The random process concept

The concept of random process is based on enlarging the random variable concept to include time.

Since a random variable x is, by its definition, a function of the possible outcomes s of an experiment, now it becomes a function of both s and t.

$$x(t,s)$$

$x(t,s)$ is random process.

~~A random process $X(t,s)$ represents a random variable when t is fixed and s is a var~~

A random process $x(t,s)$ represents a family or ensemble of time function when t and s are variable.

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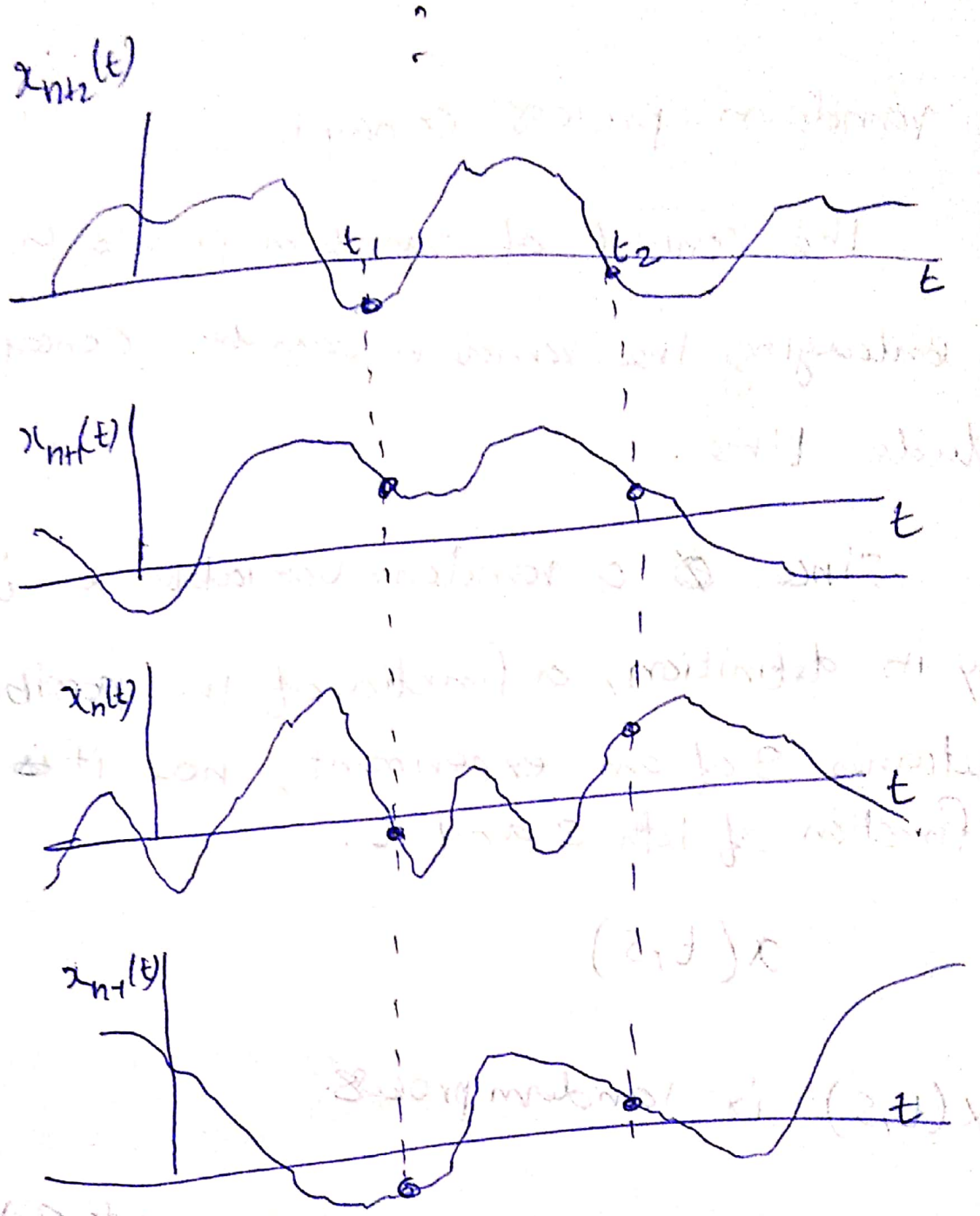

few member of an ensemble.

* Each member time function is called a sample function, ensemble member or realization of the process.

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Thus a random process also represents a single time function when t is a variable and s is fixed at a specific value (out come)

Also a random process represents a random variable when t is fixed and s is a variable.

For example, the random variable $X(t_1, s) = X(t_1)$ is obtained from the process when time is frozen at the value t_1.

X_1 denotes the random variable associated with the process $X(t)$ at time t_1.

X_1 corresponds to a vertical "slice" through the ensemble at time t_1.

The expected value of X_1 is called the ensemble average or expected or mean value of the random process (at time t_1)

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Since t_1 may have various values. the mean value of a process may not be constant. It may be a function of time.

Classification of Process.

4 Classes Random process.

If x is continuous and t can have any of a continum of values, then $x(t)$ is called a continous random process.

To classify random processes according to the characteristics of t and the random variable $X = x(t)$ at time t.

Discrete random process

Corresponds to the random variable x having only discrete values while t is continuous.

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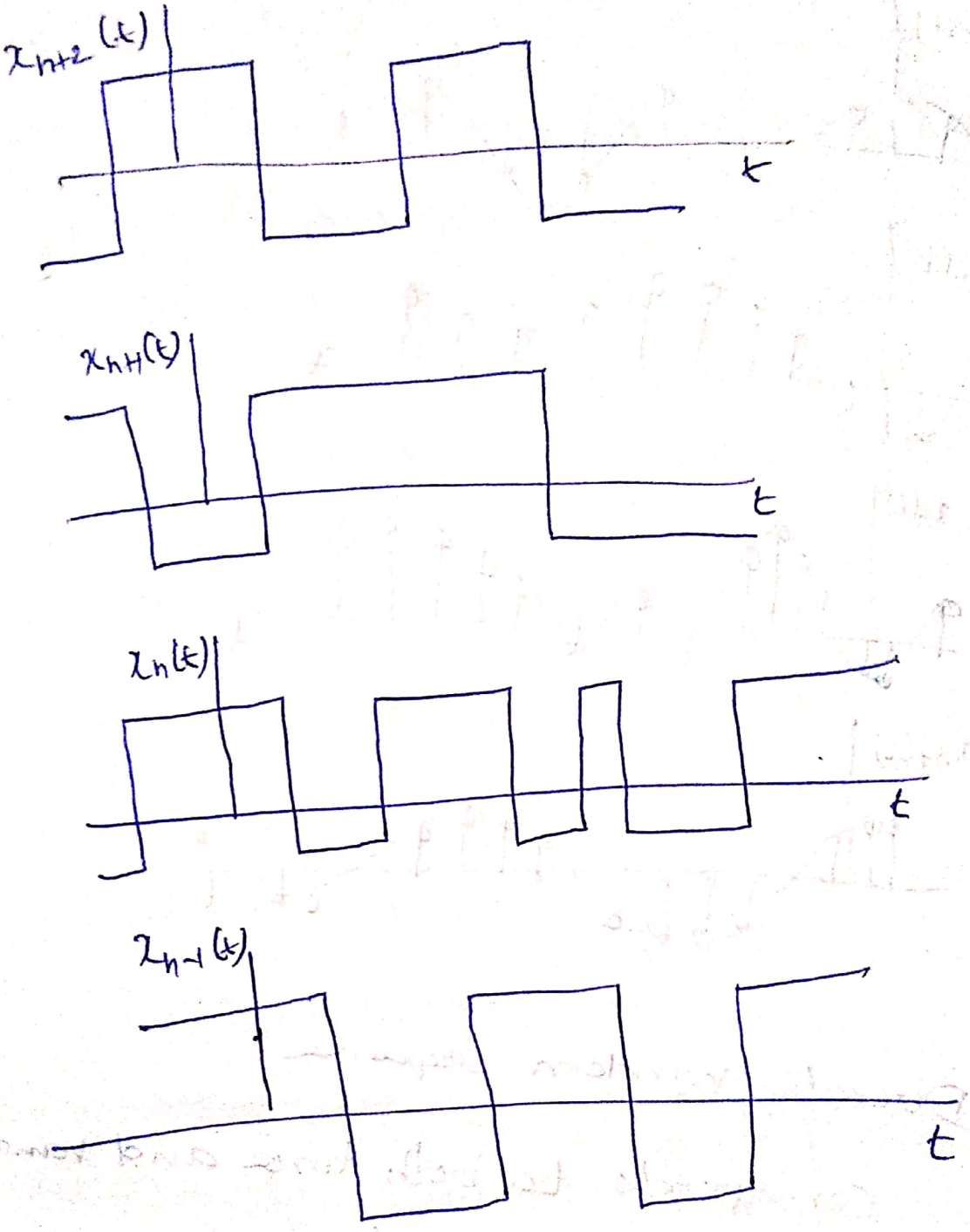

A random process for which X is continuous but time has only discrete values is called continous random sequence (or) discrete time random process.

A continous random sequence is defined at only discrete (sample) times. Its sample function are called DT random signal.

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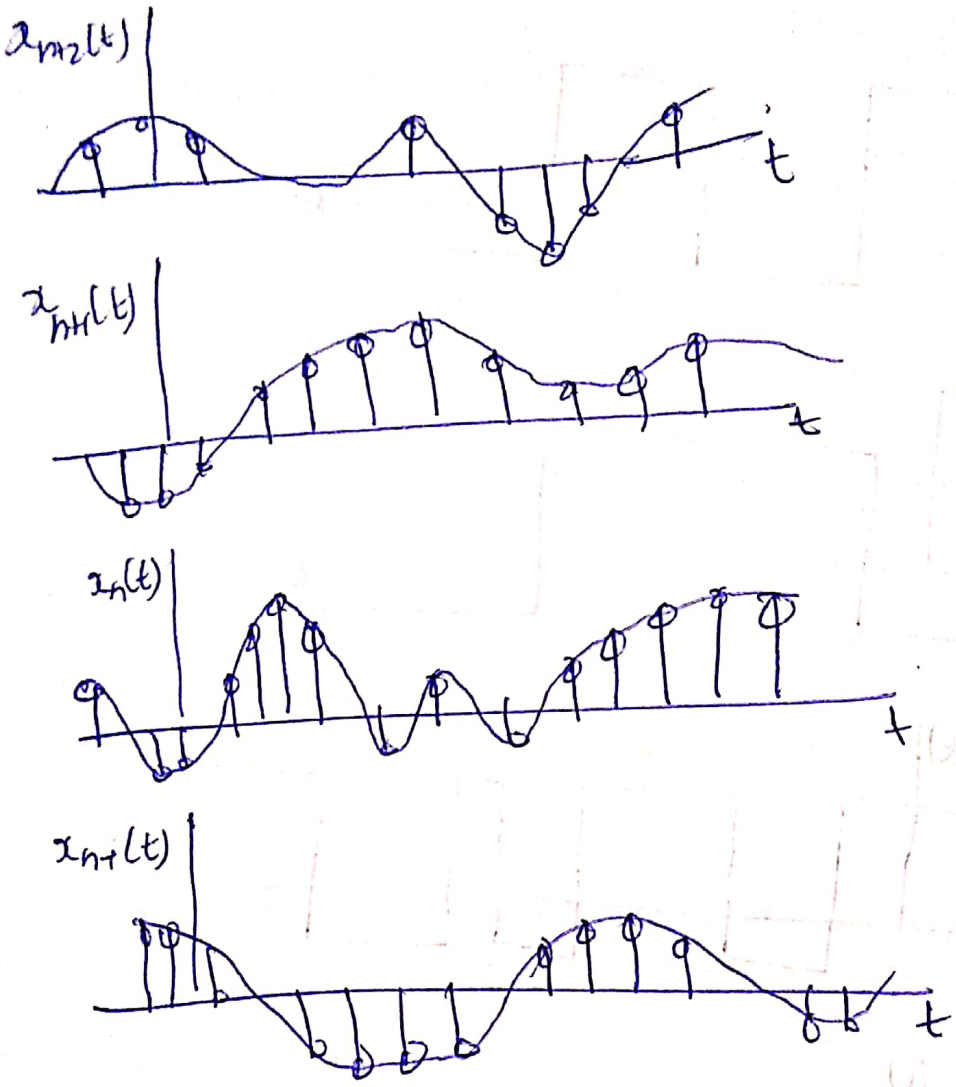

Discrete random sequence

corresponds to both time and random variable being discrete.

It can derive from rounding off samples of DT random process (continuous random sequence).

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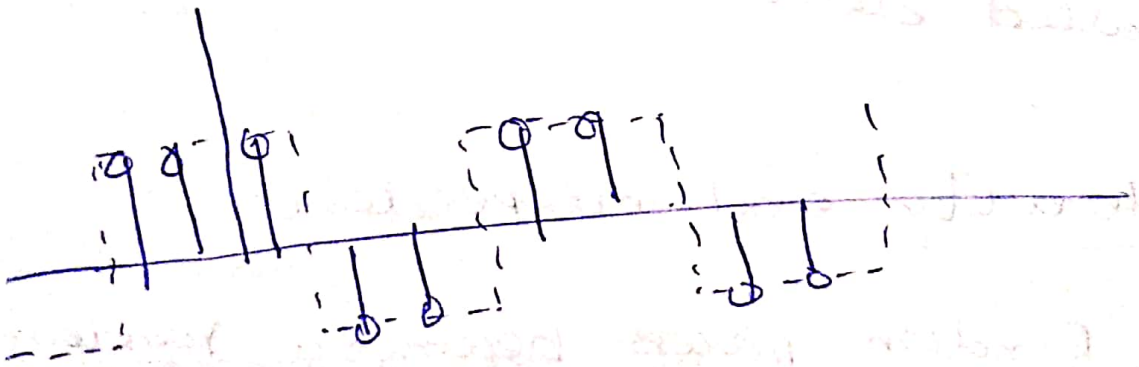
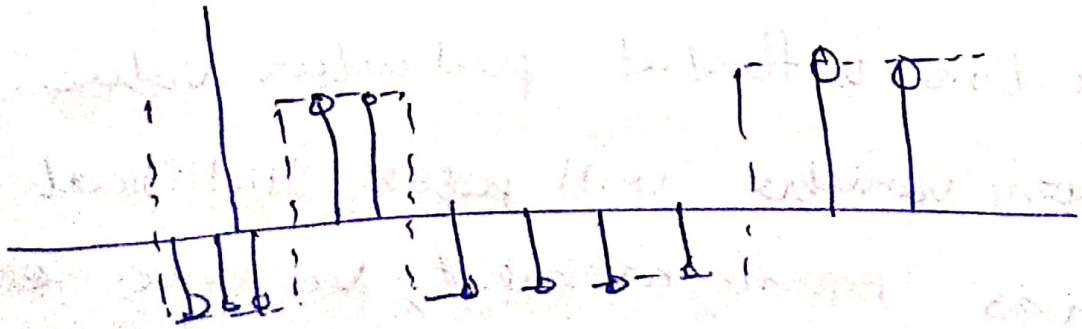
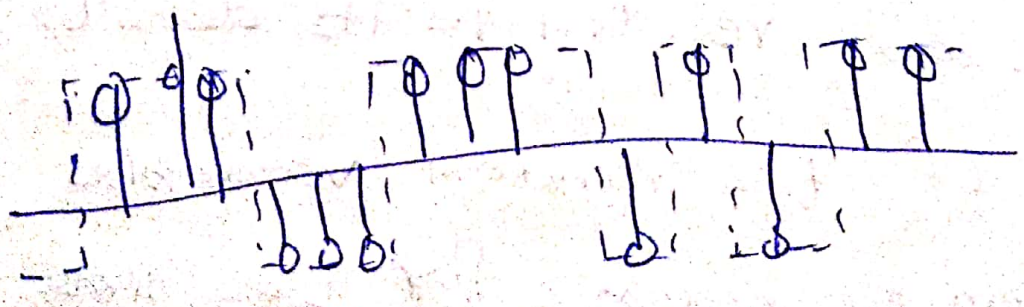
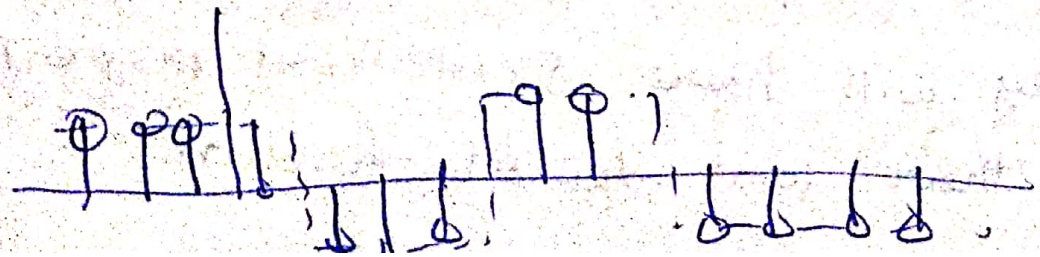
The rounding consists of choosing a discrete amplitude from a finite set of discrete amplitudes, that most closely equals each sample value of the DT random process. The operation is called quantization and is necessary to convert the process to a form suitable for use in a digital computer.









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Deterministic & nondeterministic process

If future value of any sample function cannot be predicted exactly from the observed past value, called nondeterministic.

If future value of any sample function can be predicted from the past value. called deterministic.

Stationarity and independence.

Random process becomes a random variable when time is fixed at particular value. The random variable will posses statistical properties such as mean, moment, variance etc.. that are related to its density function

If two random variables are obtained from the process for two time instants, they will have statistical property related to their joint density function.

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A random process is said to be stationary if all its statistical properties do not change with time. Other processes are called non stationary.

Distribution & Density function.

To define stationarity, we first define distribution and density function, as they apply to random process $X(t)$.

For a particular time t_1, the distribution function associated with the random variable $X_1 = x(t_1)$ will be denoted as $F_X(x_1 : t_1)$

$$F_X(x_1 : t_1) = P\{X(t_1) \leq x_1\}$$

for any real number x_1.

For two random variables $X_1 = x(t_1)$ and $X_2 = x(t_2)$. the second order joint distribution function is the two dimensional extension.

$$F_X(x_1, x_2 : t_1, t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\}$$

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For N random variable $x_i = x(t_i)$ $i = 1, 2, \ldots N$ the Nth order joint distribution function is

$$F_x(x_1, x_2 \ldots x_N : t_1, t_2 \ldots t_N) = P[X(t_1) \leq x_1 \ldots X(t_n) < x_n]$$

Joint density function of interest are found from the appropriate derivatives.

$$f_x(x_1 : t_1) = d F_x(x_1 : t_1) / dx_1$$

$$f_x(x_1, x_2 : t_1; t_2) = \partial^2 F_x(x_1, x_2 : t_1, t_2) / \partial x_1 \partial x_2$$

$$f_x(x_1, x_2 \ldots x_N : t_1, t_2 \ldots t_N) = \partial^N F_x(x_1 \ldots x_n : t_1 \ldots t_N) / \partial x_1 \ldots \partial x_N$$

Statistical Independance.

Two processes $X(t)$ and $Y(t)$ are statistically independent, if the random variable group $X_1(t_1), X(t_2) \ldots X(t_N)$ is independent of the group $Y(t_1), Y(t_2) \ldots Y(t_N)$ for any choice of time $t_1, t_2 \ldots t_N, t_1', t_2' \ldots t_M'$

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$$f_{XY}(x_1 \ldots x_N, y_1 \ldots y_N : t_1 \ldots t_N, t_1' \ldots t_N')$$

$$= f_X(x_1 \ldots x_N : t_1 \ldots t_N)\, f_Y(y_1 \ldots y_m : t_1' \ldots t_m')$$

First order stationary Processes.

A random process is called stationary to order one, if its first order density function does not change with a shift in time orgin.

$$f_X(x_1 : t_1) = f_X(x_1 : t_1 + \Delta)$$

The process mean value $E[X(t)]$ is a constant

$$E[X(t)] = \bar{X} = \text{constant.}$$

The mean values of the random variable $X_1 = X(t_1)$ and $X_2 = X(t_2)$

$$E[X_1] = E[X(t_1)] = \int_{-\infty}^{\infty} x_1 f_X(x_1 : t_1)\, dx_1$$

For X_2

$$E[X_2] = E[X(t_2)] = \int_{-\infty}^{\infty} x_1 f_X(x_1 : t_2)\, dx_1$$

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by letting $t_2 = t_1 + \Delta$

$$E[x(t_1+\Delta)] = E[x(t_1)]$$

which must be a constant because t_1 and Δ are arbitrary.

Second-order & wide sense stationarity

A process is called stationary to order two if its second-order density function satisfies

$$f_X(x_1, x_2 ; t_1, t_2) = f_X(x_1, x_2 ; t_1+\Delta, t_2+\Delta)$$

time difference. $\Delta = t_2 - t_1$

Correlation

$E[x_1, x_2] = E[x(t_1)\, x(t_2)]$ of a random process will be a function of t_1 and t_2.

$R_{xx}(t_1, t_2)$ is autocorrelation function of the random process. $x(t)$

$$R_{xx}(t_1, t_2) = E[x(t_1)\, x(t_2)]$$

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The auto correlation function of a second order stationary process is a function only if time difference and not absolute time.

$$\tau = t_2 - t_1$$

$$R_{xx}(t_1, t_1+\tau) = E[x(t_1)x(t_1+\tau)] = R_{xx}(\tau)$$

The wide sense stationary process defined as that for which two conditions are true

$$E[x(t)] = \bar{x} = \text{constant}$$

$$E[x(t)\, x(t+\tau)] = R_{xx}(\tau)$$

A process stationary to order 2 is clearly wide sense stationary.

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Show that the random process

$$X(t) = A\cos(\omega_0 t + \theta)$$

is wide sense stationary if it is assumed that A and ω_0 constants and θ is uniformly distributed Random variable on the interval $(0, 2\pi)$.

The mean value is

$$E[X(t)] = \int_0^{2\pi} A\cos(\omega_0 t + \theta)\frac{1}{2\pi}\,d\theta = 0$$

The auto correlation function $t_1 = t$ and $t_2 = t + \tau$

$$R_{XX}(t_1; t_1+\tau) = E[X(t_1)\,X(t_1+\tau)]$$

$$R_{XX}(t, t+\tau) = E[A\cos(\omega_0 t + \theta)\,A\cos(\omega_0 t + \omega_0\tau + \theta)]$$

$$2\cos A\cos B = \cos(A-B) + \cos(A+B)$$

$$= \frac{A^2}{2}E[\cos(\omega_0\tau) + \cos(2\omega_0 t + \omega_0\tau + 2\theta)]$$

$$= \frac{A^2}{2}\cos(\omega_0\tau) + \frac{A^2}{2}E[\cos(2\omega_0 t + \omega_0\tau + 2\theta)]$$

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The second term is zero, the auto correlation depends only on τ and mean value is constant.

So $x(t)$ is wide sense stationary.

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Correlation function.

two random variables $X_1 = x(t_1)$ and $X_2 = x(t_2)$ defined by the process at times t_1 and t_2

$$R_{xx}(t_1, t_2) = E[x(t_1)\, x(t_2)]$$

assume $t_1 = \tau_1$ and $t_2 = t_1 + \tau$

$$R_{xx}(t, t+\tau) = E[x(t)\, x(t+\tau)]$$

If $x(t)$ is a wide sense stationary,

$R_{xx}(t, t+\tau)$ must be a function only of time difference $\tau = t_2 - t_1$

$$\therefore R_{xx}(\tau) = E[x(t)\, x(t+\tau)]$$

properties

(1) $|R_{xx}(\tau)| \leq R_{xx}(0)$ $R_{xx}(\tau)$ is bounded by its value at the origin.

(2) $R_{xx}(-\tau) = R_{xx}(\tau)$ $\rightarrow$ even symmetry.

(3) $R_{xx}(0) = E[x^2(t)]$ $\rightarrow$ is called power in the process.

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(4) If $E[x(t)] = \overline{x} \neq 0$ and $x(t)$ is ergodic with no periodic components, then

$$\lim_{|\tau| \to \infty} R_{xx}(\tau) = \overline{x}^2$$

(5) If $x(t)$ has periodic components then $R_{xx}(\tau)$ will have a periodic component with the same period.

(6) If $x(t)$ is ergodic, zero mean and has no periodic components then

$$\lim_{|\tau| \to \infty} R_{xx}(\tau) = 0$$

(7) $R_{xx}(\tau)$ cannot have an arbitrary shape.

(1) Given the auto correlation function, for a stationary ergodic process with no periodic components is

$$R_{xx}(\tau) = 25 + \frac{4}{1+6\tau^2}$$

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Using property - 4

$$\lim_{|\tau| \to \infty} R_{xx}(\tau) = \bar{x}^2$$

$$R_{xx}(\tau) = \lim_{\tau \to \infty} 25 + \frac{4}{1+6\tau^2}$$

$$\bar{x}^2 = 25$$

$$\bar{x} = \pm 5$$

$$\sigma_x^2 = E[x^2(t)] - [E[x(t)]]^2$$

Using property -3

$$R_{xx}(0) = E[x^2(t)]$$

$$R_{xx}(0) = 25 + 4 = 29$$

$$\sigma_x^2 = 29 - 25 = 4$$

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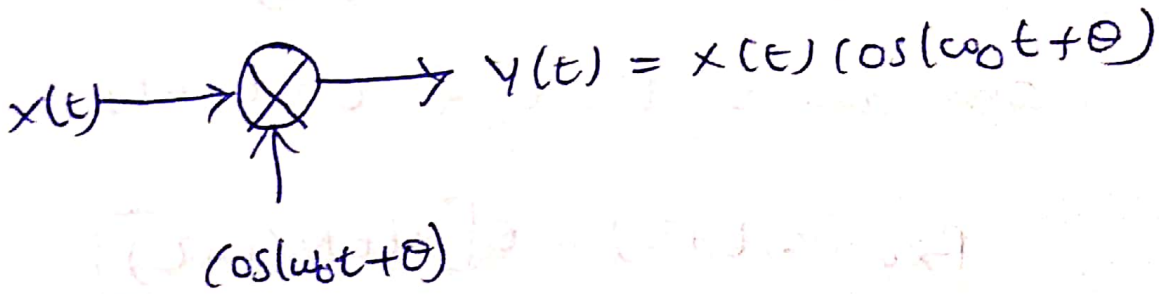
Let $X(t)$ be a wide-sense stationary random process with autocorrelation function.

$$R_{XX}(\tau) = e^{-a(\tau)}$$

where $a > 0$ is a constant. We assume $X(t)$ "amplitude modulates" a carrier $\cos(\omega_0 t + \theta)$



where ω_0 is a constant and θ is a random variable uniform on $(-\pi, \pi)$ that is statistically independent of $X(t)$. Determine the autocorrelation function of $Y(t)$

$$R_{YY}(t, t+\tau) = E[Y(t)\,Y(t+\tau)]$$

$$= E[X(t)\,X(t+\tau)\cos(\omega_0 t + \theta)\cos(\omega_0 t + \omega_0 \tau + \theta)]$$

$$= R_{XX}(\tau)\frac{1}{2}E[\cos(\omega_0 \tau) + \cos(2\omega_0 t + \omega_0 \tau + 2\theta)]$$

$$\because E[\cos(2\omega_0 t + \omega_0 \tau + 2\theta)] = \int_{-\pi}^{\pi} \frac{1}{2\pi}[\cos 2\omega_0 t + \omega_0 \tau + 2\theta]\,d\theta$$

$$= 0$$

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$$R_{YY}(t, t+\tau) = \frac{1}{2} R_{XX}(\tau) \cos\omega_0 \tau$$

$$= R_{YY}(\tau)$$

Cross correlation Function and Its properties.

The cross-correlation function of two random process $x(t)$ and $y(t)$

and assume $t_1 = t$ & $\tau = t_2 - t_1$

$$R_{XY}(t, t+\tau) = E[x(t) y(t+\tau)]$$

If $x(t)$ and $Y(t)$ are atleast jointly wide sense Stationary $R_{XY}(t, t+\tau)$ is independent of absolute time.

$$R_{XX}(\tau) = E[x(t) Y(t+\tau)]$$

If

$$R_{XY}(t, t+\tau) = 0$$

then $x(t)$ and $Y(t)$ are called orthogonal processes.

If the two processes are statistically independent, the cross-correlation function becomes

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$$R_{xy}(t, t+\tau) = E[X(t)]\,E[Y(t+\tau)]$$

$X(t)$ & $Y(t)$ are atleast wide sense stationary

$$R_{xx}(\tau) = \bar{X}\,\bar{Y}$$

which is constant.

<u>properties</u>

(1) $R_{xy}(-\tau) = R_{yx}(\tau)$

(2) $|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0)\,R_{yy}(0)}$

(3) $|R_{xy}(\tau)| \leq \frac{1}{2}\left[R_{xx}(0) + R_{yy}(0)\right]$

The geometric mean of two positive numbers cannot exceed their arithmetic mean

$$\sqrt{R_{xx}(0)\,R_{yy}(0)} \leq \frac{1}{2}\left[R_{xx}(0) + R_{yy}(0)\right]$$

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Let two random processes $X(t)$ and $Y(t)$ be defined by

$$X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

$$Y(t) = B\cos(\omega_0 t) - A\sin(\omega_0 t)$$

where A and B are random variables and ω_0 is a constant. $X(t)$ is wide-sense stationary if A & B are uncorrelated zero mean random variables with the same variance. With these same constraints on A and B, $Y(t)$ is also wide-sense stationary. Find the ~~same~~ cross correlation function $R_{xy}(t, t+\tau)$ and also show that $X(t)$ & $Y(t)$ are jointly wide sense stationary

$$R_{xy}(t, t+\tau) = E[X(t)\,Y(t+\tau)]$$

$$= E[AB\cos(\omega_0 t)\cos(\omega_0 t + \omega_0 \tau)$$

$$+ B^2 \sin(\omega_0 t)\cos(\omega_0 t + \omega_0 \tau) - A^2 \cos(\omega_0 t)\sin(\omega_0 t + \omega_0 \tau)$$

$$- AB\sin(\omega_0 t)\sin(\omega_0 t + \omega_0 \tau)$$

$$= E[AB]\cos(2\omega_0 t + \omega_0 \tau) + E(B^2)\sin(\omega_0 t)\cos(\omega_0 t + \omega$$

$$- E[A^2]\cos(\omega_0 t)\sin(\omega_0 t + \omega_0 \tau)$$

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$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

Since A and B are assumed to be zero mean, uncorrelated random variables $E[AB]=0$ also A & B are assumed to ~~be~~ have equal variances $E[A^2] = E[B^2] = \sigma^2$

$$R_{xy}(t, t+\tau) = -\sigma^2 \sin(\omega_0 \tau)$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y.$$

Thus $x(t)$ and $y(t)$ are jointly wide sense stationary because $R_{xy}(t, t+\tau)$ depends only on τ.

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Covariance function

Covariance of two random variable,

$$C_{xy} = E[(x-\bar{x})(y-\bar{y})] .$$

can be extended to random process.

The autocovariance function is defined by

$$C_{xx}(t, t+\tau) = E[\{x(t) - E[x(t)]\}\{x(t+\tau) - E[x(t+\tau)]\}] \quad \text{--- (1)}$$

$$R_{xx}(t, t+\tau) = E[x(t)x(t+\tau)]$$

$$= E[\{x(t)x(t+\tau) - x(t)\cancel{E[x(t+\tau)]} - E[x(t)]x(t+\tau) + E[x(t)]\cancel{E[x(t+\tau)]}\}]$$

$$= E[x(t)x(t+\tau)] - E[x(t)]E[x(t+\tau)]$$

$$= R_{xx}(t, t+\tau) - E[x(t)]E[x(t+\tau)] \quad \text{--- (2)}$$

Similarly

The Cross Covariance function for two processes $x(t)$ and $y(t)$

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$$C_{xy}(t,t+\tau) = E[\{x(t) - E[x(t)]\}\{y(t+\tau) - E[y(t+\tau)]\}]$$

$$C_{xy}(t,t+\tau) = R_{xy}(t,t+\tau) - E[x(t)]E[y(t+\tau)] \quad (3)$$

If $x(t)$ and $y(t)$ are at least wide-sense stationary eqn (2) & (2) becomes

$$\boxed{\begin{aligned} R_{xy}(t,t+\tau) &= E[x(t)]E[y(t+\tau)]. \\ R_{xy}(\tau) &= \bar{x}\,\bar{y} \end{aligned}}$$

$$C_{xx}(\tau) = R_{xx}(\tau) - \bar{x}^2 \quad \text{——} \quad (4)$$

$$C_{xy}(\tau) = R_{xy}(\tau) - \bar{x}\bar{y} \quad \text{——} \quad (5)$$

The variance of a random process in eqn (1) is general with $\tau = 0$.

For ~~wel~~-WSS process, variance does not depend on time and It is given by eqn (4) with $\tau = 0$

$$\sigma_x^2 = E[\{x(t) - E[x(t)]\}^2] = R_{xx}(0) - \bar{x}^2$$

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For two random processes, if

$$C_{xy}(t, t+\tau) = 0$$

x & y are uncorrelated.

So the equation (3) becomes.

$$R_{xy}(t, t+\tau) = E[x(t)]\, E[y(t+\tau)]$$

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Given the random process

$$Y(t) = X(t)\cos(\omega_0 t + \theta)$$

where $X(t)$ is a wide sense stationary random process that amplitude modulates a carrier of constant angular frequency ω_0 with a random phase θ independent of $X(t)$ and uniformly distributed on $(-\pi, \pi)$

(a) Find $E[Y(t)]$

(b) Find the autocorrelation function of $Y(t)$

(c) Is $Y(t)$ wide-sense stationary?

(a) $E[Y(t)] = E[X(t)\cos(\omega_0 t + \theta)]$

$= E[X(t)] \cdot E[\cos(\omega_0 t + \theta)]$

$= E[X(t)] \cdot \int_{-\pi}^{\pi} \cos(\omega_0 t + \theta) \frac{1}{2\pi} d\theta$

$$\int_{-\pi}^{\pi} \cos(\omega_0 t + \theta) \frac{1}{2\pi} d\theta = \frac{1}{2\pi} \int_{\pi}^{\pi} \cos\omega_0 t \cos\theta \, d\theta - \int_{-\pi}^{\pi} \sin\omega_0 t \sin\theta \, d\theta$$

$$= \frac{1}{2\pi}\left[[\sin\theta]_{-\pi}^{\pi} \cos\omega_0 t \; [\cos\omega_0 t] [\cos\theta]_{-\pi}^{\pi} \right]$$

$$= \frac{1}{2\pi}\left[0 \times \cos\omega_0 t + \cos\omega_0 t \, (1-1) \right] = 0$$

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$E[Y(t)] = E[X(t)] \cdot 0 = 0$

(b) $R_{YY}(t, t+\tau) = E[X(t)\cos(\omega_0 t + \theta)$
$\times X(t+\tau)\cos(\omega_0 t + \omega_0 \tau + \theta)]$

$= E[X(t)X(t+\tau)] \frac{E}{2}[\cos(\omega_0 \tau)$
$+ \cos(2\omega_0 t + 2\theta + \omega_0 \tau)$

$= \frac{1}{2} R_{XX}(\tau) \cdot \cos \omega_0 \tau = R_{YY}(\tau)$

(c) yes. $Y(t)$ is WSS because $E[Y(t)] = 0$ is constant while $R_{YY}(t, t+\tau) = R_{YY}(\tau)$ does not depends on t.

(2) Statistically independent, zero-mean random processes $X(t)$ and $Y(t)$ have autocorrelation function

$$R_{XX}(\tau) = e^{-|\tau|}$$ and

$$R_{YY}(\tau) = \cos(2\pi\tau)$$

respectively.

(a) Find the autocorrelation function of the sum $W_1(t) = X(t) + Y(t)$

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(b) Find the auto correlation function of the difference $W_2(t) = x(t) - y(t)$

(c) Find the cross-correlation function of $W_1(t)$ and $W_2(t)$

(a) $R_{W_1W_1}(t, t+\tau) = E[\{x(t)+y(t)\}\{x(t+\tau)+y(t+\tau)\}]$

$= E[x(t)x(t+\tau)] + E[x(t).y(t+\tau)] + E[y(t)x(t+\tau)]$

$+ E[y(t)\,y(t+\tau)]$

$= R_{xx}(\tau) + R_{yy}(\tau) + R_{xy}(t, t+\tau) + R_{yx}(t, t+\tau)$

here $x(t)$ and $Y(t)$ are independent

$\therefore R_{xy}(t, t+\tau) = R_{yx}(t, t+\tau) = 0$

Thus

$R_{W_1W_1}(t, t+\tau) = R_{xx}(\tau) + R_{yy}(\tau)$

$= e^{-|\tau|} + \cos 2\pi\tau$

(b) $R_{W_2W_2}(t, t+\tau) = E[\{x(t)-y(t)\}\{x(t+\tau)-y(t+\tau)\}]$

$= R_{xx}(\tau) + R_{yy}(\tau)$

$= \cos 2\pi\tau + e^{-|\tau|}$

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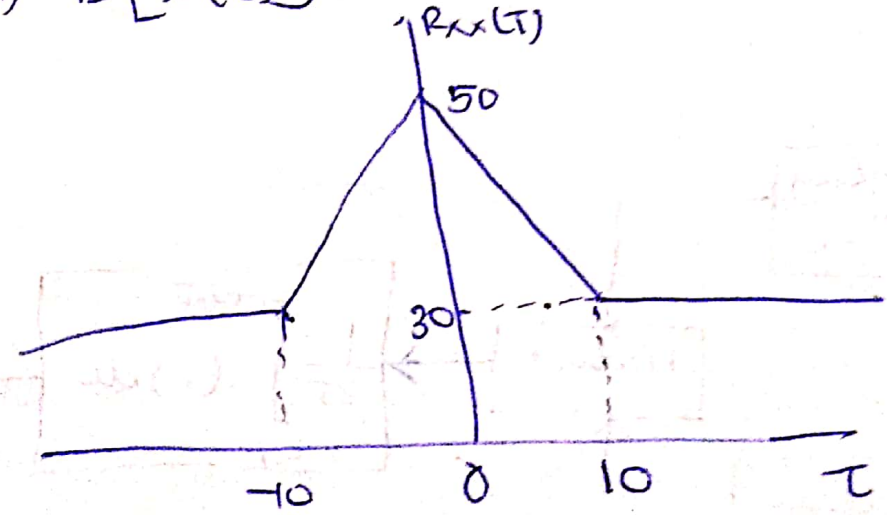
(c) $R_{W_1 W_2}(t, t+\tau) = E\left[\{x(t)+y(t)\}\{x(t+\tau)-y(t+\tau)\}\right]$

$= R_{xx}(\tau) - R_{yy}(\tau) = e^{-|\tau|} - \cos 2\pi\tau$

(3) For a stationary ergodic random process having the autocorrelation function shown in Figure.

Find (a) $E[x(t)]$ (b) $E[x^2(t)]$ and (c) σ_x^2



(a) $E[x(t)]$

using the property

If $E[x(t)] = \bar{x} \neq 0$ and $x(t)$ is ergodic with no periodic components.

$$\lim_{|\tau| \to \infty} R_{xx}(\tau) = \bar{x}^2$$

$$\bar{x}^2 = 30, \quad \bar{x} = \sqrt{30}$$

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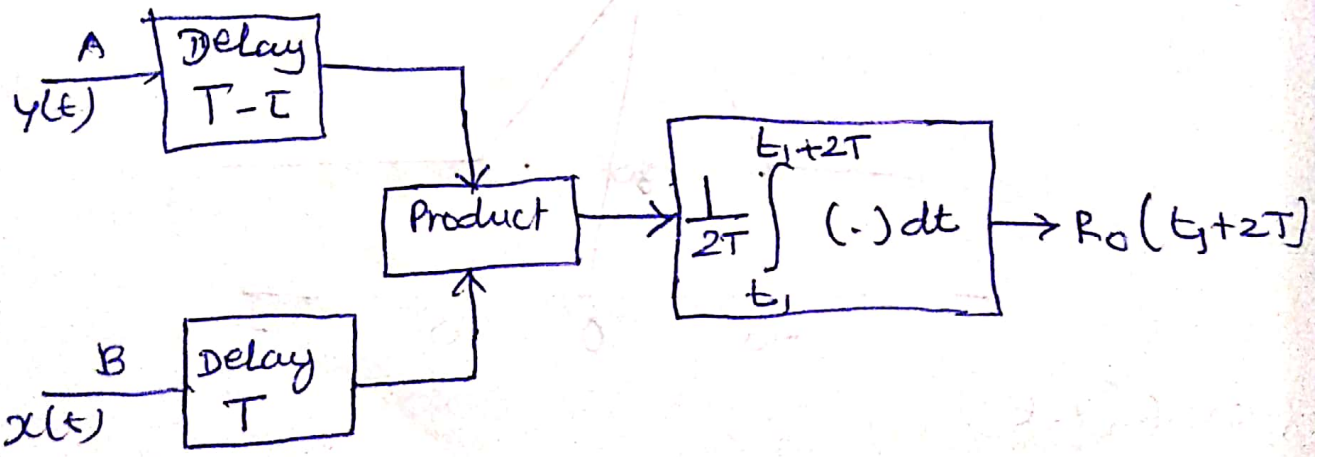
(2) power in the process

$$E[x^2(t)] = \overline{x^2} = R_{xx}(0) = 50$$

(3) $$\sigma_x^2 = \overline{x^2} - \bar{x}^2$$

$$= 50 - 30 = 20$$

Measurement of correlation function.



Block diagram of a possible system for measuring the approximate time cross-correlation function of two jointly ergodic random process $x(t)$ and $y(t)$. Sample functions $x(t)$ and $y(t)$ are delayed by T and $T-\tau$ respectively. And the product of the delayed waveform is formed.

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This product is then integrated to form the output which equals the integral at time t_1+2T, where t_1 is arbitary and $2T$ is the integration period.

If we assume that $x(t)$ and $y(t)$ exist atleast during the interval $-T < t$ and t_1 is an arbitrary time except $0 \leq t_1$, then the o/p is easily found to be.

$$R_0(t_1+2T) = \frac{1}{2T}\int_{t_1-T}^{t_1+T} x(t)\, y(t+\tau)\, dt$$

If we choose $t_1 = 0$, and assume T is Large.

$$R_0(2T) = \frac{1}{2T}\int_{-T}^{T} x(t)\, y(t+\tau)\, dt$$

$$= R_{xx}(\tau)$$

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(1) given Random process

$$x(t) = A\cos(\omega_0 t + \theta)$$

A and ω_0 are constant and θ represent a specific value of the random variable. And ~~taces~~ we connect points A and B together and use the system to measure the autocorrelation function of the process

$$R_0(2T) = \frac{1}{2T}\int_{-T}^{T} x(t)\, y(t+\tau)\, dt.$$

$$R_0(2T) = \frac{1}{2T}\int_{-T}^{T} A^2 \cos(\omega_0 t + \theta)\cos(\omega_0 t + \theta + \omega_0 \tau)\, dt$$

$$2\cos A \cos B = \cos(A-B) + \cos(A+B)$$

$$R_0(2T) = \frac{A^2}{2T}\int_{-T}^{T} \frac{1}{2}\left[\cos(\omega_0 \tau) + \cos(2\omega_0 t + 2\theta + 2\omega_0 \tau)\right] dt$$

$$= \frac{A^2}{4T}\left[\cos\omega_0\tau\,[t]_{-T}^{T} + \int_{-T}^{T} \cos 2\omega_0 t \cdot \cos(2\theta + 2\omega_0 t)\, dt\right.$$

$$- \int_{-T}^{T} \sin 2\omega_0 t \sin(2\theta + 2\omega_0 t)\, dt$$

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$$= \frac{A^2}{4T}\left[\cos\omega_0\tau \cdot 2T + \cos(2\theta + \omega_0\tau)\left[\frac{\sin 2\omega_0 t}{2\omega_0}\right]_{-T}^{T}\right.$$

$$\left. + \sin(2\theta + 2\omega_0\tau)\left[\frac{-\cos 2\omega_0 t}{2\omega_0}\right]_{-T}^{T}\right]$$

$$= \frac{A^2}{4T}\left[\cos\omega_0\tau \cdot 2T + 2 \times \cos(2\theta + \omega_0\tau) \cdot \frac{\sin 2\omega_0 T}{2\omega_0}\right]$$

$$= \frac{A^2}{2}\cos(\omega_0\tau) + \frac{A^2}{4T} \times \cos(2\theta + \omega_0\tau)\frac{\sin 2\omega_0 T}{\omega_0}$$

$$= \frac{A^2}{2}\cos(\omega_0\tau) + \frac{A^2}{2}\cos(2\theta + \omega_0\tau) \cdot \frac{\sin(2\omega_0 T)}{2\omega_0 T}$$

$$= R_{xx}(\tau) + \epsilon(T)$$

$R_{xx}(\tau) = \frac{A^2}{2}\cos(\omega_0\tau)$ is the true auto-correlation function of $x(t)$

$$\epsilon(t) = \frac{A^2}{2}\cos(\omega_0\tau + 2\theta)\frac{\sin(2\omega_0 T)}{2\omega_0 T}$$

is the error term.

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Gaussian Random processes

Consider a continuous random process which define N random variables $X_1 = X(t_1) \ldots X_i$

$X_1 = X(t_1) \ldots \quad X_i = X(t_i) \ldots \quad X_N = X(t_N)$

Corresponding to N time instants $t_1 \ldots t_i \ldots t_N$

If, for any $N = 1, 2 \ldots$ and any time $t_1 \ldots t_N$ these random variables are jointly gaussian, that is they have a joint density

$$f_X(x_1, \ldots x_N; t_1 \ldots t_N) = \frac{\exp\{-(1/2)[x-\bar{x}]^t [C_X]^{-1} [x-\bar{x}]\}}{\sqrt{(2\pi)^N |[C_X]|}}$$

$$[x-\bar{x}] = \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \\ \vdots \\ x_N - \bar{x}_N \end{bmatrix}$$

$$[C_X] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & & & \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix}$$

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The mean values $\bar{x}_i$ of $x(t_i)$ are

$$\bar{x}_i = E[x_i] = E[x(t_i)]$$

The elements of the covariance matrix $[C_x]$

$$C_{ik} = C_{x_i x_k} =$$

$$= E[(x_i - \bar{x}_i)(x_k - \bar{x}_k)]$$

$$= E[\{x(t_i) - E[x(t_i)]\}\{x(t_k) - E[x(t_k)]\}]$$

$$= C_{xx}(t_i, t_k)$$

$$C_{xx}(t_i, t_k) = R_{xx}(t_i, t_k) - E[x(t_i)]E[x(t_k)]$$

If the gaussian process is not stationary the mean and autocovariance functions will depend on absolute time.

~~The mean is~~ The process is wide-sense stationary the mean will be constant

$$\bar{x}_i = E[x(t_i)] = \bar{x} \quad (\text{constant})$$

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The autocovariance and autocorrelation functions will depend only on time difference and not absolute time

$$C_{xx}(t_i, t_k) = C_{xx}(t_k - t_i)$$

$$R_{xx}(t_i, t_k) = R_{xx}(t_k - t_i)$$

(1) A Gaussian random process has an autocorrelation function

$$R_{xx}(\tau) = 6 \exp(-|\tau|/2)$$

Determine a covariance matrix for the random variables $x(t)$, $x(t+1)$, $x(t+2)$ and $x(t+3)$

The covariance matrix of the x_i $i = 1,2,3,4$ will be a 4×4 matrix with elements

$$C_{ij} = E[x_i x_j] = E[x(t+i-1)\, x(t+j-1)]$$

$$= R_{xx}(j-i)$$

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Thus

$$C_{ij} = R_{XX}(j-i) = 6\exp\left[-|j-i|/2\right]$$

$$[C] = 6\begin{bmatrix} 1 & e^{-1/2} & e^{-1} & e^{-3/2} \\ e^{-1/2} & 1 & e^{-1/2} & e^{-1} \\ e^{-1} & e^{-1/2} & 1 & e^{-1/2} \\ e^{-3/2} & e^{-1} & e^{-1/2} & 1 \end{bmatrix}$$

(2) A Gaussian random process is known to be wide sense stationary with a mean of $\overline{X} = 4$ and autocorrelation function

$$R_{XX}(\tau) = 25\, e^{-3|\tau|}$$

we seek to specify the joint density function for three random variable $x(t_i)$, $i = 1, 2, 3$ defined at times $t_i = t_0 + [(i-1)/2]$, with t_0 a constant.

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Here

$$t_k - t_i = (k-i)/2 \quad \text{and } k = 1, 2, 3 \ldots$$

$$R_{xx}(t_k - t_i) = 25e^{-3|k-i|/2}$$

$$C_{xx}(t_k - t_i) = R_{xx}(t_k - t_i) - \bar{x}^2$$

$$\boxed{C_{xx}(t_k - t_i) = 25e^{-3|k-i|/2} - 16}$$

$$[C_x] = \begin{bmatrix} (25-16) & (25e^{-3/2}-16) & (25e^{-6/2}-16) \\ (25e^{-3/2}-16) & (25-16) & (25e^{-3/2}-16) \\ (25e^{-6/2}-16) & (25e^{-3/2}-16) & (25-16) \end{bmatrix}$$

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Poisson Random process

The probability of exactly k occurrences over a time interval $(0,t)$ is

$$P[x(t)=k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad k = 0, 1, 2 \ldots$$

The probability density of the number of occurrences is

$$f_x(x) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k)$$

The mean and variance of a poisson random variable

$$E[x(t)] = \int_{-\infty}^{\infty} x f_x(x)\,dx = \int_{-\infty}^{\infty} x \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k)\,dx$$

$$= \sum_{k=0}^{\infty} \frac{k(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t \cancel{e^{\lambda t}} \cancel{e^{-\lambda t}}$$

$$= \lambda t \qquad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

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$$E[x^2(t)] = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx = \sum_{k=0}^{\infty} \frac{k^2 (\lambda t)^k e^{-\lambda t}}{k!}$$

$$= \lambda t\, e^{-\lambda t}(1+\lambda t)\, e^{\lambda t}$$

$$= \lambda t(1+\lambda t)$$

$$\boxed{\begin{aligned} & e^{-\lambda t}\left[0 + \lambda t + \frac{4(\lambda t)^2}{2!} + \frac{9(\lambda t)^3}{3!} + \frac{16(\lambda t)^4}{4!} + \cdots\right] \\ & e^{-\lambda t}\left[1 + \frac{4\lambda t}{2!} + \frac{9(\lambda t)^2}{3!} + \frac{16(\lambda t)^3}{4!} + \cdots\right] \end{aligned}}$$

Joint Probability Density

To determine the joint probability density function for the poisson process at times $0 < t_1 < t_2$,

The probability of k_1 event occurences over $(0, t_1)$

$$P[x(t_1) = k_1] = \frac{(\lambda t_1)^{k_1} e^{-\lambda t_1}}{k_1!} \quad k_1 = 0, 1, 2, \ldots \quad (1)$$

The conditional probability of k_2 occurrences over $(0, t_2)$ given that k_1 events occurred over $(0, t_1)$

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The probability that $k_2 - k_1$ events occurred over (t_1, t_2) is

$$P[x(t_2) = k_2 \mid x(t_1) = k_1] = \frac{[\lambda(t_2 - t_1)]^{k_2 - k_1} e^{-\lambda(t_2 - t_1)}}{(k_2 - k_1)!} \quad (2)$$

for $k_2 \geq k_1$. The joint probability of k_2 occurrences at time t_2 and k_1 occurrences at time t_1 is product of (1) & (2)

$$P(k_1, k_2) = P[x(t_2) = k_2 \mid x(t_1) = k_1] \cdot P[x(t_1) = k_1]$$

$$= \frac{(\lambda t_1)^{k_1} \lambda(t_2 - t_1)^{k_2 - k_1} e^{-\lambda t_2}}{k_1!\,(k_2 - k_1)!} \qquad k_2 \geq k_1$$

The joint density becomes

$$f_x(x_1, x_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=k_1}^{\infty} P(k_1, k_2)\, \delta(x_1 - k_1)\, \delta(x_2 - k_2)$$

for the process random variables $X(t_1) = X_1$, and $X(t_2) = X_2$.

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Complex Random Processes.

If the complex random variable to include time, the result is called complex random process $Z(t)$

$$Z(t) = X(t) + jY(t)$$

where $X(t)$ and $Y(t)$ are real processes.

① $Z(t)$ is called stationary if $X(t)$ and $Y(t)$ are jointly stationary.

② If $X(t)$ and $Y(t)$ are jointly WSS, then $Z(t)$ is said to be WSS.

If Two complex processes $Z_1(t)$ and $Z_j(t)$ are jointly WSS, if each is WSS and their cross correlation function is a function of time differences only and not absolute time.

The mean value of $Z(t)$ is

$$E[Z(t)] = E[X(t)] + jE[Y(t)]$$

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Autocorrelation function is

$$R_{zz}(t, t+\tau) = E\left[z^*(t)\, z(t+\tau)\right]$$

$*$ indicates complex conjugate.

Autocovariance function is

$$C_{zz}(t, t+\tau) = E\left[\{z(t) - E[z(t)]\} * \{z(t+\tau) - E[z(t+\tau)]\}\right]$$

If $z(t)$ is atleast WSS, the mean value becomes a constant

$$\bar{z} = \bar{x} + j\bar{y}$$

the correlation functions are independent of absolute time

$$R_{zz}(t, t+\tau) = R_{zz}(\tau)$$

$$C_{zz}(t, t+\tau) = C_{zz}(\tau)$$

For two complex processes $z_i(t)$ and $z_j(t)$, Cross-correlation and Cross-covariance functions are

$$R_{z_i z_j}(t, t+\tau) = E\left[z_i^*(t)\, z_j(t+\tau)\right]. \quad i \neq j$$

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and

$$C_{z_i z_j}(t, t+\tau) = E\left[\{z_i(t) - E[z_i(t)]\} * \{z_j(t+\tau) - E[z_j(t+\tau)]\}\right] \quad i \neq j$$

If the two processes are atleast jointly WSS

$$R_{z_i z_j}(t, t+\tau) = R_{z_i z_j}(\tau) \quad i \neq j$$

$$C_{z_i z_j}(t, t+\tau) = C_{z_i z_j}(\tau) \quad i \neq j$$

$z_i(t)$ and $z_j(t)$ are said to uncorrelated process if $C_{z_i z_j}(t, t+\tau) = 0 \quad i \neq j$.

Orthogonal processes if

$$R_{z_i z_j}(t, t+\tau) = 0 \quad i \neq j$$

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(1) Aircraft arrive at an airport according to a poisson process at a rate of 12 per hour. All aircraft are handled by one air traffic controller. If the controller takes a 2 minute coffee break, what is the probability that he will miss one or more arriving aircraft

$$P(\text{miss 1 or more aircraft}) = 1 - P(\text{miss } 0)$$

$$= 1 - P(0 \text{ arrive})$$

$$= 1 - \frac{(\lambda t)^0}{0!} e^{-\lambda t} = 1 - e^{-\lambda t}$$

$$\lambda = \frac{12}{60} \quad t = 2 \text{ min}$$

$$= 1 - e^{-12/60 \times 2} = 1 - e^{-0.4} = 0.3297$$

(2) Telephone calls are initiated through an exchange at the average rate of 75 per minute and are described by a poisson process. Find the probability that more than 3 calls are initiated in any 5 second period.

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$$P(\text{more than } 3 \text{ calls}) = 1 - P(\leq 3)$$

$$= 1 - P(0 \text{ calls}) - P(1 \text{ calls}) - P(2 \text{ calls}) - P(3 \text{ calls})$$

& $\lambda = 75/60 \quad t = 5$

$$P(0 \text{ calls}) = \frac{\left(\frac{75}{60} \times 5\right)^0 e^{-75/60 \times 5}}{0!} = e^{-75/60 \times 5}$$

$$P(1 \text{ calls}) = \frac{\left(\frac{75}{60} \times 5\right)^1 e^{-\frac{75}{60} \times 5}}{1!} = \frac{75}{60} \times 5 \, e^{-\frac{75}{60} \times 5}$$

$$P(2 \text{ calls}) = \frac{\left(\frac{75 \times 5}{60}\right)^2 e^{-\frac{75}{60} \times 5}}{2!} = \frac{1}{2}\left(\frac{75 \times 5}{60}\right)^2 e^{-\frac{75}{60} \times 5}$$

$$P(3 \text{ calls}) = \frac{\left(\frac{75 \times 5}{60}\right)^3 e^{-75/60 \times 5}}{3!} = \frac{1}{6}\left(\frac{75 \times 5}{60}\right) e^{-\frac{75}{60} \times 5}$$

$$P(\text{more than } 3 \text{ calls}) = 1 - e^{75/60 \times 5}\left(1 + \frac{75 \times 5}{60} + \left(\frac{75 \times 5}{60}\right)^2 \times \frac{1}{2} + \left(\frac{75 \times 5}{60}\right)^3 \times \frac{1}{6}\right)$$

$$= 0.8697$$

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(3) A small store has two check out lanes that develop waiting lines if more than two customers arrive in any one minute interval. Assume that a poisson process describes the number of customers that arrive for check out. Find the probability ~~that~~ of a waiting line if the average rate of customer arrivals is (a) 2 per minutes (b) 1 per minutes (c) $\frac{1}{2}$ per minutes.

$$P(\text{waiting line}) = P(\geq 3 \text{ arrive})$$

$$= 1 - P(0 \text{ arrive}) - P(1 \text{ arrive}) - P(2 \text{ arrive})$$

(a) $\lambda = 2/\text{minutes}$

$$P(\text{line}) = 1 - e^{-2}\left[\frac{[2(1)]^0}{0!} + \frac{[2(1)]^1}{1!} + \frac{[2(1)]^2}{2!}\right]$$

$$= 1 - 5e^{-2} = 0.3233$$

(b) $\lambda = 1/\text{min}$

$$P \text{ line} = 1 - e^{-1}\left[\frac{[1(1)]^0}{0!} + \frac{(1)^1}{1!} + \frac{1}{2!}\right]$$

$$= 1 - 2.5e^{-1} = 0.0803$$

(c) $\lambda = \frac{1}{2}$ per minute

$$P(\text{line}) = 1 - e^{-1/2}\left[\frac{(1/2)^0}{0!} + \frac{(1/2)^1}{1!} + \frac{(1/2)^2}{2!}\right] = 0.0144$$

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A complex random process is defined by

$$z(t) = \exp(j\Omega t)$$

where Ω is a zero mean random variable uniformly distributed on the interval from $\omega_0 - \Delta\omega$ to $\omega_0 + \Delta\omega$. where ω_0 & $\Delta\omega$ are positive constants. Find

(a) the mean value

(b) the autocorrelation function of $z(t)$

(c) Is $z(t)$ wide-sense stationary.

$$\boxed{E[g(z)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(z) \cdot f_{x,y}(x,y)\, dx\, dy}$$

$$E[z(t)] = \int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega} e^{j\Omega_0 t}\, d\Omega \times \frac{1}{2\Delta\omega}$$

$$= \frac{1}{2\Delta\omega}\left[\frac{e^{j\Omega t}}{t}\right]_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega}$$

$$= \frac{1}{2\Delta\omega t}\left[e^{j\omega_0 t} \cdot e^{j\Delta\omega t} - e^{j\omega_0 t} e^{-j\Delta\omega t}\right]$$

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$$= \frac{e^{j\omega_0 t}}{\Delta\omega t}\left[\frac{e^{j\Delta\omega t} - e^{-j\Delta\omega t}}{2}\right]$$

$$\frac{e^{j\theta} - e^{-j\theta}}{2} = \sin\theta$$

$$\boxed{E[z(t)] = e^{j\omega_0 t} \cdot \frac{\sin\Delta\omega t}{\Delta\omega t}}$$

$$R_{zz}(t, t+\tau) = E[z^*(t) z(t+\tau)]$$

(b) $R_{zz}(t, t+\tau) = E\left[\cancel{e^{-j\Omega t}} \cdot \cancel{e^{j\Omega t}} \cdot e^{j\Omega\tau}\right]$

$$= E\left[e^{j\Omega\tau}\right]$$

$$= e^{j\omega_0\tau} \frac{\sin\Delta\omega\tau}{\Delta\omega\tau} = R_{zz}(\tau)$$

(c) $z(t)$ is not a wide-sense stationary because the mean value depends on t.

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② Let $x(t)$ and $y(t)$ be statistically independent wide sense stationary real processes having the same auto correlation function $R(\tau)$. Define the complex process

$$Z(t) = x(t)\cos(\omega_0 t) + j\,y(t)\sin(\omega_0 t)$$

where ω_0 is positive constant. Find the auto correlation function $Z(t)$. Is $Z(t)$ WSS

$$R_{ZZ}(t, t+\tau) = E\left[Z(t)^* Z(t+\tau)\right]$$

$$= E\left[\{x(t)\cos(\omega_0 t) - j\,y(t)\sin(\omega_0 t)\}\right.$$
$$\left.\cdot\{x(t+\tau)\cos(\omega_0 t+\omega_0\tau) + j\,y(t+\tau)\sin(\omega_0 t+\omega_0\tau)\}\right]$$

$$= E\left[x(t)x(t+\tau)\cos\omega_0 t\cos(\omega_0 t+\omega_0\tau)\right.$$
$$+ j\,x(t)y(t+\tau)\cos\omega_0 t\sin(\omega_0 t+\omega_0\tau)$$
$$- j\,y(t)x(t+\tau)\sin\omega_0 t\cos(\omega_0 t+\omega_0\tau)$$
$$\left.+ j\,y(t)y(t+\tau)\sin\omega_0 t\sin(\omega_0 t+\omega_0\tau)\right]$$

$$2\cos x\cos y = \cos(x-y) + \cos(x+y)$$
$$2\sin x\sin y = \cos(x-y) - \cos(x+y)$$
$$2\sin x\cos y = \sin(x-y) + \sin(x+y)$$

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$= R(\tau)\cos\omega_0\tau + j\bar{X}\bar{Y}\sin(\omega_0\tau)$

The mean value

$$E[Z(t)] = \bar{X}\cos\omega_0 t + j\bar{Y}\sin(\omega_0 t)$$

Thus, $Z(t)$ is not WSS. because the mean is a function of t.

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