Classification of Materials

- Until now we have considered electric field in vacuum or free space (ε_0). What about E field in other materials?

- Materials can now be broadly classified based on their conductivity σ measured in mhos/meter (℧/m) or Siemens/meter (S/m).

- Conductors are those materials with abundance of free electrons and insulators (or dielectrics) with least number.

- Conductivity of materials changes with temperature and is inversely proportional $\sigma \propto 1/T$ (^{0}K)

- Materials when cooled to very low temperatures ($\approx 0^0$ K) using liquid helium or liquid nitrogen, exhibit perfect conductivity (i.e infinite conductivity). Such materials are called Superconductors

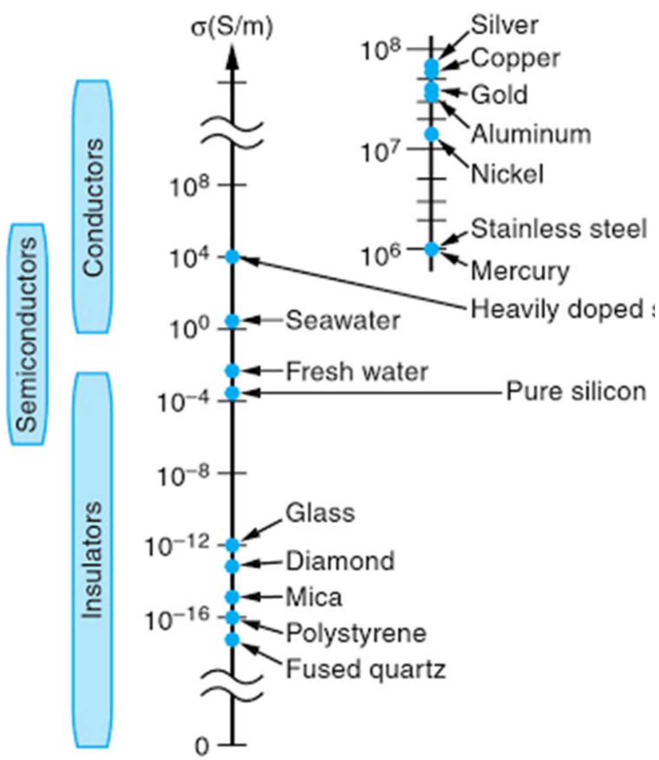
Classification of Materials

- A material with high conductivity ($\sigma >> 1$) is referred to as **metals**.
- A material with small conductivity ($\sigma << 1$) is referred to as insulator (or **dielectric**).
- A material with conductivity lies between those of metals and insulators are called **semiconductors**.

➢ **Copper and aluminium are metals.**

➢ **Silicon and germanium are semiconductor**

➢ **Glass and rubber are insulators.**



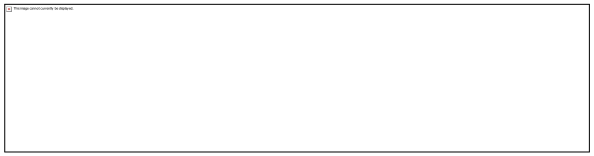
Conductivity Chart (at room temperature)

Current & Current Density

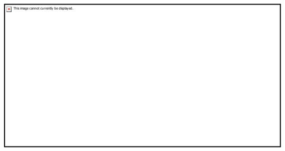
- Current (ampere) can be defined as the flow of electric charge (Coulombs) through a given area per unit time (seconds)

$$I = \frac{Q}{t}$$

- Just as we related charge Q to charge density ρ, we will define current density J as Current per unit surface (Surface Current A/m^2) or Current per unit volume (A/m^3)



The total current flowing through a surface is



- Again there are three types of current (1) Convection Current (2) Conduction Current and (3) Displacement Current

Current & Current Density

Displacement current density is the time varying phenomenon that allows current between plates of the capacitor

Convection current density involves movement of charged particles through vacuum, air or other non conductive media in response to applied electric field $\boxed{J = \rho_v v \left(A/m^2\right)}$

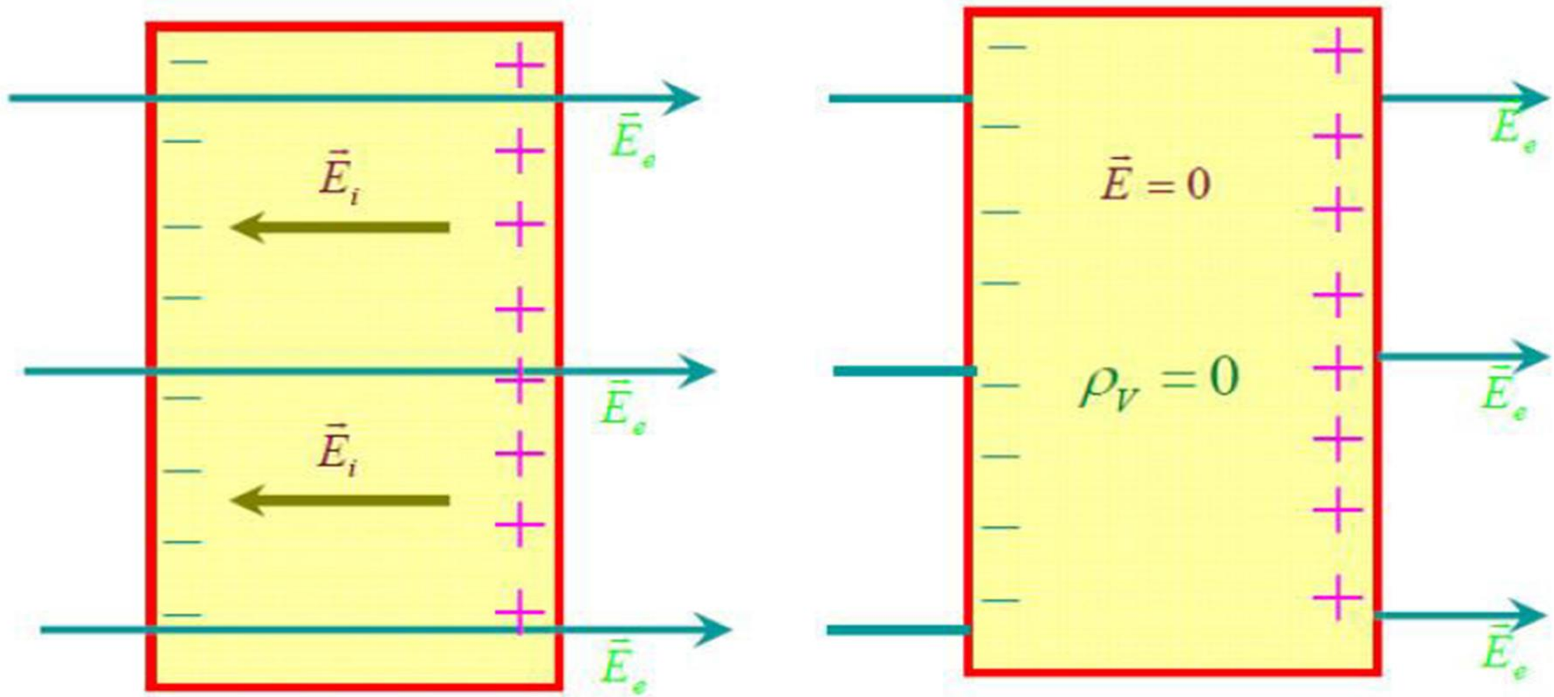
Eg- beam of electrons in vacuum tubes.

Conduction current involves movement of charged particles through conductive media in response to applied field

$\boxed{J = \sigma E}$ ← Ohm's Law

Conductor

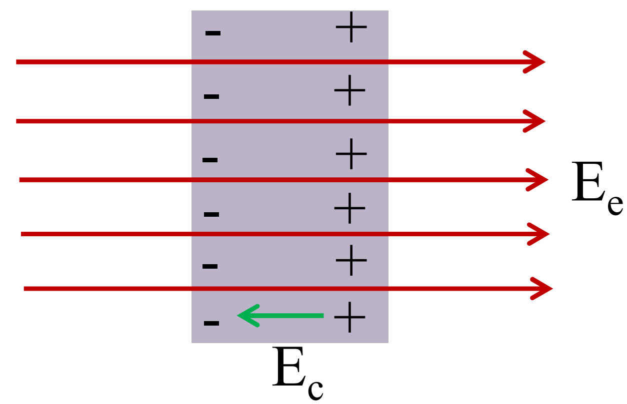
- A conductor has abundance of charge that is free to move.
- **A perfect conductor ($\sigma=\infty$) can not contain an electrostatic field within it.**



Isolated Conductor

Electric Field Inside a Conductor

- Inside a conductor $E_e = E_c \Rightarrow E = 0$



- Charge Density $\rho = 0$, from Gauss's Law

$$\nabla \bullet E = \frac{\rho}{\varepsilon_0} = 0 \because E = 0$$

- Also there is abundant charge inside a conductor, the positive and negative charges are uniformly distributed making the charge density zero $\rho = 0$

- Charge, if any, resides only on the surface

- A conductor is an equipotential body. If a and b are two points within or on the surface of the conductor, then

$$V(b) - V(a) = -\int_a^b E \bullet dl = 0 \Rightarrow V(a) = V(b)$$

D5.1. Given the vector current density $J = 10\rho^2 z\, \mathbf{a}_\rho - 4\rho \cos^2\phi\, \mathbf{a}_\phi$ mA/m^2 :

(a) find the current density at $P(\rho = 3, \phi = 30^\circ, z = 2)$;

(b) determine the total current flowing outward through the circular band $\rho = 3$, $0 < \phi < 2\pi$, $2 < z < 2.8$.

Ans. $180a_\rho - 9a_\phi$ mA/m^2 ; 3.26 A

D5.2. Current density is given in cylindrical coordinates as $J = -10^6 z^{1.5}\mathbf{a}_z$ A/m^2 in the region $0 \le \rho \le 20\mu m$; for $\rho \ge 20\mu m$, $J = 0$.

(a) Find the total current crossing the surface $z = 0.1$ m in the $\mathbf{a}_z$ direction.

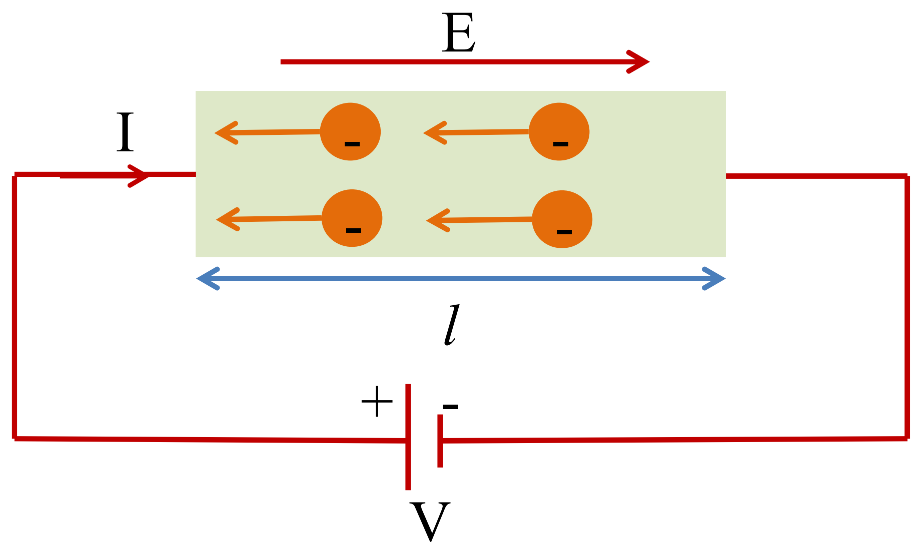
(b) If the charge velocity is 2×10^6 m/s at $z = 0.1$ m, find ρ_v there.

(c) If the volume charge density at $z = 0.15$ m is -2000 C/m^3 , find the charge velocity there.

Ans. $-39.7\mu A$; -15.8 mC/m^3 ; 29.0

Electric Field Inside a Conductor

- There is no static equilibrium when the conductor is connected to an external source. Hence E ≠ 0



Then $E = \dfrac{V}{l}$ (since E = dV/dx)

Assuming the conductor having uniform cross-section S $J = \dfrac{I}{S}$

But we also know that J = σ E ⇒ $\dfrac{I}{S} = \sigma E = \dfrac{\sigma V}{l}$

Hence $\boxed{R = \dfrac{l}{\sigma S} = \dfrac{\rho_c l}{S}}$ is the resistance offered by the conductor to the current due to an external source. Where ρ_C is resistivity of the material

- **To derive Resistance:**

The magnitude of electric field is given by $E = \dfrac{V}{l}$

Assuming conductor has uniform cross section of area *S*, $J = \dfrac{I}{S}$

$$\frac{I}{S} = \sigma \mathrm{E} = \frac{\sigma V}{l}$$

$$R = \frac{V}{I} = \frac{l}{\sigma S}$$

$$\boxed{R = \frac{l}{\sigma S}}$$

$$R = \frac{l}{\sigma S} = \frac{\rho_C l}{S}, \quad \rho_C = \frac{1}{\sigma} \Rightarrow \text{Resistivity}$$

If the cross section of the conductor is not uniform:

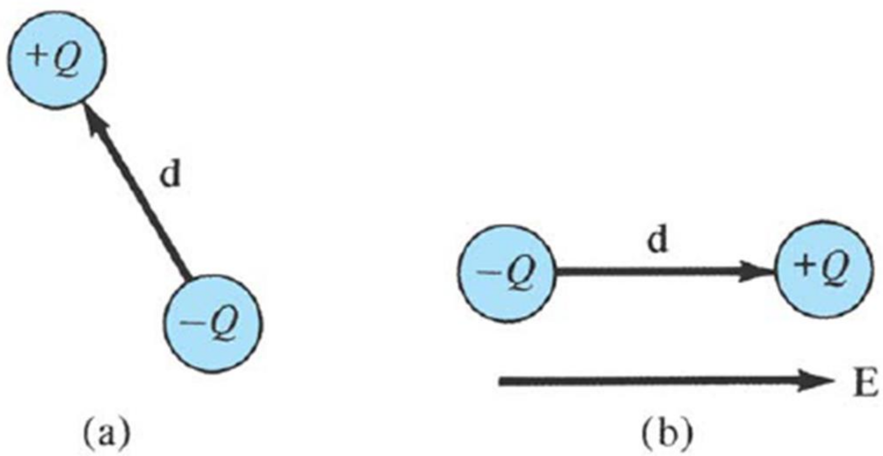
$$\boxed{R = \frac{V}{I} = \frac{-\int \mathrm{E} \cdot d\mathbf{l}}{\int \sigma \mathrm{E} \cdot d\mathrm{S}}}$$

The power *P* (in watts): $P = \int \mathrm{E} \cdot \mathrm{J}\, dv \quad \text{or} \quad P = I^2 R$

Dielectrics

Two groups of dielectrics:

- ***Nonpolar:*** nonpolar dielectric molecules do not posses *dipoles* until the application of electric field.
- Examples: hydrogen, oxygen, nitrogen,
- ***Polar:*** molecules have built-in permanent *dipoles* that are randomly oriented. When external **E** is applied, dipole moments are aligned parallel with **E**.

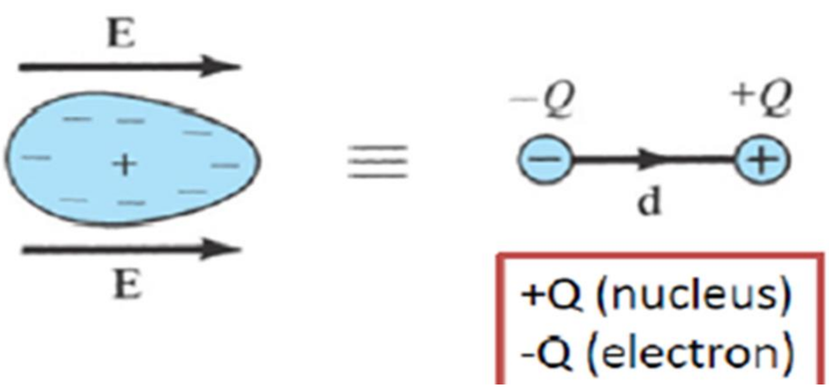


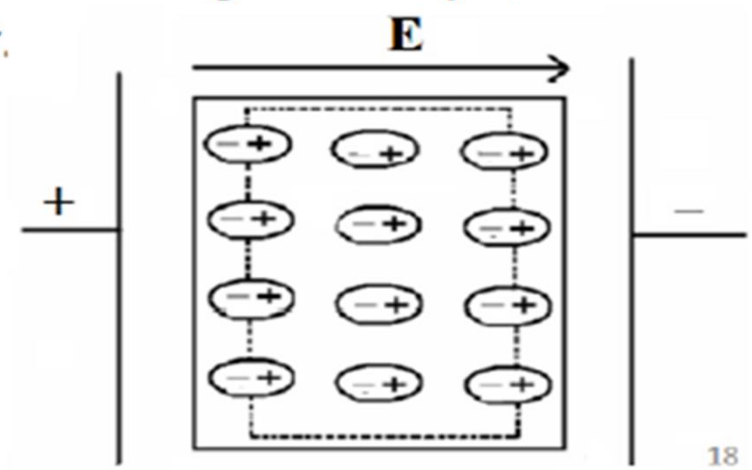
Polarization of a **polar** molecule: **(a)** permanent dipole (**E** = 0), **(b)** alignment of permanent dipole (**E** ≠ 0).

- Examples: water, sulfer dioxide

Polarization in Dielectrics

- In dielectric materials, charges are not able to move about freely, they are bound by finite forces. (displacement will take place when external force is applied).
- Atoms or molecules are electrically neutral since positive and negative charges have equal amounts.
- When Electric field is applied, positive charges move in the direction of **E**, and negative charges move in the opposite direction.
- The molecules are deformed from their original shape, and each molecule gets some ***dipole moment***.





Polarization & Dipole Moment

- The **dipole moment** is

$$\boxed{\vec{P} = Q\vec{d}}$$

Where **d** is the distance vector from –Q to +Q of the dipole.

- If there are *N* dipoles, the total dipole moment due to the

electric field is:

$$\boxed{Q_1 d_1 + Q_2 d_2 + \cdots + Q_N d_N = \sum_{k=1}^{N} Q_k d_k}$$

- As a measure of intensity of polarization, define **Polarization P** (in coulombs per meter squared) as the dipole moment per unit volume of the dielectric:

$$P = \frac{\vec{P}}{\Delta v} \qquad \boxed{P = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{N} Q_k d_k}{\Delta v}}$$

Electric Field in Dielectrics

- For some dielectrics P is proportional to E

$$\boxed{P = \chi_e \varepsilon_0 E}$$

- Where χ_e is the electric susceptibility of the material and is a measure of the sensitivity of the dielectric to external electric field

$$\boxed{D = \varepsilon_0 E + P = \varepsilon_0 E + \chi_e \varepsilon_0 E = \varepsilon_0 E(1 + \chi_e)}$$

Now we can write the electric flux density as $D = \varepsilon_0 (1 + \chi_e) E$

Where $\varepsilon_r = (1 + \chi_e) = \dfrac{\varepsilon}{\varepsilon_0}$ is called relative permittivity or dielectric constant of the medium

- In an non ideal dielectric, application of high external electric fields will tear the electrons from atoms thus making it behave like conductor. The minimum electric field where dielectric breakdown occurs is defined as dielectric strength.

Boundary Conditions

- ❑ If the electric field extends across a region consisting of two or more material media, the conditions the field must satisfy at the interface of these medium are called *Boundary Conditions*

- ❑ Pre-requisite ? – If the fields are known on one side of the interface, and the material parameters are known, then field can be evaluated on the other side of interface using boundary conditions

➢ What kind of material interfaces do we have ?

- ❑ Dielectric – Dielectric
- ❑ Conductor – Dielectric
- ❑ Conductor – free space
- ❑ Dielectric – free space

What conditions does the electric fields satisfy ?

$$\oint_C E \bullet dL = 0 \ \ \& \int_S D \bullet dS = Q_v$$

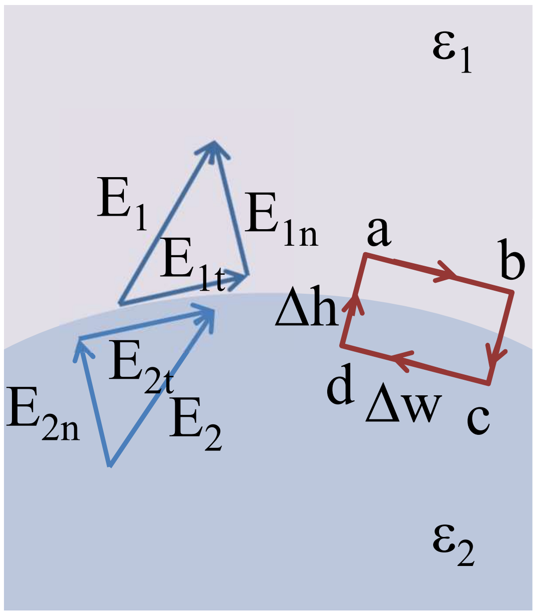
What components they have ?

E_t, D_t, E_n, D_n

Tangential and Normal components

Dielectric – Dielectric Boundary Condition

- Consider an interface formed by two different dielectrics of permittivity $\varepsilon_1 = \varepsilon_0\varepsilon_{r1}$ & $\varepsilon_2 = \varepsilon_0\varepsilon_{r2}$



The electric fields on either side can be written as

$$E_1 = E_{1t} + E_{1n} \ \& \ E_2 = E_{2t} + E_{2n}$$

Now taking the closed loop integral

$$\oint_{abcda} E \bullet dL = 0 = E_{1t}\Delta w - E_{1n}\frac{\Delta h}{2}$$

$$-E_{2n}\frac{\Delta h}{2} - E_{2t}\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2}$$

$$\Rightarrow \left(E_{1t} - E_{2t}\right)\Delta w = 0$$

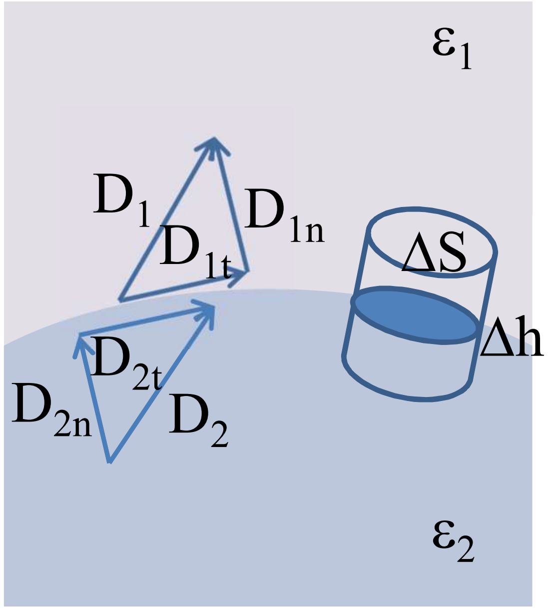
Condition 1

$$\boxed{E_{1t} = E_{2t}}$$

The tangential component of E-field is continuous across the interface

Dielectric – Dielectric Interface

- Likewise the electric flux density can be written as $D_{1t} = \varepsilon_1 E_{1t}$ & $D_{2t} = \varepsilon_2 E_{2t}$



Condition 2

$$\boxed{\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}}$$

The tangential component of D is discontinuous across the interface

- Now applying Gauss's law across the interface, the contribution from sides is zero ($\Delta h = 0$), we have

$$\Delta Q = \rho_s \,\Delta S = D_{1n}\,\Delta S - D_{2n}\,\Delta S \qquad \hat{a}_{12}.(D_1 - D_2)$$

Condition 3

$$\boxed{D_{1n} - D_{2n} = \rho_s}$$

The normal components of electric flux are discontinuous and equal to the charge present at the interface

Condition 4

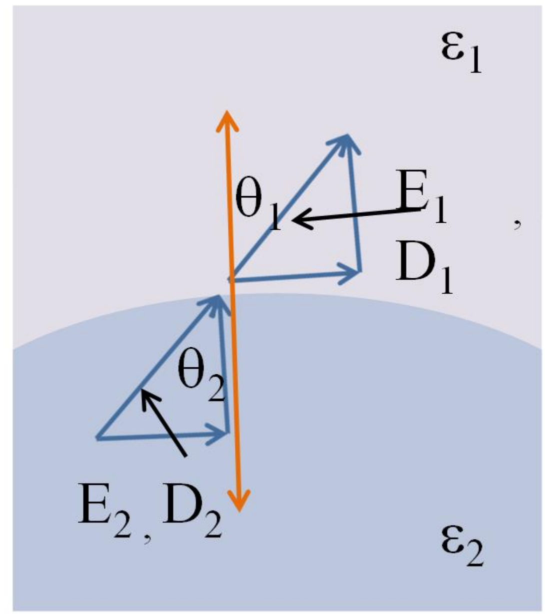
$$\boxed{\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}}$$

Assuming $\rho_s = 0$ the normal components of the electric field are discontinuous

Dielectric – Dielectric Boundary Condition

- If θ_1 is the angle E_1 & D_1 make with the normal of material 1 and θ_2 the angle of E_2 & D_2 with normal of material 2, then

$E_{1t} = E_1 \sin\theta_1$ & $E_{2t} = E_2 \sin\theta_2$



- Applying condition 1, we have

$$E_1 \sin\theta_1 = E_2 \sin\theta_2 \qquad (1)$$

- Applying condition 3 or 4 and considering no charge at interface $\rho_s = 0$, we have

$D_{1n} = \varepsilon_1 E_1 \cos\theta_1$ & $D_{2n} = \varepsilon_2 E_2 \cos\theta_2$

Dividing (1) with this condition we have

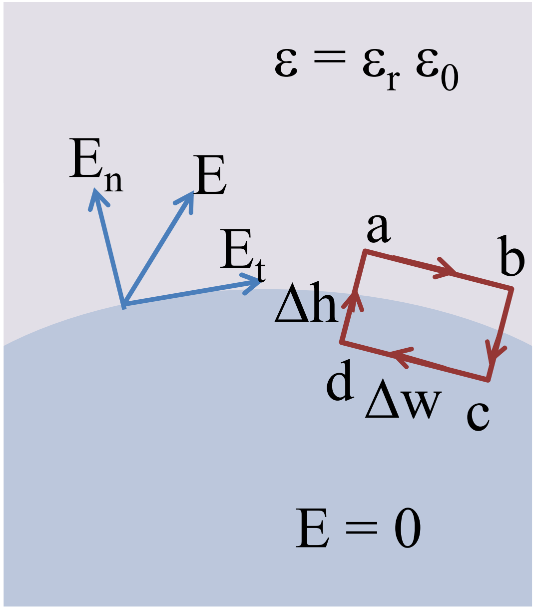
Law of refraction of electric field at the boundary free of charge. Where $\varepsilon_1 = \varepsilon_{r1} \varepsilon_0$ & $\varepsilon_2 = \varepsilon_{r2} \varepsilon_0$

Conductor – Dielectric Interface

$$\oint_{abcda} E \bullet dL = 0 = E_t \Delta w - E_n \frac{\Delta h}{2}$$

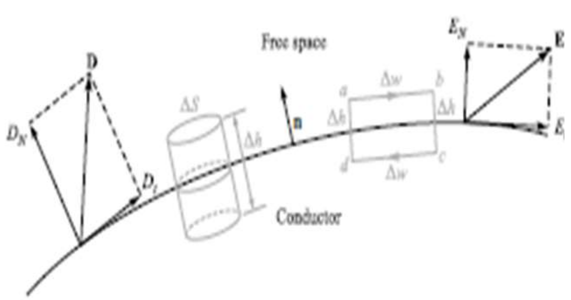
$$-0\left(\frac{\Delta h}{2}\right) - 0(\Delta w) - 0\frac{\Delta h}{2} + E_n \frac{\Delta h}{2}$$

$$\Rightarrow E_t = 0 \quad \textbf{as } \Delta h \to 0$$



- Now applying Gauss's law across the interface, the contribution from sides is zero ($\Delta h = 0$), we have

$$\Delta Q = \rho_s \,\Delta S = D_n \,\Delta S - 0 \bullet \Delta S \qquad \Rightarrow D_n = \frac{\Delta Q}{\Delta S} = \rho_s$$



Condition 1

$$\boxed{\rho_v = 0, E = 0}$$

Condition 2

$$\boxed{D_t = \varepsilon_0 \varepsilon_r E_t = 0 \quad \& \quad D_n = \varepsilon_0 \varepsilon_r E_n = \rho_s}$$

Conductor / Dielectric – Free Space Interface

- The conductor - free space interface may be considered similar to conductor - dielectric interface with $\varepsilon_r = 1$

Condition Conductor – Free Space

$$\boxed{D_t = \varepsilon_0 E_t = 0 \quad \& \quad D_n = \varepsilon_0 E_n = \rho_s}$$

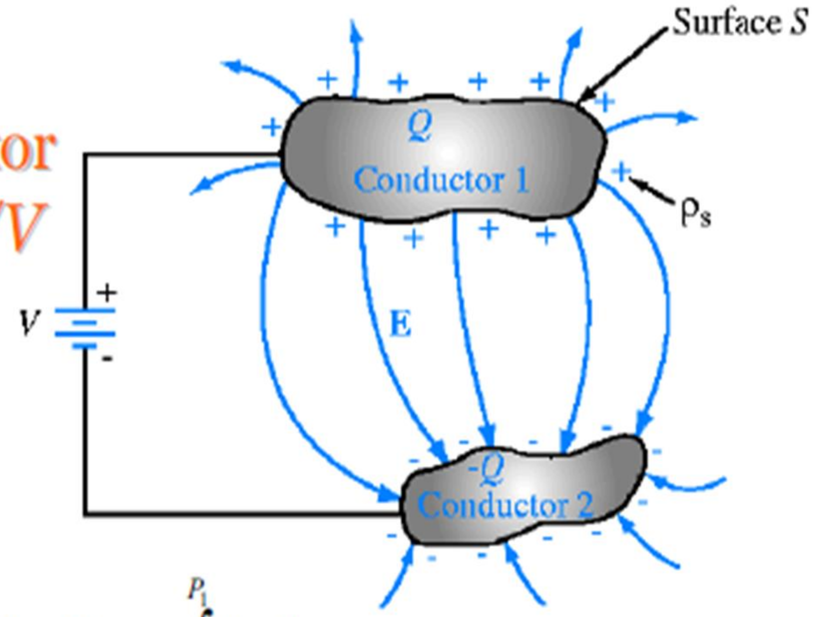
- The dielectric - free space interface may be considered similar to dielectric - dielectric interface with $\varepsilon_1 = \varepsilon_0$ substituted in the conditions therein

D5.9. Let Region 1 ($z < 0$) be composed of a uniform dielectric material for which $\varepsilon_r = 3.2$, while Region 2 ($z > 0$) is characterized by $\varepsilon_r = 2$. Let D1 = −30ax + 50ay + 70az nC/m^2 and find: (a) D_{N1}; (b) $\mathbf{D}_{t1}$; (c) D_{t1}; (d) D_1; (e) θ_1; (f) P_1.

Ans. 70 nC/m^2 ; $-30a_x + 50a_y$ nC/m^2 ; 58.3 nC/m^2 ; 91.1 nC/m^2 ; 39.8 ◦ ; $-20.6a_x + 34.4a_y + 48.1a_z$ nC/m

Capacitance

- Concept of Capacitance
  - Capacitance of a two-conductor capacitor is defined as $C = Q/V$ (C/V or F)
  - $E_t = 0$ while $E_n = \rho_s/\varepsilon$ on the surface of the conductor



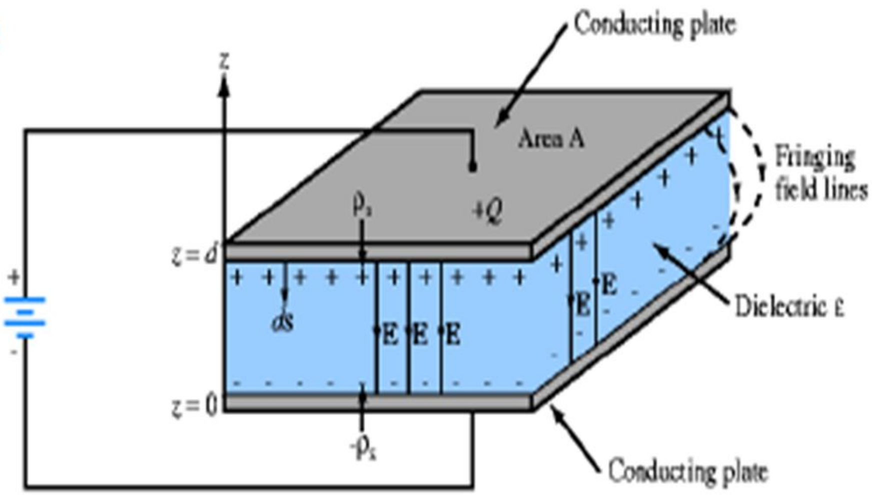
$$Q = \iint_S \rho_s ds = \iint_S \varepsilon \hat{\mathbf{n}} \cdot \mathbf{E} ds = \iint_S \varepsilon \mathbf{E} \cdot d\mathbf{s}, \quad \text{and} \quad V = V_{12} = -\int_{P_2}^{P_1} \mathbf{E} \cdot d\mathbf{l}$$

$$\boxed{C = \frac{Q}{V} = \frac{\iint_S \varepsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_{P_2}^{P_1} \mathbf{E} \cdot d\mathbf{l}}, \quad \text{and} \quad R = \frac{V}{I} = \frac{-\int_{P_2}^{P_1} \mathbf{E} \cdot d\mathbf{l}}{\iint_S \sigma \mathbf{E} \cdot d\mathbf{s}} \quad \text{so that} \quad RC = \frac{\varepsilon}{\sigma}.}$$

- # Example 8

Question: Derive capacitance C of a parallel-plate capacitor comprised of two parallel plates each of surface area A and separated by a distance d. The capacitor is filled with a dielectric material with permittivity ε. Also, determine the breakdown voltage if $d = 1$ cm and the dielectric material is quartz.

Solution: Assume the $+Q$ and $-Q$ on the upper and lower plates, respectively. The charge density will be $\rho_s = Q/A$. Hence,



$$\mathbf{E} = -\hat{\mathbf{z}}E = -\hat{\mathbf{z}}\left(\rho_s/\varepsilon\right) = -\hat{\mathbf{z}}\left(Q/\varepsilon A\right)$$

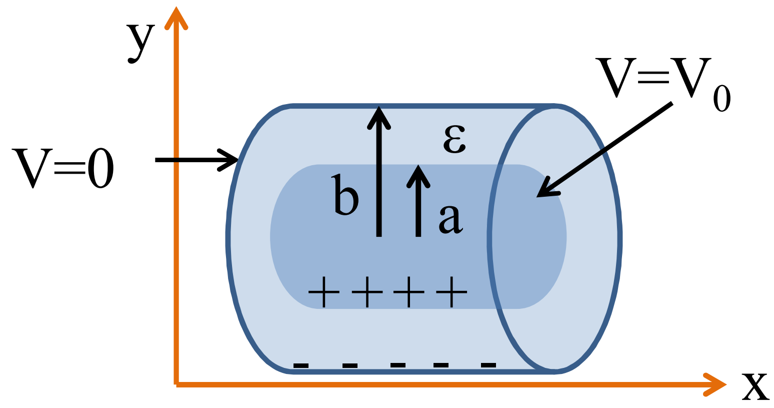
$$V = -\int_0^d \mathbf{E}\cdot d\mathbf{l} = -\int_0^d \left(-\hat{\mathbf{z}}E\right)\left(\hat{\mathbf{z}}dz\right) = Ed,$$

$$C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{\varepsilon A}{d}.$$

Apparently, $V = V_{br}$ when $E = E_{ds}$. As $E_{ds} = 30$ (MV/m) for quartz, hence the breakdown voltage is $V_{br} = E_{ds}d = 30\times10^6\times10^{-2} = 3\times10^5$ V.

Coaxial Cylindrical Capacitor

- We consider that V is given.
- Solving Laplace's equation in 1D



$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right)=0 \qquad V=V_0\frac{\ln\left(b/\rho\right)}{\ln\left(b/a\right)}$$

Now we find E = -∇V $\quad E = V_0/\rho\left[\dfrac{1}{\ln(b/a)}\right]a_\rho$

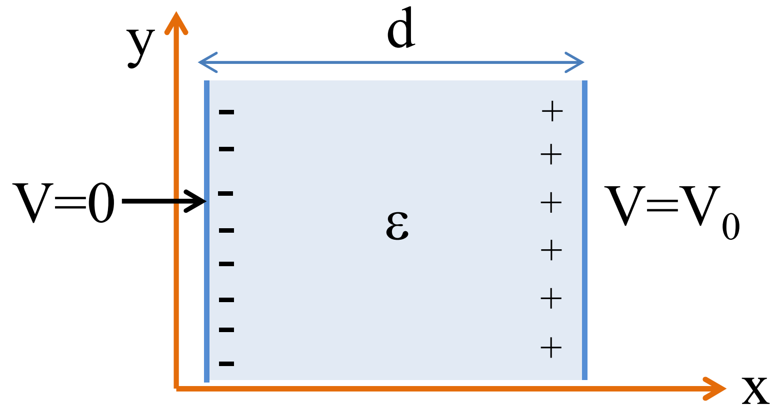
Then we find D = εE $\quad D_N = \varepsilon V_0/a\left[\dfrac{1}{\ln(b/a)}\right]=\rho_s$

From which we can have $\quad Q=\int_s D_N dS = \dfrac{\varepsilon V_0 2\pi a L}{a\ln(b/a)}$

The capacitance C can now be written as $\quad \boxed{C=\dfrac{|Q|}{V}=\dfrac{2\pi\varepsilon L}{\ln(b/a)}}$

Parallel Plate Capacitor

- We consider that V is given.
- Solving Laplace's equation in 1D



$$V = V_0 \frac{x}{d}$$

Now we find E = -∇V $\quad E = -\frac{V_0}{d} a_x$

Then we find D = εE $\quad D = -\varepsilon \frac{V_0}{d} a_x$

Where $D_N = -\varepsilon \frac{V_0}{d} = \rho_S$

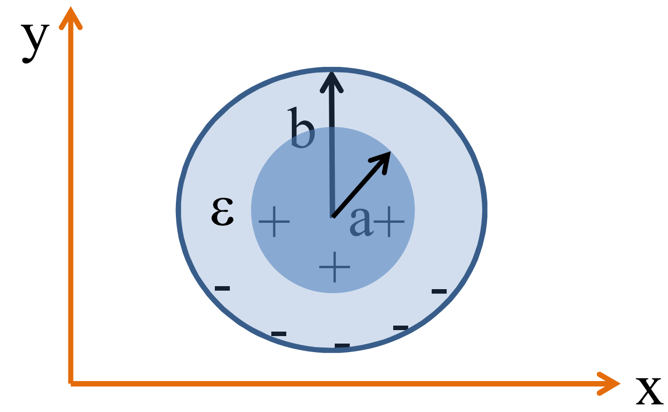
From which we can have $\quad Q = \int_s \frac{-\varepsilon V_0}{d} dS = -\varepsilon \frac{V_0 S}{d}$

The capacitance C can now be written as

$$\boxed{C = \frac{|Q|}{V} = \frac{\varepsilon S}{d}}$$

Coaxial Spherical Capacitor

- We solve this using Gauss's Law, if the charge density is given



$$Q = \varepsilon \oint E \bullet dS = \varepsilon E_r 4\pi r^2$$

Now we have $E = \dfrac{Q}{4\pi\varepsilon r^2} a_r$

Then we find $V = -\int_b^a E \bullet dL = \dfrac{Q}{4\pi\varepsilon}\left[\dfrac{1}{a} - \dfrac{1}{b}\right]$

The capacitance C can now be written as $\boxed{C = \dfrac{|Q|}{V} = \dfrac{4\pi\varepsilon}{\left[\dfrac{1}{a} - \dfrac{1}{b}\right]}}$

- <u>Electrostatic Potential Energy</u>
  - The voltage υ across a capacitor C is related to charge q by $\upsilon = q/C$.
  - The work is thus $dW_e = \upsilon dq = q dq/C$, or

$$W_e = \int_0^Q \frac{q}{C}\,dq = \frac{1}{2}\frac{Q^2}{C} \overset{C=Q/V}{=} \frac{1}{2}CV^2 \overset{\substack{V=Ed \\ C=\varepsilon A/d}}{=} \frac{1}{2}\frac{\varepsilon A}{d}(Ed)^2 = \frac{1}{2}\varepsilon E^2 (Ad) = \frac{1}{2}\varepsilon E^2 v$$

  - Electrostatic energy per unit volume and energy

$$\boxed{w_e = \frac{W_e}{v} = \frac{1}{2}\varepsilon E^2, \quad (\mathrm{J/m^3}) \text{ or } W_e = \frac{1}{2}\iiint_V \varepsilon E^2\,dv \quad (\mathrm{J}).}$$

    - The half value comes from the mean square evaluation

D6.1. Find the relative permittivity of the dielectric material present in a parallel-plate capacitor if:

(a) $S = 0.12\ m^2$, $d = 80\mu m$, $V_0 = 12$ V, and the capacitor contains $1\mu J$ of energy;

(b) the stored energy density is $100\ J/m^3$, $V_0 = 200$ V, and $d = 45\mu m$;

(c) $E = 200$ kV/m and $\rho_S = 20\mu C/m^2$.

Ans. 1.05; 1.14; 11.3

D6.2. Determine the capacitance of:

(a) a 1-ft length of 35B/U coaxial cable, which has an inner conductor 0.1045 in. in diameter, a polyethylene dielectric (ọr = 2.26 from Table C.1), and an outer conductor that has an inner diameter of 0.680 in.;

(b) a conducting sphere of radius 2.5 mm, covered with a polyethylene layer 2 mm thick, surrounded by a conducting sphere of radius 4.5 mm;

(c) two rectangular conducting plates, 1 cm by 4 cm, with negligible thickness, between which are three sheets of dielectric, each 1 cm by 4 cm, and 0.1 mm thick, having dielectric constants of 1.5, 2.5, and 6.

Ans. 20.5 pF; 1.41 pF; 28.7 pF

D6.3. A conducting cylinder with a radius of 1 cm and at a potential of 20 V is parallel to a conducting plane which is at zero potential. The plane is 5 cm distant from the cylinder axis. If the conductors are embedded in a perfect dielectric for which $\varepsilon_r = 4.5$, find:

(a) the capacitance per unit length between cylinder and plane;

(b) ρ_S,max on the cylinder.

Ans. 109.2 pF/m; 42.6 nC/m^2