CHAPTER 3

ELECTRIC FLUX DENSITY, GAUSS'S LAW, AND DIVERGENCE

After drawing a few of the fields described in the previous chapter and becoming familiar with the concept of the streamlines which show the direction of the force on a test charge at every point, it is difficult to avoid giving these lines a physical significance and thinking of them as *flux* lines. No physical particle is projected radially outward from the point charge, and there are no steel tentacles reaching out to attract or repel an unwary test charge, but as soon as the streamlines are drawn on paper there seems to be a picture showing "something" is present.

It is very helpful to invent an *electric flux* which streams away symmetrically from a point charge and is coincident with the streamlines and to visualize this flux wherever an electric field is present.

This chapter introduces and uses the concept of electric flux and electric flux density to solve again several of the problems presented in the last chapter. The work here turns out to be much easier, and this is due to the extremely symmetrical problems which we are solving.

3.1 ELECTRIC FLUX DENSITY

About 1837 the Director of the Royal Society in London, Michael Faraday, became very interested in static electric fields and the effect of various insulating materials on these fields. This problem had been bothering him during the past ten years when he was experimenting in his now famous work on induced electromotive force, which we shall discuss in Chap. 10. With that subject completed, he had a pair of concentric metallic spheres constructed, the outer one consisting of two hemispheres that could be firmly clamped together. He also prepared shells of insulating material (or dielectric material, or simply *dielectric*) which would occupy the entire volume between the concentric spheres. We shall not make immediate use of his findings about dielectric materials, for we are restricting our attention to fields in free space until Chap. 5. At that time we shall see that the materials he used will be classified as ideal dielectrics.

His experiment, then, consisted essentially of the following steps:

1. With the equipment dismantled, the inner sphere was given a known positive charge.
2. The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.
3. The outer sphere was discharged by connecting it momentarily to ground.
4. The outer space was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured.

Faraday found that the total charge on the outer sphere was equal in *magnitude* to the original charge placed on the inner sphere and that this was true regardless of the dielectric material separating the two spheres. He concluded that there was some sort of "displacement" from the inner sphere to the outer which was independent of the medium, and we now refer to this flux as *displacement*, *displacement flux*, or simply *electric flux*.

Faraday's experiments also showed, of course, that a larger positive charge on the inner sphere induced a correspondingly larger negative charge on the outer sphere, leading to a direct proportionality between the electric flux and the charge on the inner sphere. The constant of proportionality is dependent on the system of units involved, and we are fortunate in our use of SI units, because the constant is unity. If electric flux is denoted by Ψ (psi) and the total charge on the inner sphere by Q, then for Faraday's experiment

$$\boxed{\Psi = Q}$$

and the electric flux Ψ is measured in coulombs.

We can obtain more quantitative information by considering an inner sphere of radius a and an outer sphere of radius b, with charges of Q and $-Q$, respectively (Fig. 3.1). The paths of electric flux Ψ extending from the inner sphere to the outer sphere are indicated by the symmetrically distributed streamlines drawn radially from one sphere to the other.

At the surface of the inner sphere, Ψ coulombs of electric flux are produced by the charge $Q(=\Psi)$ coulombs distributed uniformly over a surface having an area of $4\pi a^2\,\text{m}^2$. The density of the flux at this surface is $\Psi/4\pi a^2$ or $Q/4\pi a^2\,\text{C/m}^2$, and this is an important new quantity.

Electric flux density, measured in coulombs per square meter (sometimes described as "lines per square meter," for each line is due to one coulomb), is given the letter **D**, which was originally chosen because of the alternate names of *displacement flux density* or *displacement density*. Electric flux density is more descriptive, however, and we shall use the term consistently.

The electric flux density **D** is a vector field and is a member of the "flux density" class of vector fields, as opposed to the "force fields" class, which includes the electric field intensity **E**. The direction of **D** at a point is the direction of the flux lines at that point, and the magnitude is given by the number of flux lines crossing a surface normal to the lines divided by the surface area.

Referring again to Fig. 3.1, the electric flux density is in the radial direction and has a value of

$$\mathbf{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\mathbf{a}_r \qquad \text{(inner sphere)}$$

$$\mathbf{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\mathbf{a}_r \qquad \text{(outer sphere)}$$

and at a radial distance r, where $a \le r \le b$,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

If we now let the inner sphere become smaller and smaller, while still retaining a charge of Q, it becomes a point charge in the limit, but the electric flux density at a point r meters from the point charge is still given by

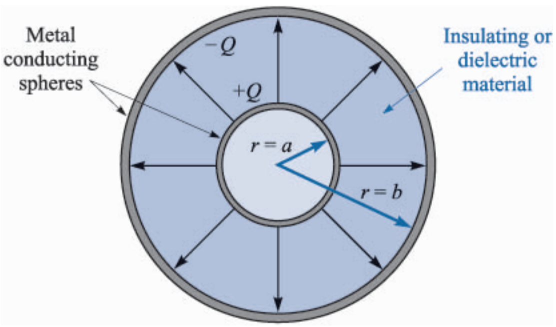

FIGURE 3.1
The electric flux in the region between a pair of charged concentric spheres. The direction and magnitude of **D** are not functions of the dielectric between the spheres.

$$\boxed{\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r} \tag{1}$$

for Q lines of flux are symmetrically directed outward from the point and pass through an imaginary spherical surface of area $4\pi r^2$.

This result should be compared with Sec. 2.2, Eq. (10), the radial electric field intensity of a point charge in free space,

$$\boxed{\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r}$$

In free space, therefore,

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E}} \qquad \text{(free space only)} \tag{2}$$

Although (2) is applicable only to a vacuum, it is not restricted solely to the field of a point charge. For a general volume charge distribution in free space

$$\boxed{\mathbf{E} = \int_{\text{vol}} \frac{\rho_v dv}{4\pi\epsilon_0 R^2}\mathbf{a}_r} \qquad \text{(free space only)} \tag{3}$$

where this relationship was developed from the field of a single point charge. In a similar manner, (1) leads to

$$\boxed{\mathbf{D} = \int_{\text{vol}} \frac{\rho_v\, dv}{4\pi R^2}\mathbf{a}_r} \tag{4}$$

and (2) is therefore true for any free-space charge configuration; we shall consider (2) as defining **D** in free space.

As a preparation for the study of dielectrics later, it might be well to point out now that, for a point charge embedded in an infinite ideal dielectric medium, Faraday's results show that (1) is still applicable, and thus so is (4). Equation (3) is not applicable, however, and so the relationship between **D** and **E** will be slightly more complicated than (2).

Since **D** is directly proportional to **E** in free space, it does not seem that it should really be necessary to introduce a new symbol. We do so for several reasons. First, **D** is associated with the flux concept, which is an important new idea. Second, the **D** fields we obtain will be a little simpler than the corresponding **E** fields, since ϵ_0 does not appear. And, finally, it helps to become a little familiar with **D** before it is applied to dielectric materials in Chap. 5.

Let us consider a simple numerical example to illustrate these new quantities and units.

Example 3.1

We wish to find **D** in the region about a uniform line charge of 8 nC/m lying along the z axis in free space.

Solution. The **E** field is

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0 \rho}\mathbf{a}_\rho = \frac{8 \times 10^{-9}}{2\pi(8.854 \times 10^{-12})\rho}\mathbf{a}_\rho = \frac{143.8}{\rho}\mathbf{a}_\rho \text{ V/m}$$

At $\rho = 3$ m, $\mathbf{E} = 47.9\mathbf{a}_\rho$ V/m.

Associated with the **E** field, we find

$$\mathbf{D} = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho = \frac{8 \times 10^{-9}}{2\pi\rho}\mathbf{a}_\rho = \frac{1.273 \times 10^{-9}}{\rho}\mathbf{a}_\rho \text{ C/m}^2$$

The value at $\rho = 3$ m is $\mathbf{D} = 0.424\mathbf{a}_\rho$ nC/m.

The total flux leaving a 5-m length of the line charge is equal to the total charge on that length, or $\Psi = 40$ nC.

✔ **D3.1.** Given a 60-μC point charge located at the origin, find the total electric flux passing through: (a) that portion of the sphere $r = 26$ cm bounded by $0 < \theta < \frac{\pi}{2}$ and $0 < \phi < \frac{\pi}{2}$; (b) the closed surface defined by $\rho = 26$ cm and $z = \pm 26$ cm; (c) the plane $z = 26$ cm.

Ans. 7.5 μC; 60 μC; 30 μC

✔ **D3.2.** Calculate **D** in rectangular coordinates at point $P(2, -3, 6)$ produced by: (a) a point charge $Q_A = 55$ mC at $Q(-2, 3, -6)$; (b) a uniform line charge $\rho_{LB} = 20$ mC/m on the x axis; (c) a uniform surface charge density $\rho_{SC} = 120\ \mu\text{C/m}^2$ on the plane $z = -5$ m.

Ans. $6.38\mathbf{a}_x - 9.57\mathbf{a}_y + 19.14\mathbf{a}_z\ \mu\text{C/m}^2$; $-212\mathbf{a}_y + 424\mathbf{a}_z\ \mu\text{C/m}^2$; $60\mathbf{a}_z\ \mu\text{C/m}^2$

3.2 GAUSS'S LAW

The results of Faraday's experiments with the concentric spheres could be summed up as an experimental law by stating that the electric flux passing through any imaginary spherical surface lying between the two conducting spheres is equal to the charge enclosed within that imaginary surface. This enclosed charge is distributed on the surface of the inner sphere, or it might be concentrated as a point charge at the center of the imaginary sphere. However, since one coulomb of electric flux is produced by one coulomb of charge, the inner conductor might just as well have been a cube or a brass door key and the total induced charge on the outer sphere would still be the same. Certainly the flux density would change from its previous symmetrical distribution to some unknown configuration, but $+Q$ coulombs on any inner conductor would produce an induced charge of $-Q$ coulombs on the surrounding sphere. Going one step further, we could now replace the two outer hemispheres by an empty (but completely closed) soup can. Q coulombs on the brass door key would produce $\Psi = Q$ lines of electric flux and would induce $-Q$ coulombs on the tin can.[1]

These generalizations of Faraday's experiment lead to the following statement, which is known as *Gauss's law*:

> The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

The contribution of Gauss, one of the greatest mathematicians the world has ever produced, was actually not in stating the law as we have above, but in providing a mathematical form for this statement, which we shall now obtain.

Let us imagine a distribution of charge, shown as a cloud of point charges in Fig. 3.2, surrounded by a closed surface of any shape. The closed surface may be the surface of some real material, but more generally it is any closed surface we wish to visualize. If the total charge is Q, then Q coulombs of electric flux will pass through the enclosing surface. At every point on the surface the electric-flux-density vector $\mathbf{D}$ will have some value $\mathbf{D}_S$, where the subscript S merely reminds us that $\mathbf{D}$ must be evaluated at the surface, and $\mathbf{D}_S$ will in general vary in magnitude and direction from one point on the surface to another.

We must now consider the nature of an incremental element of the surface. An incremental element of area ΔS is very nearly a portion of a plane surface, and the complete description of this surface element requires not only a statement of its magnitude ΔS but also of its orientation in space. In other words, the incremental surface element is a vector quantity. The only unique direction which may be associated with $\Delta\mathbf{S}$ is the direction of the normal to that plane which is tangent to the surface at the point in question. There are, of course, two

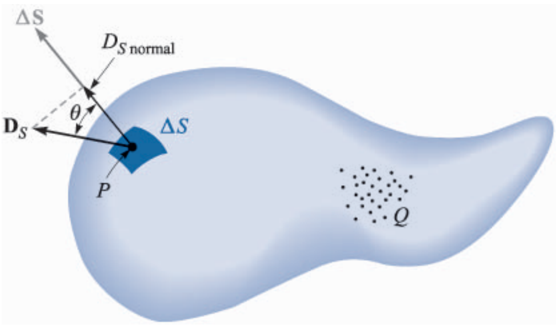

FIGURE 3.2
The electric flux density $\mathbf{D}_S$ at P due to charge Q. The total flux passing through $\Delta\mathbf{S}$ is $\mathbf{D}_S \cdot \Delta\mathbf{S}$.

[1] If it were a perfect insulator, the soup could even be left in the can without any difference in the results.

such normals, and the ambiguity is removed by specifying the outward normal whenever the surface is closed and "outward" has a specific meaning.

At any point P consider an incremental element of surface ΔS and let $\mathbf{D}_S$ make an angle θ with ΔS, as shown in Fig. 3.2. The flux crossing $\Delta \mathbf{S}$ is then the product of the normal component of $\mathbf{D}_S$ and $\Delta \mathbf{S}$,

$$\Delta\Psi = \text{flux crossing } \Delta\mathbf{S} = D_{S,norm}\Delta = D_S S \cos\theta \Delta S = \mathbf{D}_S \cdot \Delta\mathbf{S}$$

where we are able to apply the definition of the dot product developed in Chap. 1.

The *total* flux passing through the closed surface is obtained by adding the differential contributions crossing each surface element $\Delta\mathbf{S}$,

$$\Psi = \int d\Psi = \oint_{\substack{\text{closed}\\ \text{surface}}} \mathbf{D}_S \cdot d\mathbf{S}$$

The resultant integral is a *closed surface integral*, and since the surface element $d\mathbf{S}$ always involves the differentials of two coordinates, such as $dx\,dy$, $\rho\, d\phi\, d\rho$, or $r^2 \sin\theta\, d\theta\, d\phi$, the integral is a double integral. Usually only one integral sign is used for brevity, and we shall always place an S below the integral sign to indicate a surface integral, although this is not actually necessary since the differential $d\mathbf{S}$ is automatically the signal for a surface integral. One last convention is to place a small circle on the integral sign itself to indicate that the integration is to be performed over a *closed* surface. Such a surface is often called a *gaussian surface*. We then have the mathematical formulation of Gauss's law,

$$\boxed{\Psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \text{charge enclosed} = Q} \qquad (5)$$

The charge enclosed might be several point charges, in which case

$$Q = \Sigma Qn$$

or a line charge,

$$Q = \int \rho_L \, dL$$

or a surface charge,

$$Q = \int_S \rho_S dS \qquad \text{(not necessarily a closed surface)}$$

or a volume charge distribution,

$$Q = \int_{\text{vol}} \rho_v \, dv$$

The last form is usually used, and we should agree now that it represents any or all of the other forms. With this understanding Gauss's law may be written in terms of the charge distribution as

$$\boxed{\oint_S \mathbf{D}_S \cdot d\mathbf{S} = \int_{\text{vol}} \rho_v \, dv} \tag{6}$$

a mathematical statement meaning simply that the total electric flux through any closed surface is equal to the charge enclosed.

To illustrate the application of Gauss's law, let us check the results of Faraday's experiment by placing a point charge Q at the origin of a spherical coordinate system (Fig. 3.3) and by choosing our closed surface as a sphere of radius a. The electric field intensity of the point charge has been found to be

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r$$

and since

$$\mathbf{D} = \epsilon_0 \mathbf{E}$$

we have, as before,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

At the surface of the sphere,

$$\mathbf{D}_S = \frac{Q}{4\pi a^2}\mathbf{a}_r$$

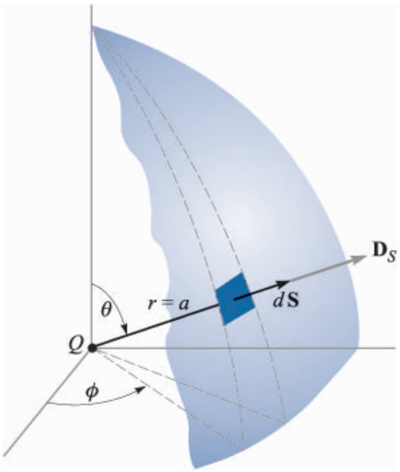

FIGURE 3.3
Application of Gauss's law to the field of a point charge Q on a spherical closed surface of radius a. The electric flux density $\mathbf{D}$ is everywhere normal to the spherical surface and has a constant magnitude at every point on it.

The differential element of area on a spherical surface is, in spherical coordinates from Chap. 1,

$$dS = r^2 \sin\theta \, d\theta \, d\phi = a^2 \sin\theta \, d\theta \, d\phi$$

or

$$d\mathbf{S} = a^2 \sin\theta \, d\theta \, d\phi$$

The integrand is

$$\mathbf{D}_S \cdot d\mathbf{S} = \frac{Q}{4\pi a^2} a^2 \sin\theta \, d\theta \, d\phi \mathbf{a}_r \cdot \mathbf{a}_r = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

leading to the closed surface integral

$$\int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \sin\theta \, d\theta \, d\phi$$

where the limits on the integrals have been chosen so that the integration is carried over the entire surface of the sphere once.[2] Integrating gives

$$\int_0^{2\pi} \frac{Q}{4\pi} (-\cos\theta)_0^{\pi} d\phi = \int_0^{2\pi} \frac{Q}{2\pi} d\phi = Q$$

and we obtain a result showing that Q coulombs of electric flux are crossing the surface, as we should since the enclosed charge is Q coulombs.

The following section contains examples of the application of Gauss's law to problems of a simple symmetrical geometry with the object of finding the elelctric field intensity.

✔ **D3.3.** Given the electric flux density, $\mathbf{D} = 0.3r^2 \mathbf{a}_r \, \text{nC/m}^2$ in free space: (*a*) find $\mathbf{E}$ at point $P(r = 2, \theta = 25^\circ, \phi = 90^\circ)$; (*b*) find the total charge within the sphere $r = 3$; (*c*) find the total electric flux leaving the sphere $r = 4$.

Ans. $135.5\mathbf{a}_r$ V/m; 305 nC; 965 nC

✔ **D3.4.** Calculate the total electric flux leaving the cubical surface formed by the six planes $x, y, z = \pm 5$ if the charge distribution is: (*a*) two point charges, 0.1 μC at $(1, -2, 3)$ and $\frac{1}{7}$ μC at $(-1, 2, -2)$; (*b*) a uniform line charge of π μC/m at $x = -2$, $y = 3$; (*c*) a uniform surface charge of 0.1 μC/m^2 on the plane $y = 3x$.

Ans. 0.243 μC; 31.4 μC; 10.54 μC

[2] Note that if θ and ϕ both cover the range from 0 to 2π, the spherical surface is covered twice.

3.3 APPLICATION OF GAUSS'S LAW: SOME SYMMETRICAL CHARGE DISTRIBUTIONS

Let us now consider how we may use Gauss's law,

$$\boxed{Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S}}$$

to determine $\mathbf{D}_S$ if the charge distribution is known. This is an example of an integral equation in which the unknown quantity to be determined appears inside the integral.

The solution is easy if we are able to choose a closed surface which satisfies two conditions:

1. $\mathbf{D}_S$ is everywhere either normal or tangential to the closed surface, so that $\mathbf{D}_S \cdot d\mathbf{S}$ becomes either $D_S dS$ or zero, respectively.
2. On that portion of the closed surface for which $\mathbf{D}_S \cdot dS$ is not zero, $D_S =$ constant.

This allows us to replace the dot product with the product of the scalars D_S and dS and then to bring D_S outside the integral sign. The remaining integral is then $\int_S dS$ over that portion of the closed surface which $\mathbf{D}_S$ crosses normally, and this is simply the area of this section of that surface.

Only a knowledge of the symmetry of the problem enables us to choose such a closed surface, and this knowledge is obtained easily by remembering that the electric field intensity due to a positive point charge is directed radially outward from the point charge.

Let us again consider a point charge Q at the origin of a spherical coordinate system and decide on a suitable closed surface which will meet the two requirements listed above. The surface in question is obviously a spherical surface, centered at the origin and of any radius r. $\mathbf{D}_S$ is everywhere normal to the surface; D_S has the same value at all points on the surface.

Then we have, in order,

$$\begin{aligned} Q &= \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \oint_{\text{sph}} D_S dS \\ &= D_S \oint_{\text{sph}} dS = D_S \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} r^2 \sin\theta \, d\theta \, d\phi \\ &= 4\pi r^2 D_S \end{aligned}$$

and hence $$D_S = \frac{Q}{4\pi r^2}$$

Since r may have any value and since $\mathbf{D}_S$ is directed radially outward,

$$\boxed{\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r \qquad \mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r}$$

which agrees with the results of Chap. 2. The example is a trivial one, and the objection could be raised that we had to know that the field was symmetrical and directed radially outward before we could obtain an answer. This is true, and that leaves the inverse-square-law relationship as the only check obtained from Gauss's law. The example does, however, serve to illustrate a method which we may apply to other problems, including several to which Coulomb's law is almost incapable of supplying an answer.

Are there any other surfaces which would have satisfied our two conditions? The student should determine that such simple surfaces as a cube or a cylinder do not meet the requirements.

As a second example, let us reconsider the uniform line charge distribution ρ_L lying along the z axis and extending from $-\infty$ to $+\infty$. We must first obtain a knowledge of the symmetry of the field, and we may consider this knowledge complete when the answers to these two questions are known:

1. With which coodinates does the field vary (or of what variables is D a function)?
2. Which components of **D** are present?

These same questions were asked when we used Coulomb's law to solve this problem in Sec. 2.5. We found then that the knowledge obtained from answering them enabled us to make a much simpler integration. The problem could have been (and was) worked without any consideration of symmetry, but it was more difficult.

In using Gauss's law, however, it is not a question of using symmetry to simplify the solution, for the application of Gauss's law depends on symmetry, and *if we cannot show that symmetry exists then we cannot use Gauss's law* to obtain a solution. The two questions above now become "musts."

From our previous discussion of the uniform line charge, it is evident that only the radial component of D is present, or

$$\mathbf{D} = D_\rho \mathbf{a}_\rho$$

and this component is a function of ρ only.

$$D_\rho = f(\rho)$$

The choice of a closed surface is now simple, for a cylindrical surface is the only surface to which D_ρ is everywhere normal and it may be closed by plane surfaces normal to the z axis. A closed right circular cylindrical of radius ρ extending from $z = 0$ to $z = L$ is shown in Fig. 3.4.

We apply Gauss's law,

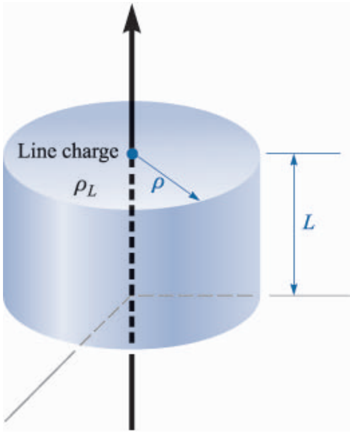

FIGURE 3.4
The gaussian surface for an infinite uniform line charge is a right circular cylinder of length L and radius ρ. **D** is constant in magnitude and everywhere perpendicular to the cylindrical surface; **D** is parallel to the end faces.

$$Q = \oint_{\text{cyl}} \mathbf{D}_S \cdot d\mathbf{S} = D_S \int_{\text{sides}} dS + 0 \int_{\text{top}} dS + 0 \int_{\text{bottom}} dS$$

$$= D_S \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz = D_S 2\pi\rho L$$

and obtain
$$D_S = D_\rho = \frac{Q}{2\pi\rho L}$$

In terms of the charge density ρ_L, the total charge enclosed is

$$Q = \rho_L L$$

giving
$$D_\rho = \frac{\rho_L}{2\pi\rho}$$

or
$$E_\rho = \frac{\rho_L}{2\pi\epsilon_0\rho}$$

Comparison with Sec. 2.4, Eq. (20), shows that the correct result has been obtained and with much less work. Once the appropriate surface has been chosen, the integration usually amounts only to writing down the area of the surface at which **D** is normal.

The problem of a coaxial cable is almost identical with that of the line charge and is an example which is extremely difficult to solve from the standpoint of Coulomb's law. Suppose that we have two coaxial cylindrical conductors, the inner of radius a and the outer of radius b, each infinite in extent (Fig. 3.5). We shall assume a charge distribution of ρ_S on the outer surface of the inner conductor.

Symmetry considerations show us that only the D_ρ component is present and that it can be a function only of ρ. A right circular cylinder of length L and

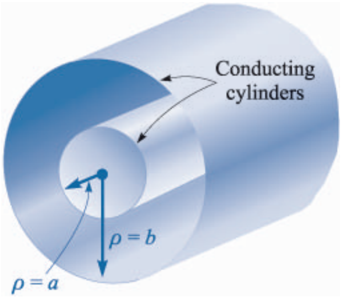

FIGURE 3.5
The two coaxial cylindrical conductors forming a coaxial cable provide an electric flux density within the cylinders, given by $D_\rho = a\rho_S/\rho$.

radius ρ, where $a < \rho < b$, is necessarily chosen as the gaussian surface, and we quickly have

$$Q = D_S 2\pi\rho L$$

The total charge on a length L of the inner conductor is

$$Q = \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho_S a \, d\phi \, dz = 2\pi a L \rho_S$$

from which we have

$$D_S = \frac{a\rho_S}{\rho} \qquad \mathbf{D} = \frac{a\rho_S}{\rho}\mathbf{a}_\rho \qquad (a < \rho < b)$$

This result might be expressed in terms of charge per unit length, because the inner conductor has $2\pi a\rho_S$ coulombs on a meter length, and hence, letting $\rho_L = 2\pi a \rho_S$,

$$\boxed{\mathbf{D} = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho}$$

and the solution has a form identical with that of the infinite line charge.

Since every line of electric flux starting from the charge on the inner cylinder must terminate on a negative charge on the inner surface of the outer cylinder, the total charge on that surface must be

$$Q_{\text{outer cyl}} = -2\pi a L \rho_{S,\text{inner cyl}}$$

and the surface charge on the outer cylinder is found as

$$2\pi b L \rho_{S,\text{outer cyl}} = -2\pi a L \rho_{S,\text{inner cyl}}$$

or

$$\rho_{S,\text{outer cyl}} = -\frac{a}{b}\rho_{S,\text{inner cyl}}$$

What would happen if we should use a cylinder of radius ρ, $\rho > b$, for the gaussian surface? The total charge enclosed would then be zero, for there are equal and opposite charges on each conducting cylinder. Hence

$$0 = D_S 2\pi\rho L \qquad (\rho > b)$$
$$D_S = 0 \qquad (\rho > b)$$

An identical result would be obtained for $\rho < a$. Thus the coaxial cable or capacitor has no external field (we have proved that the outer conductor is a "shield"), and there is no field within the center conductor.

Our result is also useful for a *finite* length of coaxial cable, open at both ends, provided the length L is many times greater than the radius b so that the unsymmetrical conditions at the two ends do not appreciably affect the solution. Such a device is also termed a *coaxial capacitor*. Both the coaxial cable and the coaxial capacitor will appear frequently in the work that follows.

Perhaps a numerical example can illuminate some of these results.

Example 3.2

Let us select a 50-cm length of coaxial cable having an inner radius of 1 mm and an outer radius of 4 mm. The space between conductors is assumed to be filled with air. The total charge on the inner conductor is 30 nC. We wish to know the charge density on each conductor, and the **E** and **D** fields.

Solution. We begin by finding the surface charge density on the inner cylinder,

$$\rho_{S,\text{inner cyl}} = \frac{Q_{\text{inner cyl}}}{2\pi a L} = \frac{30 \times 10^{-9}}{2\pi(10^{-3})(0.5)} = 9.55 \quad \mu\text{C/m}^2$$

The negative charge density on the inner surface of the outer cylinder is

$$\rho_{S,\text{outer cyl}} = \frac{Q_{\text{outer cyl}}}{2\pi b L} = \frac{-30 \times 10^{-9}}{2\pi(4 \times 10^{-3})(0.5)} = -2.39 \quad \mu\text{C/m}^2$$

The internal fields may therefore be calculated easily:

$$D_\rho = \frac{a\rho_S}{\rho} = \frac{10^{-3}(9.55 \times 10^{-6})}{\rho} = \frac{9.55}{\rho} \quad \text{nC/m}^2$$

and

$$E_\rho = \frac{D_\rho}{\epsilon_0} = \frac{9.55 \times 10^{-9}}{8.854 \times 10^{-12}\rho} = \frac{1079}{\rho} \quad \text{V/m}$$

Both of these expressions apply to the region where $1 < \rho < 4$ mm. For $\rho < 1$ mm or $\rho > 4$ mm, **E** and **D** are zero.

✔ **D3.5.** A point charge of 0.25 μC is located at $r = 0$, and uniform surface charge densities are located as follows: 2 mC/m^2 at $r = 1$ cm, and -0.6 mC/m^2 at $r = 1.8$ cm. Calculate **D** at: (*a*) $r = 0.5$ cm; (*b*) $r = 1.5$ cm; (*c*) $r = 2.5$ cm. (*d*) What uniform surface charge density should be established at $r = 3$ cm to cause $\mathbf{D} = 0$ at $r = 3.5$ cm?

Ans. $796\mathbf{a}_r$ μC/m^2; $977\mathbf{a}_r$ μC/m^2; $40.8\mathbf{a}_r$ μC/m^2; -28.3 μC/m^2

3.4 APPLICATION OF GAUSS'S LAW: DIFFERENTIAL VOLUME ELEMENT

We are now going to apply the methods of Guass's law to a slightly different type of problem—one which does not possess any symmetry at all. At first glance it might seem that our case is hopeless, for without symmetry a simple gaussian surface cannot be chosen such that the normal component of **D** is constant or zero everywhere on the surface. Without such a surface, the integral cannot be evaluated. There is only one way to circumvent these difficulties, and that is to choose such a very small closed surface that **D** is *almost* constant over the surface, and the small change in **D** may be adequately represented by using the first two terms of the Taylor's-series expansion for **D**. The result will become more nearly correct as the volume enclosed by the gaussian surface decreases, and we intend eventually to allow this volume to approach zero.

This example also differs from the preceding ones in that we shall not obtain the value of **D** as our answer, but instead receive some extremely valuable information about the way **D** varies in the region of our small surface. This leads directly to one of Maxwell's four equations, which are basic to all electromagnetic theory.

Let us consider any point P, shown in Fig. 3.6, located by a cartesian coordinate system. The value of **D** at the point P may be expressed in cartesian components, $\mathbf{D}_0 = D_{x0}\mathbf{a}_x + D_{y0}\mathbf{a}_y + D_{z0}\mathbf{a}_z$. We choose as our closed surface the small rectangular box, centered at P, having sides of lengths Δx, Δy, and Δz, and apply Gauss's law,

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = Q}$$

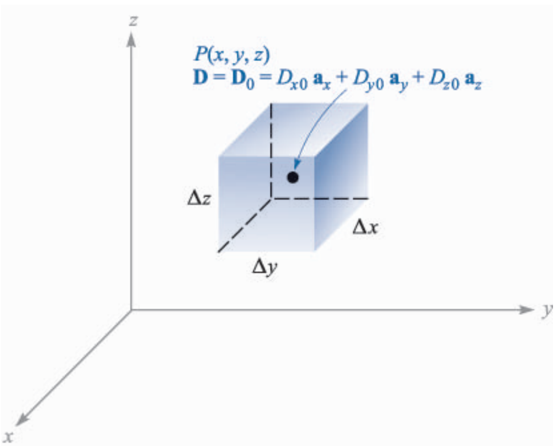

FIGURE 3.6
A differential-sized gaussian surface about the point P is used to investigate the space rate of change of **D** in the neighborhood of P.

In order to evaluate the integral over the closed surface, the integral must be broken up into six integrals, one over each face,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

Consider the first of these in detail. Since the surface element is very small, **D** is essentially constant (over *this* portion of the entire closed surface) and

$$\begin{aligned}\int_{\text{front}} &\doteq \mathbf{D}_{\text{front}} \cdot \Delta\mathbf{S}_{\text{front}} \\ &\doteq \mathbf{D}_{\text{front}} \cdot \Delta y\, \Delta z\, \mathbf{a}_x \\ &\doteq D_{\text{x,front}} \Delta y\, \Delta z\end{aligned}$$

where we have only to approximate the value of D_x at this front face. The front face is at a distance of $\Delta x/2$ from P, and hence

$$\begin{aligned}D_{x,\text{front}} &\doteq D_{x0} + \frac{\Delta x}{2} \times \text{ rate of change of } D_x \text{ with } x \\ &\doteq D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\end{aligned}$$

where D_{x0} is the value of D_x at P, and where a partial derivative must be used to express the rate of change of D_x with x, since D_x in general also varies with y and z. This expression could have been obtained more formally by using the constant term and the term involving the first derivative in the Taylor's-series expansion for D_x in the neighborhood of P.

We have now

$$\int_{\text{front}} \doteq \left(D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y\, \Delta z$$

Consider now the integral over the back surface,

$$\begin{aligned}\int_{\text{back}} &\doteq \mathbf{D}_{\text{back}} \cdot \Delta\mathbf{S}_{\text{back}} \\ &\doteq \mathbf{D}_{\text{back}} \cdot (-\Delta y\, \Delta z\, \mathbf{a}_x) \\ &\doteq -D_{x,\text{back}} \Delta y\, \Delta z\end{aligned}$$

and

$$D_{x,\text{back}} \doteq D_{x0} - \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}$$

giving

$$\int_{\text{back}} \doteq \left(-D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y\, \Delta z$$

If we combine these two integrals, we have

$$\int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x} \Delta x \, \Delta y \, \Delta z$$

By exactly the same process we find that

$$\int_{\text{right}} + \int_{\text{left}} \doteq \frac{\partial D_y}{\partial y} \Delta x \, \Delta y \, \Delta z$$

and

$$\int_{\text{top}} + \int_{\text{bottom}} \doteq \frac{\partial D_z}{\partial z} \Delta x \, \Delta y \, \Delta z$$

and these results may be collected to yield

$$\oint_S \mathbf{D} \cdot d\mathbf{S} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \, \Delta y \, \Delta z$$

or

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v \tag{7}$$

The expression is an approximation which becomes better as Δv becomes smaller, and in the following section we shall let the volume Δv approach zero. For the moment, we have applied Gauss's law to the closed surface surrounding the volume element Δv and have as a result the approximation (7) stating that

$$\boxed{\text{Charge enclosed in volume } \Delta v \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \times \text{volume } \Delta v} \tag{8}$$

Example 3.3

Find an approximate value for the total charge enclosed in an incremental volume of $10^{-9}\ \text{m}^3$ located at the origin, if $\mathbf{D} = e^{-x} \sin y \ \mathbf{a}_x - e^{-x} \cos y \ \mathbf{a}_y + 2z\mathbf{a}_z \ \text{C/m}^2$.

Solution. We first evaluate the three partial derivatives in (8):

$$\frac{\partial D_x}{\partial x} = -e^{-x} \sin y$$

$$\frac{\partial D_y}{\partial y} = e^{-x} \sin y$$

$$\frac{\partial D_z}{\partial z} = 2$$

At the origin, the first two expressions are zero, and the last is 2. Thus, we find that the charge enclosed in a small volume element there must be approximately $2\Delta v$. If Δv is $10^{-9}\ \text{m}^3$, then we have enclosed about 2 nC.

✔ **D3.6.** In free space, let $\mathbf{D} = 8xyz^4\mathbf{a}_x + 4x^2z^4\mathbf{a}_y + 16x^2yz^3$ pC/m^2. (*a*) Find the total electric flux passing through the rectangular surface $z = 2$, $0 < x < 2$, $1 < y < 3$, in the $\mathbf{a}_z$ direction. (*b*) Find $\mathbf{E}$ at $P(2, -1, 3)$. (*c*) Find an approximate value for the total charge contained in an incremental sphere located at $P(2, -1, 3)$ and having a volume of 10^{-12} m^3.

Ans. 1365 pC; $-146.4\mathbf{a}_x + 146.4\mathbf{a}_y - 195.2\mathbf{a}_z$ V/m; -2.38×10^{-21} C

3.5 DIVERGENCE

We shall now obtain an exact relationship from (7), by allowing the volume element Δv to shrink to zero. We write this equation as

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \doteq \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \frac{Q}{\Delta v}$$

or, as a limit

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v}$$

where the approximation has been replaced by an equality. It is evident that the last term is the volume charge density ρ_v, and hence that

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \rho_v \tag{9}$$

This equation contains too much information to discuss all at once, and we shall write it as two separate equations,

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} \tag{10}$$

and

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \rho_v \tag{11}$$

where we shall save (11) for consideration in the next section.

Equation (10) does not involve charge density, and the methods of the previous section could have been used on any vector $\mathbf{A}$ to find $\oint_S \mathbf{A} \cdot d\mathbf{S}$ for a small closed surface, leading to

$$\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} \tag{12}$$

where $\mathbf{A}$ could represent velocity, temperature gradient, force, or any other vector field.

This operation appeared so many times in physical investigations in the last century that it received a descriptive name, *divergence*. The divergence of $\mathbf{A}$ is defined as

$$\boxed{\text{Divergence of } \mathbf{A} = \text{div } \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}} \tag{13}$$

and is usually abbreviated div **A**. The physical interpretation of the divergence of a vector is obtained by describing carefully the operations implied by the right-hand side of (13), where we shall consider **A** as a member of the flux-density family of vectors in order to aid the physical interpretation.

> The divergence of the vector flux density **A** is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.

The physical interpretation of divergence afforded by this statement is often useful in obtaining qualitative information about the divergence of a vector field without resorting to a mathematical investigation. For instance, let us consider the divergence of the velocity of water in a bathtub after the drain has been opened. The net outflow of water through *any* closed surface lying entirely within the water must be zero, for water is essentially incompressible and the water entering and leaving different regions of the closed surface must be equal. Hence the divergence of this velocity is zero.

If, however, we consider the velocity of the air in a tire which has just been punctured by a nail, we realize that the air is expanding as the pressure drops, and that consequently there is a net outflow from any closed surface lying within the tire. The divergence of this velocity is therefore greater than zero.

A positive divergence for any vector quantity indicates a *source* of that vector quantity at that point. Similarly, a negative divergence indicates a *sink*. Since the divergence of the water velocity above is zero, no source or sink exists.[3] The expanding air, however, produces a positive divergence of the velocity, and each interior point may be considered a source.

Writing (10) with our new term, we have

$$\boxed{\text{div } \mathbf{D} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right)} \tag{14}$$

This expression is again of a form which does not involve the charge density. It is the result of applying the definition of divergence (13) to a differential volume element in *cartesian coordinates*.

If a differential volume unit $\rho\, d\rho\, d\phi\, dz$ in cylindrical coordinates, or $r^2 \sin\theta\, dr\, d\theta\, d\phi$ in spherical coordinates, had been chosen, expressions for diver-

[3] Having chosen a differential element of volume within the water, the gradual decrease in water level with time will eventually cause the volume element to lie above the surface of the water. At the instant the surface of the water intersects the volume element, the divergence is positive and the small volume is a source. This complication is avoided above by specifying an integral point.

gence involving the components of the vector in the particular coordinate system and involving partial derivatives with respect to the variables of that system would have been obtained. These expressions are obtained in Appendix A and are given here for convenience:

$$\boxed{\text{div }\mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \qquad \text{(cartesian)}} \tag{15}$$

$$\boxed{\text{div }\mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \qquad \text{(cylindrical)}} \tag{16}$$

$$\boxed{\text{div }\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi} \qquad \text{(spherical)}} \tag{17}$$

These relationships are also shown inside the back cover for easy reference.

It should be noted that the divergence is an operation which is performed on a vector, but that the result is a scalar. We should recall that, in a somewhat similar way, the dot, or scalar, product was a multiplication of two vectors which yielded a scalar product.

For some reason it is a common mistake on meeting divergence for the first time to impart a vector quality to the operation by scattering unit vectors around in the partial derivatives. Divergence merely tells us *how much* flux is leaving a small volume on a per-unit-volume basis; no direction is associated with it.

We can illustrate the concept of divergence by continuing with the example at the end of the previous section.

Example 3.4

Find div $\mathbf{D}$ at the origin if $\mathbf{D} = e^{-x}\sin y\,\mathbf{a}_x - e^{-x}\cos y\,\mathbf{a}_y + 2z\mathbf{a}_z$.

Solution. We use (14) or (15) to obtain

$$\begin{aligned}\text{div }\mathbf{D} &= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \\ &= -e^{-x}\sin y + e^{-x}\sin y + 2 = 2\end{aligned}$$

The value is the constant 2, regardless of location.

If the units of $\mathbf{D}$ are C/m^2, then the units of div $\mathbf{D}$ are C/m^3. This is a volume charge density, a concept discussed in the next section.

D3.7. In each of the following parts, find a numerical value for div $\mathbf{D}$ at the point specified: (*a*) $\mathbf{D} = (2xyz - y^2)\mathbf{a}_x + (x^2z - 2xy)\mathbf{a}_y + x^2y\mathbf{a}_z$ C/m^2 at $P_A(2, 3, -1)$; (*b*) $\mathbf{D} = 2\rho z^2 \sin^2\phi\,\mathbf{a}_\rho + \rho z^2 \sin 2\phi\,\mathbf{a}_\phi + 2\rho^2 z \sin^2\phi\,\mathbf{a}_z$ C/m^2 at $P_B(\rho = 2, \phi = 110^\circ, z = -1)$; (*c*) $\mathbf{D} = 2r\sin\theta\cos\phi\,\mathbf{a}_r + r\cos\theta\cos\phi\,\mathbf{a}_\theta - r\sin\phi\,\mathbf{a}_\phi$ at $P_C(r = 1.5, \theta = 30^\circ, \phi = 50^\circ)$.

Ans. -10.00; 9.06; 2.18

3.6 MAXWELL'S FIRST EQUATION (ELECTROSTATICS)

We now wish to consolidate the gains of the last two sections and to provide an interpretation of the divergence operation as it relates to electric flux density. The expressions developed there may be written as

$$\text{div}\,\mathbf{D} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} \tag{18}$$

$$\text{div}\,\mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \tag{19}$$

and

$$\text{div}\,\mathbf{D} = \rho_v \tag{20}$$

The first equation is the definition of divergence, the second is the result of applying the definition to a differential volume element in cartesian coordinates, giving us an equation by which the divergence of a vector expressed in cartesian coordinates may be evaluated, and the third is merely (11) written using the new term div $\mathbf{D}$. Equation (20) is almost an obvious result if we have achieved any familiarity at all with the concept of divergence as defined by (18), for given Gauss's law,

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = Q$$

per unit volume

$$\frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} = \frac{Q}{\Delta v}$$

As the volume shrinks to zero,

$$\lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v}$$

we should see div $\mathbf{D}$ on the left and volume charge density on the right,

$$\boxed{\text{div}\,\mathbf{D} = \rho_v} \tag{20}$$

This is the first of Maxwell's four equations as they apply to electrostatics and steady magnetic fields, and it states that the electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there. This equation is aptly called the *point form of Gauss's law*. Gauss's law relates the flux leaving any closed surface to the charge enclosed, and Maxwell's first equation makes an identical statement on a per-unit-volume basis for a vanishingly small volume, or at a point. Remembering that the divergence may be expressed as the sum of three partial derviatves, Maxwell's first equation is also described as the differential-equation form of Gauss's law, and conversely, Gauss's law is recognized as the integral form of Maxwell's first equation.

As a specific illustration, let us consider the divergence of **D** in the region about a point charge Q located at the origin. We have the field

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

and make use of (17), the expression for divergence in spherical coordinates given in the previous section:

$$\text{div } \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(D_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}$$

Since D_θ and D_ϕ are zero, we have

$$\text{div } \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{Q}{4\pi r^2}\right) = 0 \qquad (\text{if } r \neq 0)$$

Thus, $\rho_v = 0$ everywhere except at the origin where it is infinite.

The divergence operation is not limited to electric flux density; it can be applied to any vector field. We shall apply it to several other electromagnetic fields in the coming chapters.

✔ **D3.8.** Determine an expression for the volume charge density associated with each **D** field following: (*a*) $\mathbf{D} = \frac{4xy}{z}\mathbf{a}_x + \frac{2x^2}{z}\mathbf{a}_y - \frac{2x^2y}{z^2}\mathbf{a}_z$; (*b*) $\mathbf{D} = z\sin\phi\,\mathbf{a}_\rho + z\cos\phi\,\mathbf{a}_\phi + \rho\sin\phi\,\mathbf{a}_z$; (*c*) $\mathbf{D} = \sin\theta\sin\phi\,\mathbf{a}_r + \cos\theta\sin\phi\,\mathbf{a}_\theta + \cos\phi\,\mathbf{a}_\phi$.

Ans. $\frac{4y}{z^3}(x^2 + z^2)$; 0; 0.

3.7 THE VECTOR OPERATOR ∇ AND THE DIVERGENCE THEOREM

If we remind ourselves again that divergence is an operation on a vector yielding a scalar result, just as the dot product of two vectors gives a scalar result, it seems possible that we can find something which may be dotted formally with **D** to yield the scalar

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Obviously, this cannot be accomplished by using a dot *product*; the process must be a dot *operation*.

With this in mind, we define the *del operator* ∇ as a *vector operator*,

$$\boxed{\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z} \tag{21}$$

Similar *scalar operators* appear in several methods of solving differential equations where we often let D replace d/dx, D^2 replace d^2/dx^2, and so forth.[4] We agree on defining ∇ (pronounced "del") that it shall be treated in every way as an ordinary vector with the one important exception that partial derivatives result instead of products of scalars.

Consider $\nabla \cdot \mathbf{D}$, signifying

$$\nabla \cdot \mathbf{D} = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right) \cdot (D_x\mathbf{a}_x + D_y\mathbf{a}_y + D_z\mathbf{a}_z)$$

We first consider the dot products of the unit vectors, discarding the six zero terms and having left

$$\nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(D_x) + \frac{\partial}{\partial y}(D_y) + \frac{\partial}{\partial z}(D_z)$$

where the parentheses are now removed by operating or differentiating:

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

This is recognized as the divergence of $\mathbf{D}$, so that we have

$$\boxed{\text{div } \mathbf{D} = \nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}}$$

The use of $\nabla \cdot \mathbf{D}$ is much more prevalent than that of div $\mathbf{D}$, although both usages have their advantages. Writing $\nabla \cdot \mathbf{D}$ allows us to obtain simply and quickly the correct partial derivatives, but only in cartesian coordinates, as we shall see below. On the other hand, div $\mathbf{D}$ is an excellent reminder of the physical interpretation of divergence. We shall use the operator notation $\nabla \cdot \mathbf{D}$ from now on to indicate the divergence operation.

The vector operator ∇ is used not only with divergence, but willl appear in several other very important operations later. One of these is ∇u, where u is any scalar field, and leads to

---

[4] This scalar operator D, which will not appear again, is not to be confused with the electric flux density.

$$\nabla u = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z \right) u = \frac{\partial u}{\partial x}\mathbf{a}_x + \frac{\partial u}{\partial y}\mathbf{a}_y + \frac{\partial u}{\partial z}\mathbf{a}_z$$

The ∇ operator does not have a specific form in other coordinate systems. If we are considering $\mathbf{D}$ in cylindrical coordinates, then $\nabla \cdot \mathbf{D}$ still indicates the divergence of $\mathbf{D}$, or

$$\nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

where this expression has been taken from Sec. 3.5. We have no form for ∇ itself to help us obtain this sum of partial derivatives. This means that ∇u, as yet unnamed but easily written above in cartesian coordinates, cannot be expressed by us at this time in cylindrical coordinates. Such an expression will be obtained when ∇u is defined in Chap. 4.

We shall close our discussion of divergence by presenting a theorem which will be needed several times in later chapters, the *divergence theorem*. This theorem applies to any vector field for which the appropriate partial derivatives exist, although it is easiest for us to develop it for the electric flux density. We have actually obtained it already and now have little more to do than point it out and name it, for starting from Gauss's law,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

and letting

$$Q = \int_{\text{vol}} \rho_v dv$$

and then replacing ρ_v by its equal,

$$\nabla \cdot \mathbf{D} = \rho_v$$

we have
$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_v dv = \int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv$$

The first and last expressions constitute the divergence theorem,

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv} \tag{22}$$

which may be stated as follows:

> The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.

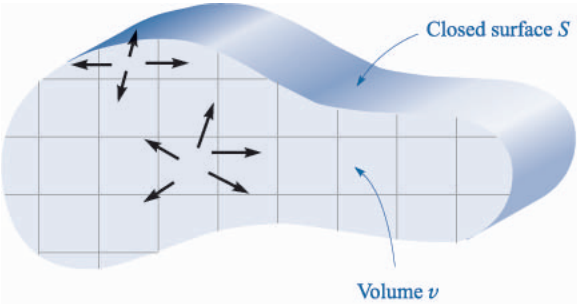

FIGURE 3.7
The divergence theorem states that the total flux crossing the closed surface is equal to the integral of the divergence of the flux density throughout the enclosed volume. The volume is shown here in cross section.

Again, we emphasize that the divergence theorem is true for any vector field, although we have obtained it specifically for the electric flux density **D**, and we shall have occasion later to apply it to several different fields. Its benefits derive from the fact that it relates a triple integration *throughout some volume* to a double integration *over the surface* of that volume. For example, it is much easier to look for leaks in a bottle full of some agitated liquid by an inspection of the surface than by calculating the velocity at every internal point.

The divergence theorem becomes obvious physically if we consider a volume v, shown in cross section in Fig. 3.7, which is surrounded by a closed surface S. Division of the volume into a number of small compartments of differential size and consideration of one cell show that the flux diverging from such a cell *enters*, or *converges* on, the adjacent cells unless the cell contains a portion of the outer surface. In summary, the divergence of the flux density throughout a volume leads, then, to the same result as determining the net flux crossing the enclosing surface.

Let us consider an example to illustrate the divergence theorem.

Example 3.5

Evaluate both sides of the divergence theorem for the field $\mathbf{D} = 2xy\mathbf{a}_x + x^2\mathbf{a}_y$ C/m^2 and the rectangular parellelepiped formed by the planes $x = 0$ and 1, $y = 0$ and 2, and $z = 0$ and 3.

Solution. Evaluating the surface integral first, we note that **D** is parallel to the surfaces at $z = 0$ and $z = 3$, so $\mathbf{D} \cdot d\mathbf{S} = 0$ there. For the remaining four surfaces we have

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_0^3 \int_0^2 (\mathbf{D})_{x=0} \cdot (-dy\,dz\,\mathbf{a}_x) + \int_0^3 \int_0^2 (\mathbf{D})_{x=1} \cdot (dy\,dz\,\mathbf{a}_x)$$
$$+ \int_0^3 \int_0^1 (\mathbf{D})_{y=0} \cdot (-dx\,dz\,\mathbf{a}_y) + \int_0^3 \int_0^1 (\mathbf{D})_{y=2} \cdot (dx\,dz\,\mathbf{a}_y)$$

$$= -\int_0^3\int_0^2 (D_x)_{x=0}dy\,dz + \int_0^3\int_0^2 (D_x)_{x=1}dy\,dz$$
$$-\int_0^3\int_0^1 (D_y)_{y=0}dx\,dz + \int_0^3\int_0^1 (D_y)_{y=2}dx\,dz$$

However, $(D_x)_{x=0} = 0$, and $(D_y)_{y=0} = (D_y)_{y=2}$, which leaves only

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_0^3\int_0^2 (D_x)_{x=1}dy\,dz = \int_0^3\int_0^2 2y\,dy\,dz$$
$$= \int_0^3 4\,dz = 12$$

Since
$$\nabla\cdot\mathbf{D} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) = 2y$$

the volume integral becomes

$$\int_{\text{vol}} \nabla\cdot\mathbf{D}\,dv = \int_0^3\int_0^2\int_0^1 2y\,dx\,dy\,dz = \int_0^3\int_0^2 2y\,dy\,dz$$
$$= \int_0^3 4\,dz = 12$$

and the check is accomplished. Remembering Gauss's law, we see that we have also determined that a total charge of 12 C lies within this parallelepiped.

✔ **D3.9.** Given the field $\mathbf{D} = 6\rho\sin\frac{1}{2}\phi\,\mathbf{a}_\rho + 1.5\rho\cos\frac{1}{2}\phi\,\mathbf{a}_\phi$ C/m^2, evaluate both sides of the divergence theorem for the region bounded by $\rho = 2$, $\phi = 0$, $\phi = \pi$, $z = 0$, and $z = 5$.

Ans. 225; 225

SUGGESTED REFERENCES

1. Kraus, J. D. and D. A. Fleisch: "Electromagnetics," 5th ed., McGraw-Hill Book Company, New York, 1999. The static electric field in free space is introduced in chap. 2.
2. Plonsey, R., and R. E. Collin: "Principles and Applications of Electromagnetic Fields," McGraw-Hill Book Company, New York, 1961. The level of this text is somewhat higher than the one we are reading now, but it is an excellent text to read next. Gauss's law appears in the second chapter.
3. Plonus, M. A.: "Applied Electromagnetics," McGraw-Hill Book Company, New York, 1978. This book contains rather detailed descriptions of many practical devices that illustrate electromagnetic applications. For example, see the discussion of xerography on pp. 95–98 as an electrostatics application.
4. Skilling, H. H.: "Fundamentals of Electric Waves," 2d ed., John Wiley & Sons, Inc., New York, 1948. The operations of vector calculus are well

illustrated. Divergence is discussed on pp. 22 and 38. Chapter 1 is interesting reading.

5. Thomas, G. B., Jr., and R. L. Finney: (see Suggested References for Chap. 1). The divergence theorem is developed and illustrated from several different points of vew on pp. 976–980.

PROBLEMS

3.1 An empty metal paint can is placed on a marble table, the lid is removed, and both parts are discharged (honorably) by touching them to ground. An insulating nylon thread is glued to the center of the lid, and a penny, a nickel, and a dime are glued to the thread so that they are not touching each other. The penny is given a charge of +5 nC, and the nickel and dime are discharged. The assembly is lowered into the can so that the coins hang clear of all walls, and the lid is secured. The outside of the can is again touched momentarily to ground. The device is carefully disassembled with insulating gloves and tools. (*a*) What charges are found on each of the five metallic pieces? (*b*) If the penny had been given a charge of +5 nC, the dime a charge of −2 nC, and the nickel a charge of −1 nC, what would the final charge arrangement have been?

3.2 A point charge of 12 nC is located at the origin. Four uniform line charges are located in the $x = 0$ plane as follows: 80 nC/m at $y = -1$ and −5 m, −50 nC/m at $y = -2$ and −4 m. (*a*) Find **D** at $P(0, -3, 2)$. (*b*) How much electric flux crosses the plane $y = -3$, and in what direction? (*c*) How much electric flux leaves the surface of a sphere, 4 m in radius, centered at $C(0, -3, 0)$?

3.3 The cylindrical surface $\rho = 8$ cm contains the surface charge density, $\rho_S = 5e^{-20|z|}$ nC/m^2. (*a*) What is the total amount of charge present? (*b*) How much electric flux leaves the surface $\rho = 8$ cm, 1 cm $< z <$ 5 cm, $30^\circ < \phi < 90^\circ$?

3.4 The cylindrical surfaces $\rho = 1$, 2, and 3 cm carry uniform surface charge densities of 20, −8, and 5 nC/m^2, respectively. (*a*) How much electric flux passes through the closed surface $\rho = 5$ cm, $0 < z < 1$ m? (*b*) Find **D** at P(1 cm, 2 cm, 3 cm).

3.5 Let $\mathbf{D} = 4xy\mathbf{a}_x + 2(x^2 + z^2)\mathbf{a}_y + 4yz\mathbf{a}_z$ C/m^2 and evaluate surface integrals to find the total charge enclosed in the rectangular parallelepiped $0 < x < 2$, $0 < y < 3$, $0 < z < 5$ m.

3.6 Two uniform line charges, each 20 nC/m, are located at $y = 1$, $z = \pm 1$ m. Find the total electric flux leaving the surface of a sphere having a radius of 2 m, if it is centered at: (*a*) $A(3, 1, 0)$; (*b*) $B(3, 2, 0)$.

3.7 Volume charge density is located in free space as $\rho_v = 2e^{-1000r}$ nC/m^3 for $0 < r < 1$ mm, and $\rho_v = 0$ elsewhere. (*a*) Find the total charge enclosed by the spherical surface $r = 1$ mm. (*b*) By using Gauss's law, calculate the value of D_r on the surface $r = 1$ mm.

3.8 Uniform line charges of 5 nC/m are located in free space at $x = 1$, $z = 1$, and at $y = 1$, $z = 0$. (*a*) Obtain an expression for $\mathbf{D}$ in cartesian coordinates at $P(0, 0, z)$. (*b*) Plot $|\mathbf{D}|$ versus z at P, $-3 < z < 10$.

3.9 A uniform volume charge density of 80 μC/m^3 is present throughout the region 8 mm $< r <$ 10 mm. Let $\rho_v = 0$ for $0 < r < 8$ mm. (*a*) Find the total charge inside the spherical surface $r = 10$ mm. (*b*) Find D_r at $r = 10$ mm. (*c*) If there is no charge for $r > 10$ mm, find D_r at $r = 20$ mm.

3.10 Let $\rho_S = 8\,\mu\text{C/m}^2$ in the region where $x = 0$ and $-4 < z < 4$ m, and let $\rho_S = 0$ elsewhere. Find $\mathbf{D}$ at $P(x, 0, z)$, where $x > 0$.

3.11 In cylindrical coordinates, let $\rho_v = 0$ for $\rho < 1$ mm, $\rho_v = 2 \sin 2000 \pi\rho$ nC/m^3 for 1 mm $< \rho <$ 1.5 mm, and $\rho_v = 0$ for $\rho > 1.5$ mm. Find $\mathbf{D}$ everywhere.

3.12 A nonuniform volume charge density, $\rho_v = 120r$ C/m^3, lies within the spherical surface $r = 1$ m, and $\rho_v = 0$ elsewhere. (*a*) Find D_r everywhere. (*b*) What surface charge density ρ_{S2} should be on the surface $r = 2$ so that $D_{r|r=2^-} = 2D_{r|r=2^+}$? (*c*) Make a sketch of D_r vs r for $0\ r < 5$ with both distributions present.

3.13 Spherical surfaces at $r = 2$, 4, and 6 m carry uniform surface charge densities of 20 nC/m^2, -4 nC/m^2, and ρ_{S0}, respectively. (*a*) Find $\mathbf{D}$ at $r = 1$, 3, and 5 m. (*b*) Determine ρ_{S0} such that $\mathbf{D} = 0$ at $r = 7$ m.

3.14 If $\rho_v = 5$ nC/m^3 for $0 < \rho < 1$ mm and no other charges are present: (*a*) find D_ρ for $\rho < 1$ mm; (*b*) find D_ρ for $\rho > 1$ mm. (*c*) What line charge ρ_L at $\rho = 0$ would give the same result for part *b*?

3.15 Volume charge density is located as follows: $\rho_v = 0$ for $\rho < 1$ mm and for $\rho > 2$ mm, $\rho_v = 4\rho\,\mu\text{C/m}^3$ for $1 < \rho < 2$ mm. (*a*) Calculate the total charge in the region, $0 < \rho < \rho_1$, $0 < z < L$, where $1 < \rho_1 < 2$ mm. (*b*) Use Gauss's law to determine D_ρ at $\rho = \rho_1$. (*c*) Evaluate D_ρ at $\rho = 0.8$ mm, 1.6 mm, and 2.4 mm.

3.16 Given the electric flux density, $\mathbf{D} = 2xy\mathbf{a}_x + x^2\mathbf{a}_y + 6z^3\mathbf{a}_z$ C/m^2: (*a*) use Gauss's law to evaluate the total charge enclosed in the volume $0 < x, y, z < a$; (*b*) use Eq. (8) to find an approximate value for the above charge. Evaluate the derivatives at $P(a/2, a/2, a/2)$. (*c*) Show that the results of parts *a* and *b* agree in the limit as $a \to 0$.

3.17 A cube is defined by $1 < x, y, z < 1.2$. If $\mathbf{D} = 2x^2y\mathbf{a}_x + 3x^2y^2\mathbf{a}_y$ C/m^2: (*a*) apply Gauss's law to find the total flux leaving the closed surface of the cube; (*b*) evaluate $\dfrac{\partial D_x}{\partial x} + \dfrac{\partial D_y}{\partial y} + \dfrac{\partial D_z}{\partial z}$ at the center of the cube. (*c*) Estimate the total charge enclosed within the cube by using Eq. (8).

3.18 Let a vector field be given by $\mathbf{G} = 5x^4y^4z^4\mathbf{a}_y$. Evaluate both sides of Eq. (8) for this $\mathbf{G}$ field and the volume defined by $x = 3$ and 3.1, $y = 1$ and 1.1, and $z = 2$ and 2.1. Evaluate the partial derivatives at the center of the volume.

3.19 A spherical surface of radius 3 mm is centered at $P(4, 1, 5)$ in free space. Let $\mathbf{D} = x\mathbf{a}_x$ C/m^2. Use the results of Sec. 3.4 to estimate the net electric flux leaving the spherical surface.

3.20 A cube of volume a^3 has its faces parallel to the cartesian coordinate surfaces. It is centered at $P(3, -2, 4)$. Given the field $\mathbf{D} = 2x^3\mathbf{a}_x$ C/m^2: (*a*) calculate div $\mathbf{D}$ at P; (*b*) evaluate the fraction in the rightmost side of Eq. (13) for $a = 1$ m, 0.1 m, and 1 mm.

3.21 Calculate the divergence of $\mathbf{D}$ at the point specified if $\mathbf{D} =$: (*a*) $\dfrac{1}{z^2}[10xyz\mathbf{a}_x + 5x^2z\mathbf{a}_y + (2z^3 - 5x^2y)]$ at $P(-2, 3, 5)$; (*b*) $5z^2\mathbf{a}_\rho + 10\rho z\mathbf{a}_z$ at $P(3, -45^\circ, 5)$; (*c*) $2r\sin\theta\sin\phi\mathbf{a}_r + r\cos\theta\sin\phi\mathbf{a}_\theta + r\cos\phi\mathbf{a}_\phi$ at $P(3, 45^\circ, -45^\circ)$.

3.22 Let $\mathbf{D} = 8\rho\sin\phi\mathbf{a}_\rho + 4\rho\cos\phi\mathbf{a}_\phi$C/m^2. (*a*) Find div $\mathbf{D}$. (*b*) Find the volume charge density at $P(2.6, 38^\circ, -6.1)$. (*c*) How much charge is located inside the region defined by $0 < \rho < 1.8$, $20^\circ < \phi < 70^\circ$, $2.4 < z < 3.1$?

3.23 (*a*) A point charge Q lies at the origin. Show that div $\mathbf{D} = 0$ everywhere except at the origin. (*b*) Replace the point charge with a uniform volume charge density ρ_{v0} for $0 \le r \le a$. Relate ρ_{v0} to Q and a so that the total charge is the same. Find div $\mathbf{D}$ everywhere.

3.24 Inside the cylindrical shell, $3 < \rho < 4$ m, the electric flux density is given as $5(\rho - 3)^3\mathbf{a}_\rho$ C/m^2. (*a*) What is the volume charge density at $\rho = 4$ m? (*b*) What is the electric flux density at $\rho = 4$ m? (*c*) How much electric flux leaves the closed surface: $3 < \rho < 4$, $0 < \phi < 2\pi$, $-2.5 < z < 2.5$? (*d*) How much charge is contained within the volume $3 < \rho < 4$, $0 < \phi < 2\pi$, $-2.5 < z < 2.5$?

3.25 Within the spherical shell, $3 < r < 4$ m, the electric flux density is given as $\mathbf{D} = 5(r - 3)^3\mathbf{a}_r$ C/m^2. (*a*) What is the volume charge density at $r = 4$? (*b*) What is the electric flux density at $r = 4$? (*c*) How much electric flux leaves the sphere $r = 4$? (*d*) How much charge is contained within the sphere $r = 4$?

3.26 Given the field, $\mathbf{D} = \dfrac{5\sin\theta\cos\phi}{r}\mathbf{a}_r$ C/m^2, find: (*a*) the volume charge density; (*b*) the total charge contained in the region $r < 2$ m; (*c*) the value of $\mathbf{D}$ at the surface $r = 2$; (*d*) the total electric flux leaving the surface $r = 2$.

3.27 Let $\mathbf{D} = 5r^2\mathbf{a}_r$ mC/m^2 for $r < 0.08$ m, and $\mathbf{D} = 0.1\mathbf{a}_r/r^2$ C/m^2 for $r > 0.08$ m. (*a*) Find ρ_v for $r = 0.06$ m. (*b*) Find ρ_v for $r = 0.1$ m. (*c*) What surface charge density could be located at $r = 0.08$ m to cause $\mathbf{D} = 0$ for $r > 0.08$ m?

3.28 The electric flux density is given as $\mathbf{D} = 20\rho^3\mathbf{a}_\rho$ C/m^2 for $\rho < 100$ µm, and $k\mathbf{a}_\rho/\rho$ for $\rho > 100$ µm. (*a*) Find k so that $\mathbf{D}$ is continuous at $\rho = 100$ µm. (*b*) Find and sketch ρ_v as a function of ρ.

3.29 In the region of free space that includes the volume, $2 < x, y, z < 3$, $\mathbf{D} = \dfrac{2}{z^2}(yz\mathbf{a}_x + xz\mathbf{a}_y - 2xy\mathbf{a}_z)$ C/m^2. (*a*) Evaluate the volume-integral side of the divergence theorem for the volume defined by $2 < x, y, z < 3$. (*b*) Evaluate the surface-integral side for the corresponding closed surface.

3.30 If $\mathbf{D} = 15\rho^2 \sin 2\phi \mathbf{a}_\rho + 10\rho^2 \cos 2\phi \mathbf{a}_\phi$ C/m^2, evaluate both sides of the divergence theorem for the region: $1 < \rho < 2$ m, $1 < \phi < 2$ rad, $1 < z < 2$ m.

3.31 Given the flux density, $\mathbf{D} = \dfrac{16}{r} \cos 2\theta \mathbf{a}_\theta$ C/m^2, use two different methods to find the total charge within the region $1 < r < 2$ m, $1 < \theta < 2$ rad, $1 < \phi < 2$ rad.

3.32 If $\mathbf{D} = 2r\mathbf{a}_r$ C/m^2, find the total electric flux leaving the surface of the cube, $0 \le x, y, z \le 0.4$.