Magnetostatics:

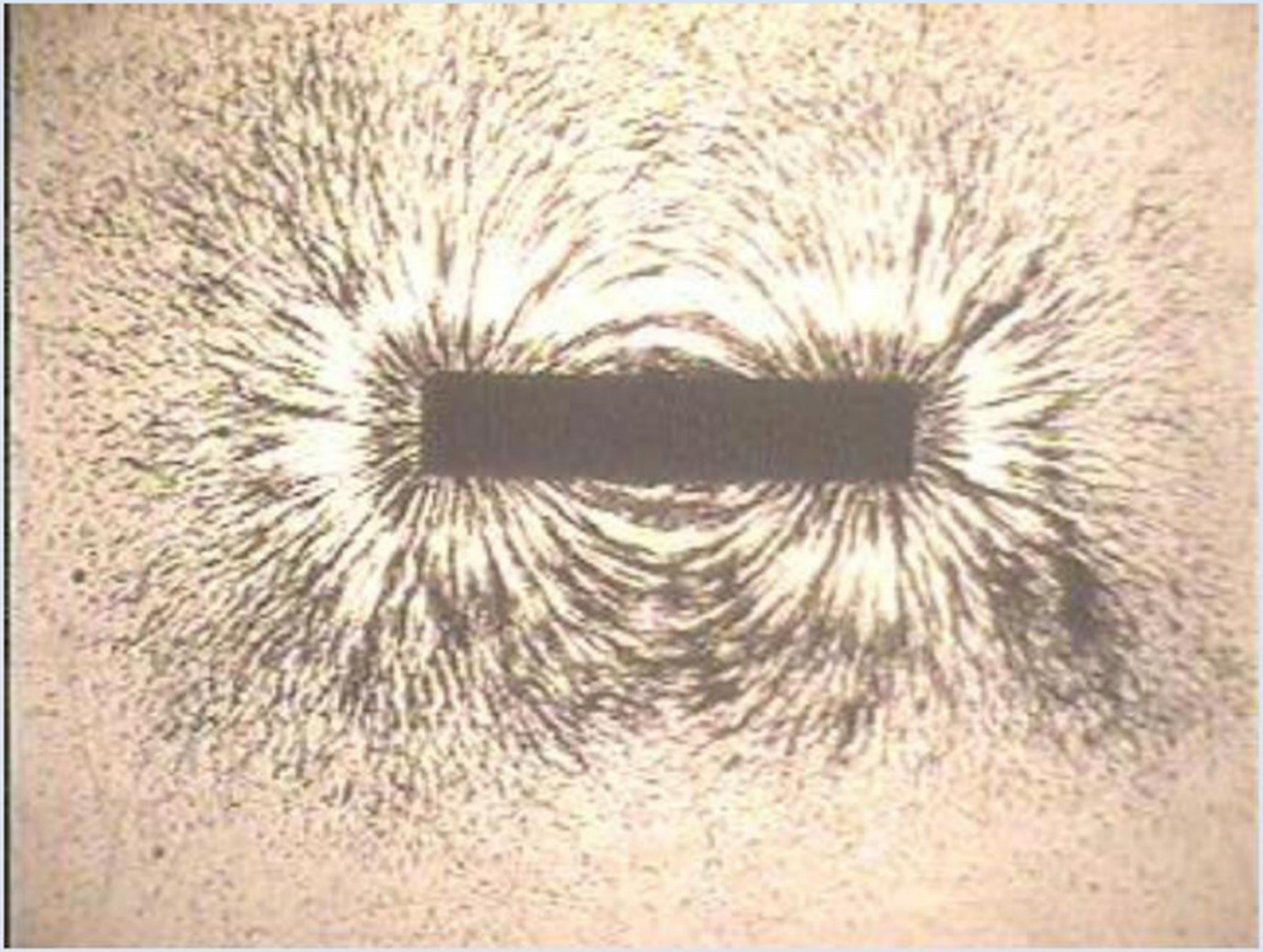
Biot-Savart's Law, Ampere's Circuit Law and Applications, Magnetic Flux Density, Magnetic Scalar and Vector Potentials, Forces due to Magnetic Fields, Magnetic Dipole, Magnetization in materials, Boundary conditions, Inductances and Magnetic Energy.

Introduction

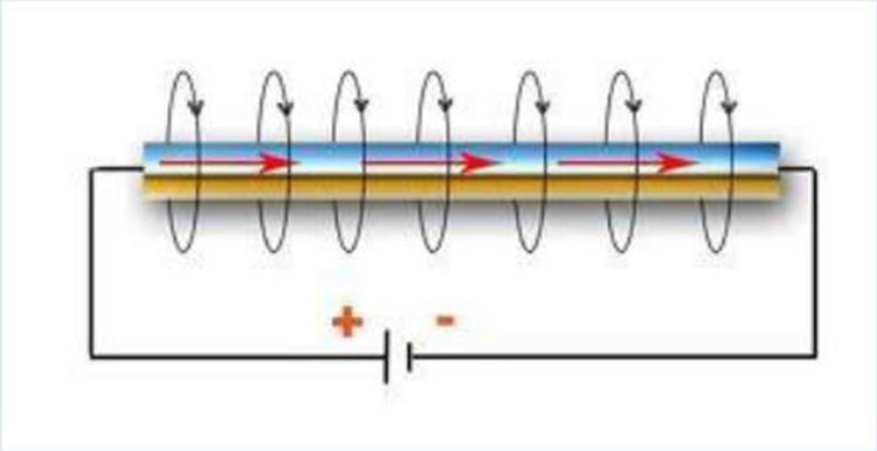
- static electric fields characterized by **E** or **D** (**D**=ε**E**) were discussed.
- considers static magnetic fields, characterised by **H** or **B** (**B**=μ**H**).

- As we have noticed, a distribution of ***static*** or ***stationary*** **charges** produces static electric field.
- If the charges move at a constant rate (**direct current- DC**), a static magnetic field is produced (**magnetostatic field**).
- Static magnetic field are also produced by **stationary permanent magnets**.

Magnet and Magnetic Field



The iron filings form circles around the **wire** along the **magnetic field**



Magnetic Field Around **Current** Carrying **Wires**

A compass needle is deflected by the **direct current** flowing in a conductor

Electrostatic Fields have dual equations for magnetostatic fields

Term	Electric	Magnetic
Basic laws	$\mathbf{F} = \frac{Q_1 Q_2}{4\pi\varepsilon_r^2}\mathbf{a}_r$	$d\mathbf{B} = \frac{\mu_o I\, d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2}$
	$\oint \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$	$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc}$
Force law	$\mathbf{F} = Q\mathbf{E}$	$\mathbf{F} = Q\mathbf{u} \times \mathbf{B}$
Source element	dQ	$Q\mathbf{u} = I\, d\mathbf{l}$
Field intensity	$E = \frac{V}{\ell}$ (V/m)	$H = \frac{I}{\ell}$ (A/m)
Flux density	$\mathbf{D} = \frac{\Psi}{S}$ (C/m^2)	$\mathbf{B} = \frac{\Psi}{S}$ (Wb/m^2)
Relationship between fields	$\mathbf{D} = \varepsilon\mathbf{E}$	$\mathbf{B} = \mu\mathbf{H}$
Potentials	$\mathbf{E} = -\nabla V$	$\mathbf{H} = -\nabla V_m\ (\mathbf{J} = 0)$
	$V = \int \frac{\rho_L dl}{4\pi\varepsilon r}$	$\mathbf{A} = \int \frac{\mu\, I\, d\mathbf{l}}{4\pi R}$
Flux	$\Psi = \int \mathbf{D} \cdot d\mathbf{S}$	$\Psi = \int \mathbf{B} \cdot d\mathbf{S}$
	$\Psi = Q = CV$	$\Psi = LI$
	$I = C\frac{dV}{dt}$	$V = L\frac{dI}{dt}$
Energy density	$w_E = \frac{1}{2}\mathbf{D} \cdot \mathbf{E}$	$w_m = \frac{1}{2}\mathbf{B} \cdot \mathbf{H}$
Poisson's equation	$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$	$\nabla^2 \mathbf{A} = -\mu\mathbf{J}$

Biot- Savart Law

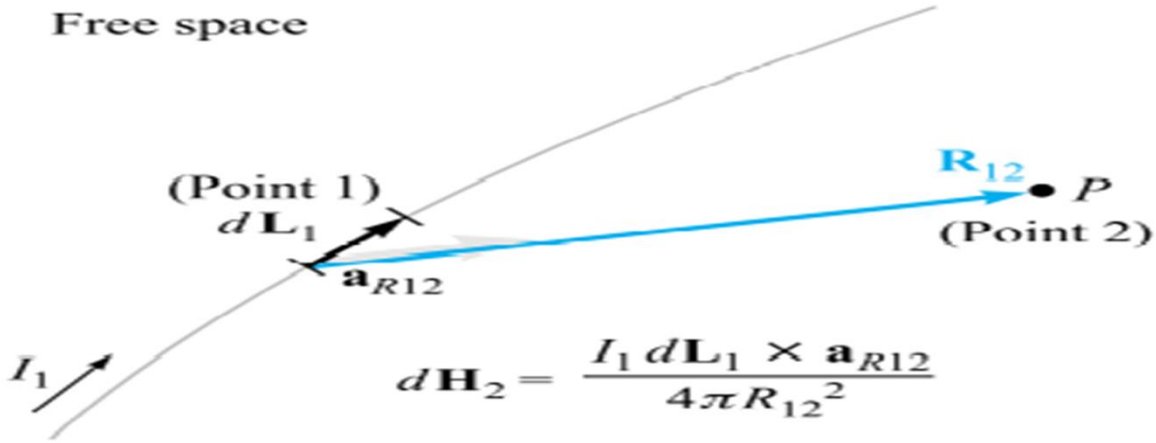
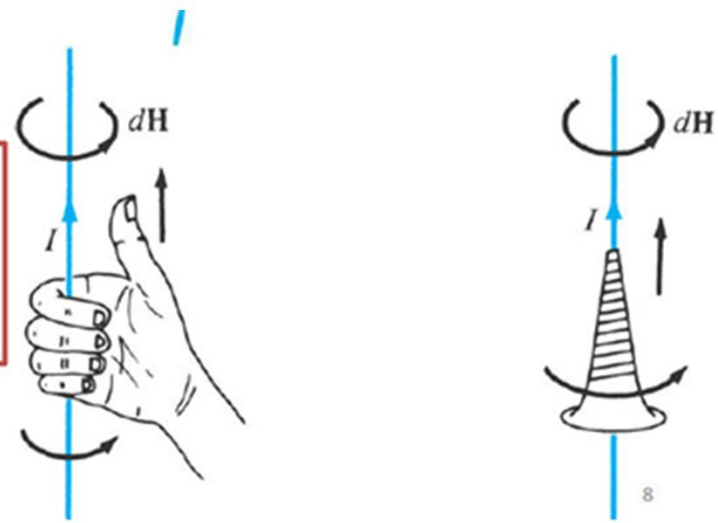
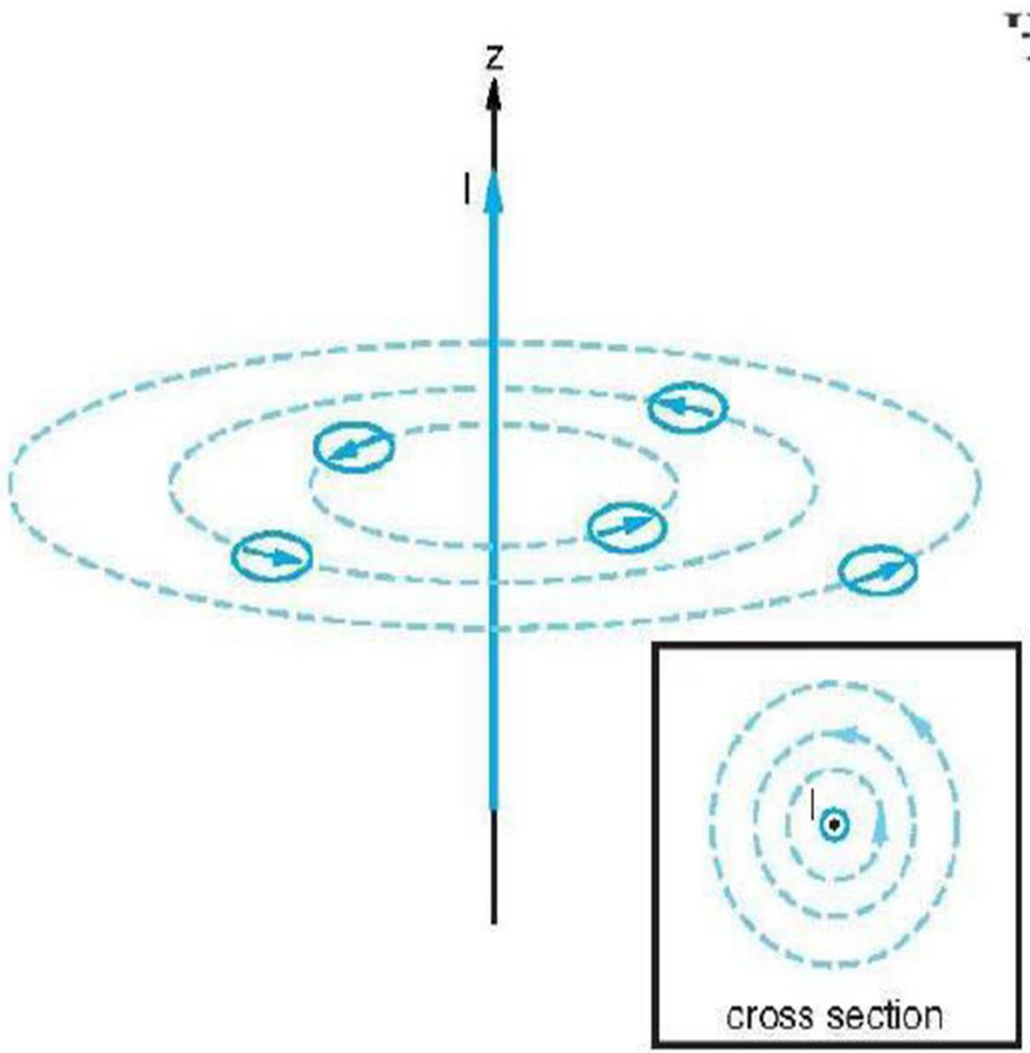


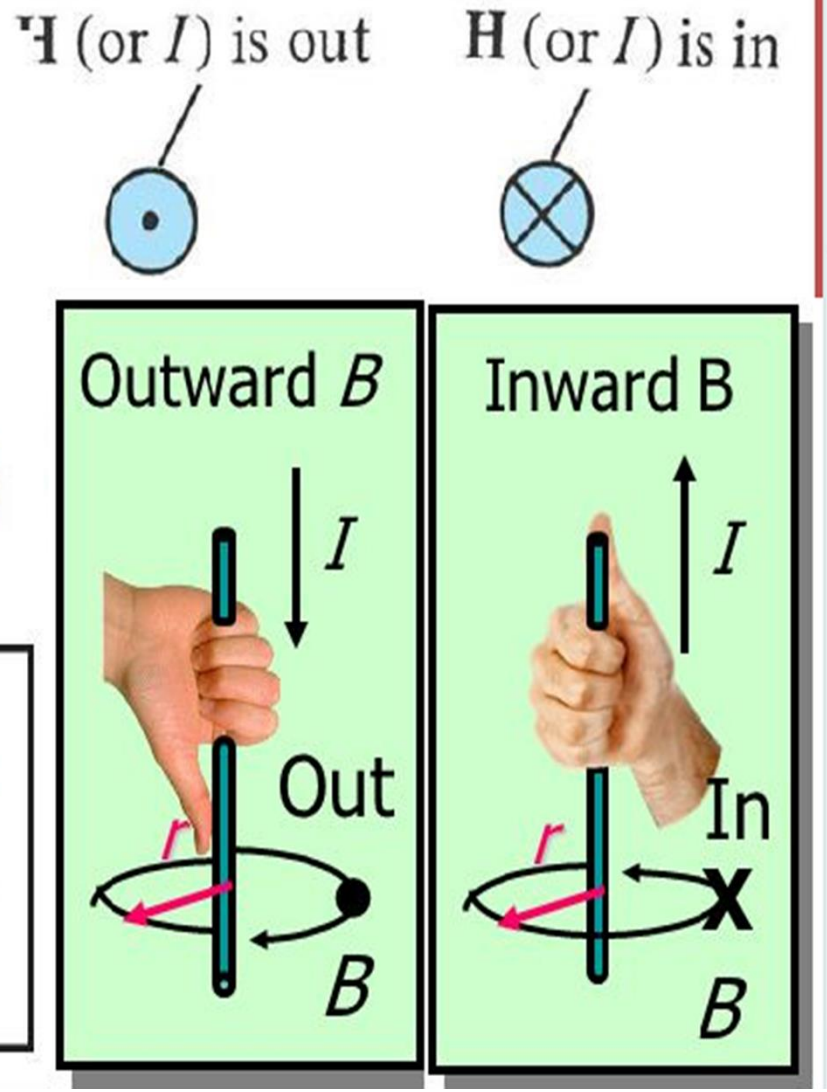
Figure 7.1 The law of Biot-Savart expresses the magnetic field intensity $d\mathbf{H}_2$ produced by a differential current element $I_1 d\mathbf{L}_1$. The direction of $d\mathbf{H}_2$ is into the page.

The direction of *dH* can be determined by the right-hand rule or right-handed screw rule.



The direction of magnetic field intensity **H** (or current **I**) can be represented by a small circle with a dot or cross sign depending on whether **H** (or **I**) is out of, or into the page.



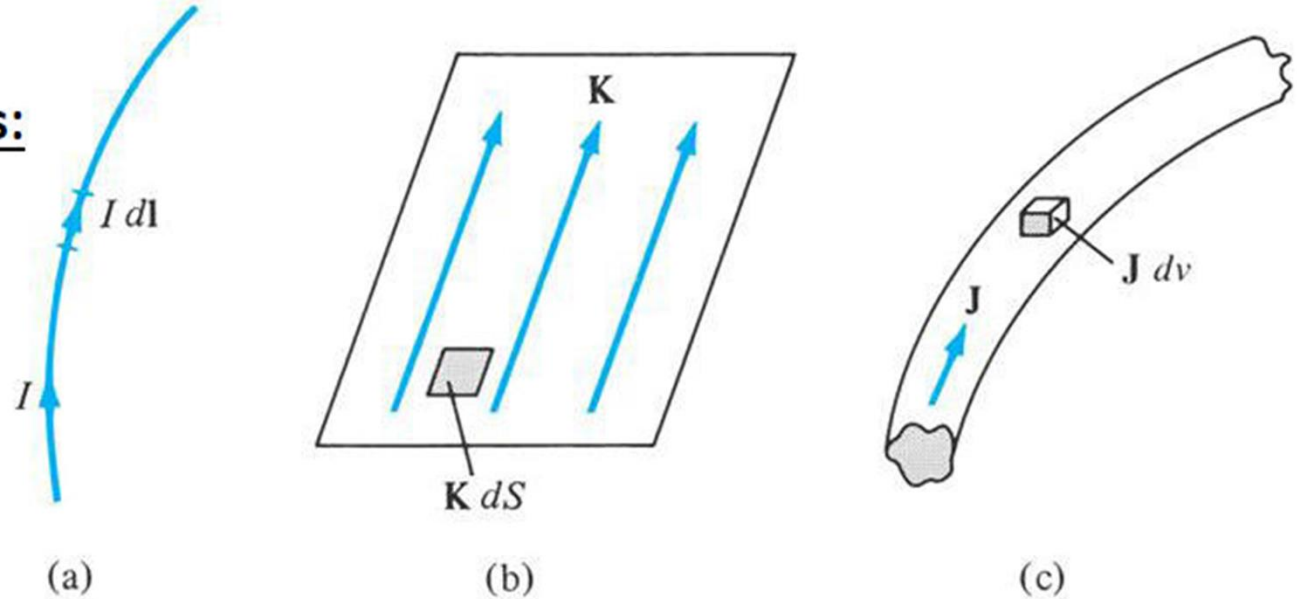


Biot-Savart's Law

For different current distributions:

Current distributions:

(a) line current,
(b) surface current,
(c) volume current.



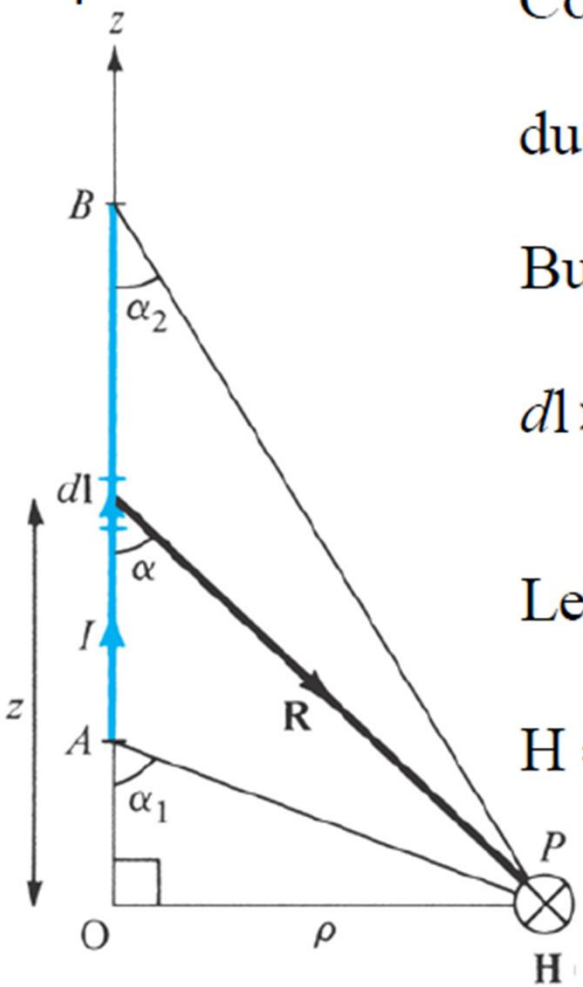
$$H = \int_L \frac{I\, d\mathbf{l} \times a_R}{4\pi R^2} \quad \text{(line current)}$$

$$H = \int_S \frac{\mathrm{K}\, d\mathrm{S} \times a_R}{4\pi R^2} \quad \text{(surface current)} \quad \text{(K: surface current density)}$$

$$H = \int_v \frac{\mathrm{J}\, dv \times a_R}{4\pi R^2} \quad \text{(volume current)} \quad \text{(J: volume current density)}$$

Magnetic Field of straight Conductor

Consider conductor of finite length AB, carrying current from point A to point B.



$\tan\alpha = \rho / z$

$\rightarrow z = \rho\cot\alpha$

$1+\cot^2 x = \operatorname{cosec}^2 x$

Consider the contribution $d\mathrm{H}$ at P

due to $d\mathbf{l}$ at (0,0,z): $d\mathrm{H} = \dfrac{I\ d\mathbf{l}\times\mathrm{R}}{4\pi R^3}$,

But $d\mathbf{l} = dz\ a_z$ and $\mathrm{R} = \rho a_\rho - z a_z$, *so*,

$d\mathbf{l}\times\mathrm{R} = \rho dz\ \mathrm{a}_\phi$, Hence $\mathrm{H} = \displaystyle\int \frac{I\rho dz}{4\pi\left[\rho^2+z^2\right]^{3/2}}\mathrm{a}_\phi$

Letting $z = \rho\cot\alpha$, $dz = -\rho\operatorname{cosec}^2\alpha\, d\alpha$,

$$\mathrm{H} = -\frac{I}{4\pi}\int_{\alpha 1}^{\alpha 2}\frac{\rho^2\operatorname{cosec}^2\alpha\, d\alpha}{\rho^3\operatorname{cosec}^3\alpha}a_\phi = -\frac{I}{4\pi\rho}a_\phi\int_{\alpha 1}^{\alpha 2}\sin\alpha\, d\alpha$$

or $\boxed{\mathrm{H} = \dfrac{I}{4\pi\rho}\left(\cos\alpha_2 - \cos\alpha_1\right)a_\phi}$

$$\boxed{\mathrm{H}=\frac{I}{4\pi\rho}\left(\cos\alpha_2-\cos\alpha_1\right)a_\phi}$$

When the conductor is semi-infinite, so that point A is now at O(0,0,0) while B is at $(0,0,\infty)$, $\alpha_1=90°$, $\alpha_2=0°$:

$$\boxed{\mathrm{H}=\frac{I}{4\pi\rho}a_\phi}$$

When the conductor is of infinite length, point A is at $(0,0,-\infty)$ while B is at $(0,0,\infty)$,$\alpha_1=180°$, $\alpha_2=0°$:

$$\boxed{\mathrm{H}=\frac{I}{2\pi\rho}a_\phi}$$

A simple approach to determine a_ϕ :

$$\boxed{a_\phi=a_l\times a_\rho}$$

where a_l is is a unit vector along the line current, and a_ρ is a unit vector along the perpendicular line from the line current to the field point.

D7.1. Given the following values for P_1, P_2, and $I_1\Delta L_1$, calculate ΔH_2:

(a) $P_1(0, 0, 2)$, $P_2(4, 2, 0)$, $2\pi a_z \mu A\cdot m$;

(b) $P_1(0, 2, 0)$, $P_2(4, 2, 3)$, $2\pi a_z \mu A\cdot m$;

(c) $P_1(1, 2, 3)$, $P_2(-3, -1, 2)$, $2\pi(-a_x + a_y + 2a_z)\ \mu A\cdot m$.

Ans. $-8.51a_x + 17.01a_y$ nA/m; $16a_y$ nA/m; $18.9a_x - 33.9a_y + 26.4a_z$ nA/m

D7.2. A current filament carrying 15 A in the $\mathbf{a}_z$ direction lies along the entire z axis. Find **H** in rectangular coordinates at:

(a) $P_A(\sqrt{20}, 0, 4)$;

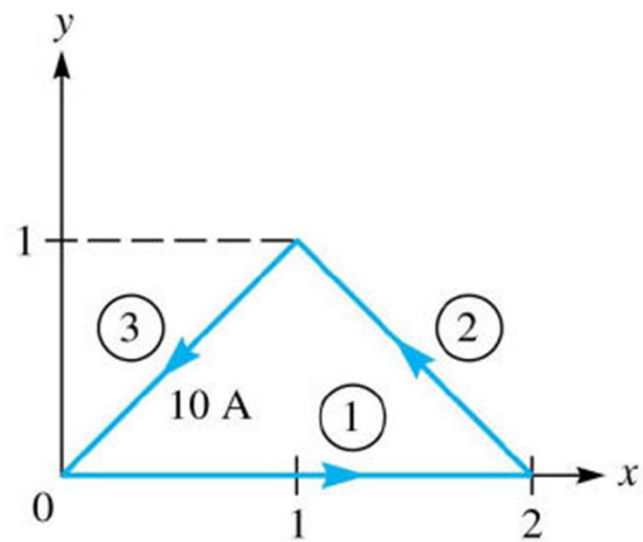
(b) $P_B(2, -4, 4)$.

Ans. $0.534\mathbf{a}_y$ A/m; $0.477\mathbf{a}_x + 0.239\mathbf{a}_y$ A/m

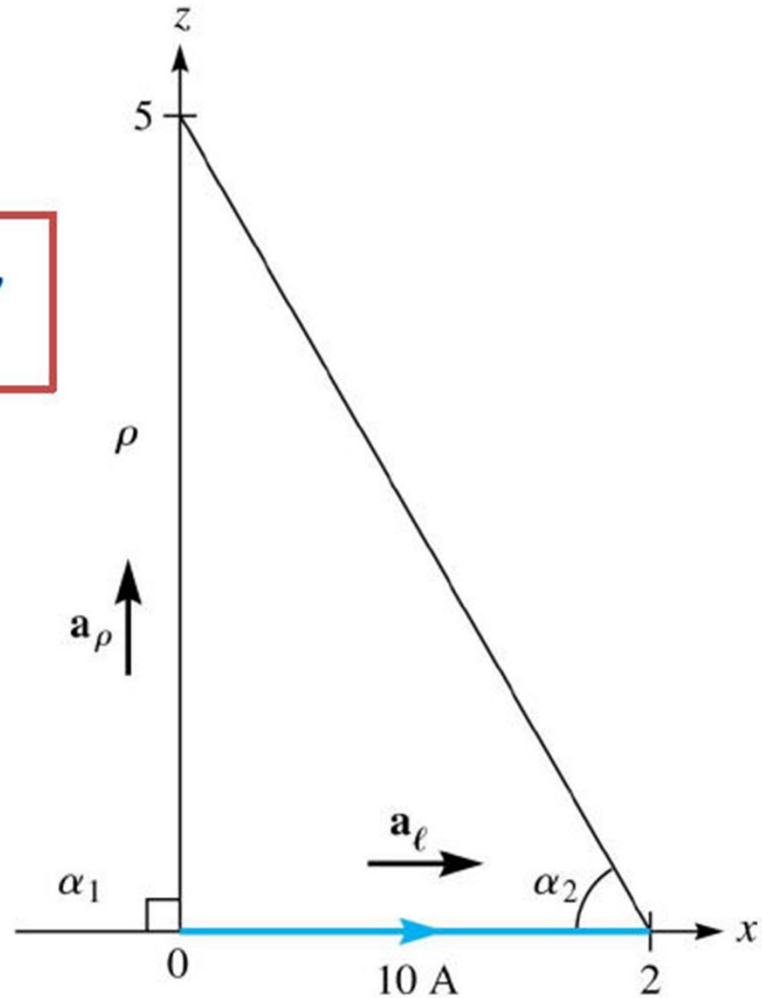
Example 7.1

The conducting triangular loop in the figure carries a current of 10 A. Find **H** at (0,0,5) due to side 1 of the loop.

(a) conducting triangular loop,
(b) side 1 of the loop.



(a)



(b)

Example 7.1 Solution

Applying H= $\frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)a_\phi$, that is valid for any straight, thin, current carrying conductor. α_1 , α_2, and a_ϕ are found as follows:

$\cos\alpha_1 = \cos 90^\circ = 0,\ \cos\alpha_2 = \frac{2}{\sqrt{29}},\ \rho=5$

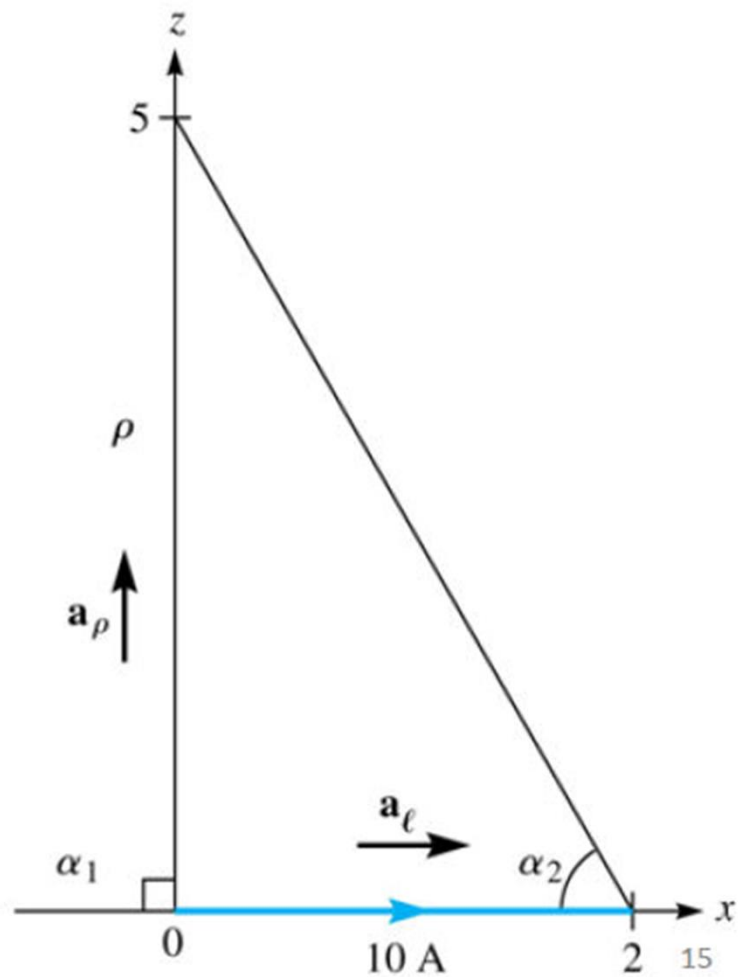
$a_\phi = a_l \times a_\rho,$ but $a_l = a_x,$ and $a_\rho = a_z$

$so\ \ a_\phi = a_x \times a_z = -a_y$, Hence

$$\mathrm{H}_1 = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)a_\phi$$

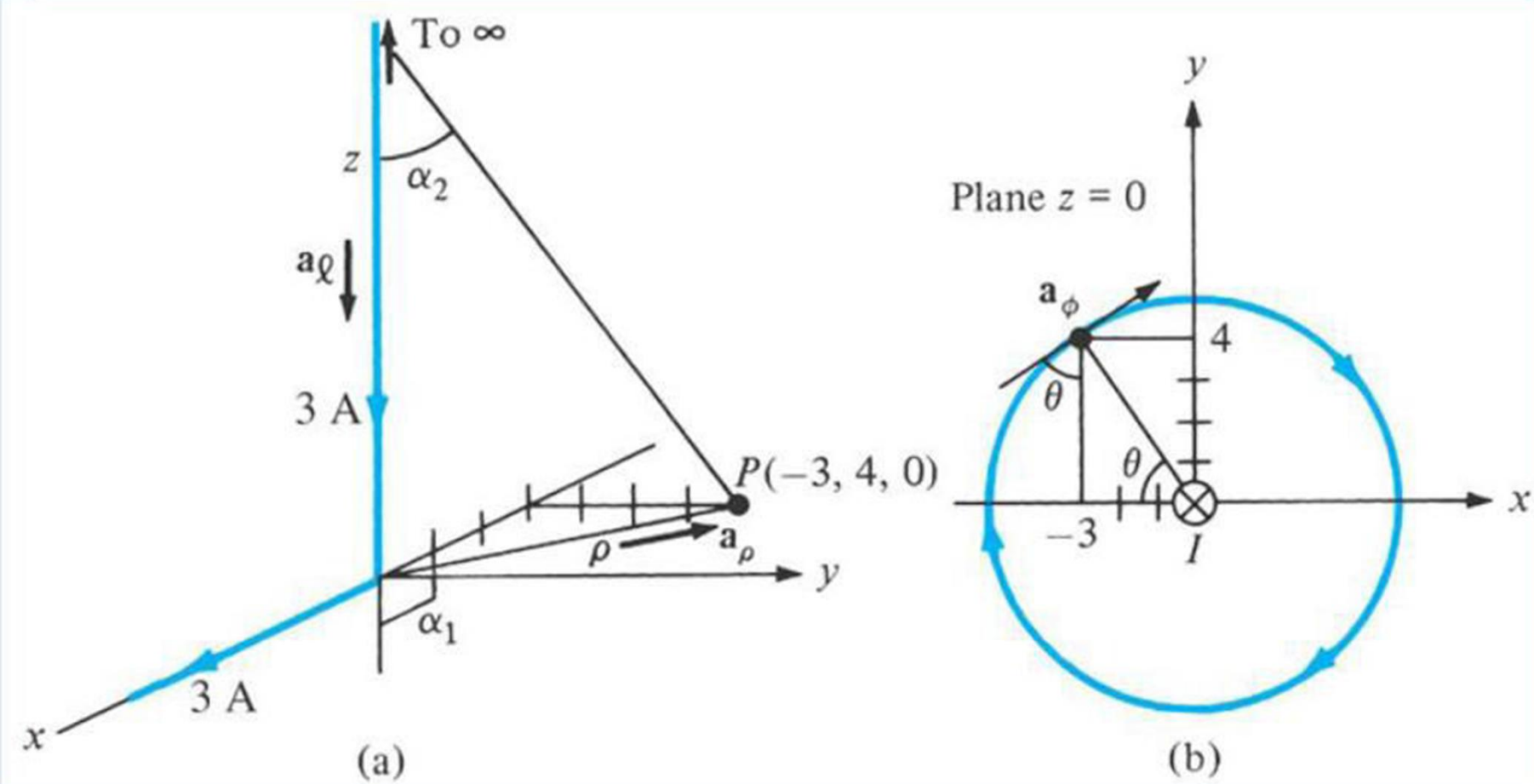
$$\mathrm{H}_1 = \frac{10}{4\pi(5)}\left(\frac{2}{\sqrt{29}} - 0\right)(-a_y)$$

$$\mathrm{H}_1 = -59.1a_y \text{ mA/m}$$



Example 7.2

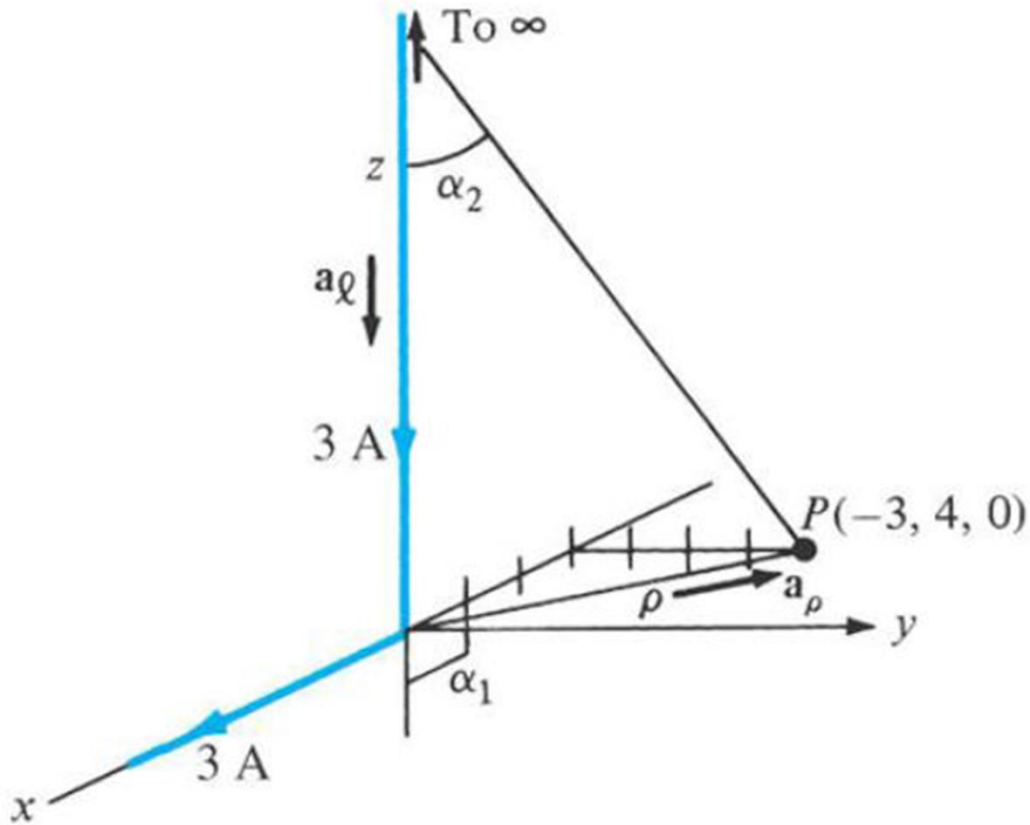
Find **H** at (-3,4,0) due to the current filament shown in the figure.



(a) current filament along semi-infinite *x*- and *z*-axes, $\mathbf{a}_\ell$ and $\mathbf{a}_\rho$ for $\mathbf{H}_2$ only; **(b)** determining $\mathbf{a}_\rho$ for $\mathbf{H}_2$.

Example 7.2 Solution

Let $H=H_1+H_2$ at P(-3,4,0), where H_1 is due to current filament along x-axis, and H_2 is due to current filament along z-axis.



$$H_2 = \frac{I}{4\pi\rho}\left(\cos\alpha_2 - \cos\alpha_1\right)a_\phi$$

At P(-3,4,0), $\rho=(9+16)^{1/2} = 5$,

$\alpha_1=90^\circ$, $\alpha_2=0^\circ$

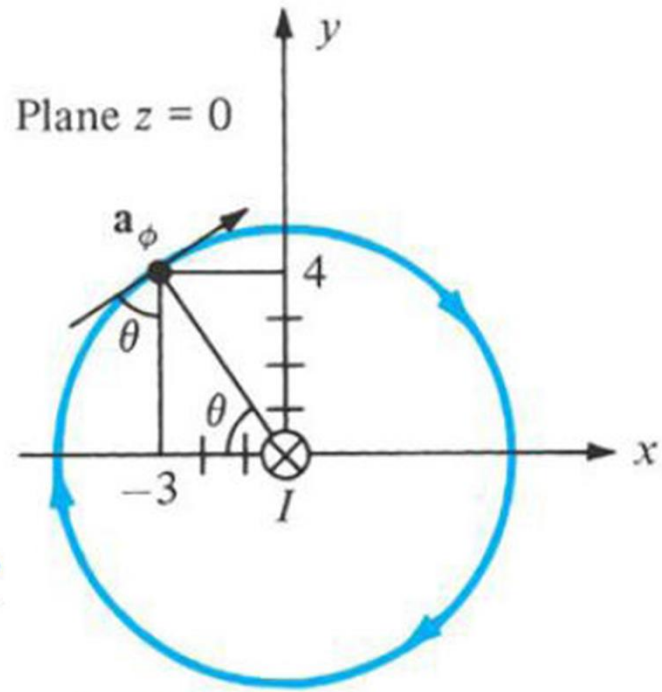
$a_\phi = a_l \times a_\rho$,

but $a_l = -a_z$,

and $a_\rho = \frac{-3}{5}a_x + \frac{4}{5}a_y$

$$a_\phi = -a_z \times \left(\frac{-3}{5}a_x + \frac{4}{5}a_y\right) = \frac{4}{5}a_x + \frac{3}{5}a_y$$

or alternatively, from figure b,



$$a_\phi = \sin\theta a_x + \cos\theta a_y = \frac{4}{5}a_x + \frac{3}{5}a_y$$

Thus, $\mathrm{H}_2 = \dfrac{3}{4\pi(5)}(1-0)\dfrac{(4a_x + 3a_y)}{5}$

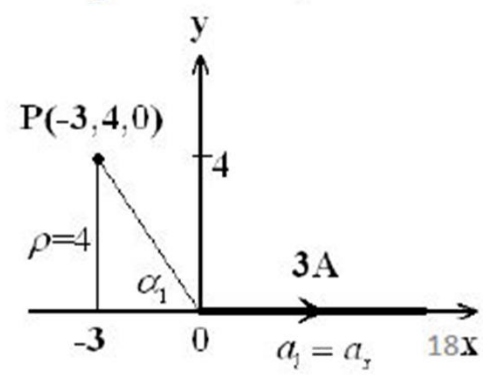
$\mathrm{H}_2 = 38.2a_x + 28.65a_y$ mA/m

a_ϕ in this case is the negative of the regular a_ϕ of cylindrical coordinates: $\mathrm{H}_2 = \dfrac{3}{4\pi(5)}(1-0)(-a_\phi) = -47.75a_\phi$ mA/m

<u>To find H_1 at P:</u>

ρ=4, $\alpha_2 = 0^o$, $\cos\alpha_1 = 3/5$, and $a_\phi = a_l \times a_\rho = a_x \times a_y = a_z$. Hence,

$$\mathrm{H}_1 = \frac{3}{4\pi(4)}\left(1 - \frac{3}{5}\right)a_z = 23.88a_z \text{ mA/m}$$

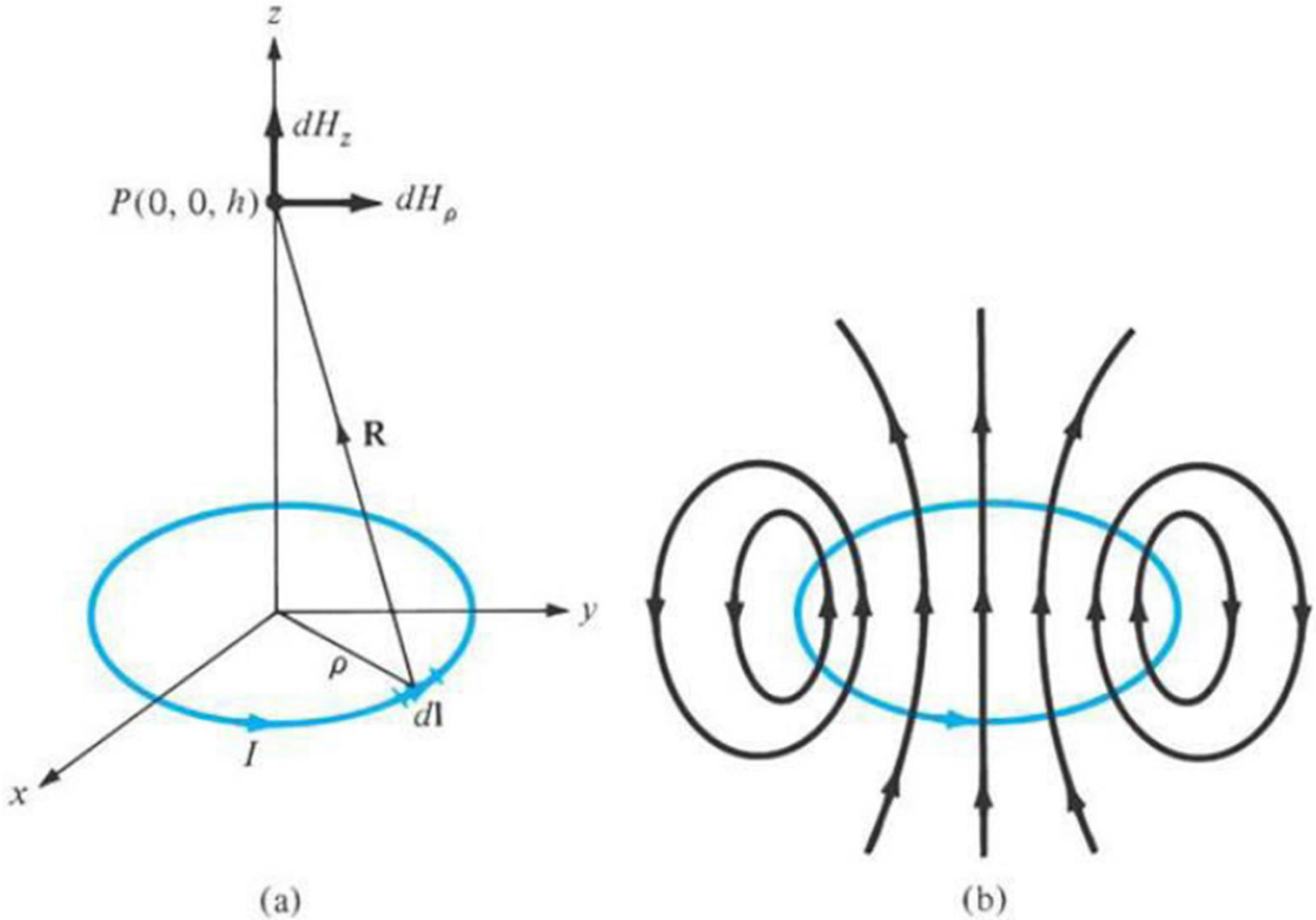


Thus H=H_1+H_2 $= 38.2a_x + 28.65a_y + 23.88a_z$ mA/m

or H= $-47.75a_\phi + 23.88a_z$ mA/m

Example 7.3

A circular loop located at $X^2+y^2=9$, z=0 carries a direct current of 10 A along a_Φ. Determine H at (0,0,4) and (0,0,-4).



(a) circular current loop, **(b)** flux lines due to the current loop.

Example 7.3 - solution

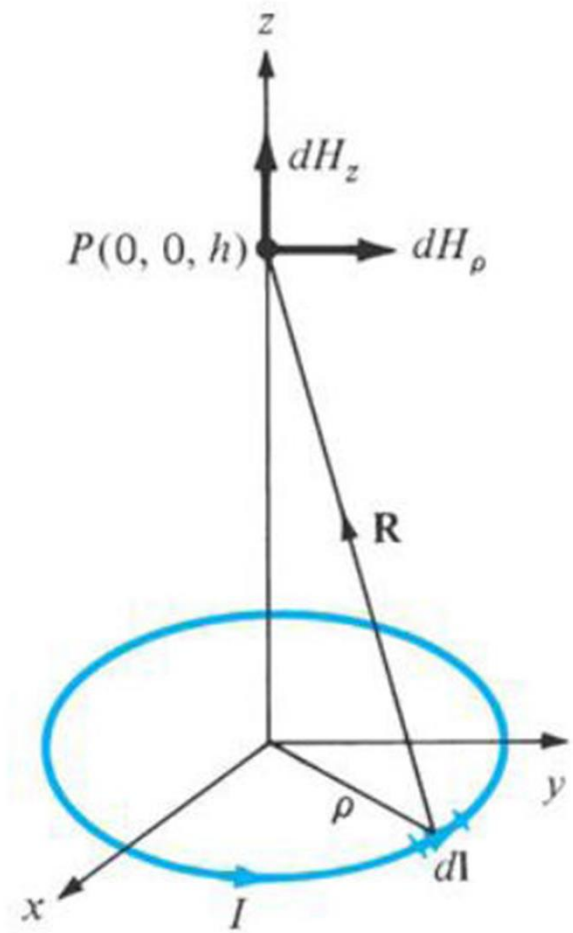
The magnetic field intensity dH at point P(0,0,h) contributed by current element Idl is :

$$d\mathrm{H} = \frac{I\ d\mathrm{l} \times \mathrm{R}}{4\pi R^3}$$

where $d\mathrm{l} = \rho d\phi a_\phi$, $\mathrm{R}=(0,0,h)-(x,y,0)$

$$\mathrm{R} = -\rho a_\rho + h a_z, \text{ and}$$

$$d\mathrm{l} \times \mathrm{R} = \begin{vmatrix} a_\rho & a_\phi & a_z \\ 0 & \rho d\phi & 0 \\ -\rho & 0 & h \end{vmatrix} = \rho h\ d\phi\ a_\rho + \rho^2 d\phi\ a_z$$



Hence,

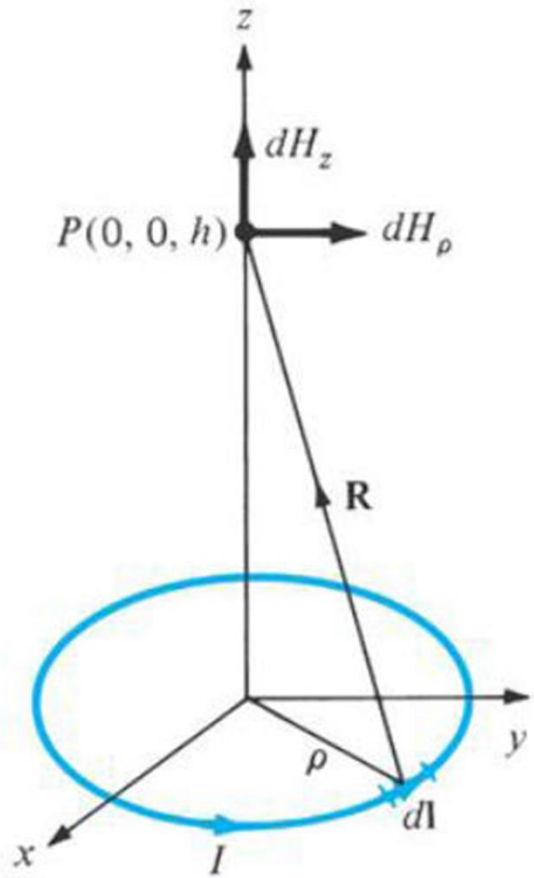
$$d\mathrm{H} = \frac{I}{4\pi\left[\rho^2 + h^2\right]^{3/2}}\left(\rho h\ d\phi\ a_\rho + \rho^2 d\phi\ a_z\right) = dH_\rho a_\rho + dH_z a_z$$

By symmetry, the contributions along a_ρ add up to zero $\rightarrow \mathrm{H}_\rho = 0$

Example 7.3 - solution

Thus, $H=\int dH_z a_z = \int_0^{2\pi} \frac{I\rho^2 d\phi\, a_z}{4\pi\left[\rho^2+h^2\right]^{3/2}} = \frac{I\rho^2 2\pi\, a_z}{4\pi\left[\rho^2+h^2\right]^{3/2}}$

or $H=\frac{I\rho^2}{2\left[\rho^2+h^2\right]^{3/2}} a_z$



(a) Substituting I=10 A, ρ=3, h=4 gives

$$H(0,0,4)=\frac{10(3)^2\, a_z}{2[9+16]^{3/2}} = 0.36\, a_z \quad \text{A/m}$$

(b) replacing h by -h,

$$d\mathbf{l}\times \mathbf{R} = -\rho h\, d\phi\, a_\rho + \rho^2 d\phi\, a_z$$

(z-component remains the same)

Hence, $H(0,0,4) = H(0,0,-4) = 0.36\, a_z \quad \text{A/m}$

AMPERE'S CIRCUIT LAW - MAXWELL'S EQUATION

Ampere's circuit law states that the line integral of the tangential component of **H** around a closed path is the same as the net current $\mathbf{I}_{\mathbf{enc}}$ enclosed by the path.

In other words, the circulation of H equals I_{enc}; that is,

$$\boxed{\oint \mathrm{H} \cdot d\mathrm{l} = I_{\text{enc}}}$$ (integral form of 3rd Maxwell's equation)

Apply Stoke's theorem,

$$I_{\text{enc}} = \oint_L \mathrm{H} \cdot d\mathrm{l} = \int_S (\nabla \times \mathrm{H}) \cdot d\mathrm{S}$$

But, $I_{\text{enc}} = \int_S \mathrm{J} \cdot d\mathrm{S}$

$\rightarrow$ $\boxed{\nabla \times \mathrm{H} = \mathrm{J}}$ (differential form of third Maxwell's equation)

AMPERE'S CIRCUIT LAW - MAXWELL'S EQUATION

$$\boxed{\oint \mathrm{H} \cdot d\mathrm{l} = I_{\mathrm{enc}}}$$

$\rightarrow \oint_L \mathrm{H} \cdot d\mathrm{l} \rightarrow$ apply to Amperian path (L).

$\rightarrow I_{\mathrm{enc}} \rightarrow$ current enclosed by amperian path.

I_{enc} is found if $\mathrm{J}(\mathrm{A/m^2})$ is known: $I_{\mathrm{enc}} = \int_S \mathrm{J} \cdot d\mathrm{S}$

or if $\mathrm{K}(\mathrm{A/m})$ is known for surface current density.

APPLICATIONS OF AMPER'S LAW

A. Infinite Line Current

Consider infinite filamentary current ***I*** along the z-axis. To determine H at point *P*, allow **Amperian** path passing through ***P***, such that **H** is constant provided **ρ** is constant.

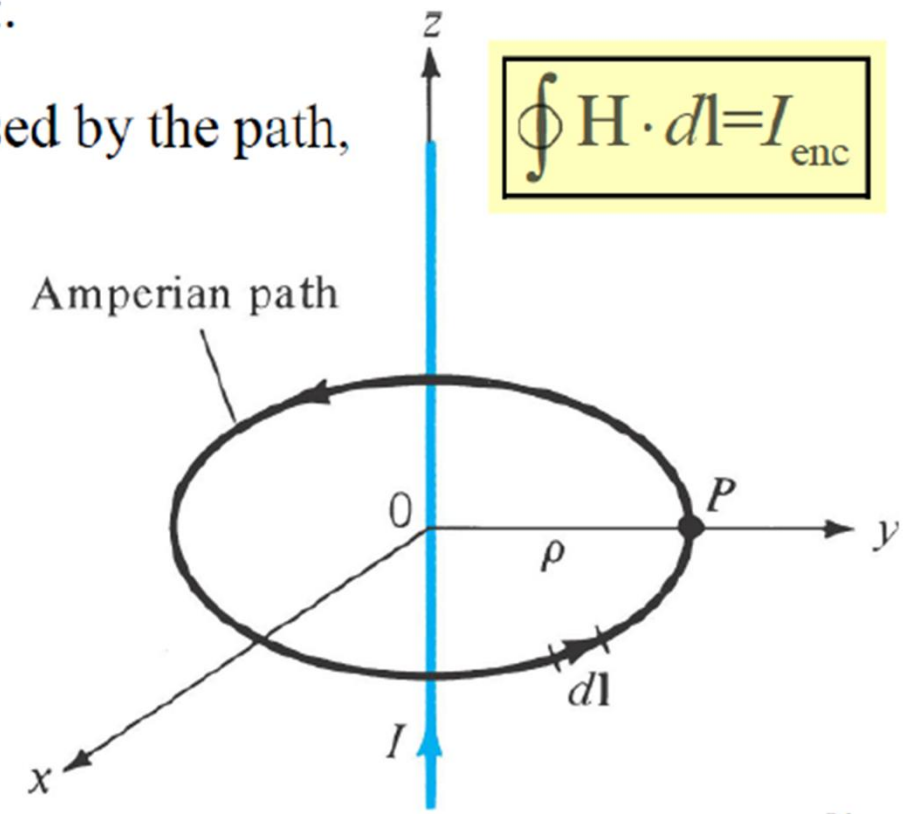
$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc}$$

The whole current I is enclosed by the path, according to Amper's law:

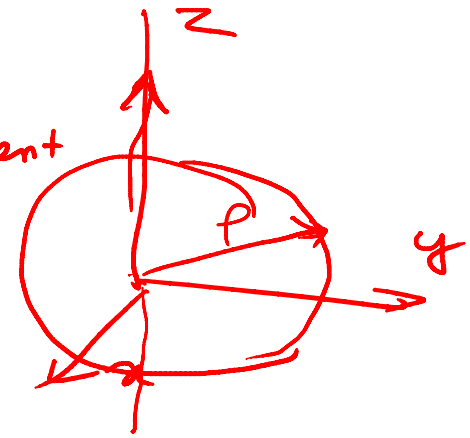
$$I = \int H_\phi \mathbf{a}_\phi \cdot \rho \, d\phi \, \mathbf{a}_\phi$$

$$I = H_\phi \int_0^{2\pi} \rho \, d\phi = H_\phi (2\pi\rho)$$

or $$\boxed{\mathbf{H} = \frac{I}{2\pi\rho} \mathbf{a}_\phi}$$



H-field for a current element



Infinite element

$$\oint \vec{H} \cdot \vec{dl} = I_{enc}$$

$$H = H_\phi a_\phi, \qquad \vec{dl} = \rho \, d\phi \, a_\phi$$

$$\oint \vec{H} \cdot \vec{dl} = \int H_\phi \hat{a}_\phi \cdot \rho \, d\phi \, \hat{a}_\phi = I_{enc} = I$$

$$= H_\phi \rho \cdot \int_0^{2\pi} d\phi$$

$$= H_\phi \rho \cdot [\phi]_0^{2\pi} = H_\phi \rho \cdot 2\pi = I_{enc}$$

$$H_\phi = \frac{I_{enc}}{2\pi\rho}, \qquad H = \frac{I}{2\pi\rho} \hat{a}_\phi$$

B. Infinite Sheet of Current

Consider an infinite current sheet in the z=0 plane with uniform current desnity $K = K_y a_y$ A/m. Applying Ampere's law to the rectangular closed 1-2-3-4-1 path gives: $\oint H \cdot dl = I_{enc} = K_y b$

$H = \frac{1}{2} K \times \hat{a}_N$

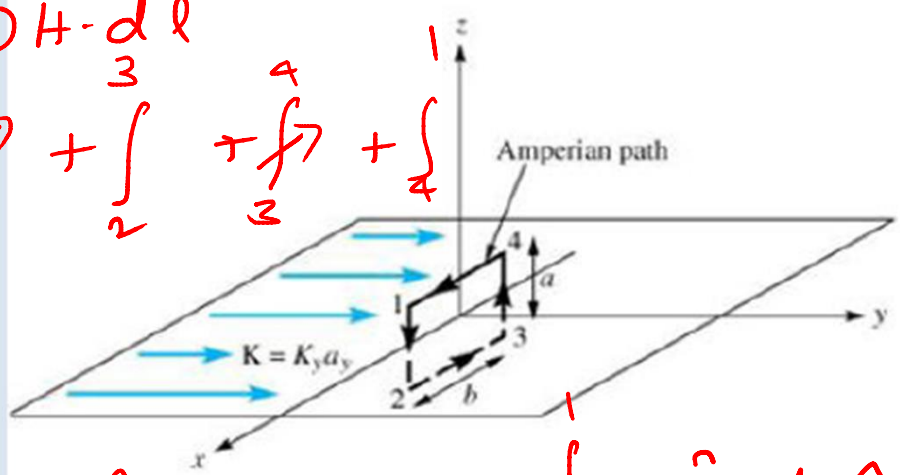
→ Consider the sheet as comprising of filaments

→ Consider dH above and below the sheet due to a pair of filamentary currents.

$2 H_x b = K_y b, \quad H_x = \frac{1}{2} K_y$

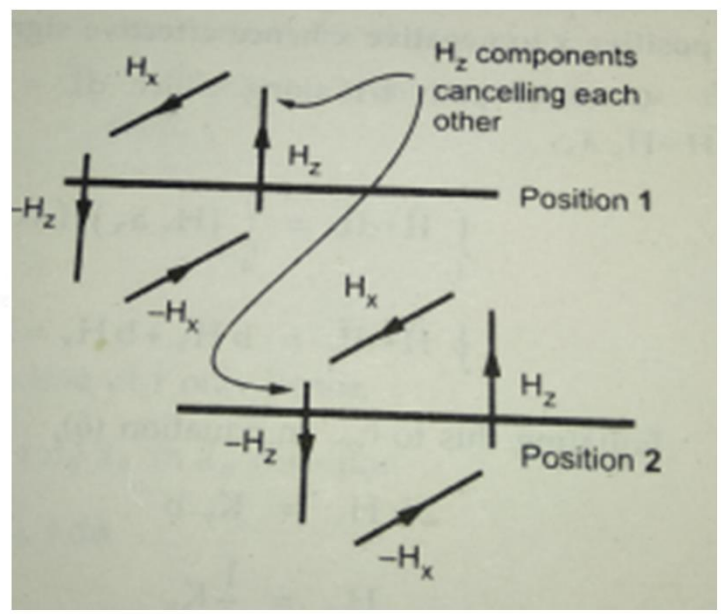
$\oint H \cdot dl$

$\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1$



$\int_2^3 H_x \hat{a}_x \cdot (-dx \hat{a}_x) + \int_4^1 H_x \hat{a}_x \cdot dx \hat{a}_x$

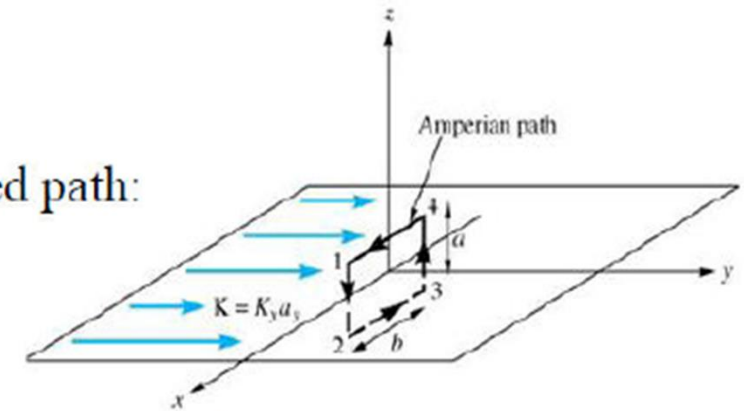
$H_x b + H_x b = 2 H_x b$



→ The resultant dH has only an x-component.

→ H on one side is the negative of that on the other side.

$$\mathrm{H}=\begin{cases} H_0 a_x & z>0 \\ -H_0 a_x & z<0 \end{cases}$$

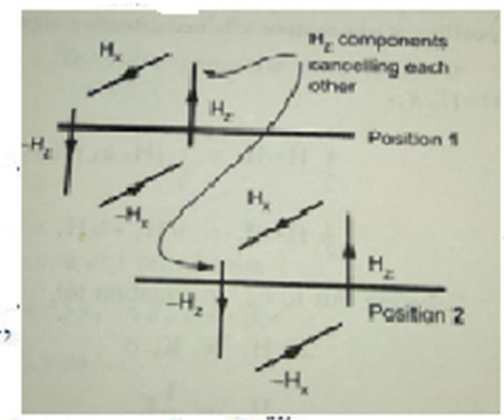


Evaluating the line integral of H along the closed path:

$$\oint \mathrm{H}.d\mathrm{l} = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \right) \mathrm{H}.d\mathrm{l} = I_{\mathrm{enc}} = K_y b$$

$$0(-a)+(-H_0)(-b)+0(a)+H_0(b)=2H_0 b = K_y b \rightarrow H_0 = \frac{1}{2}K_y$$

$$\mathrm{H}=\begin{cases} \frac{1}{2}K_y a_x & z>0 \\ -\frac{1}{2}K_y a_x & z<0 \end{cases}$$



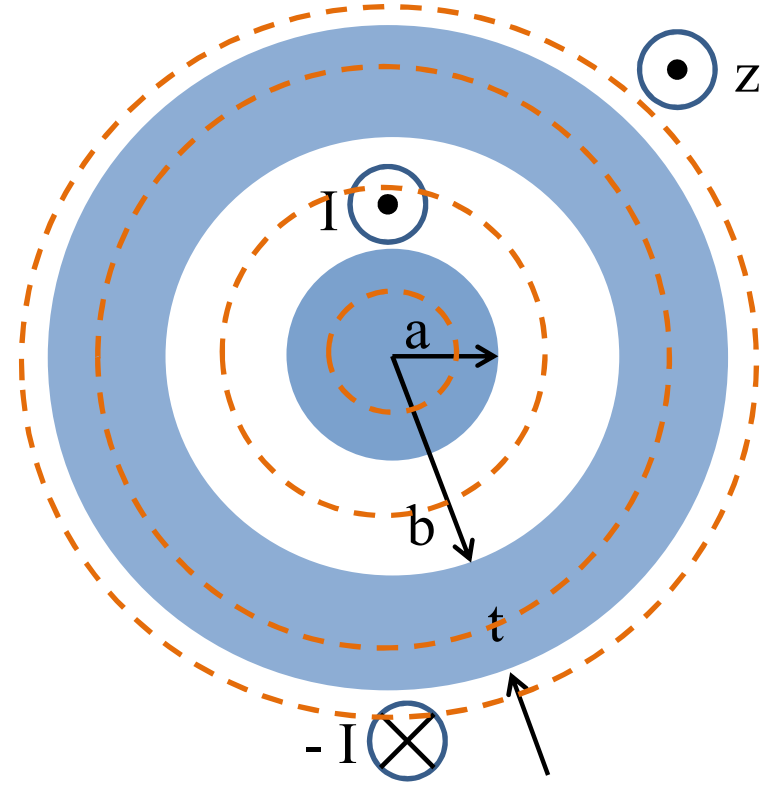
In general, for an infinite sheet of current density K A/m,

$$\boxed{\mathrm{H}=\frac{1}{2}\mathrm{K}\times \mathrm{a_n}}$$ where $\mathrm{a_n}$ is a unit normal vector directed from the current sheet to the point of interest

Infinite Volume Current

Infinitely Long Coaxial Line

- We consider a coaxial transmission line with inner & outer conductors carrying $+I$ & $-I$ currents respectively



For the region $0 \le \rho \le a$

$$\oint H \bullet dL = I_{enc} = \int J \bullet dS$$

$$\Rightarrow J = \frac{I}{\pi a^2} a_z \quad \& \quad dS = \rho d\phi d\rho a_z$$

For the volume of inner conductor, we have

$$I_{enc} = \int J \bullet dS = \frac{I\rho^2}{a^2}$$

Evaluation LHS we have $\oint H \bullet dL = H_\phi \int_L dl = H_\phi 2\pi\rho \Rightarrow H_\phi = \dfrac{I\rho}{2\pi a^2}$

Infinite Volume Current

For the region a ≤ ρ ≤ b $\oint H \bullet dL = I_{enc} = I$

$$\Rightarrow H_\phi = \frac{I}{2\pi\rho}$$

For the region b ≤ ρ ≤ b+t $\oint H \bullet dL = H_\phi 2\pi\rho = I_{enc}$

$I_{enc} = I + \int J \bullet dS$ But we have $J = -\frac{I}{\pi\left[(b+t)^2 - b^2\right]} a_z$

$$\therefore I_{enc} = I\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right] \qquad H_\phi = \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

- For the region ρ ≥ b+t $\oint H \bullet dL = I - I = 0$ as it encloses both lines with $+I$ & $-I$

Infinite Volume Current

- For the given infinite coaxial transmission line we can write H as

$$H = \begin{cases} \dfrac{\mathrm{I}\rho}{2\pi \mathrm{a}^2}\mathrm{a}_\phi & \forall 0 \le \rho \le a \\ \\ \dfrac{I}{2\pi\rho}\mathrm{a}_\phi & \forall a \le \rho \le b \\ \\ \dfrac{I}{2\pi\rho}\left[1 - \dfrac{\rho^2 - b^2}{t^2 + 2bt}\right]a_\phi & \forall b \le \rho \le b + t \\ \\ 0 & \forall \rho \ge b + t \end{cases}$$

$$\oint \vec{H}\cdot d\vec{l} = I_{encl}$$

$$\lim_{\Delta s \to 0} \oint \frac{H\cdot dl}{\Delta s}\hat{a}_n = \lim_{\Delta s \to 0} \frac{I_{encl}}{\Delta s}\hat{a}_N$$

Circulation $\nabla \times H = J$

$$\oint H\cdot dl = J\Delta s$$

Stokes theorem

$$\oint H\cdot dl = \int (\nabla \times H)\cdot ds$$

Curl

rectangular

$$\nabla \times H = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

Cylindrical

$$\nabla \times H = \frac{1}{\rho}\begin{bmatrix} a_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_\rho & \rho H_\phi & H_z \end{bmatrix}$$

Spherical Coordinate

$$\nabla \times H = \frac{1}{r^2 \sin\theta}\begin{bmatrix} a_r & r a_\theta & r\sin\theta\, a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_r & r H_\theta & r\sin\theta\, H_\phi \end{bmatrix}$$

D7.3. Express the value of H in rectangular components at P(0, 0.2, 0) in the field of:

(a) a current filament, 2.5 A in the $\mathbf{a}_z$ direction at x = 0.1, y = 0.3;

(b) a coax, centered on the z axis, with a = 0.3, b = 0.5, c = 0.6, I = 2.5 A in the $\mathbf{a}_z$ direction in the center conductor;

(c) three current sheets, $2.7\mathbf{a}_x$ A/m at y = 0.1, $-1.4\mathbf{a}_x$ A/m at y = 0.15, and $-1.3\mathbf{a}_x$ A/m at y = 0.25.

Ans. 1.989ax – 1.989ay A/m; –0.884ax A/m; 1.300az A/m

(0, 0.2, 0)

P(0, 0.2, 0)

$$\rho = \sqrt{x^2+y^2} = \sqrt{0+(0.2)^2}$$

$$\rho = 0.2$$

$$H = \frac{I}{2\pi\rho} a_\phi$$

$$= (0, 0.2) - (0.1, 0.3)$$

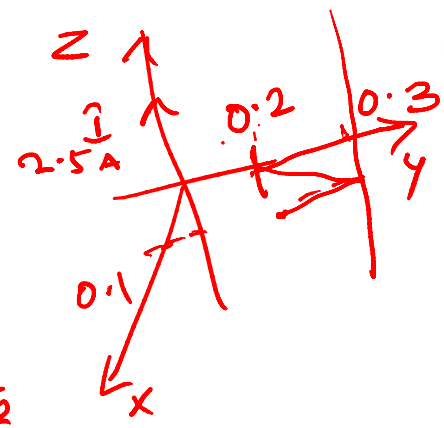
$$= -0.1, -0.1$$

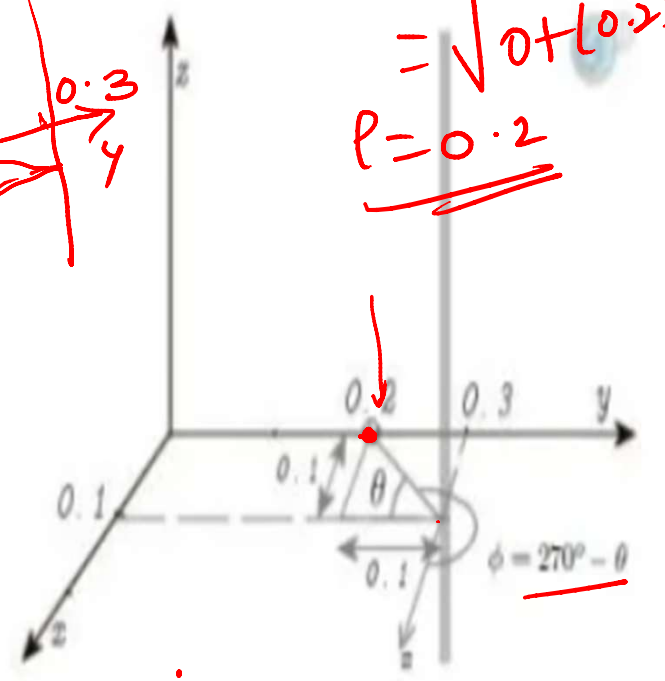
$$\rho = \sqrt{(0.1)^2+(0.1)^2} = \frac{\sqrt{2}}{10} = 0.141$$

$$\hat{a}_\phi = -\sin\phi\,\hat{a}_x + \cos\phi\,\hat{a}_y$$

$$\phi = 270^\circ - \theta = 270^\circ - \tan^{-1}\left(\frac{y}{x}\right)$$

$$= 270^\circ - \tan^{-1}\left(\frac{0.1}{0.1}\right) \quad \phi = 225^\circ$$





$$H = \frac{1}{2} K \times \hat{a}_n$$

$$H_1 = \frac{1}{2}(2.7\,\hat{a}_x \times \hat{a}_y) = 1.35\,\hat{a}_z$$

$$H_2 = \frac{1}{2}(-1.4\,\hat{a}_x \times \hat{a}_y) = -0.7\,\hat{a}_z$$

$$H_3 = \frac{1}{2}(-1.3\,\hat{a}_x \times -\hat{a}_y) = 0.65\,\hat{a}_z$$

$$\vec{H}_T = \vec{H}_1 + \vec{H}_2 + \vec{H}_3 = 1.3\,\hat{a}_z \text{ A/m}$$

D7.5. Calculate the value of the vector current density:

(a) in rectangular coordinates at $P_A(2, 3, 4)$ if

$\mathbf{H} = x^2 z\, \mathbf{a}_y - y^2 x \mathbf{a}_z$;

(b) in cylindrical coordinates at $P_B(1.5, 90^\circ, 0.5)$ if $H = 2/\rho\, (\cos 0.2\phi)\mathbf{a}_\rho$;

(a) in spherical coordinates at $P_C(2, 30^\circ, 20^\circ)$ if $H = 1/\sin\theta\, a_\theta$.

Ans. $-16a_x + 9a_y + 16a_z$ A/m^2 ; $0.055a_z$ A/m2 ; a_ϕ A/m^2

D7.5 $\quad J = \nabla \times H \quad , \quad P_a(2,3,4)$

a) $J = \nabla \times H = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_y \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x^2 z & -y^2 x \end{vmatrix}$

$$J = \hat{a}_x \left[\frac{\partial}{\partial y}(-y^2 x) - \frac{\partial}{\partial z} x^2 z\right] - \hat{a}_y \left[\frac{\partial}{\partial x}(-y^2 x) - 0\right]$$
$$+ \hat{a}_z \left[\frac{\partial}{\partial x} x^2 z - 0\right]$$

$$= \hat{a}_x (-2xy - x^2) - \hat{a}_y (-y^2) + \hat{a}_z (2xz)$$

$$J = -16\hat{a}_x + 9\hat{a}_y + 16\hat{a}_z$$

b) $J = \nabla \times H = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_\rho & \rho H_\phi & H_z \end{vmatrix}$

$$= \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & -\rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \frac{2}{\rho}(\cos 0.2\phi) & 0 & 0 \end{vmatrix}$$

$$= \frac{1}{\rho}\left\{ a_\rho (0) - \rho\hat{a}_\phi \left(0 - \frac{\partial}{\partial z} \frac{2}{\rho} \cos 0.2\phi\right) + \hat{a}_z \right.$$

$$\left. = \quad a_z\left[-\frac{\partial}{\partial \phi} \left(\frac{2}{\rho} \cos(-0.2\phi)\right)\right] \right\}$$

$$= \frac{-1}{\rho} \frac{2}{\rho} (-\sin 0.2\phi) \cdot 0.2 \; \hat{a}_z \qquad \phi = 90^\circ, \; \rho = 1.5$$

$$= \frac{-2}{(1.5)^2} (\sin 0.2 \times 90^\circ) \times 0.2 \; \hat{a}_z$$

$$J = 0.0549 \; A/m^2$$

$$J = (\nabla \times H) = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\,\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_r & rH_\theta & r\sin\theta\,H_\phi \end{vmatrix}$$

$$J = \hat{a}_\phi \ \ A/m^2$$

D7.4.

(a) Evaluate the closed line integral of H about the rectangular path $P_1(2, 3, 4)$ to $P_2(4, 3, 4)$ to $P_3(4, 3, 1)$ to $P_4(2, 3, 1)$ to P_1, given $\mathbf{H} = 3z\mathbf{a}_x - 2x^3\mathbf{a}_z$ A/m.

(b) Determine the quotient of the closed line integral and the area enclosed by the path as an approximation to $(\nabla \times \mathbf{H})_y$.

(c) Determine $(\nabla \times \mathbf{H})_y$ at the center of the area. Ans. 354 A; 59 A/m^2; 57 A/m^2

a) $$\oint H\cdot dl = \int_{\substack{x=2\\z=4}}^{4} 3z\,dx + \int_{\substack{z=4\\x=4}}^{1} (-2x^3)\,dz + \int_{\substack{x=4\\z=1}}^{2} 3z\,dx + \int_{\substack{z=1\\x=2}}^{4} (-2x^3)\,dz$$

$$\oint H\cdot dl = 24 + 384 - 6 - 48 = 354\text{ A}$$

b) $$\frac{\oint H\cdot dl}{\Delta S} \cong (\nabla\times H)_y \qquad \Delta S = xz = 2\times 3 = 6$$

$$(\nabla\times H)_y = \frac{\oint H\cdot dl}{\Delta S} = \frac{354}{6} = 59\text{ A/m}^2$$

c) $$(\nabla\times H)_y = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3z & 0 & -2x^3 \end{vmatrix} = (6x^2+3) \rightarrow$$

at center

$$x = \frac{2+4}{2} = 3$$

$$z = \frac{1+4}{2} = 2.5$$

$x = 2.5,\ z = 3,$

$$(\nabla\times H)_y = 57\text{ A/m}^2$$

D7.6. Evaluate both sides of Stokes' theorem for the field $\mathbf{H} = 6xy\mathbf{a}_x - 3y^2\mathbf{a}_y$ A/m and the rectangular path around the region, $2 \le x \le 5$, $-1 \le y \le 1$, $z = 0$. Let the positive direction of d**S** be $\mathbf{a}_z$.

Ans. −126 A; −126 A

$$\oint H \cdot dl = \int (\nabla \times H) \cdot ds$$

$$\oint H \cdot dl =$$

D7.7. A solid conductor of circular cross section is made of a homogeneous nonmagnetic material. If the radius a = 1 mm, the conductor axis lies on the z axis, and the total current in the $\mathbf{a}_z$ direction is 20 A, find:

(a) H_ϕ at $\rho = 0.5$ mm;

(b) B_ϕ at $\rho = 0.8$ mm;

(c) the total magnetic flux per unit length inside the conductor;

(d) the total flux for $\rho < 0.5$ mm;

(e) the total magnetic flux outside the conductor.

Ans. 1592 A/m; 3.2 mT; 2 μWb/m; 0.5μWb:∞

Magnetic Flux Density

•Electric flux density and Electric field intensity are related by **D=ε_0 E** in free space.

•Similarly, the **magnetic flux density B** is related to the **magnetic field intensity H** by:

$$\boxed{\mathrm{B} = \mu_0 \mathrm{H}}$$

•Where μ_0 is known as the ***permeability*** *of free space*.

$$\boxed{\mu_0 = 4\pi \times 10^{-7} \quad \mathrm{H/m}}$$

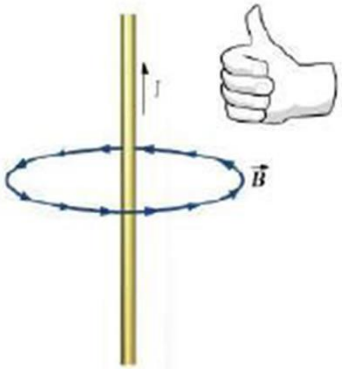
•The magnetic flux through a surface is given by

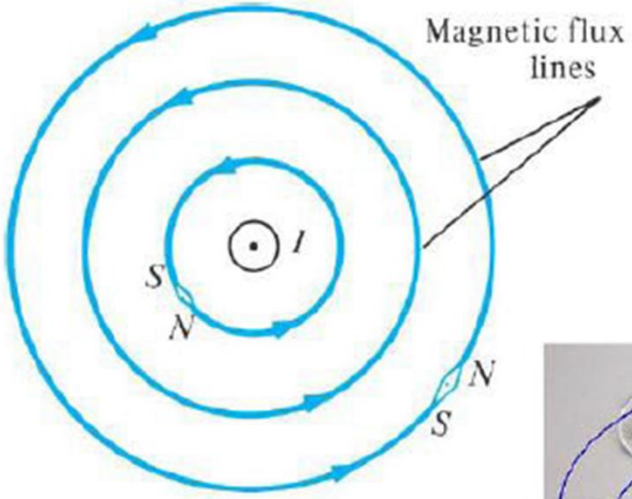
$$\boxed{\Psi = \int_S \mathrm{B} \cdot d\mathrm{S}}$$

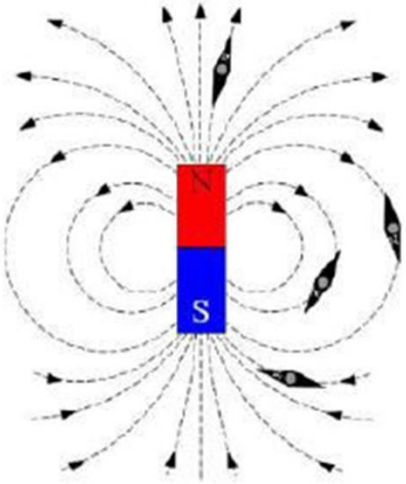
•Where **Ψ** is in webers (Wb), B is in (**Wb/m²**) or **teslas** (T).

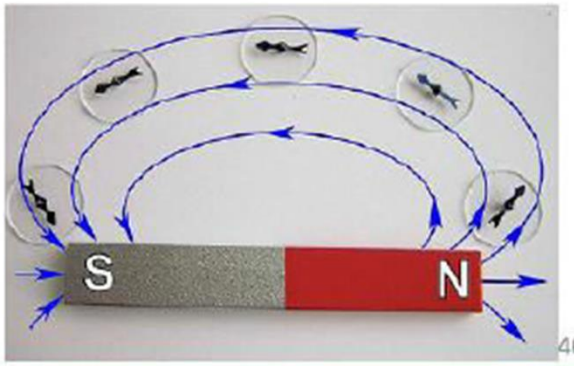
Magnetic Flux Lines

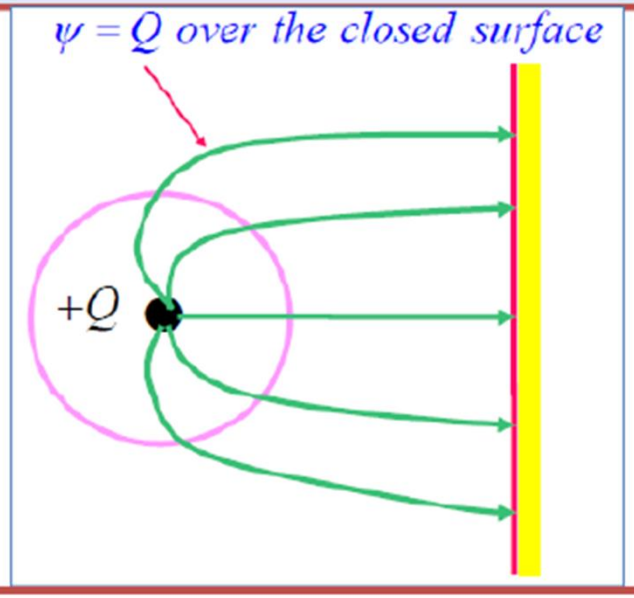
- Magnetic flux line is a path to which **B** is tangential at every point on the line.
- Each flux line is **closed** and has no beginning or end.

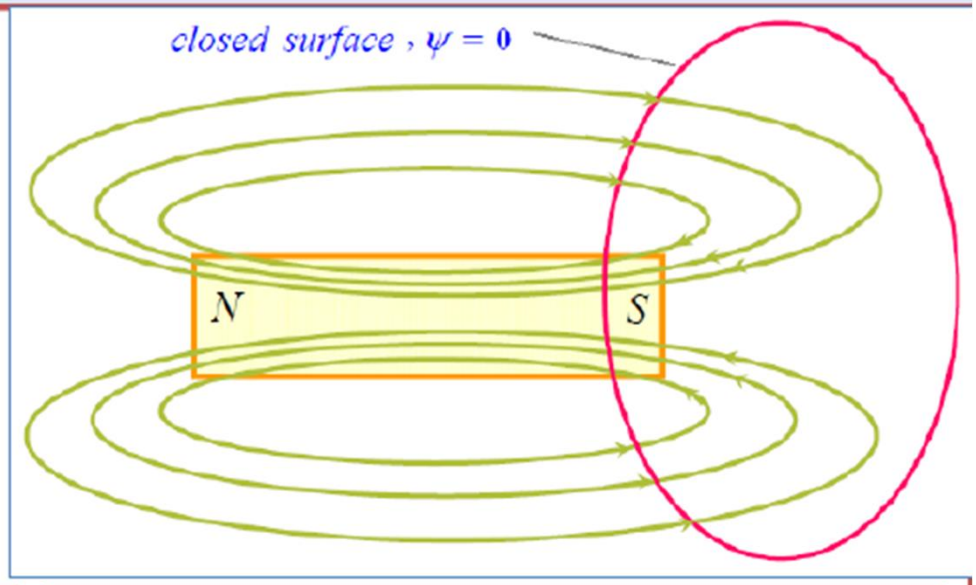












•In an electrostatic field the flux crossing a closed surface is the same as the charge enclosed.

$$\Psi = \oint_S \mathrm{D} \cdot d\mathrm{S} = Q$$

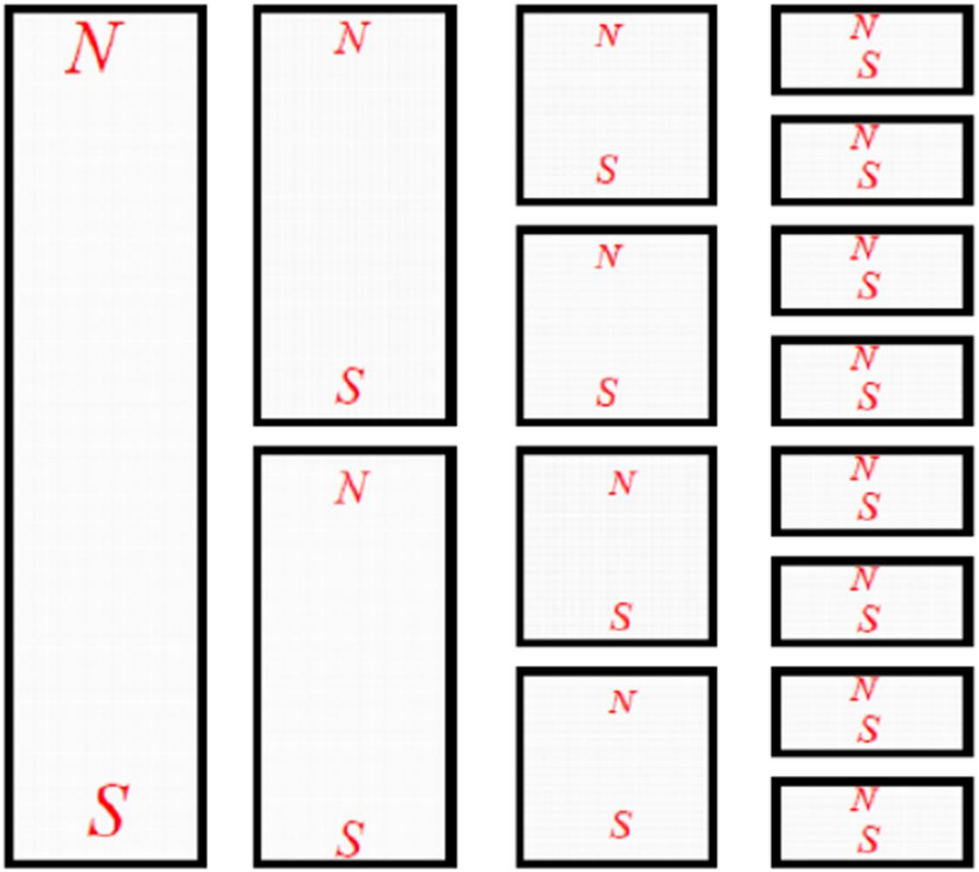
•So it is possible to have an isolated electric charge and the flux lines produced by it need not be closed.

•Unlike electric flux lines, magnetic flux lines always close upon themselves.

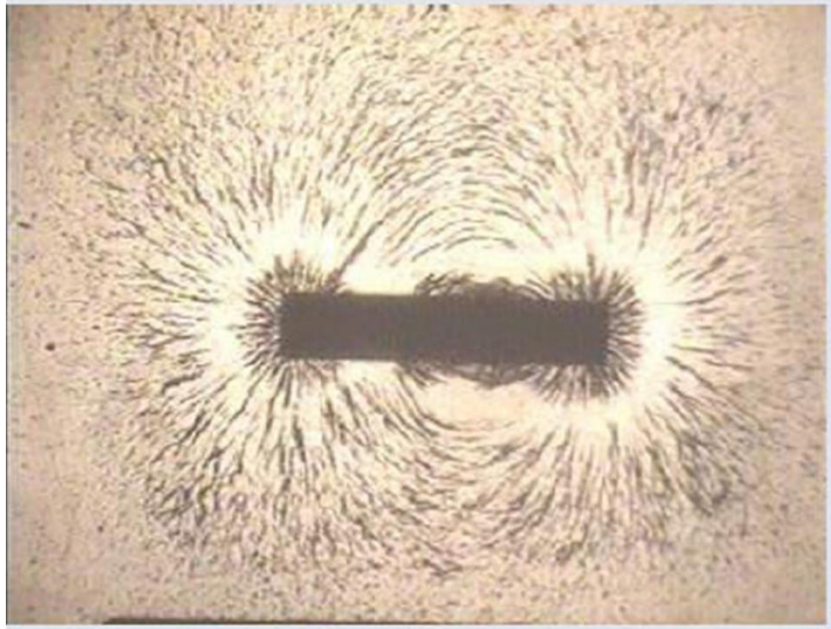
•This is because it is not possible to have isolated magnetic poles (or magnetic charges).

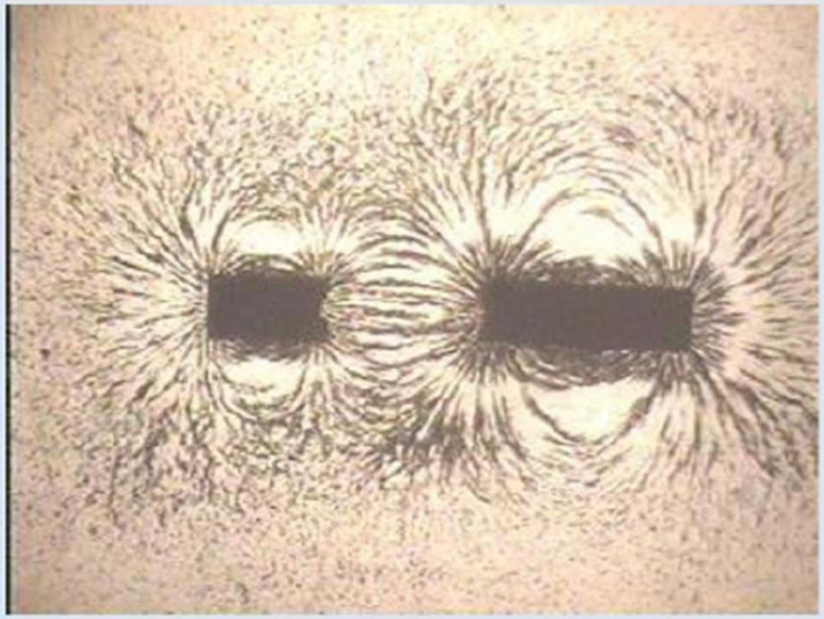
$$\Psi = \oint_S \mathrm{B} \cdot d\mathrm{S} = 0$$

It is not possible to isolate the north and south poles of a magnet.



Broken Magnet





Gauss's Law for magnetostatic fields

The total flux through a closed surface in a magnetic field is zero.

$$\boxed{\oint \mathrm{B} \cdot d\mathrm{S} = 0}$$

Applying divergence theorem,

$$\oint_S \mathrm{B} \cdot d\mathrm{S} = \int_v \nabla \cdot \mathrm{B}\, dv = 0$$

Or

$$\boxed{\nabla \cdot \mathrm{B} = 0}$$

This is the **Maxwell's fourth equation**. **It states that magnetostatic fields have no sources or sinks.**

Maxwell's Equations for Static Fields

Differential form	*Integral form*	*Derived from*
$\vec{\nabla} \cdot \vec{D} = \rho_V$	$\oint_S \vec{D} \cdot \overrightarrow{dS} = \int_V \rho_V dV$	*Gauss's Law*
$\vec{\nabla} \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot \overrightarrow{dS} = 0$	*Nonexistance of magnetic Monopole*
$\vec{\nabla} \times \vec{E} = 0$	$\oint_L \vec{E} \cdot \overrightarrow{dl} = 0$	*Conservativeness of Electrostatic field*
$\vec{\nabla} \times \vec{H} = \vec{J}$	$\oint_L \vec{H} \cdot \overrightarrow{dl} = \int_S \vec{J} \cdot \overrightarrow{dS}$	*Ampere's Law*

Magnetic Scalar and Vector Potentials

*In electrostatics electric field intensity and potential are related by:

$$\mathrm{E} = -\nabla V$$

*Similar to this we can relate magnetic field intensity with two magnetic potentials:

- **Magnetic scalar potential (V_m)**
- **Magnetic vector potential (A)**

*Magnetic scalar potential $\mathbf{V_m}$ is related to **H** by the relation:

$$\boxed{\mathrm{H} = -\nabla V_m} \qquad \text{if J=0}$$

$\mathrm{J}=\nabla\times\mathrm{H}=\nabla\times(-\nabla V_m)=0$ (since for any scalar, $\nabla\times(\nabla V)=0$)

so the magnetic scalar potential $\mathrm{V_m}$ is only defined in the region where J=0.

V_m satisfies Laplace's equation $\nabla^2 V_m = 0$ (J=0)

Magnetic Vector Potential

Since for a magnetostatic field $\nabla \cdot B=0$

For any vector $\nabla \cdot (\nabla \times A)=0$

we can define *magnetic vector potential* A such that

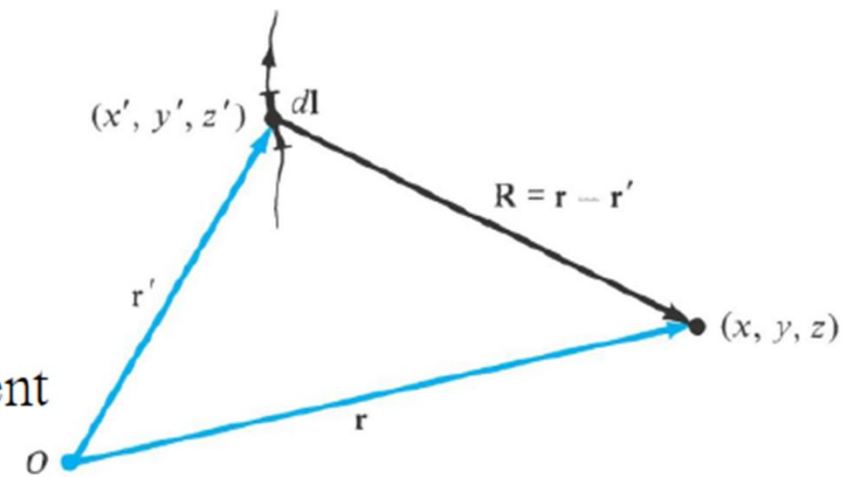
$$\boxed{B=\nabla \times A}$$

In many EM problems it is more convenient to first find A and then find B from it.

$$\boxed{A = \int_L \frac{\mu_0 I d\mathbf{l}}{4\pi R}}$$ for line current

$$\boxed{A = \int_S \frac{\mu_0 K dS}{4\pi R}}$$ for surface current

$$\boxed{A = \int_v \frac{\mu_0 \mathrm{J} dv}{4\pi R}}$$ for volume current



Magnetic flux from vector potential

The magnetic fluc through a given area can be found from

$$\boxed{\Psi = \int_S \mathrm{B} \cdot d\mathrm{S}}$$

Applying Stoke's theorem, we obtain,

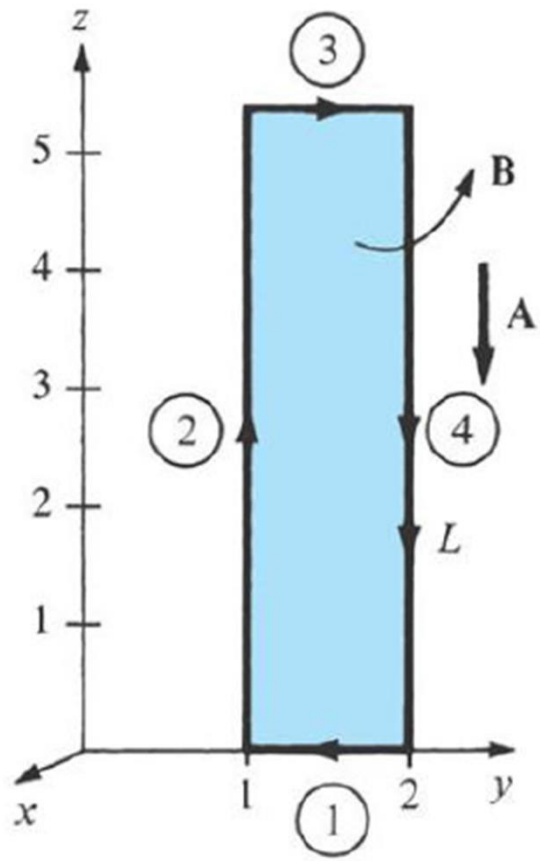
$$\Psi = \int_S \mathrm{B} \cdot d\mathrm{S} = \int_S (\nabla \times \mathrm{A}) \cdot d\mathrm{S} = \oint_L \mathrm{A} \cdot d\mathrm{l}$$

$$\boxed{\Psi = \oint_L \mathrm{A} \cdot d\mathrm{l}}$$

Example 7.7

Given the magnetic vector potential $A=-\rho^2/4\ a_z$ Wb/m, calculate the total magnetic flux crossing the surface $\Phi=\pi/2$, $1\le\rho\le2$ m, $0\le z\le5$ m.

Example 7.7 Solution



Method 1

$$\mathbf{B}=\nabla\times\mathbf{A}=-\frac{\partial A_z}{\partial\rho}\mathbf{a}_\phi=\frac{\rho}{2}\mathbf{a}_\phi,\qquad d\mathbf{S}=d\rho\,dz\,\mathbf{a}_\phi$$

Hence,

$$\Psi=\int\mathbf{B}\cdot d\mathbf{S}=\frac{1}{2}\int_{z=0}^{5}\int_{\rho=1}^{2}\rho d\rho\,dz=\frac{1}{4}\rho^2\Big|_1^2(5)=\frac{15}{4}$$

$$\Psi=3.75\text{ Wb}$$

Example 7.7 Solution Continued

$A=-\rho^2/4\ a_z$

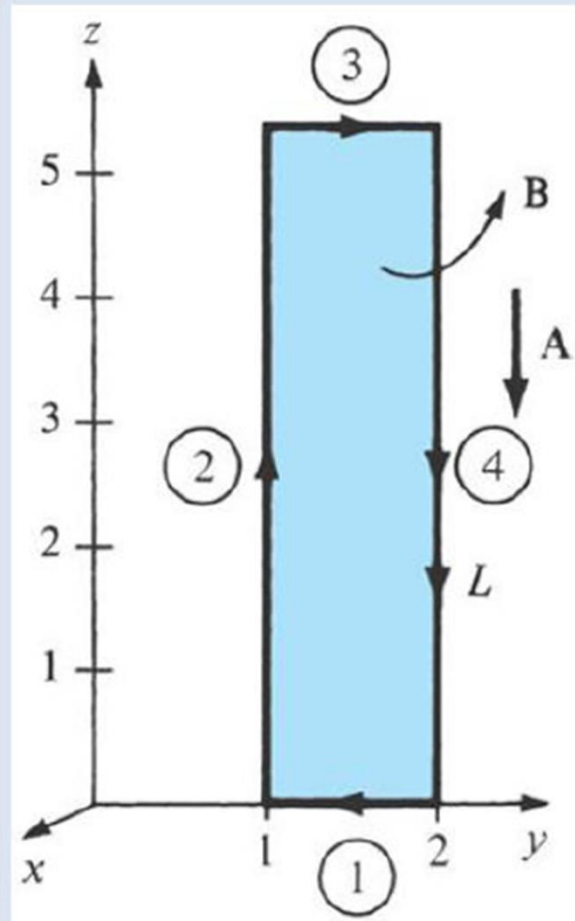
Method 2

$$\Psi = \oint_L \mathbf{A}\cdot d\mathbf{l} = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4$$

where L is the path bounding surface S.

Since A has only z-component,

$$\Psi_1 = 0 = \Psi_3$$

$$\Psi = \Psi_2 + \Psi_4 = -\frac{1}{4}\left[(1)^2\int_0^5 dz + (2)^2\int_5^0 dz\right]$$

$$= -\frac{1}{4}(1-4)(5) = \frac{15}{4} = 3.75\ \text{Wb}$$