Impulse-train Sampling

One type of sampling that satisfies the Sampling Theorem is called impulse-train sampling. This type of sampling is achieved by the use of a periodic impulse train multiplied by a continuous time signal, $x(t)$. The periodic impulse train, $p(t)$ is referred to as the sampling function, the period, T, is referred to as the sampling period, and the fundamental frequency of $p(t)$,

$$\omega_s = \frac{2\pi}{T},$$

is the sampling frequency. We define $x_p(t)$ by the equation,

$$x_p(t) = x(t)p(t)$$, where

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Graphically, this equation looks as follows,

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x(t) ----------> x --------> x_p(t)
                 ^
                 |
                 |
p(t) = Σ_{n=-∞}^{∞} δ(t − nT)
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By using linearity and the sifting property, $x_p(t)$ can be represented as follows,

$$x_p(t) = x(t)p(t)$$
$$= x(t)\sum_{n=-\infty}^{\infty} \delta(t-nT)$$
$$= \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT)$$
$$= \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

Now, in the time domain, $x_p(t)$ looks like a group of shifted deltas with magnitude equal to the value of $x(t)$ at that time, nT, in the original function. In the frequency domain, $X_p(\omega)$ looks like shifted copies of the original $X(\omega)$ that repeat every ω_s, except that the magnitude of the copies is 1/T of the magnitude of the original $X(\omega)$.

Why does $X_p(\omega)$ look like copies of the original $X(\omega)$?

This answer can be found simply by using the Fourier Transform of the $X_p(\omega)$.

$$X_p(\omega) = F(x(t)p(t))$$
$$= \frac{1}{2\pi} X(\omega) * P(\omega)$$
$$= \frac{1}{2\pi} X(\omega) * \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - \omega_s), a_k = \frac{1}{T}$$
$$= \sum_{k=-\infty}^{\infty} \frac{1}{T} X(\omega - k\omega_s)$$

From the above equation, it is obvious that $X_p(\omega)$ is simply shifted copies of the original function (as can be seen by the $X(\omega - k\omega_s)$) that are divided by T (as can be seen by 1/T.

How to recover $x(t)$

In order to recover the original function, $x_p(t)$, we can simply low-pass filter $x_p(t)$ as long as the filter,

$$H(\omega) = \begin{cases} T, & |\omega| < \omega_c \\ 0, & else \end{cases}$$

with some ω_c satisfying, $\omega_m < \omega_c < \omega_s - \omega_m$. Also, the low-pass filter must have a gain of T. This can be represented graphically as shown below,

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filter
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$x_p(t)$ -------> $H(\omega)$ -------> $x_r(t)$,

where $x_r(t)$ represents the recovered original function.

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