Analysis of circuits in which the source voltage or current is time-varying. We are particularly interested in sinusoidally time-varying excitation

A sinusoid is a signal that has the form of the sine or cosine function.

A sinusoidal current is usually referred to as *alternating current* (*ac*). Such a current reverses at regular time intervals and has alternately positive and negative values. Circuits driven by sinusoidal current or voltage sources are called *ac circuits.*

Sinusoids

Consider the sinusoidal voltage

$$\boxed{v(t) = V_m \sin \omega t}$$

where

$\omega = 2\pi f$

$f = \frac{1}{T}$ (Hz)

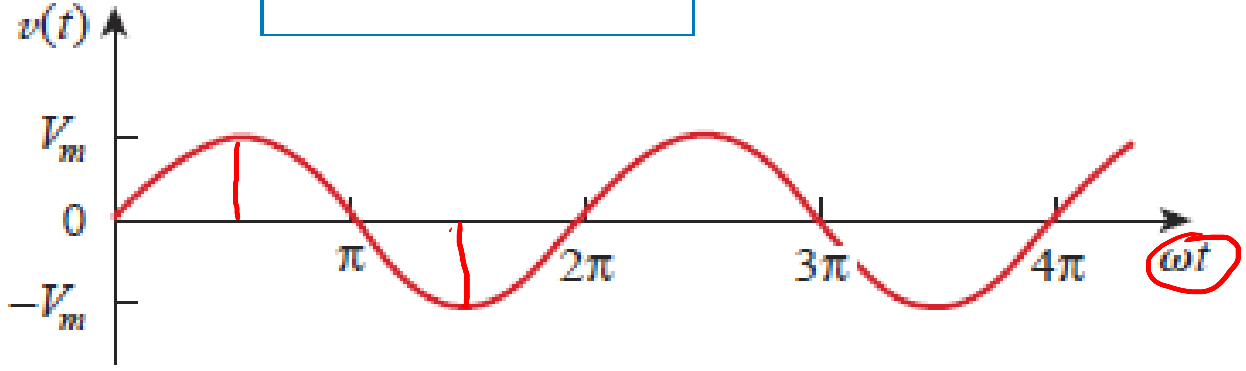
$\omega = \frac{2\pi}{T}$

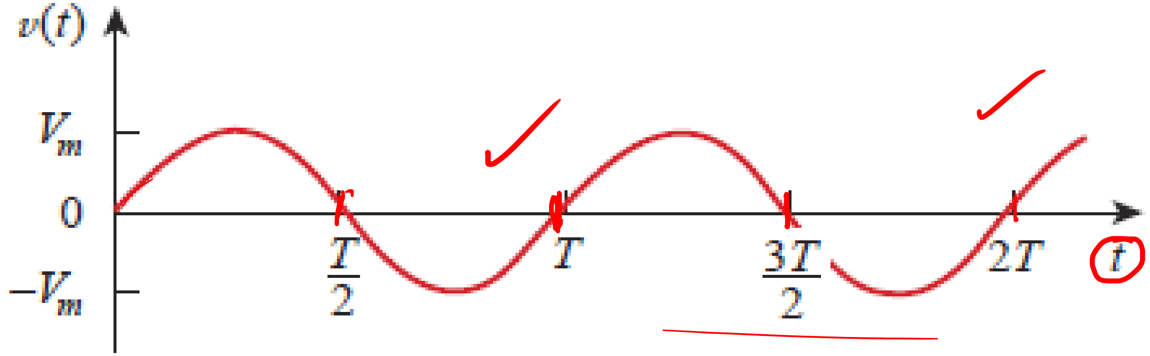
V_m = the *amplitude* of the sinusoid

ω = the *angular frequency* in radians/s

ωt = the *argument* of the sinusoid

$$\boxed{T = \frac{2\pi}{\omega}}$$





Time

The fact that $v(t)$ repeats itself every T seconds is shown by replacing t by $t + T$

$$v(t+T) = V_m \sin\omega(t+T) = V_m \sin\omega\left(t + \frac{2\pi}{\omega}\right)$$

$$= V_m \sin(\omega t + 2\pi) = V_m \sin\omega t = v(t)$$

$$\boxed{v(t+T) = v(t)}$$

that is, v has the same value at $t + T$ as it does at t and $v(t)$ is said to be *periodic*. In general,

A **periodic function** is one that satisfies $f(t) = f(t + nT)$, for all t and for all integers n.

$$\boxed{f = \frac{1}{T}}$$

$$\omega = 2\pi f$$

While ω is in radians per second (rad/s), f is in hertz (Hz).

Let us now consider a more general expression for the sinusoid,

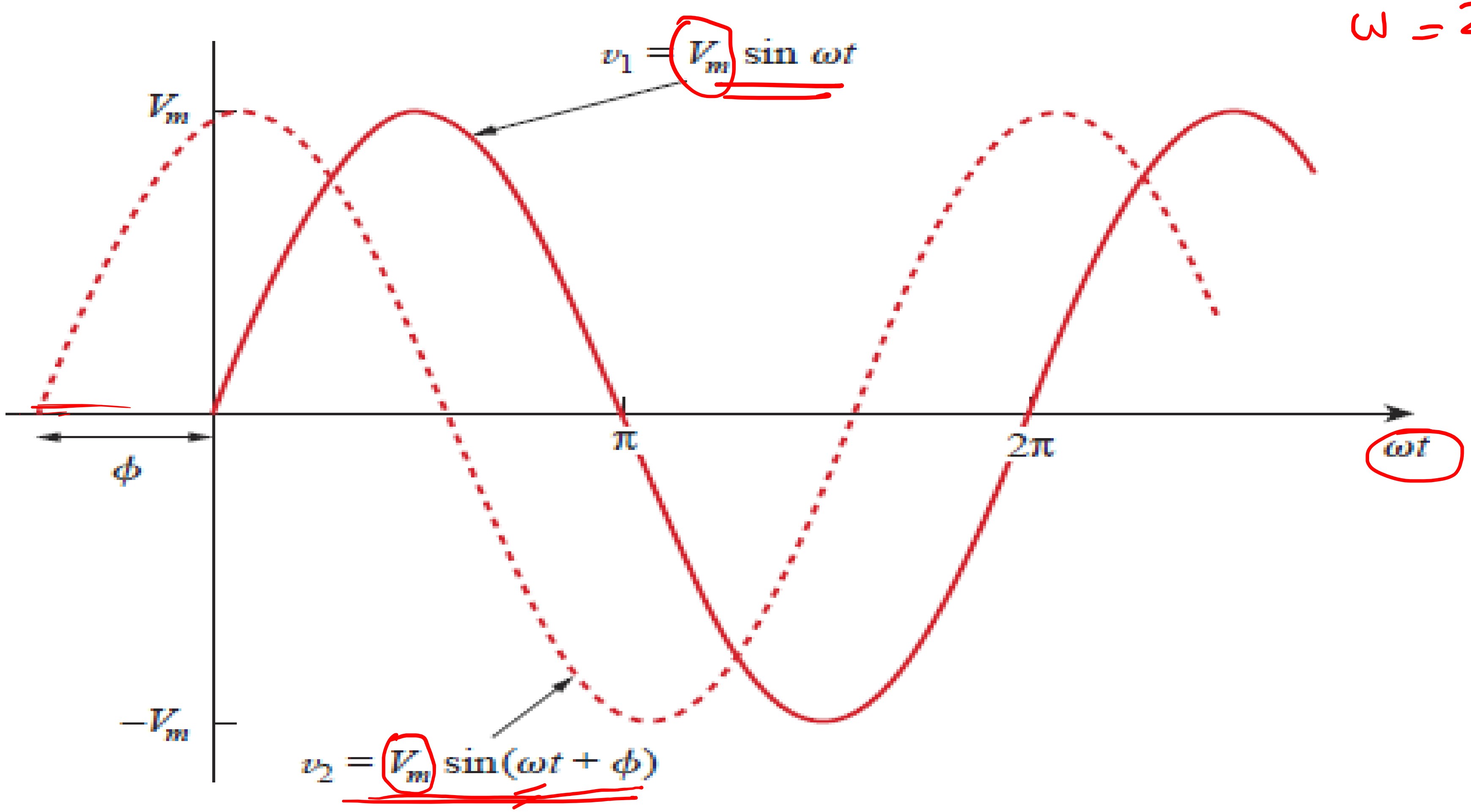
$\phi = 0$

$$v(t) = V_m \sin(\omega t + \phi)$$

where $(\omega t + \phi)$ is the argument and ϕ is the *phase*. Both argument and phase can be in radians or degrees.

Let $v_1(t) = V_m \sin \omega t$ and $v_2(t) = V_m \sin(\omega t + \phi)$

$\omega = 2\pi f$



If $\phi \neq 0$, we also say that v_1 and v_2 are *out of phase*. If $\phi = 0$, then v_1 and v_2 are said to be *in phase*; they reach their minima and maxima at exactly the same time. We can compare v_1 and v_2 in this manner because they operate at the same frequency; they do not need to have the same amplitude.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

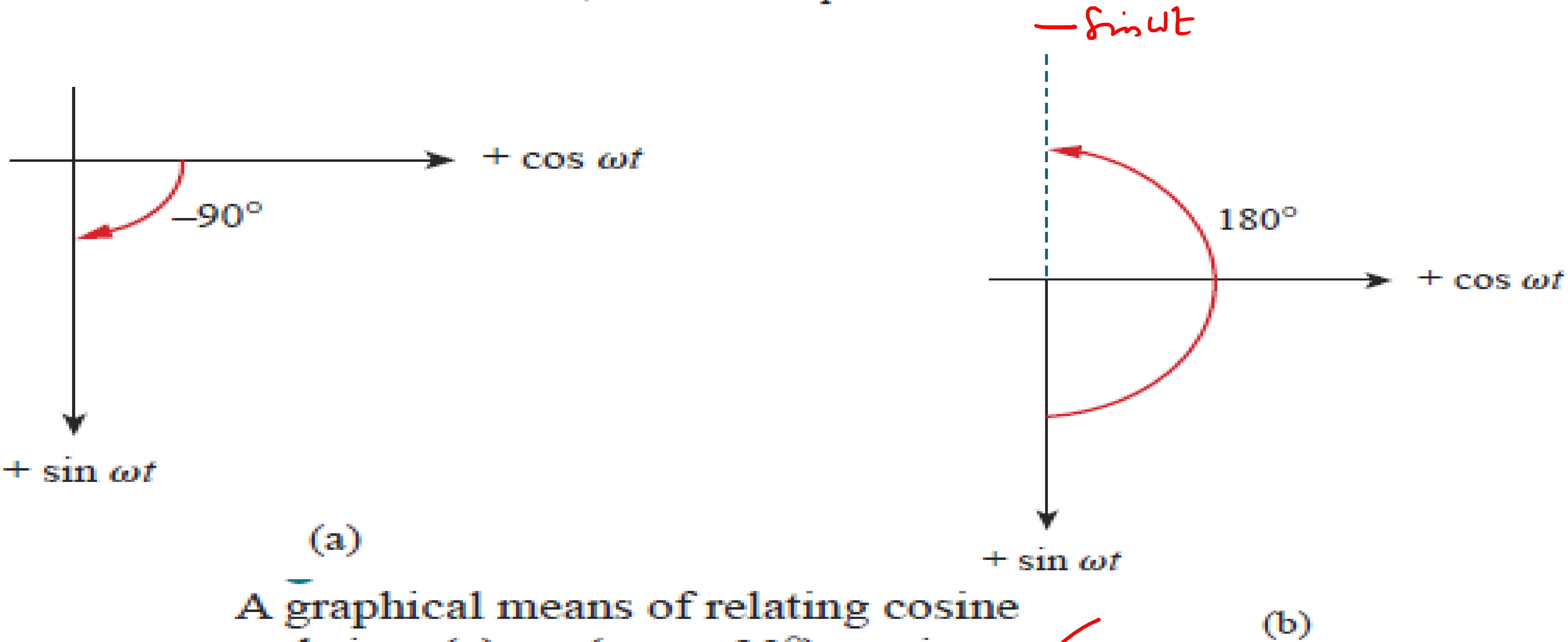
With these identities, it is easy to show that

$$\begin{aligned}
\sin(\omega t \pm 180^\circ) &= -\sin \omega t \\
\cos(\omega t \pm 180^\circ) &= -\cos \omega t \\
\sin(\omega t \pm 90^\circ) &= \pm \cos \omega t \\
\cos(\omega t \pm 90^\circ) &= \mp \sin \omega t
\end{aligned}$$

Using these relationships, we can transform a sinusoid from sine form to cosine form or vice versa.

The horizontal axis represents the magnitude of cosine, while the vertical axis (pointing down) denotes the magnitude of sine. Angles are measured positively counterclockwise from the horizontal, as usual in polar coordinates.

$4\hat{i}+3\hat{j}-\hat{k}$



A graphical means of relating cosine and sine: (a) $\cos(\omega t - 90°) = \sin \omega t$, (b) $\sin(\omega t + 180°) = -\sin \omega t$.

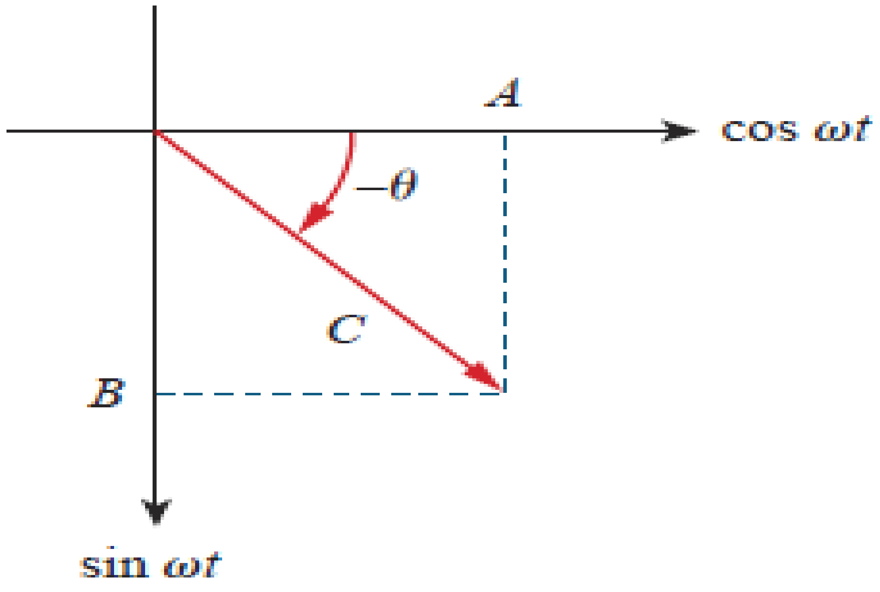
The graphical technique can also be used to add two sinusoids of the same frequency when one is in sine form and the other is in cosine form.

$$A\cos\omega t + B\sin\omega t = C\cos(\omega t - \theta)$$

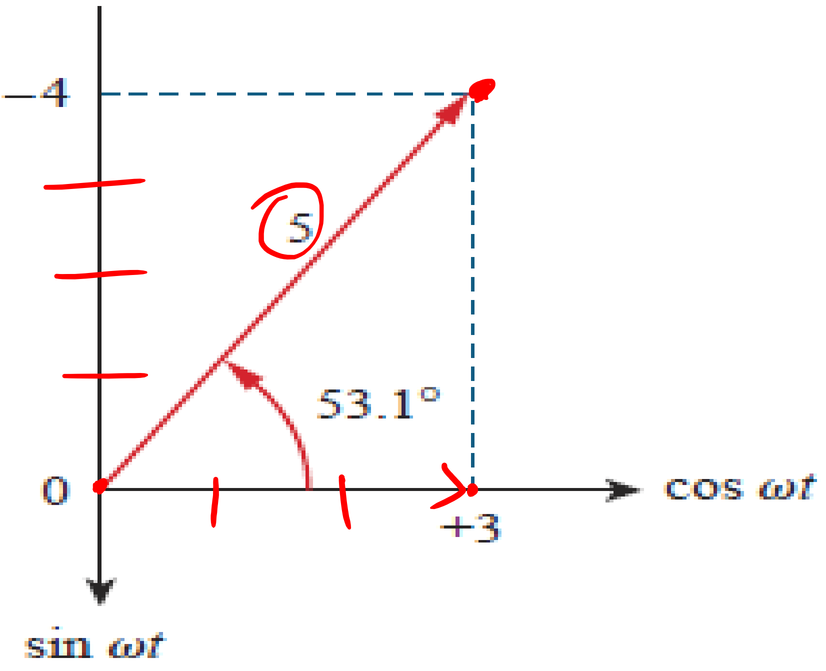
sin ωt

$$C = \sqrt{A^2 + B^2}, \qquad \theta = \tan^{-1}\frac{B}{A}$$

$$\sqrt{3^2 + (-4)^2} = \sqrt{25}$$



(a)



(b)

(a) Adding $A\cos\omega t$ and $B\sin\omega t$, (b) adding $3\cos\omega t$ and $-4\sin\omega t$.

$$\boxed{3\cos\omega t - 4\sin\omega t = 5\cos(\omega t + 53.1^\circ)}$$

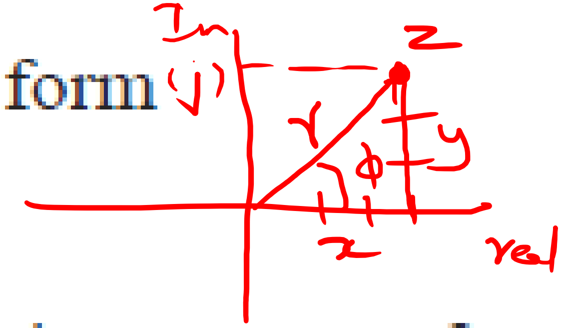
Phasors

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions.

> A **phasor** is a complex number that represents the amplitude and phase of a sinusoid.

- Phasors provide a simple means of analyzing linear circuits excited by sinusoidal sources; solutions of such circuits would be intractable otherwise.
- The notion of solving ac circuits using phasors was first introduced by Charles Steinmetz in 1893

A complex number z can be written in rectangular form



$$z = x + jy \rightarrow r\angle\phi$$

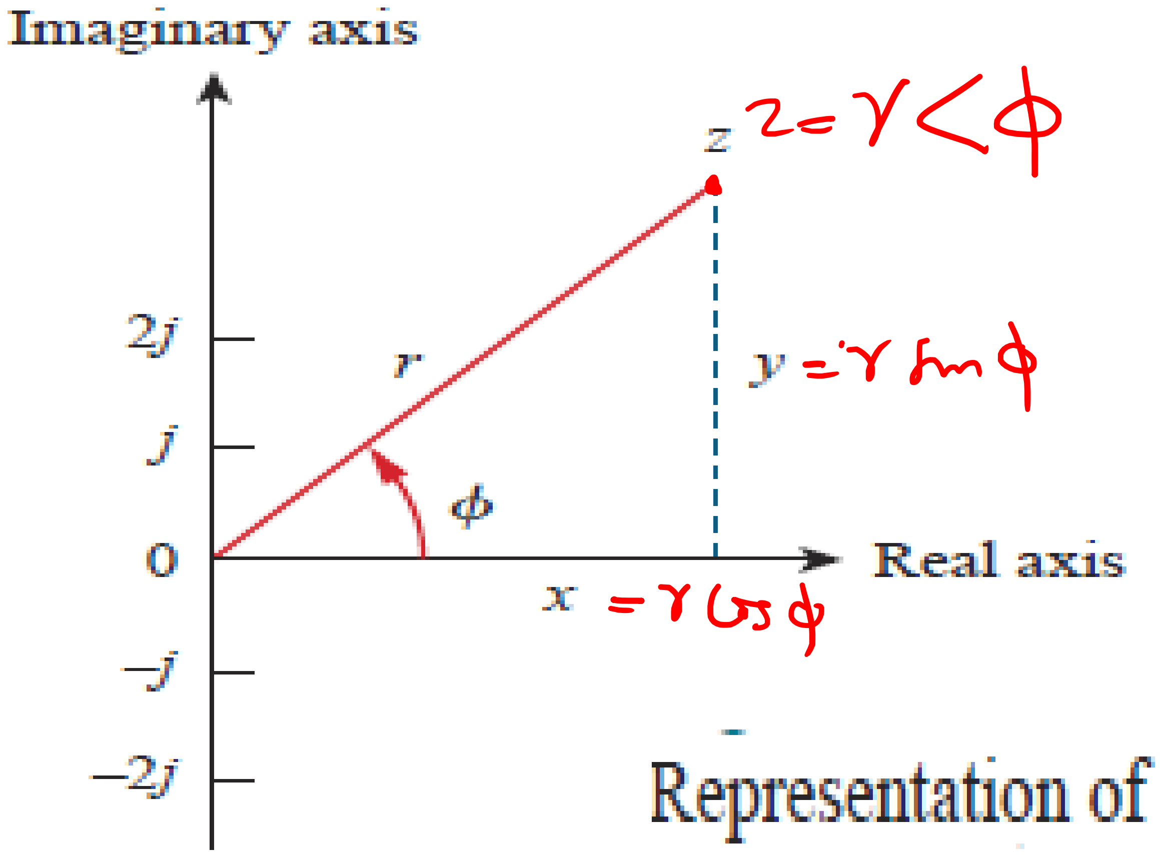
where $j = \sqrt{-1}$; x is the real part of z; y is the imaginary part of z.

The complex number z can also be written in polar or exponential form as

$$\rightarrow \quad z = r\angle\phi = re^{j\phi}$$

where r is the magnitude of z, and ϕ is the phase of z. We notice that z can be represented in three ways:

$z = x + jy$	Rectangular form
$z = r\angle\phi$	Polar form
✓ $z = re^{j\phi}$	Exponential form



$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

$z = r\cos\phi + j\, r\sin\phi$

$= r\,[\cos\phi + j\sin\phi]$

$re^{j\phi}$

Representation of a complex number $z = x + jy = r\underline{/\phi}$.

On the other hand, if we know r and ϕ, we can obtain x and y as

$$x = r\cos\phi, \qquad y = r\sin\phi$$

$$\boxed{z = x + jy = r\underline{/\phi} = r(\cos\phi + j\sin\phi)}$$

$$z = x + jy = r\angle\phi, \qquad z_1 = x_1 + jy_1 = r_1\angle\phi_1$$

$$z_2 = x_2 + jy_2 = r_2\angle\phi_2$$

Addition:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction:

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication:

$$z_1 z_2 = r_1 r_2 \angle\phi_1 + \phi_2$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle\phi_1 - \phi_2$$

Reciprocal:

$$\frac{1}{z} = \frac{1}{r}\angle-\phi$$

Square Root:

$$\sqrt{z} = \sqrt{r}\angle\phi/2$$

Complex Conjugate:

$$z^* = x - jy = r\angle-\phi = re^{-j\phi}$$

$z_1 = r_1 < \phi_1$

$z_2 = r_2 \angle \phi_2$

$z_1 z_2 = r_1 r_2 < \phi_1 + \phi_2$

$$\frac{1}{j} = -j$$

$z = (x + jy)$

$z^* = (x - jy)$

The idea of phasor representation is based on Euler's identity. In general,

$$\boxed{e^{\pm j\phi} = \cos\phi \pm j\sin\phi}$$

$$\cos\phi = \mathrm{Re}(e^{j\phi})$$
$$\sin\phi = \mathrm{Im}(e^{j\phi})$$

Given a sinusoid $v(t) = V_m \cos(\omega t \mp \phi)$,

$$v(t) = V_m \cos(\omega t + \phi) = \mathrm{Re}(V_m e^{j(\omega t + \phi)})$$

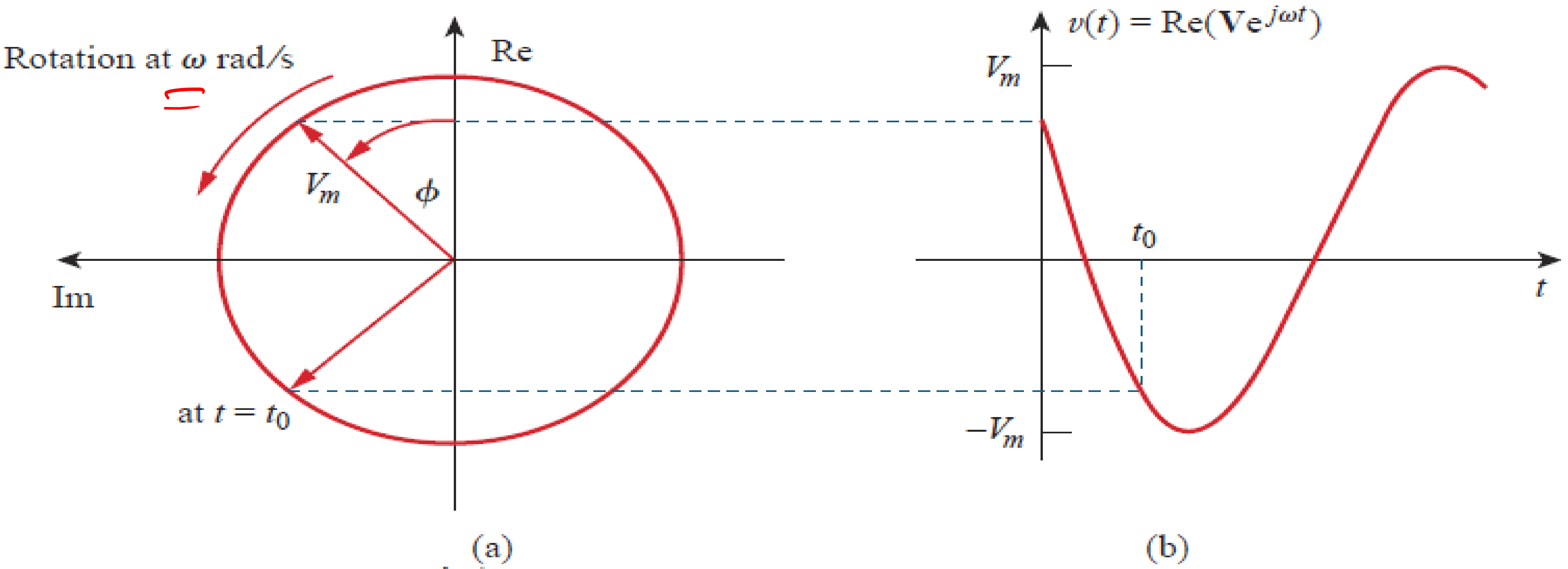
$$v(t) = \mathrm{Re}(V_m e^{j\phi} e^{j\omega t})$$

$$\boxed{v(t) = \mathrm{Re}(\mathbf{V} e^{j\omega t})}$$

a phasor is a complex representation of the magnitude and phase of a sinusoid

$$\mathbf{V} = V_m e^{j\phi} = V_m \underline{/\phi}$$

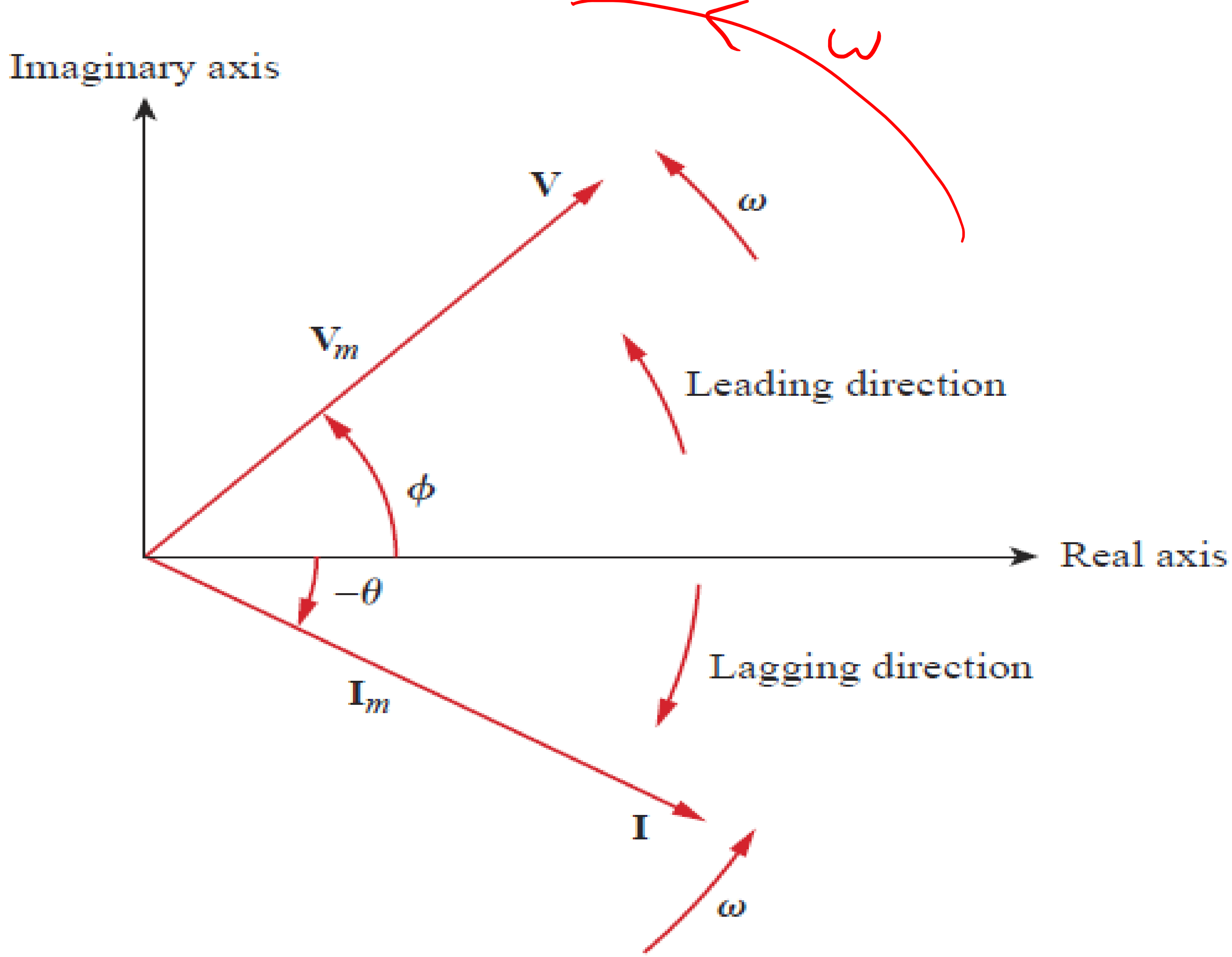
$\mathbf{V}$ is thus the *phasor representation* of the sinusoid $v(t)$



Representation of $\mathbf{V}e^{j\omega t}$: (a) sinor rotating counterclockwise, (b) its projection on the real axis, as a function of time.

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \angle \phi$$

(Time-domain representation) (Phasor-domain representation)



A phasor diagram showing $\mathbf{V} = V_m \underline{/\phi}$ and $\mathbf{I} = I_m \underline{/-\theta}$

Time domain representation	Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \phi - 90^\circ$
$I_m \cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle \theta - 90^\circ$

$$v(t) = \mathrm{Re}(\mathbf{V}e^{j\omega t}) = V_m \cos(\omega t + \phi)$$

$$\mathbf{V} = V_m e^{j\phi} = V_m \underline{/\phi}$$

$$\frac{dv}{dt} = -\omega V_m \sin(\omega t + \phi) = \omega V_m \cos(\omega t + \phi + 90^\circ)$$

$$= \mathrm{Re}(\omega V_m e^{j\omega t} e^{j\phi} e^{j90^\circ}) = \mathrm{Re}(j\omega \mathbf{V} e^{j\omega t})$$

This shows that the derivative $v(t)$ is transformed to the phasor domain as $j\omega\mathbf{V}$

$$\underset{\textbf{(Time domain)}}{\frac{dv}{dt}} \quad \Leftrightarrow \quad \underset{\textbf{(Phasor domain)}}{j\omega\mathbf{V}}$$

Similarly, the integral of $v(t)$ is transformed to the phasor domain as $\mathbf{V}/j\omega$

$$\underset{\textbf{(Time domain)}}{\int v\,dt} \quad \Leftrightarrow \quad \underset{\textbf{(Phasor domain)}}{\frac{\mathbf{V}}{j\omega}}$$

Evaluate these complex numbers:

(a) $(40\underline{/50^\circ} + 20\underline{/-30^\circ})^{1/2}$

$$\left[40\angle 50^\circ + 20\angle -30^\circ\right]^{1/2} \qquad r\angle\phi$$

(b) $\dfrac{10\underline{/-30^\circ} + (3 - j4)}{(2 + j4)(3 - j5)^*}$

$$\left[(40\cos 50^\circ + j40\sin 50^\circ) + [20\cos(-30) + j\sin(-30)]\right]^{1/2}$$

Solution:

(a) Using polar to rectangular transformation,

$$40\underline{/50^\circ} = 40(\cos 50^\circ + j\sin 50^\circ) = 25.71 + j30.64$$

$$20\underline{/-30^\circ} = 20[\cos(-30^\circ) + j\sin(-30^\circ)] = 17.32 - j10$$

$$\tan^{-1} y/x$$

Adding them up gives

$$40\underline{/50^\circ} + 20\underline{/-30^\circ} = 43.03 + j20.64 = 47.72\underline{/25.63^\circ} \qquad = r\angle\phi$$

$$x + jy$$

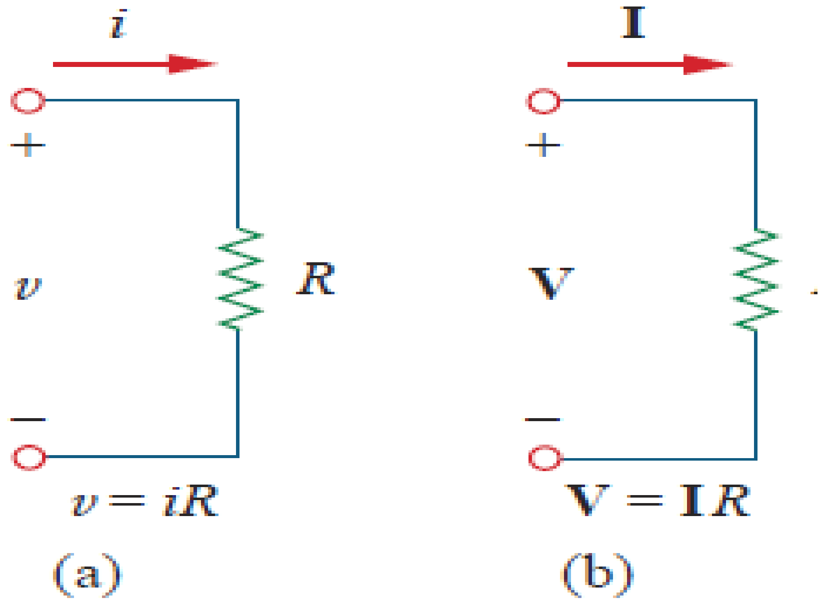
$$r = \sqrt{x^2 + y^2}$$

The differences between $v(t)$ and $\mathbf{V}$ should be emphasized:

1. $v(t)$ is the *instantaneous or time domain* representation, while $\mathbf{V}$ is the *frequency or phasor domain* representation.
2. $v(t)$ is time dependent, while $\mathbf{V}$ is not. (This fact is often forgotten by students.)
3. $v(t)$ is always real with no complex term, while $\mathbf{V}$ is generally complex.

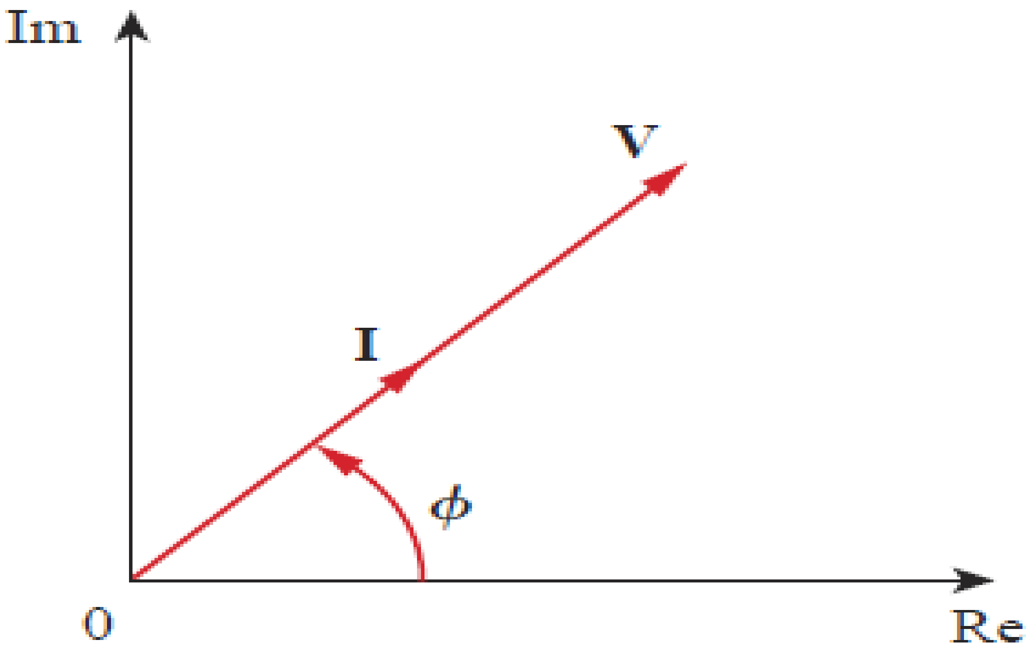
Finally, we should bear in mind that phasor analysis applies only when frequency is constant; it applies in manipulating two or more sinusoidal signals only if they are of the same frequency.

Phasor Relationships for Circuit Elements



If the current through a resistor R is
$i = I_m \cos(\omega t + \phi)$,
the voltage across it is given by Ohm's law as

$$v = iR = RI_m \cos(\omega t + \phi)$$



The phasor form of this voltage is

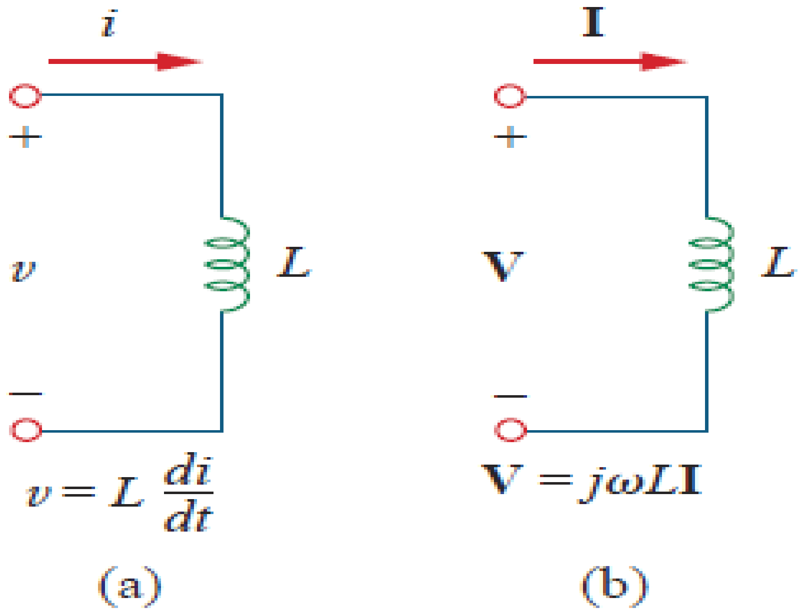
$$\mathbf{V} = RI_m \underline{/\phi}$$

$$\mathbf{I} = I_m \underline{/\phi}.$$

$$\mathbf{V} = R\mathbf{I}$$

Voltage-current relation for the resistor in the phasor domain continues to be Ohm's law, as in the time domain

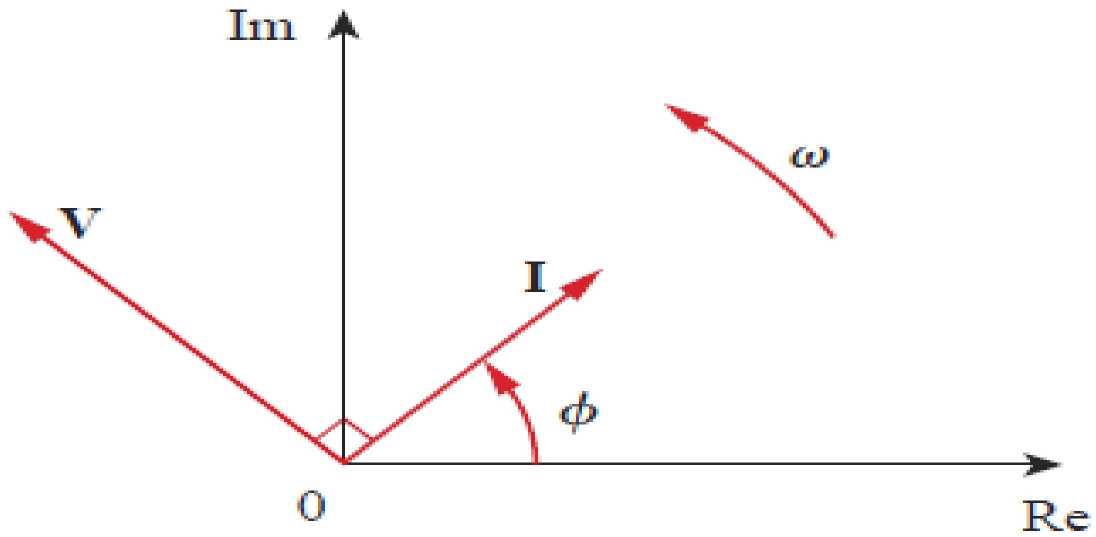
voltage and current are in phase



$$i = I_m \cos(\omega t + \phi)$$

The voltage across the inductor is

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi)$$



$$-\sin A = \cos(A + 90^\circ)$$

$$v = \omega L I_m \cos(\omega t + \phi + 90^\circ)$$

which transforms to the phasor

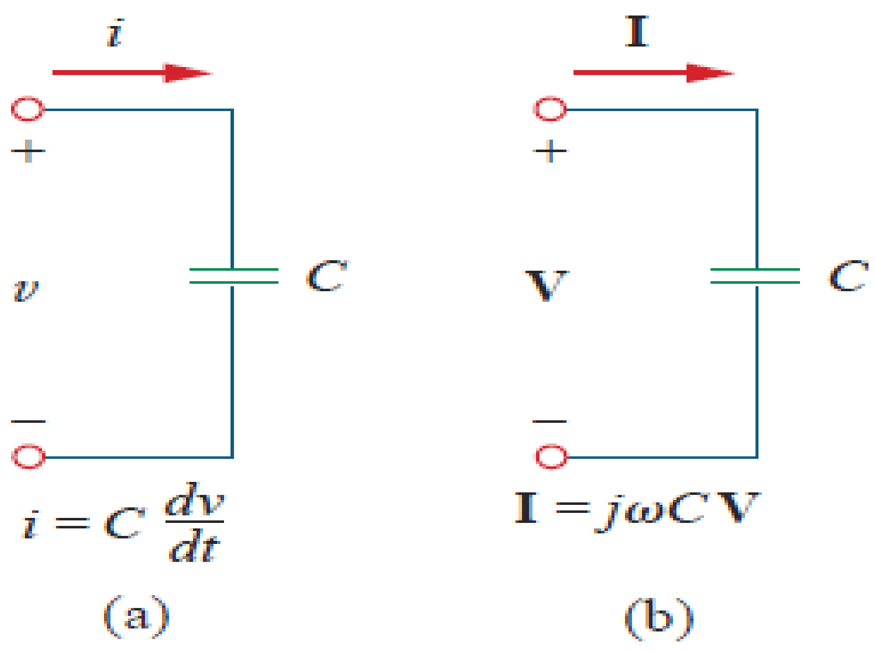
$$\mathbf{V} = \omega L I_m e^{j(\phi + 90^\circ)} = \omega L I_m e^{j\phi} e^{j90^\circ} = \omega L I_m \underline{/\phi + 90^\circ}$$

But $I_m \underline{/\phi} = \mathbf{I}$, $e^{j90^\circ} = j$. Thus, $\mathbf{V} = j\omega L\mathbf{I}$

showing that the voltage has a magnitude of $\omega L I_m$ and a phase of $(\emptyset + 90^0)$The voltage and current are out of phase. Specifically, the current lags the voltage by90^0

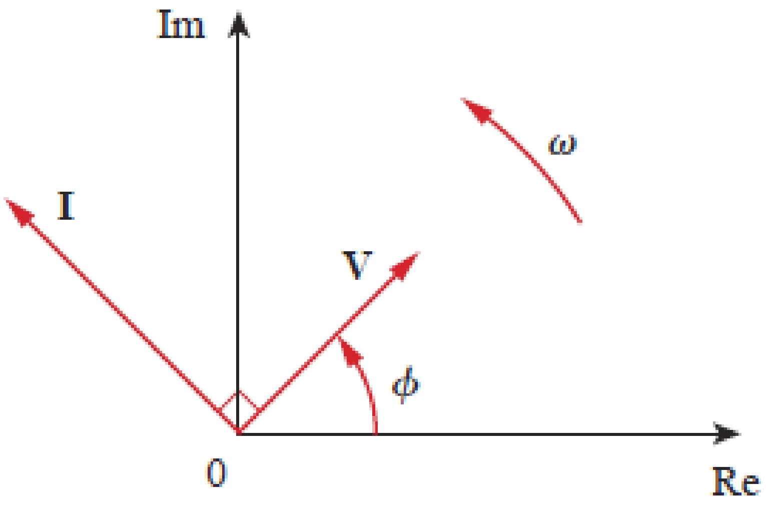
$$v = V_m \cos(\omega t + \phi)$$

The current through the capacitor is



$$i = C\frac{dv}{dt}$$

$$\mathbf{I} = j\omega C\mathbf{V} \quad \Rightarrow \quad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$



showing that the current and voltage are 90^0 out of phase. To be specific,the current leads the voltage by 90^0

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\dfrac{dv}{dt}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$

Impedance and Admittance

Current relations for the three passive elements as

$$\mathbf{V} = R\mathbf{I}, \qquad \mathbf{V} = j\omega L\mathbf{I}, \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

we obtain Ohm's law in phasor form for any type of element as

$$\boxed{\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \qquad \text{or} \qquad \mathbf{V} = \mathbf{Z}\mathbf{I}}$$

where $\mathbf{Z}$ is a frequency-dependent quantity known as *impedance*, measured in ohms.

The impedance $\mathbf{Z}$ of a circuit is the ratio of the phasor voltage $\mathbf{V}$ to the phasor current $\mathbf{I}$, measured in ohms (Ω).

Impedances and admittances of passive elements.

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \dfrac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \dfrac{1}{j\omega L}$
C	$\mathbf{Z} = \dfrac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

As a complex quantity, the impedance may be expressed in rectangular form as

$$\mathbf{Z} = R + jX$$

where $R = \text{Re}\,\mathbf{Z}$ is the *resistance* and $X = \text{Im}\,\mathbf{Z}$ is the *reactance*.

The reactance X may be positive or negative.

- Impedance is inductive when *X* is positive

impedance $\mathbf{Z} = R + jX$ is said to be *inductive* or lagging since current lags voltage

- Impedance is capacitive when *X* is negative

impedance $\mathbf{Z} = R - jX$ is capacitive or leading because current leads voltage

The impedance, resistance, and reactance are all measured in ohms.

The impedance may also be expressed in polar form as

$$\mathbf{Z} = |\mathbf{Z}| \angle \theta$$

$$\boxed{\mathbf{Z} = R + jX = |\mathbf{Z}| \angle \theta}$$

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \qquad \theta = \tan^{-1} \frac{X}{R}$$

$$R = |\mathbf{Z}| \cos \theta, \qquad X = |\mathbf{Z}| \sin \theta$$

It is sometimes convenient to work with the reciprocal of impedance, known as ***admittance***.

The **admittance** $\mathbf{Y}$ is the reciprocal of impedance, measured in siemens (S).

The admittance **Y** of an element (or a circuit) is the ratio of the phasor current through it to the phasor voltage across it, or

$$\boxed{\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}}}$$

As a complex quantity, we may write **Y** as

$$\boxed{\mathbf{Y} = G + jB}$$

where $G = \operatorname{Re}\mathbf{Y}$ is called the *conductance* and $B = \operatorname{Im}\mathbf{Y}$ is called the *susceptance*. Admittance, conductance, and susceptance are all expressed in the unit of siemens (or mhos).

$$G + jB = \frac{1}{R + jX}$$

By rationalization,

$$G + jB = \frac{1}{R + jX} \cdot \frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + X^2}$$

Equating the real and imaginary parts gives

$$G = \frac{R}{R^2 + X^2}, \qquad B = -\frac{X}{R^2 + X^2}$$

showing that $G \neq 1/R$ as it is in resistive circuits. Of course, if $X = 0$, then $G = 1/R$.