Instantaneous and Average Power

The *instantaneous power p*(*t*) absorbed by an element is the product of the instantaneous voltage *v*(*t*) across the element and the instantaneous current *i*(*t*) through it.

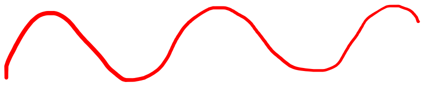
$P = V \times I$

Time varying signals

$V(t)$

$i(t)$

$$\boxed{p(t) = v(t)i(t)}$$



t

The instantaneous power (in watts) is the power at any instant of time.

$P = \frac{\text{Work}}{\text{time}}$

$\frac{\text{energy}}{\text{time}}$

It is the rate at which an element absorbs energy

$$v(t) = V_m \cos(\omega t + \theta_v)$$

$$i(t) = I_m \cos(\omega t + \theta_i)$$

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$$

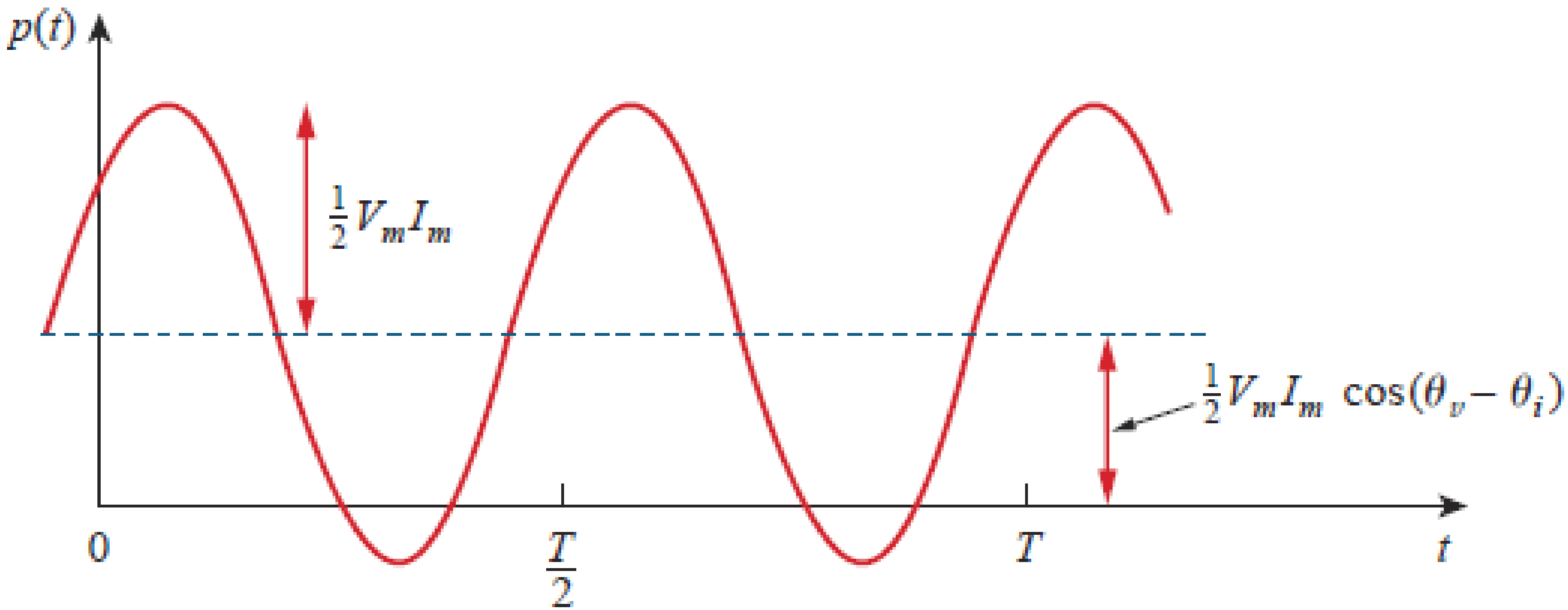
We apply the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

$$p(t) = \underbrace{\frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)}_{1} + \underbrace{\frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)}_{2}$$

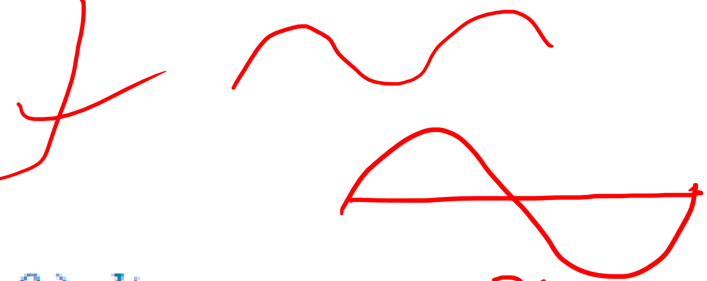
- The first part is constant or time independent. Its value depends on the phase difference between the voltage and the current.
- The second part is a sinusoidal function whose frequency is which is twice the angular frequency 2ω of the voltage or current.

$$p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$



The **average power**, in watts, is the average of the instantaneous power over one period.

$$\boxed{P = \frac{1}{T}\int_0^T p(t)\,dt}$$



360

T

$$P = \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\,dt$$

$$+ \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)\,dt$$

$$= \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\frac{1}{T}\int_0^T dt$$

$$+ \frac{1}{2}V_m I_m \frac{1}{T}\int_0^T \cos(2\omega t + \theta_v + \theta_i)\,dt$$

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \frac{1}{T} \int_0^T dt + \frac{1}{2} V_m I_m \frac{1}{T} \int_0^T \cos(2\omega t + \theta_v + \theta_i)\, dt$$

- **The first integrand is constant, and the average of a constant is the same constant. The second integrand is a sinusoid.**

- **The average of a sinusoid over its period is zero because the area under the sinusoid during a positive half-cycle is canceled by the area under it during the following negative half-cycle.**

$$\boxed{P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)}$$

$e^{j\phi}$

$e^{-j\phi}$

$$\boxed{P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)}$$

$$\boxed{\frac{1}{2}\mathbf{VI}^* = \frac{1}{2} V_m I_m \underline{/\theta_v - \theta_i}}$$

$$= \underline{\frac{1}{2} V_m I_m[\cos(\theta_v - \theta_i)} + j \sin(\theta_v - \theta_i)]$$

* Complex conjugate

$$\boxed{P = \frac{1}{2}\mathrm{Re}[\mathbf{VI}^*] = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)}$$

$$P = \frac{1}{2}\text{Re}[\mathbf{VI}^*] = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$$

When $\theta_v = \theta_i$, the voltage and current are in phase. This implies a purely resistive circuit or resistive load R, and

$$P = \frac{1}{2}V_m I_m = \frac{1}{2}I_m^2 R = \frac{1}{2}|\mathbf{I}|^2 R$$

When $\theta_v - \theta_i = \pm 90^\circ$, we have a purely reactive circuit,

$$P = \frac{1}{2}V_m I_m \cos 90^\circ = 0$$

A resistive load (R) absorbs power at all times, while a reactive load (L or C) absorbs zero average power.

Example

Given that $v(t) = 120\cos(377t + 45^\circ)$ V and $i(t) = 10\cos(377t - 10^\circ)$ A

find the instantaneous power and the average power absorbed by the passive linear network

Solution:

The instantaneous power is given by

$$p = vi = 1200\cos(377t + 45^\circ)\cos(377t - 10^\circ)$$

Applying the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

gives

$$p = 600[\cos(754t + 35^\circ) + \cos 55^\circ]$$

or

$$p(t) = 344.2 + 600\cos(754t + 35^\circ) \text{ W}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}120(10)\cos[45^\circ - (-10^\circ)]$$

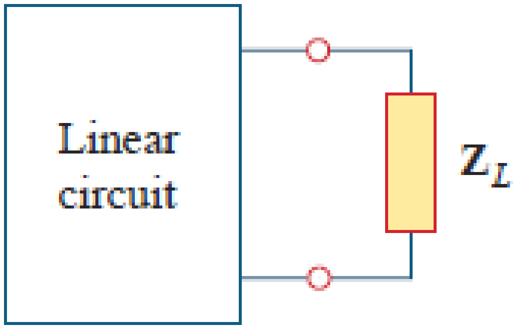
$$= 600\cos 55^\circ = 344.2 \text{ W}$$

which is the constant part of $p(t)$ above.

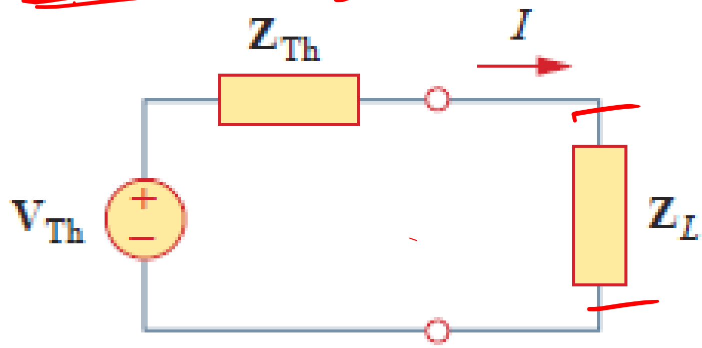
Maximum Average Power Transfer

R_{TH}

Impedence.



(a)



(b)

$(x+yj)$

$(2+3j)$

Thevenin impedance $\mathbf{Z}_{\text{Th}}$ and the load impedance $\mathbf{Z}_L$ are

AC

$$\mathbf{Z}_{\text{Th}} = R_{\text{Th}} + jX_{\text{Th}}$$

$$\mathbf{Z}_L = R_L + jX_L$$

Thevenin
Norton
Mesh
Nodal
Superposit

The current through the load is

$$\mathbf{I} = \frac{\mathbf{V}_{\text{Th}}}{\mathbf{Z}_{\text{Th}} + \mathbf{Z}_L} = \frac{\mathbf{V}_{\text{Th}}}{(R_{\text{Th}} + jX_{\text{Th}}) + (R_L + jX_L)}$$

The average power delivered to the load is

$$P = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{|\mathbf{V}_{\mathrm{Th}}|^2 R_L/2}{(R_{\mathrm{Th}} + R_L)^2 + (X_{\mathrm{Th}} + X_L)^2}$$

$$\frac{\partial P}{\partial X_L} = -\frac{|\mathbf{V}_{\mathrm{Th}}|^2 R_L (X_{\mathrm{Th}} + X_L)}{[(R_{\mathrm{Th}} + R_L)^2 + (X_{\mathrm{Th}} + X_L)^2]^2}$$

$$\frac{\partial P}{\partial R_L} = \frac{|\mathbf{V}_{\mathrm{Th}}|^2[(R_{\mathrm{Th}} + R_L)^2 + (X_{\mathrm{Th}} + X_L)^2 - 2R_L(R_{\mathrm{Th}} + R_L)]}{2[(R_{\mathrm{Th}} + R_L)^2 + (X_{\mathrm{Th}} + X_L)^2]^2}$$

Setting $\partial P/\partial X_L$ to zero gives

$$X_L = -X_{\mathrm{Th}}$$

and setting $\partial P/\partial R_L$ to zero results in

$$R_L = \sqrt{R_{\mathrm{Th}}^2 + (X_{\mathrm{Th}} + X_L)^2}$$

$$\boxed{\mathbf{Z}_L = R_L + jX_L = R_{\text{Th}} - jX_{\text{Th}} = \mathbf{Z}_{\text{Th}}^*}$$

For **maximum average power transfer**, the load impedance $\mathbf{Z}_L$ must be equal to the complex conjugate of the Thevenin impedance $\mathbf{Z}_{\text{Th}}$.

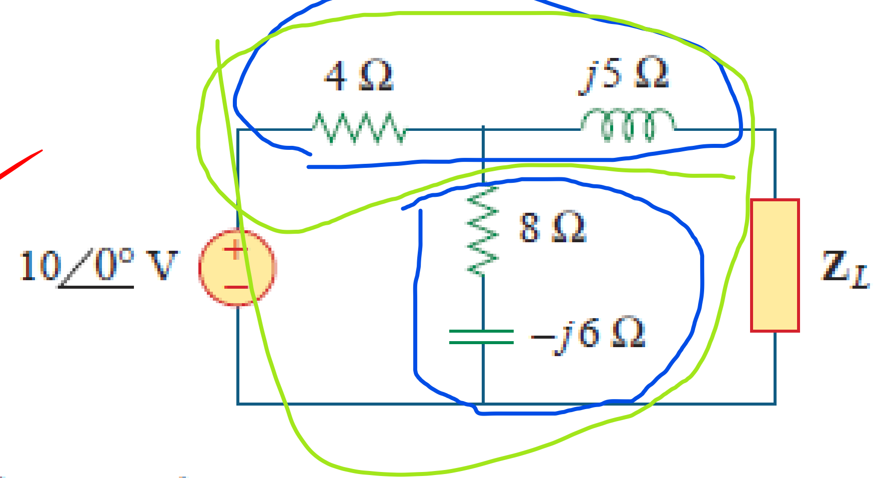
We have
$$P = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{|\mathbf{V}_{\text{Th}}|^2 R_L/2}{(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2}$$

Setting $R_L = R_{\text{Th}}$ and $X_L = -X_{\text{Th}}$

$$\boxed{P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{8R_{\text{Th}}}}$$

Example

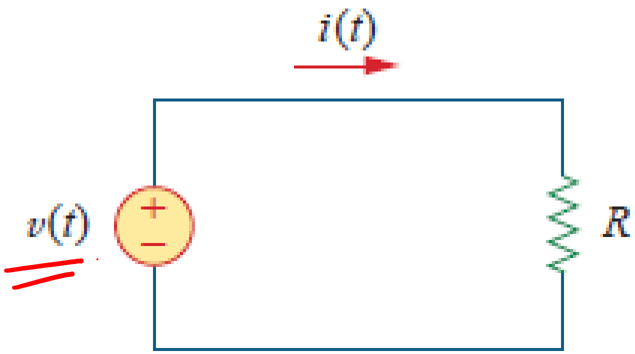
Determine the load impedance $\mathbf{Z}_L$ that maximizes the average power drawn from the circuit of Fig. 11.8. What is the maximum average power?



$$\mathbf{Z}_{\text{Th}} = j5 + 4 \parallel (8 - j6) = j5 + \frac{4(8 - j6)}{4 + 8 - j6} = 2.933 + j4.467\ \Omega$$

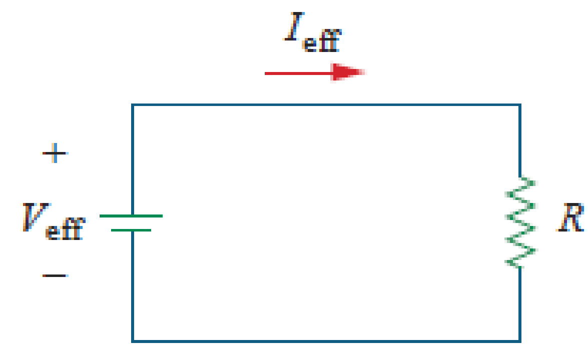
Effective or RMS Value

The **effective value** of a periodic current is the dc current that delivers the same average power to a resistor as the periodic current.



P_{av}

(a)



P_{av}

(b)

root(Mean (Sqrs))

$$P = \frac{1}{T}\int_0^T i^2 R\, dt = \frac{R}{T}\int_0^T i^2\, dt$$

$$\boxed{P = I_{\text{eff}}^2 R}$$

$$I_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T i^2\, dt} \qquad V_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T v^2\, dt} \qquad I_{\text{eff}} = I_{\text{rms}}, \qquad V_{\text{eff}} = V_{\text{rms}}$$

This indicates that the effective value is the (square) *root* of the *mean* (or average) of the *square* of the periodic signal. Thus, the effective value is often known as the *root-mean-square* value, or *rms* value for short;

For any periodic function $x(t)$ in general, the rms value is given by

$$\boxed{X_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T x^2\,dt}}$$

T

The **effective value** of a periodic signal is its root mean square (rms) value.

states that to find the rms value of $x(t)$, we first find its *square* x^2 and then find the *mean* of that, or

$$\frac{1}{T}\int_0^T x^2\,dt$$

For the sinusoid $i(t) = I_m \cos \omega t$, the effective or rms value is

$V(t) = V_m \cos \omega t$

$$I_{\rm rms} = \sqrt{\frac{1}{T}\int_0^T I_m^2 \cos^2 \omega t \, dt}$$

$$= \sqrt{\frac{I_m^2}{T}\int_0^T \frac{1}{2}(1 + \cos 2\omega t)\, dt} = \frac{I_m}{\sqrt{2}}$$

Similarly, for $v(t) = V_m \cos \omega t$,

$$V_{\rm rms} = \frac{V_m}{\sqrt{2}}$$

$I_{rms} = \frac{I_m}{\sqrt{2}}$

$V_{rms} = \frac{V_m}{\sqrt{2}}$

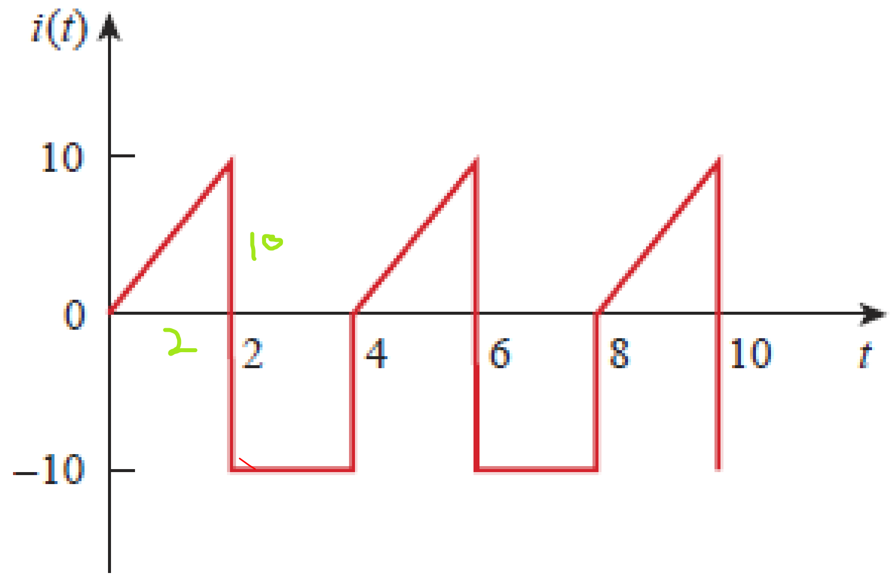
The average power can be written in terms of the rms values.

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i)$$

$$= V_{\rm rms} I_{\rm rms} \cos(\theta_v - \theta_i)$$

Example

Determine the rms value of the current waveform in Fig. 11.14. If the current is passed through a 2-Ω resistor, find the average power absorbed by the resistor.



The period of the waveform is $T = 4$. Over a period, we can write the current waveform as

$$i(t) = \begin{cases} 5t, & 0 < t < 2 \\ -10, & 2 < t < 4 \end{cases}$$

The period of the waveform is $T = 4$. Over a period, we can write the current waveform as

$$i(t) = \begin{cases} 5t, & 0 < t < 2 \\ -10, & 2 < t < 4 \end{cases}$$

The rms value is

$$I_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T i^2\,dt} = \sqrt{\frac{1}{4}\left[\int_0^2 (5t)^2\,dt + \int_2^4 (-10)^2\,dt\right]}$$

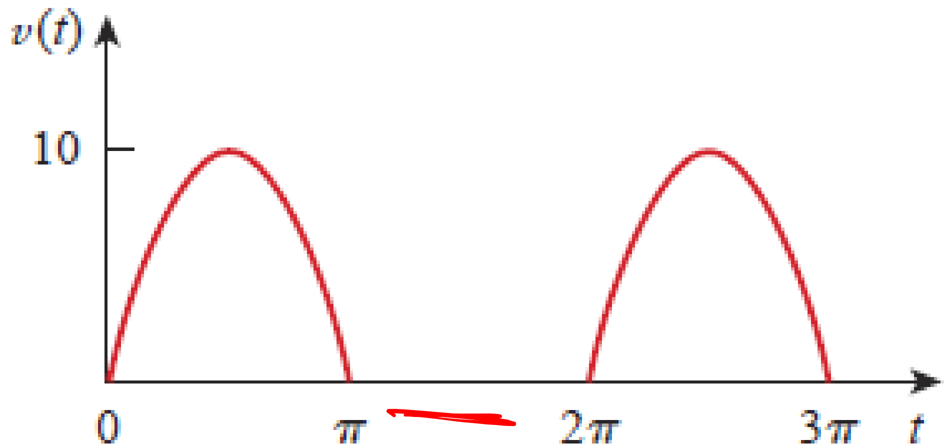
$$= \sqrt{\frac{1}{4}\left[25\frac{t^3}{3}\Big|_0^2 + 100t\Big|_2^4\right]} = \sqrt{\frac{1}{4}\left(\frac{200}{3} + 200\right)} = 8.165\text{ A}$$

The power absorbed by a 2-Ω resistor is

$$P = I_{\text{rms}}^2 R = (8.165)^2(2) = 133.3\text{ W}$$

Example

The waveform shown in Fig. 11.16 is a half-wave rectified sine wave. Find the rms value and the amount of average power dissipated in a 10-Ω resistor.



The period of the voltage waveform is $T = 2\pi$, and

$$v(t) = \begin{cases} 10 \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

The period of the voltage waveform is $T = 2\pi$, and

$$v(t) = \begin{cases} 10\sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

The rms value is obtained as

$$V_{\mathrm{rms}}^2 = \frac{1}{T}\int_0^T v^2(t)\,dt = \frac{1}{2\pi}\left[\int_0^{\pi}(10\sin t)^2\,dt + \int_{\pi}^{2\pi} 0^2\,dt\right]$$

But $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$. Hence,

$$V_{\mathrm{rms}}^2 = \frac{1}{2\pi}\int_0^{\pi}\frac{100}{2}(1 - \cos 2t)\,dt = \frac{50}{2\pi}\left(t - \frac{\sin 2t}{2}\right)\Bigg|_0^{\pi}$$

$$= \frac{50}{2\pi}\left(\pi - \frac{1}{2}\sin 2\pi - 0\right) = 25, \qquad V_{\mathrm{rms}} = 5\text{ V}$$

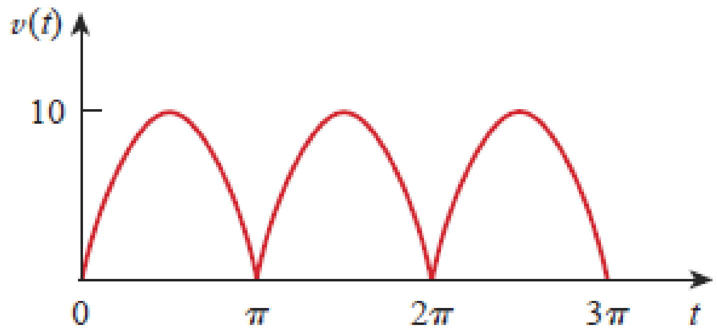
The average power absorbed is

$$P = \frac{V_{\mathrm{rms}}^2}{R} = \frac{5^2}{10} = 2.5\text{ W}$$

Practice Problem

Find the rms value of the full-wave rectified sine wave in Fig. 11.17. Calculate the average power dissipated in a 6-Ω resistor.

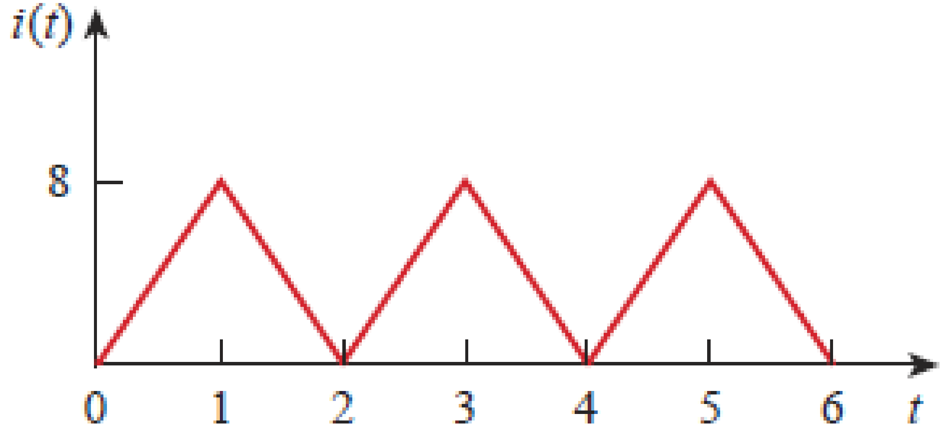
Answer: 7.071 V, 8.333 W.



Practice Problem

Find the rms value of the current waveform of Fig. 11.15. If the current flows through a 9-Ω resistor, calculate the average power absorbed by the resistor.

Answer: 4.318 A, 192 W.



Apparent Power and Power Factor

$$v(t) = V_m \cos(\omega t + \theta_v) \quad \text{and} \quad i(t) = I_m \cos(\omega t + \theta_i)$$

in phasor form, $\mathbf{V} = V_m \underline{/\theta_v}$ and $\mathbf{I} = I_m \underline{/\theta_i}$

the average power is $P = \dfrac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i)$$
$$= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i)$$

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) = S \cos(\theta_v - \theta_i)$$

We have added a new term to the equation:

$$\boxed{S = V_{\text{rms}} I_{\text{rms}}}$$

$$S = V_{rms} I_{rms}$$

The average power is a product of two terms. The product $V_{rms} I_{rms}$ is known as the *apparent power* S. The factor $\cos(\theta_v - \theta_i)$ is called the *power factor* (pf).

The **apparent power** (in VA) is the product of the rms values of voltage and current.

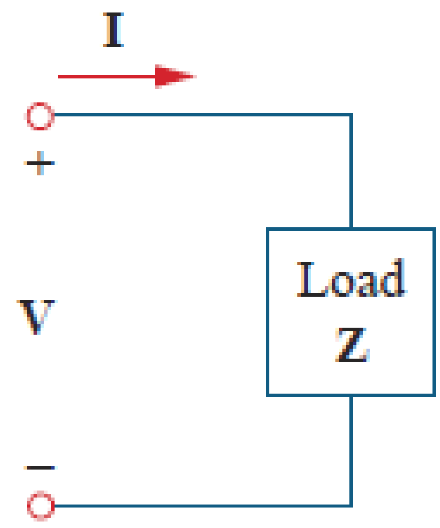
$$\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$

The angle $\theta_v - \theta_i$ is called the *power factor angle*

The power factor is dimensionless, since it is the ratio of the average power to the apparent power

Complex Power

Given the phasor form $\mathbf{V} = V_m\underline{/\theta_v}$ and $\mathbf{I} = I_m\underline{/\theta_i}$ of voltage $v(t)$ and current $i(t)$, the *complex power* $\mathbf{S}$ absorbed by the ac load



$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^*$$

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}}$$

$$\mathbf{V}_{\text{rms}} = \frac{\mathbf{V}}{\sqrt{2}} = V_{\text{rms}}\underline{/\theta_v}$$

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{I}}{\sqrt{2}} = I_{\text{rms}}\underline{/\theta_i}$$

$$\boxed{\begin{aligned}\mathbf{S} &= V_{\text{rms}}I_{\text{rms}}\underline{/\theta_v - \theta_i} \\ &= V_{\text{rms}}I_{\text{rms}}\cos(\theta_v - \theta_i) + jV_{\text{rms}}I_{\text{rms}}\sin(\theta_v - \theta_i)\end{aligned}}$$

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{V_{\text{rms}}}{I_{\text{rms}}} \angle \theta_v - \theta_i$$

Thus, $\mathbf{V}_{\text{rms}} = \mathbf{Z}\mathbf{I}_{\text{rms}}$

$$\boxed{\mathbf{S} = I_{\text{rms}}^2 \mathbf{Z} = \frac{V_{\text{rms}}^2}{\mathbf{Z}^*} = \mathbf{V}_{\text{rms}} \mathbf{I}_{\text{rms}}^*}$$

Since $\mathbf{Z} = R + jX$, $\quad \mathbf{S} = I_{\text{rms}}^2 (R + jX) = P + jQ$

$$\mathbf{S} = I_{\text{rms}}^2(R + jX) = P + jQ$$

$$P = \text{Re}(\mathbf{S}) = I_{\text{rms}}^2 R$$

$$Q = \text{Im}(\mathbf{S}) = I_{\text{rms}}^2 X$$

$$\mathbf{S} = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) + jV_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i)$$

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i), \qquad Q = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i)$$

- **The real power *P* is the average power in watts delivered to a load; It is the actual power dissipated by the load.**

- **The reactive power *Q* is a measure of the energy exchange between the source and the reactive part of the load. The unit of *Q* is the *volt-ampere reactive* (VAR) to distinguish it from the real power, whose unit is the watt.**

- **We know that energy storage elements neither dissipate nor supply power, but exchange power back and forth with the rest of the network. In the same way, the reactive power is being transferred back and forth between the load and the source. It represents a lossless interchange between the load and the source.**

1. $Q = 0$ for resistive loads (unity pf).
2. $Q < 0$ for capacitive loads (leading pf).
3. $Q > 0$ for inductive loads (lagging pf).

Thus,

Complex power (in VA) is the product of the rms voltage phasor and the complex conjugate of the rms current phasor. As a complex quantity, its real part is real power P and its imaginary part is reactive power Q.

$$\text{Complex Power} = \mathbf{S} = P + jQ = \mathbf{V}_{\text{rms}}(\mathbf{I}_{\text{rms}})^*$$

$$= V_{\text{rms}} I_{\text{rms}} \underline{/\theta_v - \theta_i}$$

$$\text{Apparent Power} = S = |\mathbf{S}| = V_{\text{rms}} I_{\text{rms}} = \sqrt{P^2 + Q^2}$$

$$\text{Real Power} = P = \text{Re}(\mathbf{S}) = S\cos(\theta_v - \theta_i)$$

$$\text{Reactive Power} = Q = \text{Im}(\mathbf{S}) = S\sin(\theta_v - \theta_i)$$

$$\text{Power Factor} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$

Example 11.11

The voltage across a load is $v(t) = 60\cos(\omega t - 10^\circ)$ V and the current through the element in the direction of the voltage drop is $i(t) = 1.5\cos(\omega t + 50^\circ)$ A. Find: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

$$v(t) = 60\cos(\omega t - 10^\circ)\text{ V}$$
$$i(t) = 1.5\cos(\omega t + 50^\circ)\text{ A.}$$

(a) For the rms values of the voltage and current, we write

$$\mathbf{V}_{\text{rms}} = \frac{60}{\sqrt{2}}\underline{/-10^\circ}, \qquad \mathbf{I}_{\text{rms}} = \frac{1.5}{\sqrt{2}}\underline{/+50^\circ}$$

The complex power is

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}} = \left(\frac{60}{\sqrt{2}}\underline{/-10^\circ}\right)\left(\frac{1.5}{\sqrt{2}}\underline{/-50^\circ}\right) = 45\underline{/-60^\circ}\text{ VA}$$

$$\mathbf{S} = \mathbf{V}_{\mathrm{rms}}\mathbf{I}^*_{\mathrm{rms}} = 45\underline{/-60^\circ}\text{ VA}$$

The apparent power is $\quad S = |\mathbf{S}| = 45\text{ VA}$

(b) We can express the complex power in rectangular form as

$$\mathbf{S} = 45\underline{/-60^\circ} = 45[\cos(-60^\circ) + j\sin(-60^\circ)] = 22.5 - j38.97$$

Since $\mathbf{S} = P + jQ$, the real power is

$$P = 22.5\text{ W}$$

while the reactive power is

$$Q = -38.97\text{ VAR}$$

(c) The power factor is

$$\text{pf} = \cos(-60^\circ) = 0.5 \text{ (leading)}$$

It is leading, because the reactive power is negative. The load impedance is

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{60\angle{-10^\circ}}{1.5\angle{+50^\circ}} = 40\angle{-60^\circ}\ \Omega$$

which is a capacitive impedance.