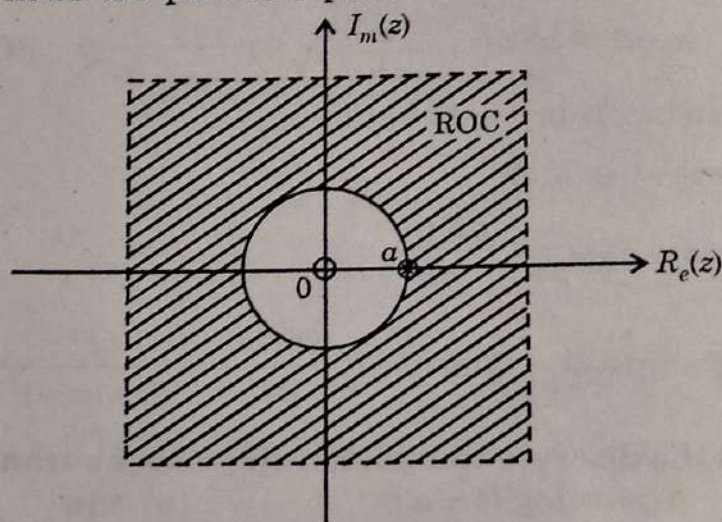


Fig. 7.12. Pole-zero plot for the causal exponential signal $x[n] = a^n \cdot u[n]$.

It may be noted that the pole at $p_1 = a$ is not included in the region of the convergence (ROC) since the z -transform does not converge at a pole.

Example 7.42. Find the z -transform of the following sequences :

(i) $x_1(n) = \{1, 2, 3, 4, 5, 0, 7\}$

(ii) $x_2(n) = \{1, 2, 3, 4, 5, 0, 7\}$

↑

Solution: (i) Let us consider the first sequence $x_1[n]$, Its samples are

$$x_1(0) = 1 \quad x_1(4) = 5$$

$$x_1(1) = 2 \quad x_1(5) = 0$$

$$x_1(2) = 3 \quad x_1(6) = 7$$

$$x_1(3) = 4$$

The z -transform of this sequence may be obtained with the help of the expression

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n}$$

Here, $x_1[n]$ has the values from $n = 0$ to $n = 6$.

Thus, the above equation becomes

$$X_1(z) = \sum_{n=0}^6 x_1(n) z^{-n}$$

Substituting the values of $x_1(n)$ and expanding, we get

$$X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4} + 0 \cdot z^{-5} + 7z^{-6}$$

$$X_1(z) = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \frac{7}{z^6}$$

Now, let us determine the region of convergence for this z -transform. $X_1(z)$ has finite values except at $z = 0$. At $z = 0$, $X_1(z)$ becomes infinite.

Hence $X_1(z)$ is convergent for all values of z except at $z = 0$.

Therefore, the region of convergence (ROC) will be:

ROC : Entire z -plane except $z = 0$.

(ii) Let us consider the second sequence $x_2[n]$. Its samples are

$$\begin{aligned} x_2(-3) &= 1 & x_2(1) &= 5 \\ x_2(-2) &= 2 & x_2(2) &= 0 \\ x_2(-1) &= 3 & x_2(3) &= 7 \\ x_2(0) &= 4 \end{aligned}$$

Here, $x_2(n)$ has the values from $n = -3$ to $n = 3$.

Thus, z -transform will be expressed as

$$X_2(z) = \sum_{n=-3}^3 x_2(n) z^{-n}$$

Substituting the values of $x_2[n]$ in above equation and expanding, we get

$$X_2(z) = 1 \cdot z^3 + 2z^2 + 3z^1 + 4z^0 + 5z^{-1} + 0 \cdot z^{-2} + 7z^{-3}$$

$$X_2(z) = z^3 + 2z^2 + 3z + 4 + 5z^{-1} + 7z^{-3}$$

$$X_2(z) = z^3 + 2z^2 + 3z + 4 + \frac{5}{z} + \frac{7}{z^3}$$

Now let us determine the region of convergence (ROC) for this z -transform. Here, $X_2(z)$ is infinite for $z = 0$ and $z = \infty$. The z -transform expressed by above equation is not convergent at $z = 0$ and $z = \infty$. Hence the region of convergence will be :

ROC : Entire z -plane except $z = 0$ and $z = \infty$. **Ans.**

Example 7.43. Find the z -transform and ROC of the following sequence :

$$x(n) = \left(-\frac{1}{3}\right)^n \cdot u(n) - \left(\frac{1}{2}\right)^n \cdot u(-n-1)$$

Solution : The z -transform of $x[n]$ will be given as

$$Z\{x(n)\} = X(z) = Z\left\{\left(-\frac{1}{3}\right)^n \cdot u(n) - \left(\frac{1}{2}\right)^n \cdot u(-n-1)\right\}$$

since z -transform satisfies linearity property, therefore we may write above equation as

$$X(z) = Z\left\{\left(-\frac{1}{3}\right)^n \cdot u(n)\right\} + Z\left\{-\left(\frac{1}{2}\right)^n \cdot u(-n-1)\right\} \quad (i)$$

Here let us use the standard results derived earlier i.e.,

$$Z\{a^n, u(n)\} = \frac{1}{1 - az^{-1}}, \text{ ROC: } |z| > \frac{1}{3}$$

Therefore,
$$Z\left\{\left(-\frac{1}{3}\right)^n \cdot u(n)\right\} = \frac{1}{1 + \frac{1}{3}z^{-1}}, \text{ ROC: } |z| > \frac{1}{3} \quad \dots(ii)$$

Also,
$$Z\{-a^n \cdot u(-n-1)\} = \frac{1}{1 - az^{-1}} \text{ ROC: } |z| < |a|$$

Therefore,
$$Z\left\{-\left(\frac{1}{2}\right)^n \cdot u(-n-1)\right\} = \frac{1}{1 - \frac{1}{2}z^{-1}} \text{ ROC: } |z| < \frac{1}{2} \quad (iii)$$

Now, using equations (ii) & (iii), the equation (i) may be written as

$$X(z) = \frac{1}{1 + \frac{1}{3}z^{-1}} + \frac{1}{1 - \frac{1}{2}z^{-1}}; \text{ ROC: } |z| > \frac{1}{3} \text{ and } |z| < \frac{1}{2}$$

The ROC of $|z| > \frac{1}{3}$ and $|z| < \frac{1}{2}$ may

also be written as $\frac{1}{3} < |z| < \frac{1}{2}$. Hence, ROC is the region between the circles of radius of $\frac{1}{3}$ and $\frac{1}{2}$. Figure 7.14 shows this ROC

Therefore, the ROC is the intersection or simply the overlap of ROC of individual functions. It is annular region

for $\frac{1}{3} < |z| < \frac{1}{2}$ looking like a disk. **Ans.**

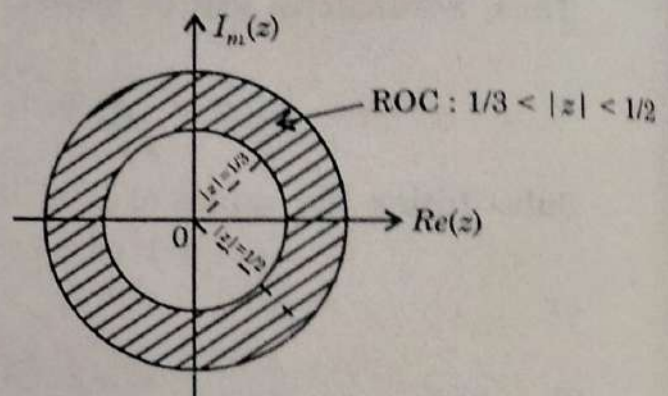


Fig. 7.13. ROC of the function of the example 7.44.

Example 7.45. Solve the difference equation of a causal discrete-time LTI system which is expressed as under :

$$y(n) + 3y(n-1) = x(n)$$

Assume that the system is initially relaxed.

Solution : We know that the transfer function $H(z)$ can be determined by taking unilateral z-transform of both sides of equation (i)

$$y(n) + 3y(n-1) = x(n) \quad \dots(i)$$

$$Z[y(n) + 3y(n-1)] = Z[x(n)]$$

or
$$Y(z) + 3z^{-1}Y(z) = X(z) \quad \dots(ii)$$

The system is initially relaxed, i.e., all the initial conditions are zero for this system,

Using equation (ii), we get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1 + 3z^{-1})} \quad \dots(iii)$$

(i) This is the transfer function of the causal LTI system described by equation

Now, assume that input is given by
 $x(n) = u(n)$
 Taking z-transform of equation (iv), we obtain

...(iv)

$$X(z) = U(z) = \frac{1}{1-z^{-1}}$$

Further, we know that

...(v)

$$Y(z) = H(z) X(z) = \frac{1}{(1+3z^{-1})(1-z^{-1})}$$

Now, Using partial fraction expression, we get

$$Y(z) = \frac{1}{(1+3z^{-1})(1-z^{-1})} = \frac{\alpha_1}{1+3z^{-1}} + \frac{\alpha_2}{(1-z^{-1})}$$

or

$$Y(z) = \frac{3/4}{(1+3z^{-1})} + \frac{1/4}{(1-z^{-1})}$$

...(vi)

Taking inverse unilateral z-transform of both sides of equation (iv), we obtain

$$Z^{-1}[Y(z)] = Z^{-1}\left[\frac{3/4}{1+3z^{-1}} + \frac{1/4}{1-z^{-1}}\right] = \frac{3}{4} Z^{-1}\left[\frac{1}{1+3z^{-1}}\right] + \frac{1}{4} Z^{-1}\left[\frac{1}{1-z^{-1}}\right]$$

or

$$y(n) = \frac{3}{4}(-3)^n u(n) + \frac{1}{4}(1)^n u(n)$$

$$y(n) = \left[\frac{1}{4} + \frac{3}{4}(-3)^n\right] u(n)$$

This is the required solution for given difference equation. Ans.

Example 7.45. Find the step response of a system which is expressed by

$$y(n) = Ay(n-1) + x(n), \quad -1 < A < 1$$

When the initial condition is $y(-1) = 1$.

Solution: By taking one-sided z-transform of both sides of the given difference equation, we get

$$Y(z) = A[z^{-1} Y(z) + y(-1)] + X(z) = A[z^{-1} Y(z) + 1] + X(z)$$

$$\text{or } Y(z) [1 - Az^{-1}] = A + X(z)$$

...(i)

However, for step response, $x(n) = u(n)$

$$\text{Then, we have } X(z) = Z[u(n)] = \frac{1}{1-z^{-1}}$$

...(ii)

Now, substituting equation (ii) in equation (i), we obtain

$$Y(z) [1 - Az^{-1}] = A + \frac{1}{1-z^{-1}}$$

or

$$Y(z) = \frac{A}{1-Az^{-1}} + \frac{1}{(1-z^{-1})(1-Az^{-1})}$$

...(iii)

Using partial fraction expansion, we get

$$Y(z) = \frac{A}{1-Az^{-1}} + \frac{B_1}{1-z^{-1}} + \frac{B_2}{1-Az^{-1}}$$

...(iv)

$$1 - z^{-1} = 0$$

$$B_1 = \frac{1}{1 - Az^{-1}} = \frac{1}{1 - A}$$

or

$$z = 1$$

$$1 - Az^{-1} = 0$$

$$B_2 = \frac{1}{1 - z^{-1}} = \frac{1}{1 - \frac{1}{A}} = \frac{A}{A - 1} = -\frac{A}{1 - A}$$

or

$$z^{-1} = 1/A$$

Substituting the value of B_1 and B_2 in equation (iv), we get

$$y(z) = \frac{A}{1 - Az^{-1}} + \left(\frac{1}{1 - A} \right) + \left(\frac{-A}{1 - A} \right) \frac{1}{1 - Az^{-1}} \quad \dots(v)$$

Taking the inverse z-transform of equation (v), we get

$$y(n) = A^{n+1}u(n) + \frac{1}{(1 - A)}u(n) + \frac{-1}{(1 - A)}A^{n+1}u(n)$$

or

$$y(n) = A^{n+1}u(n) + \left[\frac{1 - A^{n+1}}{1 - A} \right] u(n)$$

or

$$y(n) = \frac{1}{1 - A} [1 - A^{n+2}] u(n) \quad \text{Ans.}$$

Example 7.46. Solve the following difference equation by using z-transform method

$$x(n + 2) + 3x(n + 1) + 2x(n) = 0$$

Given that initial conditions are $x(0) = 0$ and $x(1) = 1$.

Solution: Given difference equation is

$$x(n + 2) + 3x(n + 1) + 2x(n) = 0 \quad \dots(i)$$

Taking the z-transform of both sides of the above equation, we obtain

$$[z^2X(z) - z^2x(0) - zx(1)] + 3[zX(z) - zx(0)] + 2[X(z)] = 0$$

$$\text{or } [z^2X(z) - z^2(0) - z(1)] + 3[zX(z) - z(0)] + 2[X(z)] = 0$$

$$\text{or } [z^2X(z) - z] + 3zX(z) + 2X(z) = 0$$

$$\text{or } X(z) [z^2 + 3z + 2] = z$$

or

$$X(z) = \frac{z}{z^2 + 3z + 2} \quad \dots(ii)$$

Let us now take the inverse z-transform of above z-transform by partial fraction expansion method, i.e.,

$$X(z) = \frac{z}{z^2 + 3z + 2} = \left\{ \frac{A}{(z + 1)} + \frac{B}{(z + 2)} \right\}$$

or

$$X(z) = \frac{z}{z + 1} - \frac{z}{z + 2} = \frac{1}{1 + z^{-1}} - \frac{1}{1 + 2z^{-1}} \quad \dots(iii)$$

or

$$X(z) = \frac{1}{1 + z^{-1}} - \frac{1}{1 + 2z^{-1}} \quad \dots(iv)$$

Taking the inverse z-transform of equation (iv), we obtain

$$x(n) = Z^{-1}\left[\frac{1}{1+z^{-1}}\right] - Z^{-1}\left[\frac{1}{1+2z^{-1}}\right]$$

or

$$x(n) = (-1)^n u(n) - (-2)^n u(n) \quad \text{Ans.}$$

Example 7.47. Determine the response of the following system:
 $x(n+2] - 3x(n+1] + 2x(n) = \delta(n)$

Assume that all the initial conditions are zero.

Solution: Given system is

$$x(n+2] - 3x(n+1] + 2x(n) = \delta(n) \quad \dots(i)$$

Taking one-sided z-transform of both sides of above equation, we obtain

$$z^2 X(z) - 3z^1 X(z) + 2X(z) = 1$$

or

$$X(z) [z^2 - 3z + 2] = 1$$

or

$$X(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-2)(z-1)}$$

$$= \frac{A}{z-2} + \frac{B}{z-1} = \frac{1}{z-2} - \frac{1}{z-1} \quad \dots(ii)$$

(By partial fraction expansion)

Taking inverse z-transform of both sides of equation (ii), we have

$$x(n) = Z^{-1}\left[\frac{1}{z-2}\right] - Z^{-1}\left[\frac{1}{z-1}\right] = Z^{-1}\left[\frac{z^{-1}}{1-2z^{-1}}\right] - Z^{-1}\left[\frac{z^{-1}}{1-z^{-1}}\right]$$

$$x(n) = (2)^{n-1} - (1)^{n-1}$$

or

$$x(n) = -1 - (2)^{n-1} \quad \text{Ans.}$$

Example 7.48. Determine the z-transform of the following discrete-time signals. Also find the ROC for each of the following cases:

$$(i) \ x(n) = 2^n u(n) + 3\left(\frac{1}{2}\right)^n u(n) \quad (ii) \ x(n) = 3\left(-\frac{1}{2}\right)^n u(n) - 2(3)^n u(-n-1)$$

Solution: (i) Given signal is

$$x(n) = 2^n u(n) + 3\left(\frac{1}{2}\right)^n u(n)$$

We know that Two-sided (bilateral) z-transform of $x(n)$ is defined as

$$X(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \left[2^n u(n) + 3\left(\frac{1}{2}\right)^n u(n) \right] z^{-n}$$

$$\text{or} \quad X(z) = \sum_{n=-\infty}^{\infty} 2^n u(n) z^{-n} + 3 \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n) z^{-n}$$

$$\text{or} \quad X(z) = \sum_{n=0}^{\infty} 2^n (1) z^{-n} + 3 \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n (1) z^{-n} \quad \left[\text{Since } u(n) = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \right]$$

$$\text{or } X(z) = \sum_{n=0}^{\infty} (3z^{-1})^n + 3 \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n$$

$$\text{or } X(z) = \underbrace{\frac{1}{(1-3z^{-1})}}_{\text{I-part}} + 3 \underbrace{\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)}}_{\text{II-part}}$$

$$\text{or } X(z) = \frac{\left(1-\frac{1}{2}z^{-1}\right) + 3(1-2z^{-1})}{(1-2z^{-1})\left(1-\frac{1}{2}z^{-1}\right)} = \frac{4-13z^{-1}}{(1-2z^{-1})\left(1-\frac{1}{2}z^{-1}\right)} \quad \dots(i)$$

Now, ROC of I-part : $|1-2z^{-1}| > 0$

$$\text{or } |2z^{-1}| < 1 \text{ or } |z^{-1}| < \frac{1}{2}$$

$$\text{or } \left|\frac{1}{z}\right| < \frac{1}{2} \text{ or } |z| > 2 \quad \dots(ii)$$

$$\text{ROC of II-part : } \left|1-\frac{1}{2}z^{-1}\right| > 0 \text{ or } \left|\frac{1}{2}z^{-1}\right| < 1$$

$$\text{or } |z^{-1}| < 2 \text{ or } \left|\frac{1}{z}\right| < 2 \text{ or } |z| > \frac{1}{2} \quad \dots(iii)$$

Therefore, ROC of $X(z)$, will be

$$\text{ROC : } \{|z| > 2\} \cap \left\{|z| > \frac{1}{2}\right\}$$

Thus, we have

$$\text{ROC : } \{|z| > 2\}$$

$$(ii) \text{ Given signal is } x(n) = 3\left(-\frac{1}{2}\right)^n u(n) - 2(3)^n u(-n-1)$$

We know that Two-sided z-transform of $x(n)$ is given by

$$\begin{aligned} X(z) = Z[x(n)] &= \sum_{n=-\infty}^{\infty} x(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left[3\left(-\frac{1}{2}\right)^n u(n) - 2(3)^n u(-n-1) \right] z^{-n} \end{aligned}$$

$$\text{or } X(z) = 3 \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^n u(n) z^{-n} - 2 \sum_{n=-\infty}^{\infty} (3)^n u(-n-1) z^{-n}$$

$$\text{or } X(z) = 3 \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (1) z^{-n} - 2 \sum_{n=-\infty}^{-1} (3)^n (1) z^{-n}$$

$$\left[\text{Since } u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \text{ and } u(-n-1) = \begin{cases} 1, & (-n-1) \geq 0 \rightarrow n \leq -1 \\ 0, & (-n-1) < 0 \rightarrow n > -1 \end{cases} \right]$$

or

$$X(z) = 3 \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n z^{-n} - 2 \sum_{n=-\infty}^{-1} (3)^n z^{-n}$$

or

$$\begin{aligned} X(z) &= 3 \sum_{n=0}^{\infty} \left(-\frac{1}{2} z^{-1}\right)^n - 2 \sum_{n=-\infty}^{-1} (3z^{-1})^n \\ &= 3 \sum_{n=0}^{\infty} \left(-\frac{1}{2} z^{-1}\right)^n - 2 \sum_{n=1}^{\infty} (3z^{-1})^{-n} \end{aligned}$$

or

$$\begin{aligned} X(z) &= 3 \frac{1}{1 - \left(-\frac{1}{2} z^{-1}\right)} - 2 \frac{(3z^{-1})^{-1}}{1 - (3z^{-1})^{-1}} \\ &= \frac{3}{1 + \frac{1}{2} z^{-1}} - \frac{2/3z^{-1}}{1 - \frac{1}{3z^{-1}}} = \frac{3}{1 + \frac{1}{2} z^{-1}} - \frac{2}{3z^{-1} - 1} \end{aligned}$$

or

$$X(z) = \underbrace{\frac{3}{1 + \frac{1}{2} z^{-1}}}_{\text{I-part}} - \underbrace{\frac{2}{1 - 3z^{-1}}}_{\text{II-part}}$$

...(iv)

or

$$X(z) = \frac{2(1 - 3z^{-1}) + 2\left(1 + \frac{1}{2} z^{-1}\right)}{\left(1 + \frac{1}{2} z^{-1}\right)(1 - 3z^{-1})} = \frac{4 - 5z^{-1}}{\left(1 + \frac{1}{2} z^{-1}\right)(1 - 3z^{-1})}$$

Now, ROC of I-part : $\left|1 + \frac{1}{2} z^{-1}\right| < 0$ or $\left|\frac{1}{2} z^{-1}\right| < 1$

or $|z^{-1}| < 2$ or $|z| > \frac{1}{2}$... (v)

Also, ROC of II-part : $|1 - 3z^{-1}| > 0$ or $|3z^{-1}| > 1$ or $|z| < 3$... (vi)

Therefore, ROC of $X(z)$ ROC : $\left\{|z| > \frac{1}{2}\right\} \cap \{|z| < 3\} = \frac{1}{2} < |z| < 3$ Ans.

Example 7.49. Two discrete-time signals are given as under :

$$x_1(n) = \left(\frac{1}{3}\right)^n u(n) \text{ and}$$

$$x_2(n) = \left(\frac{1}{5}\right)^n u(n)$$

Convolution of $x_1(n)$ and $x_2(n)$ is given by

$$x(n) = x_1(n) \otimes x_2(n)$$

- (i) Find $X(z)$ using convolution property of z-transform.
 (ii) Find $x(n)$ by taking inverse z-transforms of $X(z)$ by using partial fraction expansion method.

Solution : (a) Given that :

$$x(n) = x_1(n) \otimes x_2(n)$$

From convolution property of z-transform, we know

$$x(n) = x_1(n) \otimes x_2(n) \xrightarrow{Z} X(z) = X_1(z) X_2(z)$$

$$\text{or } X(z) = X_1(z) X_2(z) \quad \dots(i)$$

$$\text{where } X_1(z) = Z[x_1(n)] = Z\left[\left(\frac{1}{3}\right)^n u(n)\right] = \frac{1}{1 - \frac{1}{3}z^{-1}}, \text{ROC: } |z| > \frac{1}{3} \quad \dots(ii)$$

$$\text{and } X_2(z) = Z[x_2(n)] = Z\left[\left(\frac{1}{5}\right)^n u(n)\right] = \frac{1}{1 - \frac{1}{5}z^{-1}}, \text{ROC: } |z| > \frac{1}{5} \quad \dots(iii)$$

Substituting equations (ii) and (iii) in equation (i), we obtain

$$X(z) = X_1(z) X_2(z) = \frac{1}{\left(1 - \frac{1}{3}z^{-1}\right)} \frac{1}{\left(1 - \frac{1}{5}z^{-1}\right)} = \frac{1}{\left(1 - \frac{1}{3}z^{-1}\right)} \frac{1}{\left(1 - \frac{1}{5}z^{-1}\right)} \quad \dots(iv)$$

$$\text{ROC: } \left\{|z| > \frac{1}{3}\right\} \cap \left\{|z| > \frac{1}{5}\right\} = |z| > \frac{1}{3}$$

(ii) Using equation (iv), we have

$$X(z) = \frac{z^2}{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{5}\right)}$$

$$\text{or } \frac{X(z)}{z} = \frac{z}{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{5}\right)} = \frac{A_1}{\left(z - \frac{1}{3}\right)} + \frac{A_2}{\left(z - \frac{1}{5}\right)} \quad (\text{using partial fraction expansion})$$

$$\text{or } \frac{X(z)}{z} = \frac{5/2}{\left(z - \frac{1}{3}\right)} + \frac{-3/2}{\left(z - \frac{1}{5}\right)} = \frac{5/2z}{\left(z - \frac{1}{3}\right)} - \frac{3/2z}{\left(z - \frac{1}{5}\right)}$$

$$\frac{X(z)}{z} = \frac{5/2}{\left(1 - \frac{1}{3}z^{-1}\right)} - \frac{3/2}{\left(1 - \frac{1}{5}z^{-1}\right)}$$

Note that since both the poles are surrounded by ROC, therefore, they give only positive time sequence terms.

Now, taking inverse z-transform of both sides of equation (iii), we have

$$x(n) = Z^{-1}[X(z)] = Z^{-1}\left[\frac{5/2}{\left(1 - \frac{1}{3}z^{-1}\right)} - \frac{3/2}{\left(1 - \frac{1}{5}z^{-1}\right)}\right]$$

$$\text{or } x(n) = \frac{5}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{3}{2}\left(\frac{1}{5}\right)^n u(n) \quad \text{Ans.}$$

Example 7.50. Find the z-transform of the following sequence :

$$x(n) = \begin{cases} 2^n, & n < 0 \\ (1/2)^n, & n = 0, 2, 4 \\ (1/3)^n, & n = 1, 3, 5 \end{cases}$$

$$x(n) = \begin{cases} 2^n, & n < 0 \\ (1/2)^n, & n = 0, 2, 4 \\ (1/3)^n, & n = 1, 3, 5 \end{cases} \quad \dots(i)$$

We know that z-transform of $x(n)$ is given by

$$X(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \dots(ii)$$

Substituting equation (i) in equation (ii), we obtain

$$X(z) = \sum_{n=-\infty}^{-1} 2^n z^{-n} + \sum_{\substack{n=0 \\ (n-\text{even})}}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{\substack{n=0 \\ (n-\text{odd})}}^{\infty} \left(\frac{1}{3}\right)^n z^{-n}$$

$$\text{or } X(z) = \sum_{m=1}^{\infty} 2^{-m} z^m + \sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^{2p} z^{-2p} + \sum_{q=0}^{\infty} \left(\frac{1}{3}\right)^{2q-1} z^{-(2q+1)}$$

where $m = -n$, $p = \frac{n}{2}$ and $q = \left(\frac{n-1}{2}\right)$

Thus, we have

$$X(z) = X_1(z) + X_2(z) + X_3(z)$$

$$\text{or } X(z) = \frac{z/2}{\left(1 - \frac{z}{2}\right)} + \frac{z^2}{\left(z^2 - \frac{1}{4}\right)} + \frac{z/3}{\left(z^2 - \frac{1}{9}\right)} \quad \dots(iii)$$

Now, ROC for $X_1(z) : |z| < 2$

ROC for $X_2(z) : |z| > \frac{1}{2}$

ROC for $X_3(z) : |z| > \frac{1}{3}$

Therefore, ROC for $X(z) : \{|z| < 2\} \cap \left\{|z| > \frac{1}{2}\right\} \cap \left\{|z| > \frac{1}{3}\right\}$

$$= \frac{1}{2} < |z| < 2 \quad \text{Ans.}$$

Example 7.51. Determine the convolution $x(n)$ of the following two signals

$$x_1(n) = \{4, -2, 1\}$$

↑

$$x_2(n) = \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & \text{elsewhere} \end{cases}$$

Solution : Here, we have

$$X_1(z) = Z[x_1(n)] = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n}$$