Relationship between trigonometric Fourier series and Exponential Fourier Series

The exponential Fourier series of Periodic signal $x(t)$ is,

$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, $t_0 < t < t_0 + \frac{2\pi}{\omega_0}$

Where, $\omega_0 = \frac{2\pi}{T_0}$

$$x(t) = \sum_{k=-\infty}^{-1} c_k e^{jk\omega_0 t} + c_0 + \sum_{k=1}^{\infty} c_k e^{jk\omega_0 t}$$

Put k = -k, in the first term

$$x(t) = \sum_{k=1}^{\infty} c_{-k} e^{-jk\omega_0 t} + c_0 + \sum_{k=1}^{\infty} c_k e^{jk\omega_0 t}$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t}$$

$$= c_0 + \sum_{k=1}^{\infty} c_k \left(\cos k\omega_0 t + j \sin k\omega_0 t\right) + c_{-k}\left(\cos k\omega_0 t - j \sin k\omega_0 t\right)$$

$$= c_0 + \sum_{k=1}^{\infty} (c_k + c_{-k}) \cos k\omega_0 t + j(c_k - c_{-k}) \sin k\omega_0 t$$

The trigonometric Fourier series of $x(t)$ is,

- $x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + b_k \sin k\omega_0 t$, $t_0 < t < t_0 + \frac{2\pi}{\omega_0}$

  By comparing above two equations, we will get

$$a_0 = c_0$$
$$a_k = c_k + c_{-k}$$
$$b_k = j(c_k - c_{-k})$$
$$c_k = \frac{a_k - jb_k}{2}$$
$$c_{-k} = \frac{a_k + jb_k}{2}$$

$$|c_k| = \frac{\sqrt{{a_k}^2 + {b_k}^2}}{2}$$

$$|c_{-k}| = \frac{\sqrt{{a_k}^2 + {b_k}^2}}{2}$$

$$Angle\ of\ c_n = -\,tan^{-1}\left(\frac{b_n}{a_n}\right)$$

$$Angle\ of\ c_{-n} = tan^{-1}\left(\frac{b_n}{a_n}\right)$$

The magnitude spectrum of exponential Fourier series coefficients always has even symmetry.

The phase spectrum of exponential Fourier series coefficients always has odd symmetry.

Properties of Exponential Fourier series

The exponential Fourier series of periodic signal $x(t)$ is given by,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t},\ t_0 < t < t_0 + \frac{2\pi}{\omega_0}$$

$$c_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\, dt$$

- The exponential Fourier series coefficient of a signal can be represented as,
- $x(t) \overset{FS}{\leftrightarrow} c_k$

Property 1: Linearity

If $x_1(t) \overset{FS}{\leftrightarrow} c_k$

$x_2(t) \overset{FS}{\leftrightarrow} d_k$ then

$ax_1(t) + bx_2(t) \overset{FS}{\leftrightarrow} ac_k + bd_k$

Property 2: Time Shifting

- If $x(t) \overset{FS}{\leftrightarrow} c_k$, then
- $x(t-t_0) \overset{FS}{\leftrightarrow} e^{-jk\omega_0 t_0} c_k$

If $x(t)$ is time shifted, then the magnitude of exponential Fourier series coefficient will remain same but the phase of exponential Fourier series coefficients will change.

Property 3: Time Reversal Property

- If $x(t) \overset{FS}{\leftrightarrow} c_k$, then
- $x(-t) \overset{FS}{\leftrightarrow} c_{-k}$

  If $x(t)$ is time reversed then the exponential Fourier series coefficients are also time reversed.

Property 4: Time differentiation Property

- If $x(t) \overset{FS}{\leftrightarrow} c_k$, then
- $\frac{d}{dt}\, x(t) \overset{FS}{\leftrightarrow} jk\omega_0 c_k$

Property 5: Time Integration Property

- If $x(t) \overset{FS}{\leftrightarrow} c_k$, then
- $\int_{-\infty}^{t} x(\tau)d\tau \overset{FS}{\leftrightarrow} \frac{c_k}{jk\omega_0}$; if $c_0 = 0$

Property 6: Conjugate Symmetry Property

- If $x(t) \overset{FS}{\leftrightarrow} c_k$, then
- $x*(t) \overset{FS}{\leftrightarrow} c*_{-k}$

Property 7: Convolution Property

If $x_1(t) \overset{FS}{\leftrightarrow} c_k$

$x_2(t) \overset{FS}{\leftrightarrow} d_k$ then

$x_1(t) * x_2(t) \overset{FS}{\leftrightarrow} c_k d_k$

$The\ convolution\ always\ leads\ to\ multiplication\ and\ viceversa.$

Property 8: Multiplication Property

If $x_1(t) \overset{FS}{\leftrightarrow} c_k$

$x_2(t) \overset{FS}{\leftrightarrow} d_k$ then

$x_1(t) x_2(t) \overset{FS}{\leftrightarrow} c_k * d_k$

Property 9: Persevals Property

If $x_1(t) \overset{FS}{\leftrightarrow} c_k$

$x_2(t) \overset{FS}{\leftrightarrow} d_k$ then

$$\frac{1}{T}\int_{t_0}^{t_0+T} x_1(t) x_2{}^*(t)\,dt = \sum_{k=-\infty}^{\infty} c_k d_k^*$$