① $P_{X,Y}(x,y) = \begin{cases} \frac{1}{10} & x=1, y=1 \\ 3/10 & x=1, y=2 \\ 2/5 & x=2, y=1 \\ k & x=2, y=2. \end{cases}$

x \ y	1	2
1	1/10	3/10
2	2/5	k

(a) $\sum P_{x,y}(x,y) = 1$

$\frac{1}{10} + 3/10 + 2/5 + k = 1$

$k = 1 - 8/10 = 2/10 = 1/5.$

(b) Marginal PMF of X

$f_X(x=1) = \sum_y P_{XY}(1,y)$

$= \frac{1}{10} + 3/10 = 4/10$

(c) $f_X(x=2) = \sum_y P_{XY}(2,y) = 2/5 + 1/5 = 3/5.$

Marginal PMF of Y

$f_Y(y=1) = \sum_x P_{XY}(x,1) = 1/10 + 2/5 = 5/10$

$f_Y(y=2) = \sum_x P_{XY}(x,2) = 3/10 + 1/5 = 5/10.$

(d) The conditional pmf of x given Y.

$$* \; P(x=1/y=1) = \frac{P(x=1, y=1)}{P(y=1)} = \frac{1/10}{5/10} = 1/5$$

$$P(x=2/y=1) = \frac{P(x=2, y=1)}{P(y=1)} = \frac{2/5}{5/10} = 4/5$$

$$P(x=1/y=2) = \frac{P(x=1) y=2)}{P(y=2)} = \frac{3/10}{5/10} = 3/5$$

$$P(x=2/y=2) = \frac{P(x=2, y=2))}{P(y=2).} = \frac{1/5}{5/10} = 2/5$$

(e) The conditional pmf of y given x.

$$P(y=1/x=1) = \frac{P(x=1, y=1)}{P(x=1)} = \frac{1/10}{4/10} = 1/4$$

$$P(y=2/x=1) = \frac{P(x=1, y=2)}{P(x=1)} = \frac{3/10}{4/10} = 3/4$$

$$P(y=1/x=2) = \frac{P(x=2, y=1)}{P(x=2)} = \frac{2/5}{3/5} = 2/3$$

$$P(y=2/x=2) = \frac{P(x=2, y=2)}{P(x=2)} = \frac{1/5}{3/5} = 1/3.$$

(i) $f_{xy}(x,y) = \begin{cases} k\, e^{-(2x+3y)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\, dx\, dy = 1$$

$$\int_{0}^{\infty}\int_{0}^{\infty} k\, e^{-(2x+3y)}\, dx\, dy = 1$$

$$k \times \frac{1}{2} \times \frac{1}{3} = 1 \qquad \boxed{k = 6}$$

(ii) $f_x(x) = \int_0^{\infty} 6\, e^{-2x} e^{-3y}\, dy = 2e^{-2x}$

$f_y(y) = \int_0^{\infty} 6\, e^{-2x} e^{-3y}\, dx = 3e^{-3y}$

(iii) $P[x<1, y<0.5)$

$$\int_0^1\int_0^{0.5} 6\, e^{-2x} e^{-3y}\, dy\, dx$$

$$= 6\left[\frac{e^{-2x}}{-2}\right]_0^1 \left[\frac{e^{-3y}}{-3}\right]_0^{0.5}$$

$$= 6\left[\frac{e^{-2}-1}{-2}\right]\left[\frac{e^{-1.5}-1}{-3}\right]$$

$$= (e^{-2}-1)(e^{-1.5}-1)$$

$$= (0.135-1)(0.223-1)$$

$$= 0.6719$$

(3) $f_X(x) = 4e^{-8x} \quad x \geq 0$

$f_Y(y) = 4e^{-2y} \quad y \geq 0$

$$f_U(u) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(u-x)\, dx.$$

$$= \int_0^u 4e^{-8x} . 4e^{-2(u-x)}\, dx.$$

$$= 16e^{-2u} \int_0^u e^{-8x} . e^{2x}\, dx$$

$$= 16e^{-2u} \int_0^u e^{-6x}\, dx$$

$$f_U(u) = 16e^{-2u} \left[\frac{e^{-6x}}{-6}\right]_0^u$$

$$= \frac{16e^{-2u}}{6} \left[-e^{-6u} + 1\right]$$

$$f_U(u) = \frac{8}{3} e^{-2u} \left[1 - e^{-6u}\right]$$

(4) $\bar{X} = 0, \ \bar{Y} = -2, \ \overline{X^2} = 3, \ \overline{Y^2} = 5 \quad R_{XY} = -1$

$W = 3X - 2Y$

$V = -X + Y$

$\bar{W} = 3\bar{X} - 2\bar{Y} = 3 \times 0 - 2(-2) = 4$

$\bar{V} = -\bar{X} + \bar{Y} = 0 - 2 = -2.$

$$\overline{w^2} = \overline{(3x-2y)^2} = 9\overline{x^2} - 12\overline{xy} + 4\overline{y^2}$$

$$= 9(3) - 12(-1) + 4(5)$$

$$= 27 + 12 + 20 = 59$$

$$\overline{v^2} = \overline{(-x+y)^2} = \overline{x^2} - 2\overline{xy} + \overline{y^2}$$

$$= 3 - 2(-1) + 5 = 10$$

$$R_{WV} = E(WV) = \overline{(3x-2y)(-x+y)}$$

$$= -3x^2 + 3xy + 2xy - 2y^2$$

$$= -3\overline{x^2} + 5\overline{xy} - 2\overline{y^2}$$

$$= -3\times 3 + 5(-1) - 2(5)$$

$$R_{WV} = -9 - 5 - 10 = -24$$

(5)

$$f_{X,Y}(x,y) = 4e^{-2(x+y)} u(x)\, u(y)$$

Second order moment

$$m_{20} = \int_0^\infty \int_0^\infty x^2\, 4 e^{-2x} e^{-2y}\, dx\, dy$$

$$= 4\int_0^\infty x^2 e^{-2x}\, dx \int_0^\infty e^{-2y}\, dy.$$

$$= 4\left[\left(\frac{x^2}{-2} - \frac{2x}{4} - \frac{2}{8}\right)e^{-2x}\right]_0^\infty \times \left[\frac{e^{-2y}}{-2}\right]_0^\infty$$

$= \cancel{4} \times \frac{\cancel{2}}{8} \times \frac{1}{\cancel{2}} = \frac{1}{2}$

$\boxed{m_{02} = 1/2}$ & $m_{20} = 1/2$.

$m_{11} = 4\int_0^\infty x e^{-2x}\, dx \int_0^\infty y e^{-2y}\, dy.$

$= 4\left[\left(\frac{x}{-2} - \frac{1}{4}\right)e^{-2x}\right]_0^\infty \cdot \left(\left[\frac{y}{-2} - \frac{1}{4}\right]e^{-2y}\right)_0^\infty\Big]$

$= 4 \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{4}.$

$\boxed{m_{11} = \frac{1}{4}}$

For central moment

$m_{01} = \overline{X} = \int_0^\infty\int_0^\infty x\, 4e^{-2x}\, dx\, e^{-2y}\, dy.$

$= 4 \times \left[\left(\frac{x}{-2} - \frac{1}{4}\right)e^{-2x}\right]_0^\infty \cdot \left[\frac{e^{-2y}}{-2}\right]_0^\infty\Big]$

$= 4 \times \frac{1}{4} \times \frac{1}{2} = 1/2$

$\boxed{m_{10} = 1/2}$ & $\boxed{m_{01} = 1/2}$

$u_{20} = m_{20} - (m_{10})^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = 1/4$

$u_{02} = 1/4$

$u_{11} = R_{XY} - \overline{X}\overline{Y} = m_{11} - m_{01} m_{10} = \frac{1}{4} - 1/2 \times 1/2 = 0$