Probability

1. Theoretical

   Estimation & Detection

   P(getting 5 while throwing a dice) = 1/6

2. Empirical

   Geological records

   P(getting heavy rain in next 2 years)

3. Subjective

   Experience and Intution.

Basic Definitions

Set and Elements/outcome.

Eg : $A = \{a, b, c, d\}$

$B = \{\underbrace{e, f, g, h}_{\text{elements of set B}}\}$

$a \in A$

$a \notin B$.

SET-A

1. $A = \{a, b, c, d\}$
2. $A = \{$ first 4 letters of alphabet $\}$.

• Sample Space (Ω) = A set that has all possible outcomes

$\Omega = \{H, T\}$.

- Factorial : Determine the various ways to represent.

eg: RBG 3! ways.

Ω : {RBG, RGB, BRG, BGR, GRB, GBR}.

$\boxed{n! \text{ ways}}$

- Event : Subset of a sample space.

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$A = \{2, 4, 6\}$ → even numbers.

$[A \subset \Omega]$

Binomial

Coin : $\Omega - \{H, T\}$

2^1 $A = \{H\}$
$B = \{T\}$

2 Coins: $\Omega = \{HH, HT, TH, TT\}$ 2^2

3 Coins : $\Omega = \{\underbrace{HHH}_{A:\ 3H}, \underbrace{HHT, HTH, THH}_{B:\ 2H}, \underbrace{HTT, THT, TTH}_{C:\ 2T}, \underbrace{TTT}_{D:\ 3T}\}$

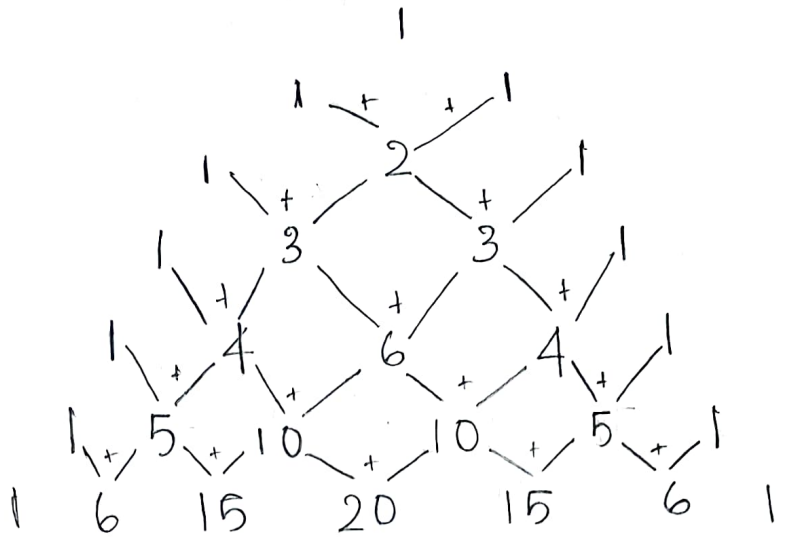
2 Coin : $(a+b)^2 = a^2 + 2ab + b^2$
3 Coin : $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
4 Coin : $(a+b)^4 = a^4 + 4a^3b + 6a^2b + 4ab^3 + b^4$

$(a+b)^n$ Pascal Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

6 times:

1	6	15	20	
6H	5H+1T	4H, 2T	3H3T	
15	6	1		
2H4T	1H5T	6T.		



6 times : $2^6 = 64$ outcomes.

$\Omega = \{2^n \text{ outcomes}\}$.

$k = \#$ of heads $\quad n-k = \#$ of tails

$$P(k \text{ heads}, n-k \text{ tails}) = \frac{\binom{n}{k}}{2^n}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Atleast one

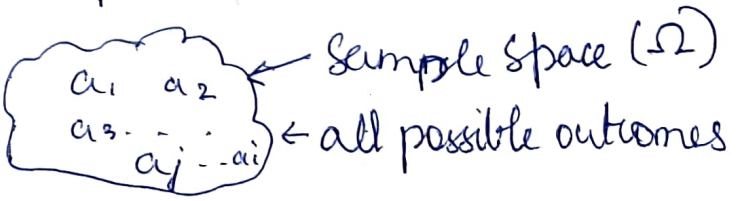
$P(\text{getting atleast 1 head}) = 1 - (P(\text{no head}))^n$

Conditional Probability

$$P(A_2 | A_1) = \frac{P(A_2 \cap A_1)}{P(A_1)}$$

Single Random Variable

When we do an experiment



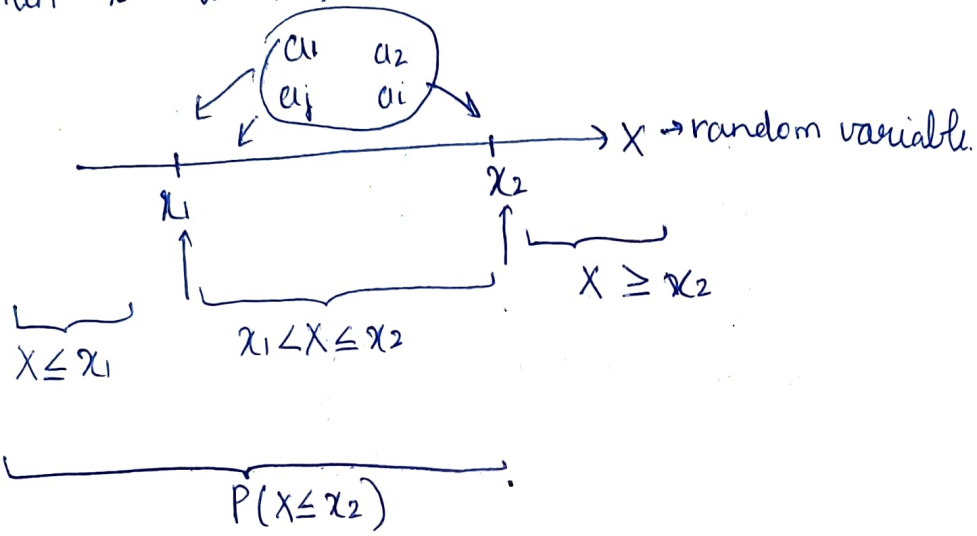
$$P(A_x) = \underbrace{P(X(a) \le x)}_{\text{variable}} = P\{x_i, x_j\}$$

value/outcome

Cummulative distribution function (CDF)

$CDF \Rightarrow F_x(x) \ge 0.$

When $x = -\infty \Rightarrow F_x(-\infty) = 0$
When $x = +\infty \Rightarrow F_x(+\infty) = 1$ } Property.



$$(X \le x_2) = (X \le x_1) \cup (x_1 < X \le x_2)$$
$$P(X \le x_2) = P(X \le x_1) + P(x_1 < X \le x_2)$$
$$F_x(x_2) = F_x(x_1) + P(x_1 < X \le x_2)$$

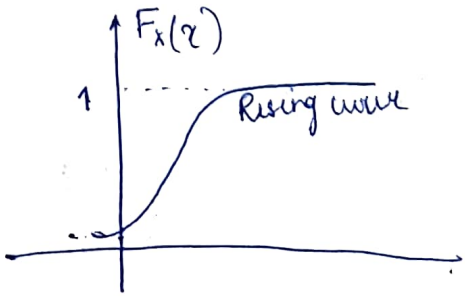
$\cup \rightarrow +$
$\cap \rightarrow \times$

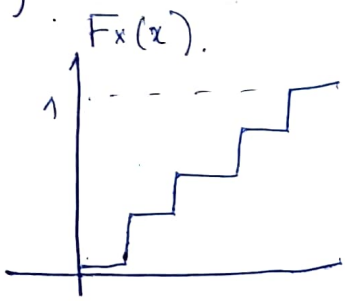
When $x_2 > x_1$

* $F_x(x_2) \ge F_x(x_1)$

$P(x_1 < X \le x_2) = F_x(x_2) - F_x(x_1) \ge 0.$

<u>Note</u> : $F_x(x) \ge 0$, (0 to 1)





Probability Density Function (PDF) $f_X(x)$

$$\boxed{\begin{aligned} &CDF \Rightarrow P(X \le x) = F_X(x) \\ &PDF \Rightarrow f_X(x) = \frac{dF_X(x)}{dx} \end{aligned}}$$

$$PDF = \frac{d}{dx}(CDF)$$

$$CDF = \int PDF \, dx$$

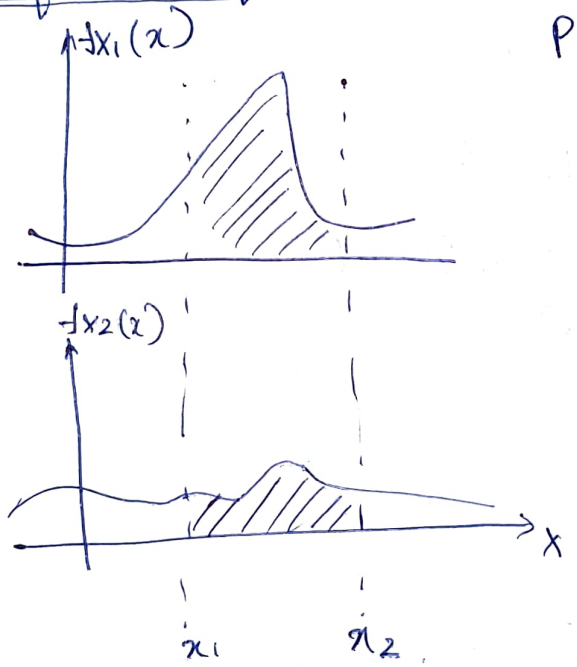
$$\boxed{f_X(x) \ge 0}$$

area under curve = 1

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

Significance of PDF

$P(x_1 < X \le x_2)$



$$F_X(x) = P(X \le x)$$

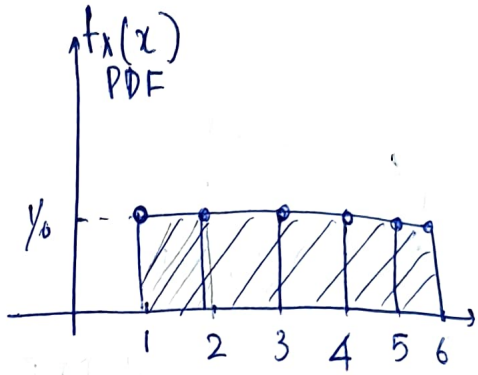
$$F_X(x|B) = P(X \le x|B)$$

$$\frac{dF_X(x|B)}{dx} = f_X(x|B)$$

Continuous RV	Discrete RV
PDF $f_X(x)$	PMF
CDF $F_X(x)$	CDF

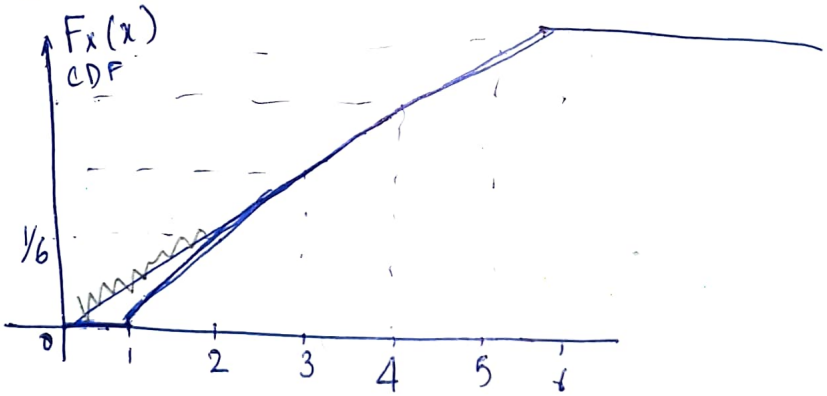
Ex: Rolling a die for one time

X	P(X)
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6
0	0
7	0



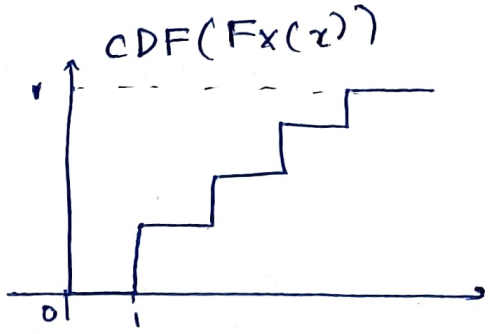
Area = $1/6 \times 6 = 1$.

Area b/w 1 and 2 = 1/6.

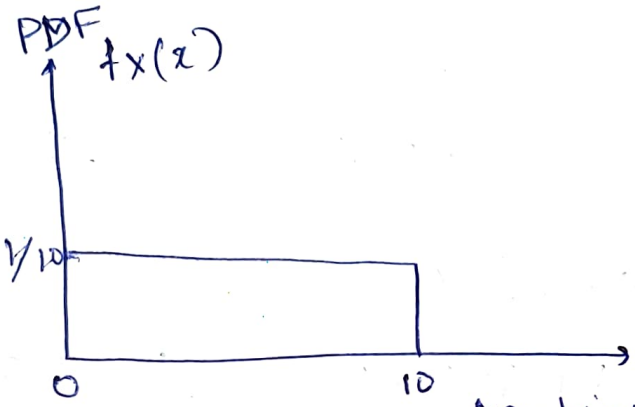


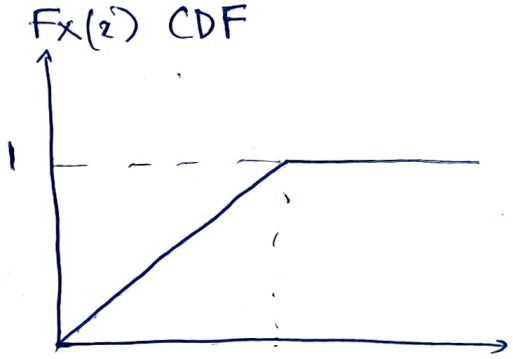
$$P(X) = \begin{cases} 0 & X < 1 \\ 1/6 & 1 \le X \le 6. \\ 0 & X > 6 \end{cases}$$

$\sum P(x) = 1$



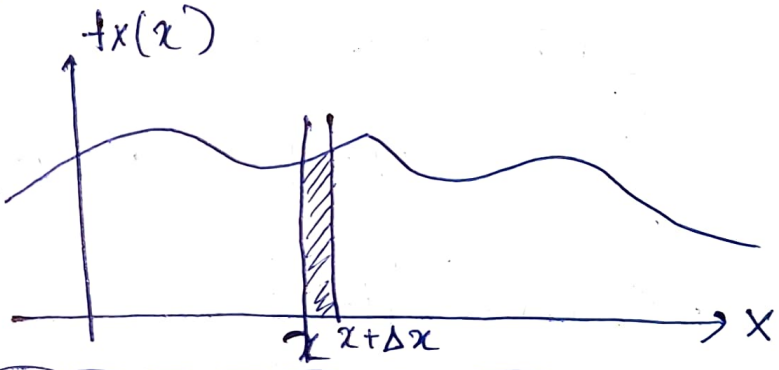
Discrete





continuous.

~~P(x <~~



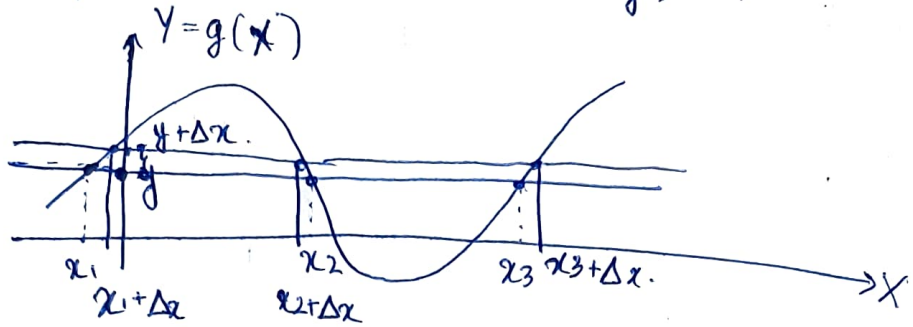
$P(x < X \leq x+\Delta x) = f_X(x) \cdot \Delta x$.

when Δx is very small

Transformation of RV

$$Y = g(X)$$

If $f_X(x)$ is known; $f_Y(y) = ?$



$$P[y < Y \le y + \Delta y] = f_Y(y).\Delta y$$

$$\Rightarrow P(x_1 < X \le x_1 + \Delta x_1) \cup P(x_2 < X \le x_2 + \Delta x_2) \cup \ldots P(x_n < X \le x_n + \Delta x_n)$$

$$= f_Y(y).\Delta y$$

$$f_Y(y)\Delta y = \sum_{i=1}^{n} P[x_i < X \le x_i + \Delta x_i].$$

$$= \sum_{i=1}^{n} f_X(x_i).|\Delta x_i| \quad , \Delta x_i \text{ can be +ve or -ve.}$$

$$f_Y(y) = \sum_{i=1}^{n} f_X(x_i) \left|\frac{\Delta x_i}{\Delta y}\right|$$

$$f_Y(y) = \sum_{i=1}^{n} f_X(x_i) \left|\frac{dx_i}{dy}\right| \quad [\Delta y \text{ smaller}]$$

$$\boxed{f_Y(y) = \sum_{i=1}^{n} f_X(x_i) \left|\frac{dx_i}{dy}\right|}$$

PDF

1. $f_X(x) \geq 0$
2. $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$
3. $F_X(x) = \int_{-\infty}^{x} f_X(z) \cdot dz$
4. $P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(x) \cdot dx$
5. $P(x < X \leq x + \Delta x) = f_X(x) \cdot \Delta x$

CDF

1. $F_X(-\infty) = 0$
2. $F_X(+\infty) = 1$
3. $0 \leq F_X(x) \leq 1$
4. $F_X(x_1) \leq F_X(x_2)$, $x_2 > x_1$
5. $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$.
6. $F_X(x^+) = F_X(x)$ $\Rightarrow P(X \leq x) = F_X(x)$

   $x + \Delta x$ (very small Δx) $P(X \leq x + \Delta x) = F_X(x)$
7. $P(X > x) = 1 - F_X(x)$.

Average value of X.

$x_1, x_2 \ldots x_n$.

$$Avg = \frac{\sum_{i=1}^{n} x_i}{n} \Rightarrow \sum_{i=1}^{n} \frac{1}{n} \cdot x_2.$$

All x_i's have equal probability.

If x_i's have diff probability,

$$Avg = \sum_{i=1}^{n} P_i \cdot x_i$$

Expected value of RV

$$E[X] = \bar{X} = \sum_{i=1}^{n} P_i x_i \rightarrow \text{discrete}$$

$$= \int_{-\infty}^{\infty} x \cdot f_X(x) \cdot dx \rightarrow \text{continuous}$$

Mean $= \mu_x \rightarrow$ First moment.

$E[X^2] \rightarrow$ Second moment.

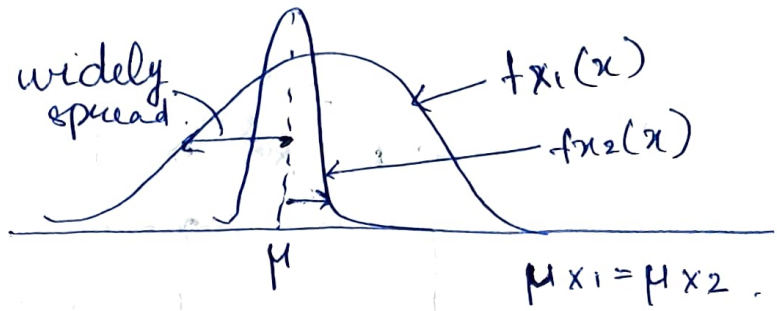
$$Y = g(X)$$

$$E[Y] = E[g(X)] = \int_x g(X) \cdot f_X(x) \cdot dx.$$

$$\left.\begin{aligned} E[X^2] &= \int_x x^2 f_X(x)\, dx \\ &\vdots \\ E[X^n] &= \int_x x^n f_X(x)\, dx \end{aligned}\right\} \text{Non-Central Moments}$$

$$\left.\begin{aligned} E[(X-\mu)^2] &= \int_x (x-\mu)^2 f_X(x)\, dx \\ &\vdots \\ E[(X-\mu)^n] &= \int_x (x-\mu)^n f_X(x)\, dx \end{aligned}\right\} \text{central moments}$$

Variance → how much the outcomes are spread.



$\sigma_{X_2}^2 < \sigma_{X_1}^2$.

$$E[(X-\mu)^2] = \sigma_X^2$$

$$E[X^2 - 2\mu X + \mu^2] = \sigma_X^2$$

$$E[X^2] - 2\mu E[X] + \mu^2 = \sigma_X^2$$

$$E[X^2] - 2\mu\mu + \mu^2 = -2\mu^2 + \mu^2.$$

$$\boxed{\sigma_X^2 = E[X^2] - [E(X)]^2}$$

$E[X^2]$ → 2nd moment

squared mean

$[E(X)]^2$ → Mean square → Square of 1st moment.

RV & Distribution

Discrete

1. Bernoulli

$$P(X=x) = \begin{cases} p & x = 1 \\ 1-p & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$p \to$ success

$1-p \to$ failure

$$\mu_X = p$$
$$\sigma_X^2 = p(1-p)$$

2. Binomial

$$P(X=x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x \in (0, n) \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_X = np$$
$$\sigma_X^2 = np(1-p)$$

3. Poison

$$P(X=x) = \begin{cases} e^{-\lambda} \dfrac{\lambda^x}{x!} & x \in (0, n) \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_X = \lambda$$
$$\sigma_X^2 = \lambda$$

Continuous

1. Uniform

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_X = \frac{b+a}{2} \qquad \sigma_X^2 = \frac{(b-a)^2}{12}$$

2. Exponential.

$$f_X(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_X = \frac{1}{\lambda}$$

$$\sigma_X^2 = \frac{1}{\lambda^2}$$

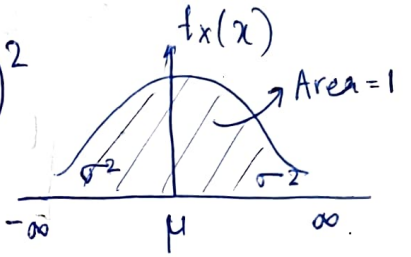
3. Gaussian/Normal

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left\{ \frac{-(x-\mu_X)^2}{2\sigma_X^2} \right\}$$

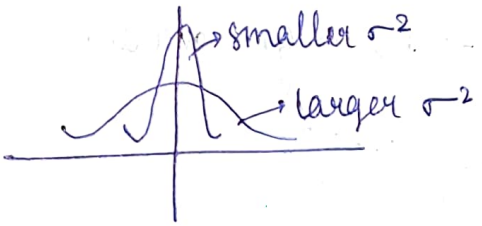
μ_X = mean

σ_X^2 = variance.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$$



Gauss RV $= \mathcal{N}(\mu, \sigma^2) = N(\mu, \sigma^2)$



$$f_X(x)\Big|_{x=0} = \frac{1}{\sqrt{2\pi\sigma^2}}$$

$$f_X(x)\Big|_{x=\pm\infty} \sim 0.$$

Tutorial - Single RV

① Current in a circuit is a continuous RV with PDF $[f_X(x)]$

$$f_X(x) = \begin{cases} 0.075x + 0.2 & 3 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

Find $E(X)$, variance, SD.

Ans: $E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$

$$= \int_3^5 (0.075x + 0.2)x \, dx$$

$$= \int_3^5 0.075x^2 + 0.2x \, dx$$

$$= \left[\frac{0.075x^3}{3} + \frac{0.2x^2}{2} \right]_3^5$$

$$= 0.075 \times \frac{125}{3} + 0.2 \times \frac{25}{2} - 0.075 \times \frac{27}{3} - 0.2 \times \frac{9}{2}$$

$= 4.05$ A $\rightarrow$ Average current

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_3^5 x^2 [0.075x + 0.2]\, dx$$

$$= 16.733$$

$$Var(X) = 16.733 - (4.05)^2 = 0.3305$$

$$SD = \sqrt{\sigma^2} = \sqrt{0.3305} = 0.5749.$$

② Compute CDF, find $P(X<4)$, $P(X>4)$, $P(3.5<X<4)$

$$P(X \le x) = F_X(x)$$

$$= \int_{-\infty}^{x} f_X(x)\,dx$$

$$= \int_{3}^{x} (0.075\overset{\text{dummy}}{t} + 0.2)\,dt$$

$$= \left[0.075\frac{t^2}{2} + 0.2t\right]_3^x$$

$$= \frac{0.075}{2}\left[x^2 - 9\right] + 0.2(x-3)$$

$$P(X<4) = \frac{0.075}{2}\left[4^2 - 9\right] + 0.2[4-3]$$

$$= 0.2625 + 0.2$$

$$= 0.4625$$

$$P(X>4) = 1 - P(X<4)$$

$$= 1 - 0.4625$$

$$= 0.5375$$

$$P(3.5<X<4) = P(X<4) - P(X<3.5)$$

$$= F_X(4) - F_X(3.5)$$

$$= 0.4625 - \frac{0.075}{2}\left[3.5^2 - 9\right] - 0.2(3.5-9)$$

$$= 0.4625 - 0.2218$$

$$= 0.2406$$

24.06% current lies b/w 3.5 and 4.

(3) Find the CDF. $f_X(x) = \begin{cases} k(1+x^2) & , -2<x<1 \\ 0 & , \text{otherwise} \end{cases}$

unknown

CDF = ∫PDF

Ans: <u>Step 1</u>: Compute the unknown k.

Property: $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\int_{-2}^{1} k(1+x^2)\,dx = 1$$

$$k\left[x + \frac{x^3}{3}\right]_{-2}^{1} = 1$$

$$k\left[1 + \frac{1}{3} + 2 + \frac{8}{3}\right] = 1$$

$$k\left[3 + \frac{9}{3}\right] = 1$$

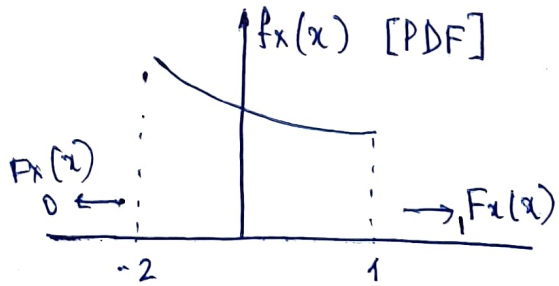
$$k[3+3] = 1$$

$$6k = 1$$

$$k = \frac{1}{6}$$

$$f_X(x) = \begin{cases} \frac{1}{6}(1+x^2) & , -2<x<1 \\ 0 & , \text{otherwise} \end{cases}$$

<u>Step 2</u>: Find the CDF



$$F_X(x) = \int_{-2}^{x} f_Z(z)\, dz$$

$$= \int_{-2}^{x} \frac{1}{6}\left[1 + z^2\right] dz$$

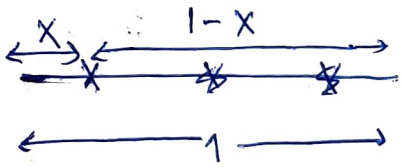
$$= \frac{1}{6}\left[z + \frac{z^3}{3}\right]_{-2}^{x}$$

$$= \frac{1}{6}\left[x + \frac{x^3}{3} + 2 + \frac{8}{3}\right]$$

$$= \frac{1}{6}\left[x + \frac{x^3}{3} + \frac{14}{3}\right]$$

$$F_X(x) = \begin{cases} 0 & , x < -2 \\ 1/6\left[x + \frac{x^3}{3} + \frac{14}{3}\right] & , -2 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

④ Breaking a 1 unit length stick into two randomnly. Find the PDF of ratio of the length of the 2 pieces



Ans: $Y = \dfrac{X}{1-X}$

PDF 'X' → uniformly distributed.

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

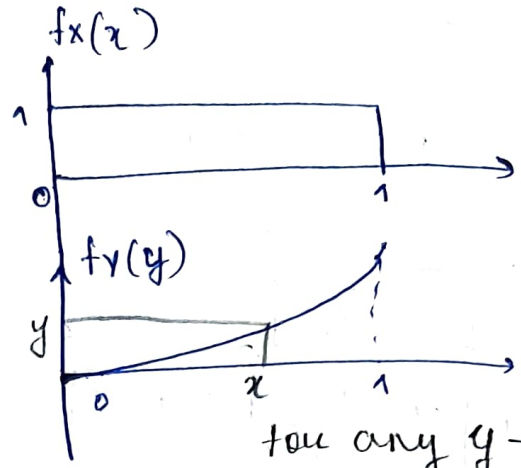
$$= \begin{cases} \frac{1}{1-0} & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

$f_X(x) = 1$.

$X \to (0-1)$

$Y \to ?$



for any $y \to 1$ x value (1 no)
$\Rightarrow n = 1$

$$Y = \frac{X}{1-X}$$

$x = 0 \quad Y = 0$

$x = 1 \quad Y = \infty$

$x = 1/2 \quad Y = 1$

$$Y = \frac{X}{1-X} \Rightarrow X = \frac{Y}{1+Y}$$

$$\left(\frac{u}{v}\right) = \frac{v\frac{du}{dy} - u\frac{dv}{dy}}{v^2}$$

$$\frac{dx}{dy} = \frac{1}{(1+y)^2}$$

$$f_Y(y) = \sum_{i=1}^{n} \left|\frac{dx}{dy}\right| \cdot f_X(x)$$

$$= \sum_{i=1}^{1} \frac{1}{(1+y)^2} \cdot 1$$

$$\boxed{f_Y(y) = \frac{1}{(1+y)^2}}$$

To prove that $f_Y(y)$ found is true.

$$\int_{-\infty}^{\infty} f_Y(y)\, dy = 1$$

$(1+y) = u$

$y = 0 \quad u = 1$

$y = \infty \quad u = \infty$

$$\int_0^{\infty} \frac{1}{(1+y)^2}\, dy = \int_1^{\infty} \frac{1}{u^2}\, du.$$

$$\int_1^{\infty} \frac{1}{u^2}\, du = \left(\frac{-1}{u}\right)_1^{\infty}$$

$$= -\frac{1}{\infty} - \left(\frac{-1}{1}\right)$$

$$= 0 + 1$$

$$= 1$$

Hence, verified.

<u>Step 1</u>: Write X in terms of Y

<u>Step 2</u>: Find $\left|\frac{dx}{dy}\right|$

<u>Step 3</u>: Write $f_Y(y) = f_X(y)\left|\frac{dx}{dy}\right|$

<u>Step 4</u>: Find limits of Y from X.

(5) Given $f_X(x) = e^{-2|x|}$, $-\infty < x < \infty$

Find $f_Y(y)$ if $Y = X^2$

<u>Step 1</u>: $y = x^2$

$x = \sqrt{y}$

<u>Step 2</u>: $\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$

<u>Step 3</u>: $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$

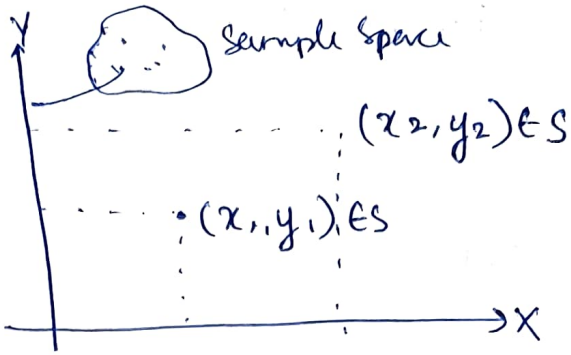
$$= e^{-2|x|} \frac{1}{2\sqrt{y}}.$$

Step 4 : $|x| = |-\infty < x < \infty|$

$= 0 < x < \infty$

$$f_Y(y) = e^{-2|\sqrt{y}|} \frac{1}{2\sqrt{y}}, \quad 0 < |y| < \infty.$$

Two Random Variable



Properties of CDF & PDF

$$CDF = F_X(x) = P(X \le x)$$
$$F_Y(y) = P(Y \le y)$$

Joint Distribution Function or Joint CDF

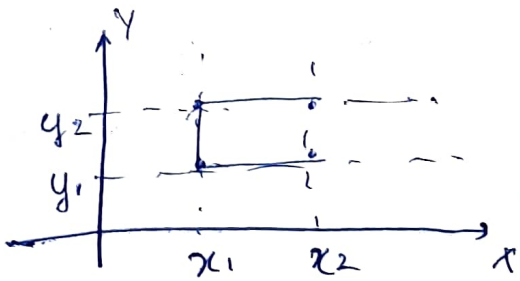
$$F_{XY}(x,y) = P[X \le x, Y \le y]$$
$$= P[X \le x \cap Y \le y]$$

1. $F_{XY}(-\infty, -\infty) = 0$
2. $F_{XY}(+\infty, +\infty) = 1$
3. $F_{XY}(x,y)$ non decreasing $(0-1)$

   $0 < F_{XY}(x,y) \le 1$
4. $F_{XY}(-\infty, y) = 0$.

   $F_{XY}(x, -\infty) = 0$.
5. $P[x_1 \le X \le x_2, y_1 \le Y \le y_2] =$

   $F_{XY}(x_2, y_2) + F_{XY}(x_1, y_1) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1)$



6. $F_{XY}(x, +\infty) = F_X(x)$
$F_{XY}(+\infty, y) = F_Y(y)$. } Marginal Distribution Function.

PDF

$$PDF = \frac{d}{dx} CDF$$

$$\boxed{f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \, \partial y}}$$

1. $f_{XY}(x,y) \geq 0$

2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1.$

3. $F_{XY}(x,y) = \int\int f_{XY}(x,y)\,dx\,dy.$
$= \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(z_1, z_2)\,dz_1\,dz_2$

4. $F_X(x) = \int_{-\infty}^{x}\left[\int_{-\infty}^{\infty}\right] f_{XY}(z_1, z_2)\,dz_2\,dz_1$

$F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{XY}(z_1, z_2)\,dz_1\,dz_2.$

5. $P(x_1 < X \leq x_2 \cap y \leq Y \leq y_2) = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{XY}(x,y)\,dx\,dy.$

6. $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy.$

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx.$

7. $P[x < X \leq x + \Delta x,\ y < Y \leq y + \Delta y]$
$= f_{XY}(x,y) \cdot \Delta x \cdot \Delta y.$

8. If X & Y are independent.

$$P[X \cap Y] = P(X) \cdot P(Y)$$

$$P[X \le x, Y \le y] = P[X \le x] \cdot P[Y \le y]$$

$$\boxed{\begin{array}{ll} \text{CDF} & F_{XY}(x,y) = F_X(x) \cdot F_Y(y) \\ \text{PDF} & f_{XY}(x,y) = f_X(x) \cdot f_Y(y) \end{array}}$$

Joint Moments

$$\mu_{nk} = m_{nk} = E[X^n Y^k]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^n y^k f_{XY}(x,y)\, dx\, dy.$$

$$\mu_{10} = E[X] = \mu_X$$

$$\mu_{01} = E[Y] = \mu_Y$$

$$\mu_{20} = E[X^2]$$

$$\mu_{02} = E[Y^2]$$

$$\mu_{11} = E[XY] = \int_x \int_y xy \cdot f_{XY}(x,y) \cdot dy\, dx.$$

$$Z = g(x,y)$$

$$E[g(x,y)] = \int_x \int_y g(x,y) f_{XY}(x,y)\, dy \cdot dx.$$

* If X & Y are independent

$$E[XY] = E[X] \cdot E[Y]$$

$$\mu_{nk} = E[(X-\mu_X)^n (Y-\mu_Y)^k]$$

$$\left.\begin{aligned}\mu_{20} &= E[(X-\mu_X)^2] = \sigma_X^2 \\ \mu_{02} &= E[(Y-\mu_Y)^2] = \sigma_Y^2\end{aligned}\right\} \text{Variance.}$$

$$\mu_{11} = E[(X-\mu_X)(Y-\mu_Y)] \quad \} \text{ Covariance}$$

Covariance : $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

$$= E[XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y]$$

$$= E[XY] - E[X]\mu_Y - E[Y]\mu_X + \mu_Y\mu_X$$

$$= E[XY] - \mu_X\mu_Y - \cancel{\mu_Y\mu_X} + \cancel{\mu_Y\mu_X}$$

$$= E[XY] - \mu_X\mu_Y$$

$\therefore$ cov can be either +ve & -ve

* If X & Y are independent.

$$Cov(XY) = E[X].E[Y] - \mu_X\mu_Y$$

$$= \mu_X\mu_Y - \mu_X\mu_Y$$

$$= 0.$$

<u>Normalized form of covariance</u>.

$var(X) = \sigma_X^2$

$var(Y) = \sigma_Y^2$

$cov(XY) = E[XY] - \mu_X\mu_Y$.

$\rho_{XY} \rightarrow$ correlation coefficient

$$\rho_{XY} = \frac{cov(X,Y)}{\sigma_X\sigma_Y}$$

$-1 \le \rho_{XY} \le 1$. $|\rho_{XY}| \le 1$.

when X & Y are indepent,

$$\rho_{XY} = 0 \Rightarrow \text{uncorrelated}$$

If X & Y are independent. they X & Y are uncorrelation

independent $\Rightarrow$ uncorrelated

uncorrelated $\not\Rightarrow$ independent

<u>Special Case</u>

If RV's are jointly gaussian,

independent $\Rightarrow$ uncorrelated

uncorrelated $\Rightarrow$ independent.

X & Y are RV

$Z = X + Y$ or $X - Y$

General : $Z = aX + bY \rightarrow$ linear combination of 2 RV

$\mu_Z, \sigma_Z^2 = ?$

$$\mu_Z = E[Z] = E[aX + bY]$$
$$= aE[X] + bE[Y]$$

$$\boxed{\mu_Z = a\mu_X + b\mu_Y}$$

$$var(Z) = E[(Z - \mu_Z)^2]$$
$$= E[\{(aX + bY) - (a\mu_X + b\mu_Y)\}^2]$$
$$= E[\{a(X - \mu_X) + b(Y - \mu_Y)\}^2]$$
$$= a^2 \underbrace{E[(X - \mu_X)^2]}_{\sigma_X^2} + 2ab \underbrace{E[(X - \mu_X)(Y - \mu_Y)]}_{cov(X,Y)}$$
$$+ b^2 \underbrace{E[(Y - \mu_Y)^2]}_{\sigma_Y^2}$$
$$= a^2\sigma_X^2 + 2ab\, cov(X,Y) + b^2\sigma_Y^2$$

$$\rho_{XY} = \frac{cov(X,Y)}{\sigma_X \sigma_Y}$$

$$cov(X,Y) = \rho_{XY}\sigma_X\sigma_Y$$

$$\boxed{\begin{aligned} var(Z) &= a^2\sigma_X^2 + 2ab\,\rho_{XY}\sigma_X\sigma_Y + b^2\sigma_Y^2 \\ var(Z) &= a^2\sigma_X^2 + 2ab\,cov(X,Y) + b^2\sigma_Y^2 \end{aligned}}$$

If X & Y are independent,

$$\rho_{XY} = 0$$

$$var(Z) = a^2\sigma_X^2 + b^2\sigma_Y^2$$

$a=1 \quad b=1 \qquad X+Y$

$$var(Z) = \sigma_X^2 + \sigma_Y^2$$

$a=1 \quad b=-1 \qquad X-Y$

$$var(Z) = \sigma_X^2 + \sigma_Y^2$$

overall variance increases or remains same.

$$var(Z) = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} \sigma_X^2 & cov(X,Y) \\ cov(X,Y) & \sigma_Y^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Characteristic Function & Moments

$$\phi_X(\omega) = E[\underbrace{e^{j\omega X}}_{g(X)}] = \int_{-\infty}^{\infty} g(X)\, f_X(x)\,dx$$

$$\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} \cdot f_X(x)\,dx.$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

$$\phi_X(\omega) = \int_{-\infty}^{\infty} \left(1 + \frac{j\omega X}{1!} + \frac{j^2\omega^2 X^2}{2!} + \dots \frac{j^k\omega^k X^k}{k!}\right) f_X(x)\,dx.$$

$$= \int_{-\infty}^{\infty} f_X(x)\,dx + \int_{-\infty}^{\infty} j\omega x\, f_X(x)\,dx + \dots$$

$$\phi_X(\omega) = 1 + \frac{j\omega}{1!} E[X] + \frac{j^2\omega^2}{2!} E[X^2] + \cdots \frac{j^k\omega^k}{k!} E[X^k].$$

$$\phi_X(\omega)\big|_{\omega=0} = 1.$$

$$\phi'_X(\omega)\big|_{\omega=0} = 0 + jE[X] + \frac{j^2}{2!} 2\omega E[X^2] + \cdots \rightarrow (1)$$

$$\phi''_X(\omega)\big|_{\omega=0} = 0 + j^2 E[X^2] + \frac{j^3}{3!} 3\omega^2 E[X^3] + \cdots \rightarrow (2)$$

From (1) $E[X] = \dfrac{\phi'_X(\omega)}{j}\Big|_{\omega=0}$

(2) $E[X^2] = \dfrac{\phi''_X(\omega)}{j^2}\Big|_{\omega=0}$

$$\boxed{\frac{1}{j^k} \frac{d^k \phi_X(\omega)}{d\omega^k} = E[X^k], \quad k = 1, 2, \ldots}$$

Joint Characteristic Function

$$\phi_{XY}(u, v) = E[e^{j(uX + vY)}]$$

$$\phi_{XY}(\omega_1, \omega_2) = E[e^{j(\omega_1 X + \omega_2 Y)}]$$

$$\boxed{E[X^m Y^n] = \frac{1}{j^{m+n}} \left[\frac{\partial^{m+n}}{\partial^m \omega_1 \, \partial^n \omega_1} (\phi_{XY}(\omega_1, \omega_2)) \right]_{\omega_1 = 0, \, \omega_2 = 0}}$$

1. $\phi_{XY}(0, 0) = 1$
2. $\phi_{XY}(\omega_1, 0) = \phi_X(\omega_1)$
3. $\phi_{XY}(0, \omega_2) = \phi_Y(\omega_2)$
4. If X and Y are independent,

$$\phi_{XY}(\omega_1, \omega_2) = \phi_X(\omega_1)\, \phi_Y(\omega_2)$$

Tutorial : Discrete 2D-RV

PDF = PMF

① X & Y are RV having PMF

$$f_{XY}(x,y) = k[3x+5y]$$

$X = 1, 2, 3 \qquad Y = 0, 1, 2$

Find marginal Density function of X & Y, $P(X=x_i \mid Y=2)$, $P(X \le 2 \mid Y \le 1)$.

Ans :

X\Y	0	1	2
1	3k	8k	13k
2	6k	11k	16k
3	9k	14k	19k.

$$\sum_x \sum_y P(x,y) = 1$$

$$3k+8k+13k+6k+11k+16k+9k+14k+19k = 1$$

$$k = \frac{1}{99}$$

Joint PMF

X\Y	0	1	2	$f_X(x)$
1	3/99	8/99	13/99	24/99
2	6/99	11/99	16/99	33/99
3	9/99	14/99	19/99.	42/99
$f_Y(y)$	18/99	33/99	48/99	

X	1	2	3
$P(X=x)$	$24/99$	$33/99$	$42/99$

$f_X(x)$

Y	0	1	2
$P(Y=y)$	$18/99$	$33/99$	$48/99$

$f_Y(y)$

$$P(X=x_i \mid Y=2) = \frac{P(X=x_i \cap Y=2)}{P(Y=2)}$$

$$P(Y=2) = \frac{13}{99} + \frac{16}{99} + \frac{19}{99} = \frac{48}{99}$$

$$P(X=x_i \mid Y=2) = \frac{P(X=1 \cap Y=2)}{P(Y=2)} + \frac{P(X=2 \cap Y=2)}{P(Y=2)} + \frac{P(X=3 \cap Y=2)}{P(Y=2)}$$

$$= \frac{13/99 + 16/99 + 19/99}{48/99}$$

$$P(X=1 \mid Y=2) = \frac{P(X=1 \cap Y=2)}{P(Y=2)}$$

$$= \frac{13/99}{48/99} = \frac{13}{48}$$

$$P(X=2 \mid Y=2) = \frac{P(X=2 \cap Y=2)}{P(Y=2)}$$

$$= \frac{16/99}{48/99} = \frac{16}{48}$$

$$P(X=3 \mid Y=2) = \frac{P(X=3 \cap Y=2)}{P(Y=2)}$$

$$= \frac{19/99}{48/99} = \frac{19}{48}$$

$X=x_i$	1	2	3
$P(X=x_i \mid Y=2)$	$\frac{13}{48}$	$\frac{16}{48}$	$\frac{19}{48}$

$$P(X \le 2 \mid Y \le 1) = \frac{P(X \le 2 \cap Y \le 1)}{P(Y \le 1)}$$

$$= \frac{3/99 + 8/99 + 6/99 + 11/99}{\frac{3+8+6+11+9+14}{99}}$$

$$= \frac{28}{51}$$

(2) Covariance and correlation coefficient of 2D discrete RV

Joint PMF of X & Y,

$$f_{XY}(x,y) = \frac{x+y}{32} \qquad X = 1, 2 \quad Y = 1, 2, 3, 4.$$

Find ρ_{XY}

Ans:

X \ Y	1	2	3	4	$f_X(x)$
1	2/32	3/32	4/32	5/32	14/32
2	3/32	4/32	5/32	6/32	18/32
$f_Y(y)$	5/32	7/32	9/32	11/32	

$$\rho_{XY} = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y}$$

$$\mathrm{cov}(X,Y) = E[XY - \mu_X \mu_Y]$$

$$E[XY] = \sum_x \sum_y xy\, P(x,y)$$

$$= (1)(1)\frac{2}{32} + (1)(2)\frac{3}{32} + (1)(3)\frac{4}{32} + (1)(4)\frac{5}{32}$$
$$+ (2)(1)\frac{3}{32} + (2)(2)\frac{4}{32} + (2)(3)\frac{5}{32} + (2)(4)\frac{6}{32}$$

$$= \frac{140}{32}.$$

$$\mu_X = E[X] = \sum x\, P(x)$$

$$= 1 \cdot \frac{14}{32} + 2 \cdot \frac{18}{32} = \frac{14}{32} + \frac{36}{32} = \frac{50}{32}.$$

$$\mu_Y = E[Y] = \sum y\, P(x)$$

$$= 1 \cdot \frac{5}{32} + 2 \cdot \frac{7}{32} + 3 \cdot \frac{9}{32} + 4 \cdot \frac{11}{32}$$

$$= \frac{5}{32} + \frac{14}{32} + \frac{27}{32} + \frac{44}{32}$$

$$= \frac{90}{32}$$

$$E[X^2] = \sum x^2 P(x)$$

$$= 1^2\left(\frac{14}{32}\right) + (2^2)\left(\frac{18}{32}\right) = \frac{86}{32}$$

$$E[Y^2] = 1^2\left(\frac{5}{32}\right) + 2^2\left(\frac{7}{32}\right) + 3^2\left(\frac{9}{32}\right) + 4^2\left(\frac{11}{32}\right)$$

$$= \frac{290}{32}$$

$$\sigma_X^2 = E[X^2] - (E[X])^2$$

$$= \frac{86}{32} - \left(\frac{50}{32}\right)^2$$

$$= \frac{63}{256}$$

$$\sigma_Y^2 = E[Y^2] - (E(Y))^2$$

$$= \frac{290}{32} - \left(\frac{90}{32}\right)^2$$

$$= \frac{295}{256}$$

$$\sigma_X = \sqrt{\sigma_X^2} = 0.496$$

$$\sigma_Y = \sqrt{\sigma_Y^2} = 1.073$$

$$\text{cov}(X, Y) = E[XY - \mu_X \mu_Y]$$

$$= E[XY] - \mu_X \mu_Y$$

$$= \frac{140}{32} - \frac{50}{32} \times \frac{90}{32} = \frac{-5}{256}.$$

$$|\rho_{XY}| = \frac{cov(X,Y)}{\sigma_X \sigma_Y}$$

$$= \frac{\frac{-5}{256}}{0.496 \times 1.073}$$

$$= |-0.0367|$$

$$|\rho_{XY}| = 0.0367$$

Tutorial - 2D Continuous RV

① X & Y are RV's have joint PDF:

$$f_{XY}(x,y) = kxy\, e^{-(x^2+y^2)}, \quad x>0,\ y>0$$

Verify that X & Y are independent.

Ans: Step 1: Find the unknown

Step 2: Verify independent $[f_{XY}(x,y) = f_X(x) \cdot f_Y(y)]$

To find k,

joint PDF: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

$$\int_0^{\infty}\int_0^{\infty} kxy\, e^{-(x^2+y^2)}\,dy\,dx = 1$$

$$\int_0^{\infty} kx\left[\frac{y^2}{2}\,\frac{e^{-(x^2+y^2)}}{-2y}\right]_0^{\infty} dx = 1$$

$x^2 = u$

$2x\,dx = du$

$$k\int_0^{\infty} x e^{-x^2}dx \cdot \int_0^{\infty} y\, e^{-y^2}dy = 1$$

$$k\int_0^{\infty} \frac{du}{2}\, e^{-u} \int_0^{\infty} \frac{dv}{2}\, e^{-v} = 1$$

$$\frac{k}{4}\left[\frac{e^{-u}}{-1}\right]_0^{\infty}\left[\frac{e^{-v}}{-1}\right]_0^{\infty} = 1$$

$$\frac{k}{4}[1][1] = 1$$

$$k = 4$$

Marginal Density function:

$$f_X(x) = \int_0^\infty f_{XY}(x,y)\,dy$$

$$= -2xe^{-x^2}$$

$$f_Y(y) = \int_0^\infty f_{XY}(x,y)\,dx$$

$$= -2ye^{-y^2}$$

$$\cancel{f_{XY}(x,y)} \quad f_X(x) \times f_Y(y) = 4xye^{-(x^2+y^2)}$$

$$= f_{XY}(x,y)$$

$\therefore$ X & Y are independent

② Joint PDF of X & Y is

$$f_{XY}(x,y) = 25e^{-5y}, \quad 0 < x < 0.2, \quad y > 0$$

Find ρ_{XY}

Ans: $$E[X] = \int_0^{0.2}\int_0^\infty x \cdot f_{XY}(x,y)\,dy\,dx$$

$$= \int_0^{0.2}\int_0^\infty x \cdot 25e^{-5y}\,dy\,dx$$

$$= \int_0^{0.2}\left[25x\frac{e^{-5y}}{-5}\right]_0^\infty dx$$

$$= \int_0^{0.2} \cancel{25}^{5}x\left(\frac{1}{\cancel{5}}\right) dx$$

$$= 5\left[\frac{x^2}{2}\right]_0^{0.2}$$

$$= 0.1$$

$$\int \frac{e^{-5y}}{-5}$$

$$E[Y] = \int_0^{0.2}\int_0^{\infty} y \cdot 25e^{-5y}\, dy\, dx$$

$$= \int_0^{0.2} 25\left[y\frac{e^{-5y}}{-5} - \frac{e^{-5y}}{25}\right]_0^{\infty} dx$$

$$= \int_0^{0.2} \cancel{25}\left[\frac{1}{\cancel{25}}\right] dx$$

$$= [x]_0^{0.2}$$

$$= 0.2$$

$$E[XY] = \int_0^{0.2} x \int_0^{\infty} y\, 25e^{-5y}\, dy\, dx$$

$$= \int_0^{0.2} x\, 25\left[0 + \frac{1}{25}\right] dx$$

$$= \left[\frac{x^2}{2}\right]_0^{0.2}$$

$$= 0.02.$$

$$\text{cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$= 0.02 - 0.2 \times 0.1$$

$$= 0.$$

$$\Rightarrow \rho_{XY} = 0$$

$\therefore$ X & Y are uncorrelated.

(3) X & Y are independent RV's

$\sigma_X^2 = 36 \quad \sigma_Y^2 = 16.$

find ρ_{UV} if $U = X+Y$, $V = X-Y$.

Ans: $E[U] = E[X+Y] = E[X] + E[Y]$

$E[V] = E[X-Y] = E[X] - E[Y]$

$E[UV] = E[(X+Y)(X-Y)] = E[X^2 - Y^2] = E[X^2] - E[Y^2]$

$$E[U^2] = E[(X+Y)^2] = E[X^2] + E[Y^2] + 2[E[XY]$$

$$E[V^2] = E[(X-Y)^2] = E[X^2] + E[Y^2] - 2E[XY]$$

$$\begin{aligned}
cov(UV) &= E[UV] - E[U]E[V] \\
&= E[X^2] - E[Y^2] - [(E[X] + E[Y])(E[X] - E[Y])] \\
&= E[X^2] - E[Y^2] - [(E[X])^2 - (E[Y])^2] \\
&= E[X^2] - E[Y^2] - (E[X])^2 + (E[Y])^2 \\
&= \sigma_X^2 - \sigma_Y^2 \\
&= 36 - 16 \\
&= 20.
\end{aligned}$$

$$\rho_{UV} = \frac{cov(UV)}{\sigma_U \sigma_V}$$

$$\begin{aligned}
\sigma_U^2 &= E[U^2] - [E[U]]^2 \\
&= E[X^2] + E[Y^2] + 2E[XY] - \{E[X] + E[Y]\}^2 \\
&= E[X^2] + E[Y^2] + 2\cancel{E[XY]} - (E[X])^2 - (E(Y))^2 - 2\cancel{E[X]E[Y]} \\
&= E[X^2] + E[Y^2] - (E[X])^2 - (E[Y])^2 \\
&= \sigma_X^2 + \sigma_Y^2
\end{aligned}$$

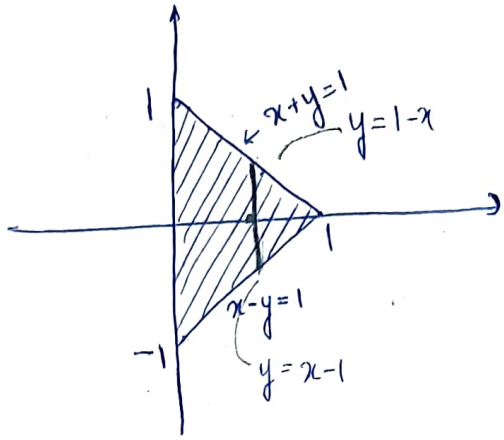
$$\sigma_V^2 = \sigma_X^2 + \sigma_Y^2$$

$$\sigma_U^2 = 52 \qquad \sigma_V^2 = 52$$

$$\rho_{UV} = \frac{cov(UV)}{\sigma_U \sigma_V} = \frac{20}{\sqrt{52}\sqrt{52}}$$

$$\rho_{UV} = 0.3846.$$

④



$f_{XY}(x,y) = \begin{cases} 1 \\ 0 \text{ otherwise.} \end{cases}$

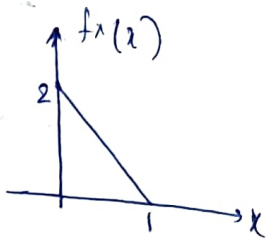
X, Y ∈ shaded area.

Verify the independency & correlation of X & Y.

Ans: condition

$f_{XY}(x,y) = f_X(x)\, f_Y(y) \rightarrow$ independent.

$\rho_{XY} = 0 \rightarrow$ uncorrelated



$$f_X(x) = \int f_{XY}(x,y)\, dy.$$

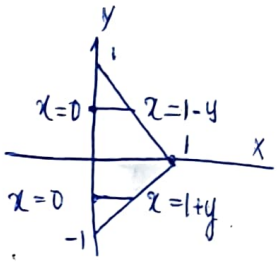
$$= \int_{x-1}^{1-x} 1 . dy.$$

$$= [y]_{x-1}^{1-x}$$

$$= 1 - x - x + 1$$

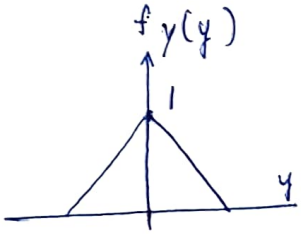
$$= 2 - 2x.$$

$$= 2(1-x)$$



$$f_Y(y) = \begin{cases} \int_0^{1-y} f_{XY}(x,y)\, dx \\ \int_0^{1+y} f_{XY}(x,y)\, dx. \end{cases}$$

$$= \begin{cases} 1-y & 0 < y < 1 \\ 1+y & -1 < y < 0. \end{cases}$$



$f_{XY}(x,y) = 1$ (given)

$f_X(x) . f_Y(y) = 2(1-x)(1 \pm y)$

$f_{XY}(x,y) \neq f_X(x) f_Y(y) \Rightarrow$ X & Y are not independent

$$\rho_{XY} = \frac{cov(X,Y)}{\sigma_X \sigma_Y}$$

$$= \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y}$$

$$E[Y] = \int y \cdot f_Y(y)\, dy$$

$$= \int_0^1 (1-y)\, y\, dy + \int_{-1}^0 (1+y) \cdot y\, dy.$$

$$= \int_0^1 (1-y)\, y\, dy - \int_0^1 (1-y)\, y\, dy.$$

$$= 0$$

$$E[XY] = \iint xy\, f_{XY}(x,y)\, dy\, dx$$

$$= \int_0^1 \int_{y=x-1}^{1-x} y \cdot 1\, dy\, dx$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_{x-1}^{1-x} dx$$

$$= \int_0^1 \frac{(1-x)^2 - (x-1)^2}{2}\, dx$$

$$= 0.$$

$cov(XY) = 0$

$\rho_{XY} = 0 \Rightarrow$ ~~th~~ X & Y are uncorrelated.

Characteristics function

Poisson dist - discrete RV

① Find the characteristics of X and find the moments.

$$P[X=k] = e^{-\lambda} \cdot \frac{\lambda^k}{k!}, \quad k = 0, 1, 2 \ldots$$

$$\phi_x(\omega) = E[e^{j\omega x}]$$

$$= \sum_{k=0}^{\infty} e^{j\omega k} \cdot P[X=k]$$

$$= \sum_{k=0}^{\infty} e^{j\omega k} \cdot e^{-\lambda} \frac{\lambda^k}{k!} \qquad e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda \cdot e^{j\omega})^k}{k!}$$

$$= e^{-\lambda} e^{\lambda e^{j\omega}}$$

$$= e^{-\lambda(1-e^{j\omega})}$$

$$\phi_x(\omega) = e^{-\lambda(1-e^{j\omega})}$$ → characteristic function of poisson dist.

To find the moments:

$$\phi'_x(\omega) = e^{-\lambda(1-e^{j\omega})} \cdot \frac{d}{d\omega}(-\lambda(1-e^{j\omega}))$$

$$= e^{-\lambda(1-e^{j\omega})} \lambda j e^{j\omega}$$

$$\phi'_x(\omega=0) = \lambda j$$

$$\frac{1}{j} \phi'_x(\omega=0) = \lambda \longrightarrow E[X].$$

$$\phi''_x(\omega) = \frac{d}{d\omega}\left[j\lambda e^{j\omega} e^{-\lambda(1-e^{j\omega})}\right]$$

$$= \lambda j \left[j \cdot e^{j\omega} \cdot e^{-\lambda(1-e^{j\omega})} + e^{j\omega} \cdot j\lambda \cdot e^{-\lambda(1-e^{j\omega})}\right]$$

$$\phi_x''(\omega=0) = \lambda j\,[j(e^0+\lambda e^0)]$$
$$= j^2\lambda[1+\lambda]$$
$$= j^2[\lambda+\lambda^2]$$

$$\frac{\phi_x''(\omega=0)}{j^2} = \lambda+\lambda^2 \rightarrow E[X^2]$$

$$var(X) = E[X^2]-(E[X])^2$$
$$= \lambda+\cancel{\lambda^2}-\cancel{\lambda^2}$$
$$= \lambda$$

(2) X is a RV with poison $P(\lambda_1)$, $Y \sim P(\lambda_2)$. X & Y are independent. If $Z=X+Y$, find $f_Z(z)$ using characteristics function.

Property of cha. function: $\phi_{XY}(\omega_1,\omega_2) = \phi_X(\omega_1)\,\phi_Y(\omega_2)$ if X & Y are independent.

$$\phi_Z(\omega) = E[e^{j\omega X}].\,E[e^{j\omega Y}]$$

$$E[e^{j\omega X}] = \phi_X(\omega) = e^{-\lambda_1(1-e^{j\omega})}$$
$$E[e^{j\omega Y}] = \phi_Y(\omega) = e^{-\lambda_2(1-e^{j\omega})}$$

$$\phi_Z(\omega) = e^{-\lambda_1(1-e^{j\omega})}\cdot e^{-\lambda_2(1-e^{j\omega})}$$
$$= e^{-(\lambda_1+\lambda_2)(1-e^{j\omega})}$$
$$= e^{-\lambda(1-e^{j\omega})}$$

$$f_Z(z) = P(\lambda_1+\lambda_2)$$

If X & Y are independent poisson, then X+Y is a poisson distribution with $(\lambda_1+\lambda_2)$ as the variable.

$$Z \sim P(\lambda_1+\lambda_2).$$

③ If $Z = X - Y$. X & $Y \sim P(\lambda_1)$ & $P(\lambda_2)$ what is $\phi_Z(\omega)$ or $f_Z(z)$

$$\phi_X(\omega) = e^{-\lambda_1(1-e^{j\omega})}$$

$$\phi_Y(\omega) = e^{-\lambda_2(1-e^{-j\omega})}$$

$$\phi_Z(\omega) = e^{-\lambda_1(1-e^{j\omega})} \cdot e^{-\lambda_2(1-e^{-j\omega})}$$

$$= e^{-\lambda_1 + \lambda_1 e^{j\omega} - \lambda_2 + \lambda_2 e^{-j\omega}}$$

$$= e^{-(\lambda_1+\lambda_2) + \lambda_1 e^{j\omega} + \lambda_2 e^{-j\omega}}$$

$\phi_Z(\omega) = f_Z(z)$ is not a poison distribution.

$Z = X - Y$ is not poission.

④ If $\phi_{XY}(\omega_1, \omega_2) = \exp(-2\omega_1^2 - 8\omega_2^2)$, find $E[X]$, $E[Y]$, ρ_{XY}.

$$E[X^m Y^n] = \frac{1}{j^{m+n}} \left[\frac{\partial^{m+n} \phi_{XY}(\omega_1 \omega_2)}{\partial^m \omega_1 \partial^n \omega_2} \right]_{\omega_1 = 0, \omega_2 = 0}$$

$$E[X] = \frac{1}{j} \left[\frac{\partial \phi_{XY}(\omega_1, \omega_2)}{\partial \omega_1} \right]_{\omega_1 = 0, \omega_2 = 0}$$

$$E[Y] = \frac{1}{j} \left[\frac{\partial \phi_{XY}(\omega_1 \omega_2)}{\partial \omega_2} \right]_{\omega_1 = 0, \omega_2 = 0}$$

$$E[XY] = \frac{1}{j^2} \left[\frac{\partial^2 \phi_{XY}(\omega_1 \omega_2)}{\partial \omega_1 \partial \omega_2} \right]_{\omega_1 = 0 \ \omega_2 = 0}$$

$E[X] = 0$

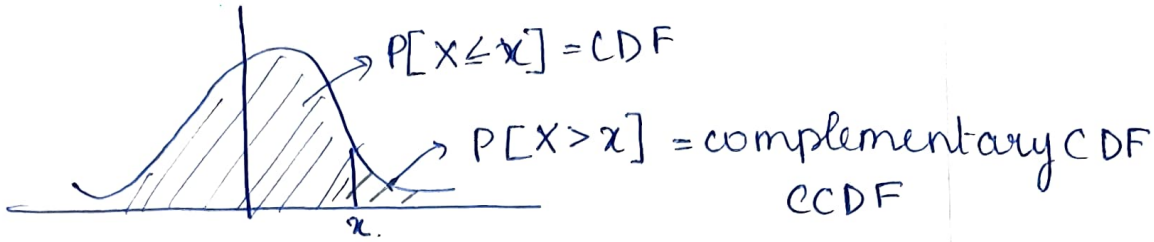
$E[Y] = 0$.

$E[XY] = 0$

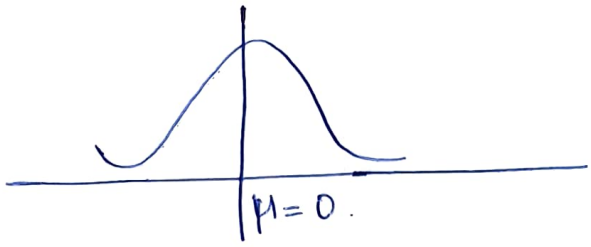
$\rho_{XY} = 0$. $\Rightarrow$ X & Y are uncorrelated.

Gaussian RV - 1D.

CDF : $P[X \le x] = \int_{-\infty}^{x} f_X(x) \cdot dx$



* Gaussian is said to be normal when $\mu = 0$



$N(0,1) \Rightarrow$ zero mean & unit variance normal dist.

CDF of normal : $F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$ $\quad N(0,1)$. $\mu = 0 \; \sigma^2 = 1$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Let $u = \frac{x-\mu}{\sigma}$

$x \to -\infty \qquad u \to -\infty$.

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x-\mu/\sigma} e^{-u^2/2} \cdot \sigma du.$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-u^2/2} du}_{F_U(u)}$$

$$\boxed{F_X(x) = F\left[\frac{X-\mu}{\sigma}\right]}$$

CCDF : $\boxed{Q(x) = \int_{x}^{\infty} f_X(x) \cdot dx}$

$\boxed{F_X(x) = 1 - Q(x)}$

$Z = g(X,Y) \rightarrow \sqrt{X^2+Y^2}$

$X+Y$ $\quad X-Y \quad \frac{X}{Y}$

one function two RV

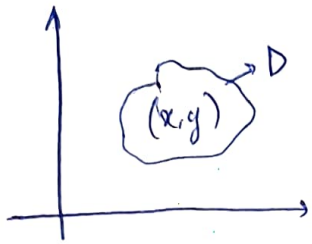
$Z = g(X,Y)$

$W = h(X,Y)$

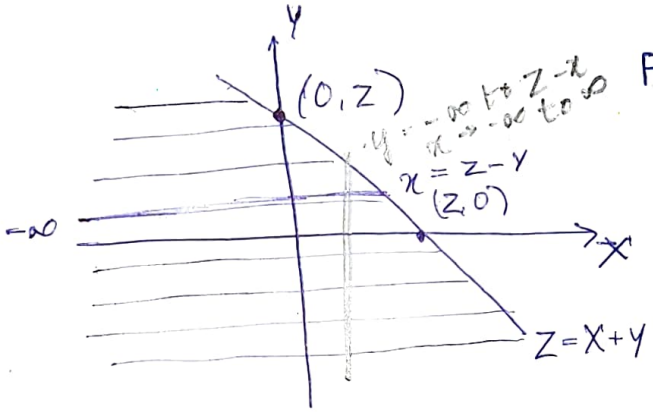
$f_{ZW}(z,w)$

Two function

Two RV



$$P[(x,y) \in D] = \iint_{(x,y \in D)} f_{XY}(x,y)\, dx\, dy.$$



$$F_Z(z) = P[Z \le z]$$
$$= P[X+Y \le z]$$

$x \rightarrow -\infty$ to $z-y$

$y \rightarrow -\infty$ to $+\infty$.

$$\cancel{f_{XY}(x}\ P[X+Y \le Z] = \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_{XY}(x,y)\, dx\, dy = F_Z(z)$$

$$\frac{d}{dz} F_Z(z) = \frac{d}{dz}\int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_{XY}(x,y)\, dx\, dy.$$

Leibnitz Rule:

$$\boxed{\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\, dt = \frac{dv(x)}{dx} f(x, v(x)) - \frac{du(x)}{dx} f(x, u(x)) + \int_{u(x)}^{v(x)} \frac{\partial f(x,t)}{\partial x}\, dt.}$$

$$\frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} \frac{d}{dz} \int_{-\infty}^{z-y} f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} \frac{d}{dz}(z-y)\, f(z-y, y)\,dx - \frac{d}{dz}(-\infty) \cdot f(-\infty, y)$$

$$+ \int_{-\infty}^{z-y} \frac{\partial f(x,y)}{\partial z}\,dx \cdot dy$$

$$f_Z(z) = \int_{-\infty}^{\infty} 1 \cdot f(z-y, y)\,dy$$

If X & Y are independent.

$$f_Z(z) = \underbrace{\int_{-\infty}^{\infty} f(z-y) \cdot f_Y\,dy}_{\text{convolution.}}$$

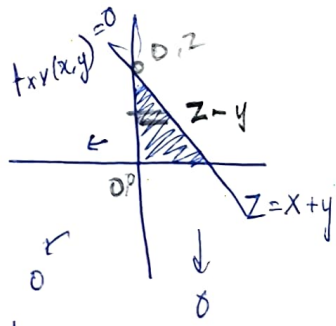
If X & Y are independent RV, then the PDF of their sum is the convolution of marginal PDF's.

<u>Special Case:</u>

$Z = X + Y$ $\quad f_{XY}(x,y)$ is non-zero

$f_{XY}(x,y) = 0 \quad X < 0$

$\qquad\qquad\qquad Y < 0$



$$F_Z(z) = P(X+Y \le z) = \int_0^z \int_0^{z-y} f_{XY}(x,y)\,dx\,dy.$$

Leibnitz for outer integral.

$$= \frac{d}{dz} \int_{x=0}^{z-z} f_{XY}(x,z)\,dx - \frac{d0}{dz} \int_0^{z-0} f(x,0)\,dx + \int_0^z \frac{\partial}{\partial z} \int_0^{z-y} f_{XY}(x,y)\,dx\,dy.$$

$$\boxed{f_Z(z) = \int_0^z f_{XY}(z-y, y)\,dy.}$$

If X & Y are independent exponential distributions of $Z = X+Y$, find $f_Z(z)$

Given

$$X \sim f(x,\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$ ← Exponential PDF

$$Y \sim f(y,\lambda) = \begin{cases} \lambda e^{-\lambda y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

X & Y are independent.

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

$$f_Z(z) = \int_0^z \lambda \cdot e^{-\lambda x} \cdot \lambda e^{-\lambda y}\, dy$$

$$= \int_0^z \lambda \cdot e^{-(z-y)} \cdot \lambda e^{-\lambda y}\, dy$$

$$= \int_0^z \lambda^2 \cdot e^{-\lambda(z-\cancel{y}+\cancel{y})}\, dy$$

$$= \int_0^z \lambda^2 e^{-\lambda z}\, dy$$

$$= \lambda^2 e^{-\lambda z} \int_0^z dy$$

$$= \lambda^2 e^{-\lambda z} \left[y^2 \right]_0^z$$

$$\boxed{f_Z(z) = z \cdot \lambda^2 e^{-\lambda z}, \quad z > 0}$$

Inference: Sum of independent exp RV's is Gamma distribution.

Gamma: $$f_X(x) = \begin{cases} \dfrac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$\lambda, \alpha > 0$

If $\alpha = 2$. $f_X(x) = \lambda^2 \cdot x \cdot e^{-\lambda x}$, $\Gamma(2) = 1$.

$\Gamma(1) = 1$.

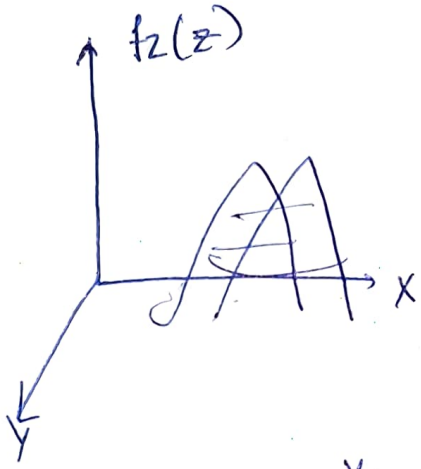
$\alpha = 1$ $f_X(x) = \lambda e^{-\lambda x}$ gamma $\rightarrow$ exponential.

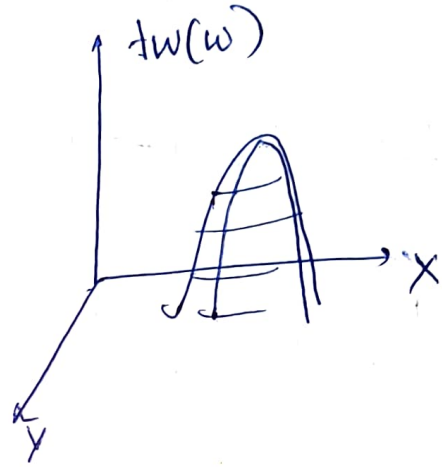
Two Function - Two RV

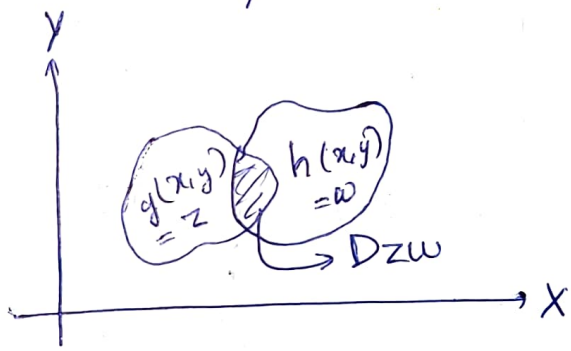
$Z = g(X, Y)$, $W = h(X, Y)$

Joint PDF of Z & W = $f_{ZW}(z, w) = ?$

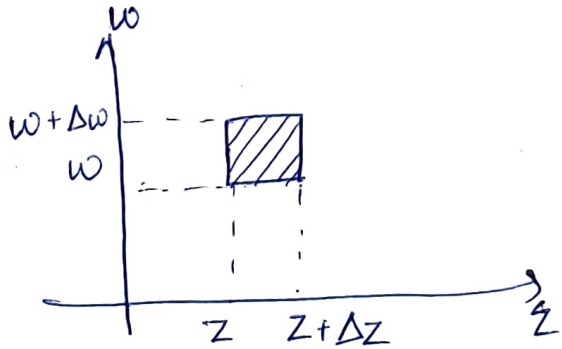
CDF of $(ZW) = F_{ZW}(z, w) = P[Z \le z, W \le w]$

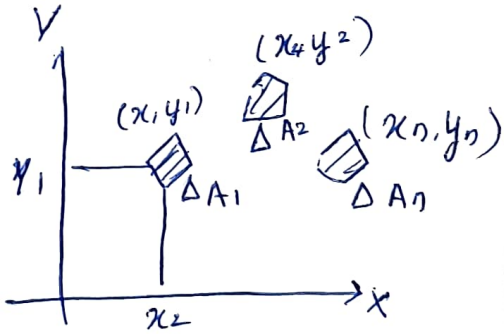






$$f_{ZW}(z, w) = \iint_{x, y \in D_{zw}} f_{XY}(x, y)\, dy\, dx .$$





$Z = g(X, Y)$ $\rightarrow$ $x_1 y_1 \ldots x_n y_n$

$W = h(X, Y)$

$$P[Z < Z \le Z + \Delta Z,\ W < W \le W + \Delta W] = \sum_i P(x_i < X \le x_i + \Delta x_i,\ y_i < Y \le y_i + \Delta y_i)$$

$$f_{ZW}(z, w) \cdot \Delta z \cdot \Delta w = \sum_i f_{XY}(x_i, y_i)\, \Delta A_i(x_i, y_i)$$

$$f_{ZW}(z, w) = \sum_i f_{XY}(x_i, y_i) \left| \frac{\Delta A_i(x_i, y_i)}{\Delta z\, \Delta w} \right|$$

$$\left| \frac{\Delta A_i(x_i, y_i)}{\Delta z\, \Delta w} \right| \rightarrow \text{Jacobian Matrix.}$$

$$|J| = \begin{vmatrix} \frac{\partial g(x,y)}{\partial x} & \frac{\partial g(x,y)}{\partial y} \\ \frac{\partial h(x,y)}{\partial x} & \frac{\partial h(x,y)}{\partial y} \end{vmatrix} = J(g, h) \text{ or } J(z, w).$$

$$\boxed{f_{ZW}(z, w) = \sum_i f_{XY}(x_i, y_i)\, |J(x_i, y_i)|}$$

or

$$\sum_i f_{XY}(x_i, y_i) \cdot \frac{1}{|J(w, z)|}$$

$$Z = g(X, Y) \qquad x_1 = g^{-1}(z, w)$$
$$W = h(X, Y) \qquad y_1 = h^{-1}(z, w)$$

① X & Y are independent Gaussian RV, with zero mean & equal variance.

$$X \,\&\, Y \sim N(0, \sigma^2)$$

$$Z = \sqrt{X^2+Y^2} \quad , \quad \theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

find $f_{Z\theta}(z, \theta) = ?$ and verify the independance of Z & θ.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{1}{2}\left(\frac{X-\mu_X}{\sigma}\right)^2}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{y^2}{2\sigma^2}}$$

joint PDF of X & Y

$$f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$$

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

$$f_{Z\theta}(z, \theta) = \sum_i f_{XY}(x, y) \cdot |J|$$

$$Z = \sqrt{X^2+Y^2} \qquad \theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

$$Y = X \tan\theta$$

$$Z = \sqrt{X^2 + X^2(\tan^2\theta)}$$

$$= \sqrt{X^2(1+\tan^2\theta)}$$

$$= X\sqrt{\frac{1}{\cos^2\theta}}$$

$$Z = \frac{X}{\cos\theta}$$

$$X = Z\cos\theta$$

$$Y = Z\sin\theta.$$

$$\left.\begin{array}{ll} X_1 = Z\cos\theta & Y_1 = Z\sin\theta \\ X_2 = -Z\cos\theta & Y_2 = -Z\sin\theta \end{array}\right\} \text{2 roots} \quad i = 2$$

$$|J| = \begin{vmatrix} \frac{\partial g_1}{\partial z} & \frac{\partial g_1}{\partial \theta} \\ \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{Z\cos\theta}{\partial z} & \frac{Z\cos\theta}{\partial \theta} \\ \frac{Z\sin\theta}{\partial z} & \frac{Z\sin\theta}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos & -Z\sin\theta \\ \sin\theta & Z\cos\theta \end{vmatrix}$$

$$= Z\cos^2\theta - (-Z\sin^2\theta \cdot Z)$$

$$= Z(\cos^2\theta + \sin^2\theta)$$

$$= Z$$

$$|J| = Z$$

$$f_{Z\Theta}(z,\theta) = \sum_i |J| . f_X(x_i\ y_i)$$

$$= Z\frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}} + Z\frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

$$= \cancel{2}Z \cdot \frac{1}{\cancel{2}\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

$$\boxed{f_{Z\Theta}(z,\theta) = \frac{Z}{\pi\sigma^2} e^{-\frac{Z^2}{2\sigma^2}}}$$

$$Z > 0$$

$$|\theta| < \frac{\pi}{2}$$

Marginal PDF of z, θ.

$$f_{z\theta}(z,\theta) = f_z(z)\, f_\theta(\theta)$$

$$f_\theta(\theta) = \int_0^\infty f_{z\theta}(z\theta).dz \quad , z>0.$$

$$= \int_0^\infty \frac{z}{\pi\sigma^2} e^{-z^2/2\sigma^2} . dz.$$

~~Let u~~ Let $u = z^2/2\sigma^2$ $\quad du = \frac{\cancel{2}z\,dz}{\cancel{2}\sigma^2}$ $\quad u=0 \quad z=0$

$u=\infty \quad z=\infty$

$$du = \frac{z\,dz}{\sigma^2}.$$

$$f_\theta(\theta) = \int_0^\infty \frac{1}{\pi} e^{-u} du.$$

$$= \frac{1}{\pi}\left[\frac{e^{-u}}{-1}\right]_0^\infty$$

$$\boxed{f_\theta(\theta) = \frac{1}{\pi}} \qquad |\theta| < \pi/2$$

uniform distribution: $f_x(x) = \frac{1}{b-a}$.

$$= \frac{1}{\pi/2 - (-\pi/2)}$$

$$= \frac{1}{\pi}$$

$|\theta| < \frac{\pi}{2}$ satisfies the ~~condition~~ PDF.

$$f_z(z) = \int_{-\pi/2}^{+\pi/2} f_{z\theta}(z,\theta)\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{z}{\pi\sigma^2} e^{-z^2/2\sigma^2} d\theta.$$

$$= \frac{z}{\pi\sigma^2} e^{-z^2/2\sigma^2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right]$$

Ref $v = \frac{4z^2}{2\sigma^2}$

$$\boxed{f_Z(z) = \frac{z}{\sigma^2} \cdot e^{-z^2/2\sigma^2}}$$ Rayleigh Distribution.

$z = \sqrt{X^2+Y^2}$ $\Rightarrow$ $f_Z(z)$ is Rayleigh, if X & Y are independent Gaussian RV, with zero mean & same variance.

$\theta = \tan^{-1}\left(\frac{Y}{X}\right)$ is an uniform dist.

(2) $f_X(x) = e^{-x}u(x)$

$f_Y(y) = e^{-y}u(y)$

If $u = \frac{X}{X+Y}$, $v = X+Y$ find $f_{UV}(u,v)$ & verify u and v are independent.

$$f_{UV}(u,\theta) = \sum_i |J| f_{XY}(x_i, y_i).$$

$$f_{XY} = e^{-x}u(x) \cdot e^{-y}u(y).$$

$$= e^{-(x+y)} \cdot 1.$$

$$\boxed{f_{XY} = e^{-(x+y)}} \quad X, Y \geq 0.$$

$u = \frac{X}{X+Y}$ $\qquad$ $V = X+Y.$

$X = V-Y$

$u(X+Y) = X.$ $\qquad$ $Y = V-X$

$u(X+V-X) = X.$ $\qquad$ $Y = V-X$

$\cancel{X = uv}$ $\qquad$ $Y = V-uV$

$\boxed{X = uv}$ $\qquad$ $\boxed{Y = v(1-u)}$

Roots = 1.

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial uv}{\partial u} & \frac{\partial uv}{\partial v} \\ \frac{\partial v(1-u)}{\partial u} & \frac{\partial v(1-u)}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix}$$

$$= v.$$

$$\boxed{f_{uv}(u,v) = e^{-(\underbrace{x+y}_{v})} \cdot v, \quad \begin{matrix} 0 \le u \le 1 \\ 0 \le v < \infty \end{matrix}}$$

$X = uv \Rightarrow X \ge 0 \qquad uv \ge 0$

$u \ge 0$

$v \ge 0$

$Y \ge 0, \quad v(1-u) \ge 0.$

$1-u \ge 0.$

$1 \ge u.$

$0 \le u \le 1$

$0 \le v \le \infty.$

$f_u(u) = 1 \qquad f_v(v) = e^{-v} \cdot v.$

$f_{uv}(u,v) = 1 \cdot e^{-v} \cdot v \longrightarrow$ independent.

Bivariate - Gaussian RV

$\downarrow$

2D.

univariate (1D) : $f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_x}\right)^2}$

$= [\text{Scalar}]_{1\times 1}$.

Multivariate (P)

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix} \qquad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix}$$

$$\sigma^2 = \text{covariance} = \Sigma = \begin{bmatrix} \sigma_{11} & \cdots & \cdots & \sigma_{1p} \\ \vdots & \sigma_{22} & & \\ & & \ddots & \\ \sigma_{p1} & \cdots & \cdots & \sigma_{pp} \end{bmatrix}$$

$Z = ax + by$

$W = cx + dy$.

$$\begin{bmatrix} Z \\ W \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

X_1 & X_2 $\rightarrow$ 2 Gaussian RV's

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \qquad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \qquad [C_X] = \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}$$

<u>Univariate</u>

$$f_X(x) = \left[\frac{1}{\sqrt{2\pi\sigma^2}}\right]\left[e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma_x^2}}\right]$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \underbrace{e^{-\frac{1}{2}\left[(x-\mu)(\sigma_x^2)^{-1}(x-\mu)\right]}}_{e^{-\frac{1}{2}\left([(x-\mu)[\Sigma]^{-1}[x-\mu]^T\right)}}$$

Bivariate Joint Gaussian PDF:

$$f_{X_1X_2}(x_1,x_2) = \frac{1}{2\pi(\sigma_1^2\sigma_2^2(1-\rho_{12}^2))^{1/2}} \exp\left\{\frac{-1}{2(1-\rho_{12}^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho_{12}\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]\right\}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{12}^2}} \exp\left\{\frac{-1}{2(1-\rho_{12}^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho_{12}\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]\right\}$$

Linear Transformation of Gaussian variables

$$[Y] = [T][X]$$

$$[\bar{Y}] = [T][\bar{X}]$$

$$[Y-\bar{Y}] = [T][X-\bar{X}]$$

$$[C_Y] = E[(Y-\bar{Y})(Y-\bar{Y})^T]$$

$$= E[T(X-\bar{X})(X-\bar{X})^T T^T]$$

$$= T\,E[(X-\bar{X})(X-\bar{X})^T]\,T^T$$

$$= [T][C_X][T]^T$$

$$\boxed{[C_Y] = [T][C_X][T]^T}$$

$[C_Y]$ ↓ Covariance of Y

$[C_X]$ ↳ Covariance of X

Central Limit Theorem (CLT)

When $X_1, \ldots X_N$ are identical & independent RV

IID ⇒ $\bar{X}_i = \bar{X}$ → same mean

$\sigma_{X_i}^2 = \sigma_X^2$ → same variance

$$\boxed{\begin{aligned} &E[Y_N] = N\bar{X} \\ &E[(Y_N)^2 - (N\bar{X})^2] = \sigma_Y^2 = N\sigma_X^2 \end{aligned}}$$

$E[X_1] = E[X_2] = \ldots \quad E[X_n] = \mu$

$Var(X_1) = \ldots \quad Var[X_n] = \sigma^2$

Sample Sum $(S_n) = \sum_{i=1}^{n} X_i$

$S_n = X_1 + X_2 + \ldots X_n$

$E[S_n] = n\mu$

$Var(S_n) = n\sigma^2$

$S_n \sim N(n\mu, n\sigma^2)$.

Sample Mean $\bar{X} = \frac{S_n}{n}$

$$E[\bar{X}] = \frac{1}{n}[E[X_1] + \ldots E[X_n]]$$

$$= \frac{1}{n}[\mu + \ldots \mu n]$$

$$= \frac{n\mu}{n}$$

$$\boxed{E[\bar{X}] = \mu}$$

$$Var[\bar{X}] = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n^2}[Var(X_1) + \ldots Var(X_n)]$$

$$= \frac{1}{n^2}[n\sigma^2]$$

$$\boxed{Var[\bar{X}] = \frac{\sigma^2}{n} \quad SD \text{ of } \bar{X} = \frac{\sigma}{\sqrt{n}}}$$

$\bar{X} \rightarrow$ Gaussian RV $\quad \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

Standard Normal Distribution : $\boxed{Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}}$

Tutorial: Linear Transformation of Gaussian RV & CLT

① X_1 & X_2 zero mean, $\sigma_{X_1}^2 = 4$, $\sigma_{X_2}^2 = 9$. Gaussian RV
$X_1 \sim N(0,4)$, $X_2 \sim N(0,9)$.
Linearly transformed to

$$Y_1 = X_1 - 2X_2 \qquad Y_2 = 3X_1 + 4X_2$$

find mean, variance, covariance & corr coeff of Y_1 & Y_2.

Ans: Transformation matrix $[T]$

$$[T] = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$

$$[C_X] = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix}$$

$$[\bar{Y}] = [\bar{X}][T] = 0.$$

$$[C_Y] = [T][C_X][T]^T$$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & -66 \\ -66 & 252 \end{bmatrix}$$

$\sigma_{Y_1}^2 = 28$ $\qquad$ $\sigma_{Y_1 Y_2} = C_{Y_1 Y_2} = -66.$

$\sigma_{Y_2}^2 = 252$

$$\rho_{Y_1 Y_2} = \frac{-66}{\sqrt{\sigma_{Y_1}^2 \sigma_{Y_2}^2}} = -0.7857.$$

$$-1 \le \rho \le 1.$$

$$f_{Y_1Y_2}(y_1;y_2) = \frac{1}{2\pi\sqrt{1-\rho_{12}^2}} \exp\left\{ \frac{-1}{2(1-\rho_{12}^2)} \begin{bmatrix} y_1-0 & y_2-0 \end{bmatrix} [\Sigma]^{-1} \begin{bmatrix} y_1-0 \\ y_2-0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 28 & -66 \\ -66 & 252 \end{bmatrix}^{-1} = \begin{bmatrix} 0.0933 & 0.0244 \\ 0.0244 & 0.0104 \end{bmatrix}$$

$$f_{Y_1Y_2}(y_1\ y_2) = \frac{1}{2\pi\sqrt{1-(-0.785))^2}} \exp\left\{ \frac{-1}{2(1-(-0.7857)^2} \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 0.0933 \\ 0.2244 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\}$$

② X_1, X_2 are Gaussian RV's

$$\bar{X} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad C_X = \begin{bmatrix} 5 & -2/\sqrt{5} \\ -2/\sqrt{5} & 4 \end{bmatrix} \quad T = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$

Find $[\bar{Y}]$, $[C_Y]$, $\rho_{Y_1Y_2}$.

Ans: $$[\bar{Y}] = T[\bar{X}] = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}$$

$$[C_Y] = [T][C_X][T]^T$$

$$= \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 5 & -2/\sqrt{5} \\ -2/\sqrt{5} & 4 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5.1056 & 3.3820 \\ 3.3810 & 4.3556 \end{bmatrix}$$

$$\rho_{Y_1Y_2} = \frac{3.3820}{\sqrt{5.1056 \times 4.3556}} = 0.7172$$

③ (CLT) Life time of a certain bulb is RV.
mean life time = 1200 hrs
std. dev = 250 hrs
Using CLT, find the prob that the avg lifetime of 60 bulbs exceeds 1250 hrs.

Ans: X_i - life time of bulb

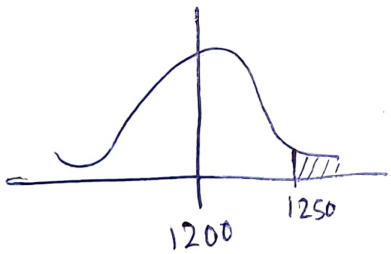
$X_1, X_2 \ldots X_{60}$ $\qquad n = 60$

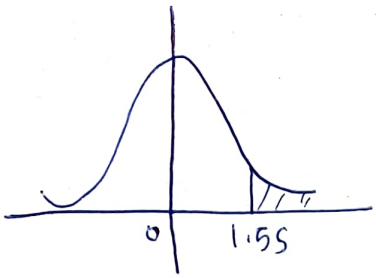
$$\bar{X} = \frac{X_1 + X_2 \ldots}{60}$$

$E[X_i] = 1200$ $\qquad SD(\bar{X}) = 250$

$P(\bar{X} > 1250 \text{ hrs})$ $\qquad \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{1250 - 1200}{250/\sqrt{60}} = 1.55$$





$0.5 - P(0 < Z < 1.55)$

$0.5 - 0.4394$

0.0606

$P(\bar{X} > 1250) = 0.0606$

6% chance is possible.

④ CLT. X_i poisson with $\lambda = 2$ $i = 1, 2 \ldots n$

$S_n = X_1 + X_2 \ldots X_n$, $n = 75$.

$P(120 \leq S_n \leq 160)$.

S_n normal dist. $N(n\mu, n\sigma^2)$

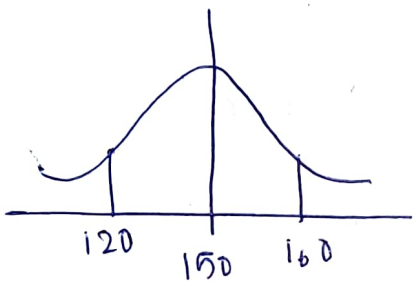
poisson $\mu = \lambda$

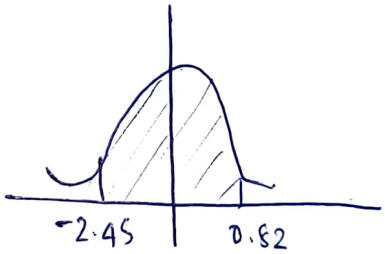
$\sigma^2 = \lambda$

$\mu_{S_n} = n\lambda = 75 \times 2 = 150$

$\sigma_{S_n}^2 = n\lambda = 150$

SD $S_n = \sqrt{n\lambda} = 12.25$.



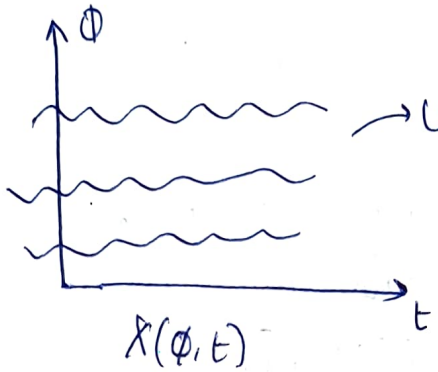


$$Z = \frac{X - \mu_{S_n}}{\sigma_{S_n}} = \frac{120 - 150}{12.25} = -2.45$$

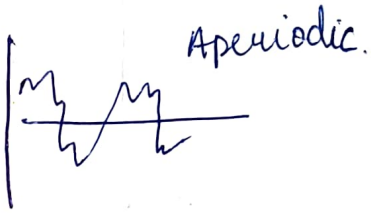
$$Z = \frac{160 - 150}{12.25} = 0.82$$

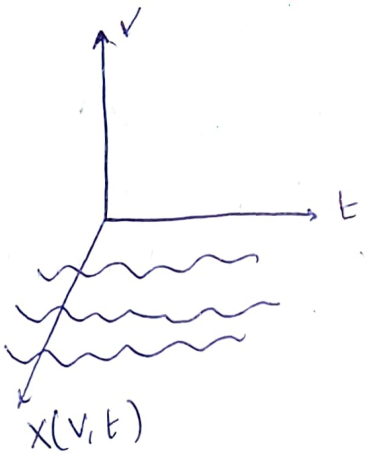
$$P(-2.45 < Z < 0) + P(0 < Z < 0.82) = 0.4929 + 0.2882$$
$$= 0.8811.$$

$P(120 \leq S_n \leq 160) = 0.8811$.



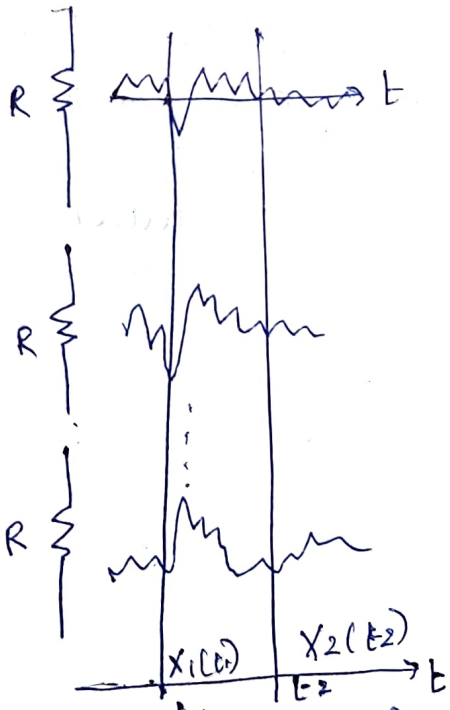
→ collection of diff realization is called stochastic process





Random process

$f_X(x) \rightarrow f_X(x, t)$



"Time Avg" $= \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cdot dt$

$X_1(t_1) \rightarrow$ Random variable

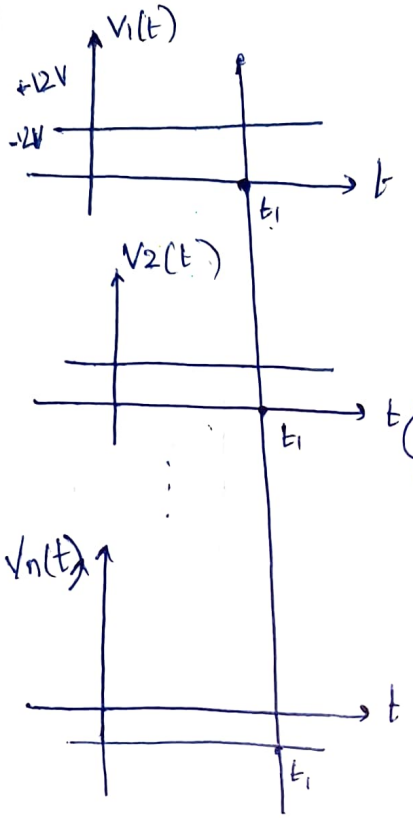
freeze the time axis at t_1 & find avg → ensemble avg.

I order characterization

$f_X(x,t)$ - RV. X is fnctn of time

II order charac.

$f_{X_1X_2}(x_1, x_2, t_1, t_2)$ - RV X_1 & X_2 are fnctn of time



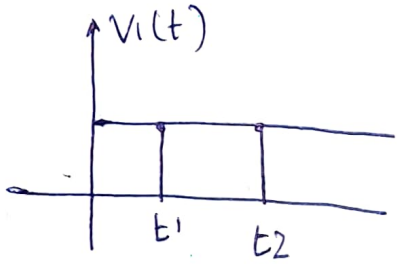
$E[V_1(t)] \neq 0$. $E[V_1(t)] \neq 0$ $E[V_n(t)] \neq$

time avg

$E[V(t_1)] = 0$.

Ensemble avg

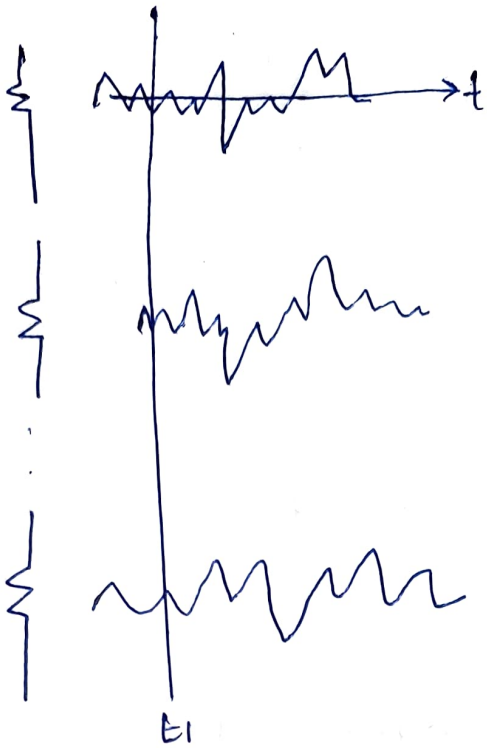
If time avg $\neq$ ensemble avg, then what we have done is called non-ergodic process



$V_1(t_1) = V_1(t_2)$

If outcome of any exp ~~d~~ remains constant with time

→ stationary process



$E[V_1(t)] = 0$

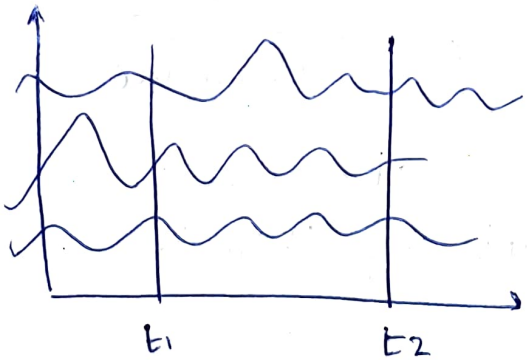
$E[V_2(t)] = 0$

$E[V_n(t)] = 0$

$E[V(t_1)] = 0$

If ensemble avg = time avg
→ ergodic process

$V_1(t_1) \neq V_2(t_2)$
non stationary process



$X_i(t_i)$ → Random variable

$X_1(t_1)$ $\quad$ $X_2(t_2)$

$f_{X_1}(x_1, t_1)$ $\quad$ $f_{X_2}(x_2, t_2)$.

I order charac. $f_X(x, t)$

II order charac, $f_{X_1 X_2}(x_1, x_2, t_1, t_2)$

$\vdots$

n^{th} order charac, $f_{X_1 X_2 \ldots X_n}(x_1, x_2 \ldots x_n, t_1, t_2 \ldots t_n)$

From I order charact.

$$E[X(t)] = \int_x x \cdot f_X(x, t) \cdot dx$$

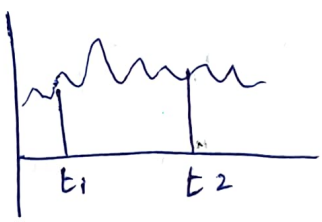
$$= \varphi(t)$$

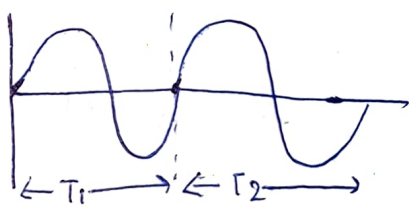
$$= \mu_X(t) \rightarrow \text{func. of time}$$

From II order charac.

$$E[X(t_1) X(t_2)] = \int_{x_2} \int_{x_1} x_1 x_2 f_{X_1 X_2}(x_1, x_2, t_1, t_2) dx_1 dx_2$$

$$= \varphi(t_1, t_2) \rightarrow \text{auto correlation}$$





$$\underbrace{Cov[X_1(t), X_2(t)]}_{\text{auto covariance}} = \underbrace{E[X_1(t_1) X_2(t_2)]}_{\text{auto correlation}} - \mu_{X_1}(t_1) \mu_{X_2}(t_2)$$

$\rightarrow R_{XX}(t_1 t_2)$

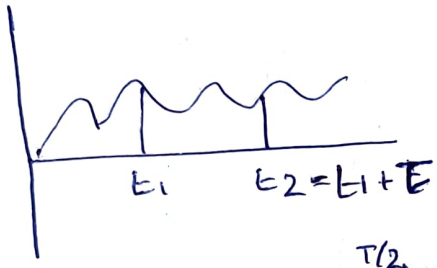
$$C_{XX}(t_1 t_2) = R_{XX}(t_1 t_2) - \mu_X(t_1) \mu_X(t_2)$$

* Notes

1. Mean of the process is function of time $\mu_x(t)$
2. Auto correlation function of time $R_{xx}(t_1 t_2)$

$$\text{Energy } (E) = \lim_{T\to\infty} \int_{-\infty}^{\infty} |X(t)|^2 \, dt$$

$$\text{Power } (P) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |X(t)|^2 dt$$

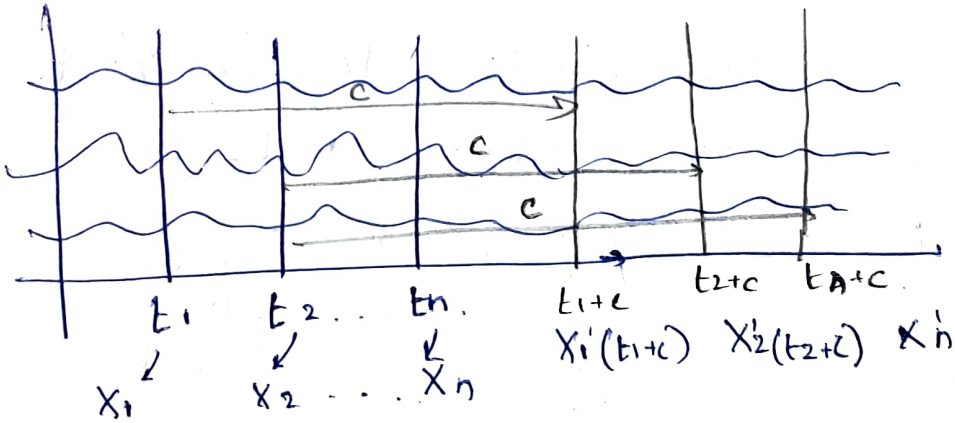


$$\overline{X^2} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t_1)\, X(t_1+\tau)\, dt = R_{xx}(\tau)$$

$$\tau = 0, \quad \frac{1}{T} \int X(t_1)^2 \rightarrow \text{avg power.}$$

$$\sqrt{R_{xx}(0)} \rightarrow \text{RMS}$$

Stationary Behaviour of Stochastic Process



$$f_{X_1 X_2 \ldots X_n}(x_1, x_2 \ldots x_n, t_1, t_2 \ldots t_n)$$

$$f_{X_1' X_2' \ldots X_n'}(x_1, x_2 \ldots x_n, t_1+c, t_2+c \ldots t_n+c).$$

If it is a stationary process,

$$f_{X_1 X_2 \ldots X_n}(x_1, x_2 \ldots x_n, t_1, t_2 \ldots t_n) \equiv f_{X_1' X_2' \ldots X_n'}(x_1, x_2 \ldots x_n, t_1+c, t_2+c, t_n+c)$$

Strict sense stationary process (SSS).

SSS - true for any t_i, n, c → any time shift

t_i → any time instant

n → any no of variable

1st order charac of SSS $(n=1)$

$$f_{X_1}(x_1, t) = f_{X_1'}(x_1, t+c)$$

$c = -t$ $\quad f_{X_1}(x_1, t) = f_{X_1'}(x_1)$

$$E[X_1'(t)] = \underbrace{\int_{x_1} x_1 f_{X_1'}(x_1)\, dx_1}_{\mu_{X_1'}}$$

If the process is SSS, its 1st order realization is constant

$$\mu_X = \text{constant}.$$

2nd order charac of SSS $(n=2)$

$$f_{X_1 X_2}(x_1, x_2, t_1, t_2) = f_{X_1' X_2'}(x_1, x_2, t_1+c, t_2+c)$$

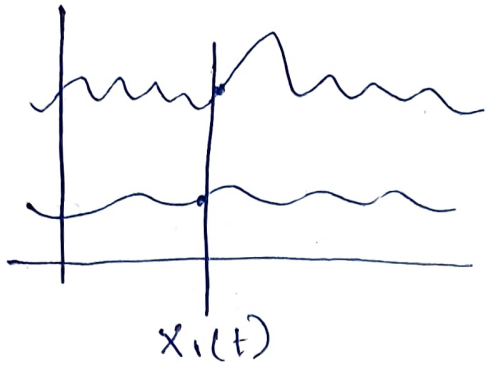
$c = -t_2$ $\quad = f_{X_1' X_2'}(x_1, x_2, t_1 - t_2)$

Auto correlation funct $(t_1 - t_2)$

or

$(t_2 - t_1)$

Wide Sense Stationary Process (WSS)



If 1) $E[X(t)] = \mu_X \rightarrow$ constant

2) $E[X(t_1)X(t_2)] = R_{XX}(t_1 - t_2)$

$X(t)$ is a WSS.

If SSS $\Rightarrow$ WSS

SSS $\not\Leftarrow$ WSS reverse not true.

<u>Special case</u>:

For Gaussian Random Process,

SSS $\Leftrightarrow$ WSS.

① $X(t) = A\sin(\omega t + \theta)$

$A \ni \mu_A, \sigma_A^2$ $\theta \sim U[-\pi, +\pi]$

A & θ are independent, verify $X(t)$ is WSS.

Ans: $E[X(t)] = E[A\sin(\omega t + \theta)]$

$= E[A]\,E[\sin(\omega t + \theta)]$ Because A & θ are independent.

$= \mu_A \int_\theta f_\theta(\theta) \cdot \sin(\omega t + \theta)\, d\theta$

$= \mu_A \int_{-\pi}^{\pi} \frac{1}{2\pi} \sin(\omega t + \theta)\, d\theta.$

$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$= \frac{\mu_A}{2\pi}\left[-\cos(\omega t + \theta)\right]_{-\pi}^{\pi}$

$= \frac{-\mu_A}{2\pi}\left[\cos(\omega t + \pi) - \cos(\omega t - \pi)\right]$

$= 0$

$E[X(t)]$ is a constant.

$\sin A \sin B = \frac{1}{2}[\cos(A+B) - \cos(A+B)]$

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$= E[A^2 \sin(\omega t_1 + \theta)\sin(\omega t_2 + \theta)]$$

$$= E[A^2]\, E[\sin(\omega t_1 + \theta)\sin(\omega t_2 + \theta)]$$

$$= E[A^2] \cdot \frac{1}{2} E[\cos(\omega(t_1 - t_2)) - \cos(\omega(t_1 + t_2) + 2\theta)]$$

$$= E[A^2] \cdot \frac{1}{2}\{E[\cos(\omega(t - t_2))] - E[\cos(\omega(t_1 + t_2) + 2\theta)]\}$$

$$= E[A^2] \cdot \frac{1}{2}\{\cos(\omega(t_1 - t_2)) - 0\}$$

$$= \frac{E[A^2]}{2}\cos[\omega(t_1 - t_2)]$$

$\therefore X(t)$ is a WSS process.

(2) $X(t) = e^{-Yt}$, $Y \sim U[0, b]$

Verify whether $X(t)$ is WSS.

Ans: $E[X(t)] = E[e^{-Yt}]$

$$= \int_0^b \frac{1}{b} e^{-Yt}\, dy$$

$$= \frac{1}{b}\left[\frac{-e^{-Yt}}{t}\right]_0^b$$

$$= \frac{1}{bt}\left[-e^{-bt} - (-e^0)\right]$$

$$= \frac{1}{bt}\left[1 - e^{-bt}\right]$$

$E[X(t)]$ is not a constant

$$E[e^{-Yt_1} e^{-Yt_2}] = \int_0^b \frac{1}{b} e^{-y(t_1+t_2)} dy$$

$$= \frac{1}{b}\left[\frac{-e^{-y(t_1+t_2)}}{(t_1+t_2)}\right]_0^b$$

$$= \frac{1}{b(t_1+t_2)}\left[1 - e^{-b(t_1+t_2)}\right]$$

$X(t)$ is not a WSS process

Gaussian RV Process

$$WSS \Leftrightarrow SSS$$

① $X(t) = Y\cos\omega t + Z\sin\omega t$

$Y, Z \sim N(0, \sigma^2)$ independent

Verify $X(t)$ is SSS in 2nd order.

Ans: $$f_{YZ}(y,z) = \frac{1}{2\pi\sigma_Y\sigma_Z\sqrt{1-\rho_{YZ}^2}} \cdot \exp\left\{\frac{-1}{2(1-\rho_{YZ}^2)}\left[\frac{y\cdot\mu_Y}{\sigma_Y^2} - \frac{2\rho_{YZ}\cdot yz}{\sigma_Z\sigma_Y} + \left(\frac{z\cdot\mu_Z}{\sigma_Z^2}\right)\right]\right\}$$

$$E[X(t)] = E[Y\cos\omega t + Z\sin\omega t]$$

$$= E[Y\cos\omega t] + E[Z\sin\omega t]$$

$$= \cos\omega t\, E[Y] + \sin\omega t\, E[Z]$$

$$= 0$$

~~cov~~

$$C_{XX}(t_1, t_2) = R_{XX}(t_1 t_2) - R_X(t_1 t_2) - \underset{\downarrow \atop 0}{\mu_Y \mu_Z}$$

$$\rho_{YZ} = \frac{C_{YZ}(t_1 t_2)}{\sigma_Y \sigma_Z}$$

$$= \frac{C_{YZ}(t_1 t_2)}{\sigma^2} \qquad \sigma_Y = \sigma_Z = \sigma$$

$$= \frac{E[X(t_1) X(t_2)]}{\sigma^2}$$

$$= \frac{1}{\sigma^2} E[(Y\cos\omega t_1 + Z\cos\omega t_1)(Y\cos\omega t_2 + Z\cos\omega t_2)]$$

$$= \frac{1}{\sigma^2} E\{Y^2\cos\omega t_1 \cos\omega t_2 + ZY\cos\omega t_1 \sin\omega t_2 + ZY\sin\omega t_1 \cos\omega t_2 + Z^2 \sin\omega t_1 \sin\omega t_2\}$$

$$= \frac{1}{\sigma^2} E[Y^2\cos\omega t_1 * \cos\omega t_2] + \overset{0}{E[YZ]}\cos\omega t_1 \sin\omega t_2 + \overset{0}{E[ZY]}\sin\omega t_1 \cos\omega t_2 + E[Z^2]\sin\omega t_1 \sin\omega t_2$$

$$= \frac{1}{\sigma^2} E[Y^2]\cos\omega t_1 * \cos\omega t_2 + E[Z^2]\sin\omega t_1 \sin\omega t_2$$

$$\sigma_Y^2 = E[Y^2] - (\mu_Y)^2$$

$$E[Y^2] = \sigma_Y^2 + \underset{\downarrow \atop 0}{(\mu_Y)^2}$$

$$E[Y^2] = \sigma_Y^2 \qquad E[Z^2] = \sigma_Z^2$$

$$= \frac{1}{\sigma^2}[\sigma^2\cos\omega t_1 \cos\omega t_2 + \sigma^2 \sin\omega t_1 \sin\omega t_2]$$

$$= \cos\omega t_1 \cos\omega t_2 + \sin\omega t_1 \sin\omega t_2$$

$$= \cos\omega(t_1 - t_2)$$

$$f_{YZ}(y,z) = \frac{1}{2\pi\sigma^2\sqrt{1-\cos\omega(t_1-t_2)^2}} \exp\left\{ -\frac{1}{2[1-(\cos\omega(t_1-t_2))]^2} \left[\frac{y^2}{\sigma^2} - 2\cos\omega(t_1-t_2)yz + \frac{z^2}{\sigma^2} \right] \right.$$

t_1 & $t_2 \rightarrow c$

$t_1 - t_2 \Rightarrow (t_1+c)-(t_2+c)$

$t_1 + c - t_2 - c$

$t_1 - t_2$

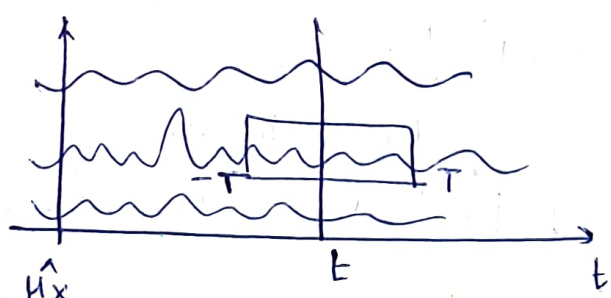
$X(t)$ is an 'SSS' process

Ergodic Process

Time Avg = Ensemble Avg.

$$\frac{1}{2T}\int_{-T}^{T} X(t)\,dt = E[X(t)]$$

$X(t)$ is "mean ergodic process"



$\hat{\mu}_X$

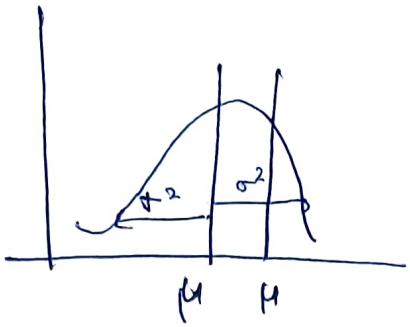
Time Avg $= \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$

$E[X(t)] = \mu_X(t) = \int_{-\infty}^{\infty} x \cdot f_X(x,t)\,dx.$

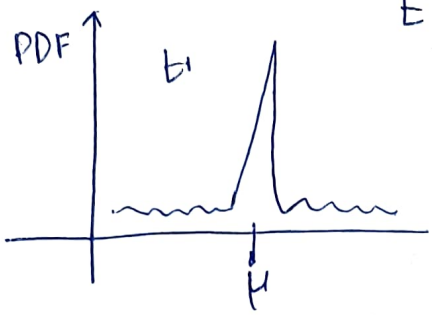
If these 2 are equal, it is a constant.

$\hat{\mu}_X = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt = \mu_X = \text{const} \rightarrow 1^{st}$ condition

$\Rightarrow \hat{\mu}_X \Rightarrow RV$

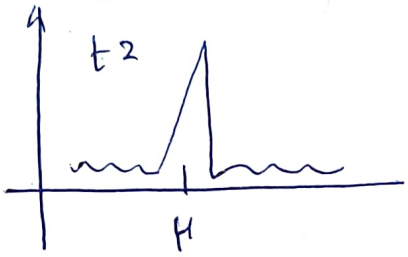


$$E[x] = \int_x x\,\delta(x-\mu)\,dx$$



$$\hat{\mu_x} = \mu_x = E[X(t)]$$

variance = 0



<u>Condition</u>

Tem Avg $(\hat{\mu_x}) = E[X(t)] = \mu_x(t) = \mu_x$

↳ RV

$E[\hat{\mu_x}] = \text{const}$.

To maintain const mean

The PDF of $\hat{\mu_x}$ should not have any duration

$\Rightarrow \sigma^2_{\hat{\mu_x}} = 0$.

① $X(t)$ stationary ergodic process with no periodic component.

$R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$, find mean & variance of $X(t)$.

Ans: Property 4: $\lim_{|\tau| \to \infty} R_{XX}(\tau) = \mu_X^2$.

$$\mu_X^2 = \lim_{|\tau| \to \infty} \underbrace{25 + \frac{4}{1+6\tau^2}}_{\tau \to 0}$$

$\mu_X^2 = 25$

$\mu_X = 5$

$\sigma_X^2 = E[X^2] - (E(X))^2$

$\tau = 0$, $E[X^2] = R_{XX}(0)$

$E[X^2] = 25 + \frac{4}{1+6.0} = 29$

$\sigma_X^2 = 29 - 25 = 4$

$\sigma_X^2(t) = 4$.

② $X(t)$ - WSS, zero mean

$R_{XX}(\tau) = 1 - \frac{|\tau|}{T}$, find mean & variance of time avg of $X(t)$ over $(0, T)$.

Is $X(t)$ mean ergodic?

$$\overline{X_T} = \frac{1}{T} \int_0^T X(t).dt$$

$E[\overline{X_T}] = E[X(t)] = 0$.

$$\frac{2}{T}\int_0^T \left(1-\frac{|\tau|}{T}\right)\left(1-\frac{|\tau|}{T}\right) d\tau$$

$$\frac{2}{T}\int_0^T \left(1-\frac{|\tau|}{T}\right)^2 d\tau$$

$$\frac{2}{T}\left[\tau + \frac{\tau^3}{3T^2} - \frac{2\tau^2}{2T}\right]_0^T$$

$$\frac{2}{T}\left[T + \frac{T^{\cancel{3}}}{3\cancel{T^2}} - \frac{2T^{\cancel{2}}}{\cancel{2}\cancel{T}}\right]$$

$$\frac{2}{T}\left[T + \frac{T}{3} - T\right]$$

$$\cancel{2} + \frac{2}{3} - \cancel{2} = \frac{2}{3}$$

$$Var[\bar{X}_T] = \frac{2}{3}$$

$$\lim_{T\to\infty}\left(Var(\bar{X}_T)\right) \neq 0.$$

$X(t)$ is not a mean ergodic process.

Ergodic in Correlation

$$Y(t) = E[X(t+\lambda)X(t)] = R_{XX}(\lambda)$$

(3) $X(t) = 10\cos(100t+\theta)$ $\quad \theta \sim U[-\pi, \pi]$

Verify $X(t)$ is ergodic in correlation

$X(t) = A\cos(\omega t+\theta)$ $\quad R_{XX}(\tau) = \frac{A^2}{2}\cos(\omega(t_1-t_2))$

$E[X(t_1)X(t_2)]$

$X(t) = 10\cos(100t+\theta)$

$R_{XX}(\tau) = \frac{10^2}{2}\cos(100(t_1-t_2)) = 50\cos(100\tau)$

$$\overline{Z_T} = \frac{1}{2T}\int_{-T}^{T} X(t+\tau)X(t)\,.\,dt = E[X(t+\tau)X(t)]$$

$$\overline{Z_T} = \frac{1}{2T}\int_{-T}^{T} 10\cos(100(t+\tau)+\theta)\cdot 10\cos(100t+\theta)\,dt$$

$$= \frac{100}{2T}\cdot\frac{1}{2}\int_{-T}^{T}\cos[100t+100\tau+\theta+100t+\theta] + \cos[\cancel{100t}+100\tau+\cancel{\theta}-\cancel{100t}-\cancel{\theta}]\,dt$$

$$= \frac{25}{T}\int_{-T}^{T}\cancel{\cos[200t+100\tau+2\theta]} + \cos[100\tau]\,dt$$

$$= \frac{25}{T}\int_{-T}^{T}\cos(100\tau)\,dt$$

$$= \frac{25}{T}\left[\cos(100\tau)t\right]_{-T}^{T}$$

$$= \frac{25}{T}\left[T\cos(100\tau) + T\cos(100\tau)\right]$$

$$= \frac{25}{\cancel{T}}\times 2\cancel{T}\cos(100\tau)$$

$$= 50\cos(100\tau)$$

$$\overline{Z_T} = R_{xx}(\tau)$$

$x(t)$ is ergodic

④ $X(t)$ - Ergodic, $R_{XX}(\tau) = 18 + \frac{2}{6+\tau^2}(1+4\cos(12\tau))$

Find i) mean ii) $X(t)$ have a periodic component, iii) a power of $X(t)$

Ans: Property : $\lim_{\tau \to \infty} R_{XX}(\tau) = \mu_X^2$

$$\lim_{\tau \to \infty} R_{XX}(\tau) = 18 + \frac{2}{6+\tau^2}(1+4\cos(12\tau))$$

$$= 18 + 0 = 18$$

$$= \mu_X^2$$

$$\mu_X = \sqrt{18}$$

$$E[X(t)] = \bar{X} = 18 \neq 0$$

$X(t)$ is ergodic with no periodic component

Property : $\lim_{\tau \to 0} R_{XX}(\tau) = R_{XX}(0) = E[X^2(t)]$

$$= 18 + \frac{2}{6+0}(1+4\cos 0)$$

$$= 18 + \frac{2}{6} \times 5$$

$$= 18 + \frac{10}{6}$$

$$= \frac{118}{6} = \frac{59}{3}$$

Cross - Correlation

$X(t)$ $\qquad\qquad$ $Y(t)$

$\downarrow$

$X(t_1) \ldots X(t_n)$ $\qquad\qquad$ $Y(t_1') \ldots Y(t_n')$

$$\boxed{R_{XY}(t_1\ t_2) = E[X(t_1)Y(t_2)]}$$

1. $R_{XY}(\tau) = R_{YX}^{*}(-\tau) \Rightarrow$ conjugate symmetry

   $R_{YX}(\tau) = R_{XY}(-\tau)$.

2. $R_{XY}(\tau) = 0 \Rightarrow$ X & Y are orthogonal

3. $R_{XY}(\tau) \xrightarrow{\text{Fourier Transform}} X(\omega)Y^{*}(\omega)$

4. $|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0) \cdot R_{YY}(0)}$

5. $R_{XY}(\tau) \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$

$$\Rightarrow \underbrace{\sqrt{R_{XX}(0)R_{YY}(0)}}_{\text{Geometric Mean}} \leq \underbrace{\frac{1}{2}[R_{XX}(0) + R_{YY}(0)]}_{\text{arithmetic mean}}$$

$a = \{4, 9\}$ $\qquad$ $GM(a) = \sqrt{4 \times 9} = 6$

$AM(a) = \frac{1}{2}[4+9] = 6.5$

Cross covariance

$$C_{XY}(t_1, t_2) = E[\{X(t_1) - \mu_X(t_1)\}\{Y(t_2) - \mu_Y(t_2)\}]$$
$$= R_{XY}(t_1, t_2) - \mu_X(t_1)\mu_Y(t_2)$$

If X & Y are orthogonal

$$C_{XY}(t_1\ t_2) = -\mu_X(t_1)\mu_Y(t_2)$$

① $X(t) = \cos(\omega t + \theta)$

$Y(t) = \sin(\omega t + \theta)$

$\theta \sim U[-\pi, \pi]$

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi \le \theta \le \pi \\ 0 & \text{otherwise} \end{cases}$$

find cross-covariance of $X(t)$ and $Y(t)$.

Ans: $E[X(t)] = E[\cos(\omega t + \theta)] = \int_{-\pi}^{\pi} \frac{1}{2\pi} \cos(\omega t + \theta)\, d\theta$

$= 0$.

$\mu_X(t) = 0$

$E[Y(t)] = E[\sin(\omega t + \theta)] = 0$

$\mu_Y(t) = 0$.

$C_{XY}(t_1\ t_2) = R_{XY}(t_1\ t_2) - \mu_X \mu_Y$

$= R_{XY}(t_1\ t_2) - 0$

$= E[\underbrace{\cos(\omega t_1 + \theta)}_{\cos A} \underbrace{\sin(\omega t_2 + \theta)}_{\sin B}]$

$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

$E[\frac{1}{2}[\sin(\omega(t_1+t_2) + 2\theta) - \sin(\omega(t_1 - t_2)]$

$C_{XY}(t_1\ t_2) = R_{XY}(t_1\ t_2)$

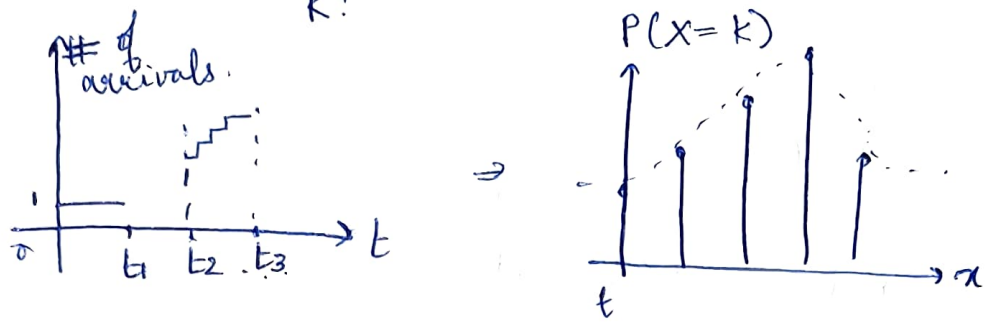
$= -\frac{1}{2} \sin(\omega(t_1 - t_2))$

Poisson Random Process

$X \sim P(\lambda)$

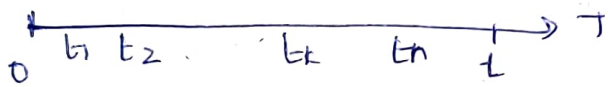
$\mu_X = \lambda \qquad \sigma_X^2 = \lambda$

$$P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0 \ldots n$$



no of arrivals $(n) \propto T$.

$$\boxed{X(t) \sim P(\lambda t)}$$



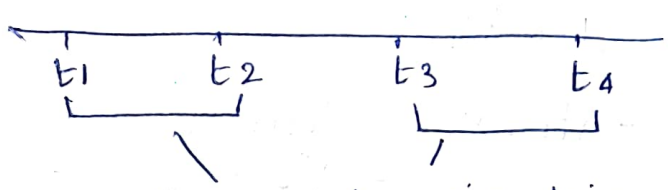
n = # of arrivals from $(0, t)$

$\boxed{X(t) = n(0,t)}$ ⇒ counting process.

Condition to be poisson process

1. $P[n(0,t) = k] \sim e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!}, \quad k = 0, \ldots n.$

2.



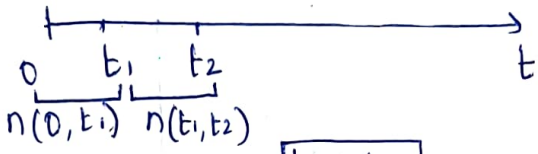
Non-overlapping time instant.

(t_1, t_2) & (t_3, t_4) are independent

If 1 & 2 satisfied, then $X(t)$ is Poisson Random process.

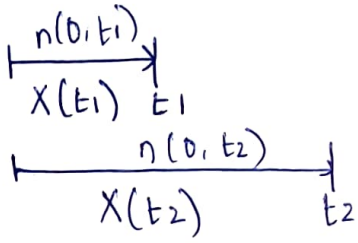
$E[X(t)] = E[n(0,t)] = \lambda t \neq WSS$

Autocorrelation



$\boxed{t_2 > t_1}$

$0 \leq x_1 \leq t \qquad t_1 < x_2 \leq t_2$



$\therefore n(t_1,t_2) = X(t_2) - X(t_1)$

$E[XY] = E[X]E[Y]$ X & Y are independent.

$E[n(0,t_1)\,n(t_1,t_2)] = E[n(0,t_1)]\,E[n(t_1,t_2)]$

$E[X(t_1)[X(t_2)-X(t_1)]] = \lambda t_1 \cdot \lambda(t_2-t_1)$

$E[X(t_1)X(t_2) - (X(t_1))^2] = \lambda t_1 \cdot \lambda(t_2-t_1)$

$\underbrace{E[X(t_1)X(t_2)]}_{R_{XX}(t_1 t_2)} - E[X^2(t_1)] = \lambda t_1 \; \lambda(t_2-t_1)$

$R_{XX}(t_1 t_2) - E[X^2(t_1)] = \lambda t_1 \lambda(t_2-t_1)$

$$E[X^2(t)] = \sigma_X^2 + (\mu_X)^2$$
$$= \lambda t + (\lambda t)^2$$
$$= \lambda t + \lambda^2 t^2$$

$$R_{XX}(t_1 t_2) - \lambda t_1 + \lambda^2 t_1^2 = \lambda t_1 + \lambda(t_2-t_1)$$
$$= \lambda^2 t_1 t_2 - \lambda^2 t_1^2$$

Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t}\, dt$$

inverse FT $\quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{+j\omega t}\, dt.$

$$f(t) \xleftrightarrow[\mathcal{F}^{-1}]{FT} F(\omega)$$

$$f(-t) = F(-\omega)$$

$$f^*(t) = F^*(-\omega)$$

$$f^*(-t) = F^*(\omega)$$

$$f(t) * g(t) = F(\omega) \times G(\omega)$$

$$x(t)\cos\omega_0 t = \frac{1}{2}\left[X(\omega+\omega_0) + X(\omega-\omega_0)\right]$$

DSBSC

$$\delta(t) \xrightarrow{FT} 1$$

$$f(t) \xrightarrow{FT} F(\omega)$$

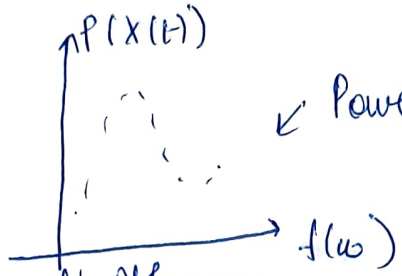
$$F(t) = 2\pi f(-\omega)$$

$$1 \rightarrow 2\pi\,\delta(\omega).$$

$$\frac{dx(t)}{dt} \xrightarrow{FT} j\omega X(\omega)$$

$R_{xx}(\tau) \rightarrow \tau = 0$

$E[X^2(t)] = R_{xx}(0)$.



Power spectrum Density = FT of autocorrelation.

Parseval's Theorem

$$\int_{-\infty}^{\infty} |f(t)|^2 dt \longleftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega.$$

energy $\xrightarrow{FT}$ Power.