CHAPTER 2

COULOMB'S LAW AND ELECTRIC FIELD INTENSITY

Now that we have formulated a new language in the first chapter, we shall establish a few basic principles of electricity and attempt to describe them in terms of it. If we had used vector calculus for several years and already had a few correct ideas about electricity and magnetism, we might jump in now with both feet and present a handful of equations, including Maxwell's equations and a few other auxiliary equations, and proceed to describe them physically by virtue of our knowledge of vector analysis. This is perhaps the ideal way, starting with the most general results and then showing that Ohm's, Gauss's, Coulomb's, Faraday's, Ampère's, Biot-Savart's, Kirchhoff's, and a few less familiar laws are all special cases of these equations. It is philosophically satisfying to have the most general result and to feel that we are able to obtain the results for any special case at will. However, such a jump would lead to many frantic cries of "Help" and not a few drowned students.

Instead we shall present at decent intervals the experimental laws mentioned above, expressing each in vector notation, and use these laws to solve a number of simple problems. In this way our familiarity with both vector analysis and electric and magnetic fields will gradually increase, and by the time we have finally reached our handful of general equations, little additional explanation will be required. The entire field of electromagnetic theory is then open to us, and we may use Maxwell's equations to describe wave propagation, radiation from antennas, skin effect, waveguides and transmission lines, and travelling-wave tubes, and even to obtain a new insight into the ordinary power transformer.

In this chapter we shall restrict our attention to *static* electric fields in *vacuum* or *free space*. Such fields, for example, are found in the focusing and deflection systems of electrostatic cathode-ray tubes. For all practical purposes, our results will also be applicable to air and other gases. Other materials will be introduced in Chap. 5, and time-varying fields will be introduced in Chap. 10.

We shall begin by describing a quantitative experiment performed in the seventeenth century.

2.1 THE EXPERIMENTAL LAW OF COULOMB

Records from at least 600 B.C. show evidence of the knowledge of static electricity. The Greeks were responsible for the term "electricity," derived from their word for amber, and they spent many leisure hours rubbing a small piece of amber on their sleeves and observing how it would then attract pieces of fluff and stuff. However, their main interest lay in philosophy and logic, not in experimental science, and it was many centuries before the attracting effect was considered to be anything other than magic or a "life force."

Dr. Gilbert, physician to Her Majesty the Queen of England, was the first to do any true experimental work with this effect and in 1600 stated that glass, sulfur, amber, and other materials which he named would "not only draw to themselves straws and chaff, but all metals, wood, leaves, stone, earths, even water and oil."

Shortly thereafter a colonel in the French Army Engineers, Colonel Charles Coulomb, a precise and orderly minded officer, performed an elaborate series of experiments using a delicate torsion balance, invented by himself, to determine quantitatively the force exerted between two objects, each having a static charge of electricity. His published result is now known to many high school students and bears a great similarity to Newton's gravitational law (discovered about a hundred years earlier). Coulomb stated that the force between two very small objects separated in a vacuum or free space by a distance which is large compared to their size is proportional to the charge on each and inversely proportional to the square of the distance between them, or

$$F = k\frac{Q_1Q_2}{R^2}$$

where Q_1 and Q_2 are the positive or negative quantities of charge, R is the separation, and k is a proportionality constant. If the International System of

Units[1] (SI) is used, Q is measured in coulombs (C), R is in meters (m), and the force should be newtons (N). This will be achieved if the constant of proportionality k is written as

$$k = \frac{1}{4\pi\epsilon_0}$$

The factor 4π will appear in the denominator of Coulomb's law but will not appear in the more useful equations (including Maxwell's equations) which we shall obtain with the help of Coulomb's law. The new constant ϵ_0 is called the *permittivity of free space* and has the magnitude, measured in farads per meter (F/m),

$$\boxed{\epsilon_0 = 8.854 \times 10^{-12} \doteq \frac{1}{36\pi} 10^{-9} \qquad \mathrm{F/m}} \tag{1}$$

The quantity ϵ_0 is not dimensionless, for Coulomb's law shows that it has the label $\mathrm{C^2/N \cdot m^2}$. We shall later define the farad and show that it has the dimensions $\mathrm{C^2/N \cdot m}$; we have anticipated this definition by using the unit F/m in (1) above.

Coulomb's law is now

$$\boxed{F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}} \tag{2}$$

Not all SI units are as familiar as the English units we use daily, but they are now standard in electrical engineering and physics. The newton is a unit of force that is equal to $0.2248\,\mathrm{lb}_f$, and is the force required to give a 1-kilogram (kg) mass an acceleration of 1 meter per second per second ($\mathrm{m/s^2}$). The coulomb is an extremely large unit of charge, for the smallest known quantity of charge is that of the electron (negative) or proton (positive), given in mks units as 1.602×10^{-19} C; hence a negative charge of one coulomb represents about 6×10^{18} electrons.[2] Coulomb's law shows that the force between two charges of one coulomb each, separated by one meter, is 9×10^9 N, or about one million tons. The electron has a rest mass of 9.109×10^{-31} kg and has a radius of the order of magnitude of 3.8×10^{-15} m. This does not mean that the electron is spherical in shape, but merely serves to describe the size of the region in which a slowly moving electron has the greatest probability of being found. All other

[1] The International System of Units (an mks system) is described in Appendix B. Abbreviations for the units are given in Table B.1. Conversions to other systems of units are given in Table B.2, while the prefixes designating powers of ten in S1 appear in Table B.3.

[2] The charge and mass of an electron and other physical constants are tabulated in Table C.4 of Appendix C.

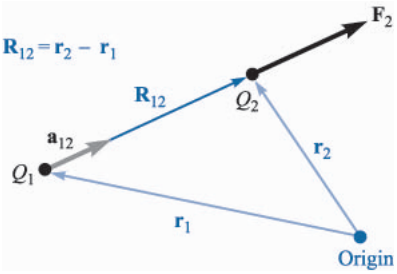

FIGURE 2.1
If Q_1 and Q_2 have like signs, the vector force $\mathbf{F}_2$ on Q_2 is in the same direction as the vector $\mathbf{R}_{12}$.

known charged particles, including the proton, have larger masses, and larger radii, and occupy a probabilistic volume larger than does the electron.

In order to write the vector form of (2), we need the additional fact (furnished also by Colonel Coulomb) that the force acts along the line joining the two charges and is repulsive if the charges are alike in sign and attractive if they are of opposite sign. Let the vector $\mathbf{r}_1$ locate Q_1 while $\mathbf{r}_2$ locates Q_2. Then the vector $\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ represents the directed line segment from Q_1 to Q_2, as shown in Fig. 2.1. The vector $\mathbf{F}_2$ is the force on Q_2 and is shown for the case where Q_1 and Q_2 have the same sign. The vector form of Coulomb's law is

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12} \tag{3}$$

where $\mathbf{a}_{12}$ = a unit vector in the direction of R_{12}, or

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\mathbf{R}_{12}}{R_{12}} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} \tag{4}$$

Example 2.1

Let us illustrate the use of the vector form of Coulomb's law by locating a charge of $Q_1 = 3 \times 10^{-4}$ C at $M(1, 2, 3)$ and a charge of $Q_2 = -10^{-4}$ C at $N(2, 0, 5)$ in a vacuum. We desire the force exerted on Q_2 by Q_1.

Solution. We shall make use of (3) and (4) to obtain the vector force. The vector $\mathbf{R}_{12}$ is

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1 = (2-1)\mathbf{a}_x + (0-2)\mathbf{a}_y + (5-3)\mathbf{a}_z = \mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z$$

leading to $|\mathbf{R}_{12}| = 3$, and the unit vector, $\mathbf{a}_{12} = \frac{1}{3}(\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z)$. Thus,

$$\mathbf{F}_2 = \frac{3 \times 10^{-4}(-10^{-4})}{4\pi(1/36\pi)10^{-9} \times 3^2}\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right)$$
$$= -30\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right) \text{ N}$$

The magnitude of the force is 30 N (or about 7 lb_f), and the direction is specified by the unit vector, which has been left in parentheses to display the magnitude of the force. The force on Q_2 may also be considered as three component forces,

$$\mathbf{F}_2 = -10\mathbf{a}_x + 20\mathbf{a}_y - 20\mathbf{a}_z$$

The force expressed by Coulomb's law is a mutual force, for each of the two charges experiences a force of the same magnitude, although of opposite direction. We might equally well have written

$$\boxed{\mathbf{F}_1 = -\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{21} = -\frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12}} \tag{5}$$

Coulomb's law is linear, for if we multiply Q_1 by a factor n, the force on Q_2 is also multiplied by the same factor n. It is also true that the force on a charge in the presence of several other charges is the sum of the forces on that charge due to each of the other charges acting alone.

✔ **D2.1.** A charge $Q_A = -20\,\mu\text{C}$ is located at $A(-6, 4, 7)$, and a charge $Q_B = 50\,\mu\text{C}$ is at $B(5, 8, -2)$ in free space. If distances are given in meters, find: (*a*) $\mathbf{R}_{AB}$; (*b*) R_{AB}. Determine the vector force exerted on Q_A by Q_B if $\epsilon_0 =$: (*c*) $10^{-9}/(36\pi)$ F/m; (*d*) 8.854×10^{-12} F/m.

Ans. $11\mathbf{a}_x + 4\mathbf{a}_y - 9\mathbf{a}_z$ m; 14.76 m; $30.76\mathbf{a}_x + 11.184\mathbf{a}_y - 25.16\mathbf{a}_z$ mN; $30.72\mathbf{a}_x + 11.169\mathbf{a}_y - 25.13\mathbf{a}_z$ mN

2.2 ELECTRIC FIELD INTENSITY

If we now consider one charge fixed in position, say Q_1, and move a second charge slowly around, we note that there exists everywhere a force on this second charge; in other words, this second charge is displaying the existence of a force *field*. Call this second charge a test charge Q_t. The force on it is given by Coulomb's law,

$$\mathbf{F}_t = \frac{Q_1 Q_t}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t}$$

Writing this force as a force per unit charge gives

$$\boxed{\frac{\mathbf{F}_t}{Q_t} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t}} \tag{6}$$

The quantity on the right side of (6) is a function only of Q_1 and the directed line segment from Q_1 to the position of the test charge. This describes a vector field and is called the *electric field intensity*.

We define the electric field intensity as the vector force on a unit positive test charge. We would not *measure* it experimentally by finding the force on a 1-C test charge, however, for this would probably cause such a force on Q_1 as to change the position of that charge.

Electric field intensity must be measured by the unit newtons per coulomb—the force per unit charge. Again anticipating a new dimensional quantity, the *volt* (V), to be presented in Chap. 4 and having the label of joules per coulomb (J/C) or newton-meters per coulomb (N·m/C), we shall at once measure electric field intensity in the practical units of volts per meter (V/m). Using a capital letter **E** for electric field intensity, we have finally

$$\boxed{\mathbf{E} = \frac{\mathbf{F}_t}{Q_t}} \tag{7}$$

$$\mathbf{E} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t} \tag{8}$$

Equation (7) is the defining expression for electric field intensity, and (8) is the expression for the electric field intensity due to a single point charge Q_1 in a vacuum. In the succeeding sections we shall obtain and interpret expressions for the electric field intensity due to more complicated arrangements of charge, but now let us see what information we can obtain from (8), the field of a single point charge.

First, let us dispense with most of the subscripts in (8), reserving the right to use them again any time there is a possibility of misunderstanding:

$$\boxed{\mathbf{E} = \frac{Q}{4\pi\epsilon_0 R^2}\mathbf{a}_R} \tag{9}$$

We should remember that R is the magnitude of the vector $\mathbf{R}$, the directed line segment from the point at which the point charge Q is located to the point at which $\mathbf{E}$ is desired, and $\mathbf{a}_R$ is a unit vector in the $\mathbf{R}$ direction.[3]

Let us arbitrarily locate Q_1 at the center of a spherical coordinate system. The unit vector $\mathbf{a}_R$ then becomes the radial unit vector $\mathbf{a}_r$, and R is r. Hence

$$\mathbf{E} = \frac{Q_1}{4\pi\epsilon_0 r^2}\mathbf{a}_r \tag{10}$$

or

$$E_r = \frac{Q_1}{4\pi\epsilon_0 r^2}$$

The field has a single radial component, and its inverse-square-law relationship is quite obvious.

---

[3] We firmly intend to avoid confusing r and $\mathbf{a}_r$ with R and $\mathbf{a}_R$. The first two refer specifically to the spherical coordinate system, whereas R and $\mathbf{a}_R$ do not refer to any coordinate system—the choice is still available to us.

Writing these expressions in cartesian coordinates for a charge Q at the origin, we have $\mathbf{R} = \mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ and $\mathbf{a}_R = \mathbf{a}_r = (x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z)/\sqrt{x^2 + y^2 + z^2}$; therefore,

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0(x^2 + y^2 + z^2)}\left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}\mathbf{a}_x + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\mathbf{a}_y + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\mathbf{a}_z\right) \tag{11}$$

This expression no longer shows immediately the simple nature of the field, and its complexity is the price we pay for solving a problem having spherical symmetry in a coordinate system with which we may (temporarily) have more familiarity.

Without using vector analysis, the information contained in (11) would have to be expressed in three equations, one for each component, and in order to obtain the equation we would have to break up the magnitude of the electric field intensity into the three components by finding the projection on each coordinate axis. Using vector notation, this is done automatically when we write the unit vector.

If we consider a charge which is *not* at the origin of our coordinate system, the field no longer possesses spherical symmetry (nor cylindrical symmetry, unless the charge lies on the z axis), and we might as well use cartesian coordinates. For a charge Q located at the source point $\mathbf{r}' = x'\mathbf{a}_x + y'\mathbf{a}_y + z'\mathbf{a}_z$, as illustrated in Fig. 2.2, we find the field at a general field point $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ by expressing $\mathbf{R}$ as $\mathbf{r} - \mathbf{r}'$, and then

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^2}\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} = \frac{Q(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^3}$$
$$= \frac{Q[(x - x')\mathbf{a}_x + (y - y')\mathbf{a}_y + (z - z')\mathbf{a}_z]}{4\pi\epsilon_0[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} \tag{12}$$

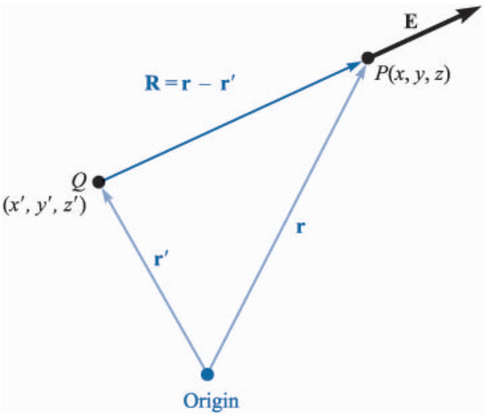

FIGURE 2.2
The vector $\mathbf{r}'$ locates the point charge Q, the vector $\mathbf{r}$ identifies the general point in space $P(x, y, z)$, and the vector $\mathbf{R}$ from Q to $P(x, y, z)$ is then $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.

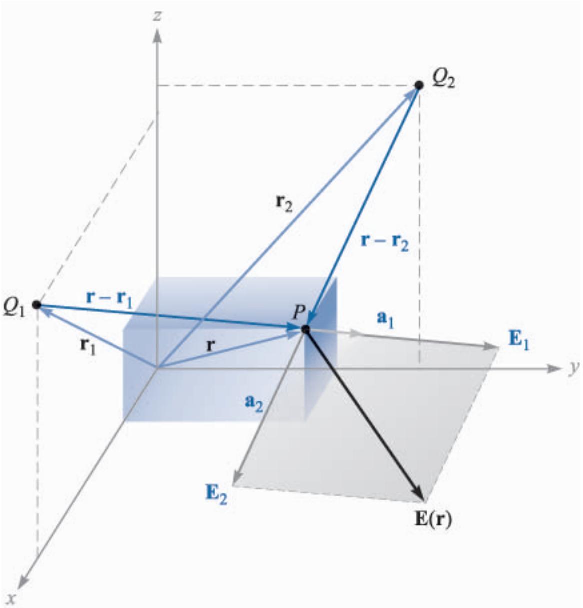

FIGURE 2.3
The vector addition of the total electric field intensity at P due to Q_1 and Q_2 is made possible by the linearity of Coulomb's law.

Earlier, we defined a vector field as a vector function of a position vector, and this is emphasized by letting $\mathbf{E}$ be symbolized in functional notation by $\mathbf{E}(\mathbf{r})$.

Equation (11) is merely a special case of (12), where $x' = y' = z' = 0$.

Since the coulomb forces are linear, the electric field intensity due to two point charges, Q_1 at $\mathbf{r}_1$ and Q_2 at $\mathbf{r}_2$, is the sum of the forces on Q_t caused by Q_1 and Q_2 acting alone, or

$$\mathbf{E}(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|^2}\mathbf{a}_1 + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|^2}\mathbf{a}_2$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are unit vectors in the direction of $(\mathbf{r}-\mathbf{r}_1)$ and $(\mathbf{r}-\mathbf{r}_2)$, respectively. The vectors $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}-\mathbf{r}_1, \mathbf{r}-\mathbf{r}_2, \mathbf{a}_1$, and $\mathbf{a}_2$ are shown in Fig. 2.3.

If we add more charges at other positions, the field due to n point charges is

$$\mathbf{E}(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|^2}\mathbf{a}_1 + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|^2}\mathbf{a}_2 + \ldots + \frac{Q_n}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_n|^2}\mathbf{a}_n \qquad (13)$$

This expression takes up less space when we use a summation sign $\sum$ and a summing integer m which takes on all integral values between 1 and n,

$$\mathbf{E}(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_m|^2}\mathbf{a}_m \qquad (14)$$

When expanded, (14) is identical with (13), and students unfamiliar with summation signs should check that result.

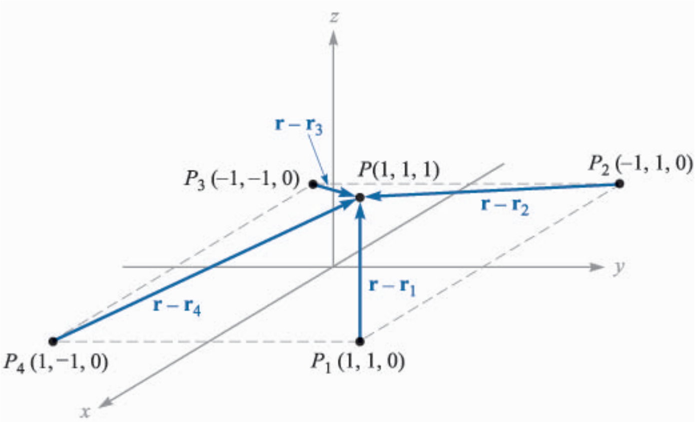

FIGURE 2.4
A symmetrical distribution of four identical 3-nC point charges produces a field at P, $\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z$ V/m.

Example 2.2

In order to illustrate the application of (13) or (14), let us find $\mathbf{E}$ at $P(1, 1, 1)$ caused by four identical 3-nC (nanocoulomb) charges located at $P_1(1, 1, 0)$, $P_2(-1, 1, 0)$, $P_3(-1, -1, 0)$, and $P_4(1, -1, 0)$, as shown in Fig. 2.4.

Solution. We find that $\mathbf{r} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$, $\mathbf{r}_1 = \mathbf{a}_x + \mathbf{a}_y$, and thus $\mathbf{r} - \mathbf{r}_1 = \mathbf{a}_z$. The magnitudes are: $|\mathbf{r} - \mathbf{r}_1| = 1$, $|\mathbf{r} - \mathbf{r}_2| = \sqrt{5}$, $|\mathbf{r} - \mathbf{r}_3| = 3$, and $|\mathbf{r} - \mathbf{r}_4| = \sqrt{5}$. Since $Q/4\pi\epsilon_0 = 3 \times 10^{-9}/(4\pi \times 8.854 \times 10^{-12}) = 26.96\,\text{V}\cdot\text{m}$, we may now use (13) or (14) to obtain

$$\mathbf{E} = 26.96\left[\frac{\mathbf{a}_z}{1}\frac{1}{1^2} + \frac{2\mathbf{a}_x + \mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2} + \frac{2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z}{3}\frac{1}{3^2} + \frac{2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2}\right]$$

or

$$\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z \ \text{V/m}$$

✔ **D2.2.** A charge of $-0.3\,\mu\text{C}$ is located at $A(25, -30, 15)$ (in cm), and a second charge of $0.5\,\mu\text{C}$ is at $B(-10, 8, 12)$ cm. Find $\mathbf{E}$ at: (*a*) the origin; (*b*) $P(15, 20, 50)$ cm.

Ans. $92.3\mathbf{a}_x - 77.6\mathbf{a}_y - 105.3\mathbf{a}_z$ kV/m; $32.9\mathbf{a}_x + 5.94\mathbf{a}_y + 19.69\mathbf{a}_z$ kV/m

✔ **D2.3.** Evaluate the sums: (*a*) $\sum_{m=0}^{5}\frac{1+(-1)^m}{m^2+1}$; (*b*) $\sum_{m=1}^{4}\frac{(0.1)^m+1}{(4+m^2)^{1.5}}$

Ans. 2.52; 0.1948

2.3 FIELD DUE TO A CONTINUOUS VOLUME CHARGE DISTRIBUTION

If we now visualize a region of space filled with a tremendous number of charges separated by minute distances, such as the space between the control grid and the cathode in the electron-gun assembly of a cathode-ray tube operating with space charge, we see that we can replace this distribution of very small particles with a smooth continuous distribution described by a *volume charge density*, just as we describe water as having a density of $1 \mathrm{~g/cm^3}$ (gram per cubic centimeter) even though it consists of atomic- and molecular-sized particles. We are able to do this only if we are uninterested in the small irregularities (or ripples) in the field as we move from electron to electron or if we care little that the mass of the water actually increases in small but finite steps as each new molecule is added.

This is really no limitation at all, because the end results for electrical engineers are almost always in terms of a current in a receiving antenna, a voltage in an electronic circuit, or a charge on a capacitor, or in general in terms of some large-scale *macroscopic* phenomenon. It is very seldom that we must know a current electron by electron.[4]

We denote volume charge density by ρ_v, having the units of coulombs per cubic meter ($\mathrm{C/m^3}$).

The small amount of charge ΔQ in a small volume Δv is

$$\Delta Q = \rho_v \Delta v \tag{15}$$

and we may define ρ_v mathematically by using a limiting process on (15),

$$\boxed{\rho_v = \lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v}} \tag{16}$$

The total charge within some finite volume is obtained by integrating throughout that volume,

$$\boxed{Q = \int_{\mathrm{vol}} \rho_v d\, v} \tag{17}$$

Only one integral sign is customarily indicated, but the differential dv signifies integration throughout a volume, and hence a triple integration. Fortunately, we may be content for the most part with no more than the indicated integration, for multiple integrals are very difficult to evaluate in all but the most symmetrical problems.

[4] A study of the noise generated by electrons or ions in transistors, vacuum tubes, and resistors, however, requires just such an examination of the charge.

Example 2.3

As an example of the evaluation of a volume integral, we shall find the total charge contained in a 2-cm length of the electron beam shown in Fig. 2.5.

Solution. From the illustration, we see that the charge density is

$$\rho_v = -5 \times 10^{-6} e^{-10^5 \rho z} \text{ C/m}^2$$

The volume differential in cylindrical coordinates is given in Sec. 1.8; therefore,

$$Q = \int_{0.02}^{0.04} \int_0^{2\pi} \int_0^{0.01} -5 \times 10^{-6} e^{-10^5 \rho z} \rho \, d\rho \, d\phi \, dz$$

We integrate first with respect to ϕ since it is so easy,

$$Q = \int_{0.02}^{0.04} \int_0^{0.01} -10^{-5} \pi e^{-10^5 \rho z} \rho \, d\rho \, dz$$

and then with respect to z, because this will simplify the last integration with respect to ρ,

$$Q = \int_0^{0.01} \left(\frac{-10^{-5}\pi}{-10^5 \rho} e^{-10^5 \rho z} \rho \, d\rho \right)_{z=0.02}^{z=0.04}$$

$$= \int_0^{0.01} -10^{-5}\pi (e^{-2000\rho} - e^{-4000\rho}) d\rho$$

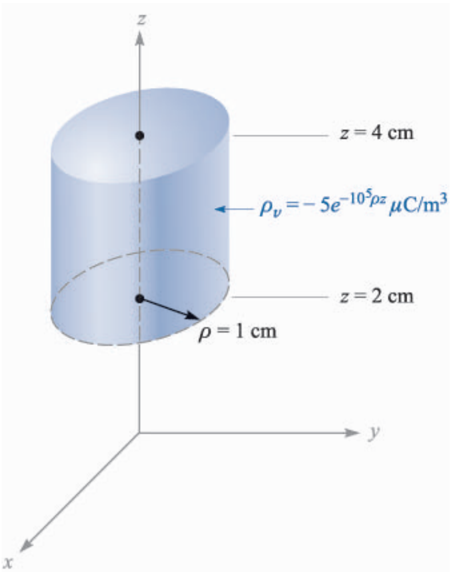

FIGURE 2.5
The total charge contained within the right circular cylinder may be obtained by evaluating

$$Q = \int_{\text{vol}} \rho_v \, dv.$$

Finally,

$$Q = -10^{-10}\pi\left(\frac{e^{-2000\rho}}{-2000} - \frac{e^{-4000\rho}}{-4000}\right)_0^{0.01}$$

$$Q = -10^{-10}\pi\left(\frac{1}{2000} - \frac{1}{4000}\right) = \frac{-\pi}{40} = 0.0785\,\text{pC}$$

where pC indicates picocoulombs.

Incidentally, we may use this result to make a rough estimate of the electron-beam current. If we assume these electrons are moving at a constant velocity of 10 percent of the velocity of light, this 2-cm-long packet will have moved 2 cm in $\frac{2}{3}$ ns, and the current is about equal to the product,

$$\frac{\Delta Q}{\Delta t} = \frac{-(\pi/40)10^{-12}}{(2/3)10^{-9}}$$

or approximately 118 μA.

The incremental contribution to the electric field intensity at $\mathbf{r}$ produced by an incremental charge ΔQ at $\mathbf{r}'$ is

$$\Delta\mathbf{E}(\mathbf{r}) = \frac{\Delta Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \frac{\rho_v \Delta v}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

If we sum the contributions of all the volume charge in a given region and let the volume element Δv approach zero as the number of these elements becomes infinite, the summation becomes an integral,

$$\mathbf{E}(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_v(\mathbf{r}')dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \tag{18}$$

This is again a triple integral, and (except in the drill problem that follows) we shall do our best to avoid actually performing the integration.

The significance of the various quantities under the integral sign of (18) might stand a little review. The vector $\mathbf{r}$ from the origin locates the field point where $\mathbf{E}$ is being determined, while the vector $\mathbf{r}'$ extends from the origin to the source point where $\rho_v(\mathbf{r}')dv'$ is located. The scalar distance between the source point and the field point is $|\mathbf{r}-\mathbf{r}'|$, and the fraction $(\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|$ is a unit vector directed from source point to field point. The variables of integration are x', y', and z' in cartesian coordinates.

✔ **D2.4.** Calculate the total charge within each of the indicated volumes: (*a*) $0.1 \le |x|, |y|, |z| \le 0.2$: $\rho_v = \frac{1}{x^3y^3z^3}$; (*b*) $0 \le \rho \le 0.1$, $0 \le \phi \le \pi$, $2 \le z \le 4$; $\rho_v = \rho^2 z^2 \sin 0.6\phi$; (*c*) universe: $\rho_v = e^{-2r}/r^2$.

Ans. 27 μC; 1.778 mC; 6.28 C

2.4 FIELD OF A LINE CHARGE

Up to this point we have considered two types of charge distribution, the point charge and charge distributed throughout a volume with a density ρ_v C/m^3. If we now consider a filamentlike distribution of volume charge density, such as a very fine, sharp beam in a cathode-ray tube or a charged conductor of very small radius, we find it convenient to treat the charge as a line charge of density ρ_L C/m. In the case of the electron beam the charges are in motion and it is true that we do not have an electrostatic problem. However, if the electron motion is steady and uniform (a dc beam) and if we ignore for the moment the magnetic field which is produced, the electron beam may be considered as composed of stationary electrons, for snapshots taken at any time will show the same charge distribution.

Let us assume a straight line charge extending along the z axis in a cylindrical coordinate system from $-\infty$ to ∞, as shown in Fig. 2.6. We desire the electric field intensity $\mathbf{E}$ at any and every point resulting from a *uniform* line charge density ρ_L.

Symmetry should always be considered first in order to determine two specific factors: (1) with which coordinates the field does *not* vary, and (2) which components of the field are *not* present. The answers to these questions then tell us which components are present and with which coordinates they *do* vary.

Referring to Fig. 2.6, we realize that as we move around the line charge, varying ϕ while keeping ρ and z constant, the line charge appears the same from every angle. In other words, azimuthal symmetry is present, and no field component may vary with ϕ.

Again, if we maintain ρ and ϕ constant while moving up and down the line charge by changing z, the line charge still recedes into infinite distance in both directions and the problem is unchanged. This is axial symmetry and leads to fields which are not functions of z.

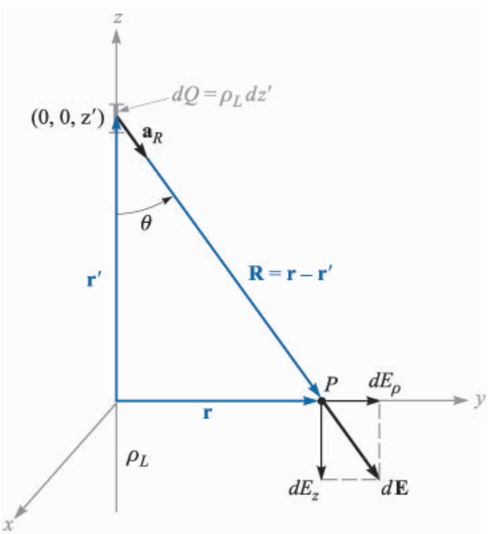

FIGURE 2.6
The contribution $d\mathbf{E} = dE_\rho \mathbf{a}_\rho + dE_z \mathbf{a}_z$ to the electric field intensity produced by an element of charge $dQ = \rho_L dz'$ located a distance z' from the origin. The linear charge density is uniform and extends along the entire z axis.

If we maintain ϕ and z constant and vary ρ, the problem changes, and Coulomb's law leads us to expect the field to become weaker as ρ increases. Hence, by a process of elimination we are led to the fact that the field varies only with ρ.

Now, which components are present? Each incremental length of line charge acts as a point charge and produces an incremental contribution to the electric field intensity which is directed away from the bit of charge (assuming a positive line charge). No element of charge produces a ϕ component of electric intensity; E_ϕ is zero. However, each element does produce an E_ρ and an E_z component, but the contribution to E_z by elements of charge which are equal distances above and below the point at which we are determining the field will cancel.

We therefore have found that we have only an E_ρ component and it varies only with ρ. Now to find this component.

We choose a point $P(0, y, 0)$ on the y axis at which to determine the field. This is a perfectly general point in view of the lack of variation of the field with ϕ and z. Applying (12) to find the incremental field at P due to the incremental charge $dQ = \rho_L dz'$, we have

$$d\mathbf{E} = \frac{\rho_L dz'(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

where

$$\mathbf{r} = y\mathbf{a}_y = \rho\mathbf{a}_\rho$$
$$\mathbf{r}' = z'\mathbf{a}_z$$

and

$$\mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho - z'\mathbf{a}_z$$

Therefore,

$$d\mathbf{E} = \frac{\rho_L dz'(\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

Since only the $\mathbf{E}_\rho$ component is present, we may simplify:

$$dE_\rho = \frac{\rho_L \rho dz'}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

and

$$E_\rho = \int_{-\infty}^{\infty} \frac{\rho_L \rho dz'}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

Integrating by integral tables or change of variable, $z' = \rho \cot\theta$, we have

$$E_\rho = \frac{\rho_L}{4\pi\epsilon_0}\rho\left(\frac{1}{\rho^2}\frac{z'}{\sqrt{\rho^2 + z'^2}}\right)_{-\infty}^{\infty}$$

and

$$\boxed{E_\rho = \frac{\rho_L}{2\pi\epsilon_0 \rho}} \tag{19}$$

This is the desired answer, but there are many other ways of obtaining it. We might have used the angle θ as our variable of integration, for $z' = \rho \cot\theta$ from Fig. 2.6 and $dz' = -\rho \csc^2\theta \, d\theta$. Since $R = \rho \csc\theta$, our integral becomes, simply,

$$dE_\rho = \frac{\rho_L \, dz'}{4\pi\epsilon_0 R^2} \sin\theta = -\frac{\rho_L \sin\theta \, d\theta}{4\pi\epsilon_0 \rho}$$

$$E_\rho = -\frac{\rho_L}{4\pi\epsilon_0 \rho} \int_\pi^0 \sin\theta \, d\theta = \frac{\rho_L}{4\pi\epsilon_0 \rho} \cos\theta \Big]_\pi^0$$

$$= \frac{\rho_L}{2\pi\epsilon_0 \rho}$$

Here the integration was simpler, but some experience with problems of this type is necessary before we can unerringly choose the simplest variable of integration at the beginning of the problem.

We might also have considered (18) as our starting point,

$$\mathbf{E} = \int_{\text{vol}} \frac{\rho_v \, dv'(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

letting $\rho_v \, dv' = \rho_L \, dz'$ and integrating along the line which is now our "volume" containing all the charge. Suppose we do this and forget everything we have learned from the symmetry of the problem. Choose point P now at a general location (ρ, ϕ, z) (Fig. 2.7) and write

$$\mathbf{r} = \rho \mathbf{a}_\rho + z \mathbf{a}_z$$

$$\mathbf{r}' = z' \mathbf{a}_z$$

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = \rho \mathbf{a}_\rho + (z - z') \mathbf{a}_z$$

$$R = \sqrt{\rho^2 + (z - z')^2}$$

$$\mathbf{a}_R = \frac{\rho \mathbf{a}_\rho + (z - z') \mathbf{a}_z}{\sqrt{\rho^2 + (z - z')^2}}$$

$$\mathbf{E} = \int_{-\infty}^{\infty} \frac{\rho_L \, dz' [\rho \mathbf{a}_\rho + (z - z') \mathbf{a}_z]}{4\pi\epsilon_0 [\rho^2 + (z - z')^2]^{3/2}}$$

$$= \frac{\rho_L}{4\pi\epsilon_0} \left\{ \int_{-\infty}^{\infty} \frac{\rho \, dz' \mathbf{a}_\rho}{[\rho^2 + (z - z')^2]^{3/2}} + \int_{-\infty}^{\infty} \frac{(z - z') \, dz' \mathbf{a}_z}{[\rho^2 + (z - z')^2]^{3/2}} \right\}$$

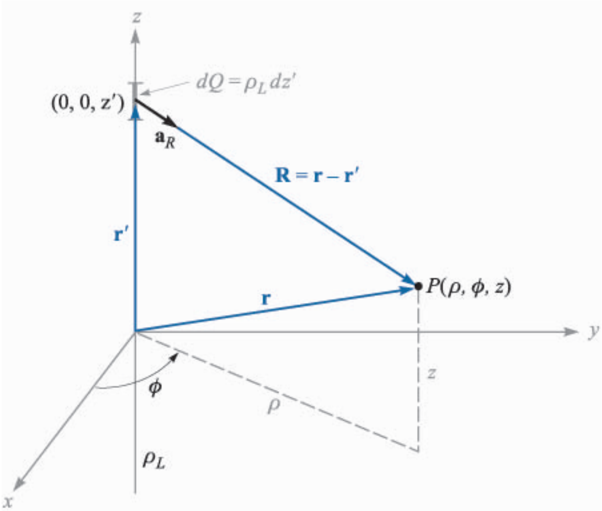

FIGURE 2.7
The geometry of the problem for the field about an infinite line charge leads to more difficult integrations when symmetry is ignored.

Before integrating a vector expression, we must know whether or not a vector under the integral sign (here the unit vectors $\mathbf{a}_\rho$ and $\mathbf{a}_z$) varies with the variable of integration (here dz'). If it does not, then it is a constant and may be removed from within the integral, leaving a scalar which may be integrated by normal methods. Our unit vectors, of course, cannot change in magnitude, but a change in direction is just as troublesome. Fortunately, the direction of $\mathbf{a}_\rho$ does not change with z' (nor with ρ, but it does change with ϕ), and $\mathbf{a}_z$ is constant always.

Hence we remove the unit vectors from the integrals and again integrate with tables or by changing variables,

$$\mathbf{E} = \frac{\rho_L}{4\pi\epsilon_0}\left\{\mathbf{a}_\rho \int_{-\infty}^{\infty} \frac{\rho\, dz'}{[\rho^2 + (z - z')^2]^{3/2}} + \mathbf{a}_z \int_{-\infty}^{\infty} \frac{(z - z')\, dz'}{[\rho^2 + (z - z')^2]^{3/2}}\right\}$$

$$= \frac{\rho_L}{4\pi\epsilon_0}\left\{\left[\mathbf{a}_\rho \rho \frac{1}{\rho^2} \frac{-(z - z')}{\sqrt{\rho^2 + (z - z')^2}}\right]_{-\infty}^{\infty} + \left[\mathbf{a}_z \frac{1}{\sqrt{\rho^2 + (z - z')^2}}\right]_{-\infty}^{\infty}\right\}$$

$$= \frac{\rho_L}{4\pi\epsilon_0}\left[\mathbf{a}_\rho \frac{2}{\rho} + \mathbf{a}_z(0)\right] = \frac{\rho_L}{2\pi\epsilon_0 \rho}\mathbf{a}_\rho$$

Again we obtain the same answer, as we should, for there is nothing wrong with the method except that the integration was harder and there were two integrations to perform. This is the price we pay for neglecting the consideration of symmetry and plunging doggedly ahead with mathematics. Look before you integrate.

Other methods for solving this basic problem will be discussed later after we introduce Gauss's law and the concept of potential.

Now let us consider the answer itself,

$$\boxed{\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0 \rho}\mathbf{a}_\rho} \tag{20}$$

We note that the field falls off inversely with the distance to the charged line, as compared with the point charge, where the field decreased with the *square* of the distance. Moving ten times as far from a point charge leads to a field only 1 percent the previous strength, but moving ten times as far from a line charge only reduces the field to 10 percent of its former value. An analogy can be drawn with a source of illumination, for the light intensity from a point source of light also falls off inversely as the square of the distance to the source. The field of an infinitely long fluorescent tube thus decays inversely as the first power of the radial distance to the tube, and we should expect the light intensity about a finite-length tube to obey this law near the tube. As our point recedes farther and farther from a finite-length tube, however, it eventually looks like a point source and the field obeys the inverse-square relationship.

Before leaving this introductory look at the field of the infinite line charge, we should recognize the fact that not all line charges are located along the z axis. As an example, let us consider an infinite line charge parallel to the z axis at $x = 6$, $y = 8$, Fig. 2.8. We wish to find $\mathbf{E}$ at the general field point $P(x, y, z)$.

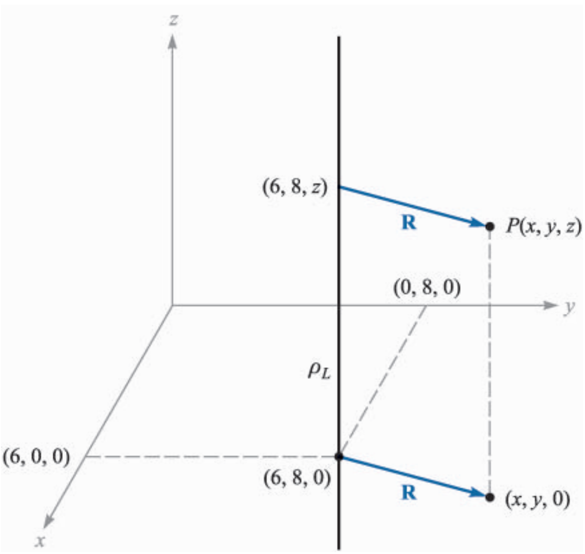

FIGURE 2.8
A point $P(x, y, z)$ is identified near an infinite uniform line charge located at $x = 6$, $y = 8$.

We replace ρ in (20) by the radial distance between the line charge and point, P, $R = \sqrt{(x-6)^2 + (y-8)^2}$, and let $\mathbf{a}_\rho$ be $\mathbf{a}_R$. Thus,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0 \sqrt{(x-6)^2 + (y-8)^2}} \mathbf{a}_R$$

where

$$\mathbf{a}_R = \frac{\mathbf{R}}{|\mathbf{R}|} = \frac{(x-6)\mathbf{a}_x + (y-8)\mathbf{a}_y}{\sqrt{(x-6)^2 + (y-8)^2}}$$

Therefore,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0} \frac{(x-6)\mathbf{a}_x + (y-8)\mathbf{a}_y}{(x-6)^2 + (y-8)^2}$$

We again note that the field is not a function of z.

In Sec. 2.6 we shall describe how fields may be sketched and use the field of the line charge as one example.

✔ **D2.5.** Infinite uniform line charges of 5 nC/m lie along the (positive and negative) x and y axes in free space. Find $\mathbf{E}$ at: (*a*) $P_A(0, 0, 4)$; (*b*) $P_B(0, 3, 4)$.

Ans. $45\mathbf{a}_z$ V/m; $10.8\mathbf{a}_y + 36.9\mathbf{a}_z$ V/m

2.5 FIELD OF A SHEET OF CHARGE

Another basic charge configuration is the infinite sheet of charge having a uniform density of ρ_S C/m^2. Such a charge distribution may often be used to approximate that found on the conductors of a strip transmission line or a parallel-plate capacitor. As we shall see in Chap. 5, static charge resides on conductor surfaces and not in their interiors; for this reason, ρ_S is commonly known as *surface charge density*. The charge-distribution family now is complete—point, line, surface, and volume, or Q, ρ_L, ρ_S, and ρ_v.

Let us place a sheet of charge in the yz plane and again consider symmetry (Fig. 2.9). We see first that the field cannot vary with y or with z, and then that the y and z components arising from differential elements of charge symmetrically located with respect to the point at which we wish the field will cancel. Hence only E_x is present, and this component is a function of x alone. We are again faced with a choice of many methods by which to evauate this component, and this time we shall use but one method and leave the others as exercises for a quiet Sunday afternoon.

Let us use the field of the infinite line charge (19) by dividing the infinite sheet into differential-width strips. One such strip is shown in Fig. 2.9. The line charge density, or charge per unit length, is $\rho_L = \rho_S\, dy'$, and the distance from

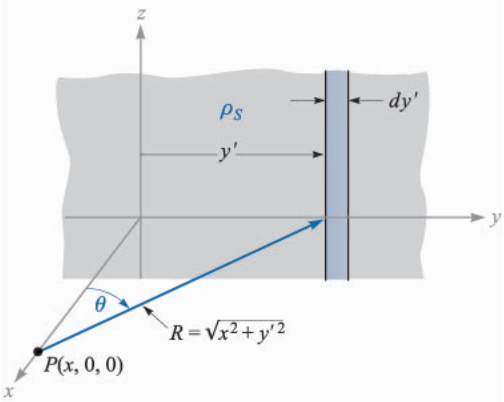

FIGURE 2.9
An infinite sheet of charge in the yz plane, a general point P on the x axis, and the differential-width line charge used as the element in determining the field at P by $d\mathbf{E} = \rho_S\, dy' \mathbf{a}_R/(2\pi\varepsilon_0 R)$.

this line charge to our general point P on the x axis is $R = \sqrt{x^2 + y'^2}$. The contribution to E_x at P from this differential-width strip is then

$$dE_x = \frac{\rho_S\, dy'}{2\pi\epsilon_0\sqrt{x^2 + y'^2}} \cos\theta = \frac{\rho_S}{2\pi\epsilon_0} \frac{x dy'}{x^2 + y'^2}$$

Adding the effects of all the strips,

$$E_x = \frac{\rho_S}{2\pi\epsilon_0}\int_{-\infty}^{\infty} \frac{x\, dy'}{x^2 + y'^2} = \frac{\rho_S}{2\pi\epsilon_0} \tan^{-1}\frac{y'}{x}\bigg]_{-\infty}^{\infty} = \frac{\rho_S}{2\epsilon_0}$$

If the point P were chosen on the negative x axis, then

$$E_x = -\frac{\rho_S}{2\epsilon_0}$$

for the field is always directed away from the positive charge. This difficulty in sign is usually overcome by specifying a unit vector $\mathbf{a}_N$, which is normal to the sheet and directed outward, or away from it. Then

$$\boxed{\mathbf{E} = \frac{\rho_S}{2\epsilon_0}\mathbf{a}_N} \tag{21}$$

This is a startling answer, for the field is constant in magnitude and direction. It is just as strong a million miles away from the sheet as it is right off the surface. Returning to our light analogy, we see that a uniform source of light on the ceiling of a very large room leads to just as much illumination on a square foot on the floor as it does on a square foot a few inches below the ceiling. If you desire greater illumination on this subject, it will do you no good to hold the book closer to such a light source.

If a second infinite sheet of charge, having a *negative* charge density $-\rho_S$, is located in the plane $x = a$, we may find the total field by adding the contribution of each sheet. In the region $x > a$,

$$\mathbf{E}_+ = \frac{\rho_S}{2\epsilon_0}\mathbf{a}_x \qquad \mathbf{E}_- = -\frac{\rho_S}{2\epsilon_0}\mathbf{a}_x \qquad \mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = 0$$

and for $x < 0$,

$$\mathbf{E}_+ = -\frac{\rho_S}{2\epsilon_0}\mathbf{a}_x \qquad \mathbf{E}_- = \frac{\rho_S}{2\epsilon_0}\mathbf{a}_x \qquad \mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = 0$$

and when $0 < x < a$,

$$\mathbf{E}_+ = \frac{\rho_S}{2\epsilon_0}\mathbf{a}_x \qquad \mathbf{E}_- = \frac{\rho_S}{2\epsilon_0}\mathbf{a}_x$$

and

$$\boxed{\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho_S}{\epsilon_0}\mathbf{a}_x} \tag{22}$$

This is an important practical answer, for it is the field between the parallel plates of an air capacitor, provided the linear dimensions of the plates are very much greater than their separation and provided also that we are considering a point well removed from the edges. The field outside the capacitor, while not zero, as we found for the ideal case above, is usually negligible.

✔ **D2.6.** Three infinite uniform sheets of charge are located in free space as follows: 3 nC/m^2 at $z = -4$, 6 nC/m^2 at $z = 1$, and -8 nC/m^2 at $z = 4$. Find **E** at the point: (*a*) $P_A(2, 5, -5)$; (*b*) $P_B(4, 2, -3)$; (*c*) $P_C(-1, -5, 2)$; (*d*) $P_D(-2, 4, 5)$.

Ans. $-56.5\mathbf{a}_z$; $283\mathbf{a}_z$; $961\mathbf{a}_z$; $56.5\mathbf{a}_z$ all V/m

2.6 STREAMLINES AND SKETCHES OF FIELDS

We now have vector equations for the electric field intensity resulting from several different charge configurations, and we have had little difficulty in interpreting the magnitude and direction of the field from the equations. Unfortunately, this simplicity cannot last much longer, for we have solved most of the simple cases and our new charge distributions must lead to more complicated expressions for the fields and more difficulty in visualizing the fields through the equations. However, it is true that one picture would be worth about a thousand words, if we just knew what picture to draw.

Consider the field about the line charge,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\rho}\mathbf{a}_\rho$$

Fig. 2.10*a* shows a cross-sectional view of the line charge and presents what might be our first effort at picturing the field—short line segments drawn here and there having lengths proportional to the magnitude of **E** and pointing in the direction of **E**. The figure fails to show the symmetry with respect to ϕ, so we try again in Fig. 2.10*b* with a symmetrical location of the line segments. The real trouble now appears—the longest lines must be drawn in the most crowded region, and this also plagues us if we use line segments of equal length but of a thickness which is proportional to **E** (Fig. 2.10*c*). Other schemes which have been suggested include drawing shorter lines to represent stronger fields (inherently misleading) and using intensity of color to represent stronger fields (difficult and expensive).

For the present, then, let us be content to show only the *direction* of **E** by drawing continuous lines from the charge which are everywhere tangent to **E**. Fig. 2.10*d* shows this compromise. A symmetrical distribution of lines (one every 45°) indicates azimuthal symmetry, and arrowheads should be used to show direction.

These lines are usually called *streamlines*, although other terms such as flux lines and direction lines are also used. A small positive test charge placed at any

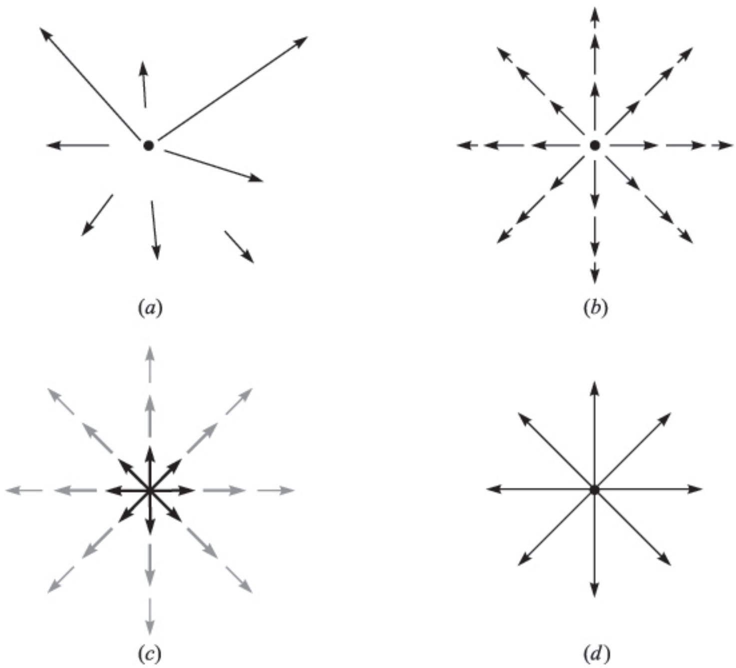

FIGURE 2.10
(*a*) One very poor sketch, (*b*) and (*c*) two fair sketches, and (*d*) the usual form of streamline sketch. In the last form, the arrows show the direction of the field at every point along the line, and the spacing of the lines is inversely proportional to the strength of the field.

point in this field and free to move would accelerate in the direction of the streamline passing through that point. If the field represented the velocity of a liquid or a gas (which, incidentally, would have to have a source at $\rho = 0$), small suspended particles in the liquid or gas would trace out the streamlines.

We shall find out later that a bonus accompanies this streamline sketch, for the magnitude of the field can be shown to be inversely proportional to the spacing of the streamlines for some important special cases. The closer they are together, the stronger is the field. At that time we shall also find an easier, more accurate method of making that type of streamline sketch.

If we attempted to sketch the field of the point charge, the variation of the field into and away from the page would cause essentially insurmountable difficulties; for this reason sketching is usually limited to two-dimensional fields.

In the case of the two-dimensional field let us arbitrarily set $E_z = 0$. The streamlines are thus confined to planes for which z is constant, and the sketch is the same for any such plane. Several streamlines are shown in Fig. 2.11, and the E_x and E_y components are indicated at a general point. Since it is apparent from the geometry that

$$\boxed{\frac{E_y}{E_x} = \frac{dy}{dx}} \tag{23}$$

a knowledge of the functional form of E_x and E_y (and the ability to solve the resultant differential equation) will enable us to obtain the equations of the streamlines.

As an illustration of this method, consider the field of the uniform line charge with $\rho_L = 2\pi\epsilon_0$,

$$\mathbf{E} = \frac{1}{\rho}\mathbf{a}_\rho$$

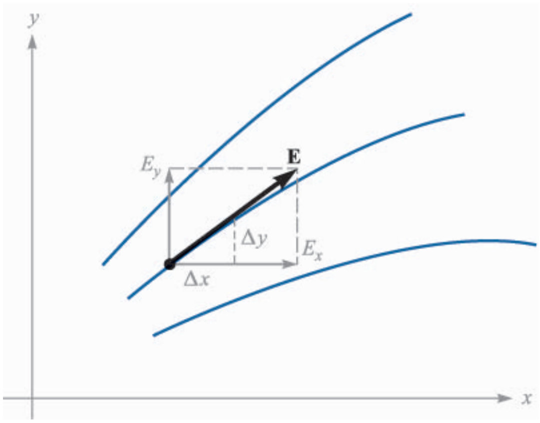

FIGURE 2.11
The equation of a streamline is obtained by solving the differential equation $E_y/E_x = dy/dx$.

In cartesian coordinates,

$$\mathbf{E} = \frac{x}{x^2 + y^2}\mathbf{a}_x + \frac{y}{x^2 + y^2}\mathbf{a}_y$$

Thus we form the differential equation

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{y}{x} \qquad \text{or} \qquad \frac{dy}{y} = \frac{dx}{x}$$

Therefore,

$$\ln y = \ln x + C_1 \qquad \text{or} \qquad \ln y = \ln x + \ln C$$

from which the equations of the streamlines are obtained,

$$y = Cx$$

If we want to find the equation of one particular streamline, say that one passing through $P(-2, 7, 10)$, we merely substitute the coordinates of that point into our equation and evaluate C. Here, $7 = C(-2)$, and $C = -3.5$, so that $y = -3.5x$.

Each streamline is associated with a specific value of C, and the radial lines shown in Fig. 2.10*d* are obtained when $C = 0, 1, -1$, and $1/C = 0$.

The equations of streamlines may also be obtained directly in cylindrical or spherical coordinates. A spherical coordinate example will be examined in Sec. 4.7.

✔ **D2.7.** Find the equation of that streamline that passes through the point $P(1, 4, -2)$ in the field $\mathbf{E} =$: (*a*) $\dfrac{-8x}{y}\mathbf{a}_x + \dfrac{4x^2}{y^2}\mathbf{a}_y$; (*b*) $2e^{5x}[y(5x+1)\mathbf{a}_x + x\mathbf{a}_y]$.

Ans. $x^2 + 2y^2 = 33$; $y^2 = 15.6 + 0.4x - 0.08\ln(x + 0.2)/1.2]$

SUGGESTED REFERENCES

1. Boast, W. B.: "Vector Fields," Harper and Row, Publishers, Incorporated, New York, 1964. This book contains numerous examples and sketches of fields.
2. Della Torre, E., and Longo, C. L.: "The Electromagnetic Field," Allyn and Bacon, Inc., Boston, 1969. The authors introduce all of electromagnetic theory with a careful and rigorous development based on a single experimental law—that of Coulomb. It begins in chap. 1.
3. Schelkunoff, S. A.: "Electromagnetic Fields," Blaisdell Publishing Company, New York, 1963. Many of the physical aspects of fields are discussed early in this text without advanced mathematics.

PROBLEMS

2.1 Four 10-nC positive charges are located in the $z = 0$ plane at the corners of a square 8 cm on a side. A fifth 10-nC positive charge is located at a point 8 cm distant from each of the other charges. Calculate the magnitude of the total force on this fifth charge for $\epsilon = \epsilon_0$.

2.2 A charge $Q_1 = 0.1\,\mu\text{C}$ is located at the origin in free space, while $Q_2 = 0.2\,\mu\text{C}$ is at $A(0.8, -0.6, 0)$. Find the locus of points in the $z = 0$ plane at which the x-component of the force on a third positive charge is zero.

2.3 Point charges of 50 nC each are located at $A(1, 0, 0)$, $B(-1, 0, 0)$, $C(0, 1, 0)$, and $D(0, -1, 0)$ in free space. Find the total force on the charge at A.

2.4 Let $Q_1 = 8\,\mu\text{C}$ be located at $P_1(2, 5, 8)$ while $Q_2 = -5\,\mu\text{C}$ is at $P_2(6, 15, 8)$. Let $\epsilon = \epsilon_0$. (*a*) Find $\mathbf{F}_2$, the force on Q_2. (*b*) Find the coordinates of P_3 if a charge Q_3 experiences a total force $\mathbf{F}_3 = 0$ at P_3.

2.5 Let a point charge $Q_1 = 25\,\text{nC}$ be located at $P_1(4, -2, 7)$ and a charge $Q_2 = 60\,\text{nC}$ be at $P_2(-3, 4, -2)$. (*a*) If $\epsilon = \epsilon_0$, find $\mathbf{E}$ at $P(1, 2, 3)$. (*b*) At what point on the y axis is $E_x = 0$?

2.6 Point charges of 120 nC are located at $A(0, 0, 1)$ and $B(0, 0, -1)$ in free space. (*a*) Find $\mathbf{E}$ at $P(0.5, 0, 0)$. (*b*) What single charge at the origin would provide the identical field strength?

2.7 A 2-μC point charge is located at $A(4, 3, 5)$ in free space. Find E_ρ, E_ϕ, and E_z at $P(8, 12, 2)$.

2.8 Given point charges of $-1\,\mu\text{C}$ at $P_1(0, 0, 0.5)$ and $P_2(0, 0, -0.5)$, and a charge of $2\,\mu\text{C}$ at the origin, find $\mathbf{E}$ at $P(0, 2, 1)$ in spherical components. Assume $\epsilon = \epsilon_0$.

2.9 A 100-nC point charge is located at $A(-1, 1, 3)$ in free space. (*a*) Find the locus of all points $P(x, y, z)$ at which $E_x = 500\,\text{V/m}$. (*b*) Find y_1 if $P(-2, y_1, 3)$ lies on that locus.

2.10 Charges of 20 and $-20\,\text{nC}$ are located at $(3, 0, 0)$ and $(-3, 0, 0)$, respectively. Let $\epsilon = \epsilon_0$. (*a*) Determine $|\mathbf{E}|$ at $P(0, y, 0)$. (*b*) Sketch $|\mathbf{E}|$ vs y at P.

2.11 A charge Q_0, located at the origin in free space, produces a field for which $E_z = 1\,\text{kV/m}$ at point $P(-2, 1, -1)$. (*a*) Find Q_0. Find $\mathbf{E}$ at $M(1, 6, 5)$ in: (*b*) cartesian coordinates; (*c*) cylindrical coordinates; (*d*) spherical coordinates.

2.12 The volume charge density $\rho_v = \rho_0 e^{-|x|-|y|-|z|}$ exists over all free space. Calculate the total charge present.

2.13 A uniform volume charge density of $0.2\,\mu\text{C/m}^2$ is present throughout the spherical shell extending from $r = 3\,\text{cm}$ to $r = 5\,\text{cm}$. If $\rho_v = 0$ elsewhere, find: (*a*) the total charge present within the shell, and (*b*) r_1 if half the total charge is located in the region $3\,\text{cm} < r < r_1$.

2.14 Let $\rho_v = 5e^{-0.1\rho}(\pi - |\phi|)\dfrac{1}{z^2 + 10}\,\mu\text{C/m}^3$ in the region $0 \le \rho \le 10$, $-\pi < \phi < \pi$, all z, and $\rho_v = 0$ elsewhere. (*a*) Determine the total charge present. (*b*) Calculate the charge within the region $0 \leq \rho \leq 4$, $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$, $-10 < z < 10$.

2.15 A spherical volume having a 2-μm radius contains a uniform volume charge density of 10^{15} C/m^3. (*a*) What total charge is enclosed in the spherical volume? (*b*) Now assume that a large region contains one of these little spheres at every corner of a cubical grid 3 mm on a side, and that there is no charge between the spheres. What is the average volume charge density throughout this large region?

2.16 The region in which $4 < r < 5$, $0 < \theta < 25^\circ$, and $0.9\pi < \phi < 1.1\pi$, contains the volume charge density $\rho_v = 10(r-4)(r-5)\sin\theta\sin\frac{1}{2}\phi$. Outside that region $\rho_v = 0$. Find the charge within the region.

2.17 A uniform line charge of 16 nC/m is located along the line defined by $y = -2$, $z = 5$. If $\epsilon = \epsilon_0$: (*a*) find **E** at $P(1, 2, 3)$; (*b*) find **E** at that point in the $z = 0$ plane where the direction of **E** is given by $\frac{1}{3}\mathbf{a}_y - \frac{2}{3}\mathbf{a}_z$.

2.18 Uniform line charges of 0.4 μC/m and −0.4 μC/m are located in the $x = 0$ plane at $y = -0.6$ and $y = 0.6$ m, respectively. Let $\epsilon = \epsilon_0$. Find **E** at: (*a*) $P(x, 0, z)$; (*b*) $Q(2, 3, 4)$.

2.19 A uniform line charge of 2 μC/m is located on the z axis. Find **E** in cartesian coordinates at $P(1, 2, 3)$ if the charge extends from: (*a*) $z = -\infty$ to $z = \infty$; (*b*) $z = -4$ to $z = 4$.

2.20 Uniform line charges of 120 nC/m lie along the entire extent of the three coordinate axes. Assuming free space conditions, find **E** at $P(-3, 2, -1)$.

2.21 Two identical uniform line charges, with $\rho_L = 75$ nC/m, are located in free space at $x = 0$, $y = \pm 0.4$ m. What force per unit length does each line charge exert on the other?

2.22 A uniform surface charge density of 5nC/m^2 is present in the region $x = 0$, $-2 < y < 2$, all z. If $\epsilon = \epsilon_0$, find **E** at: (*a*) $P_A(3, 0, 0)$; (*b*) $P_B(0, 3, 0)$.

2.23 Given the surface charge density, $\rho_S = 2$ μC/m^2 in the region $\rho < 0.2$ m, $z = 0$, and is zero elsewhere, find **E** at: (*a*) $P_A(\rho = 0, z = 0.5)$; (*b*) $P_B(\rho = 0, z = -0.5)$.

2.24 Surface charge density is positioned in free space as follows: 20 nC/m^2 at $x = -3$, −30 nC/m^2 at $y = 4$, and 40 nC/m^2 at $z = 2$. Find the magnitude of **E** at: (*a*) $P_A(4, 3, -2)$; (*b*) $P_B(-2, 5, -1)$; (*c*) $P_C(0, 0, 0)$.

2.25 Find **E** at the origin if the following charge distributions are present in free space: point charge, 12 nC, at $P(2, 0, 6)$; uniform line charge density, 3 nC/m, at $x = -2$, $y = 3$; uniform surface charge density, 0.2 nC/m^2, at $x = 2$.

2.26 A uniform line charge density of 5 nC/m is at $y = 0$, $z = 2$ m in free space, while −5 nC/m is located at $y = 0$, $z = -2$ m. A uniform surface charge density of 0.3 nC/m^2 is at $y = 0.2$ m, and −0.3 nC/m^2 is at $y = -0.2$ m. Find $|\mathbf{E}|$ at the origin.

2.27 Given the electric field $\mathbf{E} = (4x - 2y)\mathbf{a}_x - (2x + 4y)\mathbf{a}_y$, find: (*a*) the equation of that streamline passing through the point $P(2, 3, -4)$; (*b*) a unit vector $\mathbf{a}_E$ specifying the direction of **E** at $Q(3, -2, 5)$.

2.28 Let $\mathbf{E} = 5x^3\mathbf{a}_x - 15x^2y\mathbf{a}_y$, and find: (*a*) the equation of the streamline that passes through $P(4, 2, 1)$; (*b*) a unit vector $\mathbf{a}_E$ specifying the direction of $\mathbf{E}$ at $Q(3, -2, 5)$; (*c*) a unit vector $\mathbf{a}_N = (l, m, 0)$ that is perpendicular to $\mathbf{a}_E$ at Q.

2.29 If $\mathbf{E} = 20e^{-5y}(\cos 5x\mathbf{a}_x - \sin 5x\mathbf{a}_y)$, find: (*a*) $|\mathbf{E}|$ at $P(\pi/6, 0.1, 2)$; (*b*) a unit vector in the direction of $\mathbf{E}$ at P; (*c*) the equation of the direction line passing through P.

2.30 Given the electric field intensity, $\mathbf{E} = 400y\mathbf{a}_x + 400x\mathbf{a}_y$ V/m, find: (*a*) the equation of the streamline passing through point $A(2, 1, -2)$; (*b*) the equation of the surface on which $|\mathbf{E}| = 800$ V. (*c*) Sketch the streamline of part *a*. (*d*) Sketch the trace produced by the intersection of the $z = 0$ plane and the surface of part *b*.

2.31 In cylindrical coordinates with $\mathbf{E}(\rho, \phi) = \mathbf{E}_\rho(\rho, \phi)\mathbf{a}_\rho + E_\phi(\rho, \phi)\mathbf{a}_\phi$, the differential equation describing the direction lines is $E_\rho/E_\phi = d\rho/(\rho\, d\phi)$ in any $z =$ constant plane. Derive the equation of the line passing through point $P(\rho = 4, \phi = 10^\circ, z = 2)$ in the field $\mathbf{E} = 2\rho^2 \cos 3\phi\mathbf{a}_\rho + 2\rho^2 \sin 3\phi\mathbf{a}_\phi$.