VIT UNIVERSITY

SENSE

ECE 2005 -Probability Theory and Random Processes

Digital Assignment -1

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***Due Date to submit in the VTOP:* 26-02-2020**

1. If $X_1, X_2, X_3, \ldots X_n$ are Poisson variates with parameter $\lambda = 2$, use the central limit theorem to estimate $P(120 \leq S_n \leq 160)$, where $S_n = X_1 + X_2 + X_3 + \cdots + X_n$ and n=75.

2 Two random variables X and Y have means $\bar{X} = 1$ and $\bar{Y} = 2$, variances $\sigma_X^2 = 4$ and $\sigma_Y^2 = 1$, and a correlation coefficient $\rho_{XY} = 0.4$. New random variables W and V are defined by

$V = -X + 2Y$; $W = X + 3Y$

Find

a. The means
b. The variances
c. The correlations

The correlation coefficient ρ_{VW} of V and W

3 (a). A random variable X has a probability density

$$f_X(x) = \begin{cases} \left(\frac{3}{32}\right)(-x^2 + 8x - 12), & 2 \leq x \leq 6 \\ 0, & elsewhere \end{cases}$$

Find the following moments: (a) m_0 (b) m_1 (c) m_2 and (d) μ_2

(b). Find the marginal densities of X and Y using the joint density

$$f_{X,Y}(x, y) = 2\, u(x)\, u(y)\, exp\left[-\left(4y + \frac{x}{2}\right)\right]$$

4 A lab screen for the HIV virus. A person that carries the virus is screened positive in only 95% of the cases. A person who does not carry the virus is screened positive in 1% of the cases. Given that 0.5% of the population carries the virus, what is the probability that a person who has been screened positive is actually a carrier?

5 Two random variables X and Y have density function

$$f_{X,Y}(x, y) = \begin{cases} \frac{xy}{9} & ; \quad 0 < x < 2 \ and \ 0 < y < 3 \\ 0 & ; \ elsewhere \end{cases}$$

a). Show that X and Y are uncorrelated

b). Show that X and Y are also statistically independent

6 Three uncorrelated random variables X_1, X_2 and X_3 have means $\bar{X}_1 = 1$, $\bar{X}_2 = -3$ and $\bar{X}_3 = 1.5$. They have second moments $\overline{X^2}_1 = 2.5$, $\overline{X^2}_2 = 1$ and $\overline{X^2}_3 = 3.5$. Let $Y = X_1 - 2X_2 + 3X_3$ be a new random variable and find, a). The mean value b). The variance of Y

7 The power (in milliwatts) returned to a radar from a certain class of aircraft has the probability density function

$$f_P(p) = \frac{1}{10} e^{-\frac{p}{10}} u(p)$$

Suppose a given aircraft belongs to this class but is known to not produce a power larger than 15mW.

a) Find the probability density function of P conditional on $P \leq 15mW$.

b) Find the conditional mean value of P

8 Three balls are drawn at random without replacement from box containing 2 white, 3 red and 4 black balls. If X denotes the number of white balls and Y denotes the number of red balls drawn, find the probability distribution of (X,Y). Also find the following

1). P[X≤ 1] ; 2). P[Y≤ 2] ; 3). P[X≤ 1, Y≤ 2] ; 4). P[X≤1 / Y≤ 2] ; 5). P[X+Y≤ 3]

9 a) Let $Y_1 = \frac{X_1 - a}{b}, Y_2 = \frac{X_2 - c}{d}$ Show that $\rho_{Y_1,Y_2} = \rho_{X_1,X_2}$. i.e shifting and scaling does not have influence on the correlation coefficient. (5+5)

b) Let $X_i, \mathrm{i} = 1, 2, 3.....50$ are independent Random variables, each having Poisson distribution with mean (λ)=0.03. Let $Z = \sum_{i=1}^{50} X_i$. Evaluate $P[Z \geq 3]$ using central limit theorem. Compare the answer with its exact probability.

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2\. a)

$$f_X(x) = \begin{cases} \frac{3}{32}(-x^2+8x-12) & 2 \le x \le 6 \\ 0 & \text{otherwise} \end{cases}$$

$$m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\, dx$$

i) $m_0 = \int_{-\infty}^{\infty} x^0 f_X(x)\, dx = 1$

ii) $m_1 = \int_{2}^{6} x \left(\frac{3}{32}\right)(-x^2+8x-12)\, dx$

$$= \frac{3}{32}\int_{2}^{6} -x^3 + 8x^2 - 12x\, dx$$

$$= \frac{3}{32}\left[-\frac{x^4}{4} + \frac{8x^3}{3} - 6x^2\right]_2^6$$

$$= \frac{3}{32}\left[-324 + 576 - 216 + 4 - 21.33 + 24\right]$$

$$\boxed{m_1 \approx 4}$$

iii) $m_2 = \int_{-\infty}^{\infty} x^2 \left(\frac{3}{32}\right)(-x^2+8x-12)\, dx$

$$= \frac{3}{32}\int_{2}^{6} -x^4 + 8x^3 - 12x^2\, dx$$

$$= \frac{3}{32}\left[-\frac{x^5}{5} + 2x^4 - 4x^3\right]_2^6$$

$$= \frac{3}{32}\left[-1555.2 + 2592 - 864 + 6.4 - 32 + 32\right]$$

$$\boxed{m_2 = 16.8}$$

iv) $\mu_2 = m_2 - m_1^2$

$$= 16.8 - 4^2$$

$$\boxed{\mu_2 = 0.8}$$

b) $f_{V,Y}(x,y) = 2u(x)u(y)\exp\left[-\left(4y + \frac{x}{2}\right)\right]$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= 2u(x)\int_{0}^{\infty} e^{-\left(4y + \frac{x}{2}\right)}\,dy$$

$$= 2u(x)e^{-x/2}\int_{0}^{\infty} e^{-4y}\,dy$$

$$= \frac{2u(x)e^{-x/2}}{-4}\left[e^{-4y}\right]_0^{\infty} \quad -1$$

$$\boxed{f_X(x) = \frac{e^{-x/2}}{2}u(x)}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$= 2u(y)e^{-4y}\int_{0}^{\infty} e^{-x/2}\,dx$$

$$= -4u(y)e^{-4y}\left[e^{-x/2}\right]_0^{\infty} \quad -1$$

$$\boxed{f_Y(y) = 4u(y)e^{-4y}}$$

5. $f_{XY}(x,y) = \begin{cases} xy/9 & 0<x<2 \quad 0<y<3 \\ 0 & \text{else where} \end{cases}$

b) To show X & Y are statistically independent

$f_{XY}(x,y) = f_X(x)\, f_Y(y)$

$f_X(x) = \int_{-\infty}^{\infty} \frac{xy}{9}\, dy = \frac{x}{9}\int_0^3 y\, dy = \frac{x}{9}\left[\frac{y^2}{2}\right]_0^3 = \frac{x}{2}$

$f_Y(y) = \int_{-\infty}^{\infty} \frac{xy}{9}\, dx = \frac{y}{9}\int_0^2 x\, dx = \frac{y}{9}\left[\frac{x^2}{2}\right]_0^2 = \frac{2y}{9}$

$f_{XY}(x,y) = f_X(x)\cdot f_Y(y) = \frac{x}{2}\cdot\frac{2y}{9} = \frac{xy}{9}$ //

$\therefore$ X & Y are independent.

a) To show that X & Y are uncorrelated

$E[XY] = m_{11} = \int_{y=0}^{3}\int_{x=0}^{2} xy\cdot\frac{xy}{9}\, dx\, dy = \frac{1}{9}\int_{y=0}^{3} y^2\left(\int_{x=0}^{2} x^2\, dx\right) dy$

$= \frac{1}{9}\int_{y=0}^{3} y^2\times\frac{8}{3}\, dy = \frac{8}{27}\left[\frac{y^3}{3}\right]_0^3 = \frac{8}{3}$

$E[X] = m_{10} = \cancel{\int_{y=0}^{3}\int_{x=0}^{2}} = \int_{x=0}^{2} x\, f_X(x)\, dx = \int_0^2 x\cdot\frac{x}{2}\, dx$

$= \frac{1}{6}\times\left[x^3\right]_0^2 = \frac{8}{6}$ //

$E[Y] = m_{01} = \int_{y=0}^{3} y\, f_Y(y)\, dy = \int_0^3 \frac{2y^2}{9} = \frac{2}{27}\left[y^3\right]_0^3 = 2$

$E[XY] = E[X]\,E[Y] = \frac{8}{6}\times 2 = \frac{8}{3}$

Correlation coeff $= E[XY] - E[X]E[Y] = 0$

$\therefore$ X & Y are uncorrelated

8. 3 balls are drawn at random

2W 3R 4B

X → white balls $X : [0, 1, 2]$

Y → Red balls $Y : [0, 1, 2, 3]$

→ Probability distribution

let $K = \frac{1}{504}$

X \ Y	0	1	2	3
0	24K	36K	24K	6K
1	24K	24K	12K	$\times$
2	8K	6K	$\times$	$\times$

$P(0,0)$ = All black = $\frac{4}{9} \times \frac{3}{8} \times \frac{2}{7} = 24/504$

$P(0,1) = 2B\ 1R = \frac{4}{9} \times \frac{3}{8} \times \frac{3}{7} = 36/504$

$P(0,2) = 1B, 2R = \frac{4}{9} \times \frac{3}{8} \times 2/7 = 24/504$

$P(0,3) = 3R = \frac{3}{9} \times \frac{2}{8} \times \frac{1}{7} = 6/504$

$P(1,0) = 1W\ 2B = \frac{2}{9} \times \frac{4}{8} \times \frac{3}{7} = 24/504$

$P(1,1) = 1W\ 1R\ 1B = \frac{2}{9} \times \frac{3}{8} \times \frac{4}{7} = 24/504$

$P(1,2) = 1W\ 2R1 = \frac{2}{9} \times \frac{3}{8} \times \frac{2}{7} = 12/504$

$P(2,0) = 2W\ 1B = \frac{2}{9} \times \frac{1}{8} \times 4/7 = 8/504$

$P(2,1) = 2W\ 1R = \frac{2}{9} \times \frac{1}{8} \times \frac{3}{7} = 6/504$

i) $P[X \leq 1] = P(0,0) + P(0,1) + P(0,2) + P(0,3)$
$+ P(1,0) + P(1,1) + P(1,2)$
$= (24 + 36 + 24 + 6 + 24 + 24 + 12)K$
$= 150/504$

ii) $P(Y \leq 2) = P(0,0) + P(0,1) + P(0,2) + P(1,0) + P(1,1) + P(1,2) + P(2,0) + P(2,1)$

$= (24 + 36 + 24 + 24 + 24 + 12 + 8 + 6)K$

$= \frac{158}{504}$

iii) $P[X \leq 1, Y \leq 2] = P(0,0) + P(0,1) + P(0,2) + P(1,0) + P(1,1) + P(1,2)$

$= (24 + 36 + 24 + 24 + 24 + 12)K$

$= 144/504$

iv) $P\left(\frac{X \leq 1}{Y \leq 2}\right) = \frac{P(X \leq 1 \cap Y \leq 2)}{P(Y \leq 2)}$

$= \frac{144/504}{158/504} = \frac{144}{158}$

v) $P[X+Y \leq 3] = P(0,0) + P(0,1) + P(0,2) + P(0,3) + P(1,0) + P(1,1) + P(1,2) + P(2,0) + P(2,1)$

$= 1$ //

7. $f_P(p) = \frac{1}{10} e^{-P/10} u(p)$

a) $P[P \leq 15] = \int_{-\infty}^{15} f_P(p)\, dp = \int_0^{15} \frac{e^{-P/10}}{10} dp = \frac{-10}{10}\left[e^{-P/10}\right]_0^{15}$

$= 1 - e^{-1.5} \simeq 0.777 \text{ mW}$

~~Conditional mean value of~~

$f_P(P/P \leq 15\text{mW}) = \frac{f_P(P)}{\int_{-\infty}^{15} f_P(P)\, dx}$ for $p \leq 15$, $p > 0$

$= \frac{e^{-P/10}}{1 - e^{-1.5}}\left[u(P) - u(p-15)\right]$

b) $E[P/P \leq 15] = \frac{\int_{-\infty}^{15} p f_P(P)\, dp}{\int_{-\infty}^{15} f_P(P)\, dp} = \frac{\int_0^{15} Pe^{-P/10}}{10[1 - e^{-1.5}]}$

$= 5.69 \text{ mW}$

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$$m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx$$

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$= \frac{e^{-P/10}}{1-e^{-1.5}}\left[u(P) - u(p-15)\right]$

b) $E[P/P \leq 15] = \frac{\int_{-\infty}^{15} p f_P(P)\,dp}{\int_{-\infty}^{15} f_P(P)\,dp} = \frac{\int_0^{15} Pe^{-P/10}}{10[1-e^{-1.5}]}$

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