FT of Periodic signal

Let $x(t)$ be a periodic signal and $X(\omega)$ is the Fourier Transform of $x(t)$.

Then the Fourier Transform $X(\omega)$ is defined as,

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} C_n\, \delta(\omega - n\omega_0)$$

Proof:

$$1 \overset{FT}{\leftrightarrow} 2\pi\delta(\omega)$$

Amplitude Scaling

$$C_n 1 \overset{FT}{\leftrightarrow} 2\pi C_n \delta(\omega)$$

Frequency Shifting

$$C_n e^{jn\omega_0 t} \overset{FT}{\leftrightarrow} 2\pi C_n \delta(\omega - n\omega_0)$$

Applying the value in FT of aperiodic signal, we will get

$$X(\omega) = \sum_{n=-\infty}^{\infty} 2\pi C_n \delta(\omega - n\omega_0) = 2\pi \sum_{n=-\infty}^{\infty} C_n \delta(\omega - n\omega_0)$$

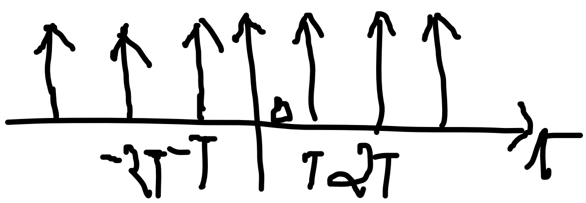
$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_o t} dt$$

Problem: Find the Fourier Transform of a train of Impulses.

Solution:

The train of impulses is a periodic signal with period T and can be written as,

$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$



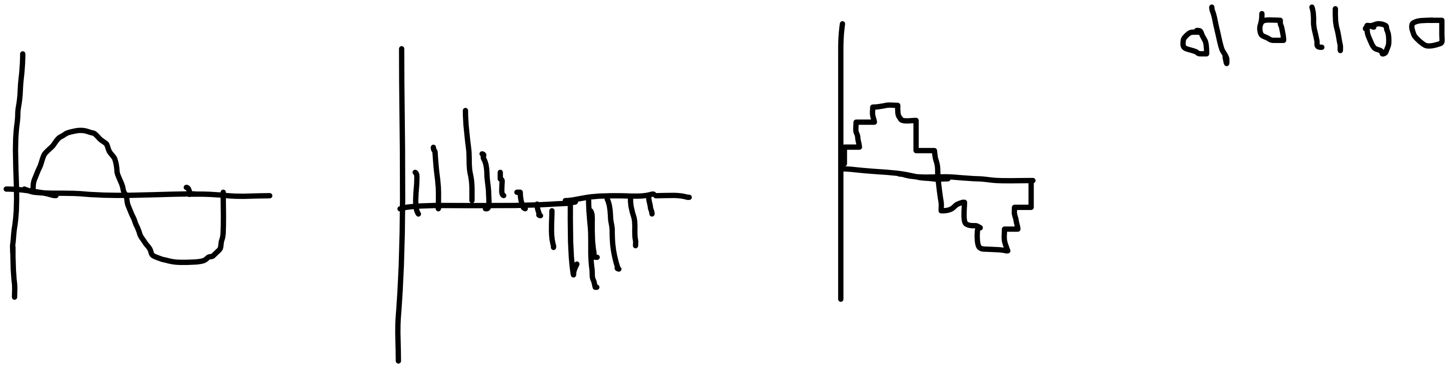
$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_o t} dt = \frac{1}{T}$$

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} \frac{1}{T}\delta(\omega - n\omega_0) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

<u>Sampling</u>

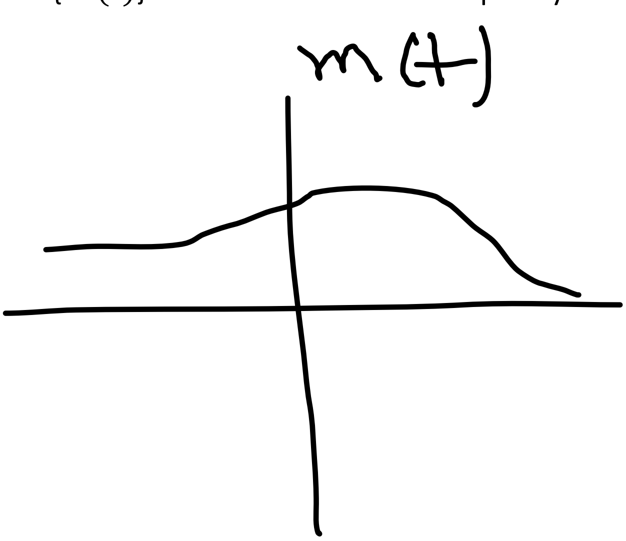
Steps for Analog to Digital Conversion:

$x(t)$ → Sampler → $x(nT)$ → Quantizer → $x_q[n]$ → Coder → Binary numbers



Sampling : Process of converting a continuous time and continuous amplitude signal to discrete time and continuous amplitude signal.

Assume a bandlimited signal, $m(t)$, as the message signal. The frequency components of the FT{ $m(t)$} will exist in a band of frequency.



Sampler: It's a multiplier that multiplies a train of impulses with the bandlimited signal. The fundamental time period or sampling period or sampling interval of the impulse train is T_s.

$$\omega_s = \frac{2\pi}{T_s} = Sampling\ angular\ frequency$$

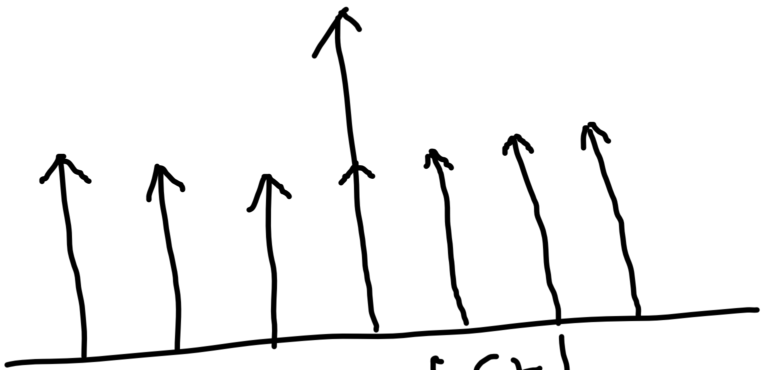
$f_s = \frac{1}{T_s}$ is known as sampling frequency

Periodic Train of impulses can be written as,

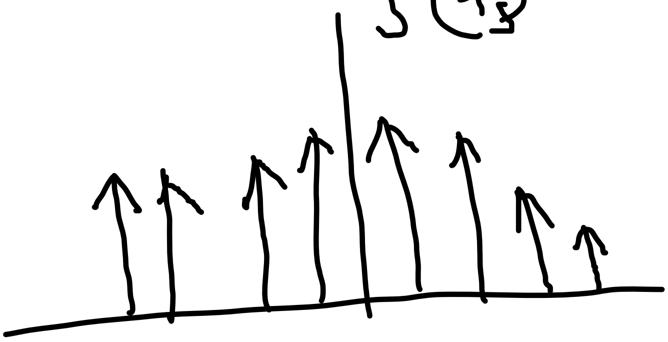
$c(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$

$s(t) = m(t) \times c(t)$

$c(t_s)$



$s(t_s)$



S(t) is the sampled signal.

The Frequency domain representation of S(t),

$S(\omega) = \frac{1}{2\pi}[M(\omega) * c(\omega)]$ (Multiplication property)

Fourier transform of a train of impulses c(t) is,

$c(\omega) = FT\{c(t)\}$

$$c(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

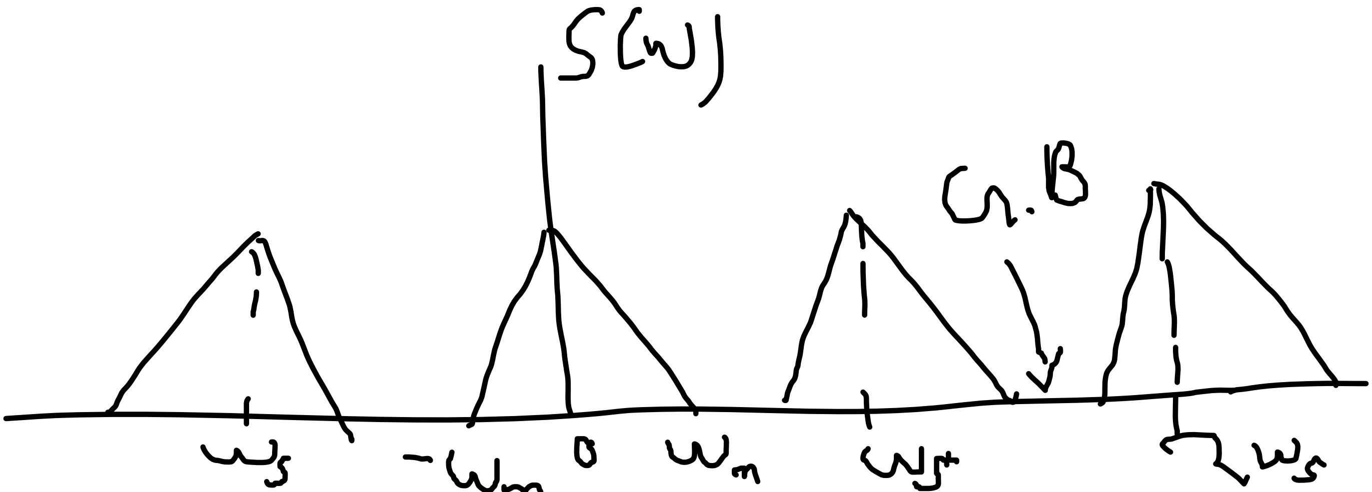
$M(\omega) = FT\{m(t)\}$

$S(\omega) = \frac{1}{2\pi}\left[M(\omega_m) * \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)\right] = \frac{1}{2\pi} \times \frac{2\pi}{T}[M(\omega_m) * \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)]$

Use Impulse property $x(t) * \delta(t - t_1) = x(t - t_1)$

$S(\omega) = \frac{1}{T}[\sum_{n=-\infty}^{\infty} M(\omega_m) * \delta(\omega - n\omega_s)] = \frac{1}{T}[\sum_{n=-\infty}^{\infty} M(\omega_m - n\omega_s)]$

$S(\omega) = \frac{1}{T}[\ldots + M(\omega_m + 2\omega_s) + M(\omega_m + \omega_s) + M(\omega_m) + M(\omega_m - \omega_s) + M(\omega_m - 2\omega_s) + \cdots]$

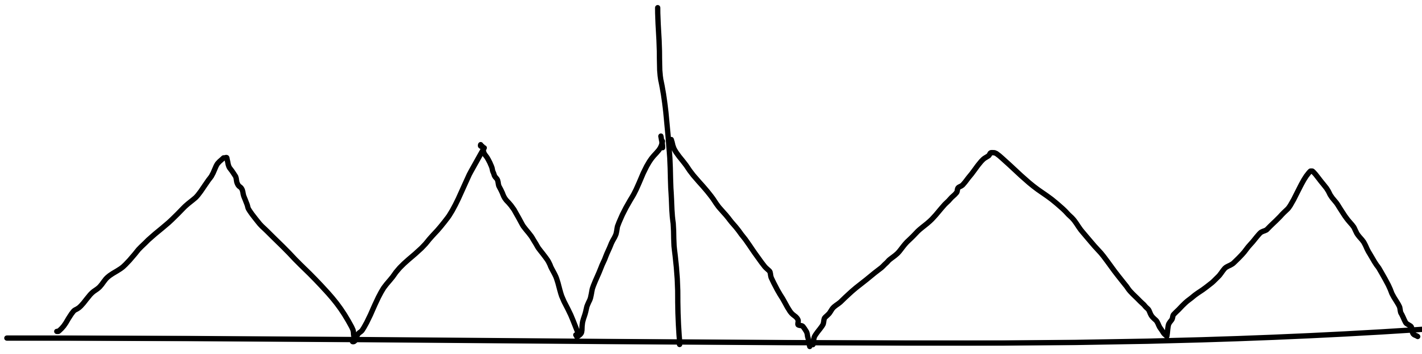


Case 1: When $\omega_s - \omega_m > \omega_m$ or $\omega_s > 2\omega_m$ then there is no overlapping.

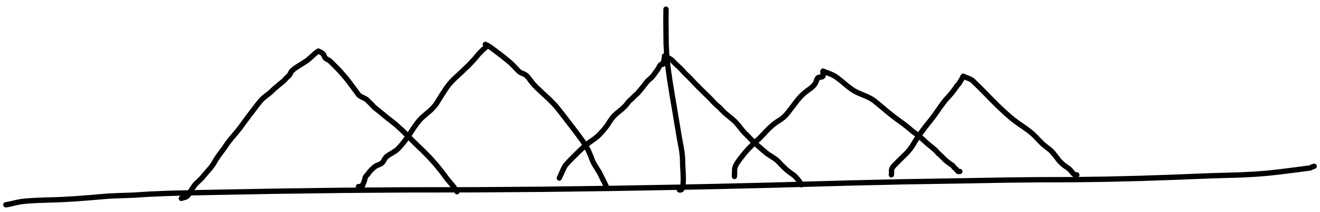
Guard Band (G.B):

It's the gap between $\omega_s - \omega_m$ andω_m.

Case 2: When, $\omega_s - \omega_m = \omega_m$ or $\omega_s = 2\omega_m$ then there is no overlapping.



Case 3: When, $\omega_s - \omega_m < \omega_m$ or $\omega_s < 2\omega_m$ then there is no overlapping.



Sampling Theorem :

A bandlimited Continuous time signal with maximum frequency $f_m(Hz)$ can be fully recovered from its samples provided that

$$F_s \geq 2f_m$$

Nyquist Rate:

The minimum rate at which a signal can be sampled and also be reconstructed from its samples is called Nyquist rate.

$$Nyquist\ rate = 2f_m$$

Nyquist interval:

$$T_s = \frac{1}{F_s} = \frac{1}{2f_m}$$

Over Sampling:

$F_s > 2f_m$

The resulted spectum will have sufficient band gap and Low Pass Filter can be used to extract the message signal.

Under Sampling:

$F_s < 2f_m$

The resulted spectum will have overlapped spectrum so it will be difficult to extract the signal.

Q1. Find the Nyquist Rate and Nyquist Interval for the following signal.

$$\boldsymbol{m(t) = cos\ 100\pi t + 2\ sin\ 200\pi t}$$

Ans:

$\mathrm{m(t) = \cos 100\pi t + 2 \sin 200\pi t}$

$m(t) = m1(t) + m2(t)$

$\omega_1 = 100\pi$

$\omega_2 = 200\pi$

$\omega_1 < \omega_2$

ω_2 is the maximum angular frequency

So $f_m = 100$ Hz

Nyquist rate : $F_s = 2f_m = 200Hz$

Nyquist interval : $T_s = \frac{1}{F_s}$=5m sec

Properties of Nyquist rate

1. Time shifting of the message signal will not change the Nyquist rate.

$$m(t \pm t_0) \rightarrow f_s$$

2. Time scaling

$$m(at) \rightarrow a \times f_s$$

3. $[m(t)]^n \rightarrow nf_s$
4. Differentiation of message signal

$$\frac{d}{dt}m(t) \rightarrow f_s$$

5. Integration of message signal

$$\int_{-\infty}^{\infty} m(t)dt \rightarrow f_s$$

6. $m(t) = m1(t) \times m2(t)$

   $f_s = f_s 1 + f_s 2$

Q2. Find the Nyquist Rate in rad/sec and in Hz.

$$\boldsymbol{m(t) = 2\ sin\ 4\pi t \times cos\ 2\pi t}$$

Ans:

$\omega_s = 2\omega_m$

$f_s = 2f_m$

$m(t) = m1(t) + m2(t)$

$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

$m(t) = \sin(4\pi t + 2\pi t) + \sin(4\pi t - 2\pi t) = \sin(6\pi t) + \sin(2\pi t)$

$\omega_1 = 6\pi$

$\omega_2 = 2\pi$

$\omega_1 > \omega_2$

So

$\omega_m = \omega_1 = 6\pi$
$\omega_s = 2\omega_m$=12 π rad/sec

$$f_s = \frac{\omega_s}{2\pi} = \frac{12\,\pi}{2\pi} = 6Hz$$

Q2. Find the Nyquist Rate in rad/sec and in Hz.

$$\boldsymbol{m(t) = \cos 200\pi t \times \cos 100\pi t}$$

Ans:

$\omega_s = 400\boldsymbol{\pi} + 200\boldsymbol{\pi} = 600\boldsymbol{\pi}\text{rad/sec}$

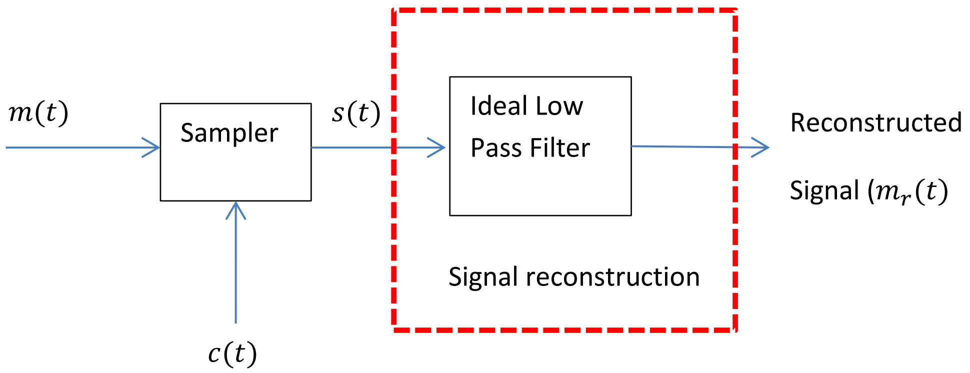
$$f_s = \frac{\omega_s}{2\pi} = \frac{600\boldsymbol{\pi}}{2\pi} = 300\ Hz$$

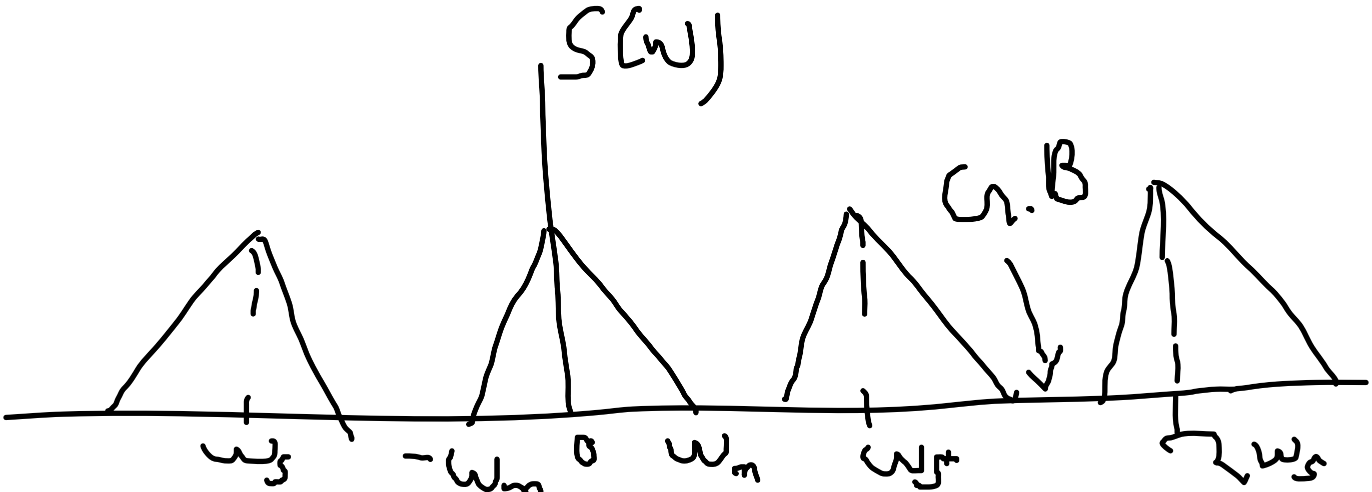
Q3. Let x(t) be a signal with Nyquist rate $\boldsymbol{\omega_s}$. Find the Nyquist rate for each of the following signals.

1. $\boldsymbol{x(t) + x(t-1)}$
2. $\boldsymbol{\frac{d}{dt}x(t)}$
3. $\boldsymbol{x^2(t)}$
4. $\boldsymbol{x(t)\cos(\omega_s t)}$

Ans:

Signal Reconstruction:





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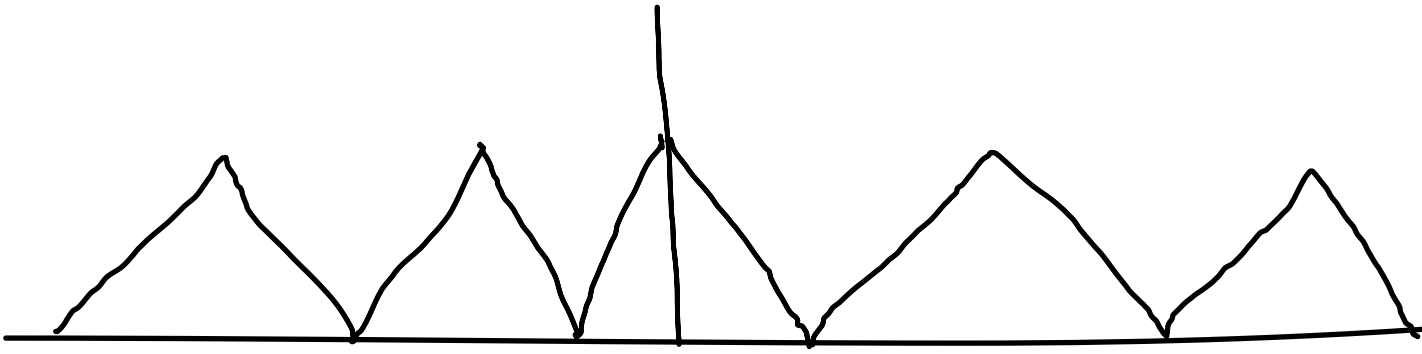


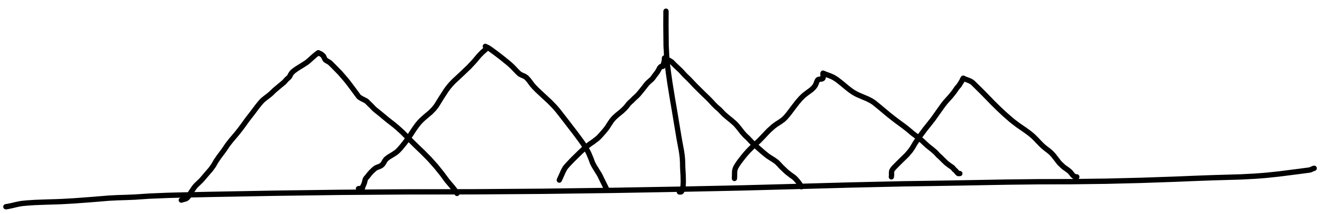
Case 3: When, $\omega_s - \omega_m < \omega_m$ or $\omega_s < 2\omega_m$ then there is no overlapping. ***This overlapping is known as Aliasing.***

Due to aliasing effect, it is not possible to recover the original signal m(t) by LPF. So due to overlapping of one reason over other, the recovered message signal is distorted.

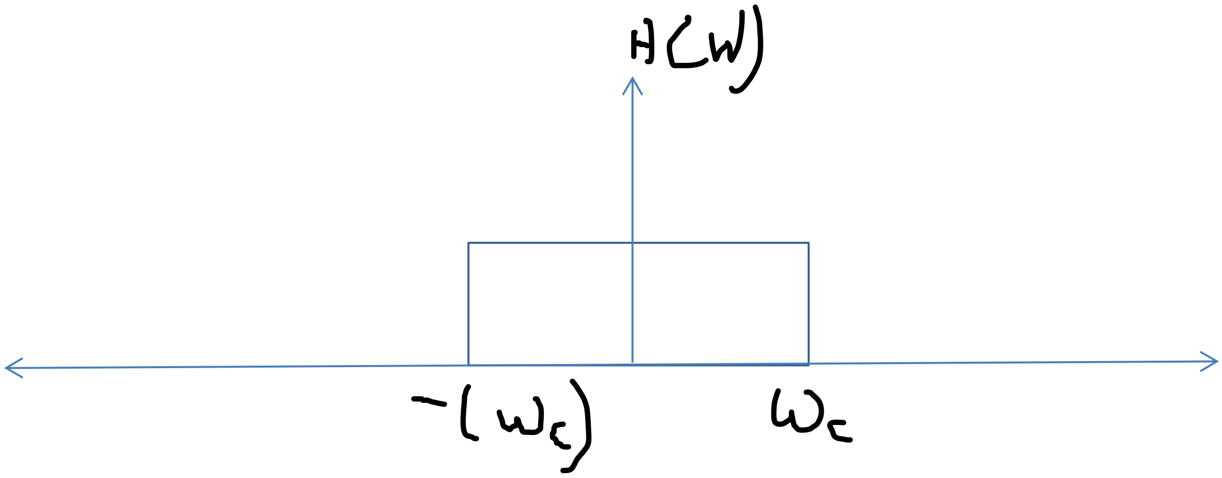
To avoid Aliasing, the message signal can be passed through LPF before sampling. This LPF is known as anti-Aliasing filter.

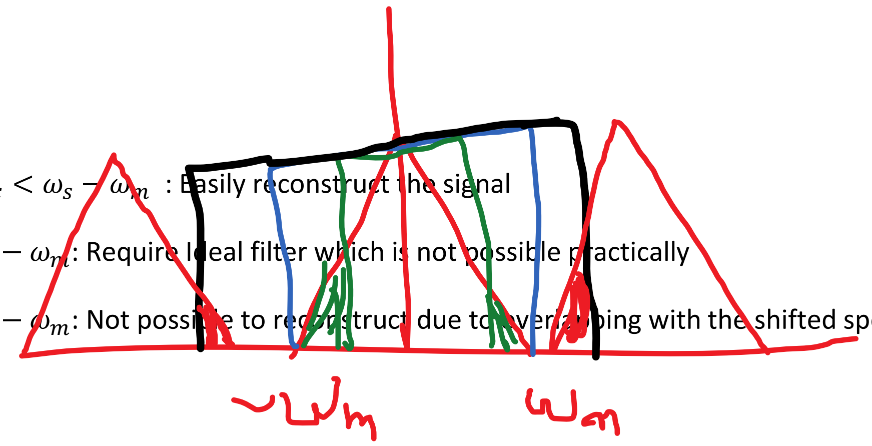
To avoid aliasing:

- $f_s \geq 2f_m$
- Use anti-Aliasing filter before sampling. This kind of filter will remove the high frequency components.



<u>Ideal Low Pass Filter property:</u>





$\omega_m < \omega_c < \omega_s - \omega_m$: Easily reconstruct the signal

$\omega_c = \omega_s - \omega_m$: Require Ideal filter which is not possible practically

$\omega_c > \omega_s - \omega_m$: Not possible to reconstruct due to overlapping with the shifted spectrum of m(t).