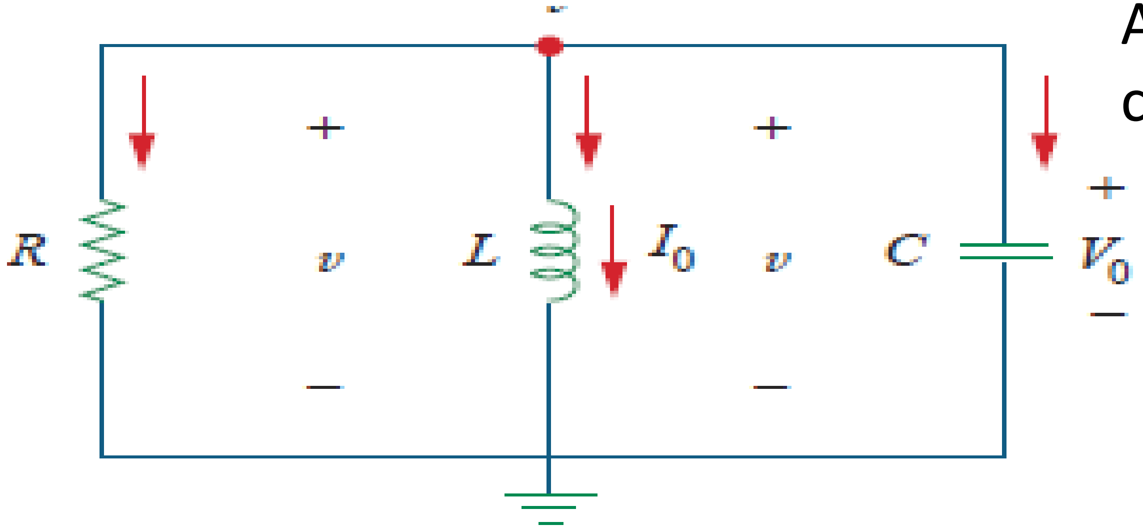
The Source-Free Parallel *RLC* Circuit

Series



Assume initial inductor current I_0 and initial capacitor voltage V_0 ,

$$i(0) = I_0 = \frac{1}{L}\int_{\infty}^{0} v(t)\,dt$$

$$v(0) = V_0$$

Since the three elements are in parallel, they have the same voltage *v* across them. According to passive sign convention, the current is entering each element; that is, the current through each element is leaving the top node.

Thus, applying KCL at the top node gives

$$\boxed{\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v\,dt + C\frac{dv}{dt} = 0}$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

We obtain the characteristic equation by replacing the first derivative by s and the second derivative by s^2

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

The roots of the characteristic equation are

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$\boxed{s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}}$$

$$\boxed{\alpha = \frac{1}{2RC}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}}$$

Overdamped Case ($\alpha > \omega_0$)

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$\alpha > \omega_0$ when $L > 4R^2C$. The roots of the characteristic equation are real and negative. The response is

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Critically Damped Case ($\alpha = \omega_0$)

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

For $\alpha = \omega_0$, $L = 4R^2C$. The roots are real and equal so that the response is

$$v(t) = (A_1 + A_2 t)e^{-\alpha t}$$

Underdamped Case ($\alpha < \omega_0$)

When $\alpha < \omega_0$, $L < 4R^2C$. In this case the roots are complex and may be expressed as

$$s_{1,2} = -\alpha \pm j\omega_d$$

where

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

The response is

$$\boxed{v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)}$$

To find constants, use initial conditions

$$\frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0$$

$$\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC}$$

Having found the voltage, the currents flowing through the components are

$$i_R = v/R \qquad v_C = C\,dv/dt.$$