Correlation

Correlation of two signals is a measure of similarity between those signals.

Continuous Convolution:

$R(\tau) = x_1 * x_2 = \int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt$=$\int_{-\infty}^{\infty} x_1(t+\tau)x_2(t)dt$

$R_{xy}(\tau) = \int_{\infty}^{\infty} x(t)y^*(t-\tau)dt = \int_{\infty}^{\infty} x(t+\tau)y^*(t)dt$, where y^* is the complex conjugate of $y(t)$.

If $x(t)$ and $y(t)$ are real,

Then $y^*(t) = \ y(t)$

$x^*(t) = \ x(t)$

Then $R_{xy}(\tau)$ can be written as,

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y(t-\tau)dt = \int_{-\infty}^{\infty} x(t+\tau)y(t)dt$$

$$R_{xy}(-\tau) = \int_{-\infty}^{\infty} x(t)y(t+\tau)dt = R_{yx}(\tau)$$

Discrete Convolution:

$R_{xy}(\tau) = x_1 * x_2 = \sum_{k=-\infty}^{\infty} x_1(n)\, x_2(n-k)$= $\sum_{k=-\infty}^{\infty} x_1(n+k)\, x_2(n)$

Correlation is of two types:

(1) Auto correlation : measure of similarity between a signal and its shifted version ($R_{11}(\tau)$)

(2) Cross Correlation : measure of similarity between two different signals ($R_{12}(\tau) or\ R_{21}(\tau)$)

Auto Correlation:

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt$$

Power signal:

$$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)y^*(t-\tau)dt = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t+\tau)y^*(t)dt$$

$R_{xy}(-\tau) = R_{yx}(\tau)$

Auto correlation of Power signal :

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t-\tau)dt$$

Energy Signal:

$$R_{xy}(\tau) = \int_{-T/2}^{T/2} x(t)y^*(t-\tau)dt = \int_{-T/2}^{T/2} x(t+\tau)y^*(t)dt$$

Auto correlation of Energy signal (at $\tau = 0$):

$$R_{xx}(\tau) = \int_{-T/2}^{T/2} x(t)x(t-\tau)dt$$

Properties of correlation and auto correlation:

1. $R_{xy}(-\tau) = R_{yx}(\tau)$
   $R_{xx}(-\tau) = R_{xx}(\tau)$, this is an even function of τ.
   $R_{xx}(\tau) = {R_{xx}}^*(-\tau)$, Auto correlation exhibits conjugate symmetry

2. For energy function:

   $R_{xx}(\tau) = \int_{-T/2}^{T/2} |x(t)|^2 dt = Energy\ of\ the\ signal$, at $\tau = 0$

   For power function:
   $R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = Power\ of\ the\ signal$, at $\tau = 0$
3. $|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0)R_{yy}(0)} = \sqrt{E_x E_y}$

   $|R_{xx}(\tau)| \leq \sqrt{R_{xx}(0)R_{xx}(0)} = E_x$ for energy signal

   Auto correlation function is maximum at $\tau = 0$.
4. If $x(t)$ is periodic with period T, then its Auto correlation function will also be periodic with same period 'T'.
   $x(t+T) = x(t)$
   $R_{xx}(\tau + T) = R_{xx}(\tau)$
5. Auto correlation function and energy spectral densities are Fourier Transform pairs i.e. $F.T[R_{xx}(\tau)] = \psi(\omega)$

6. $\psi(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau$

Problem:

Q1. Find the auto correlation function of $\boldsymbol{x(t) = e^{-3t}u(t)}$

Solution:

$$R(\tau) = \int_{\infty}^{\infty} x_1(t)x_2(t-\tau)dt$$

$$R(\tau) = x(\tau) * x(-\tau)$$