Joint Distribution.

Probability distribution fxn will be,

$$F_X(x) = P\{X \le x\}.$$

$$F_Y(y) = P\{Y \le y)$$

$$F_{XY}(x,y) = P\{X \le x \,;\, Y \le y\}.$$

For discrete random variables X and Y,

$$F_{XY}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m)\, u(x - x_n)\, u(y - y_n).$$

Joint density fxn.

$$f_{XY}(x,y) = \frac{d^2 F_{XY}(x,y)}{dx\, dy}$$

for discrete variables

$$f_{XY}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m)\, \delta(x - x_n)\, \delta(y - y_m)$$

Properties:

(i) $f_{XY}(x,y) \ge 0$

(ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1 \rightarrow$ Valid

(iii) $F_{XY}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(\varepsilon_1, \varepsilon_2)\, d\varepsilon_1, d\varepsilon_2$

(iv) $f_X(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f_{XY}(\varepsilon_1, \varepsilon_2)\, d\varepsilon_1, d\varepsilon_2$

$$f_Y(y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{XY}(\xi_1, \xi_2)\, d\xi_1\, d\xi_2$$

(v) $P[x_1 < X \leq x_2 ; y_1 < Y \leq y_2] =$

$$\int_{y_1}^{y_2} \int_{x_1}^{x_2} f_{XY}(x,y)\, dx\, dy.$$

(vi) $$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$$

Conditional distribution & Density

$$F_X(x/B) = P[X \leq x/B].$$

$$F_X(x/B) = \frac{P[X \leq x \cap B]}{P[B]}.$$

Point Conditioning.

$$F_X(x/Y=y_k) = \sum_{i=1}^{N} \frac{P(x_i, y_k)}{P(y_k)}\, u(x-x_i).$$

$$f_X(x/Y=y_k) = \sum_{i=1}^{N} \frac{P(x_i, y_k)}{P(y_k)}\, \delta(x-x_i).$$

Interval Conditioning

$$f_X(x / y_a < Y \leq y_b) = \frac{\int_{y_a}^{y_b} f_{XY}(x,y)\,dy}{\int_{y_a}^{y_b}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy}$$

Statistical Independence

$$P(A \cap B) = P(A) \cdot P(B).$$

for two events A, B defined

$$A\{X \leq x\} ; B = \{Y \leq y\}.$$

$$P[X \leq x, Y \leq y] = P(X \leq x) \cdot P(Y \leq y)$$

$$F_{XY}(x,y) = F_X(x) \cdot F_Y(y).$$

(Joint) marginal of x marginal of y

Conditional distribution for independ. variables.

$$F_X(x / Y \leq y) = \frac{F_{XY}(x,y)}{F_Y(y)}$$

Joint moments about the origin.

$$E(x^n y^k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{XY}(x,y)\,dx\,dy = m_{nk}$$

so, $m_{11} = E[XY] \rightarrow$ Cor b/w X and Y.

Joint Central Moments

$$\mu_{nk} = E\left[(X-\bar{X})^n (Y-\bar{Y})^k\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{x})^n (y-\bar{y})^k f_{XY}(x,y)\,dx\,dy$$

Here,

$$\left.\begin{aligned} \mu_{20} &= E\left[(X-\bar{X})^2\right] = \sigma_X^2 \\ \mu_{02} &= E\left[(Y-\bar{Y})^2\right] = \sigma_Y^2 \end{aligned}\right\} \text{- Second Order Central moments}$$

$$C_{XY} = \mu_{11} = E\left[(X-\bar{X})(Y-\bar{Y})\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{x})(y-\bar{y}) f_{XY}(x,y)\,dx\,dy$$

C_{XY} ↓ covariant of X and Y

Normalised Second order moment

$$\rho = \mu_{11} / \sqrt{\mu_{20}\mu_{02}} = C_{XY}/\sigma_X\sigma_Y$$

$$\rho = E\left[\frac{(X-\bar{X})}{\sigma_X} \cdot \frac{(Y-\bar{Y})}{\sigma_Y}\right] ; \quad -1 \le \rho \le 1$$

Here, ρ refers to correlation coefficient.

for N random variables.

$$\mu_{n_1 n_2 \ldots n_N} = E\left[(X_1-\bar{X}_1)^{n_1}(X_2-\bar{X}_2)^{n_2} \ldots (X_N-\bar{X}_N)^{n_N}\right]$$

$$= \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} (x_1-\bar{x}_1)^{n_1}(x_2-\bar{x}_2)^{n_2} \ldots f_{X_1X_2\ldots X_N}(x_1 \ldots x_N)\,dx_1\,dx_2 \ldots dx_N$$

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Joint characteristic fxn.

$$\Phi_{XY}(\omega_1,\omega_2) = E\left[e^{j\omega_1 X + j\omega_2 Y}\right]; \ \omega_1, \omega_2 \to \text{real no.}$$

$$\Phi_{XY}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, e^{j\omega_1 x + j\omega_2 y}\, dx\, dy.$$

from the inverse fourier transform, we get,

$$f_{XY}(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{XY}(\omega_1,\omega_2)\, e^{-j\omega_1 x - j\omega_2 y}\, d\omega_1\, d\omega_2$$

$y\left[\frac{x^2}{2}\right]_0$ Date ___ Page ___

6. (i) Sol^n

$$f_{xy}(x,y) = \begin{cases} kx, & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^1 \int_0^y f_{xy}(x,y)\,dx\,dy = 1$$

$$\int_0^1 \int_0^y kx\,dx\,dy \Rightarrow k\int_0^1 \left[\frac{x^2}{2}\right]_0^y = 1$$

$$k\left[\frac{y^3}{6}\right]_0^1 = 1$$

$$\Rightarrow \frac{k}{6} = 1 \Rightarrow k = 6.$$

So; $f_{xy}(x,6) = \begin{cases} 6x & ; 0<x<y<1 \\ 0 & ; \text{otherwise.} \end{cases}$

to get marginal Pdf;

$$f(x) = \int_x^1 f(x,y)\,dy = \int_x^1 6x\,dy = 6x\,[y]_x^1$$

$$f_X(x) = 6x(1-x); \quad 0<x<1$$

similarly for $f_Y(y) = \int_0^y f(x,y)\,dx$

$$= \int_0^y 6x\,dx$$

$$= \frac{6x^2}{2}\Big|_0^y$$

$$f_Y(y) = 3y^2 \; ; 0<y<1$$

$$E(X) = \int_0^1 x\, f(x)\, dx$$

$$= \int_0^1 x\, 6(x - x^3)\, dx.$$

$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$E(X) = 6\left[\frac{1}{3} - \frac{1}{4}\right].$$

$$= 1/2.$$

$$E(y) = \int_0^1 y\, f_Y(y)\, dy$$

$$= \int_0^1 y\, 3y^2\, dy$$

$$= 3\left[y^2/4\right]_0^1 = 3/4.$$

$$E(XY) = \int_0^1 \int_0^3 xy\, f(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^y 6x^2 y\, dx\, dy.$$

$$E[XY] = 6 \int_0^1 y \left[x^3/3\right]_0^y dy.$$

$$= \frac{6}{3} \int_0^1 y^4\, dy$$

$$= 2/5$$

$$\text{cov}(X,Y) = E(XY) - E(X)E(Y)$$
$$= \frac{2}{5} - \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)$$

$$= \frac{2}{5} - \frac{3}{8}$$
$$= \frac{1}{40}$$

$$\text{cov}(X,Y) = 1/40$$

Q(2) sol^n $Z = ax + (1-a)y \;;\; 0 \le a \le 1$

$$\text{var}(Z) = \text{var}[ax + (1-a)Y]$$

$$\text{var}(Z) = a^2 \text{var}(X) + (1-a)^2 \text{var}(Y)$$
$$\text{var}(Z) = a^2\sigma_1^2 + (1-a^2)\sigma_2^2$$

when variance is minimum,
$$\frac{d}{da}\text{var}(Z) = 0$$

$$\frac{d}{da}\left[a^2\sigma_1^2 + (1-a)^2\sigma_2^2\right] = 0$$

$$2a\sigma_1^2 - 2(1-a)\sigma_2^2 = 0$$
$$a\sigma_1^2 - a_2^2 + a\sigma_2^2 = 0$$
$$a(\sigma_1^2 + \sigma_2^2) = \sigma_2^2$$
$$a = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

$$\text{var}^{*}(z) = a^{2}(\sigma_1^2 - \sigma_2^2) + \sigma_2^2$$
$$= \frac{\sigma_2^4}{(\sigma_1^2+\sigma_2^2)^2}(\sigma_1^2 - \sigma_2^2) + \sigma_2^2$$

$$\text{var}^{*}(z) = \frac{\sigma_1^4\sigma_2^2 + 3\sigma_1^2\sigma_2^4}{\sigma_1^4 + 2\sigma_1^2\sigma_2^2 + \sigma_2^4}.$$

Q(4.) $\phi_x(-it) = M_x(t)$.

$z = ax + by + c \quad ; c \neq 0.$

$M_z(t) = M_{ax+by+c}(t) = M_{ax}(t) \cdot M_{by}(t) \cdot e^{ct}$

$\Rightarrow \exp\left[(a\mu_1 + b\mu_2)t + \frac{1}{2}(a^2\sigma_1^2 + b^2\sigma_2^2)t^2\right].$

mean & var $= 0, 1$

$M_z(t) = e^{1/2(a^2+b^2)t^2}$

$\phi_z(-it) = e^{\frac{1}{2}(a^2+b^2)(-it)^2}$

$= e^{-\frac{1}{2}(a^2+b^2)t^2}$ — (1).

z is also a random variable;

$\phi_z(-it) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$

$E(z) = 0$; $\text{var}(z) = a^2 + b^2$.

$z \sim N(0, a^2+b^2)$ — (2)

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(5) $f_Z(z) = \int_{z=-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx.$

$F_Z(z) = \int_{y=-\infty}^{z} f_Z(y)\, dy.$

Case 1: $0 \le z < 1$

$f_Z(z) = \int_0^z 1 \cdot 1\, dx = z.$

$F_Z(z) = \int_0^z y\, dy = z^2/2$

Case 2: $1 \le z \le 2.$

$f_Z(z) = \int_{z-1}^{1} 1 \cdot 1\, dx = 2-z.$

$F_Z(z) = \int_0^1 y\, dy + \int_1^z (2-y)\, dy.$

$F_Z(z) = 2z - \frac{z^2}{2} - 1$

Q. 4.6.3-

Random variables X and Y have respective density function

$$f_X(X) = \frac{1}{a}[u(x) - u(x-a)].$$

$$f_Y(Y) = b u(y) e^{-by}.$$

where $a>0$ & $b>0$. Find & sketch the density fxn $W = X+Y$ if X and Y are statistically indep.

Sol^n

$$f_W(w) = \int_{-\infty}^{\infty} f_Y(y) f_X(w-y)\,dy.$$

$$= \int_0^{\infty} b e^{-by} \frac{1}{a}[u(w-y) - u(w-y-a)]\,dy.$$

$$= \int_0^{w} \frac{b}{a} e^{-by}\,dy - \int_0^{w-a} \frac{b}{a} e^{-by}\,dy.$$

$= 0$ for $w \le a$.

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Case (i): $0 < w \le a$.

$$f_W(w) = \int_0^w \frac{b}{a} e^{-by}\,dy = \frac{1}{a}\left[1 - e^{-bw}\right].$$

Case (ii) $a < w$

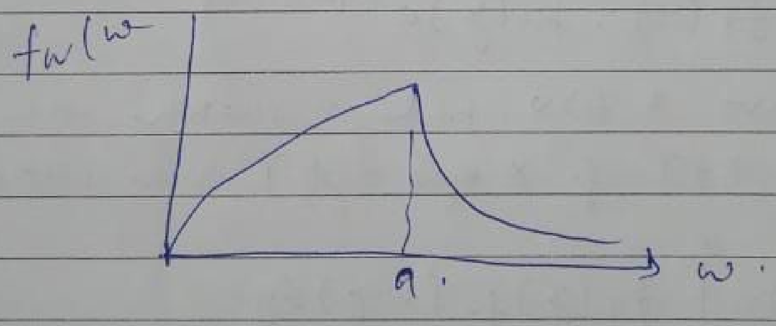
$$f_W(w) = \frac{1}{a}\left[1 - e^{-bw}\right] - \frac{b}{a}\int_0^{w-a} e^{-by}\,dy.$$

$$= \frac{e^{-bw}}{a}\left[e^{ba} - 1\right].$$

$$f_W(w) = 0 \quad ; \quad w \le 0.$$

$$= \frac{1}{a} e^{-bw}\left(e^{ba} - 1\right), \quad a < w$$

$$= \frac{1}{a} e^{-bw}\left(e^{ba} - 1\right), \quad a < w$$



Q. 4. 6. 4. Random variables X and Y have respectively density functions

$$f_X(x) = 0.1\delta(x-1) + 0.2\delta(x-2) + 0.4\delta(x-3) + 0.3\delta(x-4).$$

$$f_Y(y) = 0.4\delta(y-5) + 0.5\delta(y-6) + 0.1\delta(y-7).$$

Find and sketch the density function $W = X + Y$ if X and Y are independent.

Sol^n $f_W(w) = \int_{-\infty}^{\infty} f_Y(y)\, f_X(w-y)\, dy$

$= \int_{-\infty}^{\infty} [0.4\delta(y-5) + 0.5\delta(y-6) + 0.1\delta(y-7)]$

$[0.1\delta(w-y-1) + 0.2\delta(w-y-2) + 0.4\delta(w-y-3)$

$+ 0.3\delta(w-y-4)]\, dy.$

each integral is of form

$K \int_{-\infty}^{\infty} \delta(y-y_K)\, \delta(w-y-x_m)\, dy$ where

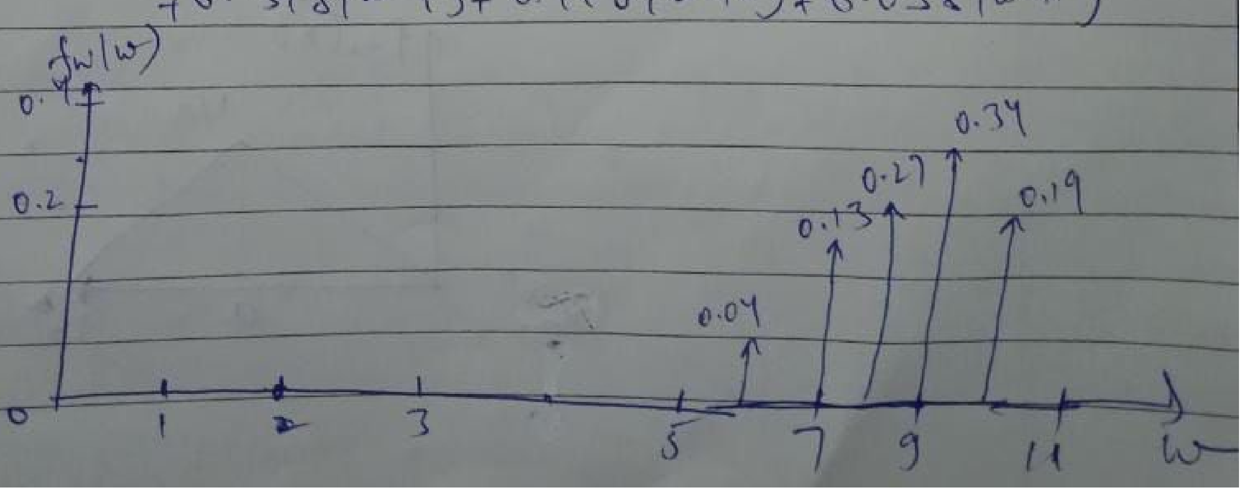
y_K and x_m are numbers & K is a constant

$w = x_m + y_K$ & the integral equals

$K\delta(w-y_K-x_m)$. Hence,

$f_W(w) = 0.04\delta(w-6) + 0.08\delta(w-7) + 0.16\delta(w-8)$
$+ 0.12\delta(w-9) + 0.05\delta(w-7) + 0.10\delta(w-8)$
$+ 0.20\delta(w-9) + 0.15\delta(w-10) + 0.01\delta(w-8)$
$+ 0.02\delta(w-9) + 0.04\delta(w-10) + 0.03\delta(w-11).$

$f_W(w) = 0.04\delta(w-6) + 0.13\delta(w-7) + 0.27\delta(w-8)$
$+ 0.34\delta(w-9) + 0.19\delta(w-10) + 0.03\delta(w-11)$



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Q-4.6-7. Three statistically independent random variables X_1, X_2 and X_3 all have the same density fxn,

$$f_{X_i}(x_i) = \frac{1}{a}[u(x_i) - u(x_i - a)] \quad i = 1, 2, 3.$$

Find and sketch the density fxn of $Y = X_1 + X_2 + X_3$ if $a > 0$ is constant.

sol(4/n)

$$W_1 = X_1 + X_2.$$

then $$f_{W_1}(w) = \int_{-\infty}^{\infty} f_{X_1}(x) \cdot f_{X_2}(w_1 - x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{a^2}[u(x) - u(x-a)][u(w_1 - x) - u(w_1 - x - a)]\,dx$$

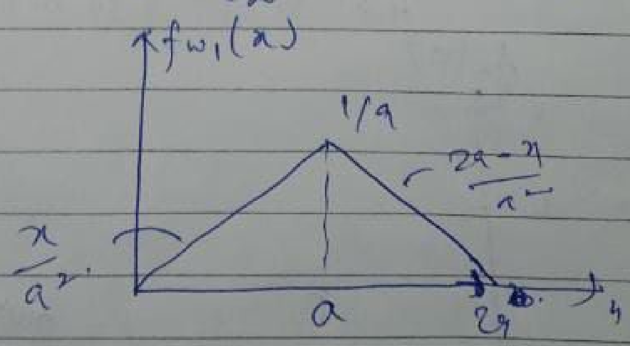
$$f_{W_1}(w_1) = 0, \quad w_1 \le 0$$

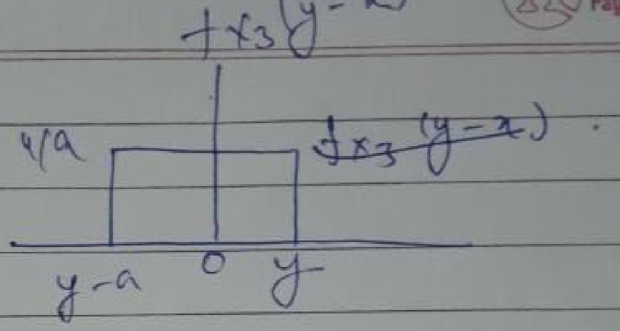
$$= w_1/a^2, \quad 0 < w_1 \le a.$$

$$= (2a - w_1)/a^2, \quad a < w_1 \le 2a.$$

$$= 0, \quad 2a < w_1.$$

let $Y = W_1 + X_3$, $f_Y(y) = \int_{-\infty}^{\infty} f_{W_1}(x)\, f_{X_3}(y - x)\,dx$





$f_Y(y) = 0, \; y < 0.$

$f(Y)(y) = \int_0^y \frac{x}{a^3}\,dx = y^2/2a^3, \quad 0 < y \le a.$

$f_Y(y) = \int_{y-a}^{a} \frac{x}{a^3}\,dx + \int_a^y \frac{(2a-x)}{a^3}\,dx.$

$$= \frac{-2y^2 + 6ay - 3a^2}{2a^3}$$

$f_Y(y) = \int_{y-a}^{2a} \frac{(2a-x)}{a^3}\,dx$

$$= \frac{(y-3a)^2}{2a^3}, \quad 2a < y \le 3a.$$

$f_Y(y) = 0, \quad 3a < y$