Module 4: Faraday’s Law and Lenz law, Maxwell’s Equations in Integral and differential form, Wave equation, Uniform plane wave propagation in lossy dielectrics, Lossless Dielectrics, Good Conductors and free space. Polarization, Power and Poynting Vector.

Time – Varying Fields

Electrodynamics $E(x,y,z,t) \leftrightarrow H(x,y,z,t)$

Maxwell's Equations

$$\boxed{\begin{aligned} &\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ &\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\ &\nabla \bullet \mathbf{D} = \rho \\ &\nabla \bullet \mathbf{B} = 0 \end{aligned}}$$

Faraday's Law

- The induced electro motive force (emf) V_{emf} in a closed circuit is equal to the time rate of change of magnetic flux linkage by the circuit

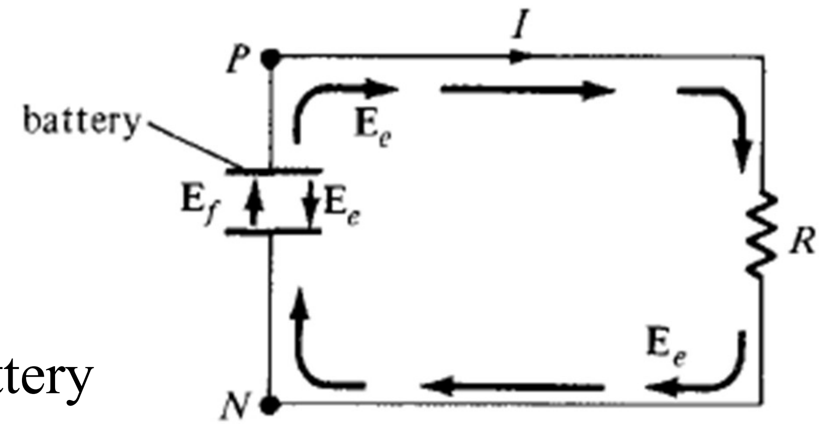
$$emf = -\frac{d\phi}{dt} = -N\frac{d\phi}{dt}$$

- The induced voltage acts in such a way as to oppose the flux producing it, hence the negative sign. Electro motive force (emf) can be defined as voltage developed by any source of electrical energy

Faraday's Law

- Electric field (E_e) is the force experienced by electric charges. Whereas electric field due to other sources is termed as electro motive force produced fields (E_{emf} or E_f)

- The total field in a closed circuit, due to a battery $E = E_f + E_e$



E_e – due to charge
E_f – due to chemical action inside battery

- Taking closed loop integral of the field, we have

$$\oint_L E \bullet dL = \oint_L E_f \bullet dL \qquad \because \oint_L E_e \bullet dL = 0$$

Inside the battery

$$\therefore V_{emf} = \int_N^P E_f \bullet dL = -\int_N^P E_e \bullet dL = IR$$

The battery generates the force which drives the electrons along the loop

Faraday's Law

- Consider a circuit with single turn, N = 1. From which we can write the induced voltage as

$$V_{emf} = -\frac{d\psi_m}{dt}$$

LHS can be written as $V_{emf} = \oint_L \mathbf{E} \bullet \mathbf{dL}$

RHS can be written as $-\frac{d\psi_m}{dt} = -\frac{d}{dt}\int_S \mathbf{B} \bullet \mathbf{dS} \quad \because \psi_m = \int_S \mathbf{B} \bullet \mathbf{dS}$

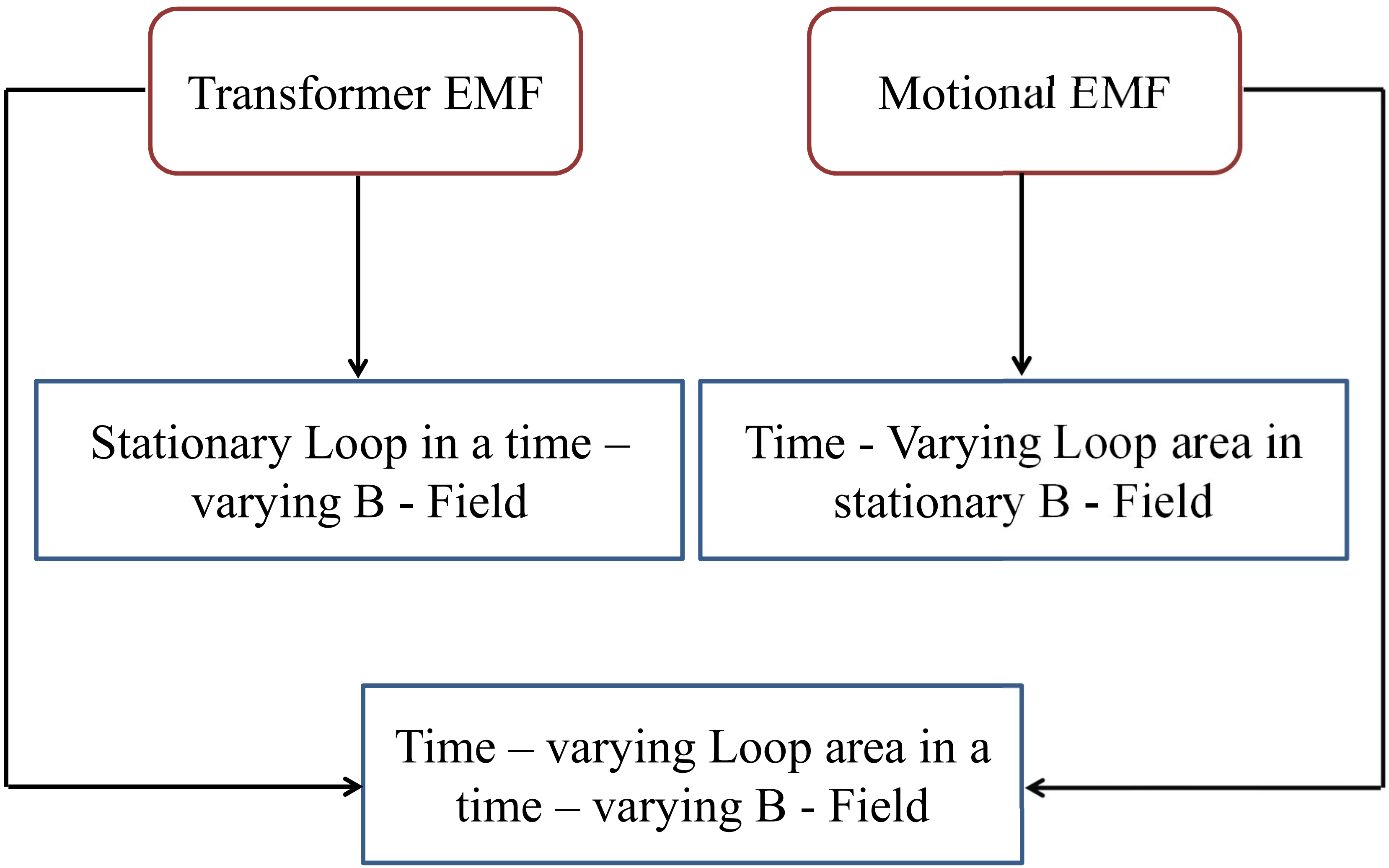
- Putting them together we have Faraday's Law of induction

$$\boxed{\oint_L \mathbf{E} \bullet \mathbf{dL} = -\frac{d}{dt}\int_S \mathbf{B} \bullet \mathbf{dS}}$$

- Applying Stoke's theorem to LHS, we can get the differential form as

$$\boxed{\nabla \times \mathbf{E} = -\frac{\mathbf{dB}}{\mathbf{d}t}}$$

Types of EMF

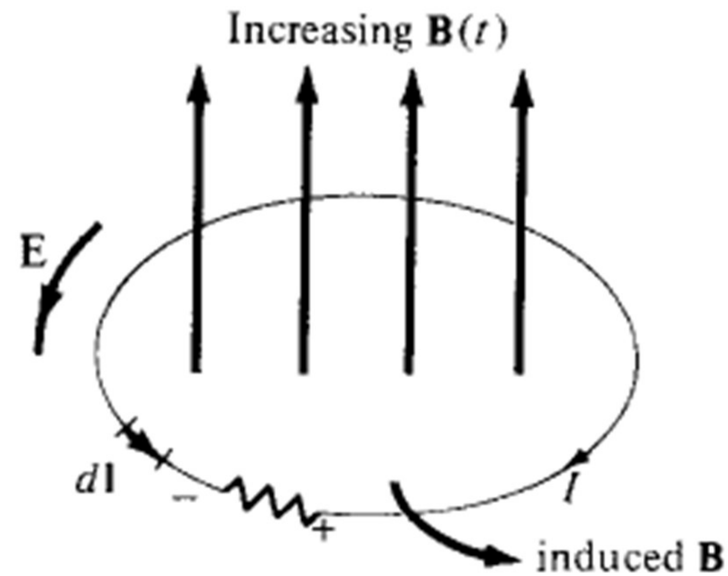


Transformer EMF
Motional EMF
Stationary Loop in a time – varying B - Field
Time - Varying Loop area in stationary B - Field
Time – varying Loop area in a time – varying B - Field

Transformer EMF

- EMF produced by a stationary conducting loop inside a time - varying magnetic field

$$V_{emf} = \int_s (\nabla \times \mathbf{E}) \bullet \mathbf{dS} = -\frac{d}{dt}\int_S \mathbf{B} \bullet \mathbf{dS}$$



$$\nabla \times \mathbf{E} = -\frac{\mathbf{dB}}{\mathbf{d}t}$$

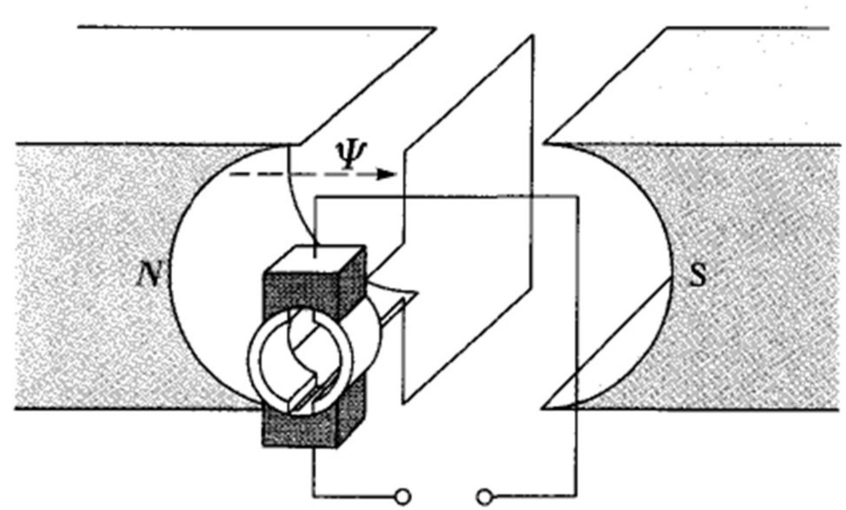
This forms the first Maxwell equation for time – varying fields

- The work done in moving a charge along closed loop in a time – varying electric field is due to the energy from the time – varying magnetic field

Motional EMF

- EMF produced by a moving conducting loop inside a stationary magnetic field

- The conducting loop consists of electric charges, the force on them due to magnetic field is



$$F_m = Q\,\mathbf{u} \times \mathbf{B}$$

The motional Electric field is

$$E_m = \frac{F_m}{Q}$$

Equating the above two equations

$$\boxed{V_{emf} = \oint_L E \bullet dL = \oint_L (u \times B) \bullet dL}$$

Moving Loop in a Time- Varying B – Field

- The EMF produced by a moving loop inside a time – varying magnetic field can be arrived from transformer and motional EMF terms defined in previous slides

$$\boxed{V_{emf} = \oint_L E \bullet dL = -\int_S \frac{\partial B}{\partial t} \bullet dS + \oint_L (u \times B) \bullet dL}$$

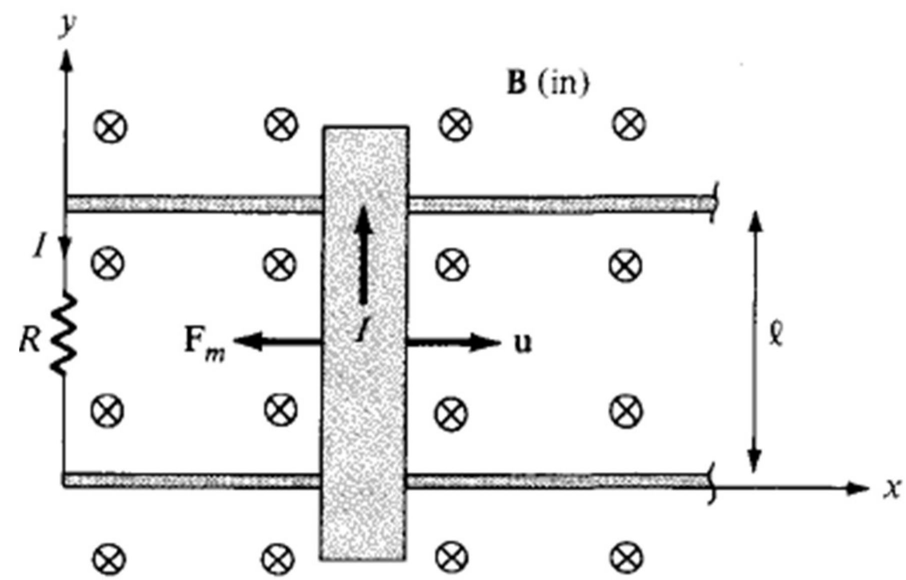
Applying Stoke's Theorem

$$\boxed{\nabla \times E = -\frac{\partial B}{\partial t} + \nabla \times (u \times B)}$$

Motional EMF

- Force on the moving loop due to static magnetic field

$$F_m = \mathrm{I}l \times \mathrm{B}$$



The induced voltage is

$$V_{emf} = uBl$$

But we have

$$\boxed{V_{emf} = \oint_L E \bullet dL = \oint_L (u \times B) \bullet dL}$$

Applying Stoke's Theorem

$$\boxed{\nabla \times \mathrm{E_m} = \nabla \times (\mathrm{u} \times \mathrm{B})}$$

Displacement Current

Taking time derivative on both sides

$$\frac{d\psi_e}{dt} = \frac{1}{\varepsilon_0}\frac{dQ}{dt}$$

Where the term $\frac{dQ}{dt} = I_d$ is the displacement current

$$\Rightarrow \varepsilon_0 \frac{d\psi_e}{dt} = I_d$$

$$\Rightarrow \varepsilon_0 \frac{d\int E \bullet dS}{dt} = I_d$$

$$\Rightarrow \varepsilon_0 A \frac{dE}{dt} = I_d \Rightarrow \frac{dD}{dt} = J_d$$

Now the Ampere's Law for time – varying fields can be written as

$$\boxed{\nabla \times H = J_c + J_d \quad \textbf{or} \quad \nabla \times H = J_c + \frac{\partial D}{\partial t}}$$

Displacement Current

Mathematically, we can also show the inconsistency of Ampere's Law

Let us write the Ampere's Law in differential form $\nabla \times H = J$

If we take divergence $\nabla \bullet \left(\nabla \times H\right) = \nabla \bullet J$

But divergence of curl (LHS) is zero, which contradicts RHS, since

Continuity equation $\nabla \bullet J = -\dfrac{\partial \rho}{\partial t}$

The above equation can be written as

$$\nabla \bullet J = -\frac{\partial \nabla \bullet D}{\partial t} \Rightarrow \nabla \bullet J = -\nabla \bullet \frac{\partial D}{\partial t}$$

From which, we have $J = -\dfrac{\partial D}{\partial t}$

We can then write Ampere's Law as $\boxed{\nabla \times H = J + \dfrac{\partial D}{\partial t}}$ which will then satisfy the divergence condition

Maxwell's & Constituent Equations

Differential Form

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \bullet \vec{D} = \rho_v$$

$$\nabla \bullet \vec{B} = 0$$

Integral Form

$$\oint \vec{E} \bullet dl = -\oiint_s \frac{\partial \vec{B}}{\partial t} \bullet dS$$

$$\oint \vec{H} \bullet dl = \oiint_s \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \bullet dS$$

$$\oiint_s \vec{D} \bullet dS = \iiint_v \rho_v dv$$

$$\oiint_s \vec{B} \bullet dS = 0$$

Material Equations

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$\vec{B} = \mu_0 \vec{H} + \vec{M}$$

$$\vec{J} = \sigma \vec{E} + \rho_v$$

Continuity Equation & Lorentz Force

$$\nabla \bullet \vec{J} = -\frac{\partial \rho}{\partial t}$$

$$F = q\left(E + u \times B\right)$$

D9.1. Within a certain region, $\varepsilon = 10^{-11}$ F/m and $\mu = 10^{-5}$ H/m. If $\mathbf{B}_x = 2\times10^{-4} \cos 10^5 t \sin 10^{-3} y$ T:

(a) use $\nabla \times H = \varepsilon\, \partial E / \partial t$ to find E;

(b) find the total magnetic flux passing through the surface x = 0, 0 < y < 40 m, 0 < z < 2 m, at t = 1 μs;

(c) find the value of the closed line integral of E around the perimeter of the given surface.

Ans. $-20\,000 \sin 10^5 t \cos 10^{-3} y\, \mathbf{a}_z$ V/m; 0.318 mWb; −3.19 V

D9.3. Find the amplitude of the displacement current density:

(a) adjacent to an automobile antenna where the magnetic field intensity of an FM signal is $\mathbf{H}_x = 0.15 \cos[3.12(3 \times 10^8 t - y)]$ A/m;

(b) in the air space at a point within a large power distribution transformer where $\mathbf{B} = 0.8 \cos[1.257\times10^{-6} (3\times10^8 t - x)]\mathbf{a}_y$ T;

(c) within a large, oil-filled power capacitor where $\varepsilon_r = 5$ and $\mathbf{E} = 0.9 \cos[1.257 \times 10^{-6} (3 \times 10^8 t - z \sqrt{5})]\mathbf{a}_x$ MV/m;

(d) in a metallic conductor at 60 Hz, if $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, $\sigma = 5.8\times107$ S/m, and $J = \sin(377t - 117.1z)\mathbf{a}_x$ MA/m^2 .

Ans. 0.468 A/m^2 ; 0.800 A/m^2 ; 0.0150 A/m^2 ; 57.6 pA/m^2

D9.2. With reference to the sliding bar shown in Figure 9.1, let d = 7 cm, $\mathbf{B} = 0.3\mathbf{a}_z$ T, and $\mathbf{v} = 0.1\mathbf{a}_y e^{20y}$ m/s. Let y = 0 at t = 0. Find:

(a) $v(t = 0)$;

(b) $y(t = 0.1)$;

(c) $v(t = 0.1)$;

(d) V_{12} at t = 0.1

Ans. 0.1 m/s; 1.12 cm; 0.125 m/s; −2.63 mV

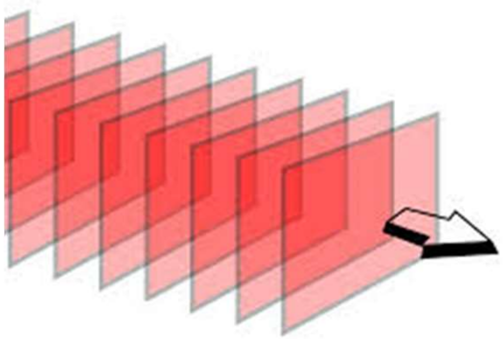
Plane Waves

- The foremost application of Maxwell's equations can be found in Plane Waves
- What is a wave ? - a disturbance or oscillation (of a physical quantity), that travels through matter or space, accompanied by a transfer of energy, with negligible or no transfer of matter
- What are the Wave-Types ? – Pressure or Longitudinal waves where the displacement of the disturbance is along the direction of propagation of the wave. Whereas in Transverse wave, it is perpendicular to the direction of propagation
- Do they require a medium ? – Longitudinal waves require a medium, since they are mechanical waves. Transverse waves require no medium and they propagate on alternating effects of electric and magnetic fields.

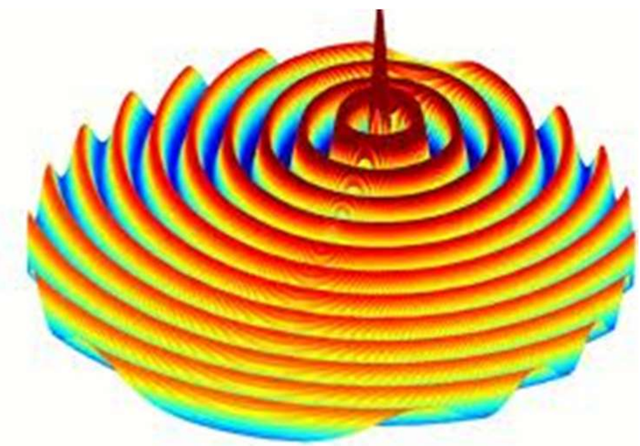
Plane Waves

- A plane wave is a constant-frequency wave whose wavefronts (surfaces of constant phase) are infinite parallel planes of constant amplitude normal to the phase velocity vector.

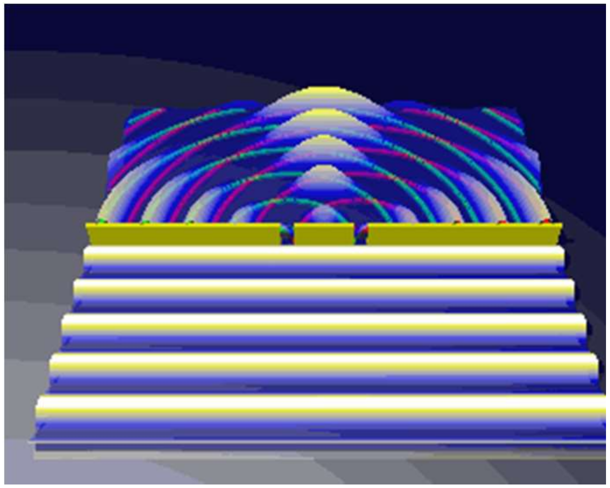
Plane Wavefront



Cylindrical Wavefront



Spherical Wavefront



Plane Waves

Taking curl of the first two equations, we have

$$\nabla\times(\nabla\times E) = -\mu\frac{\partial}{\partial t}(\nabla\times H) = -\mu\varepsilon\frac{\partial}{\partial t}\left(\frac{\partial E}{\partial t}\right)$$

$$\nabla\times(\nabla\times H) = \varepsilon\frac{\partial}{\partial t}(\nabla\times E) = -\mu\varepsilon\frac{\partial}{\partial t}\left(\frac{\partial H}{\partial t}\right)$$

Applying the vector identity $\nabla\times(\nabla\times F) = \nabla(\nabla\bullet F) - \nabla^2 F$

We have $\nabla\times(\nabla\times E) = \nabla(\nabla\bullet E) - \nabla^2 E$

$$\nabla\times(\nabla\times H) = \nabla(\nabla\bullet H) - \nabla^2 H$$

Vector Plane wave or Helmholtz equations

$$\boxed{\begin{aligned}\nabla^2 E &= \mu\varepsilon\frac{\partial^2 E}{\partial t^2}\\ \nabla^2 H &= \mu\varepsilon\frac{\partial^2 H}{\partial t^2}\end{aligned}}$$

since

$$\nabla\bullet E = 0$$

$$\nabla\bullet H = 0$$

Solutions of Wave Equation

- Phasor form of plane wave equation can be written as

$$\boxed{\begin{aligned}\nabla^2 E_s &= -\omega^2 \varepsilon\mu E_s \\ \nabla^2 H_s &= -\omega^2 \varepsilon\mu H_s\end{aligned}} \qquad \forall \frac{\partial}{\partial t} = j\omega$$

- If z is the direction of propagation, then the above equations can have a solution of the form

$$E_x(z,t) = \left(E^+ e^{j\omega t} e^{-j\beta z} + E^- e^{j\omega t} e^{j\beta z}\right)$$

$$H_y(z,t) = \left(H^+ e^{j\omega t} e^{-j\beta z} + H^- e^{j\omega t} e^{j\beta z}\right)$$

- Let us consider a part of the above solution to understand the wave properties

$$E_x(z,t) = E^+ e^{j\omega t} e^{-j\beta z} = E^+\left[\cos(\omega t - \beta z) + j\sin(\omega t - \beta z)\right]$$

$E^+ \rightarrow$ is the amplitude of the wave (here it is forward amplitude in +z direction)

Properties of a Wave

- Any wave can be characterized by the below parameters for a lossless medium

$\boxed{\omega = 2\pi f \rightarrow}$ Angular frequency of the wave in radians/sec

$\boxed{\lambda = v_p T \rightarrow}$ Wavelength (distance between two successive Crests or Troughs) (T = 1/f is period in sec)

$$\boxed{v_p = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{\omega}{\beta} = \lambda f \rightarrow}$$ Phase velocity of the wave

$$\boxed{\beta = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{v_p} = \frac{2\pi}{\lambda} \rightarrow}$$ Wave number or phase constant

$$\boxed{T = \frac{1}{f} = \frac{2\pi}{\omega} \rightarrow}$$ Period of the wave in sec, and f is frequency in Hertz

D11.1. The electric field amplitude of a uniform plane wave propagating in the az direction is 250 V/m. If E = $E_x a_x$ and ω = 1.00 Mrad/s, find: (a) the frequency; (b) the wavelength; (c) the period

Ans: 59 kHz; 1.88 km; 6.28 µs;

A 1.8GHz wave propagates in a medium characterized by μ_r=1.6, ε_r=25, σ=2.5 S/m , E field intensity is $\mathrm{E} = 0.1e^{-\alpha z}\cos(2\pi f t - \beta z)\hat{a}_x$ V/m, determine a. β b) the wavelength; (c) phase velocity