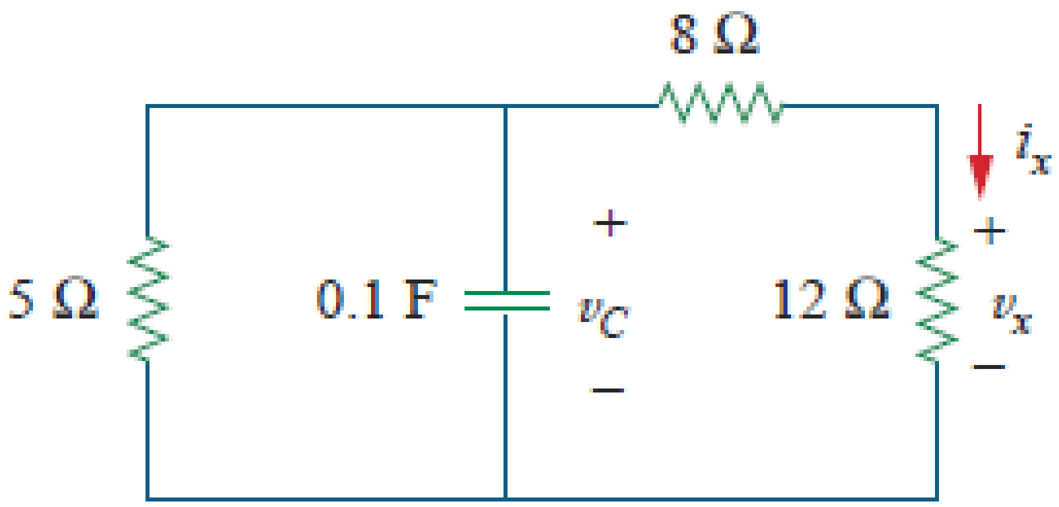
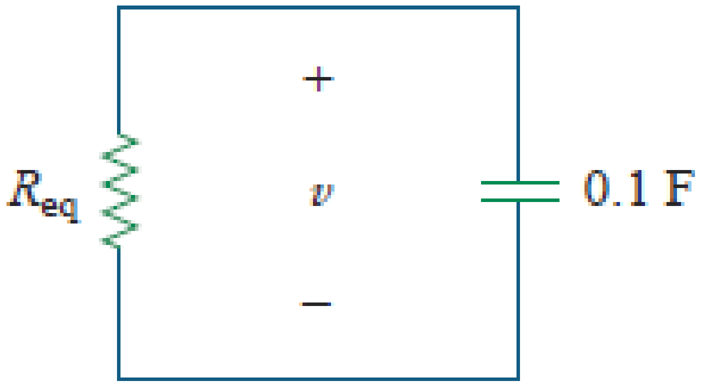
Example 7.1

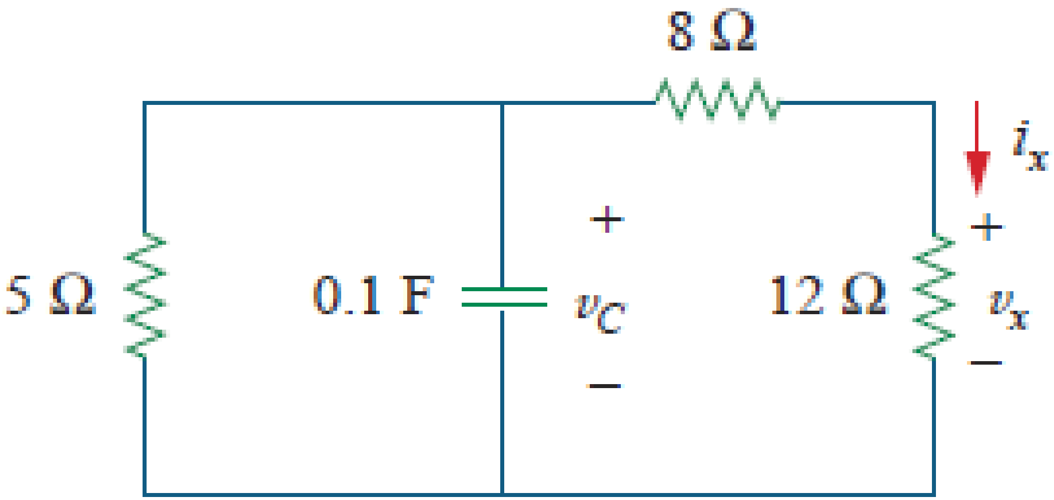
In Fig. let $v_C(0) = 15$ V. Find v_C, v_x, and i_x for $t > 0$.

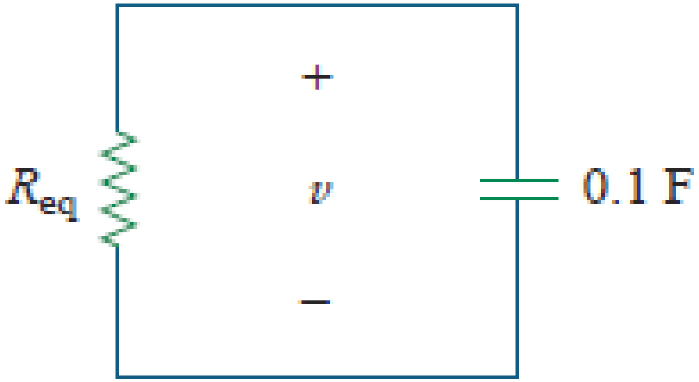


The 8-Ω and 12-Ω resistors in series can be combined to give a 20-Ω resistor. This 20-Ω resistor in parallel with the 5-Ω resistor can be combined so that the equivalent resistance is



$$R_{eq} = \frac{20 \times 5}{20 + 5} = 4\ \Omega$$





$$R_{eq} = \frac{20 \times 5}{20 + 5} = 4\ \Omega$$

$$\tau = R_{eq}C = 4(0.1) = 0.4\ \text{s}$$

$$v = v(0)e^{-t/\tau} = 15e^{-t/0.4}\ \text{V}$$

$$v_C = v = 15e^{-2.5t}\ \text{V}$$

$$v_x = \frac{12}{12 + 8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t}\ \text{V}$$

$$i_x = \frac{v_x}{12} = 0.75e^{-2.5t}\ \text{A}$$

Practice Problem 7.1

Refer to the circuit in Fig. 7.7. Let $v_C(0) = 45$ V. Determine v_C, v_x, and i_o for $t \geq 0$.

Answer: $45e^{-0.25t}$ V, $15e^{-0.25t}$ V, $-3.75e^{-0.25t}$ A.

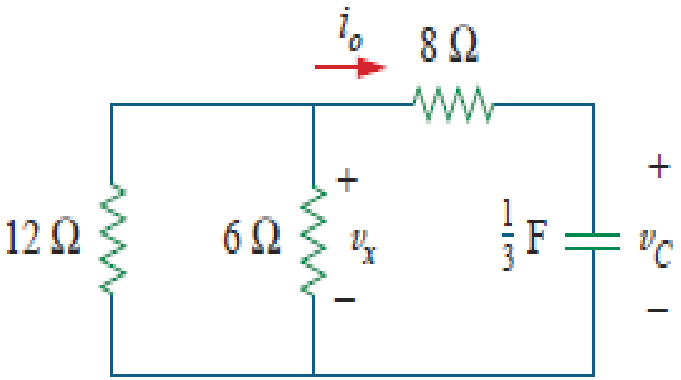
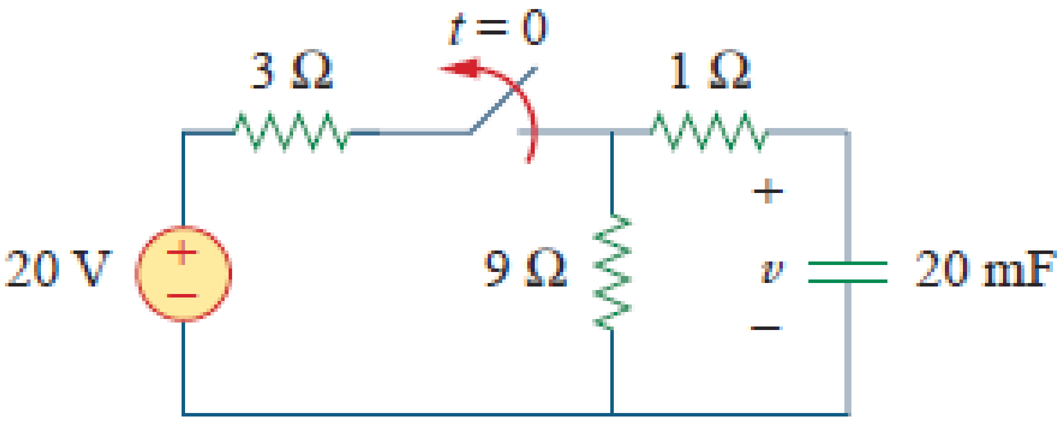


Figure 7.7

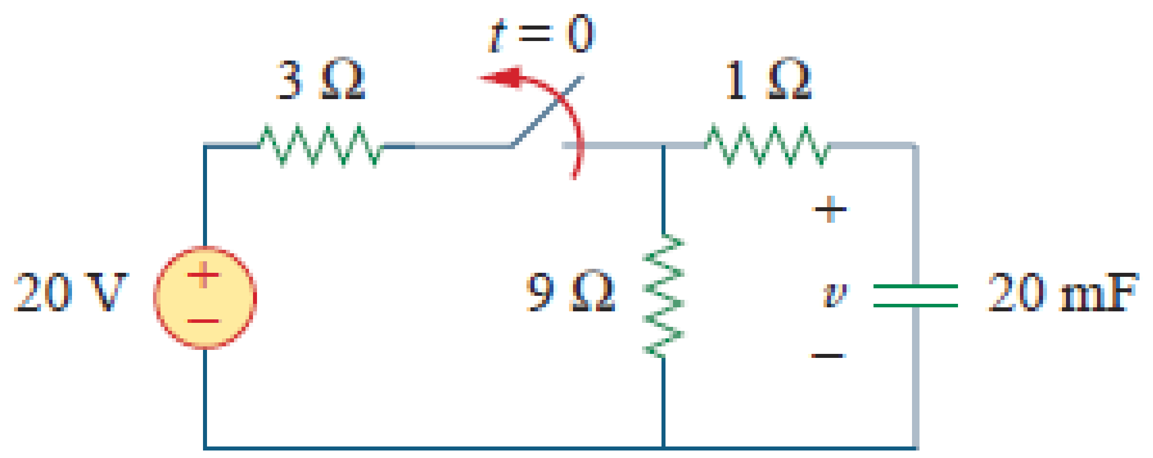
Example 7.2

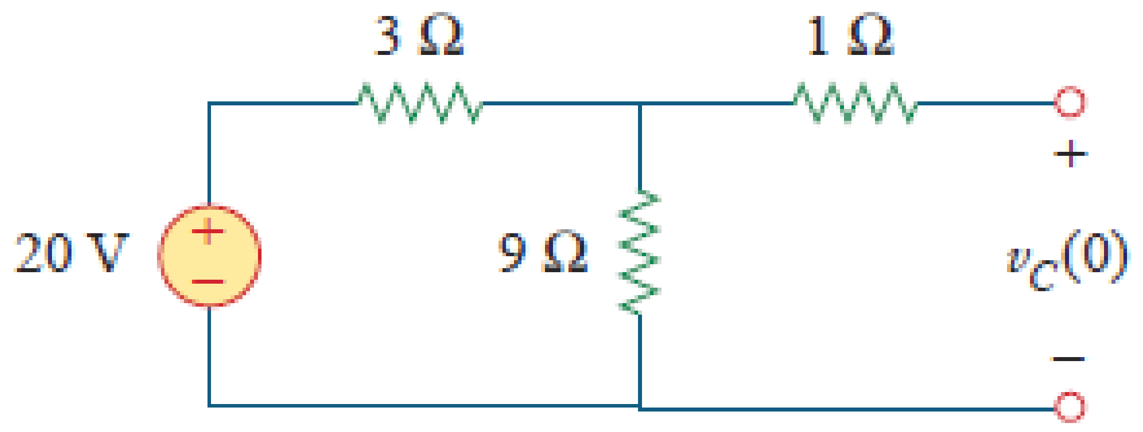
The switch in the circuit in Fig. has been closed for a long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.



For $t < 0$, the switch is closed; the capacitor is an open circuit to dc,

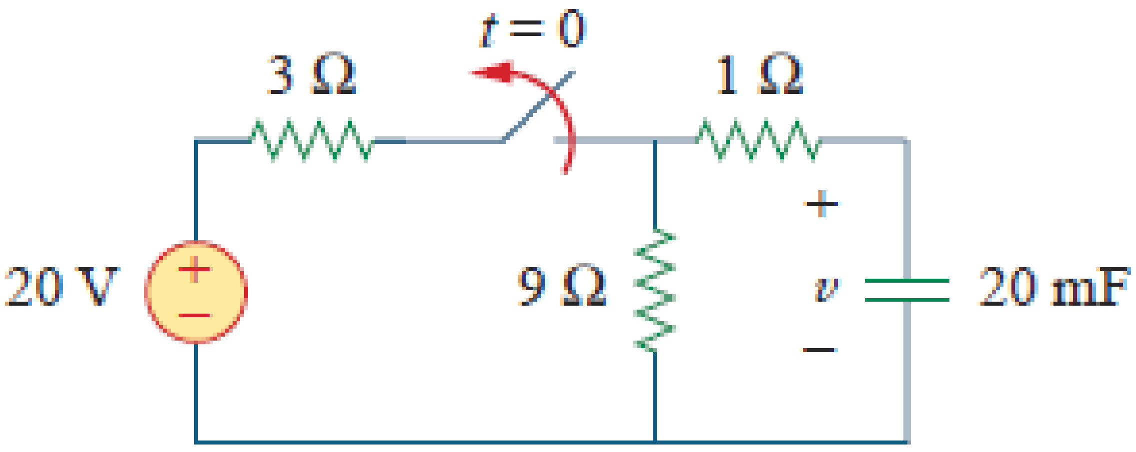
$$v_C(t) = \frac{9}{9+3}(20) = 15\ \text{V}, \qquad t < 0$$

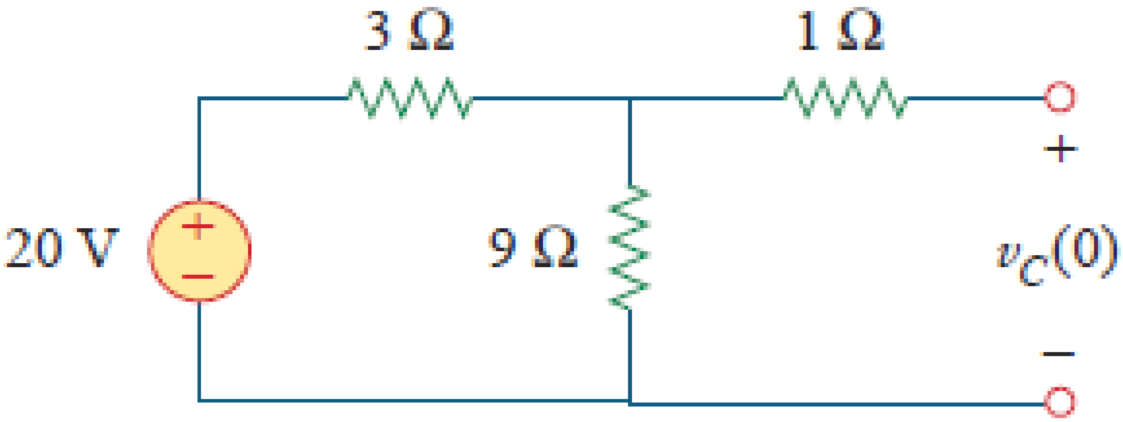




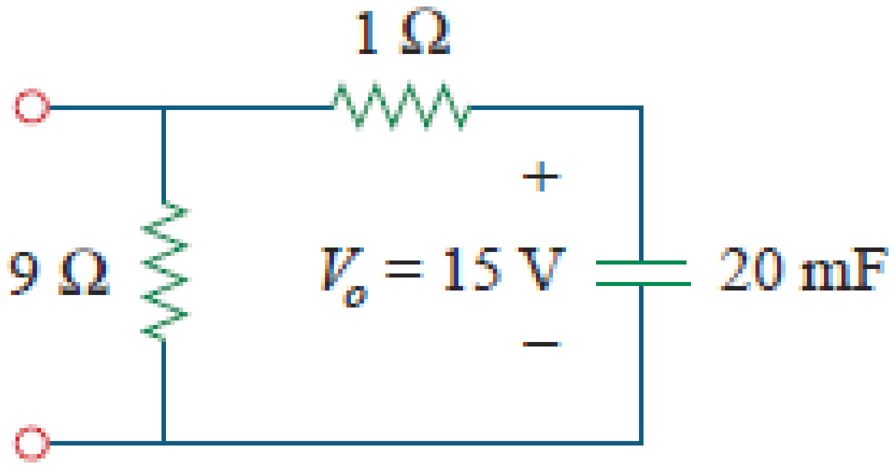
Since the voltage across a capacitor cannot change instantaneously, the voltage across the capacitor at $t = 0^-$ is the same at $t = 0$, or

$$v_C(0) = V_0 = 15 \text{ V}$$





For $t > 0$, the switch is opened, and we have the RC circuit



$$R_{eq} = 1 + 9 = 10\ \Omega$$

$$\tau = R_{eq}C = 10 \times 20 \times 10^{-3} = 0.2\ \text{s}$$

Thus, the voltage across the capacitor for $t \geq 0$ is

$$v(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0.2} \text{ V}$$

$$v(t) = 15e^{-5t} \text{ V}$$

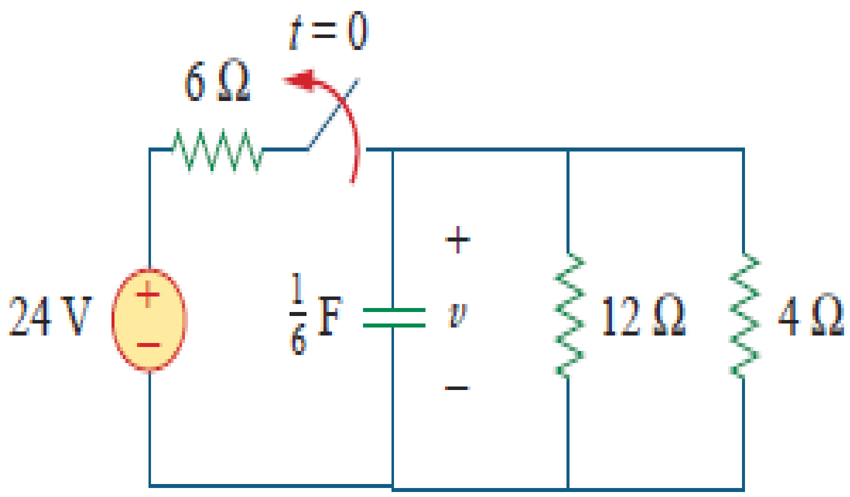
The initial energy stored in the capacitor is

$$w_C(0) = \frac{1}{2}Cv_C^2(0) = \frac{1}{2} \times 20 \times 10^{-3} \times 15^2 = 2.25 \text{ J}$$

Practice Problem 7.2

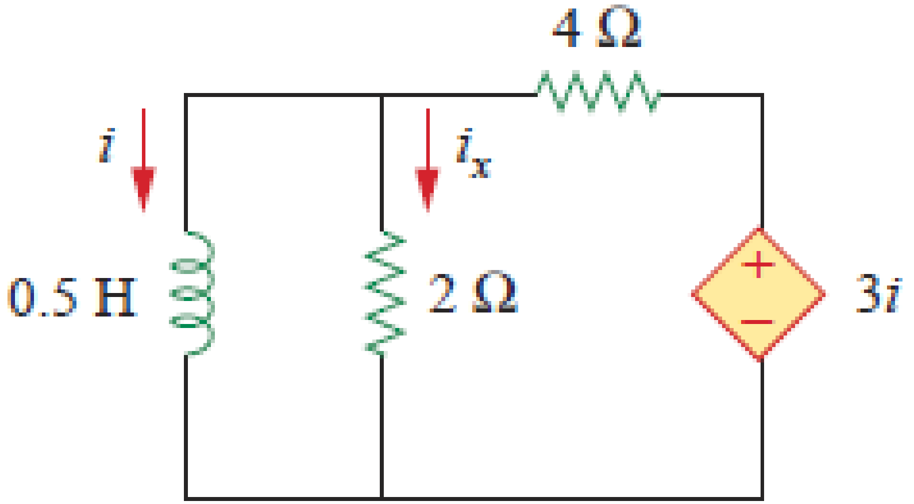
If the switch in Fig. 7.10 opens at $t = 0$, find $v(t)$ for $t \geq 0$ and $w_C(0)$.

Answer: $8e^{-2t}$ V, 5.33 J.

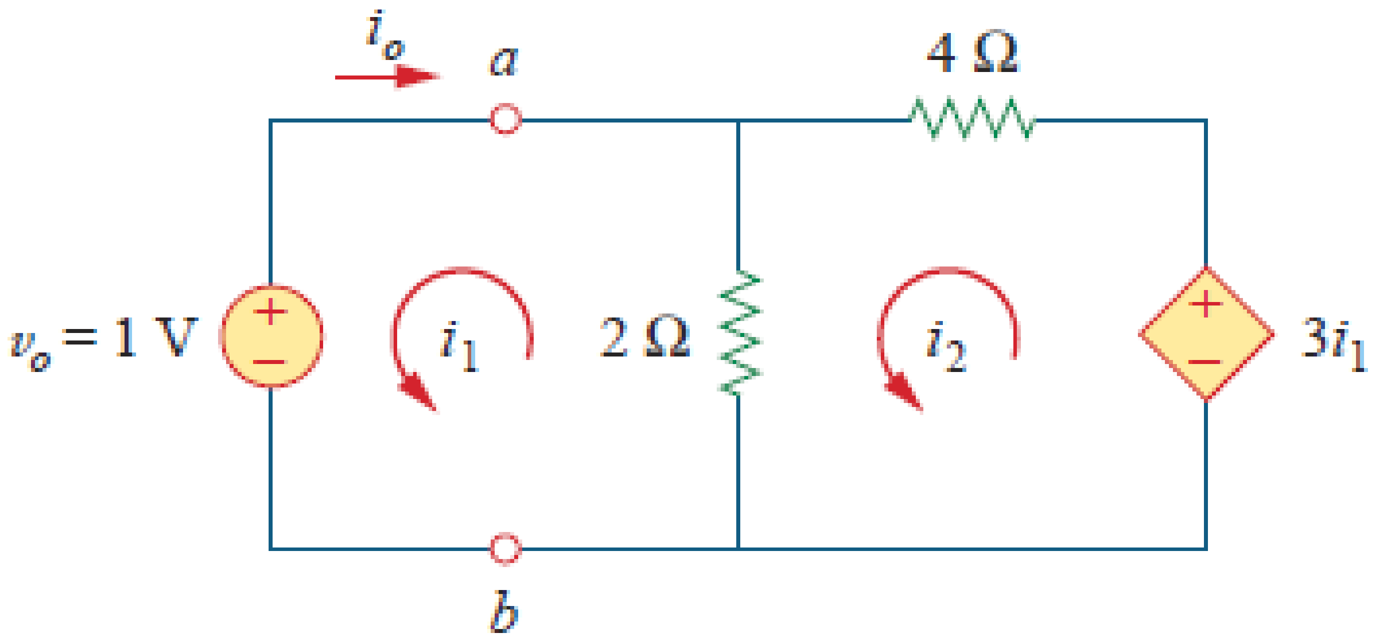


Example 7.3

Assuming that $i(0) = 10$ A, calculate $i(t)$ and $i_x(t)$ in the circuit of



METHOD 1 The equivalent resistance is the same as the Thevenin resistance at the inductor terminals. Because of the dependent source, we insert a voltage source with $v_o = 1$ V at the inductor terminals *a*-*b*, as in Fig. 7.14(a). (We could also insert a 1-A current source at the terminals.) Applying KVL to the two loops results in



$$2(i_1 - i_2) + 1 = 0 \qquad \Rightarrow \qquad i_1 - i_2 = -\frac{1}{2}$$

$$6i_2 - 2i_1 - 3i_1 = 0 \qquad \Rightarrow \qquad i_2 = \frac{5}{6}i_1$$

Substituting Eq. (7.3.2) into Eq. (7.3.1) gives

$$i_1 = -3\text{ A}, \qquad i_o = -i_1 = 3\text{ A}$$

Hence,

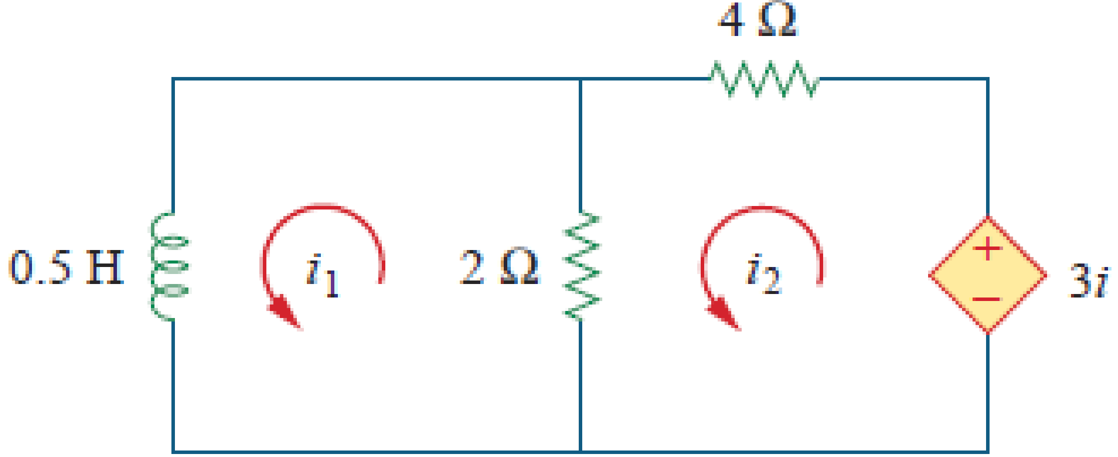
$$R_{\text{eq}} = R_{\text{Th}} = \frac{v_o}{i_o} = \frac{1}{3}\,\Omega$$

The time constant is

$$\tau = \frac{L}{R_{\text{eq}}} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}\,\text{s}$$

Thus, the current through the inductor is

$$i(t) = i(0)e^{-t/\tau} = 10e^{-(2/3)t}\,\text{A}, \qquad t > 0$$



METHOD 2 We may directly apply KVL to the circuit as in Fig. 7.14(b). For loop 1,

$$\frac{1}{2}\frac{di_1}{dt} + 2(i_1 - i_2) = 0$$

or

$$\frac{di_1}{dt} + 4i_1 - 4i_2 = 0 \tag{7.3.3}$$

For loop 2,

$$6i_2 - 2i_1 - 3i_1 = 0 \quad \Rightarrow \quad i_2 = \frac{5}{6}i_1 \tag{7.3.4}$$

Substituting Eq. (7.3.4) into Eq. (7.3.3) gives

$$\frac{di_1}{dt} + \frac{2}{3}i_1 = 0$$

Rearranging terms,

$$\frac{di_1}{i_1} = -\frac{2}{3}dt$$

Since $i_1 = i$, we may replace i_1 with i and integrate:

$$\ln i \Big|_{i(0)}^{i(t)} = -\frac{2}{3}t\Big|_0^t$$

or

$$\ln\frac{i(t)}{i(0)} = -\frac{2}{3}t$$

Taking the powers of e, we finally obtain

$$i(t) = i(0)e^{-(2/3)t} = 10e^{-(2/3)t}\ \text{A}, \qquad t > 0$$

which is the same as by Method 1.

The voltage across the inductor is

$$v = L\frac{di}{dt} = 0.5(10)\left(-\frac{2}{3}\right)e^{-(2/3)t} = -\frac{10}{3}e^{-(2/3)t}\ \text{V}$$

Since the inductor and the 2-Ω resistor are in parallel,

$$i_x(t) = \frac{v}{2} = -1.6667e^{-(2/3)t}\ \text{A}, \qquad t > 0$$

Practice Problem 7.3

Find i and v_x in the circuit of Fig. 7.15. Let $i(0) = 5$ A.

Answer: $5e^{-4t}$ V, $-20e^{-4t}$ V.

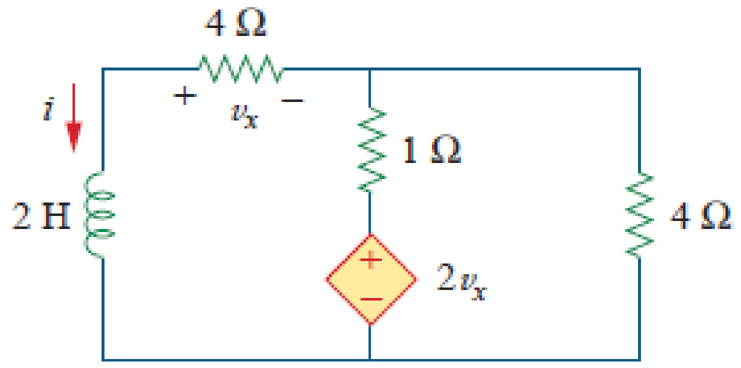
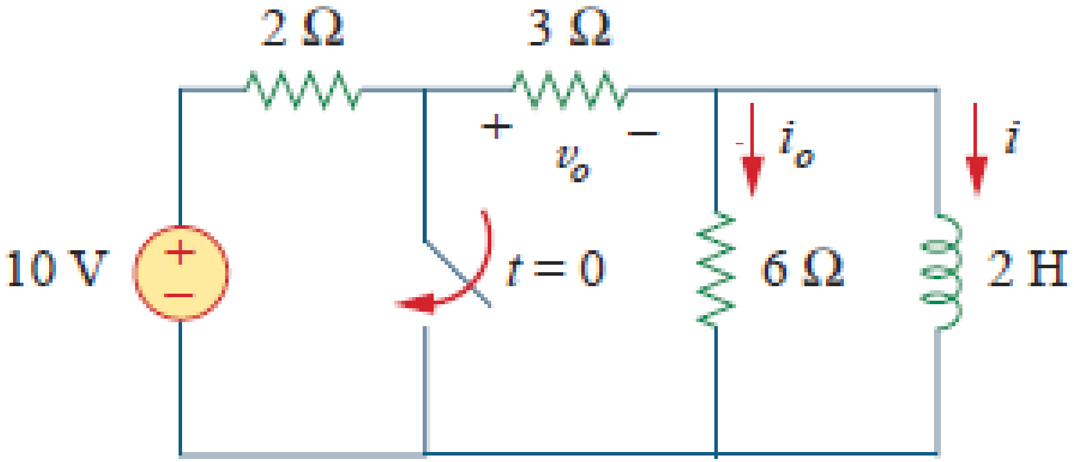


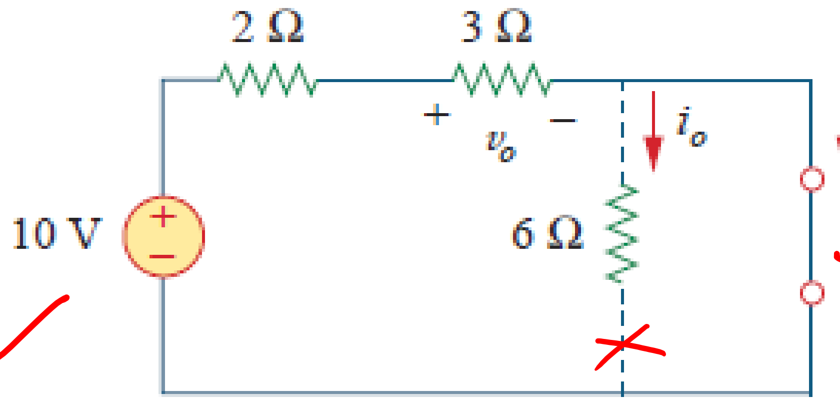
Figure 7.15
For Practice Prob. 7.3.

Example 7.5

In the circuit shown in Fig. 7.19, find i_o, v_o, and i for all time, assuming that the switch was open for a long time.



For $t < 0$, the switch is open.



(a)

Inductor acts like a short circuit

i_o, v_o ; i

$t<0$

$t=0$

$t>0$

Since the inductor acts like a short circuit to dc, the 6-Ω resistor is short-circuited, so that we have the circuit shown in Fig. (a). Hence, $i_o = 0$, and

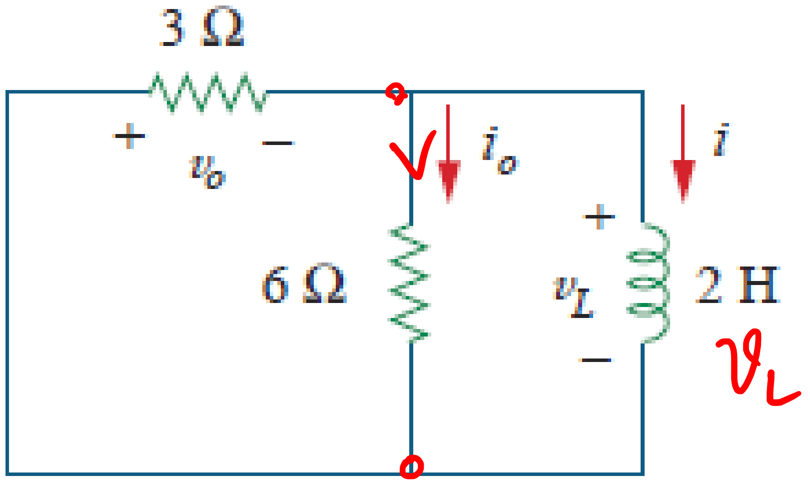
$$i(t) = \frac{10}{2+3} = 2\text{ A}, \qquad t < 0$$

$$v_o(t) = 3i(t) = 6\text{ V}, \qquad t < 0$$

Thus, $i(0) = 2$.

For $t > 0$, the switch is closed, so that the voltage source is short-circuited. We now have a source-free *RL* circuit as shown in Fig.

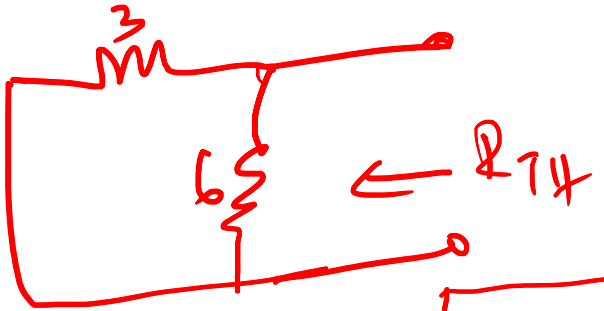
$\frac{v_L}{6}$



At the inductor terminals,

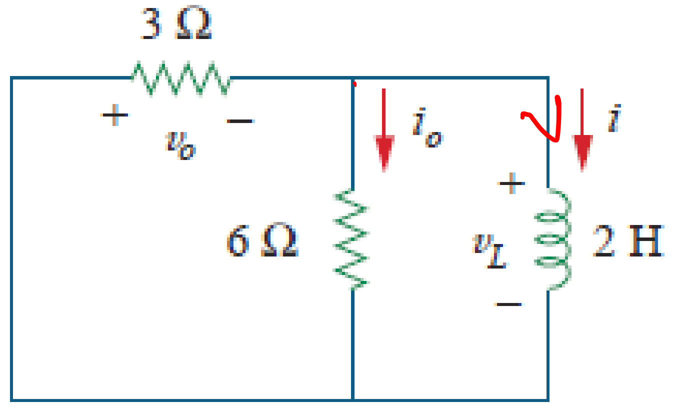
$$R_{\text{Th}} = 3 \,\|\, 6 = 2\ \Omega$$

$\frac{3\times 6}{3+6}$



$$\tau = \frac{L}{R_{\text{Th}}} = 1\ \text{s}$$

$$\boxed{i(t) = i(0)e^{-t/\tau} = 2e^{-t}\ \text{A},} \qquad t > 0$$



Since the inductor is in parallel with the 6-Ω and 3-Ω resistors,

$$v_o(t) = -v_L = -L\frac{di}{dt} = -2(-2e^{-t}) = 4e^{-t}\text{ V}, \qquad t > 0$$

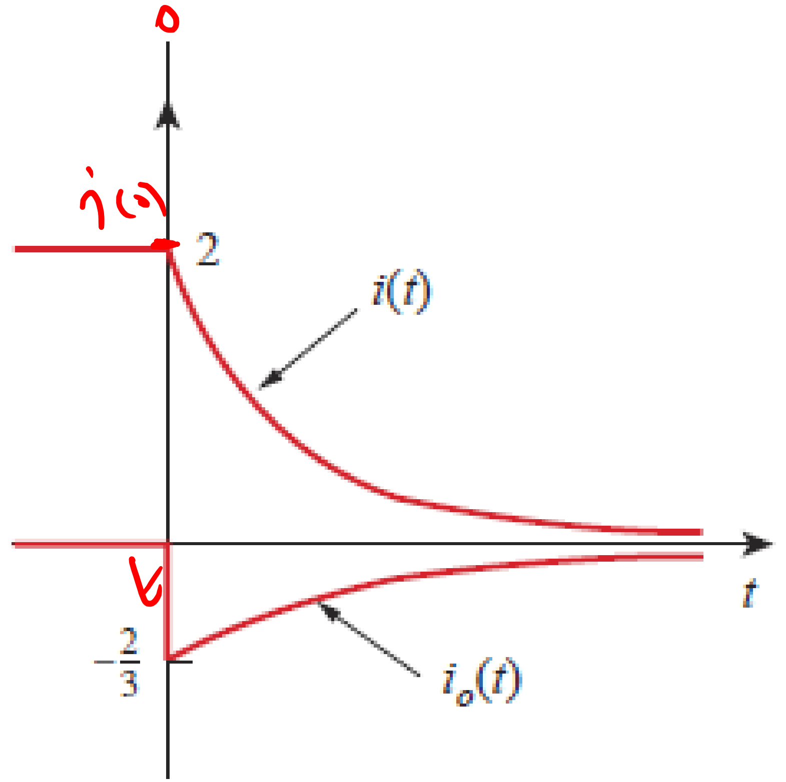
$$i_o(t) = \frac{v_L}{6} = -\frac{2}{3}e^{-t}\text{ A}, \qquad t > 0$$

Thus, for all time,

$$i_o(t) = \begin{cases} 0\text{ A}, & t < 0 \\ -\frac{2}{3}e^{-t}\text{ A}, & t > 0 \end{cases}, \qquad v_o(t) = \begin{cases} 6\text{ V}, & t < 0 \\ 4e^{-t}\text{ V}, & t > 0 \end{cases}$$

$$i(t) = \begin{cases} 2\text{ A}, & t < 0 \\ 2e^{-t}\text{ A}, & t \geq 0 \end{cases}$$

We notice that the inductor current is continuous at $t = 0$, while the current through the 6-Ω resistor drops from 0 to $-2/3$ at $t = 0$, and the voltage across the 3-Ω resistor drops from 6 to 4 at $t = 0$. We also notice that the time constant is the same regardless of what the output is defined to be. Figure 7.21 plots i and i_o.



Practice Problem 7.5

Determine i, i_o, and v_o for all t in the circuit shown in Fig. 7.22. Assume that the switch was closed for a long time. It should be noted that opening a switch in series with an ideal current source creates an infinite voltage at the current source terminals. Clearly this is impossible. For the purposes of problem solving, we can place a shunt resistor in parallel with the source (which now makes it a voltage source in series with a resistor). In more practical circuits, devices that act like current sources are, for the most part, electronic circuits. These circuits will allow the source to act like an ideal current source over its operating range but voltage-limit it when the load resistor becomes too large (as in an open circuit).

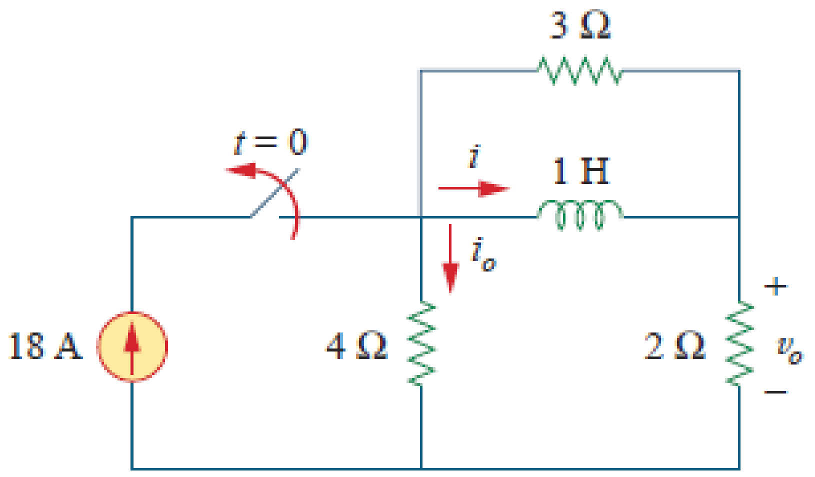


Figure 7.22
For Practice Prob. 7.5.

Answer:

$$i = \begin{cases} 12\text{ A}, & t < 0 \\ 12e^{-2t}\text{ A}, & t \geq 0 \end{cases}, \quad i_o = \begin{cases} 6\text{ A}, & t < 0 \\ -4e^{-2t}\text{ A}, & t > 0 \end{cases},$$

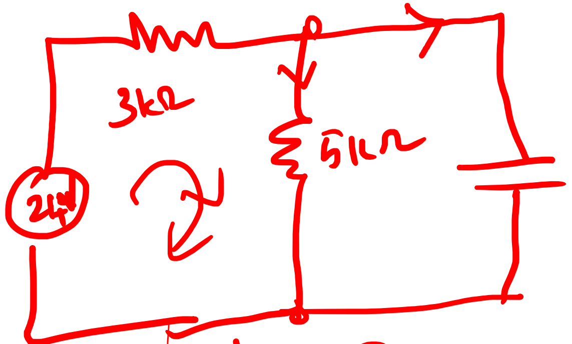
$$v_o = \begin{cases} 24\text{ V}, & t < 0 \\ 8e^{-2t}\text{ V}, & t > 0 \end{cases}$$

Step Response of an *RC* Circuit

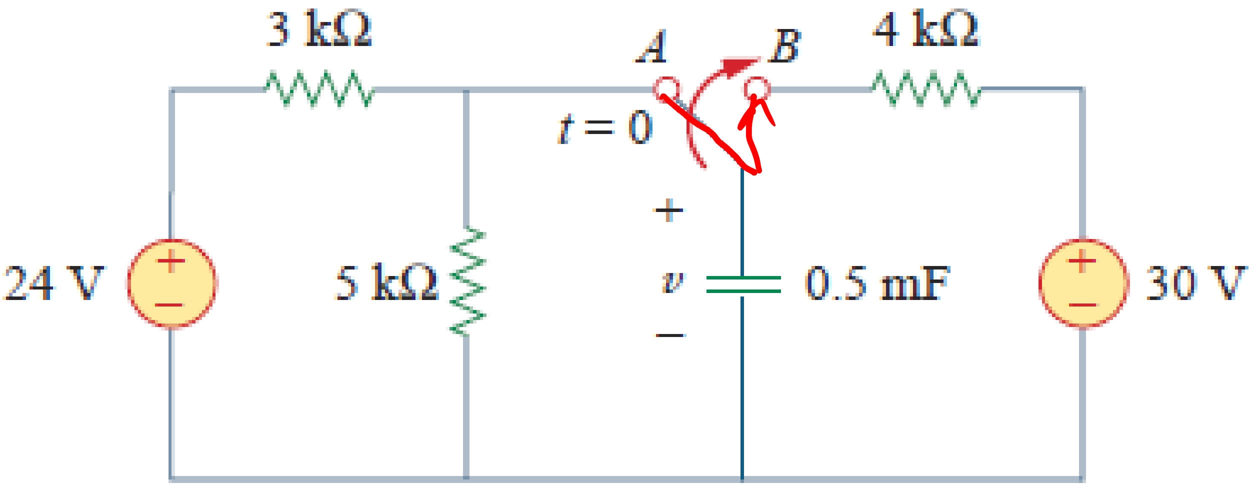
Example 7.10

$R_{TH} = R_e.$

The switch in Fig. has been in position *A* for a long time. At $t = 0$, the switch moves to *B*. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ s and 4 s.



$t<0$



$t>0$

$\frac{24}{5+3}$

For $t < 0$, the switch is at position *A*. The capacitor acts like an open circuit to dc, but v is the same as the voltage across the 5-kΩ resistor. Hence, the voltage across the capacitor just before $t = 0$ is obtained by voltage division as

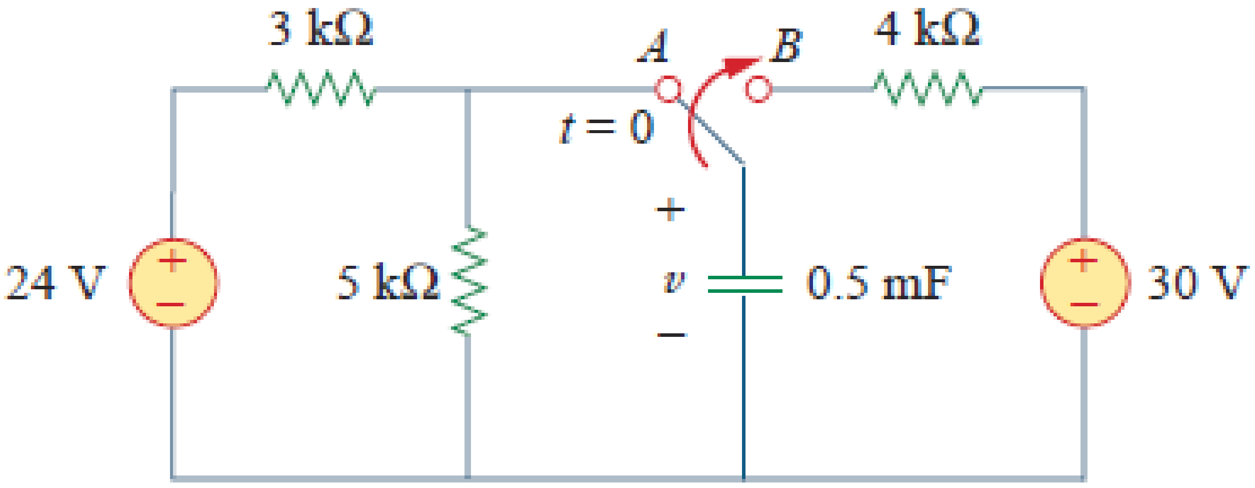
$$v(0^-) = \frac{5}{5+3}(24) = 15 \text{ V}$$

$$v(0^-) = \frac{5}{5+3}(24) = 15 \text{ V}$$

Using the fact that the capacitor voltage cannot change instantaneously,

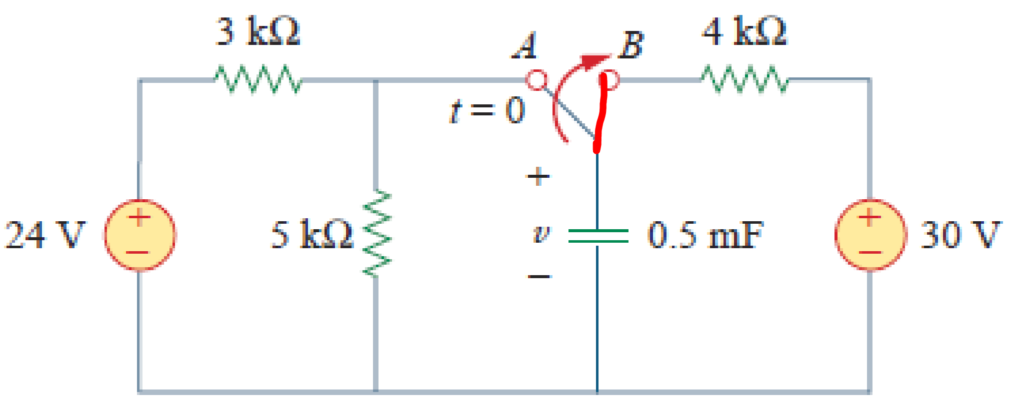
$$\boxed{v(0) = v(0^-) = v(0^+) = 15 \text{ V}}$$

Step respon



For $t > 0$, the switch is in position B. The Thevenin resistance connected to the capacitor is $R_{Th} = 4$ kΩ, and the time constant is

$$\tau = R_{Th}C = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2 \text{ s}$$



$t > 0$
1
2
3
4

$v(\infty) = 30\,V$

Since the capacitor acts like an open circuit to dc at steady state, $v(\infty) = 30$ V. Thus,

$$\boxed{v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}}$$
$$= 30 + (15 - 30)e^{-t/2} = (30 - 15e^{-0.5t})\text{ V}$$

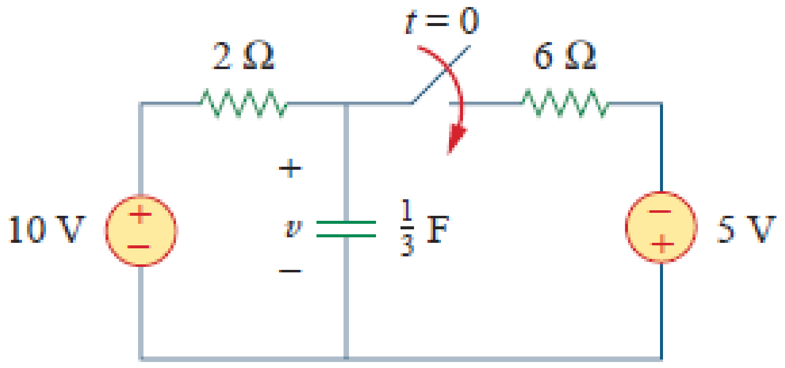
At $t = 1$,

$$v(1) = 30 - 15e^{-0.5} = 20.9\text{ V}$$

At $t = 4$,

$$v(4) = 30 - 15e^{-2} = 27.97\text{ V}$$

Practice Problem 7.10



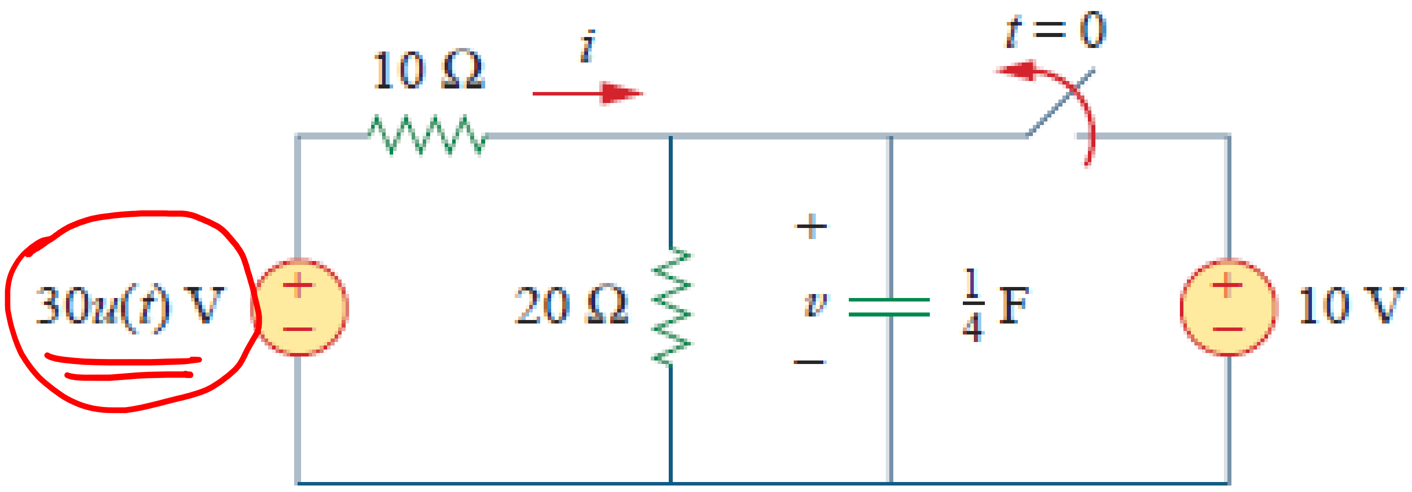
Find $v(t)$ for $t > 0$ in the circuit of Fig. 7.44. Assume the switch has been open for a long time and is closed at $t = 0$. Calculate $v(t)$ at $t = 0.5$.

Answer: $(6.25 + 3.75e^{-2t})$ V for all $t > 0$, 7.63 V.

Example 7.11

In Fig. , the switch has been closed for a long time and is opened at $t = 0$. Find i and v for all time.

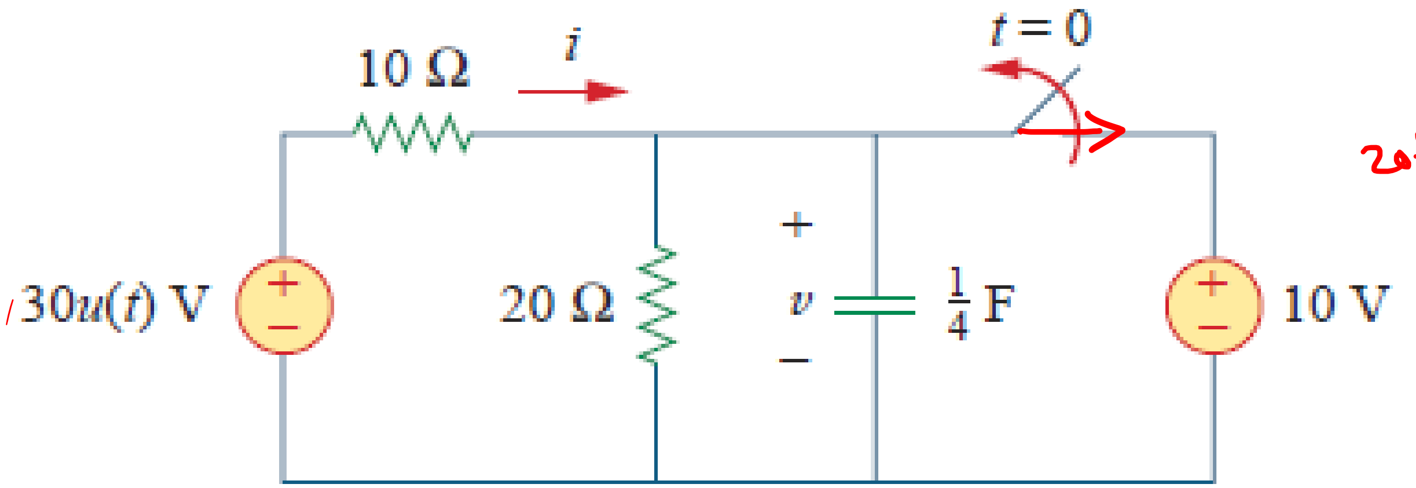
→ $t<0$ switch closed
→ $t=0$ switch opened
→ $t>0$

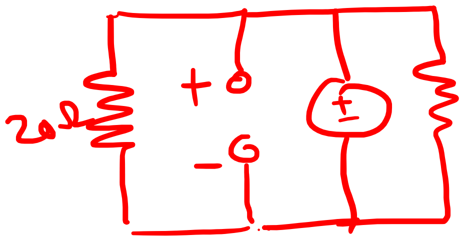


By definition of the unit step function,

$$30u(t) = \begin{cases} 0, & t < 0 \\ 30, & t > 0 \end{cases}$$

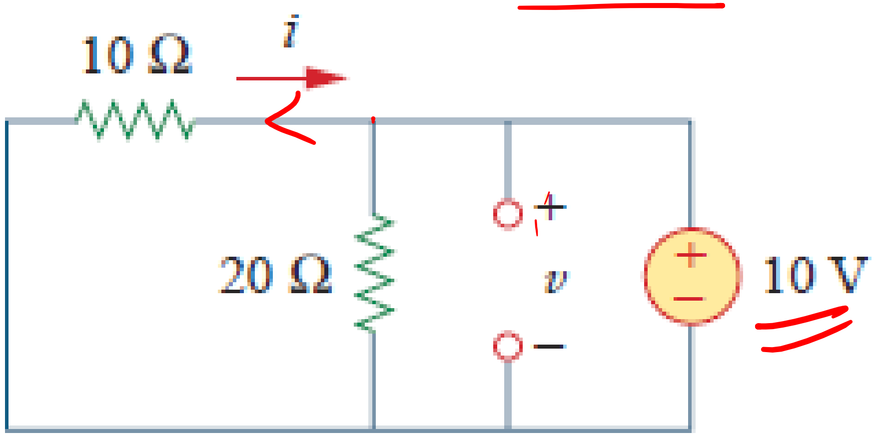
for $t < 0$





Since the switch has been closed for a long time, the capacitor voltage has reached steady state and the capacitor acts like an open circuit.

$v(0) = 10\,V = v(0^-)$



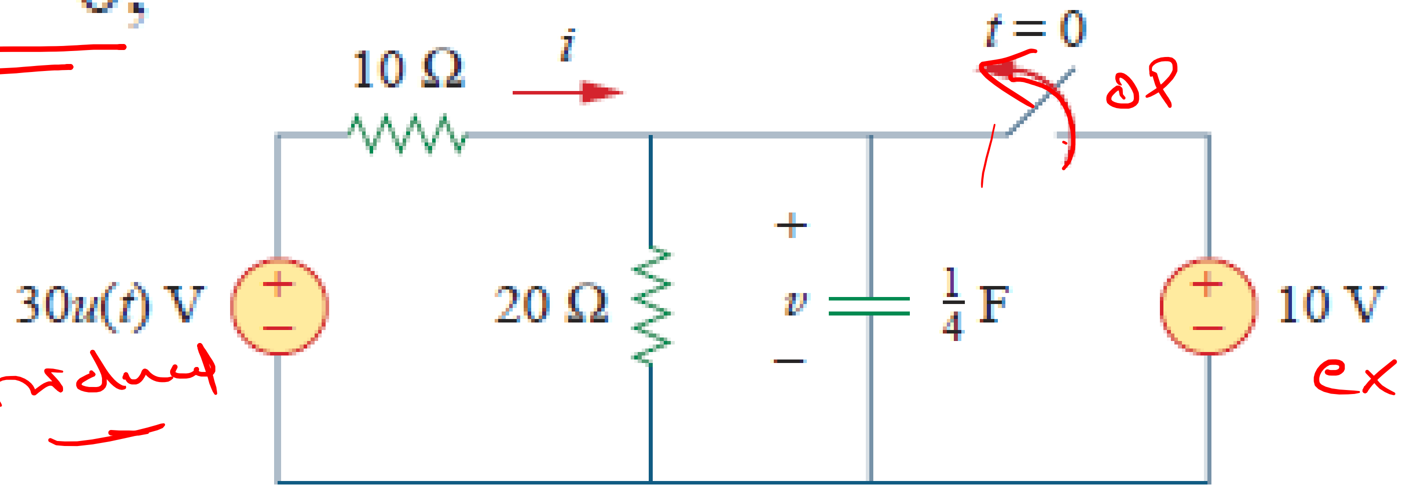
(a)

$$v = 10\text{ V}, \qquad i = -\frac{v}{10\,\Omega} = -1\text{ A} \qquad \text{for } t < 0$$

Since the capacitor voltage cannot change instantaneously,

$$\boxed{v(0) = v(0^-) = 10\text{ V}}$$

For $t > 0$,



op

introduce

excluded.

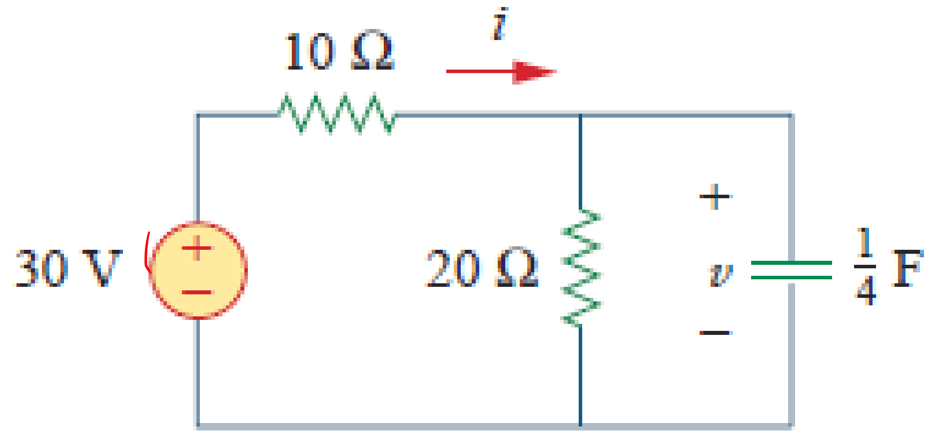
30

t=0

$$30u(t) = \begin{cases} 0, & t < 0 \\ 30, & t > 0 \end{cases}$$

$t = \infty$

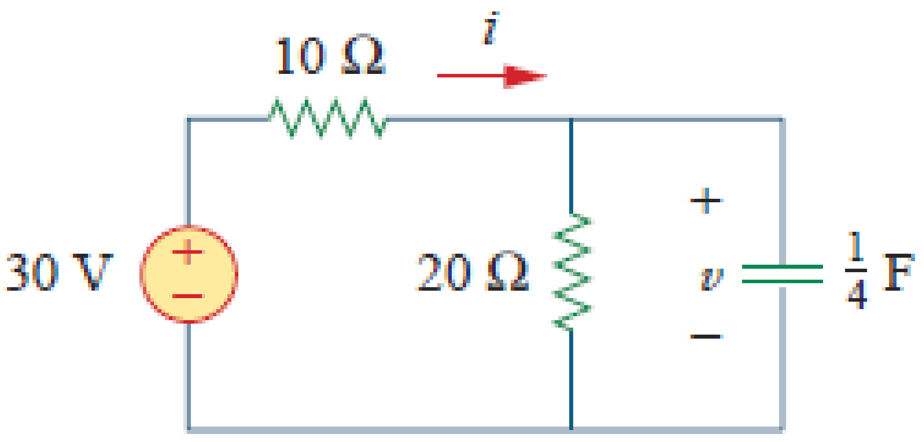
After a long time, the circuit reaches steady state and the capacitor acts like an open circuit again.



(b)

$$v(\infty) = \frac{20}{20 + 10}(30) = 20 \text{ V}$$

For $t > 0$,



(b)

$v(0)$

$v(\infty)$

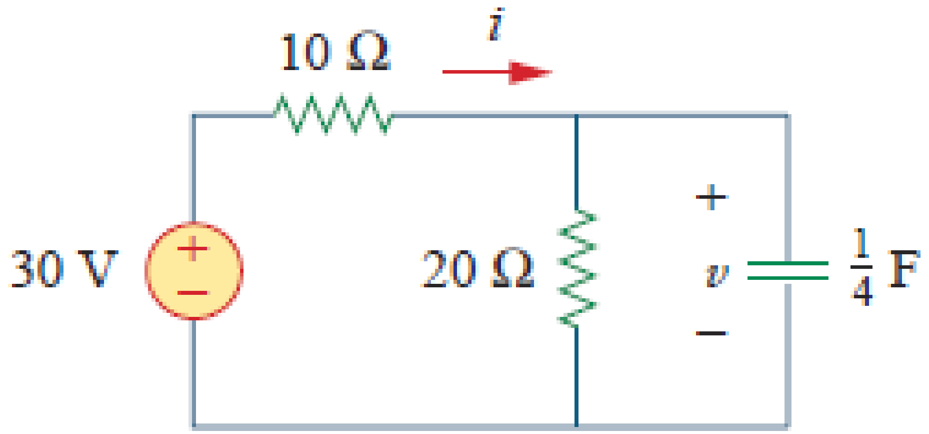
$R_{TH} = R_{eq.}$

The Thevenin resistance at the capacitor terminals is

$$R_{\text{Th}} = 10 \parallel 20 = \frac{10 \times 20}{30} = \frac{20}{3}\,\Omega$$

and the time constant is

$$\tau = R_{\text{Th}} C = \frac{20}{3} \cdot \frac{1}{4} = \frac{5}{3}\text{ s}$$



(b)

t > 0

Thus,

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$
$$= 20 + (10 - 20)e^{-(3/5)t} = (20 - 10e^{-0.6t}) \text{ V}$$

$$i = \frac{v}{20} + C\frac{dv}{dt}$$
$$= 1 - 0.5e^{-0.6t} + 0.25(-0.6)(-10)e^{-0.6t} = (1 + e^{-0.6t}) \text{ A}$$

Hence,

$$v = \begin{cases} 10 \text{ V}, & t < 0 \\ (20 - 10e^{-0.6t}) \text{ V}, & t \geq 0 \end{cases}$$

$$i = \begin{cases} -1 \text{ A}, & t < 0 \\ (1 + e^{-0.6t}) \text{ A}, & t > 0 \end{cases}$$

Practice Problem 7.11

The switch in Fig. 7.47 is closed at $t = 0$. Find $i(t)$ and $v(t)$ for all time. Note that $u(-t) = 1$ for $t < 0$ and 0 for $t > 0$. Also, $u(-t) = 1 - u(t)$.

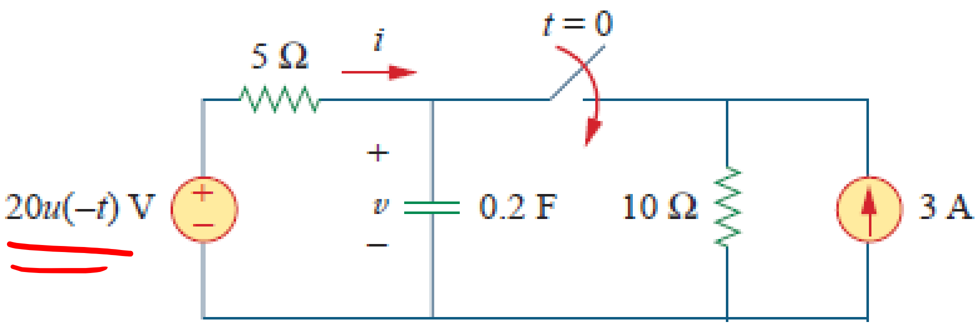


Figure 7.47
For Practice Prob. 7.11.

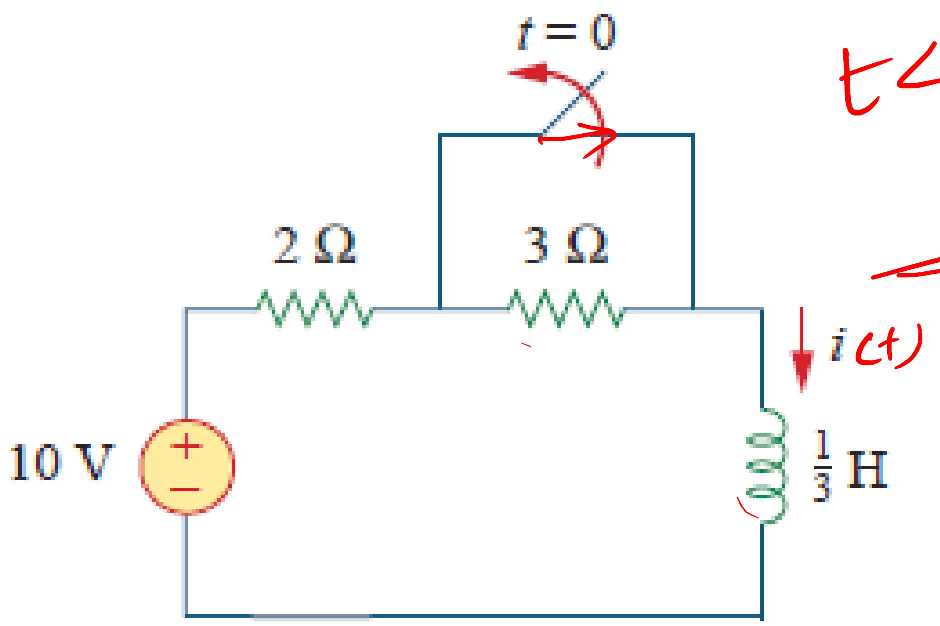
Answer: $i(t) = \begin{cases} 0, & t < 0 \\ -2(1 + e^{-1.5t})\text{ A}, & t > 0 \end{cases}$,

$$v = \begin{cases} 20\text{ V}, & t < 0 \\ 10(1 + e^{-1.5t})\text{ V}, & t > 0 \end{cases}$$

7.6 Step Response of an *RL* Circuit

Example 7.12

Find $i(t)$ in the circuit of Fig. for $t > 0$. Assume that the switch has been closed for a long time.



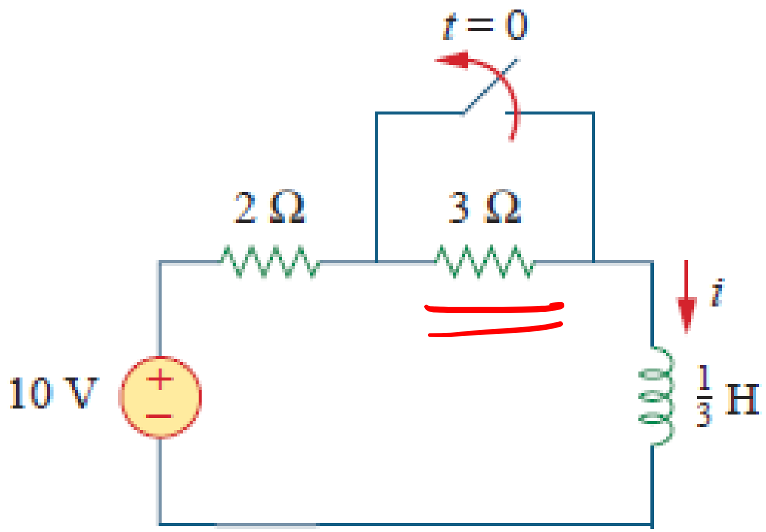
t<0

When $t < 0$, the 3-Ω resistor is short-circuited, and the inductor acts like a short circuit. The current through the inductor at $t = 0^-$ (i.e., just before $t = 0$) is

$$i(0^-) = \frac{10}{2} = 5 \text{ A}$$

Since the inductor current cannot change instantaneously,

$$i(0) = i(0^+) = i(0^-) = 5 \text{ A}$$



$i(0)$
$i(\infty)$

When $t > 0$, the switch is open. The 2-Ω and 3-Ω resistors are in series, so that

$$i(\infty) = \frac{10}{2+3} = 2\text{ A}$$

The Thevenin resistance across the inductor terminals is

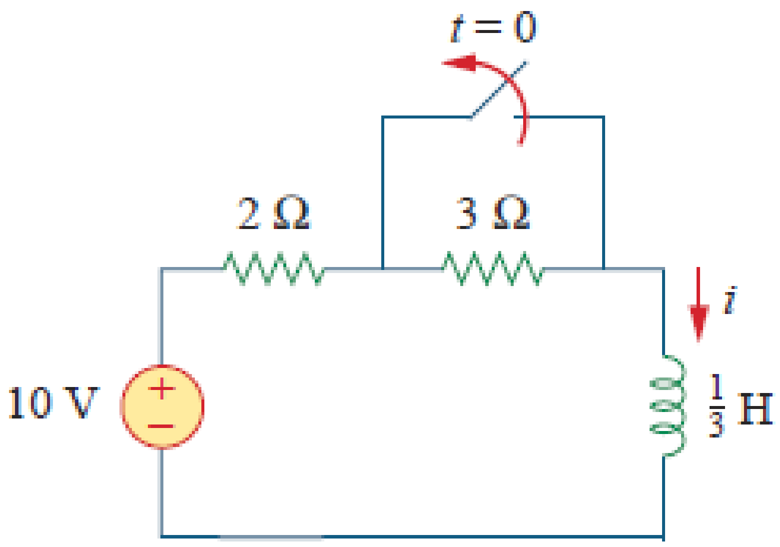
$$R_{\text{Th}} = 2 + 3 = 5\ \Omega$$

For the time constant,

$$\tau = \frac{L}{R_{\text{Th}}} = \frac{\frac{1}{3}}{5} = \frac{1}{15}\text{ s}$$

Thus,

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$
$$= 2 + (5 - 2)e^{-15t} = 2 + 3e^{-15t} \text{ A}, \quad t > 0$$



Check: In Fig. 7.51, for $t > 0$, KVL must be satisfied; that is,

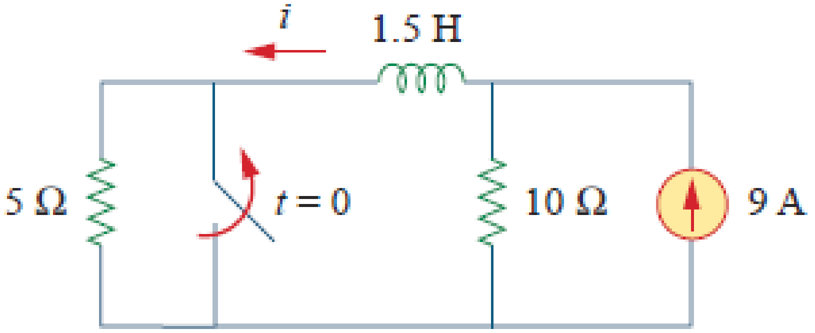
$$10 = 5i + L\frac{di}{dt}$$

$$5i + L\frac{di}{dt} = [10 + 15e^{-15t}] + \left[\frac{1}{3}(3)(-15)e^{-15t}\right] = 10$$

This confirms the result.

Practice Problem 7.12

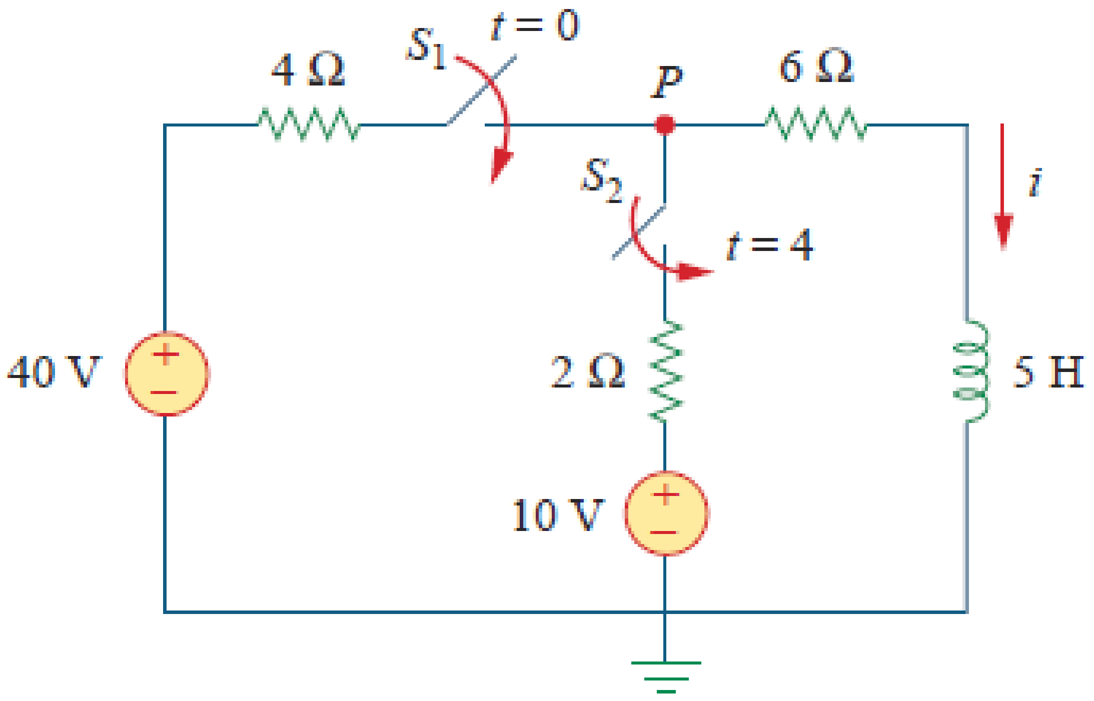
The switch in Fig. 7.52 has been closed for a long time. It opens at $t = 0$. Find $i(t)$ for $t > 0$.



Answer: $(6 + 3e^{-10t})$ A for all $t > 0$.

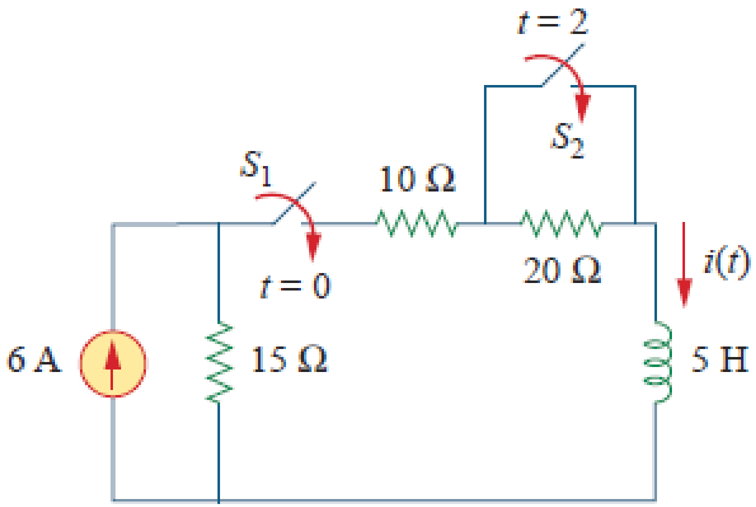
Example 7.13

At $t = 0$, switch 1 in Fig. is closed, and switch 2 is closed 4 s later. Find $i(t)$ for $t > 0$. Calculate i for $t = 2$ s and $t = 5$ s.



Practice Problem 7.13

Switch S_1 in Fig. 7.54 is closed at $t = 0$, and switch S_2 is closed at $t = 2$ s. Calculate $i(t)$ for all t. Find $i(1)$ and $i(3)$.



Answer:

$$i(t) = \begin{cases} 0, & t < 0 \\ 2(1 - e^{-9t}), & 0 < t < 2 \\ 3.6 - 1.6e^{-5(t-2)}, & t > 2 \end{cases}$$

$i(1) = 1.9997$ A, $i(3) = 3.589$ A.