$xy\left[\frac{x^2}{2}\right]_0 \; ky$

6. (i) Sol^n

$$f_{xy}(x,y) = \begin{cases} kx, & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^1\int_0^y f_{xy}(x,y)\,dx\,dy = 1$$

$$\int_0^1\int_0^y kx\,dx\,dy \Rightarrow k\int_0^1\left[\frac{x^2}{2}\right]_0^y = 1$$

$$k\left[\frac{y^3}{6}\right]_0^1 = 1$$

$$\Rightarrow \frac{k}{6} = 1 \Rightarrow k = 6.$$

So; $f_{xy}(x,6) = \begin{cases} 6x & ; 0<x<y<1 \\ 0 & ; \text{otherwise.} \end{cases}$

to get marginal Pdf;

$$f(x) = \int_x^1 f(x,y)\,dy = \int_x^1 6x\,dy = 6x\,[y]_x^1$$

$$f_X(x) = 6x(1-x); \quad 0<x<1$$

similarly for $f_Y(y) = \int_0^y f(x,y)\,dx$

$$= \int_0^y 6x\,dx$$

$$= \frac{6x^2}{2}\Big|_0^y$$

$$f_Y(y) = 3y^2 \; ; 0<y<1$$

$\int_0^{x} y \quad \frac{x^2}{2} \quad y\left(\frac{y^2}{2}\right)$

$$E(X) = \int_0^1 x\, f(x)\, dx$$

$$= \int_0^1 x\, 6(x - x^3)\, dx.$$

$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$E(X) = 6\left[\frac{1}{3} - \frac{1}{4}\right].$$

$$= 1/2.$$

$$E(y) = \int_0^1 y\, f_Y(y)\, dy$$

$$= \int_0^1 y\; 3y^2\, dy$$

$$= 3\left[y^2/4\right]_0^1 = 3/4.$$

$$E(XY) = \int_0^1 \int_0^3 xy\, f(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^y 6x^2 y\, dx\, dy.$$

$$E[XY] = 6 \int_0^1 y \left[x^3/3\right]_0^y dy.$$

$$= \frac{6}{3} \int_0^1 y^4\, dy$$

$$= 2/5.$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{2}{5} - \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)$$

$$= \frac{2}{5} - \frac{3}{8}$$

$$= \frac{1}{40}$$

$$cov(X,Y) = 1/40$$

Q(2) Sol^n $z = ax + (1-a)y$; $0 \le a \le 1$

$$var(z) = var[ax + (1-a)y]$$

$$var(z) = a^2 var(X) + (1-a)^2 var(Y)$$

$$var(z) = a^2\sigma_1^2 + (1-a^2)\sigma_2^2$$

when variance is minimum,

$$\frac{d}{da} var(z) = 0$$

$$\frac{d}{da}[a^2\sigma_1^2 + (1-a)^2\sigma_2^2] = 0$$

$$2a\sigma_1^2 - 2(1-a)\sigma_2^2 = 0$$

$$a\sigma_1^2 - a_2^2 + a\sigma_2^2 = 0$$

$$a(\sigma_1^2 + \sigma_2^2) = \sigma_2^2$$

$$a = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

$$\text{var}^{*}(z) = a^2(\sigma_1^2 - \sigma_2^2) + \sigma_2^2$$

$$= \frac{\sigma_2^4}{(\sigma_1^2+\sigma_2^2)^2}(\sigma_1^2 - \sigma_2^2) + \sigma_2^2$$

$$\text{var}^{*}(z) = \frac{\sigma_1^4\sigma_2^2 + 3\sigma_1^2\sigma_2^4}{\sigma_1^4 + 2\sigma_1^2\sigma_2^2 + \sigma_2^4}.$$

Q4. $\phi_x(-it) = M_x(t)$.

$Z = ax + by + c \; ; \; c \neq 0$.

$M_z(t) = M_{ax+by+c}(t) = M_{ax}(t) \cdot M_{by}(t) \cdot e^{ct}$.

$\Rightarrow \exp\left[(a\mu_1 + b\mu_2)t + \frac{1}{2}(a^2\sigma_1^2 + b^2\sigma_2^2)t^2\right]$.

mean & var $= 0, 1$

$M_z(t) = e^{1/2(a^2+b^2)t^2}$.

$\phi_z(-it) = e^{\frac{1}{2}(a^2+b^2)(-it)^2}$.

$= e^{-\frac{1}{2}(a^2+b^2)t^2}$ — (1).

Z is also a random variable;

$\phi_z(-it) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$

$E(z) = 0$; $\text{var}(z) = a^2 + b^2$.

$Z \sim N(0, a^2+b^2)$ — (2)

Q. 5 $f_Z(z) = \int_{x=-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$

$F_Z(z) = \int_{y=-\infty}^{z} f_Z(y)\, dy$

Case 1: $0 \le z < 1$

$f_Z(z) = \int_0^z 1 \cdot 1\, dx = z$

$F_Z(z) = \int_0^z y\, dy = z^2/2$

Case 2: $1 \le z \le 2$

$f_Z(z) = \int_{z-1}^{1} 1 \cdot 1\, dx = 2 - z$

$F_Z(z) = \int_0^1 y\, dy + \int_1^z (2-y)\, dy$

$F_Z(z) = 2z - \frac{z^2}{2} - 1$