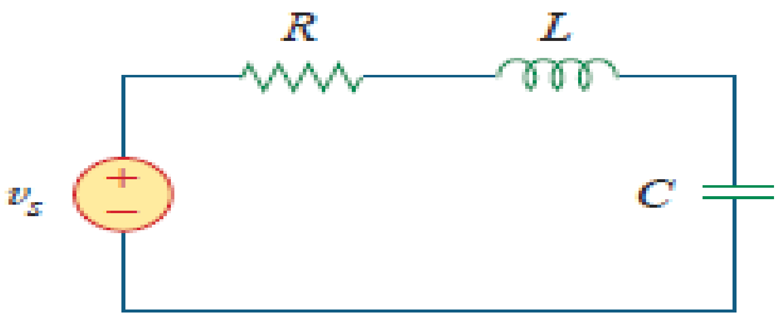
Second-Order Circuits

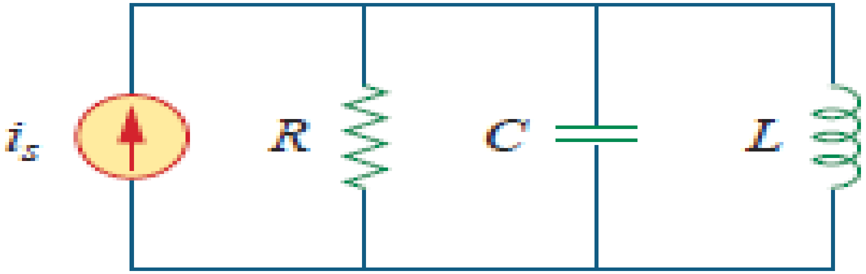
- Circuits containing two storage elements.
- *Second-order* circuits because their responses are described by differential equations that contain second derivatives.

A **second-order circuit** is characterized by a second-order differential equation. It consists of resistors and the equivalent of two energy storage elements.

Typical examples of second-order circuits:



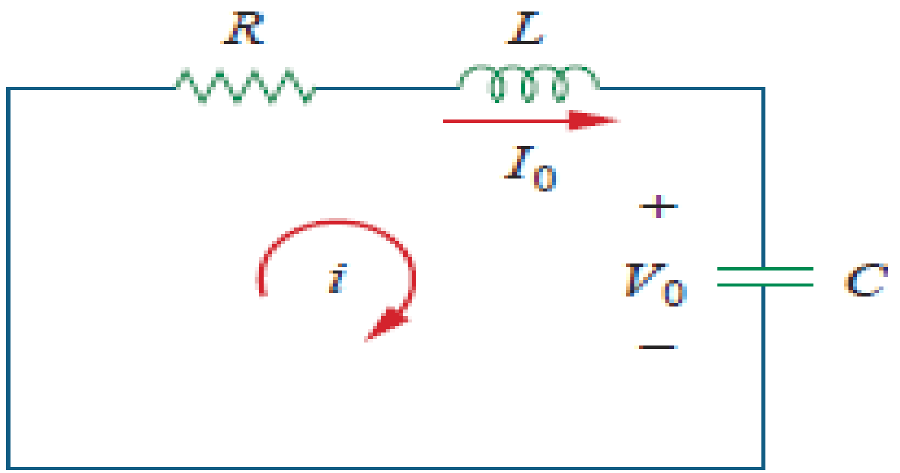
(a)



(b)

(a) series *RLC* circuit, (b) parallel *RLC*

The Source-Free Series *RLC* Circuit



The circuit is being excited by the energy initially stored in the capacitor and inductor

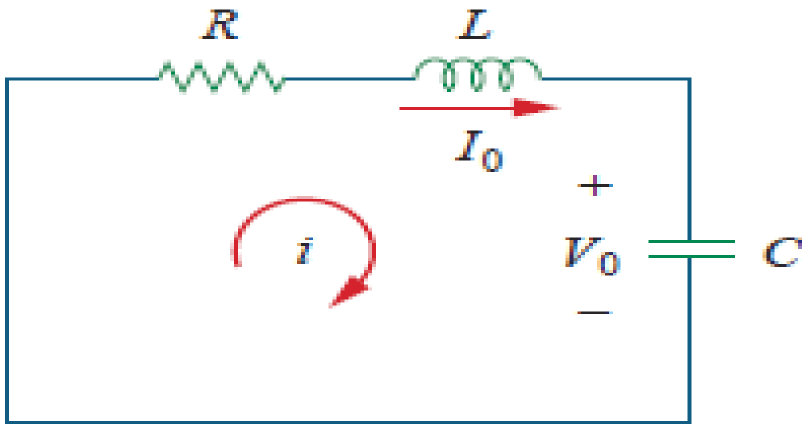
The energy is represented by the initial capacitor voltage $V_0$and initial inductor current I_0. Thus, at $t = 0$

$$v(0) = \frac{1}{C}\int_{-\infty}^{0} i\,dt = V_0$$

$$i(0) = I_0$$

Applying KVL around the loop in Fig. 8.8,

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\,dt = 0$$



$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\,dt = 0$$

To eliminate the integral, we differentiate w r to *t*

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

This is a *second-order differential equation* and to solve this Eqn

To solve such a second-order differential equation requires that we have two initial conditions, such as the initial value of *i* and its first derivative or initial values of some *i* and v

$$Ri(0) + L\frac{di(0)}{dt} + V_0 = 0 \qquad \text{or} \qquad \frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0)$$

$$i(0) = I_0 \qquad i = Ae^{st}$$

where A and s are constants to be determined.

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

$$As^2e^{st} + \frac{AR}{L}se^{st} + \frac{A}{LC}e^{st} = 0$$

$$Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0$$

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

This quadratic equation is known as the *characteristic equation* of the differential Eq. since the roots of the equation dictate the character of *i*.

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

A more compact way of expressing the roots is

$$\boxed{s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}}$$

where $$\boxed{\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}}$$

The roots s_1 and s_2 are called *natural frequencies*, measured in nepers per second (Np/s)

ω_0 is known as the *resonant frequency* or the *undamped natural frequency* (rad/s)

α is the *neper frequency* or the *damping factor*

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

The two values of *s* in indicate that there are two possible solutions for *i*

$$i_1 = A_1 e^{s_1 t}, \qquad i_2 = A_2 e^{s_2 t}$$

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where the constants A_1 and A_2 are determined from the initial values $i(0)$ and $di(0)/dt$

From following, we can infer that there are three types of solutions possible

$$\boxed{s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}}$$

1. If $\alpha > \omega_0$, we have the *overdamped* case.
2. If $\alpha = \omega_0$, we have the *critically damped* case.
3. If $\alpha < \omega_0$, we have the *underdamped* case.

Overdamped Case ($\alpha > \omega_0$)

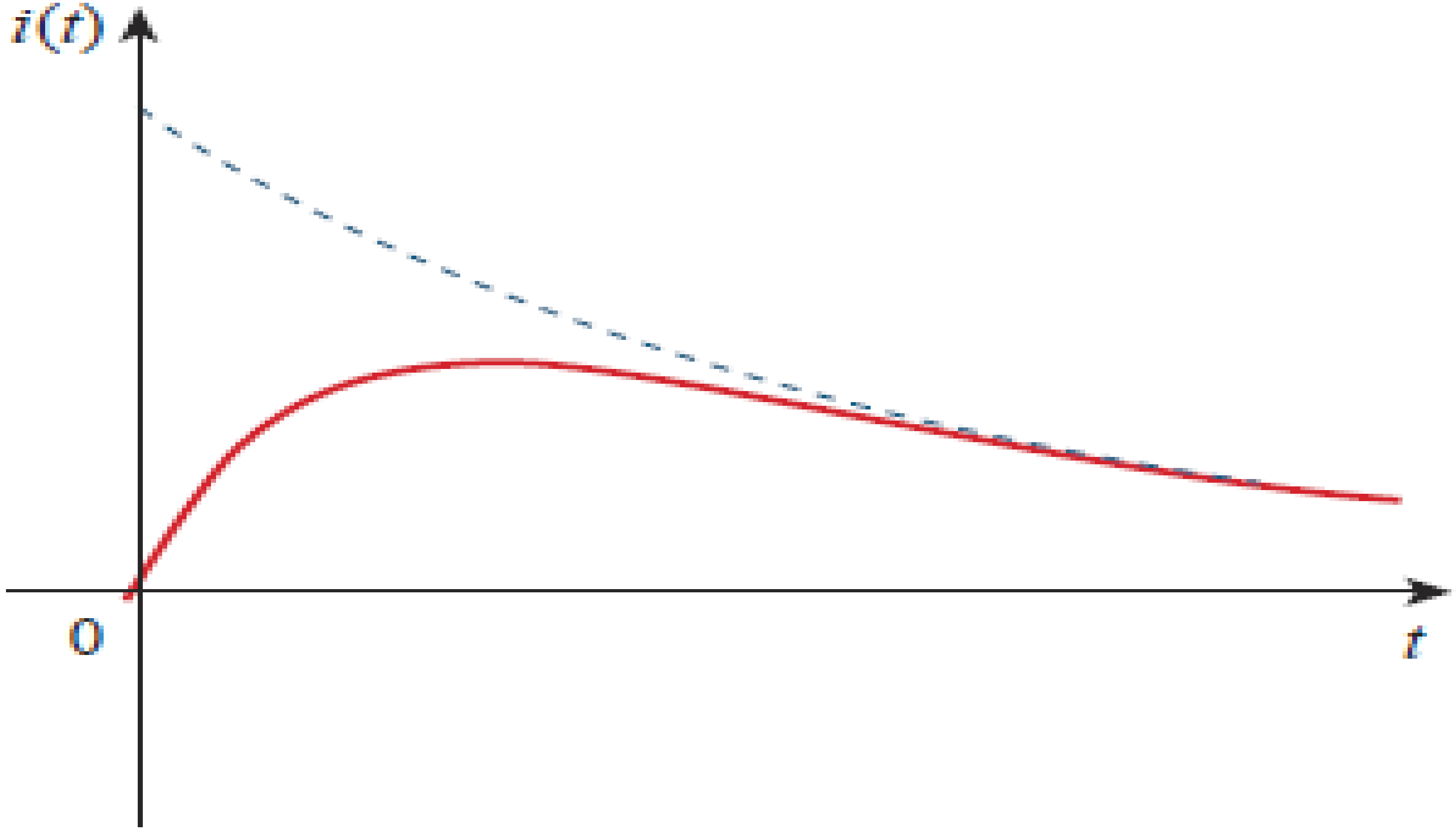
$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$
$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\boxed{s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}}$$

$\alpha > \omega_0$ implies $C > 4L/R^2$. When this happens, both roots s_1 and s_2 are negative and real.

$$\boxed{i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$

which decays and approaches zero as t increases.



(a)

(a) Overdamped response,

Critically Damped Case ($\alpha = \omega_0$)

When $\alpha = \omega_0$, $C = 4L/R^2$ and

$$s_1 = s_2 = -\alpha = -\frac{R}{2L}$$

$$i(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = A_3 e^{-\alpha t}$$

where $A_3 = A_1 + A_2$. This cannot be the solution, because the two initial conditions cannot be satisfied with the single constant A_3.

So let's begin with

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

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$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

When $\alpha = \omega_0 = R/2L$

$$\frac{d^2i}{dt^2} + 2\alpha\frac{di}{dt} + \alpha^2 i = 0$$

$$\frac{d}{dt}\left(\frac{di}{dt} + \alpha i\right) + \alpha\left(\frac{di}{dt} + \alpha i\right) = 0$$

$$\frac{df}{dt} + \alpha f = 0$$

If we let $f = \frac{di}{dt} + \alpha i$

which is a first-order differential equation with solution $f = A_1 e^{-\alpha t}$, where A_1 is a constant.

$$\frac{di}{dt} + \alpha i = A_1 e^{-\alpha t}$$

$$e^{\alpha t}\frac{di}{dt} + e^{\alpha t}\alpha i = A_1$$

$$\frac{d}{dt}(e^{\alpha t} i) = A_1$$

$$e^{\alpha t} i = A_1 t + A_2$$

Integrating both sides yields $i = (A_1 t + A_2)e^{-\alpha t}$

where A_2 is another constant. Hence, the natural response of the critically damped circuit is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term, or

$$\boxed{i(t) = (A_2 + A_1 t)e^{-\alpha t}}$$

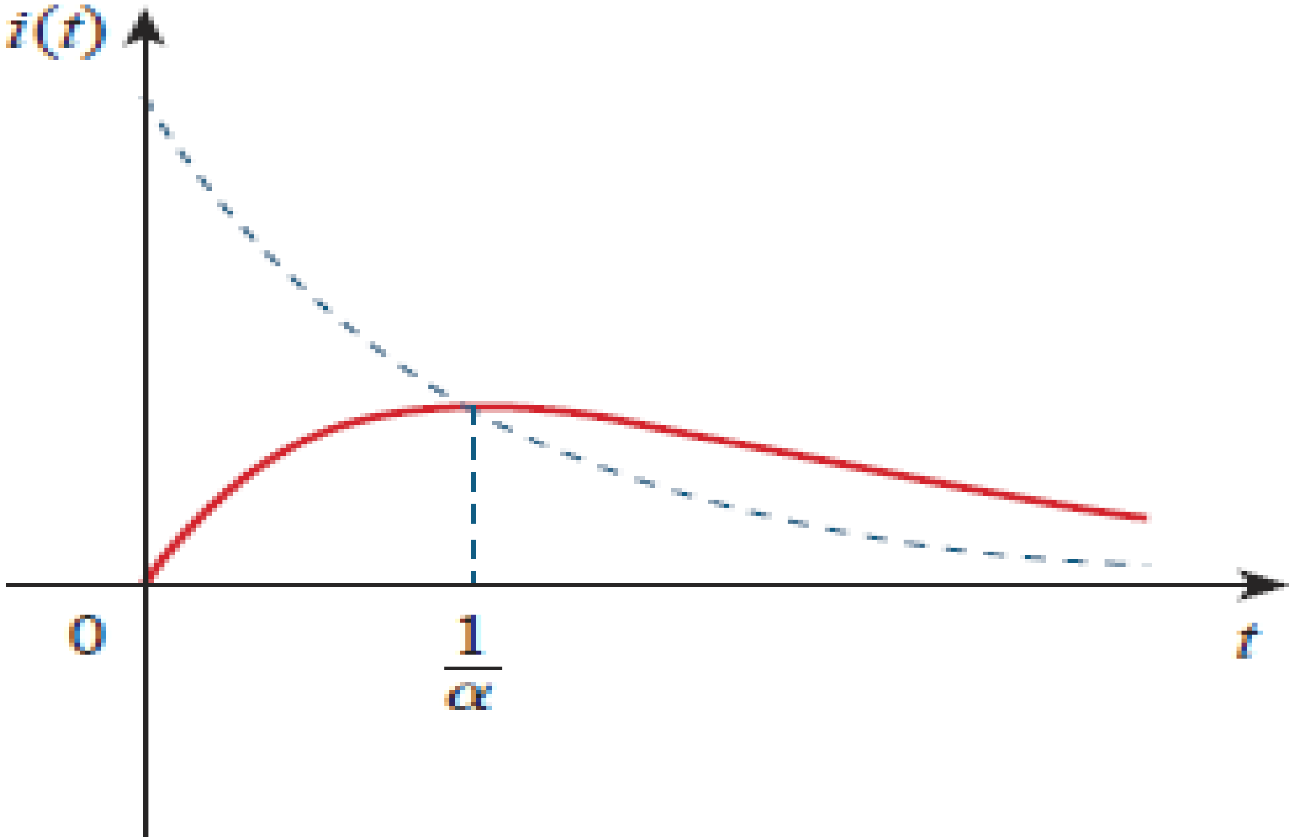


Fig. is a sketch of $i(t) = te^{-\alpha t}$, which reaches a maximum value of e^{-1}/α at $t = 1/\alpha$, one time constant, and then decays all the way to zero.

Underdamped Case ($\alpha < \omega_0$)

For $\alpha < \omega_0$, $C < 4L/R^2$. The roots may be written as

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d$$
$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d$$

where $j = \sqrt{-1}$ and $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$, which is called the *damping frequency*. Both ω_0 and ω_d are natural frequencies because they help determine the natural response; while ω_0 is often called the *undamped natural frequency*, ω_d is called the *damped natural frequency*. The natural response is

$$\begin{aligned} i(t) &= A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t} \\ &= e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}) \end{aligned}$$

Using Euler's identities,

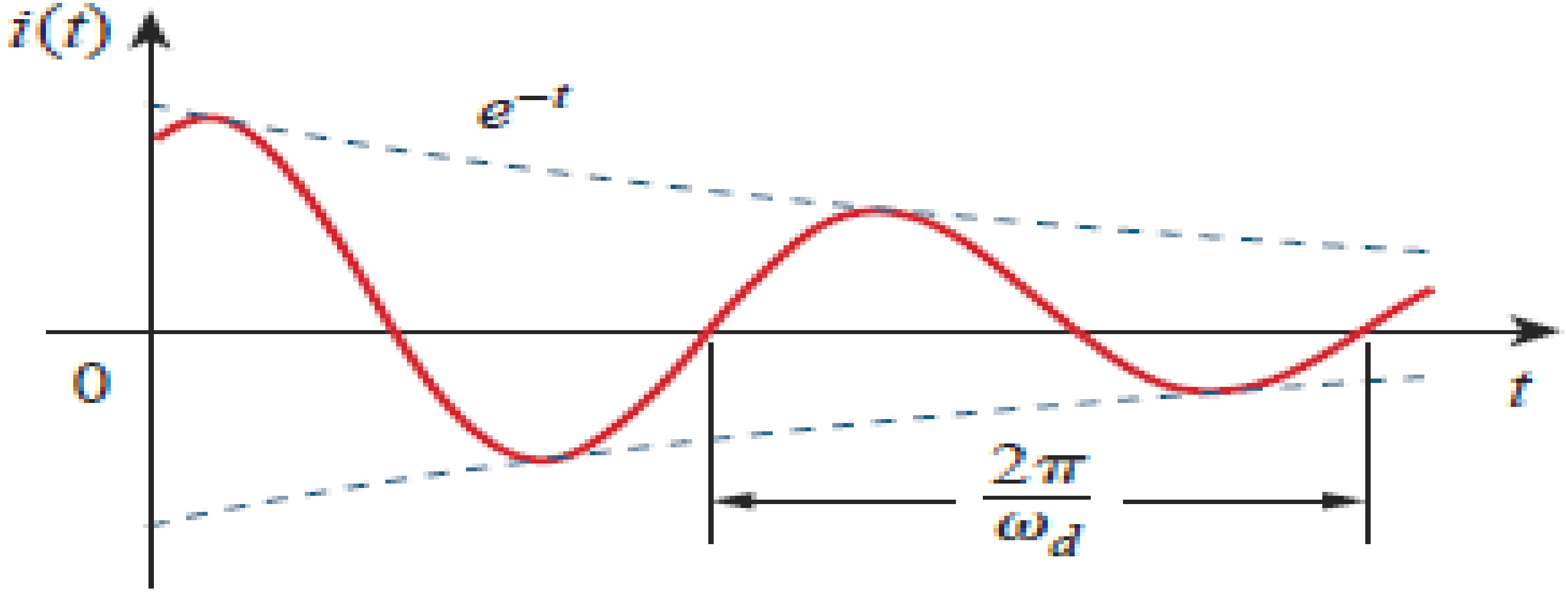
$$e^{j\theta} = \cos\theta + j\sin\theta, \qquad e^{-j\theta} = \cos\theta - j\sin\theta$$

we get

$$\begin{aligned} i(t) &= e^{-\alpha t}[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos\omega_d t - j\sin\omega_d t)] \\ &= e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t] \end{aligned}$$

Replacing constants $(A_1 + A_2)$ and $j(A_1 - A_2)$ with constants B_1 and B_2, we write

$$\boxed{i(t) = e^{-\alpha t}(B_1\cos\omega_d t + B_2\sin\omega_d t)}$$



With the presence of sine and cosine functions, it is clear that the natural response for this case is exponentially damped and oscillatory in nature. The response has a time constant of $1/\alpha$ and a period of $T = 2\pi/\omega_d$.