Exponential Fourier Series

$$f(t) = a_0 + a_1 \cos \omega_0 t + b_1 \sin \omega_0 t + a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t + a_3 \cos 3\omega_0 t + b_3 \sin 3\omega_0 t + \cdots$$

$$\boxed{f(t) = \underbrace{a_0}_{\text{dc}} + \underbrace{\sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{\text{ac}}}$$

$$\cos n\omega_0 t = \frac{1}{2}[e^{jn\omega_0 t} + e^{-jn\omega_0 t}]$$

$$\sin n\omega_0 t = \frac{1}{2j}[e^{jn\omega_0 t} - e^{-jn\omega_0 t}]$$

$$f(t) = a_0 + \frac{1}{2} \sum_{n=1}^{\infty} [(a_n - jb_n)e^{jn\omega_0 t} + (a_n + jb_n)e^{-jn\omega_0 t}]$$

If we define a new coefficient c_n so that

$$c_0 = a_0, \qquad c_n = \frac{(a_n - jb_n)}{2}, \qquad c_{-n} = c_n^* = \frac{(a_n + jb_n)}{2}$$

then $f(t)$ becomes

$$f(t) = c_0 + \sum_{n=1}^{\infty} (c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t})$$

This is the *complex* or *exponential Fourier series* representation of $f(t)$

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}\,dt$$

where $\omega_0 = 2\pi/T$, as usual. The plots of the magnitude and phase of c_n versus $n\omega_0$ are called the *complex amplitude spectrum* and *complex phase spectrum* of $f(t)$, respectively. The two spectra form the complex frequency spectrum of $f(t)$.

The **exponential Fourier series** of a periodic function $f(t)$ describes the spectrum of $f(t)$ in terms of the amplitude and phase angle of ac components at positive and negative harmonic frequencies.

The coefficients of the three forms of Fourier series (sine-cosine form, amplitude-phase form, and exponential form) are related by

$$\boxed{A_n \underline{/\phi_n} = a_n - jb_n = 2c_n}$$

or

$$c_n = |c_n| \underline{/\theta_n} = \frac{\sqrt{a_n^2 + b_n^2}}{2} \underline{/-\tan^{-1} b_n/a_n}$$

if only $a_n > 0$. Note that the phase θ_n of c_n is equal to ϕ_n.

Fourier Transform

- Fourier series enable us to represent a periodic function as a sum of sinusoids and to obtain the frequency spectrum from the series.

- The Fourier transform allows us to extend the concept of a frequency spectrum to nonperiodic functions. The transform assumes that a nonperiodic function is a periodic function with an infinite period.

- The Fourier transform is an integral representation of a nonperiodic function that is analogous to a Fourier series representation of a periodic function.

- **The Fourier transform and Laplace transform are integral transforms.**
- **They transform a function in the time domain into the frequency domain.**
- **Laplace transform can only handle circuits with inputs for $t > 0$ with initial conditions, whereas the Fourier transform can handle circuits with inputs $t < 0$**

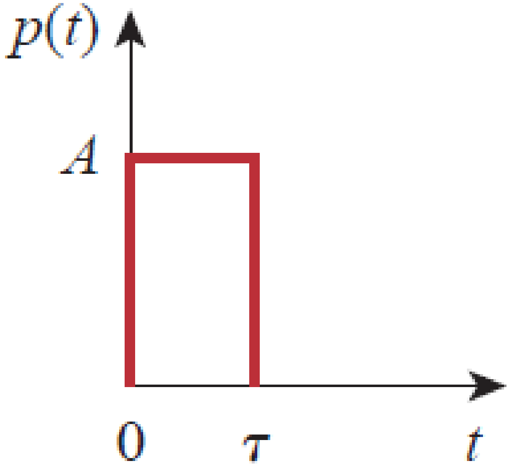
Fourier Series: **A non-sinusoidal periodic function can be represented by a Fourier series, provided that it satisfies the Dirichlet conditions.**

What happens if a function is not periodic?

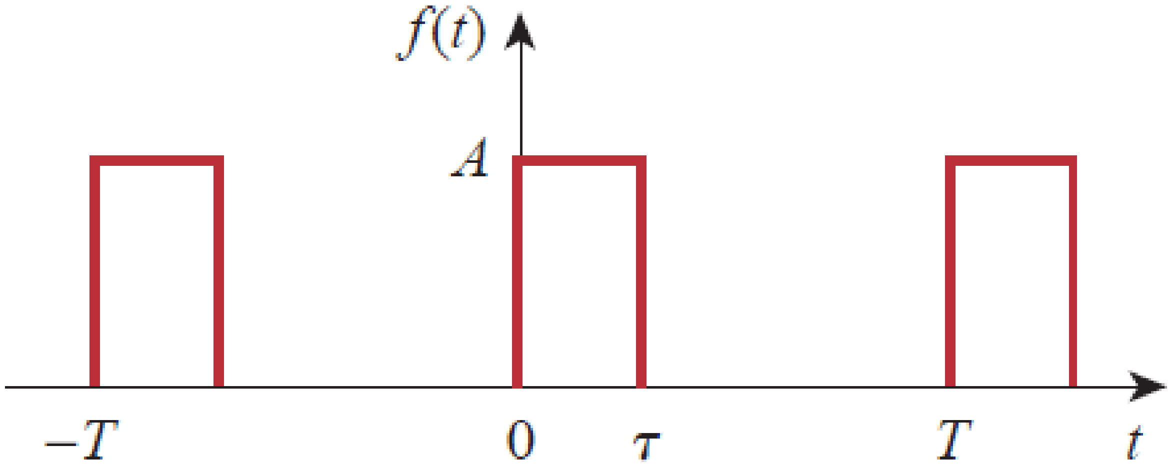
- **Unfortunately, there are many important nonperiodic functions—such as a unit step or an exponential function—that we cannot represent by a Fourier series.**

- **The Fourier transform allows a transformation from the time to the frequency domain, even if the function is not periodic.**

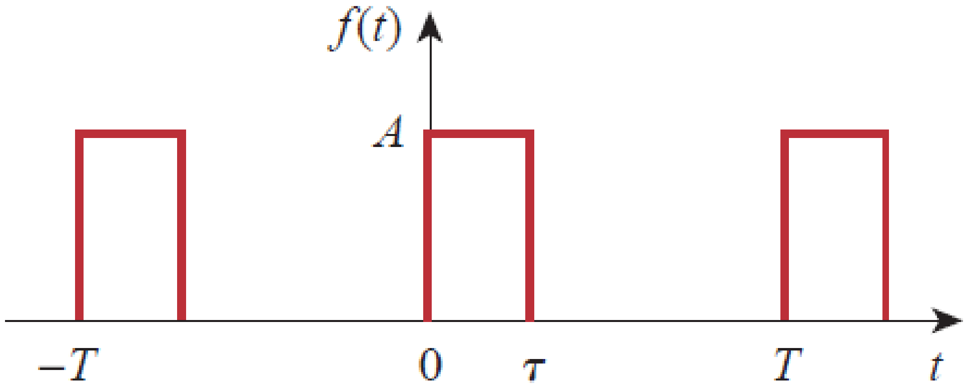
Suppose we want to find the Fourier transform of a nonperiodic function $p(t)$, shown in Fig.

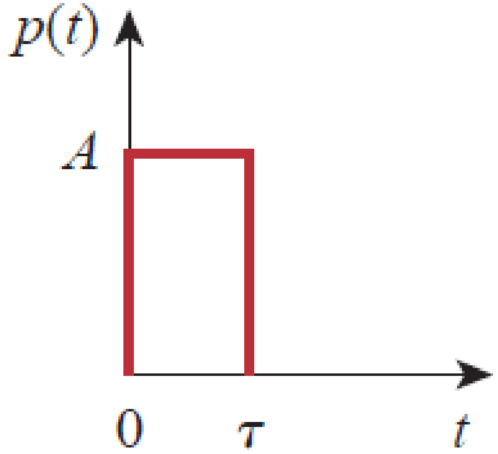


We consider a periodic function $f(t)$ whose shape over one period is the same as $p(t)$, as shown in Fig.



If we let the period $T \to \propto$, only a single pulse of width τ [the desired nonperiodic function in Fig] remains, because the adjacent pulses have been moved to infinity.

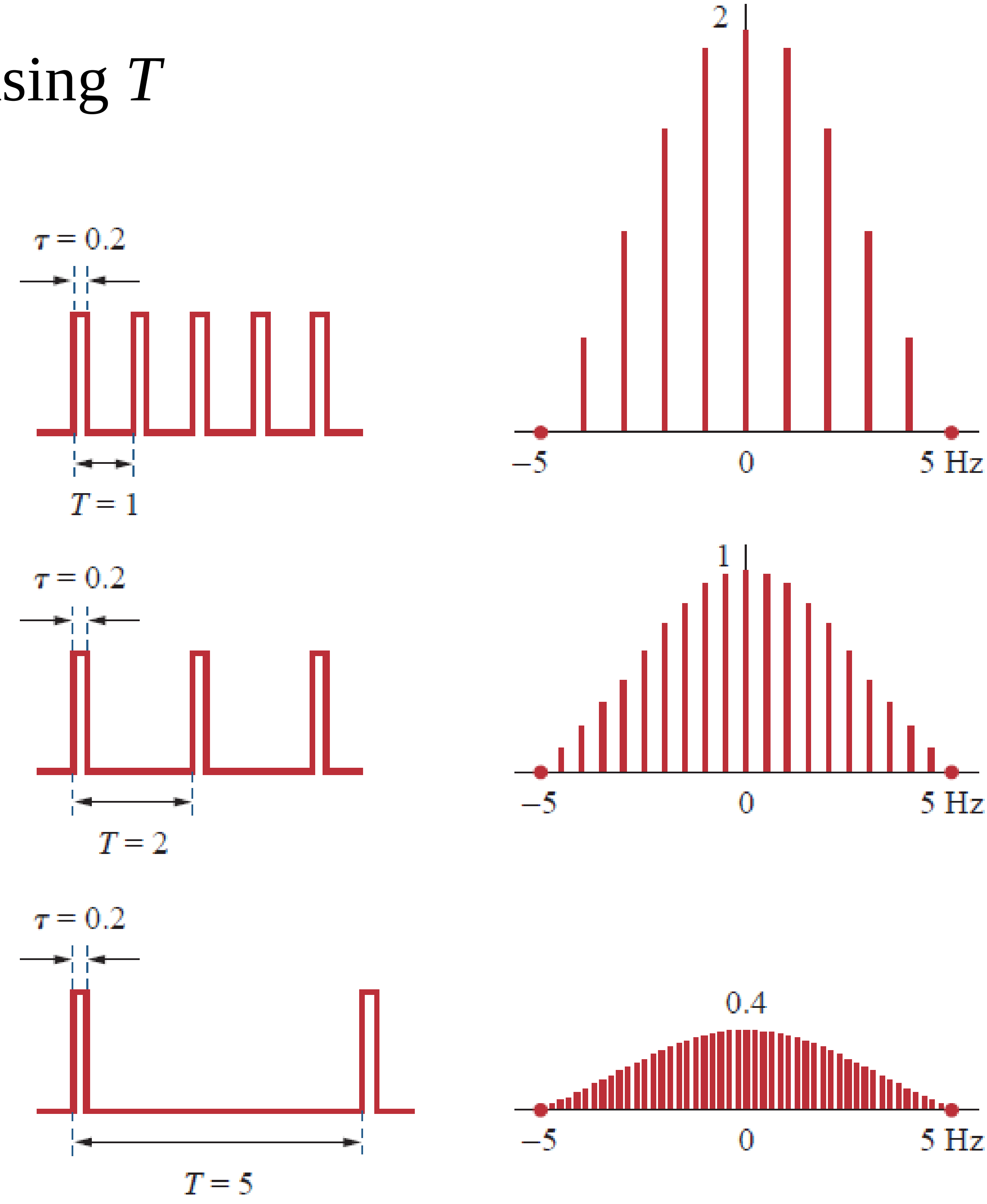


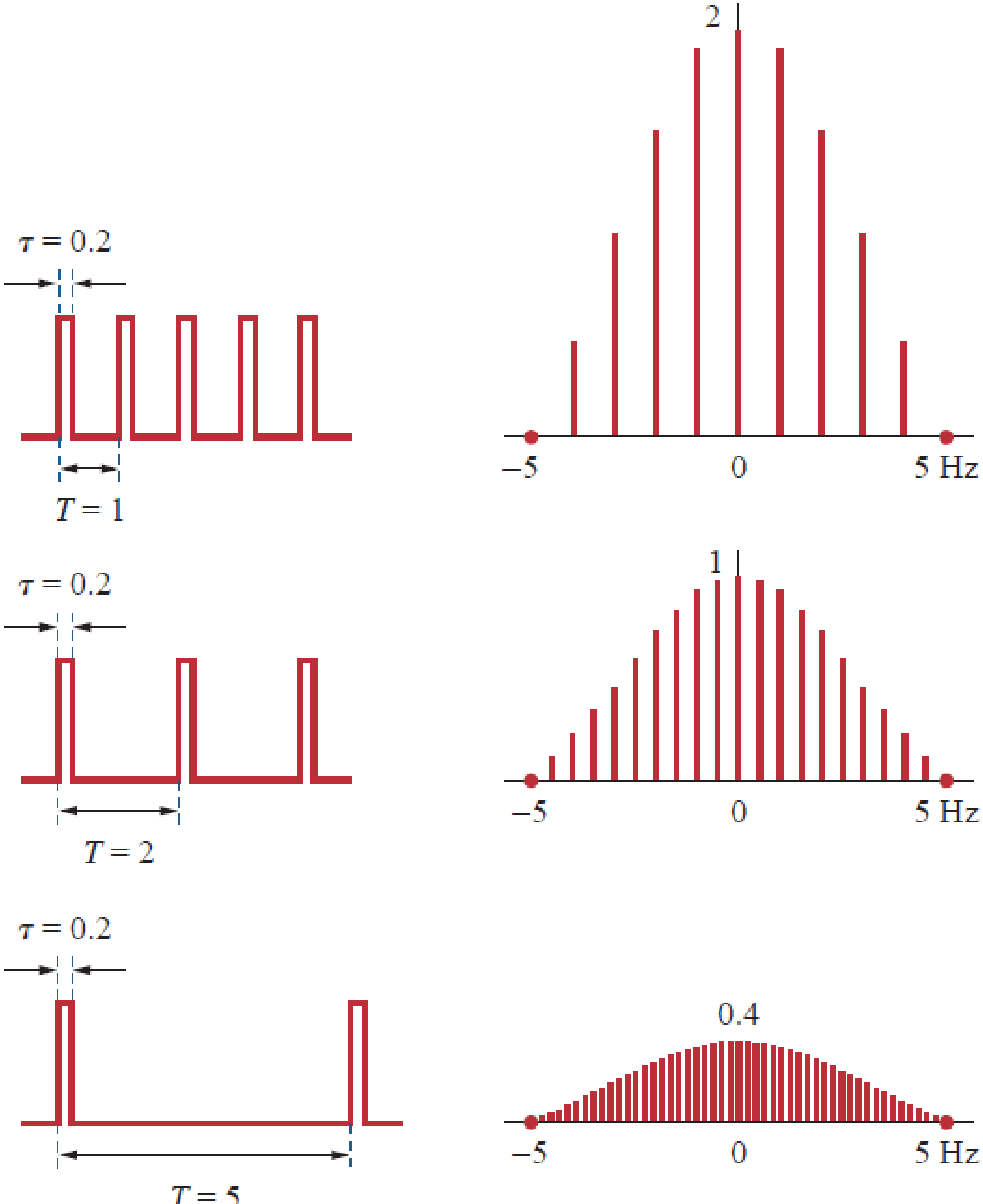


Thus, the function $f(t)$ is no longer periodic.

In other words, $f(t) = p(t)$ as $T \to \infty$.

The effect of increasing T on the spectrum.





- First, we notice that the general shape of the spectrum remains the same, and the frequency at which the envelope first becomes zero remains the same.

- However, the amplitude of the spectrum and the spacing between adjacent components both decrease, while the number of harmonics increases.

- Thus, over a range of frequencies, the sum of the amplitudes of the harmonics remains almost constant.

- As the total "strength" or energy of the components within a band must remain unchanged, the amplitudes of the harmonics must decrease as T increases.

- Since $f = \frac{1}{T}$ as T increases, f or ω decreases, so that the discrete spectrum ultimately becomes continuous.

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- As the total "strength" or energy of the components within a band must remain unchanged, the amplitudes of the harmonics must decrease as T increases.

- Since $f = \frac{1}{T}$ as T increases, f or ω decreases, so that the discrete spectrum ultimately becomes continuous.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t}\, dt$$

The fundamental frequency is

$$\omega_0 = \frac{2\pi}{T}$$

and the spacing between adjacent harmonics is

$$\Delta\omega = (n + 1)\omega_0 - n\omega_0 = \omega_0 = \frac{2\pi}{T}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \qquad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt$$

$$\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} \left[\frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt\right] e^{jn\omega_0 t} \\
&= \sum_{n=-\infty}^{\infty} \left[\frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt\right] e^{jn\omega_0 t} \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt\right] \Delta\omega e^{jn\omega_0 t}
\end{aligned}$$

If we let $T \to \infty$, the summation becomes integration, the incremental spacing $\Delta\omega$ becomes the differential separation $d\omega$, and the discrete harmonic frequency $n\omega_0$ becomes a continuous frequency ω. Thus, as $T \to \infty$,

$$\sum_{n=-\infty}^{\infty} \quad \Rightarrow \quad \int_{-\infty}^{\infty}$$

$$\Delta\omega \quad \Rightarrow \quad d\omega$$

$$n\omega_0 \quad \Rightarrow \quad \omega$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt \right] e^{j\omega t}\, d\omega$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt\right]e^{j\omega t}\,d\omega$$

The term in the brackets is known as the *Fourier transform* of $f(t)$ and is represented by $F(\omega)$. Thus,

$$\boxed{F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt}$$

where $\mathcal{F}$ is the Fourier transform operator.

The **Fourier transform** is an integral transformation of $f(t)$ from the time domain to the frequency domain.

In general, $F(\omega)$ is a complex function; its magnitude is called the *amplitude spectrum*, while its phase is called the *phase spectrum*. Thus $F(\omega)$ is the *spectrum*.

$$\boxed{f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega}$$

The function $f(t)$ and its transform $F(\omega)$ form the Fourier transform pairs:

$$f(t) \quad \Leftrightarrow \quad F(\omega)$$

since one can be derived from the other.

The Fourier transform $F(\omega)$ exists when the Fourier integral in Eq. converges. A sufficient but not necessary condition that $f(t)$ has a Fourier transform is that it be completely integrable in the sense that

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$$

To avoid the complex algebra that explicitly appears in the Fourier transform, it is sometimes expedient to temporarily replace $j\omega$ with s and then replace s with $j\omega$ at the end.

TABLE

Properties of the Fourier transform.

Property	$f(t)$	$F(\omega)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(\omega) + a_2 F_2(\omega)$
Scaling	$f(at)$	$\frac{1}{\lvert a \rvert} F\left(\frac{\omega}{a}\right)$
Time shift	$f(t - a)$	$e^{-j\omega a} F(\omega)$
Frequency shift	$e^{j\omega_0 t} f(t)$	$F(\omega - \omega_0)$

Property	$\boldsymbol{f(t)}$	$\boldsymbol{F(\omega)}$
Modulation	$\cos(\omega_0 t) f(t)$	$\frac{1}{2}[F(\omega + \omega_0) + F(\omega - \omega_0)]$
Time differentiation	$\frac{df}{dt}$	$j\omega F(\omega)$
	$\frac{d^n f}{dt^n}$	$(j\omega)^n F(\omega)$
Time integration	$\int_{-\infty}^{t} f(t)\, dt$	$\frac{F(\omega)}{j\omega} + \pi F(0)\,\delta(\omega)$
Frequency differentiation	$t^n f(t)$	$(j)^n \frac{d^n}{d\omega^n} F(\omega)$
Reversal	$f(-t)$	$F(-\omega)$ or $F^*(\omega)$
Duality	$F(t)$	$2\pi f(-\omega)$
Convolution in t	$f_1(t) * f_2(t)$	$F_1(\omega) F_2(\omega)$
Convolution in ω	$f_1(t) f_2(t)$	$\frac{1}{2\pi} F_1(\omega) * F_2(\omega)$

TABLE 18.2

Fourier transform pairs.

$f(t)$	$F(\omega)$
$\delta(t)$	1
1	$2\pi\,\delta(\omega)$
$u(t)$	$\pi\,\delta(\omega) + \dfrac{1}{j\omega}$
$u(t+\tau) - u(t-\tau)$	$2\dfrac{\sin \omega\tau}{\omega}$
$\lvert t \rvert$	$\dfrac{-2}{\omega^2}$
$\operatorname{sgn}(t)$	$\dfrac{2}{j\omega}$

$e^{-at}u(t)$	$\dfrac{1}{a + j\omega}$
$e^{at}u(-t)$	$\dfrac{1}{a - j\omega}$
$t^n e^{-at}u(t)$	$\dfrac{n!}{(a + j\omega)^{n+1}}$
$e^{-a\|t\|}$	$\dfrac{2a}{a^2 + \omega^2}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\sin \omega_0 t$	$j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$
$\cos \omega_0 t$	$\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$
$e^{-at} \sin \omega_0 t u(t)$	$\dfrac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$
$e^{-at} \cos \omega_0 t u(t)$	$\dfrac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$

Find the Fourier transform of the following functions: (a) $\delta(t - t_0)$, (b) $e^{j\omega_0 t}$, (c) $\cos\omega_0 t$.

Example 18.1

Solution:

(a) For the impulse function,

$$F(\omega) = \mathcal{F}[\delta(t - t_0)] = \int_{-\infty}^{\infty} \delta(t - t_0)e^{-j\omega t}\,dt = e^{-j\omega t_0} \tag{18.1.1}$$

where the sifting property of the impulse function in Eq. (7.32) has been applied. For the special case $t_0 = 0$, we obtain

$$\mathcal{F}[\delta(t)] = 1 \tag{18.1.2}$$

This shows that the magnitude of the spectrum of the impulse function is constant; that is, all frequencies are equally represented in the impulse function.

(c) By using the result in Eqs. (18.1.6) and (18.1.7), we get

$$\begin{aligned}\mathcal{F}[\cos\omega_0 t] &= \mathcal{F}\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] \\ &= \frac{1}{2}\mathcal{F}[e^{j\omega_0 t}] + \frac{1}{2}\mathcal{F}[e^{-j\omega_0 t}] \\ &= \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)\end{aligned} \tag{18.1.9}$$

The Fourier transform of the cosine signal is shown in Fig. 18.3.

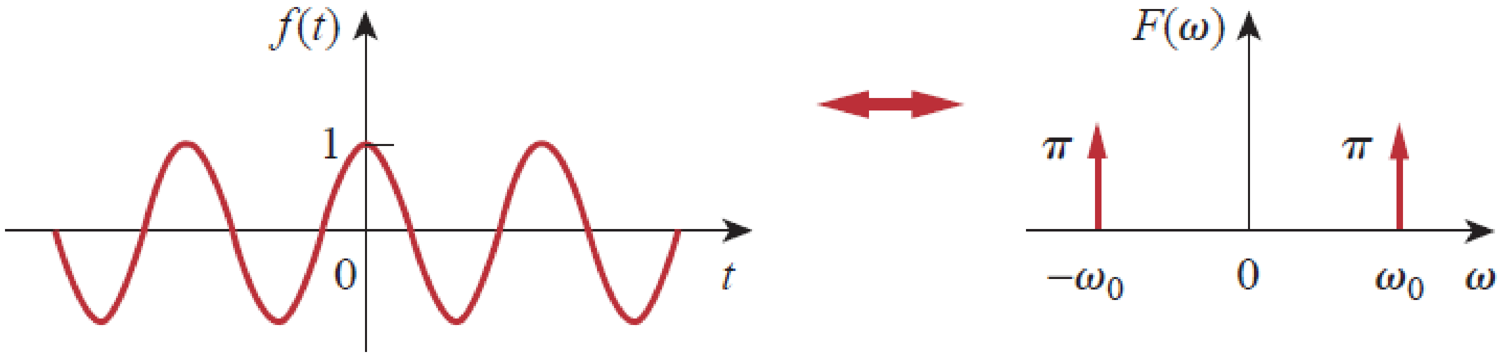


Figure 18.3
Fourier transform of $f(t) = \cos\omega_0 t$.

Example 18.2

Derive the Fourier transform of a single rectangular pulse of width τ and height A, shown in Fig. 18.4.

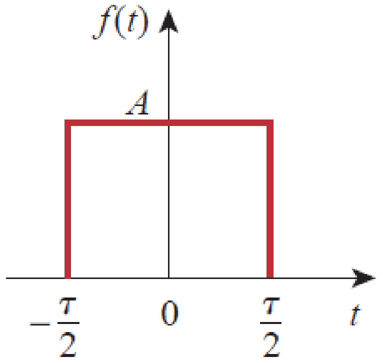


Figure 18.4
A rectangular pulse; for Example 18.2.

Solution:

$$
\begin{aligned}
F(\omega) &= \int_{-\tau/2}^{\tau/2} Ae^{-j\omega t}\, dt = -\frac{A}{j\omega} e^{-j\omega t}\Big|_{-\tau/2}^{\tau/2} \\
&= \frac{2A}{\omega}\left(\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j}\right) \\
&= A\tau \frac{\sin \omega\tau/2}{\omega\tau/2} = A\tau \operatorname{sinc} \frac{\omega\tau}{2}
\end{aligned}
$$

If we make $A = 10$ and $\tau = 2$ as in Fig. 17.27 (like in Section 17.6), then

$$
F(\omega) = 20 \operatorname{sinc} \omega
$$

whose amplitude spectrum is shown in Fig. 18.5. Comparing Fig. 18.4 with the frequency spectrum of the rectangular pulses in Fig. 17.28, we notice that the spectrum in Fig. 17.28 is discrete and its envelope has the same shape as the Fourier transform of a single rectangular pulse.

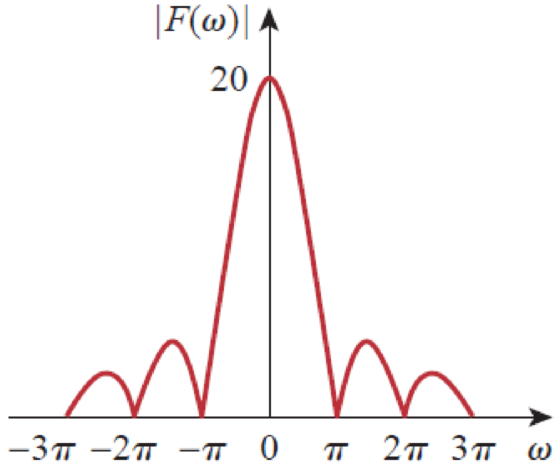


Figure 18.5
Amplitude spectrum of the rectangular pulse in Fig. 18.4: for Example 18.2.

Example 18.3

Obtain the Fourier transform of the "switched-on" exponential function shown in Fig. 18.7.

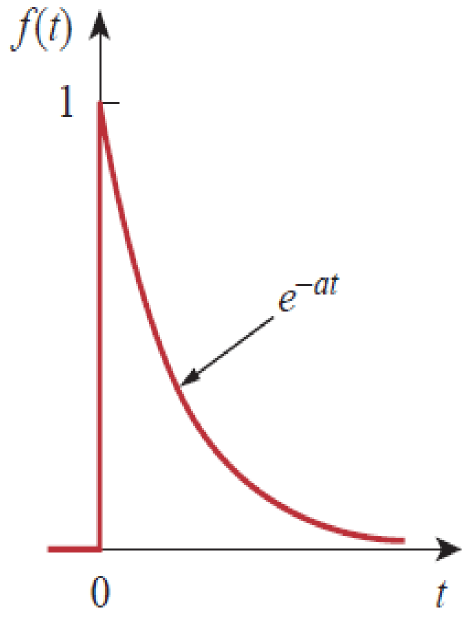


Figure 18.7
For Example 18.3.

Solution:

From Fig. 18.7,

$$f(t) = e^{-at}u(t) = \begin{cases} e^{-at}, & t > 0 \\ 0, & t < 0 \end{cases}$$

Hence,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-at}\,e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$$

$$= \frac{-1}{a + j\omega}e^{-(a+j\omega)t}\Big|_{0}^{\infty} = \frac{1}{a + j\omega}$$

Find the Fourier transform of the function in Fig. 18.14.

Example 18

Solution:

The Fourier transform can be found directly using Eq. (18.8), but it is much easier to find it using the derivative property. We can express the function as

$$f(t) = \begin{cases} 1 + t, & -1 < t < 0 \\ 1 - t, & 0 < t < 1 \end{cases}$$

Its first derivative is shown in Fig. 18.15(a) and is given by

$$f'(t) = \begin{cases} 1, & -1 < t < 0 \\ -1, & 0 < t < 1 \end{cases}$$

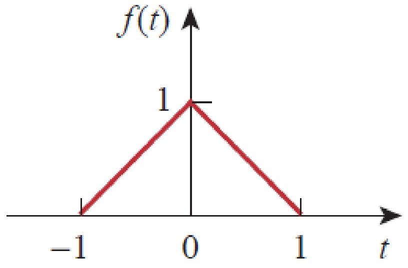


Figure 18.14
For Example 18.5.

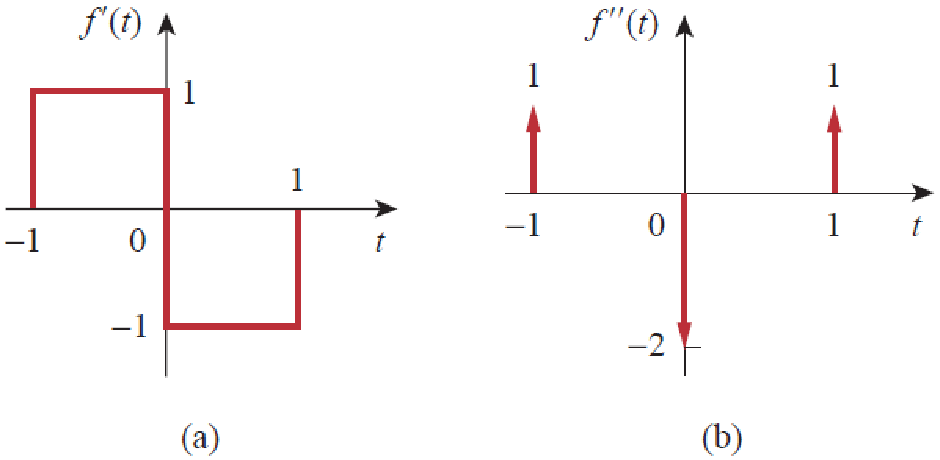


Figure 18.15
First and second derivatives of $f(t)$ in Fig. 18.14; for Example 18.5.

Its second derivative is in Fig. 18.15(b) and is given by

$$f''(t) = \delta(t + 1) - 2\delta(t) + \delta(t - 1)$$

Taking the Fourier transform of both sides,

$$(j\omega)^2 F(\omega) = e^{j\omega} - 2 + e^{-j\omega} = -2 + 2\cos\omega$$

or

$$F(\omega) = \frac{2(1 - \cos\omega)}{\omega^2}$$

Circuit Applications

- The Fourier transform generalizes the phasor technique to non periodic functions.

- Therefore, we apply Fourier transforms to circuits with non-sinusoidal excitations in exactly the same way we apply phasor techniques to circuits with sinusoidal excitations.

Thus, Ohm's law is still valid:

$$V(\omega) = Z(\omega)I(\omega)$$

where $V(\omega)$ and $I(\omega)$ are the Fourier transforms of the voltage and current and $Z(\omega)$ is the impedance.

The same expressions for the impedances of resistors, inductors, and capacitors as in phasor analysis, namely,

$$
\begin{array}{ccc}
R & \Rightarrow & R \\
L & \Rightarrow & j\omega L \\
C & \Rightarrow & \dfrac{1}{j\omega C}
\end{array}
$$

- **Transform the functions for the circuit elements into the frequency domain**
- **Take the Fourier transforms of the excitations,**
- **Use the required circuit techniques such as voltage division, source transformation,mesh analysis, node analysis, or Thevenin's theorem, to find the unknown response (current or voltage).**
- **Finally, take the inverse Fourier transform to obtain the response in the time domain**

Although the Fourier transform method produces a response that exists for $-\infty < t < \infty$, Fourier analysis cannot handle circuits with initial conditions.

$$X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow Y(\omega)$$

The transfer function is again defined as the ratio of the output response $Y(\omega)$ to the input excitation $X(\omega)$; that is,

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)}}$$

$$Y(\omega) = H(\omega)X(\omega)$$

$$Y(\omega) = H(\omega) = \mathcal{F}[h(t)]$$

indicating that $H(\omega)$ is the Fourier transform of the impulse response $h(t)$.

Example 18.7

Find $v_o(t)$ in the circuit of Fig. 18.18 for $v_i(t) = 2e^{-3t}u(t)$.

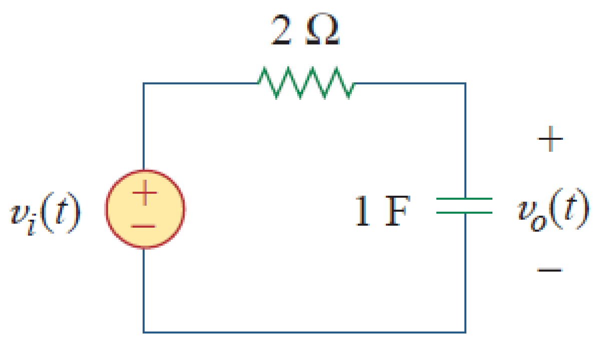


Figure 18.18
For Example 18.7.

Solution:

The Fourier transform of the input voltage is

$$V_i(\omega) = \frac{2}{3 + j\omega}$$

and the transfer function obtained by voltage division is

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{1/j\omega}{2 + 1/j\omega} = \frac{1}{1 + j2\omega}$$

Hence,

$$V_o(\omega) = V_i(\omega)H(\omega) = \frac{2}{(3 + j\omega)(1 + j2\omega)}$$

or

$$V_o(\omega) = \frac{1}{(3 + j\omega)(0.5 + j\omega)}$$

By partial fractions,

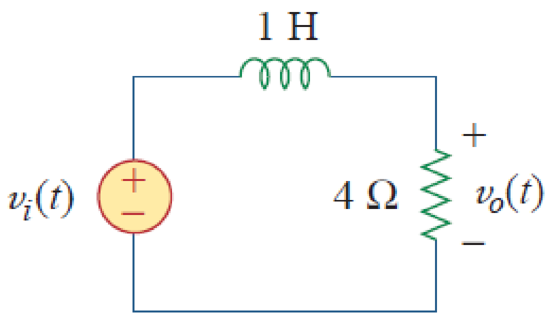
$$V_o(\omega) = \frac{-0.4}{3 + j\omega} + \frac{0.4}{0.5 + j\omega}$$

Taking the inverse Fourier transform yields

$$v_o(t) = 0.4(e^{-0.5t} - e^{-3t})u(t)$$

Practice Problem 18.7

Determine $v_o(t)$ in Fig. 18.19 if $v_i(t) = 5\,\mathrm{sgn}(t) = (-5 + 10u(t))$ V.



Answer: $-5 + 10(1 - e^{-4t})u(t)$ V.

Example 18.8

Using the Fourier transform method, find $i_o(t)$ in Fig. 18.20 when $i_s(t) = 10 \sin 2t$ A.

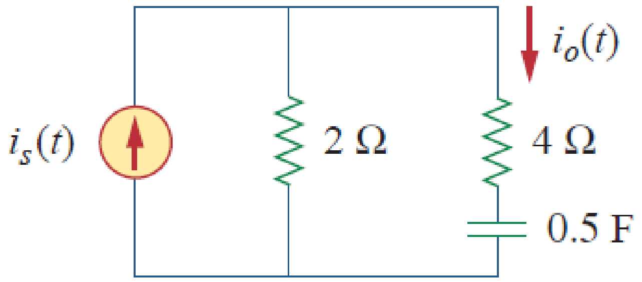


Figure 18.20
For Example 18.8.

Solution:

By current division,

$$H(\omega) = \frac{I_o(\omega)}{I_s(\omega)} = \frac{2}{2 + 4 + 2/j\omega} = \frac{j\omega}{1 + j\omega 3}$$

If $i_s(t) = 10 \sin 2t$, then

$$I_s(\omega) = j\pi 10[\delta(\omega + 2) - \delta(\omega - 2)]$$

Hence,

$$I_o(\omega) = H(\omega) I_s(\omega) = \frac{10\pi\omega[\delta(\omega - 2) - \delta(\omega + 2)]}{1 + j\omega 3}$$

The inverse Fourier transform of $I_o(\omega)$ cannot be found using Table 18.2. We resort to the inverse Fourier transform formula in Eq. (18.9) and write

$$i_o(t) = \mathcal{F}^{-1}[I_o(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{10\pi\omega[\delta(\omega - 2) - \delta(\omega + 2)]}{1 + j\omega 3} e^{j\omega t}\, d\omega$$

We apply the sifting property of the impulse function, namely,

$$\delta(\omega - \omega_0) f(\omega) = f(\omega_0)$$

or

$$\int_{-\infty}^{\infty} \delta(\omega - \omega_0) f(\omega)\, d\omega = f(\omega_0)$$

and obtain

$$i_o(t) = \frac{10\pi}{2\pi}\left[\frac{2}{1+j6}e^{j2t} - \frac{-2}{1-j6}e^{-j2t}\right]$$

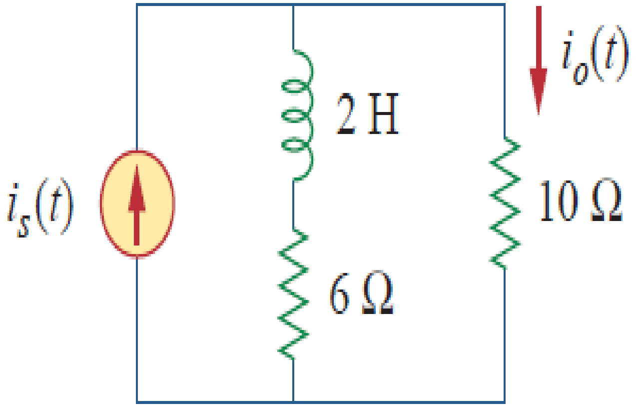
$$= 10\left[\frac{e^{j2t}}{6.082e^{j80.54^\circ}} + \frac{e^{-j2t}}{6.082\,e^{-j80.54^\circ}}\right]$$

$$= 1.644\left[e^{j(2t-80.54^\circ)} + e^{-j(2t-80.54^\circ)}\right]$$

$$= 3.288\cos(2t - 80.54^\circ)\text{ A}$$

Practice Problem 18.8

Find the current $i_o(t)$ in the circuit in Fig. 18.21, given that $i_s(t) = 5\cos 4t$ A.



Answer: $2.95\cos(4t + 26.57^\circ)$ A.