Laplace Transform

Limitations of Fourier Transform

- The FT can be used to analyse the stable system
- It cannot analyse the unstable system
- It is used to transform a time domain signal into frequency domain.

Laplace transform is used to overcome these limitations. Laplace transform is used to analyse both stable and unstable systems.

Advantages of Laplace Transform:

- It is used to analyse both stable and unstable systems in continuous time domain.
- It is used to transform a time domain signal to complex frequency domain i.e s-domain.
- It is used easier to analyse the system in s-domain.

Laplace Transform:

LT of continuous time signal $x(t)$ is defined as,

$$L\{x(t)\} = X\{s\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Where $s = \sigma + j\omega$

It exists even for non-absolutely integrable signal. It exists neither for energy nor power signals.

In $s = \sigma + j\omega$, σ is the damping factor and it provides information about stability, and ω is angular frequency, unit rad/sec.

s is a complex variable.

Bilateral Laplace Transform:

$L\{x(t)\} = X\{s\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt$ is known as bilateral Laplace Transform as the range of integration is from $[-\infty, \infty]$.

$L\{x(t)\} = X\{s\} = \int_{0}^{\infty} x(t)e^{-st}dt$ is known as unilateral Laplace Transform as the range of integration is from $[0, \infty]$.

RoC:

RoC (Region of Convergence) is the region of S plane (σ vs $j\omega$). It represents the area in which LT is finite and outside RoC the LT is infinite.

Inverse Laplace Transform:

$$\mathrm{x(t)} = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} X(s)e^{st}ds$$

Relation between LT and FT:

Let $X(s)$ is the LT of $\mathrm{x(t)}$.

Then,

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt$$

$$X(s) = FT\{x(t).e^{-\sigma t}\}$$

When $\sigma = 0$,

$X(s) = FT\{x(t)\} = X(\omega)$, where $X(\omega)$ is the FT of $\mathrm{x(t)}$ if $s = j\omega$.

This condition satisfies only when the given signal $\mathrm{x(t)}$ is absolutely integrable.

Condition for existence of Laplace Transform:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt$$

Let $x(t)e^{-j\omega t} = x_1(t)$

$$X(s) = \int_{-\infty}^{\infty} x_1(t)e^{-\sigma t}dt$$

For $X(s)$ to exist, $x_1(t)$ should be absolutely integrable. i.e.

$\int_{-\infty}^{\infty}|x_1(t)|dt < \infty \Rightarrow \int_{-\infty}^{\infty}|x(t)e^{-\sigma t}|dt < \infty$

The range of σ for which $\int_{-\infty}^{\infty}|x(t)e^{-\sigma t}|dt < \infty$ satisfies is known as RoC or Region of Convergence.

Q. Find the region of convergence or RoC of signal $x(t) = e^{2t}u(t)$

Answer:

$x(t) = e^{2t}u(t)$

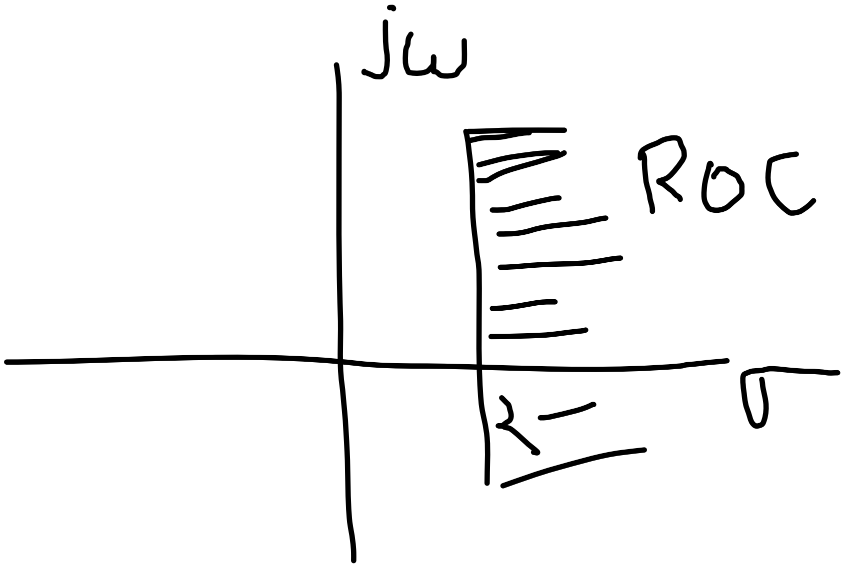
$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}|dt = \int_{-\infty}^{\infty} |e^{2t}u(t)e^{-\sigma t}|dt = \int_{0}^{\infty} |e^{2t}e^{-\sigma t}|dt$$

$\Rightarrow \int_0^\infty \left|e^{(2-\sigma)t}\right|dt$ should be less than ∞

$\int_0^\infty \left|e^{(2-\sigma)t}\right|dt < \infty$ if $(2-\sigma) < 0$

$\Rightarrow \ \sigma > 2$

RoC:



Poles and zeroes:

Zeros: Zeros of a LT are the values of s for which the numerator of the LT becomes zero.

Poles: Poles of a LT are the values of s for which the denominator of the LT becomes zero.

Ex:

$$X(s) = \frac{s+2}{(s+3)(s+4)}$$

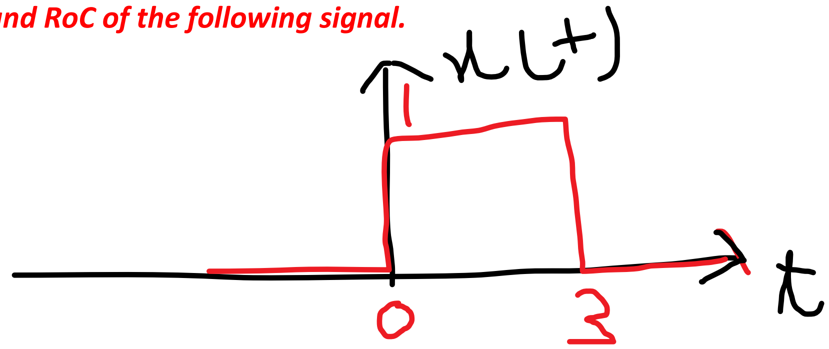
It has one zero at s=-2 and two poles at s=-3 and s=-4.

Properties of RoC:

1. RoC does not include any poles.
2. For right sided signal, RoC is right side to the right most pole.
3. For left sided signal, RoC is left side to the left most pole.

4. For the absolute integrability of a signal or the stability of a system, RoC should include imaginary axis.
5. For both sided signals, RoC is a strip in the s-plane.
6. For finite duration signals, RoC is the entire s plane excluding s=0 or +∞ or -∞

Q. Find the LT and RoC of the following signal.



Answer:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{0}^{3} 1.e^{-st}dt = \frac{1}{-s}|e^{-st}|_0^3 = \frac{e^{-3s}-1}{-s} = \frac{1-e^{-3s}}{s}$$

RoC:

For finite duration signal, RoC is the entire s-plane except 0 or +∞ or -∞

Case 1: at zero

X(0)= $\frac{1-e^{0}}{0} = \frac{0}{0}$

L'hopital's Rule:

If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{0}{0}$ **or** $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$**, then** $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$

X(0)= $\frac{3e^{-3s}}{1} = 3$ finite, so s=0 present in RoC.

X(+∞)=0 finite, so s=+∞ present in RoC.

X($-\infty$)=$\frac{\infty}{\infty}$, use **L'hopital's Rule**

X($-\infty$)= $\frac{3e^{-3s}}{1} = \infty$ which is not finite, so s=-∞ not present in RoC.

The RoC is entire s plane exluding s=$-\infty$.

Properties of LT:

Check the extra file uploaded in teams named Laplace and z transform table.pdf.

Initial Value Theorem :

$x(t) \leftrightarrow X(s)$

Then at initial value $x(0^+) = \lim_{s\to\infty} sX(s)$

It is applicable when the following conditions are satisfied.

1. Applicable only when $x(t) = 0, t < 0$
2. $x(t)$ must not contain impulse or higher order singularities at $t = 0$.

Q. Determine the initial values of $x(t)$ whose $X(s)$ is

$$\boldsymbol{X(s) = \frac{7s + 10}{s(s + 2)}}$$

Answer:

Initial Value theorem states that,

$x(t) \leftrightarrow X(s)$

Then at initial value $x(0^+) = \lim_{s\to\infty} sX(s)$

$$x(0^+) = \lim_{s\to\infty} s[\frac{\boldsymbol{7s + 10}}{\boldsymbol{s(s + 2)}}]$$

$$x(0^+) = \lim_{s\to\infty} [\frac{\boldsymbol{7s + 10}}{\boldsymbol{(s + 2)}}] = \frac{7}{1}$$

Final Value Theorem :

$x(t) \leftrightarrow X(s)$

Then at initial value $x(\infty) = \lim_{s\to 0} sX(s)$

It is applicable when the following conditions are satisfied.

1. Applicable only when $x(t) = 0, t < 0$

2. $sX(s)$ must have poles in the left half of the s-plane

Q. Determine the final values of $x(t)$ whose $X(s)$ is

$$X(s) = \frac{7s + 10}{s(s + 2)}$$

Answer:

$$x(\infty) = \lim_{s \to 0} sX(s)$$

$$x(\infty) = \lim_{s \to 0} s[\frac{7s + 10}{s(s + 2)}]$$

$$x(\infty) = \lim_{s \to 0}[\frac{7s + 10}{(s + 2)}] = \frac{10}{2} = 5$$

Inverse Laplace Transform:

Q. Find $x(t)$.

$$X(s) = \frac{10(s + 4)}{s^2(s + 2)}$$

Answer:

Using Partial fractions,

$$X(s) = \frac{10(s + 4)}{s^2(s + 2)} = \frac{A}{s^2} + \frac{B}{s} + \frac{c}{s + 2}$$

$$\frac{10(s + 4)}{s^2(s + 2)} = \frac{A(s + 2)}{s^2} + \frac{Bs(s + 2)}{s} + \frac{cs^2}{s + 2}$$

$$=> \ 10(s + 4) = \ A(s + 2) + \ Bs(s + 2) + \ cs^2$$

When s = 0,

$$10(0 + 4) = \ A(0 + 2)$$

A=20

When s = -2,

$$10(-2 + 4) = \ c(-2)^2$$

C=5

By applying the value of A and C, We will get B=-5

$$\boldsymbol{X(s) = \frac{10(s+4)}{s^2(s+2)} = \frac{A}{s^2} + \frac{B}{s} + \frac{c}{s+2} = \frac{20}{s^2} + \frac{-5}{s} + \frac{5}{s+2}}$$

$$\boldsymbol{x(t) = 20t - 5u(t) + 5e^{-2t}}$$

Laplace Transform to solve Differential equations:

Q. Use Laplace Transform to Solve Differential Equations

$$\boldsymbol{y'' - y = e^{2t}, y(0) = 0, y'(0) = 1}$$

Solution:

$$y(t) \leftrightarrow Y(s)$$

$$y'(t) \leftrightarrow sY(s) - y(0)$$

$$y''(t) \leftrightarrow s\big(sY(s) - y(0)\big) - y'(0)$$

$$L\{e^{2t}\} = \frac{1}{s-2}$$

$$s^2Y(s) - sy(0) - y'(0) - Y(s) = \frac{1}{s-2}$$

$$s^2Y(s) - 1 - Y(s) = \frac{1}{s-2}$$

$$Y(s)(s^2 - 1) = 1 + \frac{1}{s-2} = \frac{s-2+1}{s-2} = \frac{s-1}{s-2}$$

$$Y(s) = \frac{s-1}{(s-2)(s+1)(s-1)} = \frac{1}{(s-2)(s+1)}$$

$$Y(s) = \frac{1}{(s-2)(s+1)}$$

Find Inverse LT to get y(t).

Using Partial fractions,

$$Y(s) = \frac{1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

$$\frac{1}{(s-2)(s+1)} = \frac{A(s+1)}{s-2} + \frac{B(s-2)}{s+1}$$

$$1 = A(s+1) + B(s-2)$$

Putting s=-1,

$$B = -\frac{1}{3}$$

Putting s=-2,

$$A = \frac{1}{3}$$

$$Y(s) = \frac{A}{s-2} + \frac{B}{s+1} = \frac{1}{3(s-2)} - \frac{1}{3(s+1)}$$

$$y(t) = \frac{1}{3}(e^{2t} - e^{-t})$$

Z Transform :

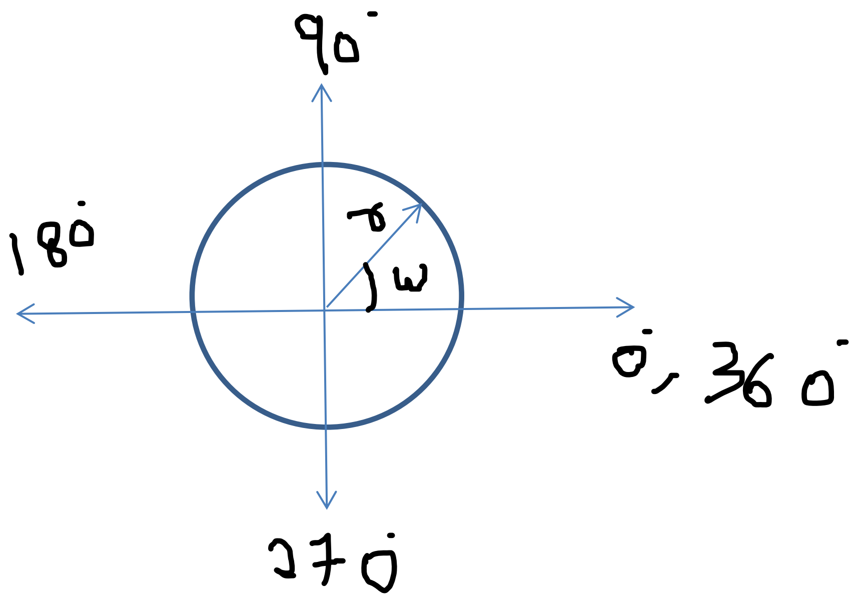
For Discrete time signals we can find the z-transform.

$$x[n] \leftrightarrow X[z]$$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Where z is a complex variable and its polar form is $z = re^{j\omega}$

Where r is the magnitude of z i.e $|z|$ and ω is the complex value or angle.



Bi-directional Z-transform:

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Unidirectional z-Transform:

$$X[z] = \sum_{0}^{\infty} x[n]z^{-n}$$

Problem: $\boldsymbol{x[n] = a^n u(n)}$***. Find the Z transform and RoC.***

Answer:

$X[z] = \sum_{n=-\infty}^{\infty} \mathrm{a^n u(n)}\, z^{-n} = \sum_{n=0}^{\infty} \mathrm{a^n}\, z^{-n} = \sum_{n=0}^{\infty} (\mathrm{a}z^{-1})^{\mathrm{n}} \; = \frac{1}{1-\mathrm{a}z^{-1}}$ when $|\mathrm{a}z^{-1}| < 1$ by using GP

$$\left|\frac{a}{z}\right| < 1$$

$$r = \; |z| > a$$

Z Transform to solve Differential equations:

$$Z(y(n)) = Y(z)$$

$$Z(y(n+1)) = z(Y(z) - y(0))$$

$$Z(y(n+2)) = z^2(Y(z) - y(0) - y(1)z^{-1})$$

$$Z(y(n-1)) = z^{-1}Y(z) - y(-1)$$

$$Z\big(y(n-2)\big) = z^{-2}Y(z) + z^{-1}y(-1) + y(-2)$$

Q. Z Transform to Solve Differential Equations

$\boldsymbol{y(n+2) + 4y(n+1) + 3y(n) = 2^n}$, $\boldsymbol{y(0) = 0}$ ***and*** $\boldsymbol{y(1) = 1}$

Solution:

$$\mathrm{y(n+2) + 4y(n+1) + 3y(n) = 2^n}$$

$$Z\big(\mathrm{y(n+2)+4y(n+1)+3y(n)}\big) = Z(2^{\mathrm{n}})$$

$$Z\big(\mathrm{y(n+2)}) + 4\mathrm{Z(y(n+1)) + 3Z(y(n)}\big) = \frac{z}{z-2}$$

$$z^2(Y(z) - y(0) - y(1)z^{-1}) + 4\,z\big(Y(z) - y(0)\big) + 3\,Y(z) = \frac{z}{z-2}$$

$$z^2(Y(z) - y(1)z^{-1}) + 4\,z\big(Y(z)\big) + 3\,Y(z) = \frac{z}{z-2}$$

$$z^2Y(z) - z + 4z\,Y(z) + 3Y(z) = \frac{z}{z-2}$$

$$Y(z)(z^2 + 4z + 3) - z = \frac{z}{z-2}$$

$$Y(z) = \frac{z^2 - z}{(z-2)(z+1)(z+3)}$$

Q. Find the Impulse response H(z) for the given system

$$\boldsymbol{y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n)}$$

Solution:

$$\mathrm{y(n)} - \frac{3}{4}\mathrm{y(n-1)} + \frac{1}{8}\mathrm{y(n-2) = x(n)}$$

$$\mathrm{Z(y(n)} - \frac{3}{4}\mathrm{y(n-1)} + \frac{1}{8}\mathrm{y(n-2)) = Z(x(n))}$$

As initial conditions are not given, assume $\mathrm{y(0) = 0}$ and $\mathrm{y(-1) = 1}$

$$\mathrm{Y(z)} - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = X(z)$$

$$\mathrm{Y(z)}(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}) = X(z)$$

$$H(z) = \frac{\mathrm{Y(z)}}{X(z)} = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

Q. Find the Impulse response H(z) for the given system

$$\boldsymbol{y(n) = 3x(n) + y(n-1)}$$

Solution:

$$\boldsymbol{H(z) = \frac{3}{1 - z^{-1}}}$$