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# CHAPTER 8

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## THE STEADY MAGNETIC FIELD

At this point the concept of a field should be a familiar one. Since we first accepted the experimental law of forces existing between two point charges and defined electric field intensity as the force per unit charge on a test charge in the presence of a second charge, we have discussed numerous fields. These fields possess no real physical basis, for physical measurements must always be in terms of the forces on the charges in the detection equipment. Those charges which are the source cause measurable forces to be exerted on other charges, which we may think of as detector charges. The fact that we attribute a field to the source charges and then determine the effect of this field on the detector charges amounts merely to a division of the basic problem into two parts for convenience.

We shall begin our study of the magnetic field with a definition of the magnetic field itself and show how it arises from a current distribution. The effect of this field on other currents, or the second half of the physical problem, will be discussed in the following chapter. As we did with the electric field, we shall confine our initial discussion to free-space conditions, and the effect of material media will also be saved for discussion in the following chapter.

The relation of the steady magnetic field to its source is more complicated than is the relation of the electrostatic field to its source. We shall find it necessary to accept several laws temporarily on faith alone, relegating their proof to

the (rather difficult) final section in this chapter. This section may well be omitted when studying magnetic fields for the first time. It is included to make acceptance of the laws a little easier; the proof of the laws does exist and is available for the disbelievers or the more advanced student.

## 8.1 BIOT-SAVART LAW

The source of the steady magnetic field may be a permanent magnet, an electric field changing linearly with time, or a direct current. We shall largely ignore the permanent magnet and save the time-varying electric field for a later discussion. Our present relationships will concern the magnetic field produced by a differential dc element in free space.

We may think of this differential current element as a vanishingly small section of a current-carrying filamentary conductor, where a filamentary conductor is the limiting case of a cylindrical conductor of circular cross section as the radius approaches zero. We assume a current  $I$  flowing in a differential vector length of the filament  $d\mathbf{L}$ . The law of Biot-Savart<sup>1</sup> then states that at any point  $P$  the magnitude of the magnetic field intensity produced by the differential element is proportional to the product of the current, the magnitude of the differential length, and the sine of the angle lying between the filament and a line connecting the filament to the point  $P$  at which the field is desired; also, the magnitude of the magnetic field intensity is inversely proportional to the square of the distance from the differential element to the point  $P$ . The direction of the magnetic field intensity is normal to the plane containing the differential filament and the line drawn from the filament to the point  $P$ . Of the two possible normals, that one is to be chosen which is in the direction of progress of a right-handed screw turned from  $d\mathbf{L}$  through the smaller angle to the line from the filament to  $P$ . Using rationalized mks units, the constant of proportionality is  $1/4\pi$ .

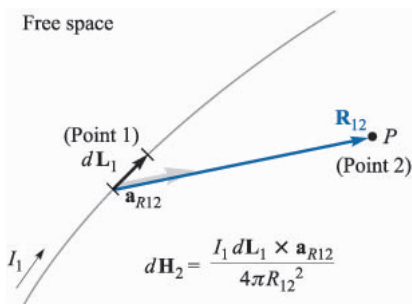
The *Biot-Savart law*, described above in some 150 words, may be written concisely using vector notation as

$$d\mathbf{H} = \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{Id\mathbf{L} \times \mathbf{R}}{4\pi R^3} \quad (1)$$

The units of the *magnetic field intensity*  $\mathbf{H}$  are evidently amperes per meter (A/m). The geometry is illustrated in Fig. 8.1. Subscripts may be used to indicate the point to which each of the quantities in (1) refers. If we locate the current element at point 1 and describe the point  $P$  at which the field is to be determined as point 2, then

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<sup>1</sup> Biot and Savart were colleagues of Ampère, and all three were professors of physics at the Collège de France at one time or another. The Biot-Savart law was proposed in 1820.

**FIGURE 8.1**

The law of Biot-Savart expresses the magnetic field intensity  $d\mathbf{H}_2$  produced by a differential current element  $I_1 d\mathbf{L}_1$ . The direction of  $d\mathbf{H}_2$  is into the page.

$$d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2} \quad (2)$$

The law of Biot-Savart is sometimes called *Ampère's law for the current element*, but we shall retain the former name because of possible confusion with Ampère's circuital law, to be discussed later.

In some aspects, the Biot-Savart law is reminiscent of Coulomb's law when that law is written for a differential element of charge,

$$d\mathbf{E}_2 = \frac{dQ_1 \mathbf{a}_{R12}}{4\pi\epsilon_0 R_{12}^2}$$

Both show an inverse-square-law dependence on distance, and both show a linear relationship between source and field. The chief difference appears in the direction of the field.

It is impossible to check experimentally the law of Biot-Savart as expressed by (1) or (2) because the differential current element cannot be isolated. We have restricted our attention to direct currents only, so the charge density is not a function of time. The continuity equation in Sec. 5.2, Eq. (5),

$$\mathbf{V} \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

therefore shows that

$$\mathbf{V} \cdot \mathbf{J} = 0$$

or upon applying the divergence theorem,

$$\oint_s \mathbf{J} \cdot d\mathbf{S} = 0$$

The total current crossing any closed surface is zero, and this condition may be satisfied only by assuming a current flow around a closed path. It is this current flowing in a closed circuit which must be our experimental source, not the differential element.

It follows that only the integral form of the Biot-Savart law can be verified experimentally,

$$\mathbf{H} = \oint \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} \quad (3)$$

Equation (1) or (2), of course leads directly to the integral form (3), but other differential expressions also yield the same integral formulation. Any term may be added to (1) whose integral around a closed path is zero. That is, any conservative field could be added to (1). The gradient of any scalar field always yields a conservative field, and we could therefore add a term  $\nabla G$  to (1), where  $G$  is a general scalar field, without changing (3) in the slightest. This qualification on (1) or (2) is mentioned to show that if we later ask some foolish questions, not subject to any experimental check, concerning the force exerted by one *differential* current element on another, we should expect foolish answers.

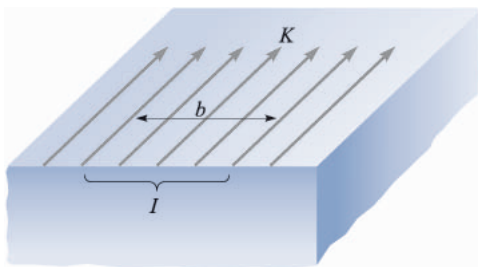
The Biot-Savart law may also be expressed in terms of distributed sources, such as current density  $\mathbf{J}$  and *surface current density*  $\mathbf{K}$ . Surface current flows in a sheet of vanishingly small thickness, and the current density  $\mathbf{J}$ , measured in amperes per square meter, is therefore infinite. Surface current density, however, is measured in amperes per meter width and designated by  $\mathbf{K}$ . If the surface current density is uniform, the total current  $I$  in any width  $b$  is

$$I = Kb$$

where we have assumed that the width  $b$  is measured perpendicularly to the direction in which the current is flowing. The geometry is illustrated by Fig. 8.2. For a nonuniform surface current density, integration is necessary:

$$I = \int K dN \quad (4)$$

where  $dN$  is a differential element of the path *across* which the current is flowing. Thus the differential current element  $I d\mathbf{L}$ , where  $d\mathbf{L}$  is in the direction of the



**FIGURE 8.2**

The total current  $I$  within a transverse width  $b$ , in which there is a *uniform* surface current density  $K$ , is  $Kb$ .

current, may be expressed in terms of surface current density  $\mathbf{K}$  or current density  $\mathbf{J}$ ,

$$I d\mathbf{L} = \mathbf{K} dS = \mathbf{J} dv \quad (5)$$

and alternate forms of the Biot-Savart law obtained,

$$\mathbf{H} = \int_s \frac{\mathbf{K} \times \mathbf{a}_R dS}{4\pi R^2} \quad (6)$$

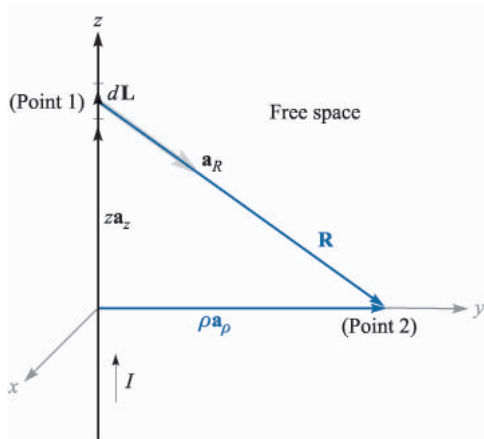
and

$$\mathbf{H} = \int_{\text{vol}} \frac{\mathbf{J} \times \mathbf{a}_R dv}{4\pi R^2} \quad (7)$$

We may illustrate the application of the Biot-Savart law by considering an infinitely long straight filament. We shall apply (2) first and then integrate. This, of course, is the same as using the integral form (3) in the first place.<sup>2</sup>

Referring to Fig. 8.3, we should recognize the symmetry of this field. No variation with  $z$  or with  $\phi$  can exist. Point 2, at which we shall determine the field, is therefore chosen in the  $z = 0$  plane. The field point  $\mathbf{r}$  is therefore  $r = \rho\mathbf{a}_\rho$ . The source point  $\mathbf{r}'$  is given by  $\mathbf{r}' = z'\mathbf{a}_z$ , and therefore

$$\mathbf{R}_{12} = \mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho - z'\mathbf{a}_z$$



**FIGURE 8.3**

An infinitely long straight filament carrying a direct current  $I$ . The field at point 2 is  $\mathbf{H} = (I/2\pi\rho)\mathbf{a}_\phi$ .

<sup>2</sup> The closed path for the current may be considered to include a return filament parallel to the first filament and infinitely far removed. An outer coaxial conductor of infinite radius is another theoretical possibility. Practically, the problem is an impossible one, but we should realize that our answer will be quite accurate near a very long straight wire having a distant return path for the current.

so that

$$\mathbf{a}_{R12} = \frac{\rho \mathbf{a}_\rho - z' \mathbf{a}_z}{\sqrt{\rho^2 + z'^2}}$$

We take  $d\mathbf{L} = dz' \mathbf{a}_z$  and (2) becomes

$$d\mathbf{H}_2 = \frac{I dz' \mathbf{a}_z \times (\rho \mathbf{a}_\rho - z' \mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

Since the current is directed toward increasing values of  $z'$ , the limits are  $-\infty$  and  $\infty$  on the integral, and we have

$$\begin{aligned} \mathbf{H}_2 &= \int_{-\infty}^{\infty} \frac{I dz' \mathbf{a}_z \times (\rho \mathbf{a}_\rho - z' \mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}} \\ &= \frac{I}{4\pi} \int_{-\infty}^{\infty} \frac{\rho dz' \mathbf{a}_\phi}{(\rho^2 + z'^2)^{3/2}} \end{aligned}$$

At this point the unit vector  $\mathbf{a}_\phi$ , under the integral sign should be investigated, for it is not always a constant, as are the unit vectors of the cartesian coordinate system. A vector is constant when its magnitude and direction are both constant. The unit vector certainly has constant magnitude, but its direction may change. Here  $\mathbf{a}_\phi$  changes with the coordinate  $\phi$  but not with  $\rho$  or  $z$ . Fortunately, the integration here is with respect to  $z'$ , and  $\mathbf{a}_\phi$  is a constant and may be removed from under the integral sign,

$$\begin{aligned} \mathbf{H}_2 &= \frac{I \rho \mathbf{a}_\phi}{4\pi} \int_{-\infty}^{\infty} \frac{dz'}{(\rho^2 + z'^2)^{3/2}} \\ &= \frac{I \rho \mathbf{a}_\phi}{4\pi} \left. \frac{z'}{\rho^2 \sqrt{\rho^2 + z'^2}} \right|_{-\infty}^{\infty} \end{aligned}$$

and

$$\mathbf{H}_2 = \frac{I}{2\pi\rho} \mathbf{a}_\phi$$

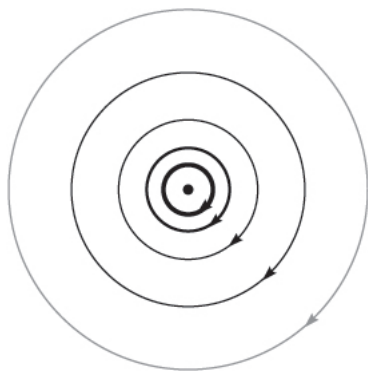
(8)

The magnitude of the field is not a function of  $\phi$  or  $z$  and it varies inversely as the distance from the filament. The direction of the magnetic-field-intensity vector is circumferential. The streamlines are therefore circles about the filament, and the field may be mapped in cross section as in Fig. 8.4.

The separation of the streamlines is proportional to the radius, or inversely proportional to the magnitude of  $\mathbf{H}$ . To be specific, the streamlines have been drawn with curvilinear squares in mind. As yet we have no name for the family of lines<sup>3</sup> which are perpendicular to these circular streamlines, but the spacing of

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<sup>3</sup> If you can't wait, see Sec. 8.6

**FIGURE 8.4**

The streamlines of the magnetic field intensity about an infinitely long straight filament carrying a direct current  $I$ . The direction of  $I$  is into the page.

the streamlines has been adjusted so that the addition of this second set of lines will produce an array of curvilinear squares.

A comparison of Fig. 8.4 with the map of the *electric* field about an infinite line *charge* shows that the streamlines of the magnetic field correspond exactly to the equipotentials of the electric field, and the unnamed (and undrawn) perpendicular family of lines in the magnetic field corresponds to the streamlines of the electric field. This correspondence is not an accident, but there are several other concepts which must be mastered before the analogy between electric and magnetic fields can be explored more thoroughly.

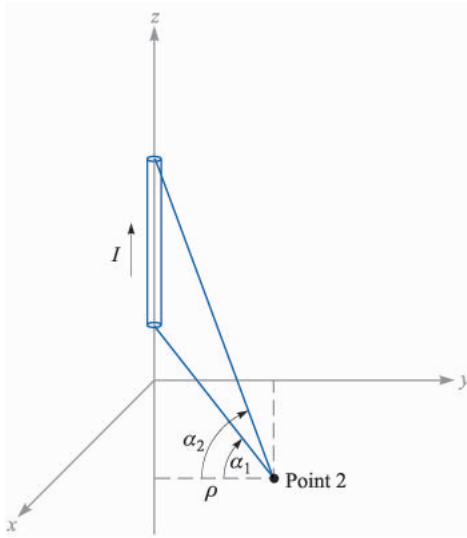
Using the Biot-Savart law to find  $\mathbf{H}$  is in many respects similar to the use of Coulomb's law to find  $\mathbf{E}$ . Each requires the determination of a moderately complicated integrand containing vector quantities, followed by an integration. When we were concerned with Coulomb's law we solved a number of examples, including the fields of the point charge, line charge, and sheet of charge. The law of Biot-Savart can be used to solve analogous problems in magnetic fields, and some of these problems now appear as exercises at the end of the chapter rather than as examples here.

One useful result is the field of the finite-length current element, shown in Fig. 8.5. It turns out (see Prob. 8 at the end of the chapter) that  $\mathbf{H}$  is most easily expressed in terms of the angles  $\alpha_1$  and  $\alpha_2$ , as identified in the figure. The result is

$$\mathbf{H} = \frac{I}{4\pi\rho} (\sin \alpha_2 - \sin \alpha_1) \mathbf{a}_\phi \quad (9)$$

If one or both ends are below point 2, then  $\alpha_1$ , or both  $\alpha_1$  and  $\alpha_2$ , are negative.

Equation (9) may be used to find the magnetic field intensity caused by current filaments arranged as a sequence of straight line segments.

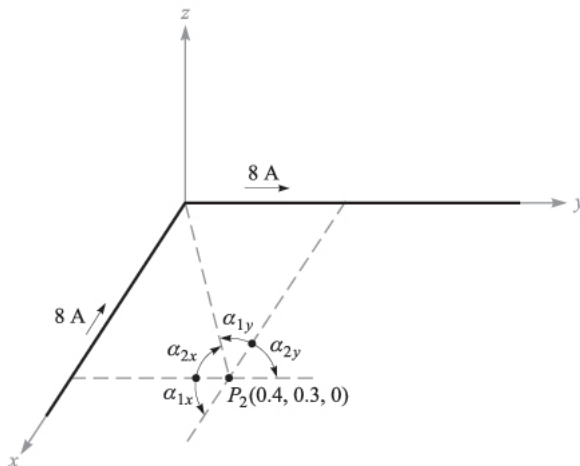
**FIGURE 8.5**

The magnetic field intensity caused by a finite-length current filament on the  $z$  axis is  $(I/4\pi\rho)(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi$ .

### Example 8.1

As a numerical example illustrating the use of (9), let us determine  $\mathbf{H}$  at  $P_2(0.4, 0.3, 0)$  in the field of an 8-A filamentary current directed inward from infinity to the origin on the positive  $x$  axis, and then outward to infinity along the  $y$  axis. This arrangement is shown in Figure 8.6.

**Solution.** We first consider the semi-infinite current on the  $x$  axis, identifying the two angles,  $\alpha_{1x} = -90^\circ$  and  $\alpha_{2x} = \tan^{-1}(0.4/0.3) = 53.1^\circ$ . The radial distance  $\rho$  is measured from the  $x$  axis, and we have  $\rho_x = 0.3$ . Thus, this contribution to  $\mathbf{H}_2$  is

**FIGURE 8.6**

The individual fields of two semi-infinite current segments are found by (9) and added to obtain  $\mathbf{H}_2$  at  $P_2$ .



$$\mathbf{H}_{2(x)} = \frac{8}{4\pi(0.3)}(\sin 53.1^\circ + 1)\mathbf{a}_\phi = \frac{2}{0.3\pi}(1.8)\mathbf{a}_\phi = \frac{12}{\pi}\mathbf{a}_\phi$$

The unit vector  $\mathbf{a}_\phi$ , must also be referred to the  $x$  axis. We see that it becomes  $-\mathbf{a}_z$ . Therefore,

$$\mathbf{H}_{2(x)} = -\frac{12}{\pi}\mathbf{a}_z \quad \text{A/m}$$

For the current on the  $y$  axis, we have  $\alpha_{1y} = -\tan^{-1}(0.3/0.4) = -36.9^\circ$ ,  $\alpha_{2y} = 90^\circ$ , and  $\rho_y = 0.4$ . It follows that

$$\mathbf{H}_{2(y)} = \frac{8}{4\pi(0.4)}(1 + \sin 36.9^\circ)(-\mathbf{a}_z) = -\frac{8}{\pi}\mathbf{a}_z \quad \text{A/m}$$

Adding these results, we have

$$\mathbf{H}_2 = \mathbf{H}_{2(x)} + \mathbf{H}_{2(y)} = -\frac{20}{\pi}\mathbf{a}_z = -6.37\mathbf{a}_z \quad \text{A/m}$$

- ✓ **D8.1.** Given the following values for  $P_1$ ,  $P_2$ , and  $I_1\Delta_1$ , calculate  $\Delta\mathbf{H}_2$ : (a)  $P_1(0, 0, 2)$ ,  $P_2(4, 2, 0)$ ,  $2\pi\mathbf{a}_z\mu\text{A} \cdot \text{m}$ ; (b)  $P_1(0, 2, 0)$ ,  $P_2(4, 2, 0)$ ,  $2\pi\mathbf{a}_z\mu\text{A} \cdot \text{m}$ ; (c)  $P_1(1, 2, 3)$ ,  $P_2(-3, -1, 2)$ ,  $2\pi(-\mathbf{a}_x + \mathbf{a}_y + 2\mathbf{a}_z)\mu\text{A} \cdot \text{m}$ .

**Ans.**  $-8.51\mathbf{a}_x + 17.01\mathbf{a}_y$  nA/m;  $16\mathbf{a}_y$  nA/m;  $3.77\mathbf{a}_x - 6.79\mathbf{a}_y + 5.28\mathbf{a}_z$  nA/m

- ✓ **D8.2.** A current filament carrying 15 A in the  $\mathbf{a}_z$  direction lies along the entire  $z$  axis. Find  $\mathbf{H}$  in cartesian coordinates at: (a)  $P_A(\sqrt{20}, 0, 4)$ ; (b)  $P_B(2, -4, 4)$ .

**Ans.**  $0.534\mathbf{a}_y$  A/m;  $0.477\mathbf{a}_x + 0.239\mathbf{a}_y$  A/m

## 8.2 AMPÈRE'S CIRCUITAL LAW

After solving a number of simple electrostatic problems with Coulomb's law, we found that the same problems could be solved much more easily by using Gauss's law whenever a high degree of symmetry was present. Again, an analogous procedure exists in magnetic fields. Here, the law that helps us solve problems more easily is known as *Ampère's circuital*<sup>4</sup> law, sometimes called Ampère's work law. This law may be derived from the Biot-Savart law, and the derivation is accomplished in Sec. 8.7. For the present we might agree to accept Ampère's circuital law temporarily as another law capable of experimental proof. As is the case with Gauss's law, its use will also require careful consideration of the symmetry of the problem to determine which variables and components are present.

Ampère's circuital law states that the line integral of  $\mathbf{H}$  about any *closed* path is exactly equal to the direct current enclosed by that path,

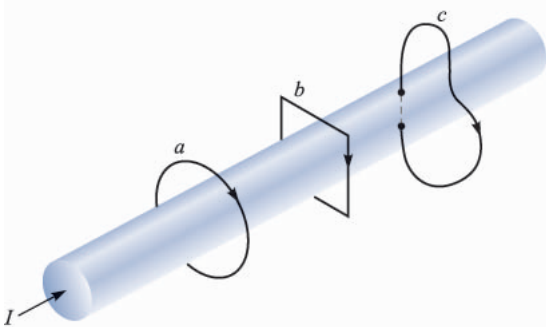
<sup>4</sup> The preferred pronunciation puts the accent on "circ-."

$$\oint \mathbf{H} \cdot d\mathbf{L} = I \quad (10)$$

We define positive current as flowing in the direction of advance of a right-handed screw turned in the direction in which the closed path is traversed.

Referring to Fig. 8.7, which shows a circular wire carrying a direct current  $I$ , the line integral of  $\mathbf{H}$  about the closed paths lettered  $a$  and  $b$  results in an answer of  $I$ ; the integral about the closed path  $c$  which passes through the conductor gives an answer less than  $I$  and is exactly that portion of the total current which is enclosed by the path  $c$ . Although paths  $a$  and  $b$  give the same answer, the integrands are, of course, different. The line integral directs us to multiply the component of  $\mathbf{H}$  in the direction of the path by a small increment of path length at one point of the path, move along the path to the next incremental length, and repeat the process, continuing until the path is completely traversed. Since  $\mathbf{H}$  will generally vary from point to point, and since paths  $a$  and  $b$  are not alike, the contributions to the integral made by, say, each micrometer of path length are quite different. Only the final answers are the same.

We should also consider exactly what is meant by the expression “current enclosed by the path.” Suppose we solder a circuit together after passing the conductor once through a rubber band, which we shall use to represent the closed path. Some strange and formidable paths can be constructed by twisting and knotting the rubber band, but if neither the rubber band nor the conducting circuit is broken, the current enclosed by the path is that carried by the conductor. Now let us replace the rubber band by a circular ring of spring steel across which is stretched a rubber sheet. The steel loop forms the closed path, and the current-carrying conductor must pierce the rubber sheet if the current is to be enclosed by the path. Again, we may twist the steel loop, and we may also deform the rubber sheet by pushing our fist into it or folding it in any way we wish. A single current-carrying conductor still pierces the sheet once, and this is the true measure of the current enclosed by the path. If we should thread the conductor once through the sheet from front to back and once from back to front, the total current enclosed by the path is the algebraic sum, which is zero.



**FIGURE 8.7**

A conductor has a total current  $I$ . The line integral of  $\mathbf{H}$  about the closed paths  $a$  and  $b$  is equal to  $I$ , and the integral around path  $c$  is less than  $I$ , since the entire current is not enclosed by the path.

In more general language, given a closed path, we recognize this path as the perimeter of an infinite number of surfaces (not closed surfaces). Any current-carrying conductor enclosed by the path must pass through every one of these surfaces once. Certainly some of the surfaces may be chosen in such a way that the conductor pierces them twice in one direction and once in the other direction, but the algebraic total current is still the same.

We shall find that the nature of the closed path is usually extremely simple and can be drawn on a plane. The simplest surface is, then, that portion of the plane enclosed by the path. We need merely find the total current passing through this region of the plane.

The application of Gauss's law involves finding the total charge enclosed by a closed surface; the application of Ampère's circuital law involves finding the total current enclosed by a closed path.

Let us again find the magnetic field intensity produced by an infinitely long filament carrying a current  $I$ . The filament lies on the  $z$  axis in free space (as in Fig. 8.3), and the current flows in the direction given by  $\mathbf{a}_z$ . Symmetry inspection comes first, showing that there is no variation with  $z$  or  $\phi$ . Next we determine which components of  $\mathbf{H}$  are present by using the Biot-Savart law. Without specifically using the cross product, we may say that the direction of  $d\mathbf{H}$  is perpendicular to the plane containing  $d\mathbf{L}$  and  $\mathbf{R}$  and therefore is in the direction of  $\mathbf{a}_\phi$ . Hence the only component of  $\mathbf{H}$  is  $H_\phi$ , and it is a function only of  $\rho$ .

We therefore choose a path to any section of which  $\mathbf{H}$  is either perpendicular or tangential and along which  $H$  is constant. The first requirement (perpendicularity or tangency) allows us to replace the dot product of Ampère's circuital law with the product of the scalar magnitudes, except along that portion of the path where  $\mathbf{H}$  is normal to the path and the dot product is zero; the second requirement (constancy) then permits us to remove the magnetic field intensity from the integral sign. The integration required is usually trivial and consists of finding the length of that portion of the path to which  $\mathbf{H}$  is parallel.

In our example the path must be a circle of radius  $\rho$  and Ampère's circuital law becomes

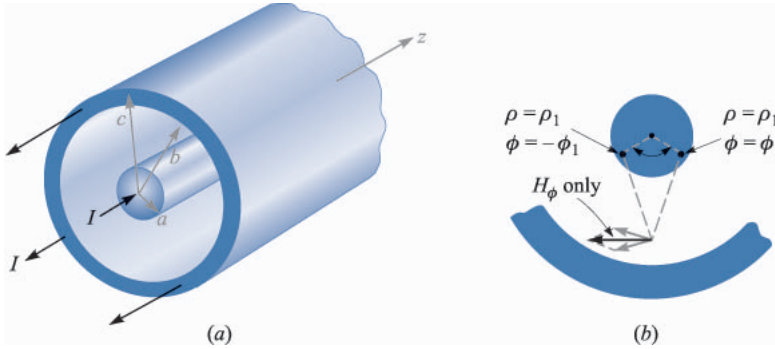
$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho d\phi = H_\phi \rho \int_0^{2\pi} d\phi = H_\phi 2\pi \rho = I$$

or

$$H_\phi = \frac{I}{2\pi\rho}$$

as before.

As a second example of the application of Ampère's circuital law, consider an infinitely long coaxial transmission line carrying a uniformly distributed total current  $I$  in the center conductor and  $-I$  in the outer conductor. The line is shown in Fig. 8.8a. Symmetry shows that  $H$  is not a function of  $\phi$  or  $z$ . In order to determine the components present, we may use the results of the previous

**FIGURE 8.8**

(a) Cross section of a coaxial cable carrying a uniformly distributed current  $I$  in the inner conductor and  $-I$  in the outer conductor. The magnetic field at any point is most easily determined by applying Ampère's circuital law about a circular path. (b) Current filaments at  $\rho = \rho_1$ ,  $\phi = \pm\phi_1$ , produces  $\mathbf{H}_\rho$  components which cancel. For the total field,  $\mathbf{H} = H_\phi \mathbf{a}_\phi$ .

example by considering the solid conductors as being composed of a large number of filaments. No filament has a  $z$  component of  $\mathbf{H}$ . Furthermore, the  $H_\rho$  component at  $\phi = 0^\circ$ , produced by one filament located at  $\rho = \rho_1$ ,  $\phi = \phi_1$ , is canceled by the  $H_\rho$  component produced by a symmetrically located filament at  $\rho = \rho_1$ ,  $\phi = -\phi_1$ . This symmetry is illustrated by Fig. 8.8b. Again we find only an  $H_\phi$  component which varies with  $\rho$ .

A circular path of radius  $\rho$ , where  $\rho$  is larger than the radius of the inner conductor but less than the inner radius of the outer conductor, then leads immediately to

$$H_\phi = \frac{I}{2\pi\rho} \quad (a < \rho < b)$$

If we choose  $\rho$  smaller than the radius of the inner conductor, the current enclosed is

$$I_{\text{encl}} = I \frac{\rho^2}{a^2}$$

and

$$2\pi\rho H_\phi = I \frac{\rho^2}{a^2}$$

or

$$H_\phi = \frac{I\rho}{2\pi a^2} \quad (\rho < a)$$

If the radius  $\rho$  is larger than the outer radius of the outer conductor, no current is enclosed and

$$H_\phi = 0 \quad (\rho > c)$$

Finally, if the path lies within the outer conductor, we have

$$2\pi\rho H_\phi = I - I\left(\frac{\rho^2 - b^2}{c^2 - b^2}\right)$$

$$H_\phi = \frac{I}{2\pi\rho} \frac{c^2 - \rho^2}{c^2 - b^2} \quad (b < \rho < c)$$

The magnetic-field-strength variation with radius is shown in Fig. 8.9 for a coaxial cable in which  $b = 3a$ ,  $c = 4a$ . It should be noted that the magnetic field intensity  $\mathbf{H}$  is continuous at all the conductor boundaries. In other words, a slight increase in the radius of the closed path does not result in the enclosure of a tremendously different current. The value of  $H_\phi$  shows no sudden jumps.

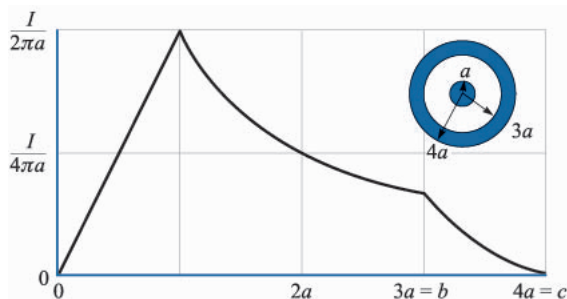
The external field is zero. This, we see, results from equal positive and negative currents enclosed by the path. Each produces an external field of magnitude  $I/2\pi\rho$ , but complete cancellation occurs. This is another example of “shielding”; such a coaxial cable carrying large currents would not produce any noticeable effect in an adjacent circuit.

As a final example, let us consider a sheet of current flowing in the positive  $y$  direction and located in the  $z = 0$  plane. We may think of the return current as equally divided between two distant sheets on either side of the sheet we are considering. A sheet of uniform surface current density  $\mathbf{K} = K_y \mathbf{a}_y$  is shown in Fig. 8.10.  $\mathbf{H}$  cannot vary with  $x$  or  $y$ . If the sheet is subdivided into a number of filaments, it is evident that no filament can produce an  $H_y$  component. Moreover, the Biot-Savart law shows that the contributions to  $H_z$  produced by a symmetrically located pair of filaments cancel. Thus,  $H_z$  is zero also; only an  $H_x$  component is present. We therefore choose the path 1-1'-2'-2-1 composed of straight-line segments which are either parallel or perpendicular to  $H_x$ . Ampère's circuital law gives

$$H_{x1}L + H_{x2}(-L) = K_y L$$

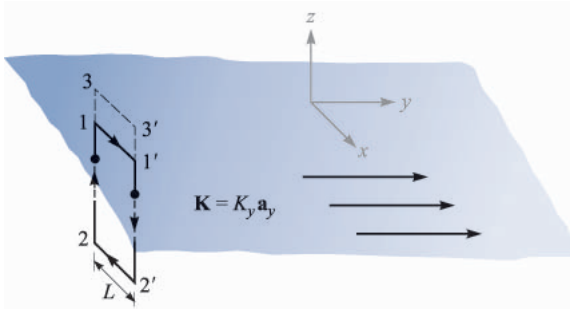
or

$$H_{x1} - H_{x2} = K_y$$



**FIGURE 8.9**

The magnetic field intensity as a function of radius in an infinitely long coaxial transmission line with the dimensions shown.

**FIGURE 8.10**

A uniform sheet of surface current  $\mathbf{K} = K_y \mathbf{a}_y$  in the  $z = 0$  plane.  $\mathbf{H}$  may be found by applying Ampère's circuital law about the paths 1-1'-2'-2-1 and 3-3'-2'-2-3.

If the path 3-3'-2'-2-3 is now chosen, the same current is enclosed, and

$$H_{x3} - H_{x2} = K_y$$

and therefore

$$H_{x3} = H_{x1}$$

It follows that  $H_x$  is the same for all positive  $z$ . Similarly,  $H_x$  is the same for all negative  $z$ . Because of the symmetry, then, the magnetic field intensity on one side of the current sheet is the negative of that on the other. Above the sheet,

$$H_x = \frac{1}{2} K_y \quad (z > 0)$$

while below it

$$H_x = -\frac{1}{2} K_y \quad (z < 0)$$

Letting  $\mathbf{a}_N$  be a unit vector normal (outward) to the current sheet, the result may be written in a form correct for all  $z$  as

$$\mathbf{H} = \frac{1}{2} \mathbf{K} \times \mathbf{a}_N \quad (11)$$

If a second sheet of current flowing in the opposite direction,  $\mathbf{K} = -K_y \mathbf{a}_y$ , is placed at  $z = h$ , (11) shows that the field in the region between the current sheets is

$$\mathbf{H} = \mathbf{K} \times \mathbf{a}_N \quad (0 < z < h) \quad (12)$$

and is zero elsewhere,

$$\mathbf{H} = 0 \quad (z < 0, z > h) \quad (13)$$

The most difficult part of the application of Ampère's circuital law is the determination of the components of the field which are present. The surest

method is the logical application of the Biot-Savart law and a knowledge of the magnetic fields of simple form.

Problem 13 at the end of this chapter outlines the steps involved in applying Ampère's circuital law to an infinitely long solenoid of radius  $a$  and uniform current density  $K_a \mathbf{a}_\phi$ , as shown in Fig. 8.11a. For reference, the result is

$$\mathbf{H} = K_a \mathbf{a}_z \quad (\rho < a) \quad (14a)$$

$$\mathbf{H} = 0 \quad (\rho > a) \quad (14b)$$

If the solenoid has a finite length  $d$  and consists of  $N$  closely wound turns of a filament that carries a current  $I$  (Fig. 8.11b), then the field at points well within the solenoid is given closely by

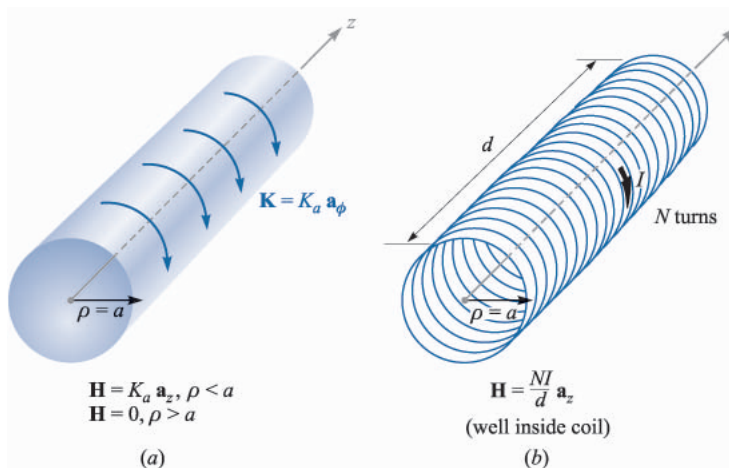
$$\mathbf{H} = \frac{NI}{d} \mathbf{a}_z \quad (\text{well within the solenoid}) \quad (15)$$

The approximation is useful if it is not applied closer than two radii to the open ends, nor closer to the solenoid surface than twice the separation between turns.

For the toroids shown in Fig. 8.12, it can be shown that the magnetic field intensity for the ideal case, Fig. 8.12a, is

$$\mathbf{H} = K_a \frac{\rho_0 - a}{\rho} \mathbf{a}_\phi \quad (\text{inside toroid}) \quad (16a)$$

$$\mathbf{H} = 0 \quad (\text{outside}) \quad (16b)$$



**FIGURE 8.11**

(a) An ideal solenoid of infinite length with a circular current sheet  $\mathbf{K} = K_a \mathbf{a}_\phi$ . (b) An  $N$ -turn solenoid of finite length  $d$ .

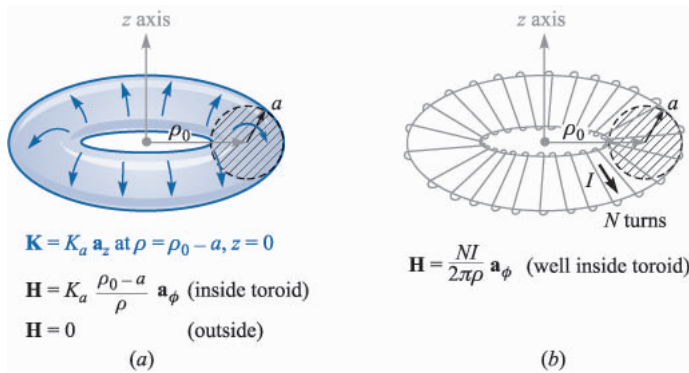


FIGURE 8.12

(a) An ideal toroid carrying a surface current  $\mathbf{K}$  in the direction shown. (b) An  $N$ -turn toroid carrying a filamentary current  $I$ .

For the  $N$ -turn toroid of Figure 8.12b, we have the good approximations,

$$\mathbf{H} = \frac{NI}{2\pi\rho} \mathbf{a}_\phi \quad (\text{inside toroid}) \quad (17a)$$

$$\mathbf{H} = 0 \quad (\text{outside}) \quad (17b)$$

as long as we consider points removed from the toroidal surface by several times the separation between turns.

Toroids having rectangular cross sections are also treated quite readily, as you can see for yourself by trying Prob. 14.

Accurate formulas for solenoids, toroids, and coils of other shapes are available in Sec. 2 of the “Standard Handbook for Electrical Engineers” (see Suggested References for Chap. 5).

- ✓ **D8.3.** Express the value of  $\mathbf{H}$  in cartesian components at  $P(0, 0.2, 0)$  in the field of: (a) a current filament, 2.5 A in the  $\mathbf{a}_z$  direction at  $x = 0.1$ ,  $y = 0.3$ ; (b) a coax, centered on the  $z$  axis, with  $a = 0.3$ ,  $b = 0.5$ ,  $c = 0.6$ ,  $I = 2.5$  A in  $\mathbf{a}_z$  direction in center conductor; (c) three current sheets,  $2.7\mathbf{a}_x$  A/m at  $y = 0.1$ ,  $-1.4\mathbf{a}_x$  A/m at  $y = 0.15$ , and  $-1.3\mathbf{a}_x$  A/m at  $y = 0.25$ .

**Ans.**  $1.989\mathbf{a}_x - 1.989\mathbf{a}_y$  A/m;  $-0.884\mathbf{a}_x$  A/m;  $1.300\mathbf{a}_z$  A/m

### 8.3 CURL

We completed our study of Gauss’s law by applying it to a differential volume element and were led to the concept of divergence. We now apply Ampère’s circuital law to the perimeter of a differential surface element and discuss the



third and last of the special derivatives of vector analysis, the curl. Our immediate objective is to obtain the point form of Ampère's circuital law.

Again we shall choose cartesian coordinates, and an incremental closed path of sides  $\Delta x$  and  $\Delta y$  is selected (Fig. 8.13). We assume that some current, as yet unspecified, produces a reference value for  $\mathbf{H}$  at the *center* of this small rectangle,

$$\mathbf{H}_0 = H_{x0}\mathbf{a}_x + H_{y0}\mathbf{a}_y + H_{z0}\mathbf{a}_z$$

The closed line integral of  $\mathbf{H}$  about this path is then approximately the sum of the four values of  $\mathbf{H} \cdot \Delta\mathbf{L}$  on each side. We choose the direction of traverse as 1-2-3-4-1, which corresponds to a current in the  $\mathbf{a}_z$  direction, and the first contribution is therefore

$$(\mathbf{H} \cdot \Delta\mathbf{L})_{1-2} = H_{y,1-2}\Delta y$$

The value of  $H_y$  on this section of the path may be given in terms of the reference value  $H_{y0}$  at the center of the rectangle, the rate of change of  $H_y$  with  $x$ , and the distance  $\Delta x/2$  from the center to the midpoint of side 1-2:

$$H_{y,1-2} \doteq H_{y0} + \frac{\partial H_y}{\partial x} \left( \frac{1}{2} \Delta x \right)$$

Thus

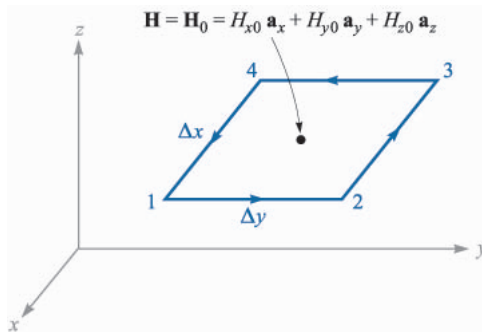
$$(\mathbf{H} \cdot \Delta\mathbf{L})_{1-2} \doteq \left( H_{y0} + \frac{1}{2} \frac{\partial H_y}{\partial x} \Delta x \right) \Delta y$$

Along the next section of the path we have

$$(\mathbf{H} \cdot \Delta\mathbf{L})_{2-3} \doteq H_{x,2-3}(-\Delta x) \doteq - \left( H_{x0} + \frac{1}{2} \frac{\partial H_x}{\partial y} \Delta y \right) \Delta x$$

Continuing for the remaining two segments and adding the results,

$$\oint \mathbf{H} \cdot d\mathbf{L} \doteq \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \Delta x \Delta y$$



**FIGURE 8.13**

An incremental closed path in cartesian coordinates is selected for the application of Ampère's circuital law to determine the spatial rate of change of  $\mathbf{H}$ .

By Ampère's circuital law, this result must be equal to the current enclosed by the path, or the current crossing any surface bounded by the path. If we assume a general current density  $\mathbf{J}$ , the enclosed current is then  $\Delta I \doteq J_z \Delta x \Delta y$ , and

$$\oint \mathbf{H} \cdot d\mathbf{L} \doteq \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \Delta x \Delta y \doteq J_z \Delta x \Delta y$$

or

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta x \Delta y} \doteq \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \doteq J_z$$

As we cause the closed path to shrink, the above expression becomes more nearly exact, and in the limit we have the equality

$$\lim_{\Delta x, \Delta y \rightarrow 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \quad (18)$$

After beginning with Ampère's circuital law equating the closed line integral of  $\mathbf{H}$  to the current enclosed, we have now arrived at a relationship involving the closed line integral of  $\mathbf{H}$  *per unit area* enclosed and the current *per unit area* enclosed, or current density. We performed a similar analysis in passing from the integral form of Gauss's law, involving flux through a closed surface and charge enclosed, to the point form, relating flux through a closed surface *per unit volume* enclosed and charge *per unit volume* enclosed, or volume charge density. In each case a limit is necessary to produce an equality.

If we choose closed paths which are oriented perpendicularly to each of the remaining two coordinate axes, analogous processes lead to expressions for the  $y$  and  $z$  components of the current density,

$$\lim_{\Delta y, \Delta z \rightarrow 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x \quad (19)$$

and

$$\lim_{\Delta z, \Delta x \rightarrow 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta z \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y \quad (20)$$

Comparing (18), (19), and (20), we see that a component of the current density is given by the limit of the quotient of the closed line integral of  $\mathbf{H}$  about a small path in a plane normal to that component and of the area enclosed as the path shrinks to zero. This limit has its counterpart in other fields of science and long ago received the name of *curl*. The curl of any vector is a vector, and any component of the curl is given by the limit of the quotient of the closed line integral of the vector about a small path in a plane normal to that component desired and the area enclosed, as the path shrinks to zero. It should be noted that

the above definition of curl does not refer specifically to a particular coordinate system. The mathematical form of the definition is

$$(\text{curl } \mathbf{H})_N = \lim_{\Delta S_N \rightarrow 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta S_N} \quad (21)$$

where  $\Delta S_N$  is the planar area enclosed by the closed line integral. The  $N$  subscript indicates that the component of the curl is that component which is *normal* to the surface enclosed by the closed path. It may represent any component in any coordinate system.

In cartesian coordinates the definition (21) shows that the  $x$ ,  $y$ , and  $z$  components of the curl  $\mathbf{H}$  are given by (18), (19), and (20), and therefore

$$\text{curl } \mathbf{H} = \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \mathbf{a}_x + \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \mathbf{a}_y + \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \mathbf{a}_z \quad (22)$$

This result may be written in the form of a determinant,

$$\text{curl } \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} \quad (23)$$

and may also be written in terms of the vector operator,

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} \quad (24)$$

Equation (22) is the result of applying the definition (21) to the cartesian coordinate system. We obtained the  $z$  component of this expression by evaluating Ampère's circuital law about an incremental path of sides  $\Delta x$  and  $\Delta y$ , and we could have obtained the other two components just as easily by choosing the appropriate paths. Equation (23) is a neat method of storing the cartesian coordinate expression for curl; the form is symmetrical and easily remembered. Equation (24) is even more concise and leads to (22) upon applying the definitions of the cross product and vector operator.

The expressions for curl  $\mathbf{H}$  in cylindrical and spherical coordinates are derived in Appendix A by applying the definition (21). Although they may be written in determinant form, as explained there, the determinants do not have one row of unit vectors on top and one row of components on the bottom, and they are not easily memorized. For this reason, the curl expansions in cylindrical

and spherical coordinates which appear below and inside the back cover are usually referred to whenever necessary.

$$\begin{aligned}\nabla \times \mathbf{H} = & \left( \frac{1}{\rho} \frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} \right) \mathbf{a}_\rho + \left( \frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} \right) \mathbf{a}_\phi \\ & + \left( \frac{1}{\rho} \frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{1}{\rho} \frac{\partial H_\rho}{\partial \phi} \right) \mathbf{a}_z \quad (\text{cylindrical})\end{aligned}\quad (25)$$

$$\begin{aligned}\nabla \times \mathbf{H} = & \frac{1}{r \sin \theta} \left( \frac{\partial(H_\phi \sin \theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi} \right) \mathbf{a}_r + \frac{1}{r} \left( \frac{1}{\sin \theta} \frac{\partial H_r}{\partial \phi} - \frac{\partial(r H_\phi)}{\partial r} \right) \mathbf{a}_\theta \\ & + \frac{1}{r} \left( \frac{\partial(r H_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta} \right) \mathbf{a}_\phi \quad (\text{spherical})\end{aligned}\quad (26)$$

Although we have described curl as a line integral per unit area, this does not provide everyone with a satisfactory physical picture of the nature of the curl operation, for the closed line integral itself requires physical interpretation. This integral was first met in the electrostatic field, where we saw that  $\oint \mathbf{E} \cdot d\mathbf{L} = 0$ . Inasmuch as the integral was zero, we did not belabor the physical picture. More recently we have discussed the closed line integral of  $\mathbf{H}$ ,  $\oint \mathbf{H} \cdot d\mathbf{L} = I$ . Either of these closed line integrals is also known by the name of “circulation,” a term obviously borrowed from the field of fluid dynamics.

The circulation of  $\mathbf{H}$ , or  $\oint \mathbf{H} \cdot d\mathbf{L}$ , is obtained by multiplying the component of  $\mathbf{H}$  parallel to the specified closed path at each point along it by the differential path length and summing the results as the differential lengths approach zero and as their number becomes infinite. We do not require a vanishingly small path. Ampère’s circuital law tells us that if  $\mathbf{H}$  does possess circulation about a given path, then current passes through this path. In electrostatics we see that the circulation of  $\mathbf{E}$  is zero about every path, a direct consequence of the fact that zero work is required to carry a charge around a closed path.

We may now describe curl as *circulation per unit area*. The closed path is vanishingly small, and curl is defined at a point. The curl of  $\mathbf{E}$  must be zero, for the circulation is zero. The curl of  $\mathbf{H}$  is not zero, however; the circulation of  $\mathbf{H}$  per unit area is the current density by Ampère’s circuital law [or (18), (19), and (20)].

Skilling<sup>5</sup> suggests the use of a very small paddle wheel as a “curl meter.” Our vector quantity, then, must be thought of as capable of applying a force to each blade of the paddle wheel, the force being proportional to the component of the field normal to the surface of that blade. To test a field for curl we dip our paddle wheel into the field, with the axis of the paddle wheel lined up with the

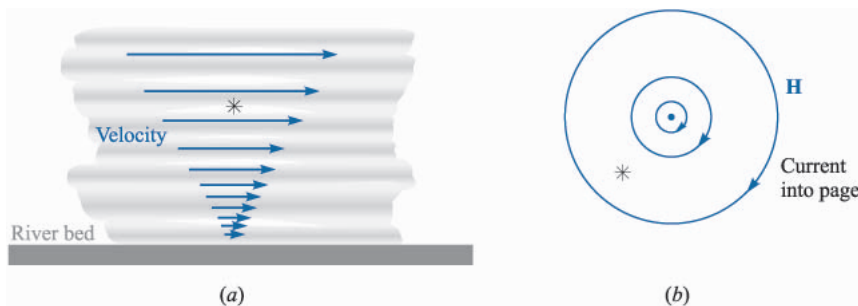
<sup>5</sup> See the Suggested References at the end of the chapter.

direction of the component of curl desired, and note the action of the field on the paddle. No rotation means no curl; larger angular velocities mean greater values of the curl; a reversal in the direction of spin means a reversal in the sign of the curl. To find the direction of the vector curl and not merely to establish the presence of any particular component, we should place our paddle wheel in the field and hunt around for the orientation which produces the greatest torque. The direction of the curl is then along the axis of the paddle wheel, as given by the right-hand rule.

As an example, consider the flow of water in a river. Fig. 8.14*a* shows the longitudinal section of a wide river taken at the middle of the river. The water velocity is zero at the bottom and increases linearly as the surface is approached. A paddle wheel placed in the position shown, with its axis perpendicular to the paper, will turn in a clockwise direction, showing the presence of a component of curl in the direction of an inward normal to the surface of the page. If the velocity of water does not change as we go up- or downstream and also shows no variation as we go across the river (or even if it decreases in the same fashion toward either bank), then this component is the only component present at the center of the stream, and the curl of the water velocity has a direction into the page.

In Fig. 8.14*b* the streamlines of the magnetic field intensity about an infinitely long filamentary conductor are shown. The curl meter placed in this field of curved lines shows that a larger number of blades have a clockwise force exerted on them but that this force is in general smaller than the counterclockwise force exerted on the smaller number of blades closer to the wire. It seems possible that if the curvature of the streamlines is correct and also if the variation of the field strength is just right, the net torque on the paddle wheel may be zero. Actually, the paddle wheel does not rotate in this case, for since  $\mathbf{H} = (I/2\pi\rho)\mathbf{a}_\phi$ , we may substitute into (25) obtaining

$$\text{curl } \mathbf{H} = -\frac{\partial H_\phi}{\partial z}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho}\mathbf{a}_z = 0$$



**FIGURE 8.14**

(*a*) The curl meter shows a component of the curl of the water velocity into the page. (*b*) The curl of the magnetic field intensity about an infinitely long filament is shown.

### Example 8.2

As an example of the evaluation of  $\text{curl } \mathbf{H}$  from the definition and of the evaluation of another line integral, let us suppose that  $\mathbf{H} = 0.2z^2\mathbf{a}_x$  for  $z > 0$ , and  $\mathbf{H} = 0$  elsewhere, as shown in Fig. 8.15. Calculate  $\oint \mathbf{H} \cdot d\mathbf{L}$  about a square path with side  $d$ , centered at  $(0, 0, z_1)$  in the  $y = 0$  plane where  $z_1 > 2d$ .

**Solution.** We evaluate the line integral of  $\mathbf{H}$  along the four segments, beginning at the top:

$$\begin{aligned}\oint \mathbf{H} \cdot d\mathbf{L} &= 0.2(z_1 + \tfrac{1}{2}d)^2 d + 0 - 0.2(z_1 - \tfrac{1}{2}d)^2 d + 0 \\ &= 0.4z_1 d^2\end{aligned}$$

In the limit as the area approaches zero, we find

$$(\nabla \times \mathbf{H})_y = \lim_{d \rightarrow 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{d^2} = \lim_{d \rightarrow 0} \frac{0.4z_1 d^2}{d^2} = 0.4z_1$$

The other components are zero, so  $\nabla \times \mathbf{H} = 0.4z_1\mathbf{a}_y$ .

To evaluate the curl without trying to illustrate the definition or the evaluation of a line integral, we simply take the partial derivative indicated by (23):

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0.2z^2 & 0 & 0 \end{vmatrix} = \frac{\partial}{\partial z}(0.2z^2)\mathbf{a}_y = 0.4z\mathbf{a}_y$$

which checks with the result above when  $z = z_1$ .

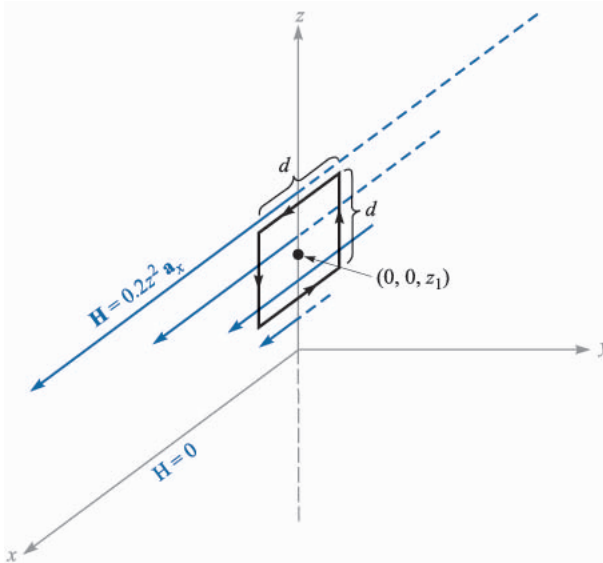


FIGURE 8.15

A square path of side  $d$  with its center on the  $z$  axis at  $z = z_1$  is used to evaluate  $\oint \mathbf{H} \cdot d\mathbf{L}$  and find  $\text{curl } \mathbf{H}$ .

Returning now to complete our original examination of the application of Ampère's circuital law to a differential-sized path, we may combine (18), (19), (20), (22), and (24),

$$\begin{aligned}\text{curl } \mathbf{H} = \nabla \times \mathbf{H} = & \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \mathbf{a}_x + \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \mathbf{a}_y \\ & + \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \mathbf{a}_z = \mathbf{J}\end{aligned}\quad (27)$$

and write the *point form of Ampère's circuital law*,

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}} \quad (28)$$

This is the second of Maxwell's four equations as they apply to non-time-varying conditions. We may also write the third of these equations at this time; it is the point form of  $\oint \mathbf{E} \cdot d\mathbf{L} = 0$ , or

$$\boxed{\nabla \times \mathbf{E} = 0} \quad (29)$$

The fourth equation appears in Sec. 8.5.

- ✓ **D8.4.** (a) Evaluate the closed line integral of  $\mathbf{H}$  about the rectangular path  $P_1(2, 3, 4)$  to  $P_2(4, 3, 4)$  to  $P_3(4, 3, 1)$  to  $P_4(2, 3, 1)$  to  $P_1$ , given  $\mathbf{H} = 3z\mathbf{a}_x - 2x^3\mathbf{a}_z$  A/m. (b) Determine the quotient of the closed line integral and the area enclosed by the path as an approximation to  $(\nabla \times \mathbf{H})_y$ . (c) Determine  $(\nabla \times \mathbf{H})_y$  at the center of the area.

**Ans.** 354 A; 59 A/m<sup>2</sup>; 57 A/m<sup>2</sup>

- ✓ **D8.5.** Calculate the value of the vector current density: (a) in cartesian coordinates at  $P_A(2, 3, 4)$  if  $\mathbf{H} = x^2z\mathbf{a}_y - y^2x\mathbf{a}_z$ ; (b) in cylindrical coordinates at  $P_B(1.5, 90^\circ, 0.5)$  if  $\mathbf{H} = \frac{2}{\rho}(\cos 0.2\phi)\mathbf{a}_\rho$ ; (c) in spherical coordinates at  $P_C(2, 30^\circ, 20^\circ)$  if  $\mathbf{H} = \frac{1}{\sin \theta}\mathbf{a}_\theta$ .

**Ans.**  $-16\mathbf{a}_x + 9\mathbf{a}_y + 16\mathbf{a}_z$  A/m<sup>2</sup>;  $0.0549\mathbf{a}_z$  A/m<sup>2</sup>;  $\mathbf{a}_\phi$  A/m<sup>2</sup>

## 8.4 STOKES' THEOREM

Although the last section was devoted primarily to a discussion of the curl operation, the contribution to the subject of magnetic fields should not be overlooked. From Ampère's circuital law we derived one of Maxwell's equations,  $\nabla \times \mathbf{H} = \mathbf{J}$ . This latter equation should be considered the point form of Ampère's circuital law and applies on a "per-unit-area" basis. In this section we shall again devote a major share of the material to the mathematical theorem known as Stokes' theorem, but in the process we shall show that we may obtain Ampère's circuital law from  $\nabla \times \mathbf{H} = \mathbf{J}$ . In other words, we are then prepared to

obtain the integral form from the point form or to obtain the point form from the integral form.

Consider the surface  $S$  of Fig. 8.16 which is broken up into incremental surfaces of area  $\Delta S$ . If we apply the definition of the curl to one of these incremental surfaces, then

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S}}{\Delta S} \doteq (\nabla \times \mathbf{H})_N$$

where the  $N$  subscript again indicates the right-hand normal to the surface. The subscript on  $d\mathbf{L}_{\Delta S}$  indicates that the closed path is the perimeter of an incremental area  $\Delta S$ . This result may also be written

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S}}{\Delta S} \doteq (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N$$

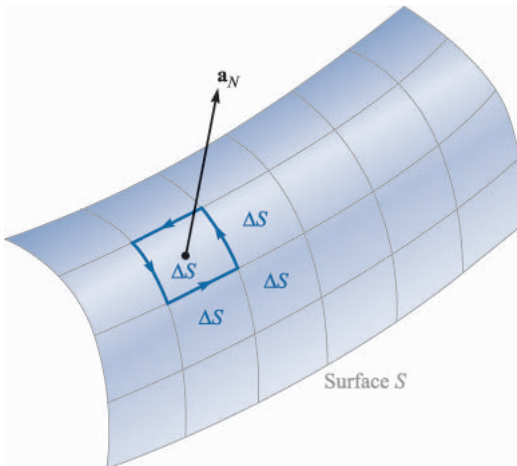
or

$$\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S} \doteq (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N \Delta S = (\nabla \times \mathbf{H}) \cdot \Delta \mathbf{S}$$

where  $\mathbf{a}_N$  is a unit vector in the direction of the right-hand normal to  $\Delta S$ .

Now let us determine this circulation for every  $\Delta S$  comprising  $S$  and sum the results. As we evaluate the closed line integral for each  $\Delta S$ , some cancellation will occur because every *interior* wall is covered once in each direction. The only boundaries on which cancellation cannot occur form the outside boundary, the path enclosing  $S$ . Therefore we have

$$\oint \mathbf{H} \cdot d\mathbf{L} \equiv \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} \quad (30)$$



**FIGURE 8.16**

The sum of the closed line integrals about the perimeter of every  $\Delta S$  is the same as the closed line integral about the perimeter of  $S$  because of cancellation on every interior path.



where  $d\mathbf{L}$  is taken only on the perimeter of  $S$ .

Equation (30) is an identity, holding for any vector field, and is known as *Stokes' theorem*.

### Example 8.3

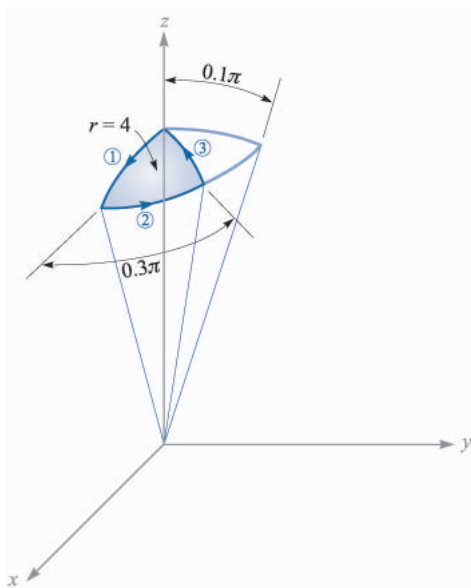
A numerical example may help to illustrate the geometry involved in Stokes' theorem. Consider the portion of a sphere shown in Fig. 8.17. The surface is specified by  $r = 4$ ,  $0 \leq \theta \leq 0.1\pi$ ,  $0 \leq \phi \leq 0.3\pi$ , and the closed path forming its perimeter is composed of three circular arcs. We are given the field  $\mathbf{H} = 6r \sin \phi \mathbf{a}_r + 18r \sin \theta \cos \phi \mathbf{a}_\phi$  and are asked to evaluate each side of Stokes' theorem.

**Solution.** The first path segment is described in spherical coordinates by  $r = 4$ ,  $0 \leq \theta \leq 0.1\pi$ ,  $\phi = 0$ ; the second one by  $r = 4$ ,  $\theta = 0.1\pi$ ,  $0 \leq \phi \leq 0.3\pi$ ; and the third by  $r = 4$ ,  $0 \leq \theta \leq 0.1\pi$ ,  $\phi = 0.3\pi$ . The differential path element  $d\mathbf{L}$  is the vector sum of the three differential lengths of the spherical coordinate system first discussed in Sec. 1.9,

$$d\mathbf{L} = dr \mathbf{a}_r + r d\theta \mathbf{a}_\theta + r \sin \theta d\phi \mathbf{a}_\phi$$

The first term is zero on all three segments of the path since  $r = 4$  and  $dr = 0$ , the second is zero on segment 2 since  $\theta$  is constant, and the third term is zero on both segments 1 and 3. Thus

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_1 H_\theta r d\theta + \int_2 H_\phi r \sin \theta d\phi + \int_3 H_\theta r d\theta$$



**FIGURE 8.17**

A portion of a spherical cap is used as a surface and a closed path to illustrate Stokes' theorem.

Since  $H_\theta = 0$ , we have only the second integral to evaluate,

$$\begin{aligned}\oint \mathbf{H} \cdot d\mathbf{L} &= \int_0^{0.3\pi} [18(4) \sin 0.1\pi \cos \phi] 4 \sin 0.1\pi d\phi \\ &= 288 \sin^2 0.1\pi \sin 0.3\pi = 22.2 \text{ A}\end{aligned}$$

We next attack the surface integral. First, we use (26) to find

$$\nabla \times \mathbf{H} = \frac{1}{r \sin \theta} (36r \sin \theta \cos \theta \cos \phi) \mathbf{a}_r + \frac{1}{r} \left( \frac{1}{\sin \theta} 6r \cos \phi - 36r \sin \theta \cos \phi \right) \mathbf{a}_\theta$$

Since  $d\mathbf{S} = r^2 \sin \theta d\theta d\phi \mathbf{a}_r$ , the integral is

$$\begin{aligned}\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} &= \int_0^{0.3\pi} \int_0^{0.1\pi} (36 \cos \theta \cos \phi) 16 \sin \theta d\theta d\phi \\ &= \int_0^{0.3\pi} 576 \left( \frac{1}{2} \sin^2 \theta \right) \Big|_0^{0.1\pi} \cos \phi d\phi \\ &= 288 \sin^2 0.1\pi \sin 0.3\pi = 22.2 \text{ A}\end{aligned}$$

Thus, the results check Stokes' theorem, and we note in passing that a current of 22.2 A is flowing upward through this section of a spherical cap.

Next, let us see how easy it is to obtain Ampère's circuital law from  $\nabla \times \mathbf{H} = \mathbf{J}$ . We merely have to dot each side by  $d\mathbf{S}$ , integrate each side over the same (open) surface  $S$ , and apply Stokes' theorem:

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} = \oint \mathbf{H} \cdot d\mathbf{L}$$

The integral of the current density over the surface  $S$  is the total current  $I$  passing through the surface, and therefore

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

This short derivation shows clearly that the current  $I$ , described as being "enclosed by the closed path," is also the current passing through any of the infinite number of surfaces which have the closed path as a perimeter.

Stokes' theorem relates a surface integral to a closed line integral. It should be recalled that the divergence theorem relates a volume integral to a closed surface integral. Both theorems find their greatest use in general vector proofs. As an example, let us find another expression for  $\nabla \cdot \nabla \times \mathbf{A}$ , where  $\mathbf{A}$  represents any vector field. The result must be a scalar (why?), and we may let this scalar be  $T$ , or

$$\nabla \cdot \nabla \times \mathbf{A} = T$$

Multiplying by  $dv$  and integrating throughout any volume  $v$ ,

$$\int_{\text{vol}} (\nabla \cdot \nabla \times \mathbf{A}) dv = \int_{\text{vol}} T dv$$

we first apply the divergence theorem to the left side, obtaining

$$\oint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_{\text{vol}} T dv$$

The left side is the surface integral of the curl of  $\mathbf{A}$  over the *closed* surface surrounding the volume  $v$ . Stokes' theorem relates the surface integral of the curl of  $\mathbf{A}$  over the *open* surface enclosed by a given closed path. If we think of the path as the opening of a laundry bag and the open surface as the surface of the bag itself, we see that as we gradually approach a closed surface by pulling on the drawstrings, the closed path becomes smaller and smaller and finally disappears as the surface becomes closed. Hence the application of Stokes' theorem to a *closed* surface produces a zero result, and we have

$$\int_{\text{vol}} T dv = 0$$

Since this is true for any volume, it is true for the differential volume  $dv$ ,

$$T dv = 0$$

and therefore

$$T = 0$$

or

$$\boxed{\nabla \cdot \nabla \times \mathbf{A} \equiv 0} \quad (31)$$

Equation (31) is a useful identity of vector calculus.<sup>6</sup> Of course, it may also be proven easily by direct expansion in cartesian coordinates.

Let us apply the identity to the non-time-varying magnetic field for which

$$\nabla \times \mathbf{H} = \mathbf{J}$$

This shows quickly that

$$\mathbf{V} \cdot \mathbf{J} = 0$$

which is the same result we obtained earlier in the chapter by using the continuity equation.

Before introducing several new magnetic field quantities in the following section, we may review our accomplishments at this point. We initially accepted the Biot-Savart law as an experimental result,

$$\mathbf{H} = \oint \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

---

<sup>6</sup> This and other vector identities are tabulated in Appendix A.3.

and tentatively accepted Ampère's circuital law, subject to later proof,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

From Ampère's circuital law the definition of curl led to the point form of this same law,

$$\nabla \times \mathbf{H} = \mathbf{J}$$

We now see that Stokes' theorem enables us to obtain the integral form of Ampère's circuital law from the point of form.

- ✓ **D8.6.** Evaluate both sides of Stokes' theorem for the field  $\mathbf{H} = 6xy\mathbf{a}_x - 3y^2\mathbf{a}_y$  A/m and the rectangular path around the region,  $2 \leq x \leq 5$ ,  $-1 \leq y \leq 1$ ,  $z = 0$ . Let the positive direction of  $d\mathbf{S}$  be  $\mathbf{a}_z$ .

**Ans.**  $-126$  A;  $-126$  A

## 8.5 MAGNETIC FLUX AND MAGNETIC FLUX DENSITY

In free space, let us define the *magnetic flux density*  $\mathbf{B}$  as

$$\mathbf{B} = \mu_0 \mathbf{H} \quad (\text{free space only}) \quad (32)$$

where  $\mathbf{B}$  is measured in webers per square meter ( $\text{Wb/m}^2$ ) or in a newer unit adopted in the International System of Units, tesla (T). An older unit that is often used for magnetic flux density is the gauss (G), where 1 T or  $1 \text{ Wb/m}^2$  is the same as 10 000 G. The constant  $\mu_0$  is not dimensionless and has the *defined value* for free space, in henrys per meter (H/m), of

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m} \quad (33)$$

The name given to  $\mu_0$  is the *permeability* of free space.

We should note that since  $\mathbf{H}$  is measured in amperes per meter, the weber is dimensionally equal to the product of henrys and amperes. Considering the henry as a new unit, the weber is merely a convenient abbreviation for the product of henrys and amperes. When time-varying fields are introduced, it will be shown that a weber is also equivalent to the product of volts and seconds.

The magnetic-flux-density vector  $\mathbf{B}$ , as the name weber per square meter implies, is a member of the flux-density family of vector fields. One of the possible analogies between electric and magnetic fields<sup>7</sup> compares the laws of Biot-Savart and Coulomb, thus establishing an analogy between  $\mathbf{H}$  and  $\mathbf{E}$ . The

<sup>7</sup> An alternate analogy is presented in Sec. 10.2.

relations  $\mathbf{B} = \mu_0 \mathbf{H}$  and  $\mathbf{D} = \epsilon_0 \mathbf{E}$  then lead to an analogy between  $\mathbf{B}$  and  $\mathbf{D}$ . If  $\mathbf{B}$  is measured in teslas or webers per square meter, then magnetic flux should be measured in webers. Let us represent magnetic flux by  $\Phi$  and define  $\Phi$  as the flux passing through any designated area,

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} \text{ Wb} \quad (34)$$

Our analogy should now remind us of the electric flux  $\Psi$ , measured in coulombs, and of Gauss's law, which states that the total flux passing through any closed surface is equal to the charge enclosed,

$$\Psi = \oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

The charge  $Q$  is the source of the lines of electric flux and these lines begin and terminate on positive and negative charge, respectively.

No such source has ever been discovered for the lines of magnetic flux. In the example of the infinitely long straight filament carrying a direct current  $I$ , the  $\mathbf{H}$  field formed concentric circles about the filament. Since  $\mathbf{B} = \mu_0 \mathbf{H}$ , the  $\mathbf{B}$  field is of the same form. The magnetic flux lines are closed and do not terminate on a "magnetic charge." For this reason Gauss's law for the magnetic field is

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \quad (35)$$

and application of the divergence theorem shows us that

$$\nabla \cdot \mathbf{B} = 0 \quad (36)$$

We have not proved (35) or (36) but have only suggested the truth of these statements by considering the single field of the infinite filament. It is possible to show that (35) or (36) follows from the Biot-Savart law and the definition of  $\mathbf{B}$ ,  $\mathbf{B} = \mu_0 \mathbf{H}$ , but this is another proof which we shall postpone to Sec. 8.7.

Equation (36) is the last of Maxwell's four equations as they apply to static electric fields and steady magnetic fields. Collecting these equations, we then have for static electric fields and steady magnetic fields

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_v \\ \nabla \times \mathbf{E} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned} \quad (37)$$

To these equations we may add the two expressions relating  $\mathbf{D}$  to  $\mathbf{E}$  and  $\mathbf{B}$  to  $\mathbf{H}$  in free space,

$$\mathbf{D} = \epsilon_0 \mathbf{E} \quad (38)$$

$$\mathbf{B} = \mu_0 \mathbf{H} \quad (39)$$

We have also found it helpful to define an electrostatic potential,

$$\mathbf{E} = -\nabla V \quad (40)$$

and we shall discuss a potential for the steady magnetic field in the following section. In addition, we have extended our coverage of electric fields to include conducting materials and dielectrics, and we have introduced the polarization  $\mathbf{P}$ . A similar treatment will be applied to magnetic fields in the next chapter.

Returning to (37), it may be noted that these four equations specify the divergence and curl of an electric and a magnetic field. The corresponding set of four integral equations that apply to static electric fields and steady magnetic fields is

$$\begin{aligned} \oint_S \mathbf{D} \cdot d\mathbf{S} &= Q = \int_{\text{vol}} \rho_v dv \\ \oint \mathbf{E} \cdot d\mathbf{L} &= 0 \\ \oint \mathbf{H} \cdot d\mathbf{L} &= I = \int_S \mathbf{J} \cdot d\mathbf{S} \\ \oint_S \mathbf{B} \cdot d\mathbf{S} &= 0 \end{aligned} \quad (41)$$

Our study of electric and magnetic fields would have been much simpler if we could have begun with either set of equations, (37) or (41). With a good knowledge of vector analysis, such as we should now have, either set may be readily obtained from the other by applying the divergence theorem of Stokes' theorem. The various experimental laws can be obtained easily from these equations.

As an example of the use of flux and flux density in magnetic fields, let us find the flux between the conductors of the coaxial line of Fig. 8.8a. The magnetic field intensity was found to be

$$H_\phi = \frac{I}{2\pi\rho} \quad (a < \rho < b)$$

and therefore

$$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi\rho} \mathbf{a}_\phi$$

The magnetic flux contained between the conductors in a length  $d$  is the flux crossing any radial plane extending from  $\rho = a$  to  $\rho = b$  and from, say,  $z = 0$  to  $z = d$

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho} \mathbf{a}_\phi \cdot d\rho dz \mathbf{a}_\phi$$

or

$$\Phi = \frac{\mu_0 I d}{2\pi} \ln \frac{b}{a} \quad (42)$$

This expression will be used later to obtain the inductance of the coaxial transmission line.

- ✓ **D8.7.** A solid conductor of circular cross section is made of a homogeneous nonmagnetic material. If the radius  $a = 1$  mm, the conductor axis lies on the  $z$  axis, and the total current in the  $\mathbf{a}_z$  direction is 20 A, find: (a)  $H_\phi$  at  $\rho = 0.5$  mm; (b)  $B_\phi$  at  $\rho = 0.8$  mm; (c) the total magnetic flux per unit length inside the conductor; (d) the total flux for  $\rho < 0.5$  mm; (e) the total magnetic flux outside the conductor.

*Ans.* 1592 A/m; 3.2 mT;  $2\mu\text{Wb}$ ;  $0.5\mu\text{Wb}$ ;  $\infty$

## 8.6 THE SCALAR AND VECTOR MAGNETIC POTENTIALS

The solution of electrostatic field problems is greatly simplified by the use of the scalar electrostatic potential  $V$ . Although this potential possesses a very real physical significance for us, it is mathematically no more than a stepping-stone which allows us to solve a problem by several smaller steps. Given a charge configuration, we may first find the potential and then from it the electric field intensity.

We should question whether or not such assistance is available in magnetic fields. Can we define a potential function which may be found from the current distribution and from which the magnetic fields may be easily determined? Can a scalar magnetic potential be defined, similar to the scalar electrostatic potential? We shall show in the next few pages that the answer to the first question is “yes,” but the second must be answered “sometimes.” Let us attack the last question first by assuming the existence of a scalar magnetic potential, which we designate  $V_m$ , whose negative gradient gives the magnetic field intensity

$$\mathbf{H} = -\nabla V_m$$

The selection of the negative gradient will provide us with a closer analogy to the electric potential and to problems which we have already solved.

This definition must not conflict with our previous results for the magnetic field, and therefore

$$\nabla \times \mathbf{H} = \mathbf{J} = \nabla \times (-\nabla V_m)$$

However, the curl of the gradient of any scalar is identically zero, a vector identity the proof of which is left for a leisure moment. Therefore we see that if  $\mathbf{H}$  is to be defined as the gradient of a scalar magnetic potential, then current density must be zero throughout the region in which the scalar magnetic potential is so defined. We then have

$$\boxed{\mathbf{H} = -\nabla V_m \quad (\mathbf{J} = 0)} \quad (43)$$

Since many magnetic problems involve geometries in which the current-carrying conductors occupy a relatively small fraction of the total region of interest, it is evident that a scalar magnetic potential can be useful. The scalar magnetic potential is also applicable in the case of permanent magnets. The dimensions of  $V_m$  are obviously amperes.

This scalar potential also satisfies Laplace's equation. In free space,

$$\nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot \mathbf{H} = 0$$

and hence

$$\mu_0 \nabla \cdot (-\nabla V_m) = 0$$

or

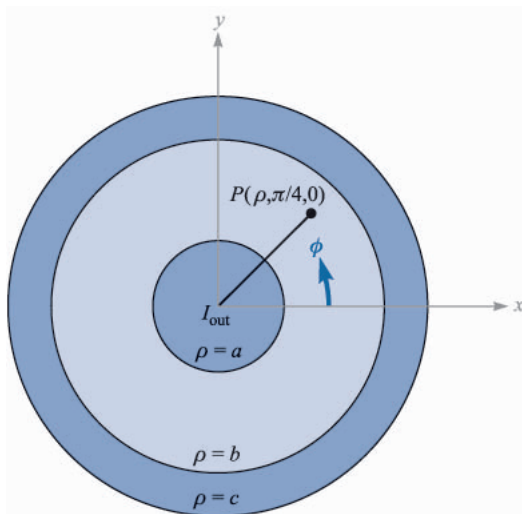
$$\boxed{\nabla^2 V_m = 0 \quad (\mathbf{J} = 0)} \quad (44)$$

We shall see later that  $V_m$  continues to satisfy Laplace's equation in homogeneous magnetic materials; it is not defined in any region in which current density is present.

Although we shall consider the scalar magnetic potential to a much greater extent in the next chapter, when we introduce magnetic materials and discuss the magnetic circuit, one difference between  $V$  and  $V_m$  should be pointed out now:  $V_m$  is not a single-valued function of position. The electric potential  $V$  is single-valued; once a zero reference is assigned, there is only one value of  $V$  associated with each point in space. Such is not the case with  $V_m$ . Consider the cross section of the coaxial line shown in Fig. 8.18. In the region  $a < \rho < b$ ,  $\mathbf{J} = 0$ , and we may establish a scalar magnetic potential. The value of  $\mathbf{H}$  is

$$\mathbf{H} = \frac{I}{2\pi\rho} \mathbf{a}_\phi$$



**FIGURE 8.18**

The scalar magnetic potential  $V_m$  is a multi-valued function of  $\phi$  in the region  $a < \rho < b$ . The electrostatic potential is always single-valued.

where  $I$  is the total current flowing in the  $\mathbf{a}_z$  direction in the inner conductor. Let us find  $V_m$  by integrating the appropriate component of the gradient. Applying (43),

$$\frac{I}{2\pi\rho} = -\nabla V_m \Big|_{\phi} = -\frac{1}{\rho} \frac{\partial V_m}{\partial \phi}$$

or

$$\frac{\partial V_m}{\partial \phi} = -\frac{I}{2\pi}$$

Thus

$$V_m = -\frac{I}{2\pi}\phi$$

where the constant of integration has been set equal to zero. What value of potential do we associate with point  $P$ , where  $\phi = \pi/4$ ? If we let  $V_m$  be zero at  $\phi = 0$  and proceed counterclockwise around the circle, the magnetic potential goes negative linearly. When we have made one circuit, the potential is  $-I$ , but that was the point at which we said the potential was zero a moment ago. At  $P$ , then,  $\phi = \pi/4, 9\pi/4, 17\pi/4, \dots$ , or  $-7\pi/4, -15\pi/4, -23\pi/4, \dots$ , or

$$V_{mP} = \frac{I}{2\pi}(2n - \tfrac{1}{4})\pi \quad (n = 0, \pm 1, \pm 2, \dots)$$

or

$$V_{mP} = I(n - \tfrac{1}{8}) \quad (n = 0, \pm 1, \pm 2, \dots)$$

The reason for this multivaluedness may be shown by a comparison with the electrostatic case. There, we know that

$$\nabla \times \mathbf{E} = 0$$

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

and therefore the line integral

$$V_{ab} = - \int_b^a \mathbf{E} \cdot d\mathbf{L}$$

is independent of the path. In the magnetostatic case, however,

$$\nabla \times \mathbf{H} = 0 \quad (\text{wherever } \mathbf{J} = 0)$$

but

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

even if  $\mathbf{J}$  is zero along the path of integration. Every time we make another complete lap around the current, the result of the integration increases by  $I$ . If no current  $I$  is enclosed by the path, then a single-valued potential function may be defined. In general, however,

$$V_{m,ab} = - \int_b^a \mathbf{H} \cdot d\mathbf{L} \quad (\text{specified path}) \quad (45)$$

where a specific path or type of path must be selected. We should remember that the electrostatic potential  $V$  is a conservative field; the magnetic scalar potential  $V_m$  is not a conservative field. In our coaxial problem let us erect a barrier<sup>8</sup> at  $\phi = \pi$ ; we agree not to select a path which crosses this plane. Therefore we cannot encircle  $I$ , and a single-valued potential is possible. The result is seen to be

$$V_m = -\frac{I}{2\pi} \phi \quad (-\pi < \phi < \pi)$$

and

$$V_{mP} = -\frac{I}{8} \quad \left( \phi = \frac{\pi}{4} \right)$$

The scalar magnetic potential is evidently the quantity whose equipotential surfaces will form curvilinear squares with the streamlines of  $\mathbf{H}$  in Fig. 8.4. This is one more facet of the analogy between electric and magnetic fields about which we will have more to say in the next chapter.

<sup>8</sup> This corresponds to the more precise mathematical term “branch cut.”

Let us temporarily leave the scalar magnetic potential now and investigate a vector magnetic potential. This vector field is one which is extremely useful in studying radiation from antennas, from apertures, and radiation leakage from transmission lines, waveguides, and microwave ovens. The vector magnetic potential may be used in regions where the current density is zero or nonzero, and we shall also be able to extend it to the time-varying case later.

Our choice of a vector magnetic potential is indicated by noting that

$$\nabla \cdot \mathbf{B} = 0$$

Next, a vector identity which we proved in Sec. 8.4 shows that the divergence of the curl of any vector field is zero. Therefore we select

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (46)$$

where  $\mathbf{A}$  signifies a *vector magnetic potential*, and we automatically satisfy the condition that the magnetic flux density shall have zero divergence. The  $\mathbf{H}$  field is

$$\mathbf{H} = \frac{1}{\mu_0} \nabla \times \mathbf{A}$$

and

$$\nabla \times \mathbf{H} = \mathbf{J} = \frac{1}{\mu_0} \nabla \times \nabla \times \mathbf{A}$$

The curl of the curl of a vector field is not zero and is given by a fairly complicated expression,<sup>9</sup> which we need not know now in general form. In specific cases for which the form of  $\mathbf{A}$  is known, the curl operation may be applied twice to determine the current density.

Equation (46) serves as a useful definition of the *vector magnetic potential*  $\mathbf{A}$ . Since the curl operation implies differentiation with respect to a length, the units of  $\mathbf{A}$  are webers per meter.

As yet we have seen only that the definition for  $\mathbf{A}$  does not conflict with any previous results. It still remains to show that this particular definition can help us to determine magnetic fields more easily. We certainly cannot identify  $\mathbf{A}$  with any easily measured quantity or history-making experiment.

We shall show in the following section that, given the Biot-Savart law, the definition of  $\mathbf{B}$ , and the definition of  $\mathbf{A}$ , then  $\mathbf{A}$  may be determined from the differential current elements by

---

<sup>9</sup>  $\nabla \times \nabla \times \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ . In cartesian coordinates, it may be shown that  $\nabla^2 \mathbf{A} \equiv \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z$ . In other coordinate systems,  $\nabla^2 \mathbf{A}$  may be found by evaluating the second-order partial derivatives in  $\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$ .

$$\mathbf{A} = \oint \frac{\mu_0 I d\mathbf{L}}{4\pi R} \quad (47)$$

The significance of the terms in (47) is the same as in the Biot-Savart law; a direct current  $I$  flows along a filamentary conductor of which any differential length  $d\mathbf{L}$  is distant  $R$  from the point at which  $\mathbf{A}$  is to be found. Since we have defined  $\mathbf{A}$  only through specification of its curl, it is possible to add the gradient of any scalar field to (47) without changing  $\mathbf{B}$  or  $\mathbf{H}$ , for the curl of the gradient is identically zero. In steady magnetic fields, it is customary to set this possible added term equal to zero.

The fact that  $\mathbf{A}$  is a vector magnetic *potential* is more apparent when (47) is compared with the similar expression for the electrostatic potential,

$$V = \int \frac{\rho_L dL}{4\pi\epsilon_0 R}$$

Each expression is the integral along a line source, in one case line charge and in the other case line current; each integrand is inversely proportional to the distance from the source to the point of interest; and each involves a characteristic of the medium (here free space), the permeability or the permittivity.

Equation (47) may be written in differential form,

$$d\mathbf{A} = \frac{\mu_0 I d\mathbf{L}}{4\pi R} \quad (48)$$

if we again agree not to attribute any physical significance to any magnetic fields we obtain from (48) until the *entire closed path in which the current flows is considered*.

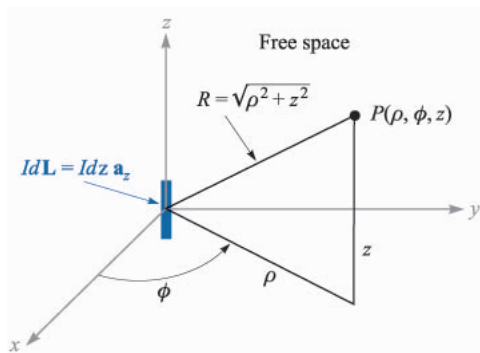
With this reservation, let us go right ahead and consider the vector magnetic potential field about a differential filament. We locate the filament at the origin in free space, as shown in Fig. 8.19, and allow it to extend in the positive  $z$  direction so that  $d\mathbf{L} = dz \mathbf{a}_z$ . We use cylindrical coordinates to find  $d\mathbf{A}$  at the point  $(\rho, \phi, z)$ :

$$d\mathbf{A} = \frac{\mu_0 I dz \mathbf{a}_z}{4\pi\sqrt{\rho^2 + z^2}}$$

or

$$d\mathbf{A}_z = \frac{\mu_0 I dz}{4\pi\sqrt{\rho^2 + z^2}} \quad dA_\phi = 0 \quad dA_\rho = 0 \quad (49)$$

We note that the direction of  $d\mathbf{A}$  is the same as that of  $I d\mathbf{L}$ . Each small section of a current-carrying conductor produces a contribution to the total vector magnetic potential which is in the same direction as the current flow in the conductor. The magnitude of the vector magnetic potential varies inversely as the distance to the current element, being strongest in the neighborhood of the

**FIGURE 8.19**

The differential current element  $I dz \mathbf{a}_z$  at the origin establishes the differential vector magnetic potential field,  $d\mathbf{A} = \frac{\mu_0 I dz \mathbf{a}_z}{4\pi \sqrt{\rho^2 + z^2}}$  at  $P(\rho, \phi, z)$ .

current and gradually falling off to zero at distant points. Skilling<sup>10</sup> describes the vector magnetic potential field as “like the current distribution but fuzzy around the edges, or like a picture of the current out of focus.”

In order to find the magnetic field intensity, we must take the curl of (49) in cylindrical coordinates, leading to

$$d\mathbf{H} = \frac{1}{\mu_0} \nabla \times d\mathbf{A} = \frac{1}{\mu_0} \left( -\frac{\partial dA_z}{\partial \rho} \right) \mathbf{a}_\phi$$

or

$$d\mathbf{H} = \frac{I dz}{4\pi} \frac{\rho}{(\rho^2 + z^2)^{3/2}} \mathbf{a}_\phi$$

which is easily shown to be the same as the value given by the Biot-Savart law.

Expressions for the vector magnetic potential  $\mathbf{A}$  can also be obtained for a current source which is distributed. For a current sheet  $\mathbf{K}$ , the differential current element becomes

$$I d\mathbf{L} = \mathbf{K} dS$$

In the case of current flow throughout a volume with a density  $\mathbf{J}$ , we have

$$I d\mathbf{L} = \mathbf{J} dv$$

In each of these two expressions the vector character is given to the current. For the filamentary element it is customary, although not necessary, to use  $I d\mathbf{L}$  instead of  $\mathbf{I} dL$ . Since the magnitude of the filamentary element is constant, we have chosen the form which allows us to remove one quantity from the integral. The alternative expressions for  $\mathbf{A}$  are then

<sup>10</sup> See the Suggested References at the end of the chapter.

$$\mathbf{A} = \int_S \frac{\mu_0 \mathbf{K} dS}{4\pi R} \quad (50)$$

and

$$\mathbf{A} = \int_{\text{vol}} \frac{\mu_0 \mathbf{J} dv}{4\pi R} \quad (51)$$

Equations (47), (50), and (51) express the vector magnetic potential as an integration over all of its sources. From a comparison of the form of these integrals with those which yield the electrostatic potential, it is evident that once again the zero reference for  $\mathbf{A}$  is at infinity, for no finite current element can produce any contribution as  $R \rightarrow \infty$ . We should remember that we very seldom used the similar expressions for  $V$ ; too often our theoretical problems included charge distributions which extended to infinity and the result would be an infinite potential everywhere. Actually, we calculated very few potential fields until the differential form of the potential equation was obtained,  $\nabla^2 V = -\rho_v/\epsilon$ , or better yet,  $\nabla^2 V = 0$ . We were then at liberty to select our own zero reference.

The analogous expressions for  $\mathbf{A}$  will be derived in the next section, and an example of the calculation of a vector magnetic potential field will be completed.

- ✓ **D8.8.** A current sheet,  $\mathbf{K} = 2.4\mathbf{a}_z$  A/m, is present at the surface  $\rho = 1.2$  in free space. (a) Find  $\mathbf{H}$  for  $\rho > 1.2$ . Find  $V_m$  at  $P(\rho = 1.5, \phi = 0.6\pi, z = 1)$  if: (b)  $V_m = 0$  at  $\phi = 0$  and there is a barrier at  $\phi = \pi$ ; (c)  $V_m = 0$  at  $\phi = 0$  and there is a barrier at  $\phi = \pi/2$ ; (d)  $V_m = 0$  at  $\phi = \pi$  and there is a barrier at  $\phi = 0$ ; (e)  $V_m = 5$  V at  $\phi = \pi$  and there is a barrier at  $\phi = 0.8\pi$ .

*Ans.*  $\frac{2.88}{\rho}\alpha_\phi$ ;  $-0.864$  V;  $2.02$  V;  $0.576$  V;  $2.70$  V

- ✓ **D8.9.** The value of  $\mathbf{A}$  within a solid nonmagnetic conductor of radius  $a$  carrying a total current  $I$  in the  $\mathbf{a}_z$  direction may be found easily. Using the known value of  $\mathbf{H}$  or  $\mathbf{B}$  for  $\rho < a$ , then (46) may be solved for  $\mathbf{A}$ . Select  $A = (\mu_0 I \ln 5)/2\pi$  at  $\rho = a$  (to correspond with an example in the next section) and find  $\mathbf{A}$  at  $\rho =$ : (a) 0; (b)  $0.25a$ ; (c)  $0.75a$ ; (d)  $a$ .

*Ans.*  $0.422 \mu$  Wb/m;  $0.416 \mu$  Wb/m;  $0.366 \mu$  Wb/m;  $0.322 \mu$  Wb/m

## 8.7 DERIVATION OF THE STEADY-MAGNETIC-FIELD LAWS

We shall now carry out our threat to supply the promised proofs of the several relationships between the magnetic field quantities. All these relationships may be obtained from the definitions of  $\mathbf{H}$ ,

$$\mathbf{H} = \oint \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} \quad (3)$$

of  $\mathbf{B}$  (in free space),

$$\mathbf{B} = \mu_0 \mathbf{H} \quad (32)$$

and of  $\mathbf{A}$ ,

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (46)$$

Let us first assume that we may express  $\mathbf{A}$  by the last equation of the preceding section,

$$\mathbf{A} = \int_{\text{vol}} \frac{\mu_0 \mathbf{J} dv}{4\pi R} \quad (51)$$

and then demonstrate the correctness of (51) by showing that (3) follows. First we should add subscripts to indicate the point at which the current element is located  $(x_1, y_1, z_1)$  and the point at which  $\mathbf{A}$  is given  $(x_2, y_2, z_2)$ . The differential volume element  $dv$  is then written  $dv_1$  and in cartesian coordinates would be  $dx_1 dy_1 dz_1$ . The variables of integration are  $x_1, y_1$ , and  $z_1$ . Using these subscripts, then,

$$\mathbf{A}_2 = \int_{\text{vol}} \frac{\mu_0 \mathbf{J}_1 dv_1}{4\pi R_{12}} \quad (52)$$

From (32) and (46) we have

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \frac{\nabla \times \mathbf{A}}{\mu_0} \quad (53)$$

To show that (3) follows from (52), it is necessary to substitute (52) into (53). This step involves taking the curl of  $\mathbf{A}_2$ , a quantity expressed in terms of the variables  $x_2, y_2$ , and  $z_2$ , and the curl therefore involves partial derivatives with respect to  $x_2, y_2$ , and  $z_2$ . We do this, placing a subscript on the del operator to remind us of the variables involved in the partial differentiation process,

$$\mathbf{H}_2 = \frac{\nabla_2 \times \mathbf{A}_2}{\mu_0} = \frac{1}{\mu_0} \nabla_2 \times \int_{\text{vol}} \frac{\mu_0 \mathbf{J}_1 dv_1}{4\pi R_{12}}$$

The order of partial differentiation and integration is immaterial, and  $\mu_0/4\pi$  is constant, allowing us to write

$$\mathbf{H}_2 = \frac{1}{4\pi} \int_{\text{vol}} \nabla_2 \times \frac{\mathbf{J}_1 dv_1}{R_{12}}$$

The curl operation within the integrand represents partial differentiation with respect to  $x_2, y_2$ , and  $z_2$ . The differential volume element  $dv_1$  is a scalar and a function only of  $x_1, y_1$ , and  $z_1$ . Consequently, it may be factored out of the curl operation as any other constant, leaving

$$\mathbf{H}_2 = \frac{1}{4\pi} \int_{\text{vol}} \left( \nabla_2 \times \frac{\mathbf{J}_1}{R_{12}} \right) dv_1 \quad (54)$$

The curl of the product of a scalar and a vector is given by an identity which may be checked by expansion in cartesian coordinates or gratefully accepted from Appendix A.3,

$$\nabla \times (S\mathbf{V}) \equiv (\nabla S) \times \mathbf{V} + S(\nabla \times \mathbf{V}) \quad (55)$$

This identity is used to expand the integrand of (54),

$$\mathbf{H}_2 = \frac{1}{4\pi} \int_{\text{vol}} \left[ \left( \nabla_2 \frac{1}{R_{12}} \right) \times \mathbf{J}_1 + \frac{1}{R_{12}} (\nabla_2 \times \mathbf{J}_1) \right] dv_1 \quad (56)$$

The second term of this integrand is zero, because  $\nabla \times \mathbf{J}_1$  indicates partial derivatives of a function of  $x_1, y_1$ , and  $z_1$ , taken with respect to the variables  $x_2, y_2$ , and  $z_2$ ; the first set of variables is not a function of the second set, and all partial derivatives are zero.

The first term of the integrand may be determined by expressing  $R_{12}$  in terms of the coordinate values,

$$R_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

and taking the gradient of its reciprocal. Problem 42 shows that the result is

$$\nabla_2 \frac{1}{R_{12}} = -\frac{R_{12}}{R_{12}^3} = -\frac{\mathbf{a}_{R12}}{R_{12}^2}$$

Substituting this result into (56), we have

$$\mathbf{H}_2 = -\frac{1}{4\pi} \int_{\text{vol}} \frac{\mathbf{a}_{R12} \times \mathbf{J}_1}{R_{12}^2} dv_1$$

or

$$\mathbf{H}_2 = \int_{\text{vol}} \frac{\mathbf{J}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2} dv_1$$

which is the equivalent of (3) in terms of current density. Replacing  $\mathbf{J}_1 dv_1$  by  $I_1 d\mathbf{L}_1$ , we may rewrite the volume integral as a closed line integral,

$$\mathbf{H}_2 = \oint \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}$$

Equation (51) is therefore correct and agrees with the three definitions (3), (32), and (46).

Next we shall continue with our mathematical orgy and prove Ampère's circuital law in point form,

$$\nabla \times \mathbf{H} = \mathbf{J} \quad (28)$$



Combining (28), (32), and (46), we obtain

$$\nabla \times \mathbf{H} = \nabla \times \frac{\mathbf{B}}{\mu_0} = \frac{1}{\mu_0} \nabla \times \nabla \times \mathbf{A} \quad (57)$$

We now need the expansion in cartesian coordinates for  $\nabla \times \nabla \times \mathbf{A}$ . Performing the indicated partial differentiations and collecting the resulting terms, we may write the result as

$$\nabla \times \nabla \times \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \quad (58)$$

where

$$\nabla^2 \mathbf{A} \equiv \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z \quad (59)$$

Equation (59) is the definition (in cartesian coordinates) of the *Laplacian of a vector*.

Substituting (58) into (57), we have

$$\nabla \times \mathbf{H} = \frac{1}{\mu_0} [\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}] \quad (60)$$

and now require expressions for the divergence and the Laplacian of  $\mathbf{A}$ .

We may find the divergence of  $\mathbf{A}$  by applying the divergence operation to (52),

$$\nabla_2 \cdot \mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_{\text{vol}} \nabla_2 \cdot \frac{\mathbf{J}_1}{R_{12}} dv_1 \quad (61)$$

and using the vector identity (44) of Sec. 4.8,

$$\nabla \cdot (S\mathbf{V}) \equiv \mathbf{V} \cdot (\nabla S) + S(\nabla \cdot \mathbf{V})$$

Thus,

$$\nabla_2 \cdot \mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_{\text{vol}} \left[ \mathbf{J}_1 \cdot \left( \nabla_2 \frac{1}{R_{12}} \right) + \frac{1}{R_{12}} (\nabla_2 \cdot \mathbf{J}_1) \right] dv_1 \quad (62)$$

The second part of the integrand is zero, because  $\mathbf{J}_1$  is not a function of  $x_2$ ,  $y_2$ , and  $z_2$ .

We have already used the result that  $\nabla_2(1/R_{12}) = -\mathbf{R}_{12}/R_{12}^3$ , and it is just as easily shown that

$$\nabla_1 \frac{1}{R_{12}} = \frac{\mathbf{R}_{12}}{R_{12}^3}$$

or that

$$\nabla_1 \frac{1}{R_{12}} = -\nabla_2 \frac{1}{R_{12}}$$

Equation (62) can therefore be written as

$$\nabla_2 \cdot \mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_{\text{vol}} \left[ -\mathbf{J}_1 \cdot \left( \nabla_1 \frac{1}{R_{12}} \right) \right] dv_1$$

and the vector identity applied again,

$$\nabla_2 \cdot \mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_{\text{vol}} \left[ \frac{1}{R_{12}} (\nabla_1 \cdot \mathbf{J}_1) - \nabla_1 \cdot \left( \frac{\mathbf{J}_1}{R_{12}} \right) \right] dv_1 \quad (63)$$

Since we are concerned only with steady magnetic fields, the continuity equation shows that the first term of (63) is zero. Application of the divergence theorem to the second term gives

$$\nabla_2 \cdot \mathbf{A}_2 = -\frac{\mu_0}{4\pi} \oint_{S_1} \frac{\mathbf{J}_1}{R_{12}} \cdot d\mathbf{S}_1$$

where the surface  $S_1$  encloses the volume throughout which we are integrating. This volume must include all the current, for the original integral expression for  $\mathbf{A}$  was an integration such as to include the effect of all the current. Since there is no current outside this volume (otherwise we should have had to increase the volume to include it), we may integrate over a slightly larger volume or a slightly larger enclosing surface without changing  $\mathbf{A}$ . On this larger surface the current density  $\mathbf{J}_1$  must be zero, and therefore the closed surface integral is zero, since the integrand is zero. Hence the divergence of  $\mathbf{A}$  is zero.

In order to find the Laplacian of the vector  $\mathbf{A}$  let us compare the  $x$  component of (51) with the similar expression for electrostatic potential,

$$A_x = \int_{\text{vol}} \frac{\mu_0 J_x dv}{4\pi R} \quad V = \int_{\text{vol}} \frac{\rho_v dv}{4\pi\epsilon_0 R}$$

We note that one expression can be obtained from the other by a straightforward change of variable,  $J_x$  for  $\rho_v$ ,  $\mu_0$  for  $1/\epsilon_0$ , and  $A_x$  for  $V$ . However, we have derived some additional information about the electrostatic potential which we shall not have to repeat now for the  $x$  component of the vector magnetic potential. This takes the form of Poisson's equation,

$$\nabla^2 V = -\frac{\rho_v}{\epsilon_0}$$

which becomes, after the change of variables,

$$\nabla^2 A_x = -\mu_0 J_x$$

Similarly, we have

$$\nabla^2 A_y = -\mu_0 J_y$$

and

$$\nabla^2 A_z = -\mu_0 J_z$$

or

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad (64)$$

Returning to (60), we can now substitute for the divergence and Laplacian of  $\mathbf{A}$  and obtain the desired answer,

$$\nabla \times \mathbf{H} = \mathbf{J} \quad (28)$$

We have already shown the use of Stokes' theorem in obtaining the integral form of Ampère's circuital law from (28) and need not repeat that labor here.

We thus have succeeded in showing that every result we have essentially pulled from thin air<sup>11</sup> for magnetic fields follows from the basic definitions of  $\mathbf{H}$ ,  $\mathbf{B}$ , and  $\mathbf{A}$ . The derivations are not simple, but they should be understandable on a step-by-step basis. It is hoped that the procedure need never be committed to memory.

Finally, let us return to (64) and make use of this formidable second-order vector partial differential equation to find the vector magnetic potential in one simple example. We select the field between conductors of a coaxial cable, with radii of  $a$  and  $b$  as usual, and current  $I$  in the  $\mathbf{a}_z$  direction in the inner conductor. Between the conductors,  $\mathbf{J} = 0$ , and therefore

$$\nabla^2 \mathbf{A} = 0$$

We have already been told (and Prob. 8.44 gives us the opportunity to check the results for ourselves) that the vector Laplacian may be expanded as the vector sum of the scalar Laplacians of the three components in cartesian coordinates,

$$\nabla^2 \mathbf{A} = \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z$$

but such a relatively simple result is not possible in other coordinate systems. That is, in cylindrical coordinates, for example,

$$\nabla^2 \mathbf{A} \neq \nabla^2 A_\rho \mathbf{a}_\rho + \nabla^2 A_\phi \mathbf{a}_\phi + \nabla^2 A_z \mathbf{a}_z$$

However, it is not difficult to show for cylindrical coordinates that the  $z$  component of the vector Laplacian is the scalar Laplacian of the  $z$  component of  $\mathbf{A}$ , or

$$\nabla^2 \mathbf{A} \Big|_z = \nabla^2 A_z \quad (65)$$

and since the current is entirely in the  $z$  direction in this problem,  $\mathbf{A}$  has only a  $z$  component. Therefore

$$\nabla^2 A_z = 0$$

---

<sup>11</sup> Free space.

or

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial A_z}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 A_z}{\partial \phi^2} + \frac{\partial^2 A_z}{\partial z^2} = 0$$

Thinking symmetrical thoughts about (51) shows us that  $A_z$  is a function only of  $\rho$ , and thus

$$\frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{dA_z}{d\rho} \right) = 0$$

We have solved this equation before, and the result is

$$A_z = C_1 \ln \rho + C_2$$

If we choose a zero reference at  $\rho = b$ , then

$$A_z = C_1 \ln \frac{\rho}{b}$$

In order to relate  $C_1$  to the sources in our problem, we may take the curl of  $\mathbf{A}$ ,

$$\nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial \rho} \mathbf{a}_\phi = -\frac{C_1}{\rho} \mathbf{a}_\phi = \mathbf{B}$$

obtain  $\mathbf{H}$ ,

$$\mathbf{H} = -\frac{C_1}{\mu_0 \rho} \mathbf{a}_\phi$$

and evaluate the line integral,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_0^{2\pi} -\frac{C_1}{\mu_0 \rho} \mathbf{a}_\phi \cdot \rho d\phi \mathbf{a}_\phi = -\frac{2\pi C_1}{\mu_0}$$

Thus

$$C_1 = -\frac{\mu_0 I}{2\pi}$$

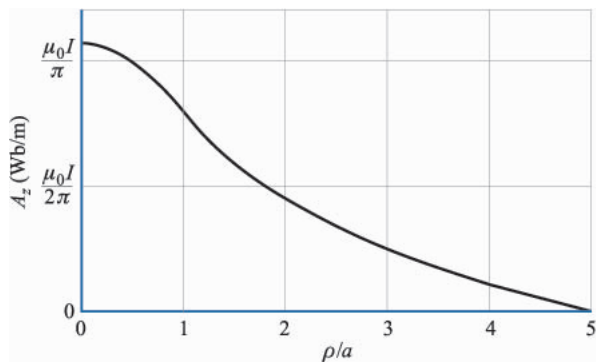
or

$$A_z = \frac{\mu_0 I}{2\pi} \ln \frac{b}{\rho} \quad (66)$$

and

$$H_\phi = \frac{I}{2\pi\rho}$$

as before. A plot of  $A_z$  versus  $\rho$  for  $b = 5a$  is shown in Fig. 8.20; the decrease of  $|\mathbf{A}|$  with distance from the concentrated current source which the inner conductor represents is evident. The results of Prob. D8.9 have also been added to the graph of Fig. 8.20. The extension of the curve into the outer conductor is left as Prob. 8.43.

**FIGURE 8.20**

The vector magnetic potential is shown within the inner conductor and in the region between conductors for a coaxial cable with  $b = 5a$  carrying  $I$  in the  $\mathbf{a}_z$  direction,  $A_z = 0$  is arbitrarily selected at  $\rho = b$ .

It is also possible to find  $A_z$  between conductors by applying a process some of us informally call “uncurling.” That is, we know  $\mathbf{H}$  or  $\mathbf{B}$  for the coax, and we may therefore select the  $\phi$  component of  $\nabla \times \mathbf{A} = \mathbf{B}$  and integrate to obtain  $A_z$ . Try it, you’ll like it!



**D8.10.** Equation (66) is obviously also applicable to the exterior of any conductor of circular cross section carrying a current  $I$  in the  $\mathbf{a}_z$  direction in free space. The zero reference is arbitrarily set at  $\rho = b$ . Now consider two conductors, each of 1-cm radius, parallel to the  $z$  axis with their axes lying in the  $x = 0$  plane. One conductor whose axis is at  $(0, 4 \text{ cm}, z)$  carries 12 A in the  $\mathbf{a}_z$  direction; the other axis is at  $(0, -4 \text{ cm}, z)$  and carries 12 A in the  $-\mathbf{a}_z$  direction. Each current has its zero reference for  $\mathbf{A}$  located 4 cm from its axis. Find the total  $\mathbf{A}$  field at: (a)  $(0, 0, z)$ ; (b)  $(0, 8 \text{ cm}, z)$ ; (c)  $(4 \text{ cm}, 4 \text{ cm}, z)$ ; (d)  $(2 \text{ cm}, 4 \text{ cm}, z)$ .

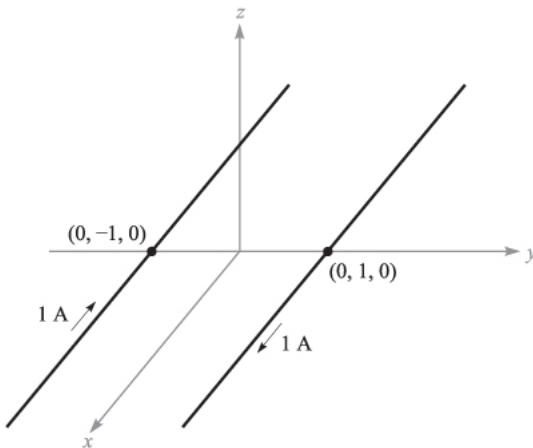
*Ans.* 0;  $2.20 \mu \text{ Wb/m}$ ; 0;  $8.93 \mu \text{ Wb/m}$

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2. Jordan, E. C., and K. G. Balmain: “Electromagnetic Waves and Radiating Systems,” 2d ed., Prentice-Hall, Inc., Englewood Cliffs, N.J., 1968. Vector magnetic potential is discussed on pp. 90–96.
3. Paul, C. R., K. W. Whites, and S. Y. Nasar: “Introduction to Electromagnetic Fields,” 3rd ed., McGraw-Hill Book Company, New York, 1998. The vector magnetic potential is presented on pp. 216–220.
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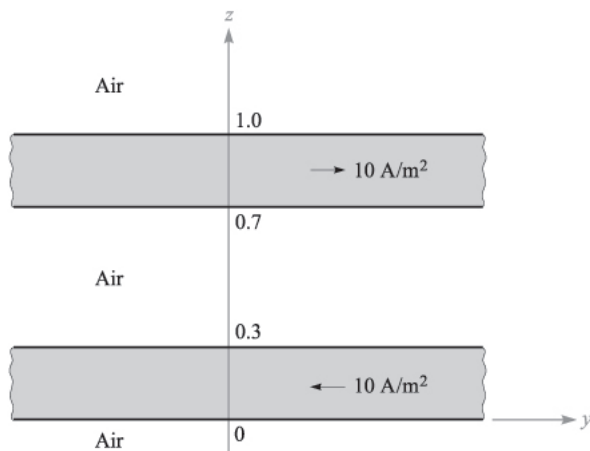
## PROBLEMS

- 8.1** (a) Find  $\mathbf{H}$  in cartesian components at  $P(2, 3, 4)$  if there is a current filament on the  $z$  axis carrying 8 mA in the  $\mathbf{a}_z$  direction. (b) Repeat if the filament is located at  $x = -1, y = 2$ . (c) Find  $\mathbf{H}$  if both filaments are present.
- 8.2** A current filament of  $3\mathbf{a}_x$  A lies along the  $x$  axis. Find  $\mathbf{H}$  in cartesian components at  $P(-1, 3, 2)$ .
- 8.3** Two semi-infinite filaments on the  $z$  axis lie in the regions  $-\infty < z < a$  and  $a < z < \infty$ . Each carries a current  $I$  in the  $\mathbf{a}_z$  direction. (a) Calculate  $\mathbf{H}$  as a function of  $\rho$  and  $\phi$  at  $z = 0$ . (b) What value of  $a$  will cause the magnitude of  $\mathbf{H}$  at  $\rho = 1, z = 0$ , to be half the value obtained for an infinite filament?
- 8.4** (a) A filament is formed into a circle of radius  $a$ , centered at the origin in the plane  $z = 0$ . It carries a current  $I$  in the  $\mathbf{a}_\phi$  direction. Find  $\mathbf{H}$  at the origin. (b) A filament of the same length is shaped into a square in the  $z = 0$  plane. The sides are parallel to the coordinate axes and a current  $I$  flows in the general  $\mathbf{a}_\phi$  direction. Again find  $\mathbf{H}$  at the origin.
- 8.5** The parallel filamentary conductors shown in Fig. 8.21 lie in free space. Plot  $|\mathbf{H}|$  versus  $y$ ,  $-4 < y < 4$ , along the line  $x = 0, z = 2$ .
- 8.6** (a) A current filament  $I$  is formed into a circle,  $\rho = a$ , in the  $z = z'$  plane. Find  $H_z$  at  $P(0, 0, z)$  if  $I$  flows in the  $\mathbf{a}_\phi$  direction. (b) Find  $H_z$  at  $P$  caused by a uniform surface current density  $\mathbf{K} = K_0\mathbf{a}_\phi$ , flowing on the cylindrical surface,  $\rho = a, 0 < z < h$ . The results of part (a) should help.
- 8.7** Given points  $C(5, -2, 3)$  and  $P(4, -1, 2)$ , a current element  $I d\mathbf{L} = 10^{-4}(4, -3, 1)$  A  $\cdot$  m at  $C$  produces a field  $d\mathbf{H}$  at  $P$ . (a) Specify the direction of  $d\mathbf{H}$  by a unit vector  $\mathbf{a}_H$ . (b) Find  $|d\mathbf{H}|$ . (c) What direction  $\mathbf{a}_1$  should  $I d\mathbf{L}$  have at  $C$  so that  $d\mathbf{H} = 0$ ?
- 8.8** For the finite-length current element on the  $z$  axis, as shown in Fig. 8.5, use the Biot-Savart law to derive Eq. (9) of Sec. 8.1.



**FIGURE 8.21**  
See Prob. 8.5.

- 8.9** A current sheet  $\mathbf{K} = 8\mathbf{a}_x$  A/m flows in the region  $-2 < y < 2$  m in the plane  $z = 0$ . Calculate  $\mathbf{H}$  at  $P(0, 0, 3)$ .
- 8.10** Let a filamentary current of 5 mA be directed from infinity to the origin on the positive  $z$  axis and then back out to infinity on the positive  $x$  axis. Find  $\mathbf{H}$  at  $P(0, 1, 0)$ .
- 8.11** An infinite filament on the  $z$  axis carries  $20\pi$  mA in the  $\mathbf{a}_z$  direction. Three uniform cylindrical current sheets are also present: 400 mA/m at  $\rho = 1$  cm,  $-250$  mA/m at  $\rho = 2$  cm, and  $-300$  mA/m at  $\rho = 3$  cm. Calculate  $H_\phi$  at  $\rho = 0.5, 1.5, 2.5$ , and  $3.5$  cm.
- 8.12** In Fig. 8.22, let the regions  $0 < z < 0.3$  m and  $0.7 < z < 1.0$  m be conducting slabs carrying uniform current densities of  $10$  A/m<sup>2</sup> in opposite directions as shown. Find  $\mathbf{H}$  at  $z =$ : (a)  $-0.2$ ; (b)  $0.2$  (c)  $0.4$ ; (d)  $0.75$ ; (e)  $1.2$  m.
- 8.13** A hollow cylindrical shell of radius  $a$  is centered on the  $z$  axis and carries a uniform surface current density of  $K_a\mathbf{a}_\phi$ . (a) Show that  $H$  is not a function of  $\phi$  or  $z$ . (b) Show that  $H_\phi$  and  $H_\rho$  are everywhere zero. (c) Show that  $H_z = 0$  for  $\rho > a$ . (d) Show that  $H_z = K_a$  for  $\rho < a$ . (e) A second shell,  $\rho = b$ , carries a current  $K_b\mathbf{a}_\phi$ . Find  $\mathbf{H}$  everywhere.
- 8.14** A toroid having a cross section of rectangular shape is defined by the following surfaces: the cylinders  $\rho = 2$  cm and  $\rho = 3$  cm, and the planes  $z = 1$  cm and  $z = 2.5$  cm. The toroid carries a surface current density of  $-50\mathbf{a}_z$  A/m on the surface  $\rho = 3$  cm. Find  $\mathbf{H}$  at the point  $P(\rho, \phi, z)$ : (a)  $P_A(1.5$  cm,  $0, 2$  cm); (b)  $P_B(2.1$  cm,  $0, 2$  cm); (c)  $P_C(2.7$  cm,  $\pi/2, 2$  cm); (d)  $3.5$  cm,  $\pi/2, 2$  cm).
- 8.15** Assume that there is a region with cylindrical symmetry in which the conductivity is given by  $\sigma = 1.5e^{-150\rho}$  kS/m. An electric field of  $30\mathbf{a}_z$  V/m is present. (a) Find  $\mathbf{J}$ . (b) Find the total current crossing the surface  $\rho < \rho_0, z = 0$ , all  $\phi$ . (c) Make use of Ampere's circuital law to find  $\mathbf{H}$ .

**FIGURE 8.22**

See Prob. 8.12.

- 8.16** The cylindrical shell,  $2 \text{ mm} < \rho < 3 \text{ mm}$ , carries a uniformly distributed total current of  $8 \text{ A}$  in the  $-\mathbf{a}_z$  direction, and a filament on the  $z$  axis carries  $8 \text{ A}$  in the  $\mathbf{a}_z$  direction. Find  $\mathbf{H}$  everywhere.
- 8.17** A current filament on the  $z$  axis carries a current of  $7 \text{ mA}$  in the  $\mathbf{a}_z$  direction, and current sheets of  $0.5\mathbf{a}_z \text{ A/m}$  and  $-0.2\mathbf{a}_z \text{ A/m}$  are located at  $\rho = 1 \text{ cm}$  and  $\rho = 0.5 \text{ cm}$ , respectively. Calculate  $\mathbf{H}$  at  $\rho =$ : (a)  $0.5 \text{ cm}$ ; (b)  $1.5 \text{ cm}$ ; (c)  $4 \text{ cm}$ ; (d) What current sheet should be located at  $\rho = 4 \text{ cm}$  so that  $\mathbf{H} = 0$  for all  $\rho > 4 \text{ cm}$ ?
- 8.18** Current density is distributed as follows:  $\mathbf{J} = 0$  for  $|y| > 2 \text{ m}$ ,  $\mathbf{J} = 8y\mathbf{a}_z \text{ A/m}^2$  for  $|y| < 1 \text{ m}$ ,  $\mathbf{J} = 8(2 - y)\mathbf{a}_z \text{ A/m}^2$  for  $1 < y < 2 \text{ m}$ ,  $\mathbf{J} = -8(2 + y)\mathbf{a}_z \text{ A/m}^2$  for  $-2 < y < -1 \text{ m}$ . Use symmetry and Ampere's law to find  $\mathbf{H}$  everywhere.
- 8.19** Calculate  $\nabla \times [\nabla(\nabla \cdot \mathbf{G})]$  if  $\mathbf{G} = 2x^2yz\mathbf{a}_x - 20y\mathbf{a}_y + (x^2 - z^2)\mathbf{a}_z$ .
- 8.20** The magnetic field intensity is given in the square region  $x = 0$ ,  $0.5 < y < 1$ ,  $1 < z < 1.5$  by  $\mathbf{H} = z^2\mathbf{a}_x + x^3\mathbf{a}_y + y^4\mathbf{a}_z \text{ A/m}$ . (a) Evaluate  $\oint \mathbf{H} \cdot d\mathbf{L}$  about the perimeter of the square region. (b) Find  $\nabla \times \mathbf{H}$ . (c) Calculate  $(\nabla \times \mathbf{H})_x$  at the center of the region. (d) Does  $(\nabla \times \mathbf{H})_x = [\oint \mathbf{H} \cdot d\mathbf{L}]/\text{Area enclosed}$ ?
- 8.21** Points  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ , and  $F$  are each  $2 \text{ mm}$  from the origin on the coordinate axis indicated in Fig. 8.23. The value of  $\mathbf{H}$  at each point is given. Calculate an approximate value for  $\nabla \times \mathbf{H}$  at the origin.
- 8.22** In the cylindrical region  $\rho \leq 0.6 \text{ mm}$ ,  $H_\phi = \frac{2}{\rho} + \frac{\rho}{2} \text{ A/m}$ , while  $H_\phi = \frac{3}{\rho} \text{ A/m}$  for  $\rho > 0.6 \text{ mm}$ . (a) Determine  $\mathbf{J}$  for  $\rho < 0.6 \text{ mm}$ . (b) Determine  $\mathbf{J}$  for  $\rho > 0.6 \text{ mm}$ . (c) Is there a filamentary current at  $\rho = 0$ ? If so, what is its value? (d) What is  $\mathbf{J}$  at  $\rho = 0$ ?
- 8.23** Given the field  $\mathbf{H} = 20\rho^2\mathbf{a}_\phi \text{ A/m}$ : (a) determine the current density  $\mathbf{J}$ ; (b) integrate  $\mathbf{J}$  over the circular surface  $\rho = 1$ ,  $0 < \phi < 2\pi$ ,  $z = 0$ , to determine the total current passing through that surface in the  $\mathbf{a}_z$  direction; (c) find the total current once more, this time by a line integral around the circular path  $\rho = 1$ ,  $0 < \phi < 2\pi$ ,  $z = 0$ .

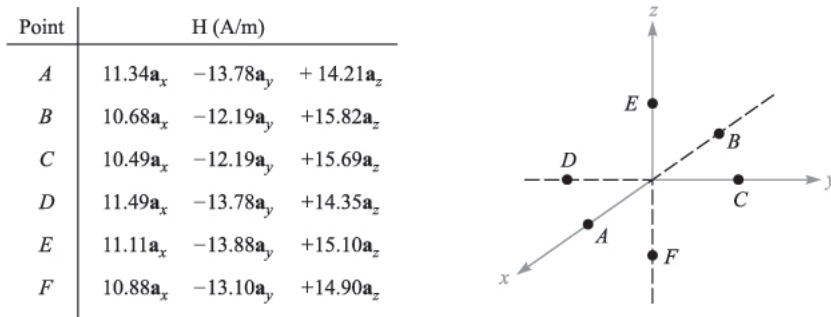


FIGURE 8.23

See Prob. 8.21.



- 8.24** Evaluate both sides of Stokes' theorem for the field  $\mathbf{G} = 10 \sin \theta \mathbf{a}_\phi$  and the surface  $r = 3$ ,  $0 \leq \theta \leq 90^\circ$ ,  $0 \leq \phi \leq 90^\circ$ . Let the surface have the  $\mathbf{a}_r$  direction.
- 8.25** Given the field  $\mathbf{H} = \frac{1}{2} \cos \frac{\phi}{2} \mathbf{a}_\rho - \sin \frac{\phi}{2} \mathbf{a}_\phi$  A/m, evaluate both sides of Stokes' theorem for the path formed by the intersection of the cylinder  $\rho = 3$  and the plane  $z = 2$ , and for the surface defined by  $\rho = 3$ ,  $0 \leq z \leq 2$ , and  $z = 0$ ,  $0 \leq \rho \leq 3$ .
- 8.26** Let  $\mathbf{G} = 15r\mathbf{a}_\phi$ . (a) Determine  $\oint \mathbf{G} \cdot d\mathbf{L}$  for the circular path  $r = 5$ ,  $\theta = 25^\circ$ ,  $0 \leq \phi \leq 2\pi$ . Evaluate  $\int_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S}$  over the spherical cap  $r = 5$ ,  $0 \leq \theta \leq 25^\circ$ ,  $0 \leq \phi \leq 2\pi$ .
- 8.27** The magnetic field intensity is given in a certain region of space as  $\mathbf{H} = \frac{x+2y}{z^2} \mathbf{a}_y + \frac{2}{z} \mathbf{a}_z$  A/m. (a) Find  $\nabla \times \mathbf{H}$ . (b) Find  $\mathbf{J}$ . (c) Use  $\mathbf{J}$  to find the total current passing through the surface  $z = 4$ ,  $1 \leq x \leq 2$ ,  $3 \leq z \leq 5$ , in the  $\mathbf{a}_z$  direction. (d) Show that the same result is obtained using the other side of Stokes' theorem.
- 8.28** Given  $\mathbf{H} = (3r^2/\sin \theta)\mathbf{a}_\theta + 54r \cos \theta \mathbf{a}_\phi$  A/m in free space: (a) find the total current in the  $\mathbf{A}_\theta$  direction through the conical surface  $\theta = 20^\circ$ ,  $0 \leq \phi \leq 2\pi$ ,  $0 \leq r \leq 5$ , by whichever side of Stokes' theorem you like the best. (b) Check the result by using the other side of Stokes' theorem.
- 8.29** A long straight nonmagnetic conductor of 0.2-mm radius carries a uniformly distributed current of 2 A dc. (a) Find  $\mathbf{J}$  within the conductor. (b) Use Ampère's circuital law to find  $\mathbf{H}$  and  $\mathbf{B}$  within the conductor. (c) Show that  $\nabla \times \mathbf{H} = \mathbf{J}$  within the conductor. (d) Find  $\mathbf{H}$  and  $\mathbf{B}$  within the conductor. (e) Show that  $\nabla \times \mathbf{H} = \mathbf{J}$  outside the conductor.
- 8.30** A solid nonmagnetic conductor of circular cross section has a radius of 2 mm. The conductor is inhomogeneous, with  $\sigma = 10^6(1 + 10^6 \rho^2)$  S/m. If the conductor is 1 m in length and has a voltage of 1 mV between its ends, find: (a)  $H$ ; (b) the total magnetic flux inside the conductor.
- 8.31** The cylindrical shell defined by  $1 \text{ cm} < \rho < 1.4 \text{ cm}$  consists of a nonmagnetic conducting material and carries a total current of 50 A in the  $\mathbf{a}_z$  direction. Find the total magnetic flux crossing the plane  $\phi = 0$ ,  $0 < z < 1$ : (a)  $0 < \rho < 1.2 \text{ cm}$ ; (b)  $1.4 \text{ cm} < \rho < 1.4 \text{ cm}$ ; (c)  $1.4 \text{ cm} < \rho < 20 \text{ cm}$ .
- 8.32** The free-space region defined by  $1 < z < 4 \text{ cm}$  and  $2 < \rho < 3 \text{ cm}$  is a toroid of rectangular cross section. Let the surface at  $\rho = 3 \text{ cm}$  carry a surface current  $\mathbf{K} = 2\mathbf{a}_z$  kA/m. (a) Specify the currents on the surfaces at  $\rho = 2 \text{ cm}$ ,  $z = 1 \text{ cm}$ , and  $z = 4 \text{ cm}$ . (b) Find  $\mathbf{H}$  everywhere. (c) Calculate the total flux within the toroid.
- 8.33** Use an expansion in cartesian coordinates to show that the curl of the gradient of any scalar field  $G$  is identically equal to zero.
- 8.34** A filamentary conductor on the  $z$  axis carries a current of 16 A in the  $\mathbf{a}_z$  direction, a conducting shell at  $\rho = 6$  carries a total current of 12 A in the  $-\mathbf{a}_z$  direction, and another shell at  $\rho = 10$  carries a total current of 4 A in

- the  $-\mathbf{a}_z$  direction. (a) Find  $\mathbf{H}$  for  $0 < \rho < 12$ . (b) Plot  $H_\phi$  versus  $\rho$ . (c) Find the total flux  $\Phi$  crossing the surface  $1 < \rho < 7$ ,  $0 < z < 1$ .
- 8.35** A current sheet,  $\mathbf{K} = 20\mathbf{a}_z$  A/m, is located at  $\rho = 2$ , and a second sheet,  $\mathbf{K} = -10\mathbf{a}_z$  A/m, is located at  $\rho = 4$ . (a) Let  $V_m = 0$  at  $P(\rho = 3, \phi = 0, z = 5)$  and place a barrier at  $\phi = \pi$ . Find  $V_m(\rho, \phi, z)$  for  $-\pi < \phi < \pi$ . (b) Let  $\mathbf{A} = 0$  at  $P$  and find  $\mathbf{A}(\rho, \phi, z)$  for  $2 < \rho < 4$ .
- 8.36** Let  $\mathbf{A} = (3y - z)\mathbf{a}_x + 2xz\mathbf{a}_y$  Wb/m in a certain region of free space. (a) Show that  $\nabla \cdot \mathbf{A} = 0$ . (b) At  $P(2, -1, 3)$ , find  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{H}$ , and  $\mathbf{J}$ .
- 8.37** Let  $N = 1000$ ,  $I = 0.8$  A,  $\rho_0 = 2$  cm, and  $a = 0.8$  cm for the toroid shown in Fig. 8.12b. Find  $V_m$  in the interior of the toroid if  $V_m = 0$  at  $\rho = 2.5$  cm,  $\phi = 0.3\pi$ . Keep  $\phi$  within the range  $0 < \phi < 2\pi$ .
- 8.38** The solenoid shown in Fig. 8.11b contains 400 turns, carries a current  $I = 5$  A, has a length of 8 cm, and a radius  $a = 1.2$  cm. (a) Find  $\mathbf{H}$  within the solenoid. (b) If  $V_m = 0$  at the origin, specify  $V_m(\rho, \phi, z)$  inside the solenoid. (c) Let  $\mathbf{A} = 0$  at the origin, and specify  $\mathbf{A}(\rho, \phi, z)$  inside the solenoid if the medium is free space.
- 8.39** Planar current sheets of  $\mathbf{K} = 30\mathbf{a}_z$  A/m and  $-30\mathbf{a}_z$  A/m are located in free space at  $x = 0.2$  and  $x = -0.2$ , respectively. For the region  $-0.2 < x < 0.2$ : (a) find  $\mathbf{H}$ ; (b) obtain an expression for  $V_m$  if  $V_m = 0$  at  $P(0.1, 0.2, 0.3)$ ; (c) find  $\mathbf{B}$ ; (d) obtain an expression for  $\mathbf{A}$  if  $\mathbf{A} = 0$  at  $P$ .
- 8.40** Let  $\mathbf{A} = (3y^2 - 2z)\mathbf{a}_x - 2x^2z\mathbf{a}_y + (x + 2y)\mathbf{a}_z$  Wb/m in free space. Find  $\nabla \times \nabla \times \mathbf{A}$  at  $P(-2, 3, -1)$ .
- 8.41** Assume that  $\mathbf{A} = 50\rho^2\mathbf{a}_z$  Wb/m in a certain region of free space. (a) Find  $\mathbf{H}$  and  $\mathbf{B}$ . (b) Find  $\mathbf{J}$ . (c) Use  $\mathbf{J}$  to find the total current crossing the surface  $0 \leq \rho \leq 1$ ,  $0 \leq \phi < 2\pi$ ,  $z = 0$ . (d) Use the value of  $H_\phi$  at  $\rho = 1$  to calculate  $\oint \mathbf{H} \cdot d\mathbf{L}$  for  $\rho = 1$ ,  $z = 0$ .
- 8.42** Show that  $\nabla_2(1/R_{12}) = -\nabla_1(1/R_{12}) = \mathbf{R}_{21}/R_{12}^3$ .
- 8.43** Compute the vector magnetic potential within the outer conductor for the coaxial line whose vector magnetic potential is shown in Fig. 8.20 if the outer radius of the outer conductor is  $7a$ . Select the proper zero reference and sketch the results on the figure.
- 8.44** By expanding Eq. (58), Sec. 8.7, in cartesian coordinates, show that (59) is correct.