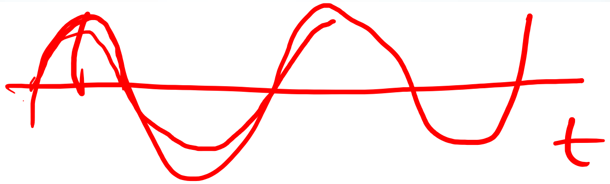
✓ Voltages and currents in a circuit with a constant frequency source

✓ let the amplitude of the sinusoidal source remain constant and vary the frequency, we obtain the circuit's *frequency response*

$\omega = \omega_0$

The frequency response of a circuit is the variation in its behavior with change in signal frequency.

$\omega \rightarrow$ constant



$\omega_1 \rightarrow 2\pi f_1$

ω_2 $\qquad f_2$

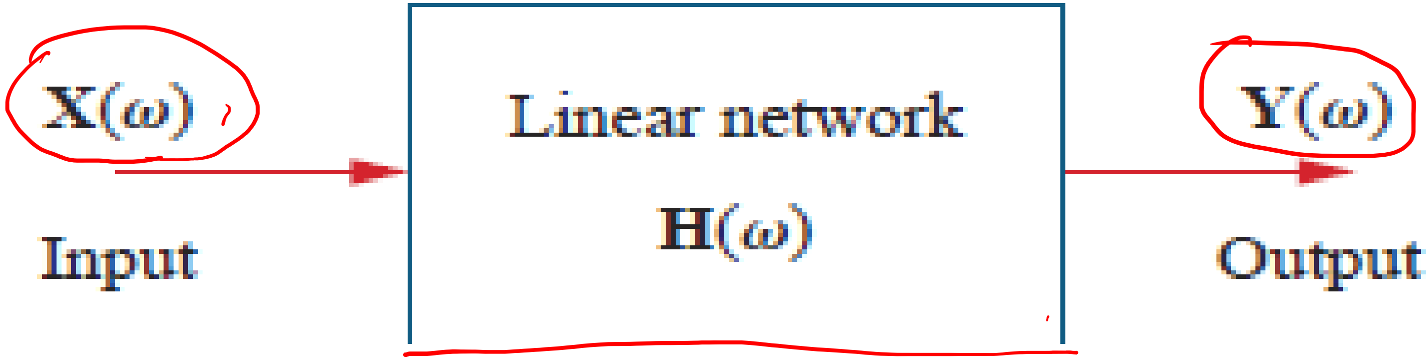
$\rightarrow$ R, L, C

series

parallel

Transfer Function

The transfer function $\mathbf{H}(\omega)$ (also called the *network function*) is a useful analytical tool for finding the frequency response of a circuit.



The **transfer function** $\mathbf{H}(\omega)$ of a circuit is the frequency-dependent ratio of a phasor output $\mathbf{Y}(\omega)$ (an element voltage or current) to a phasor input $\mathbf{X}(\omega)$ (source voltage or current).

$$\boxed{\mathbf{H}(\omega) = \frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)}}$$

Since the input and output can be either voltage or current at any place in the circuit, there are four possible transfer functions:

$$\mathbf{H}(\omega) = \text{Voltage gain} = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)}$$

Frequency domain

$$\mathbf{H}(\omega) = \text{Current gain} = \frac{\mathbf{I}_o(\omega)}{\mathbf{I}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Transfer Impedance} = \frac{\mathbf{V}_o(\omega)}{\mathbf{I}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Transfer Admittance} = \frac{\mathbf{I}_o(\omega)}{\mathbf{V}_i(\omega)}$$

where subscripts i and o denote input and output values.

Being a complex quantity, $\mathbf{H}(\omega)$ has a magnitude $H(\omega)$ and a phase ϕ; that is,

$$\boxed{\mathbf{H}(\omega) = H(\omega)\underline{/\phi}.}$$

The Decibel Scale

In communications systems, gain is measured in *bels*. Historically, the bel is used to measure the ratio of two levels of power or power gain *G*;

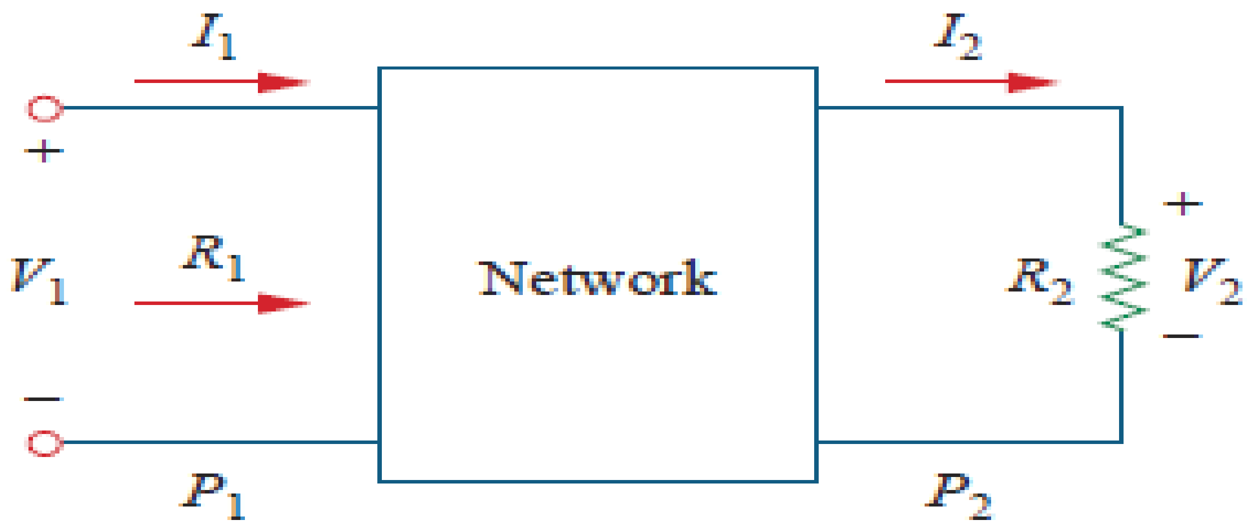
$$\boxed{G_{dB} = 10 \log_{10} \frac{P_2}{P_1}}$$

When $P_1 = P_2$, there is no change in power and the gain is 0 dB. If $P_2 = 2P_1$, the gain is

$$G_{dB} = 10 \log_{10} 2 \simeq 3 \text{ dB}$$

and when $P_2 = 0.5P_1$, the gain is

$$G_{dB} = 10 \log_{10} 0.5 \simeq -3 \text{ dB}$$



$$G_{\text{dB}} = 10 \log_{10} \frac{P_2}{P_1} = 10 \log_{10} \frac{V_2^2/R_2}{V_1^2/R_1}$$

$$= 10 \log_{10}\left(\frac{V_2}{V_1}\right)^2 + 10 \log_{10} \frac{R_1}{R_2}$$

$$G_{\text{dB}} = 20 \log_{10} \frac{V_2}{V_1} - 10 \log_{10} \frac{R_2}{R_1}$$

$R_2 = R_1$:

$$\boxed{G_{\text{dB}} = 20 \log_{10} \frac{V_2}{V_1}}$$

Instead, if $P_1 = I_1^2 R_1$ and $P_2 = I_2^2 R_2$, for $R_1 = R_2$, we obtain

$$G_{\mathrm{dB}} = 20 \log_{10} \frac{I_2}{I_1}$$

Bode Plots

Bode plots are semilog plots of the magnitude (in decibels) and phase (in degrees) of a transfer function versus frequency.

Resonant circuits

Series RLC
parallel RLC

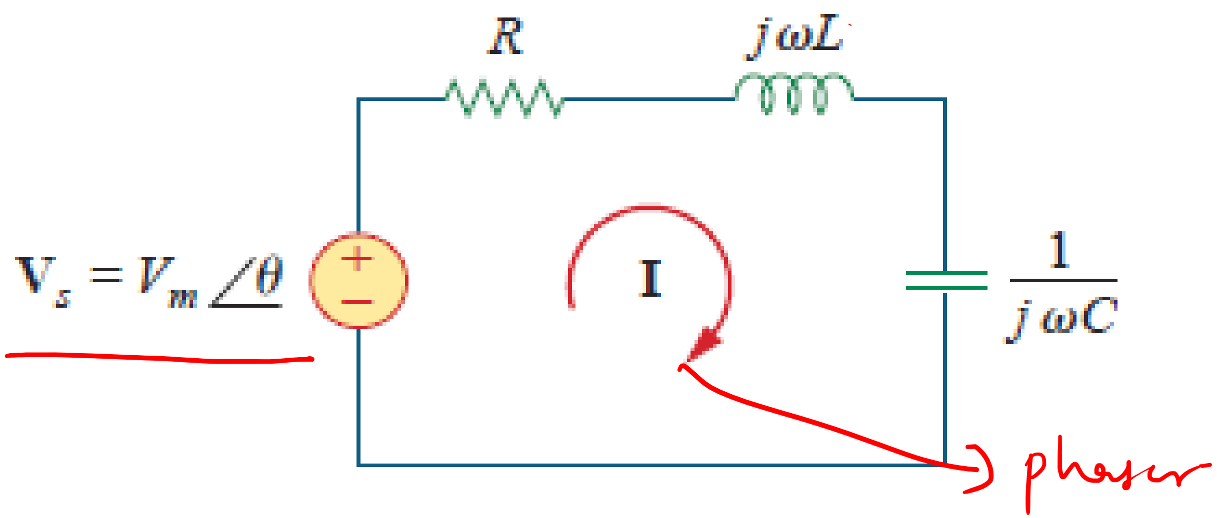
ω_0

I_n $\xrightarrow{\omega_0}$ I_n

Resonance is a condition in an *RLC* circuit in which the capacitive and inductive reactances are equal in magnitude, thereby resulting in a purely resistive impedance.

Resonant circuits (series or parallel) are useful for constructing filters, as their transfer functions can be highly frequency selective. They are used in many applications such as selecting the desired stations in radio and TV receivers.

Series Resonance



$$\mathbf{Z} = \mathbf{H}(\omega) = \frac{\mathbf{V}_s}{\mathbf{I}} = R + j\omega L + \frac{1}{j\omega C}$$

$$\mathbf{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

Resonance results when the imaginary part of the transfer function is zero, or

$$\text{Im}(\mathbf{Z}) = \omega L - \frac{1}{\omega C} = 0$$

$$\omega L = \frac{1}{\omega C}$$

ω_0

The value of ω that satisfies this condition is called the *resonant frequency* ω_0. Thus, the resonance condition is

$$\omega_0 L = \frac{1}{\omega_0 C}$$

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}}$$

$W_0 = 2\pi f_0$

$\boxed{W = W_0}$

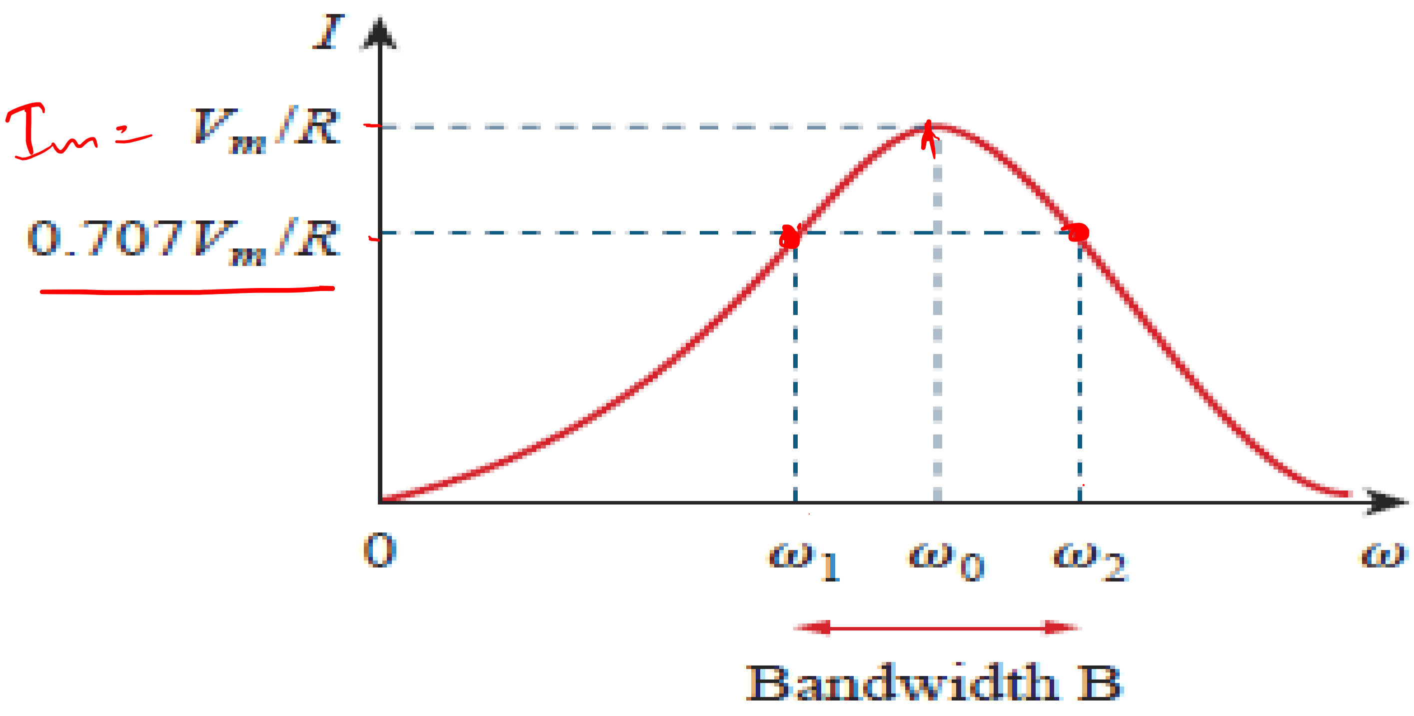
$$\text{Since } \omega_0 = 2\pi f_0, \quad f_0 = \frac{1}{2\pi\sqrt{LC}} \text{ Hz}$$

1. The impedance is purely resistive, thus, $\mathbf{Z} = R$. In other words, the LC series combination acts like a short circuit, and the entire voltage is across R.
2. The voltage $\mathbf{V}_s$ and the current $\mathbf{I}$ are in phase, so that the power factor is unity.
3. The magnitude of the transfer function $\mathbf{H}(\omega) = \mathbf{Z}(\omega)$ is minimum.
4. The inductor voltage and capacitor voltage can be much more than the source voltage.

The frequency response of the circuit's current magnitude

The frequency response of the circuit's current magnitude

$$I = |\mathbf{I}| = \frac{V_m}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$



$\omega_2 - \omega_2 = \underline{\underline{B}}$

The average power dissipated by the RLC circuit is

$$\boxed{P(\omega) = \frac{1}{2} I^2 R}$$ ✓

The highest power dissipated occurs at resonance, when $I = V_m/R$, so that

$$\boxed{P(\omega_0) = \frac{1}{2}\frac{V_m^2}{R}}$$

At certain frequencies $\omega = \omega_1, \omega_2$ the dissipated power becomes half the maximum value. Then ω_1, ω_2 are called half power frequencies

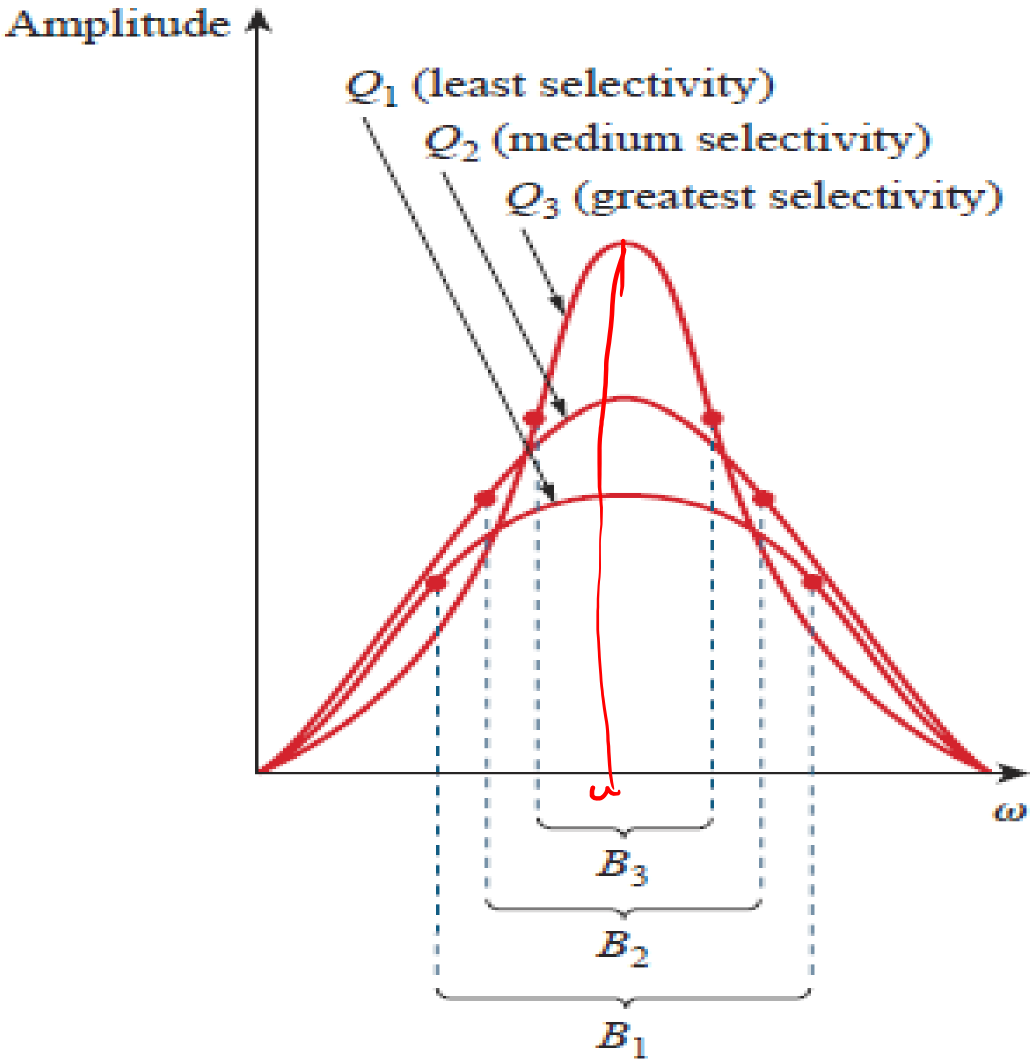
$$\checkmark\ \omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\checkmark\ \omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_0 = \sqrt{\omega_1 \omega_2}$$

$$B = \omega_2 - \omega_1$$

$P = I^2 R$



The "sharpness" of the resonance in a resonant circuit is measured quantitatively by the *quality factor* Q.

At resonance, the reactive energy in the circuit oscillates between the inductor and the capacitor

$$Q = 2\pi \frac{\frac{1}{2}LI^2}{\frac{1}{2}I^2R(1/f_0)} = \frac{2\pi f_0 L}{R}$$

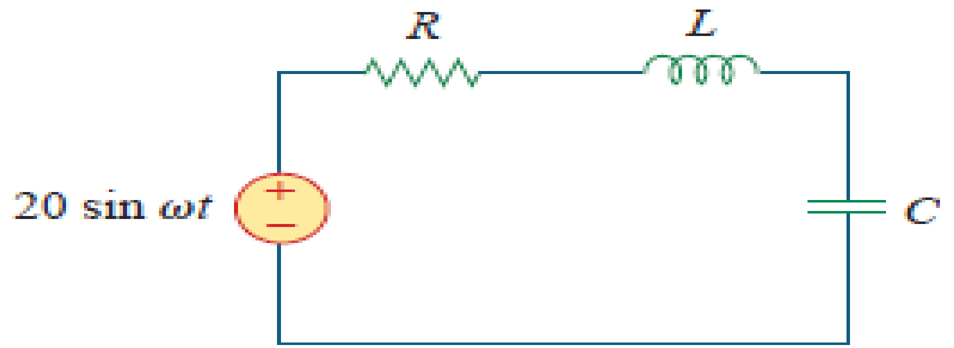
$$\boxed{Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}}$$

$$\boxed{B = \frac{R}{L} = \frac{\omega_0}{Q}}$$

The **quality factor** of a resonant circuit is the ratio of its resonant frequency to its bandwidth.

In the circuit of Fig. 14.24, $R = 2\ \Omega$, $L = 1$ mH, and $C = 0.4\ \mu$F. (a) Find the resonant frequency and the half-power frequencies. (b) Calculate the quality factor and bandwidth. (c) Determine the amplitude of the current at ω_0, ω_1, and ω_2.

Example 14.7

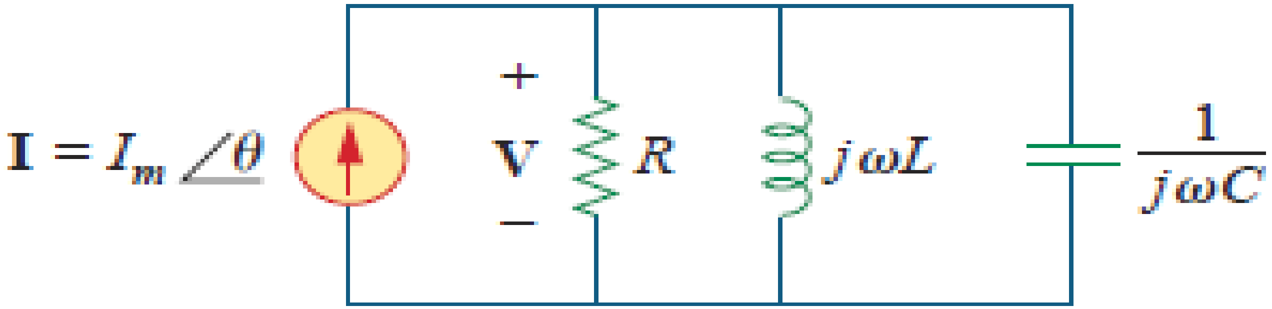


$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-3} \times 0.4 \times 10^{-6}}} = 50 \text{ krad/s}$$

$$\boxed{\begin{aligned} \omega_1 &= -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \\ \omega_2 &= \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \end{aligned}}$$

$B = \omega_1 - \omega_2$

Parallel Resonance



$$\mathbf{Y} = H(\omega) = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{R} + j\omega C + \frac{1}{j\omega L}$$

or

$$\mathbf{Y} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

Resonance occurs when the imaginary part of $\mathbf{Y}$ is zero,

$$\omega C - \frac{1}{\omega L} = 0$$

or

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}}$$

Refer Table 14.4

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$B = \omega_2 - \omega_1 = \frac{1}{RC}$$

$$Q = \frac{\omega_0}{B} = \omega_0 RC = \frac{R}{\omega_0 L}$$

$$\omega_1 = \omega_0\sqrt{1 + \left(\frac{1}{2Q}\right)^2} - \frac{\omega_0}{2Q},$$

$$\omega_2 = \omega_0\sqrt{1 + \left(\frac{1}{2Q}\right)^2} + \frac{\omega_0}{2Q}$$

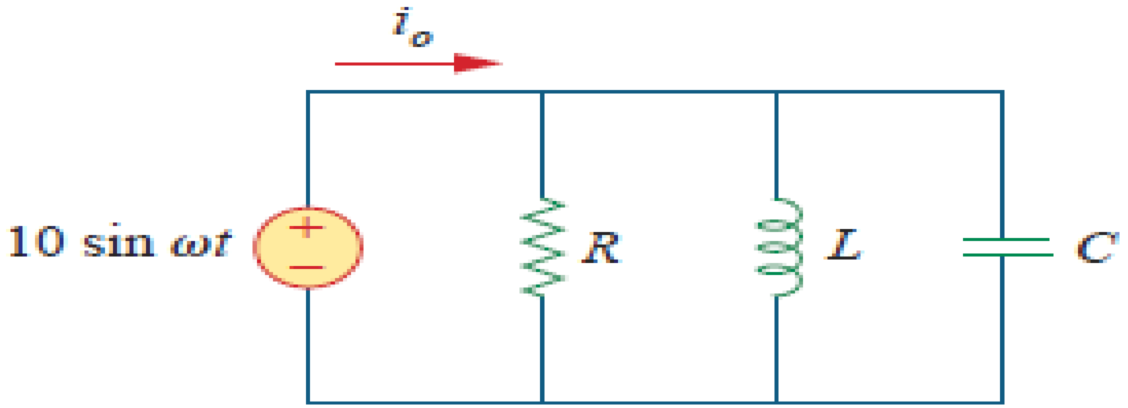
Again, for high-Q circuits ($Q \geq 10$)

$$\boxed{\omega_1 \simeq \omega_0 - \frac{B}{2}, \qquad \omega_2 \simeq \omega_0 + \frac{B}{2}}$$

Example

In the parallel RLC circuit of Fig. 14.27, let $R = 8\ \text{k}\Omega$, $L = 0.2$ mH, and $C = 8\ \mu$F. (a) Calculate ω_0, Q, and B. (b) Find ω_1 and ω_2. (c) Determine the power dissipated at ω_0, ω_1, and ω_2.

14.8



Solution:

(a)

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.2 \times 10^{-3} \times 8 \times 10^{-6}}} = \frac{10^5}{4} = 25 \text{ krad/s}$$

$$Q = \frac{R}{\omega_0 L} = \frac{8 \times 10^3}{25 \times 10^3 \times 0.2 \times 10^{-3}} = 1{,}600$$

$$B = \frac{\omega_0}{Q} = 15.625 \text{ rad/s}$$

(b) Due to the high value of Q, we can regard this as a high-Q circuit. Hence,

$$\omega_1 = \omega_0 - \frac{B}{2} = 25{,}000 - 7.812 = 24{,}992 \text{ rad/s}$$

$$\omega_2 = \omega_0 + \frac{B}{2} = 25{,}000 + 7.812 = 25{,}008 \text{ rad/s}$$

(c) At $\omega = \omega_0$, $\mathbf{Y} = 1/R$ or $\mathbf{Z} = R = 8\text{ k}\Omega$. Then

$$\mathbf{I}_o = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{10\underline{/-90^\circ}}{8{,}000} = 1.25\underline{/-90^\circ}\text{ mA}$$

Since the entire current flows through R at resonance, the average power dissipated at $\omega = \omega_0$ is

$$P = \frac{1}{2}|\mathbf{I}_o|^2 R = \frac{1}{2}(1.25 \times 10^{-3})^2(8 \times 10^3) = 6.25 \text{ mW}$$

or

$$P = \frac{V_m^2}{2R} = \frac{100}{2 \times 8 \times 10^3} = 6.25 \text{ mW}$$

At $\omega = \omega_1, \omega_2$,

$$P = \frac{V_m^2}{4R} = 3.125 \text{ mW}$$