The joint PMF of two random variables X and Y is given by

$$f_{XY}(x,y) = \begin{cases} k(2x+y) & x=1,2;\ y=1,2 \\ 0 & \text{otherwise.} \end{cases}$$

where k is a constant.

$$\sum_x \sum_y f_{xy}(x,y) = 1$$

$$\sum_{x=1}^{2} \sum_{y=1}^{2} k(2x+y) = 1$$

$$\sum_{x=1}^{2} \sum_{y=1}^{2} = k \sum_{x=1}^{2} (2x+1) + (2x+2)$$

$$= k\{(2+1) + (2+2) + (4+1) + (4+2)\}$$

$$= 18k$$

$$18k = 1 \Rightarrow k = 1/18$$

(b) The marginal PMFs are

$$f_X(x) = \sum_y f_{XY}(x,y) = \frac{1}{18} \sum_{y=1}^{2} (2x+y)$$

$$= \frac{1}{18}\{(2x+1)+(2x+2)\} = \frac{1}{18}(4x+3) \quad x=1,2$$

$$f_Y(y) = \sum_x f_{XY}(x,y) = \frac{1}{18} \sum_{x=1}^{2} 2x+y$$

$$= \frac{1}{18}\{(2+y) + (4+y) = \frac{1}{18}(2y+6) \quad y=1,2$$

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(c) since

$$P_X(x)\,P_Y(y) \neq P_{XY}(x,y)$$

we conclude that X and Y are not independent.

(2) A fair coin is tossed 3 times. Let X be a random variable that takes the value 0 if the first toss is a tail and the value 1 if the first toss is a head. Also let Y be a random variable that defines the total number of heads in the three tosses.

a. Determine the joint PMF of X and Y

b Are X and Y independent.

Solution

X takes the value 0, and 1

Y takes the values 0, 1, 2, 3

The joint PMF of X and Y can be constructed as

Sample space	value of X	value of Y
HHH	1	3
HHT	1	2
HTH	1	2
HTT	1	1
THH	0	2
THT	0	1
TTH	0	1
TTT	0	0

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$P_{XY}(0,0) = P[X=0, Y=0] = P[\{TTT\}] = 1/8$

$P_{XY}(0,1) = P[X=0, Y=1] = P[\{THT\} \cup \{TTH\}] = 1/4$

$P_{XY}(0,2) = P[X=0, Y=2] = P[\{THH\}] = 1/8$

$P_{XY}(0,3) = P[X=0, Y=3] = 0$

$P_{XY}(1,0) = P[X=1, Y=0] = 0$

$P_{XY}(1,1) = P[X=1, Y=1] = P[\{HTT\}] = 1/8$

$P_{XY}(1,2) = P[X=1, Y=2] = P[\{HTH\} \cup \{HHT\}] = 1/4$

$P_{XY}(1,3) = P[X=1, Y=3] = P[\{HHH\}] = 1/8$

(b) X & Y are independent

Consider the point $(x,y) = (0,0)$

$P_X(0) = \sum_y P_{XY}(0,y) = P_{XY}(0,0) + P_{XY}(0,1) + P_{XY}(0,2) + P_{XY}(0,3)$

$= 1/2$

$P_Y(0) = \sum_x P_{XY}(x,0) = P_{XY}(0,0) + P_{XY}(1,0) = 1/8$

$P_{XY}(0,0) = 1/8$

$P_X(0)\, P_Y(0) = \frac{1}{8} \times 1/2 = 1/16$

$\therefore P_{XY}(0,0) \neq P_X(0)\, P_Y(0)$ it is not independent.

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2. X and Y are two continuous random variables whose Joint PDF is given by

$$f_{XY}(x,y) = \begin{cases} e^{-(x+y)} & 0 \le x \le \infty \ \& \ 0 \le y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Are X and Y independent?

First evalute the marginal PDF's of X and Y:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^{\infty} e^{-(x+y)}\,dy$$

$$= \int_0^{\infty} e^{-x} e^{-y}\,dy = e^{-x}\left[\frac{e^{-y}}{-1}\right]_0^{\infty}$$

$$= e^{-x}(0-(-1)) = e^{-x}$$

$$= \begin{cases} e^{-x} & x \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^{\infty} e^{-(x+y)}\,dx$$

$$= \int_0^{\infty} e^{-x} e^{-y}\,dx = e^{-y}\int_0^{\infty} e^{-x}\,dx = e^{-y}\left[-e^{-x}\right]_0^{\infty}$$

$$= \begin{cases} e^{-y} & y \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

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Now

$$f_X(x) \cdot f_Y(y) = e^{-x} e^{-y} = e^{-(x+y)} = f_{XY}(x,y)$$

which means that X and Y are independent.

The joint PMF of two random variables X and Y is given by

$$P_{XY}(x,y) = \begin{cases} \frac{1}{18}(2x+y) & x=1,2;\ y=1,2 \\ 0 & \text{otherwise.} \end{cases}$$

a) what is the conditional PMF of Y given X?

b) what is the conditional PMF of X given Y?

we know that the marginal PMFs are given by

$$P_X(x) = \sum_y P_{XY}(x,y) = \frac{1}{18}\sum_{y=1}^{2}(2x+y) = \frac{1}{18}(2x+1)+(2x+2) = \frac{1}{18}(4x+3)$$

$x = 1,2$

$$P_Y(x) = \sum_x P_{XY}(x,y) = \frac{1}{18}\sum_{x=1}^{2} 2x+y) = \frac{1}{18}(2+y)+(4+y) = \frac{1}{18}(2y+6)$$

$y = 1,2$

The conditional PMFs are given by

$$P_{X|Y}(x|y) = \frac{P_{XY}(x,y)}{P_Y(y)} = \frac{\frac{1}{18}(2x+y)}{\frac{1}{18}(2y+6)} = \frac{2x+y}{2y+6}$$

$$P_{Y|X}(y|x) = \frac{P_{XY}(x,y)}{P_X(x)} = \frac{\frac{1}{18}(2x+y)}{\frac{1}{18}(4x+3)} = \frac{2x+y}{4x+3}$$

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Two random variables X and Y have the following Joint PDF:

$$f_{XY}(x,y) = \begin{cases} x e^{-x(y+1)} & 0 \le x < \infty;\ 0 \le y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Determine the conditional PDF of X given Y, the conditional PDF of Y given X.

the marginal PDF's

$$f_X(x) = \int_0^\infty f_{XY}(x,y)\,dy = \int_0^\infty x e^{-x(y+1)}\,dy = \int_0^\infty x e^{-xy} \cdot e^{-x}$$

$$= x e^{-x} \int_0^\infty e^{-xy}\,dy = x e^{-x} \left[-\frac{e^{-xy}}{x} \right]_0^\infty$$

$$= \frac{x e^{-x}}{x} \left[0 - (-1) \right] = e^{-x}$$

$$f_Y(y) = \int_0^\infty f_{XY}(x,y)\,dx = \int_0^\infty x e^{-x(y+1)}\,dx$$

let $u = x$ $\quad dv = e^{-x(y+1)}$

$du = dx$ $\quad v = -\dfrac{e^{-x(y+1)}}{y+1}$

$$f_Y(y) = \int_0^\infty x e^{-x(y+1)}\,dx = \left[-x \frac{e^{-x(y+1)}}{y+1} \right]_0^\infty + \frac{1}{y+1} \int e^{-x(y+1)}\,dx$$

$$+ \frac{1}{y+1} \int_0^\infty e^{-x(y+1)}\,dx$$

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$$= 0 - \frac{1}{(y+1)^2}\left[e^{-x(y+1)}\right]_0^{\infty}$$

$$= \frac{1}{(y+1)^2} \quad 0 \le y < \infty$$

The conditional PDFs are given by

$$f_{(Y|X)}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{xe^{-x(y+1)}}{e^{-x}} = \frac{xe^{-xy}\cdot \cancel{e^{-x}}}{\cancel{e^{-x}}}$$

$$= xe^{-xy} \quad 0 \le y < \infty$$

$$f_{(X|Y)}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{xe^{-x(y+1)}}{\frac{1}{(y+1)^2}}$$

$$= x(y+1)^2 e^{-x(y+1)} \quad 0 \le x < \infty$$

$$\boxed{\int u \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx}$$

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Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} x + cy^2 & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(1) Find the constant c

(2) Find $P(0 \le X \le \frac{1}{2}, 0 \le Y \le \frac{1}{2})$

To find c

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$

$$= \int_0^1\int_0^1 x + cy^2 \, dx\,dy$$

$$= \int_0^1 \left[\frac{x^2}{2} + cy^2 x\right]_0^1 dy$$

$$= \int_0^1 \frac{1}{2} + cy^2 \, dy$$

$$1 = \left[\frac{1}{2}y + \frac{cy^3}{3}\right]$$

$$1 = \left[\frac{1}{2} + \frac{c}{3}\right]$$

$$\frac{3+2c}{6} = 1 \quad \Rightarrow \quad 2c = 6-3$$

$$c = 3/2$$

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(2) To find $P(0 \le x \le \frac{1}{2}, 0 \le y \le \frac{1}{2})$

$$= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(x + \frac{3}{2} y^2\right) dx\, dy$$

$$= \int_0^{\frac{1}{2}} \left[\frac{x^2}{2} + \frac{3}{2} y^2 x\right]_0^{\frac{1}{2}} dy$$

$$= \int_0^{\frac{1}{2}} \frac{1}{8} + \frac{3}{4} y^2 \, dy$$

$$= \left[\frac{1}{8} y + \frac{\cancel{3} y^3}{4 \times \cancel{3}}\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{16} + \frac{1}{32} = \frac{2+1}{32} = \frac{3}{32} \,//$$

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let X and Y denote the proportion of two different chemicals in a sample mixture of chemicals used as an insecticide. Suppose X and Y have joint probability density given by:

$$f(x,y) = \begin{cases} 2 & 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le x+y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

1) Find the marginal density functions for X & Y

(2) Are X and Y independent?

(3) Find $P(X > \frac{1}{2} \mid Y = \frac{1}{4})$

$$f_X(x) = \int_0^{1-x} 2\,dy = \left[2y\right]_0^{1-x} = 2(1-x)$$

$$f_Y(y) = \int_0^{1-y} 2\,dx = \left[2x\right]_0^{1-y} = 2(1-y)$$

$$f_X(x) \cdot f_Y(y) = 2(1-x) \cdot 2(1-y) \neq 2 = f(x,y)$$

therefore X and Y are independent.

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$$P(X > \tfrac{1}{2} \mid Y = \tfrac{1}{4}) = \int_{1/2}^{1} f(x \mid y = \tfrac{1}{4})\, dx$$

$$= \frac{\int_{1/2}^{1} f(x, y = \tfrac{1}{4})}{f(y = \tfrac{1}{4})} = \frac{\int_{1/2}^{1} 2}{2(1 - \tfrac{1}{4})}$$

$$= \frac{[2x]_{1/2}^{1}}{2 \times (3/4)}$$

$$= \frac{2 - 1}{2 \times 3/4_2} = 2/3$$

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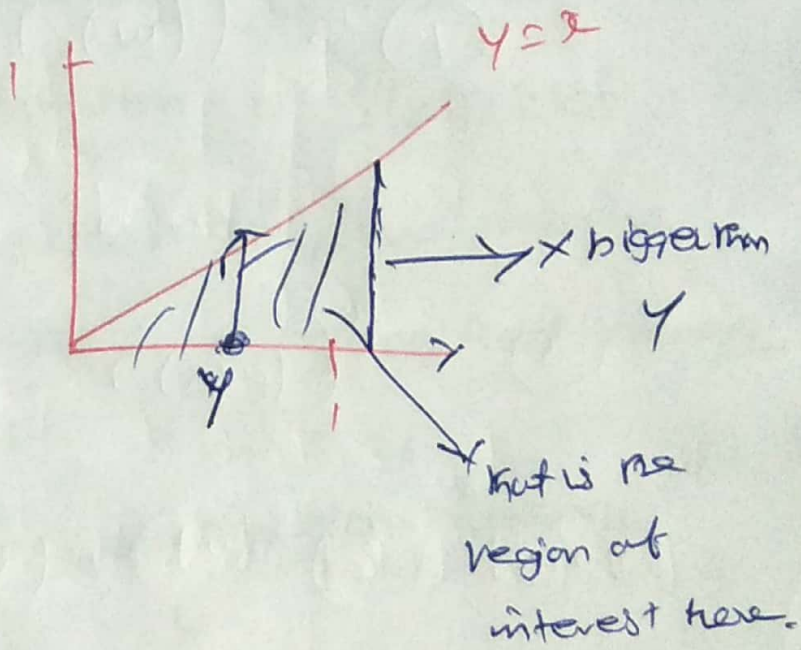
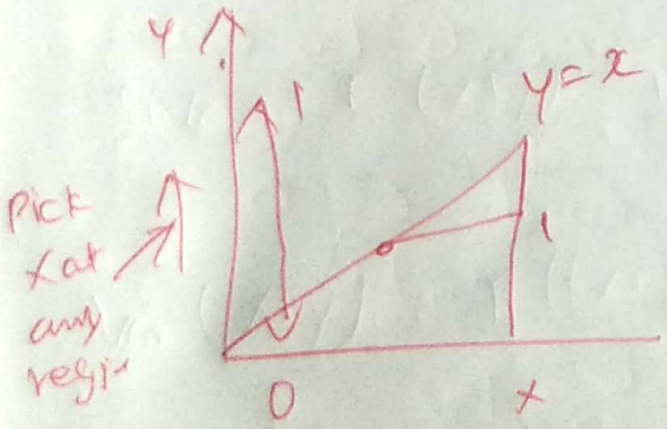
$0 \le y \le x \le 1$





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Suppose $f(x,y) = 10x^2y \quad 0 \le y \le x \le 1$

Find correlation between the random variable x & y.

$$f_x(x) = \int_0^x 10x^2y \, dy = 10x^2 \left[\frac{y^2}{2}\right]^x$$

$$= 5x^4 \quad 0 \le x \le 1$$

$$E(x) = \int_0^1 x \, f_x(x) \, dx$$

$$= \int_0^1 x \times 5x^4 \, dx = \int_0^1 5x^5 \, dx = \frac{5x^6}{6} = \frac{5}{6}$$

$$E(x^2) = \int_0^1 5x^6 \, dx = \left[\frac{5x^7}{7}\right]_0^1 = 5/7$$

$$Var(x) = E(x^2) - (E(x))^2$$

$$= \frac{5}{7} - \left(5/6\right)^2 = 0.01984$$

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Similarly

$$f_Y(y) = \int_y^1 10x^2 y \, dx = \frac{10}{3} y$$

$$= 10y \left[\frac{x^3}{3}\right]_y^1$$

$$= \frac{10y}{3}(1 - y^3) \qquad 0 \le y \le 1$$

$$E(y) = \int_0^1 y \, f_Y(y) \, dy$$

$$= \int_0^1 \frac{10}{3}(y^2 - y^5) \, dy$$

$$= \frac{10}{3}\left[\frac{1}{3} - \frac{1}{6}\right]$$

$$= \frac{10}{3} \times \frac{1}{6} = 5/9$$

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$$\cancel{Var(y)} \; E(y^2) = \int_0^1 y^2 f_Y(y)\, dy$$

$$= \int_0^1 \frac{10}{3} (y^3 - y^6)\, dy$$

$$= \frac{10}{3} \left[\frac{y^4}{4} - \frac{y^7}{7} \right]_0^1$$

$$= \frac{10}{3} \left[\frac{7-4}{28} \right] = 10/28 =$$

$$Var(y) = \{E^2(y)\} - (E(y))^2$$

$$= \frac{10}{28} - \left(\frac{5}{9}\right)^2 = \frac{10}{28} - \frac{25}{81}$$

$$= 0.3571 - 0.30864$$

$$= 0.0485$$

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$$E(xy) = \int_0^1 \int_0^x 10x^3y^2 \, dy \, dx$$

$$= \int_0^1 10x^3 \times \frac{x^3}{3} \, dx$$

$$= \left[10 \frac{x^7}{21} \right]_0^1 = 10/21$$

$$Cov(x,y) = E(xy) - E(x)E(y) = 0.1323$$

$$\rho = \frac{Cov(x,y)}{\sqrt{var(x)\, var(y)}} = 0.4265$$

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The proportions X & Y of two chemicals found in samples of an insecticide have the joint pdf

$$f(x,y) = \begin{cases} 2 & 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le x+y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

The random variable $Z = X+Y$ denotes the proportion of the insecticide due to both chemicals combined.

(1) Find $E(Z)$ and $V(Z)$

Find the correlation between X and Y and interpret its meaning.

$$E(X+Y) = \int_0^1 \int_0^{1-x} (x+y)\, 2\, dy\, dx$$

$$= \int_0^1 2\left(xy + \frac{y^2}{2}\right)_0^{1-x} dx$$

$$= \int_0^1 2\left(x(1-x) + \frac{(1-x)^2}{2}\right) dx$$

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$$= \int_0^1 2\left((x-x^2) + \frac{1+2x+x^2}{2}\right)$$

$$= \int_0^1 \cancel{2}\left(\frac{\cancel{2x}-2x^2+1+\cancel{2x}+x^2}{\cancel{2}}\right)dx$$

$$= \int_0^1 (1-x^2)\,dx = \left[x-\frac{x^3}{3}\right]_0^1$$

$$= 1-\frac{1}{3} = \frac{2}{3}$$

$$E(x+y)^2 = \int_0^1\int_0^{1-x} 2(x+y)^2\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} 2\left(x^2+2xy+y^2\right)dy\,dx$$

$$= \int_0^1 \left[2\left(x^2y+\frac{\cancel{2}xy^2}{\cancel{2}}+\frac{y^3}{3}\right)\right]_0^{1-x} dx$$

$$= 2\int_0^1 \left[x^2(1-x)+x(1-x)^2+\frac{(1-x)^3}{3}\right]$$

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$$= 2\int_0^1 x^2 - \cancel{x^3} + \cancel{x^3} - 2x^2 + x + \frac{(1-x)^3}{3}\, dx$$

$$= 2\int_0^1 -x^2 + x - \frac{x^3 - 3x + 3x^2 + 1}{3}\, dx$$

$$\cancel{= 2\int_0^1 2x^2 - 2x + 1 + x^3\, dx}$$

$$= \frac{2}{3} \times \int_0^1 \cancel{-3x^2} \cancel{+3x} - x^3 - \cancel{3x} + \cancel{3x^2} + 1\, dx$$

$$= \frac{2}{3}\int_0^1 (1 - x^3)\, dx$$

$$= \frac{2}{3}\left[x - \frac{x^4}{4}\right]_0^1$$

$$= \frac{2}{3}\left[1 - \frac{1}{4}\right]$$

$$= \frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$$

$$(x-y)^3 = x^3 - y^3 - 3x^2y + 3xy^2$$

$$(1-x)^3 = 1 - x^3 - 3x + 3x^2$$

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$$V(z) = E(z^2) - (E(z))^2$$

$$= \frac{1}{2} - \left(\frac{2}{3}\right)^2 =$$

$$= \frac{1}{2} - \frac{4}{9}$$

$$= \frac{9-8}{18} = \frac{1}{18}$$

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$$Var(x+y) = Var(x) + Var(y)$$

$$Var(w) = E(w - E(w))^2 \quad \text{--- (1)}$$

let $w = x+y$ so (1) becomes

$$Var(x+y) = E((x+y) - E(x+y))^2 \quad \text{--- (2)}$$

Since $E(x+y) = E(x) + E(y)$ (2) becomes

$$Var(x+y) = E((x+y) - (E(x) + E(y)))^2$$

$$Var(x+y) = E((x - E(x)) + (y - E(y)))^2$$

$$E(a+b)^2 = E(a+b)(a+b)$$

$$= E(a^2 + b^2 + 2ab)$$

$$= E((x - E(x))^2 + (y - E(y))^2$$

$$+ 2((x - E(x))(y - E(y)))$$

$$= E(x - E(x))^2 + E(y - E(y))^2 + 2E((x - E(x))(y - E(y)))$$

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$$= Var(X) + Var(Y) + 2E(XY - XE(Y) - YE(X) + E(X)E(Y))$$

$$= Var(X) + Var(Y) + 2(E(XY) - 2E(XY) + E(X)E(Y))$$

$$= Var(X) + Var(Y) + 2(E(X)E(Y) - E(XY))$$

$$= Var(X) + Var(Y) + 2\,Cov(X,Y)$$

Co-variance

It is a measure of the joint variability of two random variables.

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Uniform Distribution (Continuous)

The uniform distribution is one of the simplest probability distribution in statistics. It is a continuous distribution, this means that it takes values within a specified range.

The PDF for a uniform distribution taking values in the range a to b is

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

<u>Example</u>

You arrive into a building and are about to take an elevator to the your floor. Once you call the elevator it will take between 0 and 40 seconds to arrive to you. We will assume that the elevator arrives uniformly between 0 and 40 seconds after you press the button. In this case $a=0$ and $b=40$

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$$P(c \le x \le d) = \int_c^d f(x)\,dx$$

$$= \int_c^d \frac{1}{b-a}\,dx = \frac{d-c}{b-a}$$

Here, to calculate the probability that elevator takes less than 15 seconds to arrive.

Here we set $d = 15$ and $c = 0$

$$\therefore \text{probability} = \frac{15-0}{40-0} = 15/40$$

Expected value of a uniform distribution

$$E(x) = \int_a^b x\,f(x)\,dx = \int_a^b \frac{x}{b-a}\,dx$$

$$= \left[\frac{(b-a)^{\cancel{2}}}{2\,\cancel{(b-a)}}\right] = \frac{b-a}{2}$$

$$= \frac{40}{2} = 20 \text{ second}$$

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Variance

The variance of a uniform distribution is

$$Var(x) = E(x^2) - E^2(x)$$

Eqt

$$\int_a^b x^2 f(x)\,dx = \int_a^b \frac{x^2}{b-a}\,dx = \frac{(b-a)^{\cancel{3}\,2}}{3\,\cancel{(b-a)}}$$

$$= \frac{(b-a)^2}{3}$$

$$Var(x) = \frac{(b-a)^2}{3} - \left(\frac{(b-a)}{2}\right)^2$$

$$= \frac{(b-a)^2}{3} - \frac{(b-a)^2}{4}$$

$$= \frac{(b-a)^2}{12}$$

$$= \frac{(40)^2}{12} = \frac{40 \times 40}{12}$$

$$= \frac{10 \times 40}{3}$$

$$\boxed{Var(x) = \frac{400}{3}}$$

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Uniform Distribution (Discrete)

The uniform Distribution is one of the Simplest probability distributions in Statistics. It is a discrete distribution, this means that it takes a finite set of possible eg 1, 2, 3, 4, 5 and 6

The probability mass function for a uniform distribution taking one of n possible values from the set $A = \{x_1, x_2 \ldots x_n)$ is

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Example

any discrete probability mass function, we calculate probabilities by using a Summation.

$$P(x_c \leq x \leq x_d) = \sum_{i=c}^{d} f(x_i)$$

$$p(x) = \sum_{i=c}^{d} \frac{1}{n}$$

$x = 1, 2, 3 \ldots n$

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Example

Calculate the probability that the dice lands on 2 or 3 wet set $d=3$ and $d=2$

The probability

$$= \sum_{2}^{3} \frac{1}{n}$$

$$= \frac{1}{6} + \frac{1}{6} = 2/6$$

The expected value of a uniform distribution

$$E(x) = \sum_{i=1}^{n} x_i f(x_i)$$

$$= \sum_{i=1}^{n} \frac{x_i}{n} = \frac{x_1 + x_2 + \dots x_n}{n} \text{ or } \frac{x_1 + x_n}{2}$$

the expected value $= \frac{1+2+3+4+5+6}{6}$

$$= \frac{21}{6} = 7/2 = 3.5$$

(or) $\frac{6+1}{2} = 3.5$

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The variance of uniform distribution is

$$Var(x) = \frac{(b-a+1)^2 - 1}{12}$$

$$= \frac{(6-1+1)^2 - 1}{12}$$

$$\boxed{Var(x) = \frac{36-1}{12} = 35/12 = 2.9}$$

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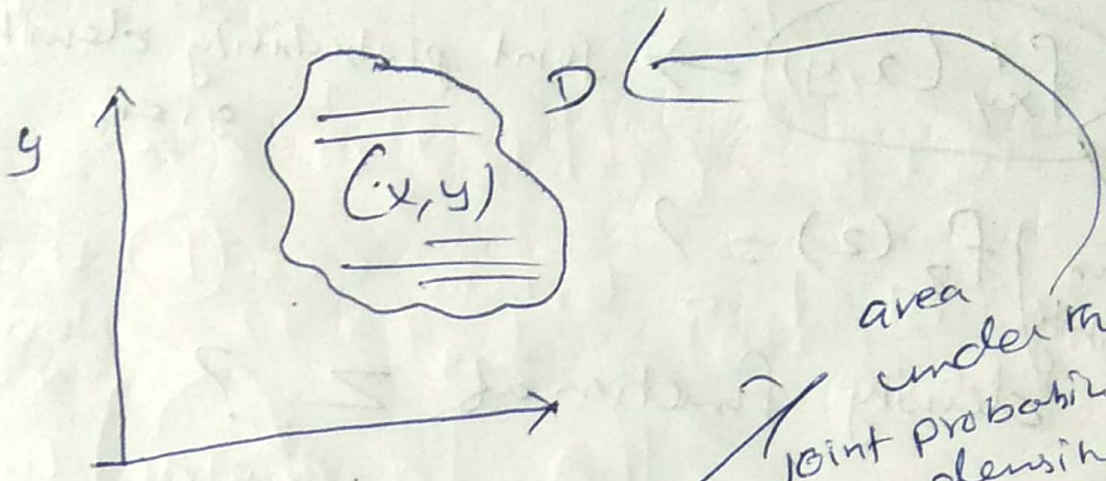
One function of two random variable.



area under the joint probability density function only over the region of

$$P[(x,y) \in D] = \iint_{(x,y)\in D} f_{XY}(x,y)\,dx\,dy$$

If you have region in 2D plane. X & Y the probability that the random variable belongs to that region.

Ex = Classical example

$$Z = g(X,Y)$$

$\sqrt{x^2+y^2}$, x^2+y^2, x/y, $\min(x,y)$, $\max(x,y)$, $x+y$, $x-y$

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$Z = X + Y$ — function given

$f'_{XY}(x, y)$ → joint probability density function given.

known quantity

$f_Z(z) = ?$

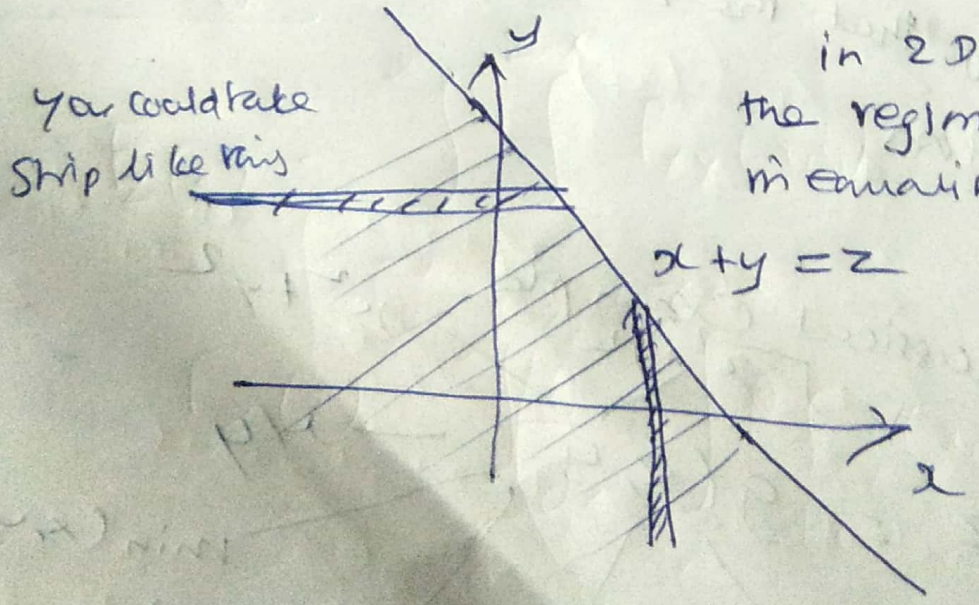
density function of Z ?

Distribution function of Z

$F_Z(z) = P[Z \leq z]$

unknown quantity

$P[Z \leq z] = P(X + Y \leq z)$



in 2D, First find out the region, where is the region inequality is satisfied.

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$$F_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_{XY}(x,y)\,dx\,dy$$

$$F_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_{XY}(x,y)\,dy\,dx$$

Leibnitz Rule

here is no derivative w.r.to y

$$G(x) = \int_{a(x)}^{b(x)} h(x,y)\,dy$$

$$\frac{dG(x)}{dx} = \frac{d}{dx} b(x)\, h(x, bx) - \frac{d}{dx} a(x)\, h(x, a(x)) + \int_{a(x)}^{b(x)} \frac{\partial h(x,y)}{\partial x}\,dy \quad \text{—— (1)}$$

partial derivative

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$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

Derivative of constant is zero.

So first two term in eqn (1) is zero.

derivative of x with respect

$$F_Z(z) = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial z} \int_{-\infty}^{z-y} f_{XY}(x,y)\, dx \right) dy$$

$$F_Z(z) = \int_{-\infty}^{\infty} 1 \cdot f_{XY}(z-y, y)\, dy.$$

<u>Similarly</u>

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(x,y)\, dy\, dx$$

$$F_Z(z) = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial z} \int_{-\infty}^{z-x} f_{XY}(x,y)\, dy \right) dx$$

$$F_Z(z) = \int_{-\infty}^{\infty} 1 \cdot f_{XY}(x, z-x)\, dx$$

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If x & y are independent.

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)\, f_Y(y)\, dy$$

$$F_Z(z) = \int_{-\infty}^{\infty} F_X(z-x)\, f_Y(x)\, dx.$$

(or)

$f_X(z) \circledast f_Y(z)$

Convolution of two density function.

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If x and y are independent random variables, both uniformly distributed on $(0,1)$, calculate the probability density of $z = x+y$

Since x, y are uniformly distributed, they have the following density functions.

$f_x(x) = 1 \quad 0 < x < 1$ & $f_y(y) = 1 \quad 0 < y < 1$

$$f_{x+y}(z) = \int_{-\infty}^{\infty} f_x(z-y)\, f_y \, dy$$

Now we goto fix the region of y.

Since $f_y = 1$ for $0 < y < 1$ then

$$f_{x+y}(z) = \int_0^1 f_x(z-y)\, dy.$$

Now the integrand is 1 when $0 < z-y < 1$

$0 \leq z-y < 1 \Rightarrow$

$0-z < z-y-z < 1-z \Rightarrow -z < -y < 1-z$

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$z-1 < y < z$

So, if $0 < z < 1$ then $0 < y < z$,

therefore

$$f_{x+y}(z) = \int_0^z dy = z$$

if $1 < z < 2$ then $z-1 < y < 1$

therefore

$$f_{x+y}(z) = \int_{z-1}^{1} dy = 1-(z-1) = 2-z$$

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$$f_{xy}(x,y) = \begin{cases} \frac{3}{4}(x-y) & 0 \le y < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $P_r(x > 1 \mid y = 1)$

$$f_{x|y}(x|y) = \frac{f_{xy}(x,y)}{f_y(y)}$$

marginal distribution of y

$$f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dx$$

$$= \int_{-\infty}^{y} f_{xy}(x,y)\,dx + \int_{y}^{2} f_{xy}(x,y)\,dx + \int_{2}^{\infty} f_{xy}(x,y)\,dx$$

$$= \int_{y}^{2} f_{xy}(x,y)\,dx$$

$$= \int_{y}^{2} \frac{3}{4}(x-y)\,dx$$

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$$= \frac{3}{4} \int_{y}^{2} (x-y)\, dx$$

$$= \frac{3}{4} \left[\frac{x^2}{2} - yx\right]_{y}^{2} = \frac{3}{4}\left[\left(\frac{\cancel{4}^{2}}{\cancel{2}} - 2y\right) - \left(\frac{y^2}{2} - y^2\right)\right]$$

$$= \frac{3}{4}\left((2-2y) + \frac{y^2}{2}\right)$$

$$= \frac{3}{4}\left(2 - 2y + \frac{y^2}{2}\right) \qquad 0 \le y \le 2$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{3/4\,(x-y)}{3/4\,(2-2y+\frac{1}{2}y^2)}$$

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{x-y}{2-2y+\frac{1}{2}y^2} & 0 \le y < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

To find $P_r(x > 1 \mid y = 1)$

$$f_{X|Y}(x|y=1) = \frac{x-1}{2-2(1)+\frac{1}{2}(1)^2} = \frac{x-1}{\frac{1}{2}}$$

$$= \begin{cases} 2(x-1) & 1 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

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$$P_r(x>1 \mid y=1) = \int_1^2 f_{x|y}(x|y=1)\,dx$$

$$= \int_1^2 2(x-1)\,dx$$

$$= 2\left(\frac{x^2}{2} - x\right)_1^2$$

$$= 2\left[\left(\frac{4}{2} - 2\right) - \left(\frac{1}{2} - 1\right)\right]$$

$$= 2 \times \frac{1}{2} = \underline{1}$$

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Conditional expectation.

We can find the expectation and variance of X with respect to this conditional distribution.

If we know that the value of Y is fixed at y then we can find the mean value of X given that Y takes the value y and also the variance of X given that $Y = y$.

$E(X|Y=y)$ is the mean value of X when Y is fixed at y.

$E(X|Y=y)$ is a number depending on y.

<u>Problem</u>

Suppose $Y = \begin{cases} 1 & \text{with probability } 1/8 \\ 2 & \text{with probability } 7/8 \end{cases}$

and $X|Y = \begin{cases} 2Y & \text{with probability } 3/4 \\ 3Y & \text{with probability } 1/4 \end{cases}$

Conditional expectation of X given $Y=y$ is a number depending on y.

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If $y=1$ then $X|(y=1) = \begin{cases} 2 & \text{with probability } 3/4 \\ 3 & \text{" } \quad 1/4 \end{cases}$

So $E(X|y=1) = 2\times 3/4 + 3\times 1/4 = 9/4$

If $y=2$ then $X|(y=2) = \begin{cases} 4 & \text{with probability } 3/4 \\ 6 & \text{with probability } 1/4 \end{cases}$

So $E(X|y=2) = 4\times 3/4 + 6\times 1/4 = \frac{18}{4}$

Thus $E(X|y=y) = \begin{cases} 9/4 & \text{if } y=1 \\ 18/4 & \text{if } y=2 \end{cases}$

So $E(X|Y=y)$ is a number depending on y.

<u>Conditional ~~variance~~</u> expectation of X given random Y ~~is~~

$E(X|Y) = \begin{cases} 9/4 & \text{if } y=1 \text{ (probability } 1/8) \\ 18/4 & \text{if } y=2 \text{ (probability } 7/8) \end{cases}$

So $E(X|Y) = \begin{cases} 9/4 & \text{with probability } 1/8 \\ 18/4 & \text{with probability } 7/8 \end{cases}$

$E(X|Y)$ is a random variable.

The total average

$E(X) = E_Y\{E(X|Y)\} = \frac{9}{4}\times\frac{1}{8} + \frac{18}{4}\times\frac{7}{8} = 4.22$

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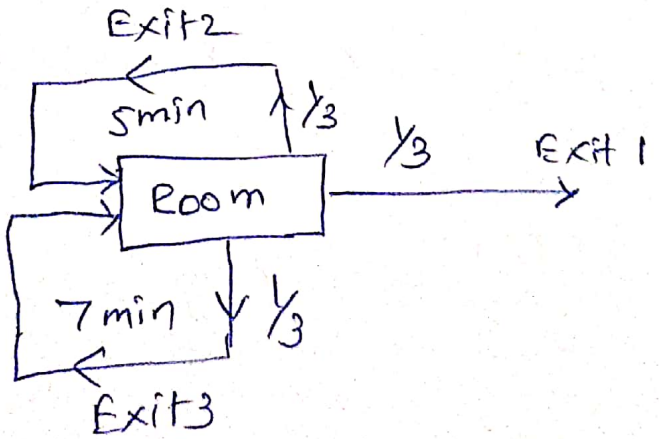
A mouse is trapped in a room with three exits at the centre of a maze.

* Exit 1 leads outside the maze after 3 minutes
* Exit 2 leads back to the room after 5 minutes
* Exit 3 leads back to the room after 7 minutes.

Every time the mouse makes a choise, it is equally likely to choose any of the three exit. what is the expected time taken for the mouse to leave the maze?

Let x = time taken for mouse to leave maze starting from room R

Let y = exit the mouse chooses first (1, 2, or 3)



$$E(x) = E_y(E(x|y))$$

$$= \sum_{y=1}^{3} E(x|y=y)\, P(y=y)$$

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$$= E(X|Y=1) \times \frac{1}{3} + E(X|Y=2) \times \frac{1}{3}$$

$$+ E(X|Y=3) \times \frac{1}{3}$$

$E(X|Y=1) = 3$ minutes

$E(X|Y=2) = 5 + E(X)$

(after 5 mins back in Room, time $E(X)$ to get out.)

$E(X|Y=3) = 7 + E(X)$

(after 5mins back in Room)

$$E(X) = 3 \times \tfrac{1}{3} + (5 + E(X)) \times \tfrac{1}{3} + (7 + E(X)) \times \tfrac{1}{3}$$

$$= 3 \times \tfrac{1}{3} + 5 \times \tfrac{1}{3} + E(X) \times \tfrac{1}{3} + 7 \times \tfrac{1}{3} + E(X) \times \tfrac{1}{3}$$

$$E(X) - \frac{2E(X)}{3} = 15 \times \tfrac{1}{3}$$

$$\frac{E(X)}{3} = \tfrac{15}{3}$$

$$\boxed{E(X) = 15 \text{ minutes}}$$

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Obtain the PDF of $Z = X + Y$, where X and Y are two independent random variable with the following PDFs

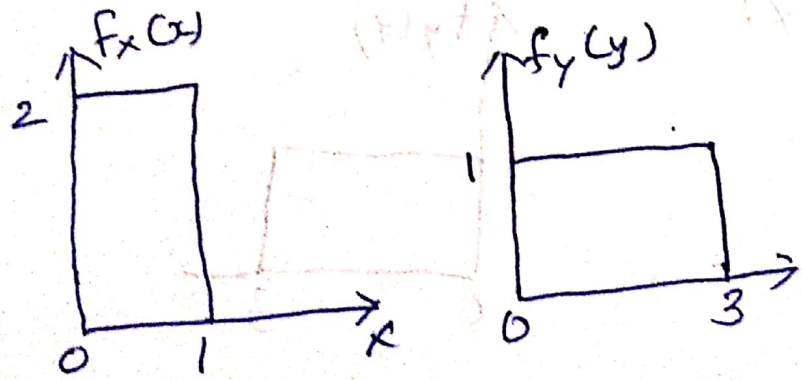
$$f_x(x) = \begin{cases} 2 & 0 < x < 1 \end{cases}$$

$$f_y(y) = \begin{cases} 1 & 0 < y < 3 \end{cases}$$

$Z = X + Y$ and the PDF is given as

$$f_Z(z) = \int_{-\infty}^{\infty} f_x(z-y)\, f_y(y)\, dy$$

two rectangular pulse



The duration of the signal $f_x(x)$ and $f_y(y)$ are given by $[t_1, T_1] = [0, 3]$ and $[t_2, T_2] = [0, 1]$

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We can conclude that the convolution of these two signal is zero

$$f_z(z) \Rightarrow f_x(x) \oplus f_y(y) = 0$$

when

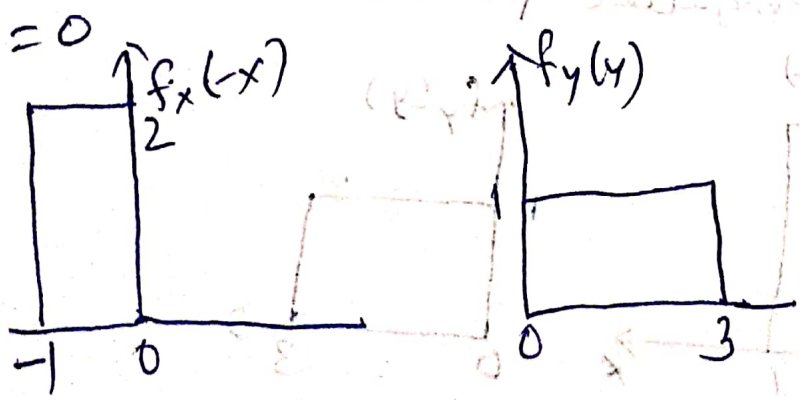
$$z < t_1 + t_2 = 0 + 0 = 0$$

and

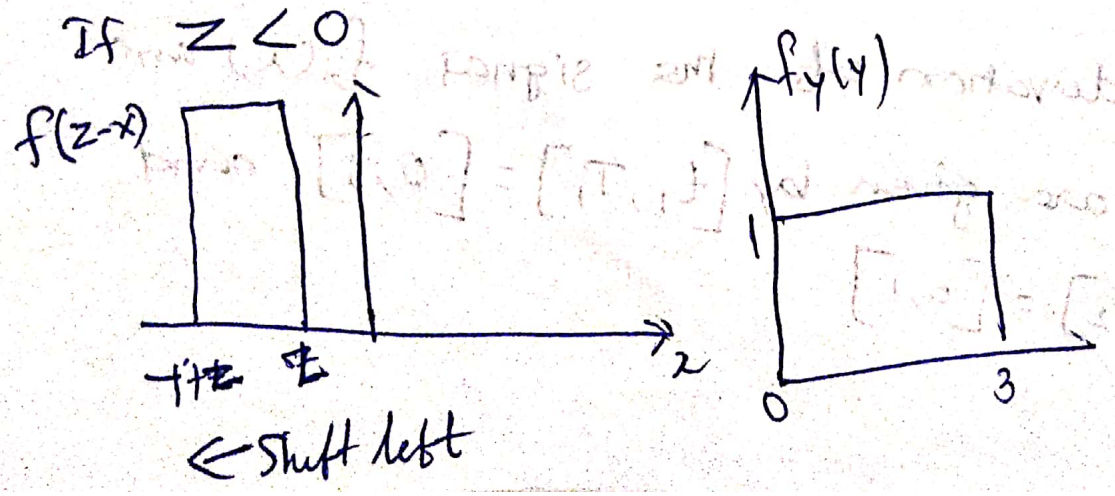
$$z \geq T_1 + T_2 = 1 + 3 = 0$$

So we need to evaluate the convolution integral in the interval $0 \leq z \leq 4$

If $z = 0$



If $z < 0$



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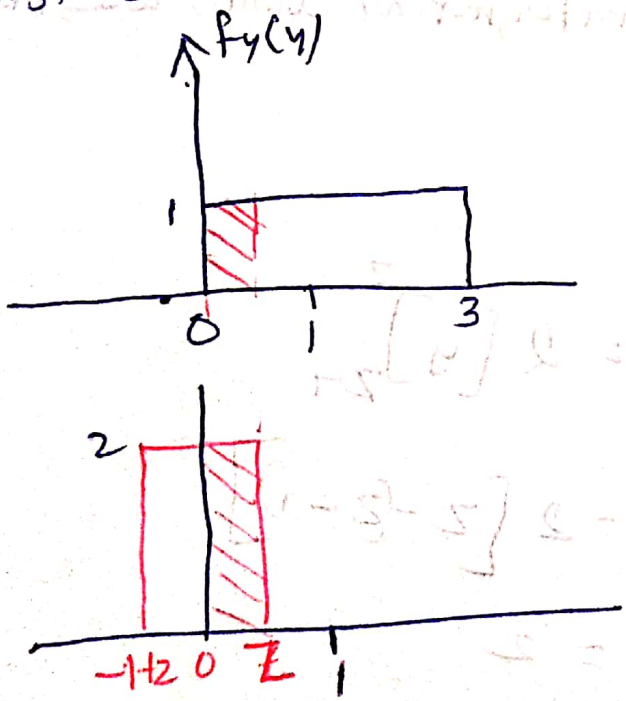
It means $z \leq 0$ produces ~~the~~ no overlapping between the signal $f_x(z-y)$ and $f_y(y)$

Thus, the convolution integral is equal to zero.

when $t > 0$

start shifting the signal $f_2(z-x)$ to the right.

first consider $0 \leq z \leq 1$



$$f_z(z) = \int_0^z 1 \times 2 \, dy = 2z \quad 0 \leq z \leq 1$$

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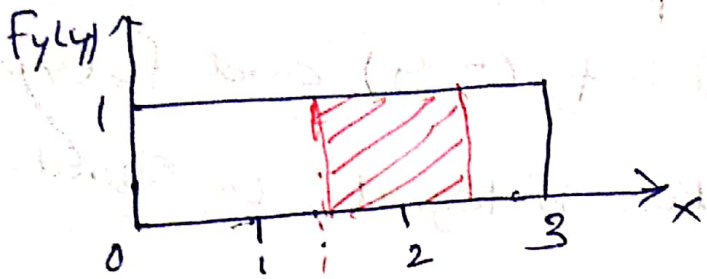
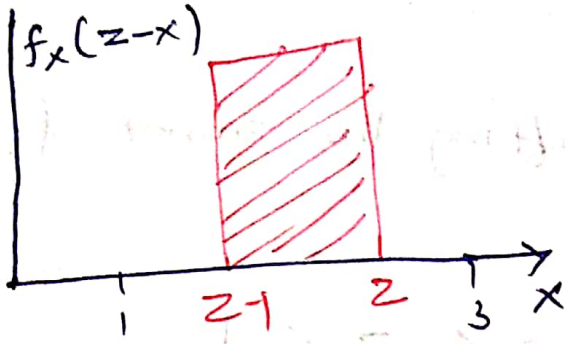
when interval $1 \le z \le 3$





the actual convolution integration limits are from $z-1$ to z

$$f_Z(z) = \int_{z-1}^{z} 2\,dy = 2\left[y\right]_{z-1}^{z}$$

$$= 2\left[z-(z-1)\right]$$

$$f_Z(z) = 2$$

for $3 \le z \le 4$

$$f_Z(z) = \int_{z-1}^{3} 2\,dy = 2\left[y\right]_{z-1}^{3}$$

$$= 2(3-z+1)$$

$$= 2(4-z) = 8-2z$$

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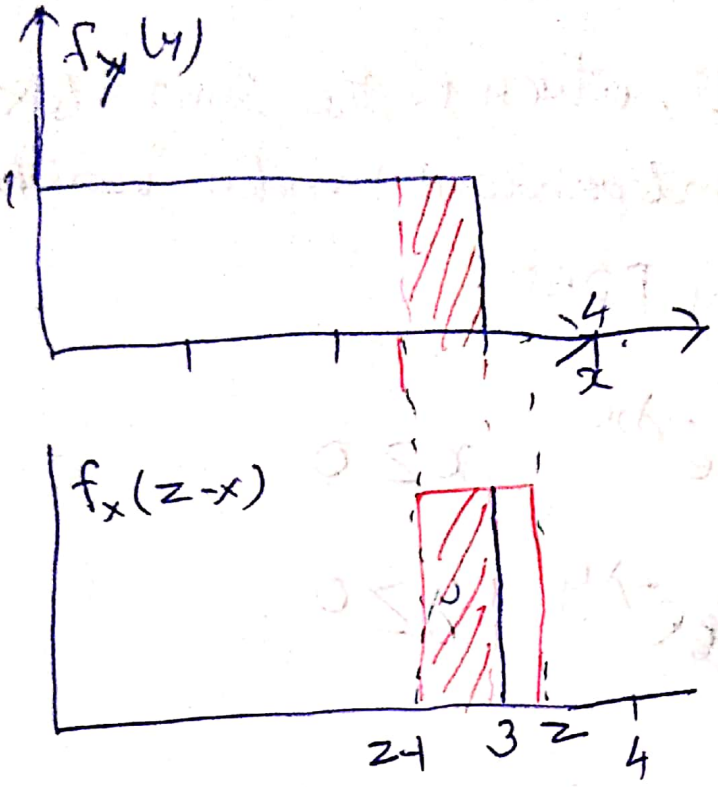
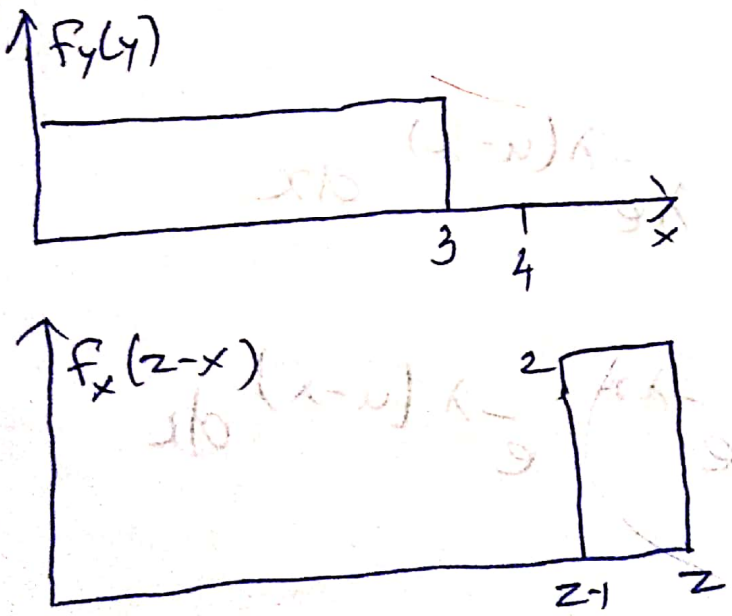


$z > 4$



$$f_z(z) = 0 \quad z > 4$$

$$f_z(z) = \begin{cases} 0 & z \le 0 \\ 2t & 0 \le z \le 1 \\ 2 & 1 \le z \le 3 \\ 8-2t & 3 \le z \le 4 \\ 0 & z \ge 4 \end{cases}$$

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Find the PDF of U, which is the sum of X and Y that are independent random variable with the following PDFs.

$$f_X(x) = \lambda e^{-\lambda x} \qquad x \geq 0$$

$$f_Y(y) = \lambda e^{-\lambda y} \qquad y \geq 0.$$

$$f_U(u) = \int_{-\infty}^{\infty} f_X(x) f_Y(u-x)\, dx.$$

$$= \int_{-\infty}^{\infty} \lambda e^{-\lambda x} \lambda e^{-\lambda(u-x)}\, dx.$$

$$= \lambda^2 \int_{-\infty}^{\infty} e^{-\lambda x} e^{-\lambda(u-x)}\, dx$$

$$= \lambda^2 \int_{-\infty}^{\infty} \cancel{e^{-\lambda x}} e^{-\lambda u} \cancel{e^{\lambda x}}\, dx$$

$$= \lambda^2 e^{-\lambda u} \int_{0}^{u} dx = \lambda^2 e^{-\lambda u} u$$

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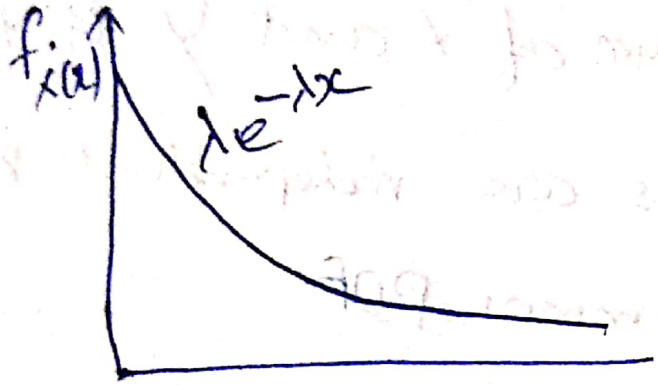
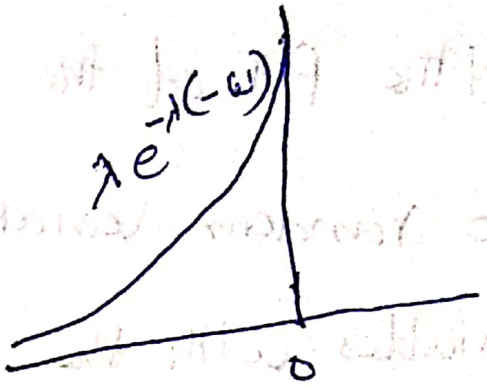
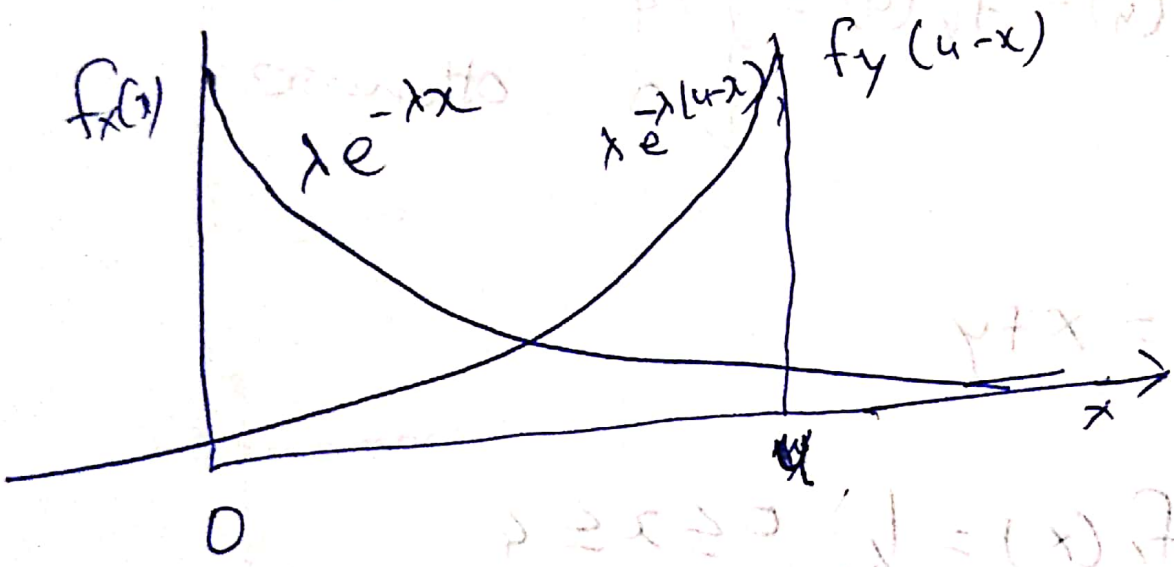
when $t = 0$







$$f_U(u) = \lambda^2 e^{-\lambda u} u \quad u \geq 0$$

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Find the PDF of the sum of x and y if the two random variables are independent random variables with the common PDF

$$f_X(u) = f_Y(u) = \begin{cases} 1/4 & 0 \le u \le 4 \\ 0 & \text{otherwise} \end{cases}$$

$S = X + Y$

$f_X(x) = 1/4 \quad 0 \le x \le 4$

$f_Y(y) = 1/4 \quad 0 \le y \le 4$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-x)\, f_Y(y)\, dy$$

$=$

lower limit $0 + 0 = 0$

upper limit $4 + 4 = 8$

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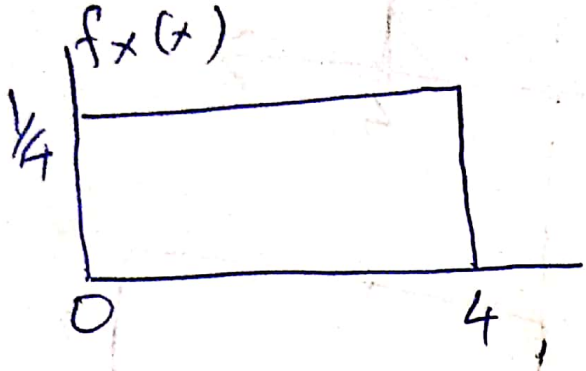
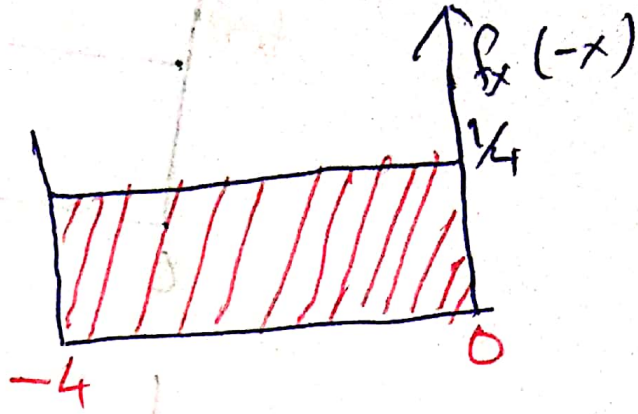
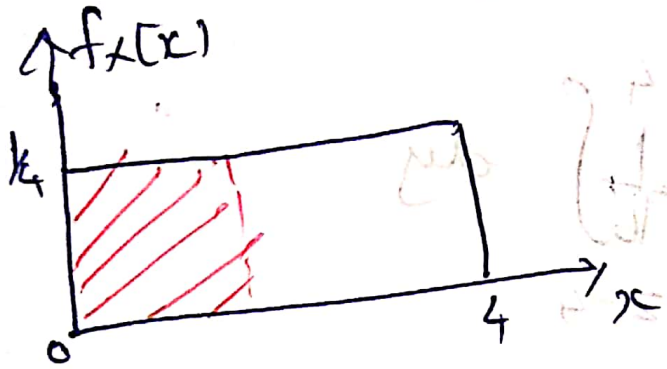
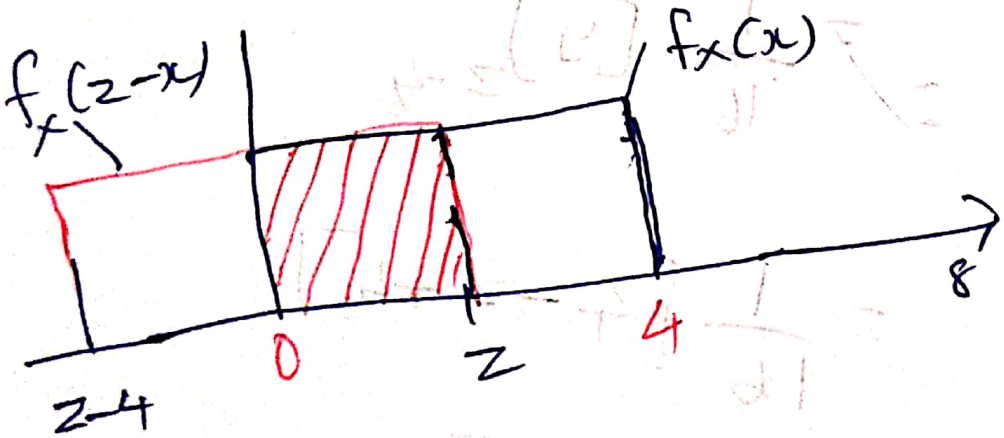
when $z = 0$





when interval $0 < z < 4$





$$f_Z(z) = \frac{1}{16}\int_0^z dy = \frac{z}{16}$$

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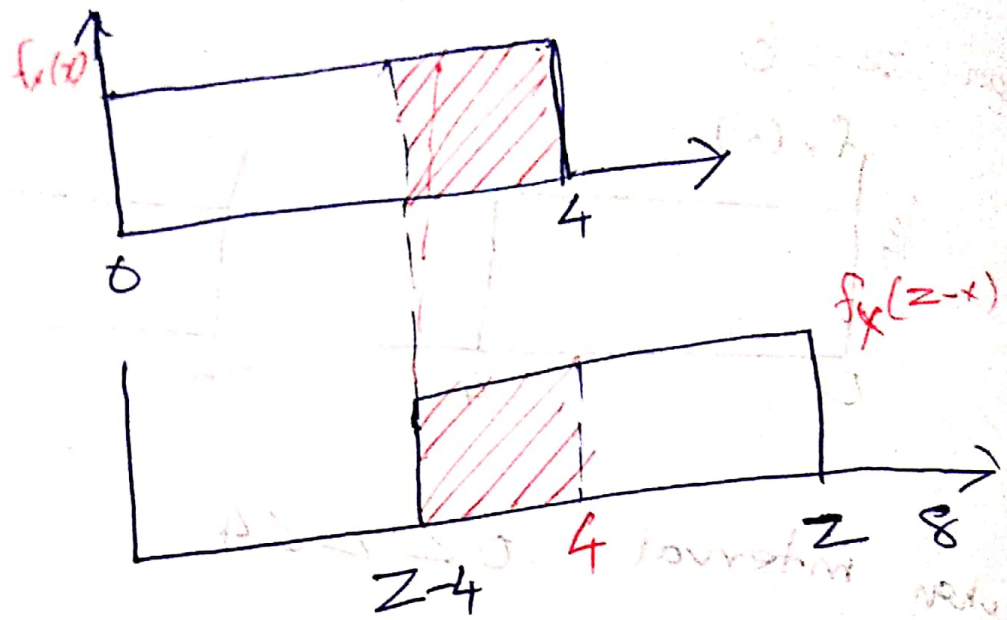
when interval between $4 < z < 8$



$$f_Z(z) = \frac{1}{16}\int_{z-4}^{4} dy$$

$$= \frac{1}{16}\left[y\right]_{z-4}^{4}$$

$$= \frac{1}{16}\left[4 - z + 4\right]$$

$$= \frac{1}{16}\left[8 - z\right]$$

$$f_Z(z) = \begin{cases} \frac{z}{16} & 0 \leq z \leq 4 \\ \frac{8-z}{16} & 4 < z < 8 \\ 0 & \text{otherwise.} \end{cases}$$

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Joint Moment about the Orgin.

If $g(x,y)$ is some function of two random variables X and Y the expected value of $g(x,y)$ is given by.

$$E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

Joint moments m_{nk} are defined by

$$m_{nk} = E[X^n Y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k\, f_{X,Y}(x,y)\, dx\, dy.$$

here

$m_{n0} = E[X^n]$ are the moments m_n of X

$m_{0k} = E[Y^k]$ are the moments of Y.

The sum $n+k$ is called Order of the moments.

here m_{02}, m_{20} and m_{11} are all second order moments of X and Y.

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The first order moments $m_{01} = E[Y] = \bar{Y}$ and $m_{10} = E[X] = \bar{X}$ are the expected values of Y & X.

The second order moment $m_{11} = E[XY]$ is called the correlation of X and Y.

$$R_{XY} = m_{11} = E[XY] = \int_{-\infty}^{\infty} xy \, f_{XY}(x,y) \, dx \, dy$$

If correlation can be written in the form

$$R_{XY} = E(X)E(Y)$$

Then X and Y are said to be uncorrelated.

Statistical independence of X and Y is sufficient to guarantee they are uncorrelated.

If $R_{XY} = 0$

for two random variable X & Y then It is called as orthogonal.

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Joint Central Moments.

For two random variables X and Y

the joint central moment

$$\mu_{nk} = E[(X-\bar{X})^n (Y-\bar{Y})^k]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{x})^n (y-\bar{y})^k f_{X,Y}(x,y)\, dx\, dy$$

The second order central moments

$$\mu_{20} = E[(X-\bar{X})^2] = \sigma_X^2$$

$$\mu_{02} = E[(Y-\bar{Y})^2] = \sigma_Y^2$$

are the variances of X and Y

The second order moment μ_{11} is called as covariance of X and Y

$$C_{XY} = \mu_{11} = E[(X-\bar{X})(Y-\bar{Y})]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{x})(y-\bar{y}) f_{X,Y}(x,y)\, dx\, dy$$

$$C_{XY} = R_{XY} - E[X]E[Y]$$

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If x and y are either independent or un correlated, then

Covariance is zero.

$$C_{xy} = 0$$

If x and y are orthogonal random variable, then

$$C_{xy} = -E[x]\,E[y] \quad x \text{ and } y \text{ orthogonal}$$

The normalized second order moment

$$\rho = \frac{\mu_{11}}{\sqrt{\mu_{20}\mu_{02}}} = \frac{C_{x,y}}{\sigma_x \sigma_y}$$

$$\rho = E\left[\frac{(x-\bar{x})}{\sigma_x}\,\frac{(y-\bar{y})}{\sigma_y}\right]$$

ρ is the correlation coefficient of x and y

$$-1 \le \rho \le 1$$

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Joint Characteristic functions.

The joint Characteristic function of two random variables X and Y is

$$\phi_{X,Y}(\omega_1, \omega_2) = E\left[e^{j\omega_1 X + j\omega_2 Y}\right]$$

where ω_1 and ω_2 are real numbers and an equivalent form is

$$\phi_{X,Y}(\omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, e^{j\omega_1 x} e^{j\omega_2 y}\, dx\, dy$$

This is two dimensional Fourier transform of the joint density function.

From the inverse Fourier transform

$$f_{X,Y}(x,y) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} \phi_{X,Y}(\omega_1, \omega_2)\, e^{-j\omega_1 x - j\omega_2 y}\, d\omega_1 d\omega_2$$

By setting either $\omega_2 = 0$ or $\omega_1 = 0$, the characteristics function of X or Y are obtained, called marginal characteristics functions.

$$|\phi_{XY}(\omega_1, \omega_2)| \leq \phi_{XY}(0,0) = 1$$

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$$\Phi_x(\omega_1) = \Phi_{x,y}(\omega_1, 0)$$

$$\Phi_y(\omega_2) = \Phi_{x,y}(0, \omega_2)$$

and Joint moments m_{nk} can be found from the joint characteristic function as

$$m_{nk} = (-j)^{n+k} \left. \frac{\partial^{n+k} \Phi_{x,y}(\omega_1, \omega_2)}{\partial \omega_1^n \, \partial \omega_2^k} \right|_{\substack{\omega_1 = 0 \\ \omega_2 = 0}}$$

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If X and Y are independent RVs

$$\phi_{XY}(\omega_1, \omega_2) = E\left(e^{j(\omega_1 X)}\right) E\left(e^{j(\omega_2 Y)}\right)$$

$$= \phi_X(\omega_1)\, \phi_Y(\omega_2)$$

$$\phi_X(\omega) = \phi_{XY}(\omega, 0)$$

$$\phi_Y(\omega) = \phi_{XY}(0, \omega)$$

Independence.

If RV X and Y are independent

then

$$E\left[e^{j(\omega_1 X + \omega_2 Y)}\right] = E\left[e^{j\omega_1 X}\right] \cdot E\left[e^{j\omega_2 Y}\right]$$

$$\phi(\omega_1, \omega_2) = \phi_X(\omega_1)\, \phi_Y(\omega_2)$$

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① The density function of two random variable X and Y is

$$f_{X,Y}(x,y) = u(x)\,u(y)\,16\,e^{-4(x+y)}$$

Find the mean value of the function

$$g(x,y) = e^{-2(x^2+y^2)}$$

Note $F(2) = 0.9773$

$$E\left(e^{-2(x^2+y^2)}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2(x^2+y^2)}\, u(x)\,u(y)\, 16\, e^{-4(x+y)}$$

$$= 16\int_0^{\infty}\int_0^{\infty} e^{-2x^2}\, e^{-2y^2}\, e^{-4x}\, e^{-4y}\, dy\, dx$$

$$= 16\int_0^{\infty} e^{-2x^2-4x}\, dx \int_0^{\infty} e^{-2y^2-4y}\, dy$$

$$= 16\int_0^{\infty} e^{-2(x^2+2x+1)}\, dx\; e^{2}\int_0^{\infty} e^{-2(y^2-2y+1)}\, e^{2}\, dy$$

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$$= 16 \times e^{4} \int_{0}^{\infty} e^{-2(x+1)^2} dx \int_{0}^{\infty} e^{-2(y+1)^2} dy$$

$m = 2(x+1)$ $\qquad x+1 = m/2$

$dm = 2dx.$

Put $x = 0 \rightarrow m = 2$

$x = \infty \rightarrow m = \infty$

$$= 16 e^{4} \int_{2}^{\infty} e^{-2(2x)}$$

$$\cancel{= 16 e^{4} \int_{2}^{\infty} e^{-2 m^2/4^2} \frac{dm}{2} \int_{2}^{\infty} e^{-l^2/2} \frac{dl}{2}}$$

$$= 16 e^{4} \left[\int_{2}^{\infty} e^{-m^2/2} dm \times \frac{1}{2} \right]^2$$

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The normalized distribution function for

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} \, d\xi$$

A function closely related to $F(x)$ is the Q-function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\xi^2/2} \, d\xi$$

For negative values

$$Q(-x) = 1 - Q(x)$$

So

$$F(x) = 1 - Q(x)$$

$$= \frac{16\pi e^4}{2} \left[\int_{2}^{\infty} \frac{e^{-m^2/2}}{\sqrt{2\pi}} \, dm \right]^2$$

$$= 8\pi e^4 \left[1 - F(2)\right]^2 = 8\pi e^4 \left[1 - 0.9772\right]^2 \simeq \underline{\underline{0.713}}$$

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(2) Find ~~the~~ all the second-order moment and central moments for the density function

$$f_{X,Y}(x,y) = u(x)\,u(y)\,16\,e^{-4(x+y)}$$

Second order moment

$$m_{20} = \int_0^\infty \int_0^\infty x^2\, 16\, e^{-4x}\, e^{-4y}\, dy\, dx$$

$$= 16 \int_0^\infty x^2 e^{-4x}\, dx \int_0^\infty e^{-4y}\, dy$$

$$\boxed{\begin{aligned} \int e^{ax}\, dx &= \frac{e^{ax}}{a} \\ \int x\, e^{ax}\, dx &= \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax} \\ \int x^2 e^{ax}\, dx &= \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right) \end{aligned}}$$

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$$= 16\left[\left(\frac{x^2}{-4} - \frac{2x}{16} - \frac{2}{64}\right)e^{-4x}\right]_0^\infty \times \left[\frac{e^{-4y}}{-4}\right]_0^\infty$$

$$= \overset{4}{\cancel{16}}\left[\frac{2}{64}\right]\left[\frac{1}{\cancel{4}}\right]$$

$$= 1/8$$

$$m_{02} = 1/8$$

$$m_{11} = \int_0^\infty\int_0^\infty xy\,16e^{-4x}e^{-4y}\,dy\,dx.$$

$$= 16\int_0^\infty x\,e^{-4x}\,dx \int_0^\infty y\,e^{-4y}\,dy$$

$$= 16\left[\frac{x}{-4} - \frac{1}{16}e^{-4x}\right]_0^\infty \left[\frac{y}{-4} - \frac{1}{16}e^{-4y}\right]_0^\infty$$

$$= 16 \times \frac{1}{16} \times \frac{1}{16}$$

$$= 1/16.$$

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For Central moment

$$m_{01} = \bar{x} = \int_{-\infty}^{\infty}\int_{0}^{\infty} x\, 16\, e^{-4x}\, e^{-4y}\, dx\, dy$$

$$= 16 \left[\left(\frac{x}{-4} - \frac{1}{16}\right)e^{-4x}\right]_0^{\infty} \left[\frac{e^{-4y}}{-4}\right]_0^{\infty}$$

$$= 16 \times \frac{1}{16} \times \frac{1}{4} = 1/4$$

$$m_{10} = \bar{y} = 1/4$$

$$\mu_{20} = \sigma_x^2 = m_{20} - (m_{10})^2 = \frac{1}{8} - 1/16 = 1/16$$

$$\mu_{02} = \sigma_y^2 = m_{02} - (m_{01})^2 = \frac{1}{8} - \frac{1}{16} = 1/16$$

$$\mu_{11} = R_{xy} - \bar{x}\bar{y}$$

$$= m_{11} - m_{10}m_{01}$$

$$= \frac{1}{16} - \frac{1}{16} = 0$$

$\mu_{11} \rightarrow$ covariance

$\mu_{11} = 0$ means x & y are uncorrelated.

x & y are independent.

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(2) For two random variables X and Y

$$f_{X,Y}(x,y) = 0.15\,\delta(x+1)\,\delta(y) + 0.18(x)\,\delta(y)$$
$$+ 0.18(x)\,\delta(y-2) + 0.4\,\delta(x-1)\,\delta(y+2)$$
$$+ 0.2\,\delta(x-1)\,\delta(y-1) + 0.05\,\delta(x-1)\,\delta(y-3)$$

(a) correlation

(b) covariance

(c) the correlation co-efficient of X & Y

(d) are X and Y either uncorrelated or orthogonal.

$$K_{XY}(x,y) = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dx\,dy$$

$$= 0.15 \times -1 \times 0 + 0.1 \times 0 \times 0 + 0.1 \times (0) \times (2)$$
$$+ 0.4\,(1)(-2) + 0.2(1)(1) + 0.05(1)(3)$$

$$= -0.8 + 0.2 + 0.15$$

$$= -0.45$$

$$m_{10} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^1 y^0\, f_{X,Y}(x,y)\,dx\,dy$$

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$$\bar{X} = 0.15(-1) + 0.1(0) + 0.1(0) + 0.4(1) + 0.2(1) + 0.05(1)$$

$$= 0.5$$

$$\bar{Y} = 0.15 \times (0) + 0.1(0) + 0.1(2) + 0.4(-2) + 0.2(1) + 0.05(3)$$

$$= -0.25$$

$$C_{xy} = E(xy) - \bar{x}\bar{y}$$

$$= -0.45 - (0.5)(-0.25)$$

$$= -0.45 - (-0.125)$$

$$= -0.325$$

$$P_{xy} = C_{xy} / \sigma_x \sigma_y$$

$$\overline{X^2} = 0.15(-1)^2 + 0.1 \times 0^2 + 0.1(0)^2 + 0.4(1)^2 + 0.2(1)^2 + 0.05(1)^2$$

$$= 0.8$$

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$$\overline{y^2} = 0.15(0)^2 + 0.1(0)^2 + 0.1(2)^2 + 0.4(-2)^2 + 0.2(1)^2 + 0.05(3)^2$$

$$= 2.65.$$

$$\sigma_x^2 = \overline{x^2} - (\bar{x})^2 = 0.8 - (0.5)^2 = 0.55$$

$$\sigma_y^2 = \overline{y^2} - (\bar{y})^2 = 2.65 - (-0.25)^2$$

$$= 2.5875$$

$$\rho_{xy} = \frac{-0.325}{\sqrt{(0.55)(2.5875)}} \simeq -0.272$$

(d) $C_{xy} \neq 0$ x & y are not uncorrelated

$R_{xy} \neq 0$ x & y " " orthogonal.

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(3) Two random variables X and Y are defined by $\overline{X} = 0$, $\overline{Y} = -1$, $\overline{X^2} = 2$, $\overline{Y^2} = 4$ and $R_{XY} = -2$. Two new random variables W and U are

$$W = 2X + Y$$

$$U = -X - 3Y$$

Find $\overline{W}$, $\overline{U}$, $\overline{W^2}$, $\overline{U^2}$ R_{WU}, σ_X^2, σ_Y^2

$$\overline{W} = \overline{2X+Y} = 2\overline{X} + \overline{Y}$$

$$= 2 \times 0 + (-1) = -1$$

$$\overline{V} = \overline{-X-3Y} = -\overline{X} - 3\overline{Y}$$

$$= 0 - 3 \times -1 = 3$$

$$\overline{W^2} = \overline{(2X+Y)^2} = 4\overline{X^2} + 4\overline{XY} + \overline{Y^2}$$

$$= 4 \times 2 + 4 \times -2 + 4$$

$$= 8 - 8 + 4 = 4$$

$$U^2 = \overline{(-X-3Y)^2} = \overline{X^2} + 6\overline{XY} + 9\overline{Y^2}$$

$$= 2 + 6 \times -2 + 4 \times 9$$

$$= 2 - 12 + 36 = 26$$

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$$R_{wu} = \overline{wu}$$

$$= \overline{(2x+y)(-x-3y)}$$

$$= -2\overline{x^2} - 7\overline{xy} - 3\overline{y^2}$$

$$= -2\times 2 - 7\times -2 - 3\times 4$$

$$= -2$$

$$\sigma_x^2 = \overline{x^2} - (\overline{x})^2$$

$$= 2$$

$$\sigma_y^2 = \overline{y^2} - (\overline{y})^2$$

$$= 4 - 1 = 3$$

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Random variables X_1 and X_2 have the joint characteristic function

$$\phi_{X_1,X_2}(\omega_1,\omega_2) = \left[(1-j2\omega_1)(1-j2\omega_2)\right]^{-N/2}$$

where $N>0$ is an integer.

(a) Find the correlation and moment m_{20} m_{02}

(b) Determine means of X_1 & X_2.

(c) What is the correlation co-efficient.

$$\frac{\partial\phi}{\partial\omega_1} = \frac{1}{(1-j2\omega_2)^{N/2}} \cdot \frac{-N/2 \times -2j}{(1-j2\omega_1)^{N/2+1}}$$

$$\frac{\partial\phi}{\partial\omega_2} = \frac{1}{(1-j2\omega_1)^{N/2}} \times \frac{jN}{(1-j2\omega_2)^{N/2+1}}$$

$$\frac{\partial^2\phi}{\partial\omega_1} = \frac{1}{(1-j2\omega_2)^{N/2}} \times \frac{jN \times (-2j) \times \frac{(N+2)}{2}}{(1-j2\omega_1)^{N/2+2}}$$

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$$\frac{\partial^2 \phi}{\partial \omega_1^2} = \frac{1}{(1-j2\omega_2)^{N/2}} \times \frac{-N \times (N+2)}{(1-j2\omega_1)^{N/2+2}}$$

$$\frac{\partial^2 \phi}{\partial \omega_2} = \frac{1}{(1-j2\omega_1)^{N/2}} \left[\frac{-N(N+2)}{(1-j2\omega_2)^{N/2+2}} \right]$$

$$m_{20} = (-j)^2 \left. \frac{\partial^2 \phi}{\partial \omega_1^2} \right|_{\substack{\omega_1=0 \\ \omega_2=0}}$$

$$= N(N+2)$$

$$m_{02} = N(N+2)$$

$$R_{x_1 x_2} = m_{11} = (-j)^2 \left. \frac{\partial \phi}{\partial \omega_1} \frac{\partial \phi}{\partial \omega_2} \right|_{\substack{\omega_1=0 \\ \omega_2=0}}$$

$$= \left[\frac{-jN}{(1-j2\omega_1)^{N/2}} \times \frac{j}{(1-j2\omega_2)^{N/2+1}} \right] \left[\frac{jN}{(1-j2\omega_2)^{N/2} (1-j2\omega_1)^{N/2+1}} \right]$$

$$= N^2$$

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$$m_{10} = -j \left.\frac{\partial f}{\partial \omega_1}\right|_{\substack{\omega_1=0 \\ \omega_2=0}} = N$$

$$m_{01} = N$$

$$\rho = \frac{C_{x_1 x_2}}{\sigma_{x_1} \sigma_{x_2}}$$

$$= \frac{R_{x_1 x_2} - \bar{x}_1 \bar{x}_2}{\sigma_x \sigma_y}$$

$$= \frac{m_{11} - m_{10} m_{01}}{\sqrt{m_{20} m_{02}}} = \frac{N^2 - N^2}{\sqrt{4N^2}} = 0$$

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Linear transformation of Gaussian Random variables.

A set of random variables $x_1, x_2 \dots x_N$ for which the joint density function of

$$f_{x_1 \dots x_n}(x_1 \dots x_n) = \frac{|[C_x]^{-1}|^{1/2}}{(2\pi)^{N/2}} \exp\left\{ \frac{-[x-\bar{x}]^t [C_x]^{-1} [x-\bar{x}]}{2} \right\}$$

$$[x-\bar{x}] = \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \\ \vdots \\ x_N \quad \bar{x}_N \end{bmatrix}$$

$$C_x = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix}$$

$$C_{ij} = E\left[(x_i - \bar{x}_i)(x_j - \bar{x}_j)\right] = \begin{cases} \sigma_x^2 & i = j \\ \sigma_{x_i x_j} & i \neq j \end{cases}$$

$$[C_x] = \begin{bmatrix} \sigma_{x_1}^2 & \rho\sigma_{x_1}\sigma_{x_2} \\ \rho\sigma_{x_1}\sigma_{x_2} & \sigma_{x_2}^2 \end{bmatrix}$$

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$$[C_x]^{-1} = \frac{1}{(1-\rho^2)\sigma_{x_1}^2\sigma_{x_2}^2}\begin{bmatrix} \sigma_{x_2}^2 & -\rho\sigma_{x_1}\sigma_{x_2} \\ -\rho\sigma_{x_1}\sigma_{x_2} & \sigma_{x_1}^2 \end{bmatrix}$$

where $|[C_x]^{-1}| = \dfrac{1}{(1-\rho^2)\sigma_{x_1}^2\sigma_{x_2}^2}$

The new variables $Y_1, Y_2, \ldots Y_N$ are,

$$Y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N$$

$$Y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N$$

$$\vdots$$

$$Y_N = a_{N1}x_1 + a_{N2}x_2 + \cdots + a_{NN}x_N$$

where the co-efficient a_{ij}, i and $j = 1, 2 \ldots N$ are real numbers

$$T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & & \vdots \\ a_{N1} & \cdots a_{N2} & \cdots & a_{NN} \end{bmatrix}$$

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$$[Y] = [T][X]$$

$$[X] = [T]^{-1}[Y]$$

where T is non singular matrix.

The $[Y - \bar{Y}] = [T][X - \bar{X}]$

$$[X - \bar{X}] = [T]^{-1}[Y - \bar{Y}]$$

Jointly gaussian of the new random variable $Y_1, Y_2 \ldots Y_N$

$$f_{Y_1 \ldots Y_N}(y_1 \ldots y_N) = \frac{\left|\,|C_Y|^{-1}\right|^{1/2}}{(2\pi)^{N/2}} \exp\left\{-\frac{(Y-\bar{Y})^T [C_Y]^{-1} [Y-\bar{Y}]}{2}\right\}$$

The new variables have means values

$$\bar{Y}_j = \sum_{k=1}^{N} a_{jk} \bar{X}_k$$

The element of the covariance matrix

$$[C_Y] = [T][C_X][T]^t.$$

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(1) Two gaussian random variables x_1 and x_2 have zero means and variance $\sigma_{x_1}^2 = 4$ and $\sigma_{x_2}^2 = 9$. Their co-variance C_{x_1, x_2} equals 3. If x_1 and x_2 are linearly transformed to new variable Y_1 and Y_2 according to

$$Y_1 = x_1 - 2x_2$$

$$Y_2 = 3x_1 + 4x_2$$

To find means, variances and covariances of Y_1 and Y_2

$$T = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \qquad [C_x] = \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix}$$

Since x_1 and x_2 are zero mean and gaussian, Y_1 & Y_2 will also be zero mean & gaussian. $\overline{Y_1} = 0$, $\overline{Y_2} = 0$

$$[C_y] = [T][C_x][T]^t$$

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$$= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4-6 & 3-18 \\ 12+12 & 9+36 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -15 \\ 24 & 45 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2+30 & -6-60 \\ 24-90 & 72+180 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & -66 \\ -66 & 252 \end{bmatrix}$$

$\sigma_{y_1}^2 = 28$ $\quad \sigma_{y_2}^2 = 252$ $\quad C_{y_1 y_2} = -66$

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Zero-mean gaussian random x_1, x_2 and x_3 having a covariance matrix

$$[C_x] = \begin{bmatrix} 4 & 2.05 & 1.05 \\ 2.05 & 4 & 2.05 \\ 1.05 & 2.05 & 4 \end{bmatrix}$$

are transformed to new variable

$$y_1 = 5x_1 + 2x_2 - x_3$$

$$y_2 = -x_1 + 3x_2 + x$$

$$y_3 = 2x_1 - x_2 + 2x_3$$

Find the co-variance of y_1, y_2 and y_3.

$$T = \begin{bmatrix} 5 & 2 & -1 \\ -1 & 3 & 1 \\ 2 & -1 & 2 \end{bmatrix}$$

$$[C_y] = [T][C_x][T]^t$$

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$$= \begin{bmatrix} 5 & 2 & -1 \\ -1 & 3 & 1 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 2.05 & 1.05 \\ 2.05 & 4 & 2.05 \\ 1.05 & 2.05 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 & 2 \\ 2 & 3 & -1 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 20+4.1-1.05 & 10.25+8-2.5 & 5.25+4.10-4 \\ -4+6.15+1.05 & -2.05+12+2.05 & -1.05+6.15+4 \\ 8-2.05+2.1 & 4.1-4+4.1 & 2.1-2.05+8 \end{bmatrix}$$

$$= \begin{bmatrix} 23.05 & 16.2 & 5.35 \\ 3.2 & 12 & 9.1 \\ 8.05 & 4.2 & 8.05 \end{bmatrix} \begin{bmatrix} 5 & -1 & 2 \\ 2 & 3 & -1 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 142.3 & 30.9 & 40.6 \\ 30.9 & 41.9 & 12.6 \\ 40.6 & 12.6 & 28.0 \end{bmatrix}$$

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