Power Density Spectrum and Its properties.

The spectral properties of a deterministic signal $x(t)$ are contained in its Fourier transform $X(\omega)$ is given by

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

The $X(\omega)$ is the spectrum of $x(t)$.

$\therefore$ unit. $\rightarrow$ volts per hertz.

when $x(t)$ is a voltage & describes the way in which relative signal voltage is distributed with frequency.

$\therefore$ The Fourier transform can be considered to be a voltage density spectrum applicable to $x(t)$.

Both amplitudes and phases of the frequency present in $x(t)$ are described by $x(\omega)$.

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$x(t)$ can be recovered by means of the inverse Fourier transform.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega.$$

The power density spectrum.

For a random process $x(t)$ let $x_T(t)$ be defined as ~~hand~~ portion of a sample function $x(t)$ that exists between $-T$ and T.

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & \text{elsewhere.} \end{cases}$$

Assume that $x_T(t)$ has bounded variation and will statisfy

$$\int_{-T}^{T} |x_T(t)|\, dt < \infty$$

and the Fourier tranform.

$$X_T(\omega) = \int_{-T}^{T} x_T(t)\, e^{-j\omega t}\, dt = \int_{-T}^{T} x(t)\, e^{-j\omega t}\, dt$$

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The energy contained in $x(t)$ in the interval $(-T, T)$

$$E(T) = \int_{-T}^{T} x_T^2(t)\,dt = \int_{-T}^{T} x^2(t)\,dt$$

Its energy must be related to $X_T(\omega)$ by parseval's theorem.

$$E(T) = \int_{-T}^{T} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(\omega)|^2\,d\omega \quad \text{—(1)}$$

(1) $\div 2T$

we can obtain the average power $P(T)$ in $x(t)$ over the interval $(-T, T)$

$$P(T) = \frac{1}{2T}\int_{-T}^{T} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T}\,d\omega$$

The average power P_{xx} for the random process

$$P_{xx} = \underset{T\to\infty}{Lt}\ \frac{1}{2T}\int_{-T}^{T} E\left[x^2(t)\right]dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}\,d\omega$$

(1) Average power P_{xx} in a random process $x(t)$ is given by the time average of its second moment:

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$$P_{xx} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E[x^2(t)]\,dt$$

$$= A\{E[x^2(t)]\}$$

For a process that is at least WSS,

$E[x^2(t)] = \overline{x^2}$, a constant and $P_{xx} = \overline{x^2}$.

(2) P_{xx} can be obtained by frequency-domain integration. If we define the power density spectrum for the random process by.

$$\mathbb{P}_{xx}(\omega) = \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

Power formula is

$$P_{xx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbb{P}_{xx}(\omega)\,d\omega.$$

Properties of the power Density spectrum.

(1) $\mathbb{P}_{xx}(\omega) \geq 0$

(2) $\mathbb{P}_{xx}(-\omega) = \mathbb{P}_{xx}(\omega)$

(3) $\mathbb{P}_{xx}(\omega)$ is real

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(4) $\frac{1}{2\pi}\int_{-\infty}^{\infty} P_{xx}(\omega)\,d\omega = A\{E[x^2(t)]\}$

(5) $P_{xx}(\omega) = \omega^2 P_{xx}(\omega)$

(6) $\frac{1}{2\pi}\int_{-\infty}^{\infty} P_{xx}(\omega)\, e^{j\omega\tau}\, d\omega = A[R_{xx}(t, t+\tau)]$

$$P_{xx}(\omega) = \int_{-\infty}^{\infty} A[R_{xx}(t, t+\tau)]\, e^{-j\omega\tau}\, d\tau \quad \text{——— (A)}$$

It states that the power density spectrum and time average of the auto correlation function form a Fourier transform pair.

And It indicates that the power spectrum and the autocorrelation function form a Fourier transform pair.

$$P_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_{xx}(\omega)\, e^{j\omega\tau}\, d\omega$$

for a WSS process

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Bandwidth of the power density Spectrum.

Assume that $X(t)$ is lowpass process, i.e the spectral components are clustered near $\omega = 0$. and have decreasing magnitudes at higher frequency, such process is called baseband.

except for the fact that the area of $P_{xx}(\omega)$ is not nearly unity, $P_{xx}(\omega)$ has a characteristics similar to a probability density function.

S.D is a measure of the spread in a density function.

The analogous quantity for the normalized power spectrum is a measure of its spread that is called rms bandwidth.

P is an even function for real process, its mean value is zero, and its S.D is the square root of its second moment.

$$\therefore \quad W_{rms}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 P_{xx}(\omega)\, d\omega}{\int_{-\infty}^{\infty} P_{xx}(\omega)\, d\omega}.$$

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① Given the power spectrum

$$P_{xx}(\omega) = \frac{10}{\left[1+(\omega/10)^2\right]^2}$$

where the 6dB Bandwidth is 10 radians per second, Find W_{rms}.

$$W_{rms}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 P_{xx}(\omega)\, d\omega}{\int_{-\infty}^{\infty} P_{xx}(\omega)\, d\omega}.$$

$$\Rightarrow \int_{-\infty}^{\infty} P_{xx}(\omega)\, d\omega = \int_{-\infty}^{\infty} \frac{10\, d\omega}{\left(1+(\omega/10)^2\right)^2}.$$

$$= \int_{-\infty}^{\infty} \frac{10}{\frac{(100+\omega^2)^2}{10^4}}\, d\omega = \int_{-\infty}^{\infty} \frac{10^5\, d\omega}{(\omega^2+100)^2}$$

$$\boxed{\int \frac{dx}{(a^2+x^2)^2} = \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right)}$$

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$$= 10^5 \left\{ \cancel{\frac{\omega}{200(\omega^2+100)}}^{\,0} \Bigg|_{-\infty}^{\infty} + \frac{1}{2000} \tan^{-1}\left(\frac{\omega}{10}\right) \Bigg|_{-\infty}^{\infty} \right\}$$

$$= 10^5 \times \frac{1}{2000} \times \left[\frac{\pi}{2} + \pi/2 \right]$$

$$= 50\pi$$

$$\int_{-\infty}^{\infty} \omega^2 P_{xx}(\omega)\, d\omega = 10^5 \int_{-\infty}^{\infty} \frac{\omega^2\, d\omega}{(\omega^2+100)^2}$$

$$\boxed{\int \frac{x^2\, dx}{(a^2+x^2)^2} = \frac{-x}{2(a^2+x^2)} + \frac{1}{2a} \tan^{-1}\left(\frac{x}{a}\right)}$$

$$= 10^5 \left\{ \cancel{\frac{-\omega}{2(\omega^2+100)}}^{\,0} \Bigg|_{-\infty}^{\infty} + \frac{1}{20} \tan^{-1}\left(\frac{\omega}{a}\right) \Bigg|_{-\infty}^{\infty} \right.$$

$$= 10^5 \times \frac{1}{20} \times \left[\pi/2 + \pi/2 \right]$$

$$= 5000\pi$$

$$\omega_{rms} = \sqrt{\frac{5000\pi}{50\pi}} = \sqrt{100} = 10 \text{ rad/sec.}$$

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Relationship b/w power spectrum and Auto Correlation function.

The inverse fourier transform of the power density spectrum is the time average of the autocorrelation function.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} P_{xx}(\omega)\, e^{j\omega\tau}\, d\omega = A\left[R_{xx}(t, t+\tau)\right]$$

$$P_{xx}(\omega) = \lim_{T\to\infty} E\left[\left|X_T(\omega)\right|^2\right] / 2T$$

$$P_{xx}(\omega) = \lim_{T\to\infty} E\left[\frac{1}{2T}\int_{-T}^{T} x(t_1)\, e^{j\omega t_1}\, dt_1 \int_{-T}^{T} x(t_2)\, e^{-j\omega t_2}\, dt_2\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E\left[x(t_1)\, x(t_2)\right] e^{-j\omega(t_2-t_1)}\, dt_2\, dt_1$$

$$\therefore\ E\left[x(t_1)\, x(t_2)\right] = R_{xx}(t_1, t_2)$$

$$-T < (t_1 \,\&\, t_2) < T$$

$$P_{xx}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xx}(t_1, t_2)\, e^{-j\omega(t_2-t_1)}\, dt_2\, dt_1$$

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The transform will exist since the power spectrum is real, nonnegative and is absolutely integrable.

The inverse transform is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} P_{xx}(\omega)\, e^{j\omega\tau}\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xx}(t, t_1)\, e^{-j\omega(t_1-t)}\, dt\, dt_1\; e^{j\omega\tau}\, d\omega$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xx}(t, t_1) \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(\tau-t_1+t)}\, d\omega\, dt_1\, dt$$

Fourier Transform of Impulse

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega x}\, d\omega = \delta(x)$$

The event symmetry of impulses, it will be recognized the inner integral as the impulse

$$2\pi\delta(\tau - t_1 + t) = 2\pi\delta(t_1 - t - \tau)$$

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So

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} P_{xx}(\omega)\, e^{j\omega\tau}\, d\omega$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xx}(t,t_1)\, \delta(t_1-t-\tau)\, dt_1\, dt$$

According to

$$\int_{-\infty}^{\infty} \delta(x)\, \phi(x)\, dx = \phi(0)$$

The definition of impulse allows the immediate evaluation of the integral over t_1:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} P_{xx}(\omega)\, e^{j\omega\tau}\, d\omega$$

$$= \underset{T\to\infty}{Lt}\, \frac{1}{2T}\int_{-T}^{T} R_{xx}(t, t+T)\, dt \qquad -T < t+\tau < T$$

It will be recognized as the time average of the process autocorrelation function

$$A\left[R_{xx}(t, t+\tau)\right] = \underset{T\to\infty}{Lt}\, \frac{1}{2T}\int_{-T}^{T} R_{xx}(t, t+\tau)\, dt$$

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The Inverse Fourier transform of the power density spectrum is the time average of the process autocorrelation function.

$$P_{xx}(\omega) = \int_{-\infty}^{\infty} A[R_{xx}(t, t+\tau)] e^{-j\omega\tau} d\tau$$

$P_{xx}(\omega)$ & $A[R_{xx}(t, t+\tau)$ form a Fourier transform pair

$$A[R_{xx}(t, t+\tau) \longleftrightarrow P_{xx}(\omega)$$

If $x(t)$ is at least WSS.

$$A[R_{xx}(t, t+\tau)] = R_{xx}(\tau)$$

$$P_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P_{xx}(\omega) e^{j\omega\tau} d\omega$$

$$R_{xx}(\tau) \longleftrightarrow P_{xx}(\omega)$$

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Cross-Power density spectrum and its properties.

Consider a real random process $w(t)$

$$w(t) = x(t) + y(t)$$

The auto correlation function of $w(t)$ is

$$R_{ww}(t, t+\tau) = E[w(t)w(t+\tau)]$$

$$= E[\{x(t) + y(t)\}\{x(t+\tau)\ y(t+\tau)\}]$$

$$= R_{xx}(t, t+\tau) + R_{yy}(t+\tau) + R_{xy}(t, t+\tau) + R_{yx}(t, t+\tau)$$

Cross power $P_{xy}(T)$ in the two processes with in the interval $(-T, T)$

$$P_{xy}(T) = \frac{1}{2T}\int_{-T}^{T} x_T(t)\, y_T(t)\, dt = \frac{1}{2T}\int_{-T}^{T} x(t)y(t)\, dt$$

Apply parseval's theorem

$$P_{xy}(T) = \frac{1}{2T}\int_{-T}^{T} x(t)\, y(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{X_T^*(\omega)\, Y_T(\omega)}{2T}\, d\omega$$

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The average cross power $\overline{P}_{xy}(T)$

$$\overline{P}_{xy}(T) = \frac{1}{2T}\int_{-T}^{T} R_{xy}(t,t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{E[X_T^*(\omega)\,Y_T(\omega)]}{2T}\,d\omega$$

The total average cross power P_{xy} by letting $T \to \infty$

$$P_{xy} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{xy}(t,t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty}\frac{E[X_T^*(\omega)\,Y_T(\omega)]}{2T}\,d\omega$$

A cross power density spectrum

$$\mathbb{P}_{xy}(\omega) = \lim_{T\to\infty}\frac{E[X_T^*(\omega)\,Y_T(\omega)]}{2T}$$

Cross power

$$P_{xy} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{P}_{xy}(\omega)\,d\omega.$$

Similarly

$$\mathbb{P}_{yx}(\omega) = \lim_{T\to\infty}\frac{E[Y_T^*(\omega)\,X_T(\omega)]}{2T}$$

$$P_{yx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{P}_{yx}(\omega)\,d\omega = P_{xy}^*$$

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properties.

① $P_{xy}(\omega) = P_{yx}(-\omega) = P^*_{xy}(\omega)$

② $Re[P_{xy}(\omega)]$ & $Re[P_{yx}(\omega)]$ are even function of ω.

③ $Im[P_{xy}(\omega)]$ and $Im[P_{yx}(\omega)]$ are odd function of ω.

④ $P_{xy}(\omega) = 0$ & $P_{yx}(\omega) = 0$ if $x(t)$ and $y(t)$ are orthogonal.

⑤ If $x(t)$ and $y(t)$ are uncorrelated and have constant means $\bar{x}$ & $\bar{y}$

$$P_{xy}(\omega) = P_{yx}(\omega) = 2\pi \bar{x}\bar{y}\,\delta(\omega)$$

⑥ $A[R_{xy}(t, t+\tau)] \Longleftrightarrow P_{xy}(\omega)$

$A[R_y(t, t+\tau)] \longleftrightarrow P_{yx}(\omega)$.

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Power spectrums for Discrete time processes and sequence

Discrete time processes.

Let $x(t)$ be a band limited random process for which samples are taken at times nT_s, $n = 0, \pm 1, \pm 2 \ldots$ to form a discrete-time process $x(nT_s)$. When sampling is the result of multiplying $x(t)$ by a periodic "sampling" pulse train consisting of rectangular pulses of short duration T_p and amplitude $1/T_p$ occurring each T_s seconds.

The autocorrelation function $R_{xx}(\tau)$ has a sampled representation.

$$R_{x_s x_s}(\tau) = R_{xx}(\tau) \frac{1}{T_p} \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{\tau - nT_s}{T_p}\right)$$

$$\simeq R_{xx}(\tau) \cdot \sum_{n=-\infty}^{\infty} \delta(\tau - nT_s)$$

(use of narrow sample pulse T_p small)

$$= \sum_{n=-\infty}^{\infty} R_{xx}(nT_s)\, \delta(\tau - nT_s)$$

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Product of the function & Impulse.

$$g(x)\,\delta(x) = g(0)\,\delta(x)$$

Sampling with very narrow pulses (impulse in the limit) is called ideal sampling

The power spectrum of the discrete-time (DT) random process

$$P_{xx}(\omega) = \sum_{n=-\infty}^{\infty} R_{xx}(nT_s)\, e^{-j\omega n T_s} \quad \text{—— (1)}$$

Eqn (1) is periodic in the variable ω with period $\omega_s = 2\pi / T_s$

The power spectrum of the DT random process is related to the power spectrum of the process $x(t)$

$$\sum_{n=-\infty}^{\infty} \delta(\tau - nT_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{jn\omega_s \tau}.$$

<u>proof</u>

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$\sum_n \delta(t-nT_s)$ is periodic with period T_s

So it has a Fourier series.

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_s t}$$

$$\omega_s = 2\pi/T_s$$

$$c_n = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} f(t)\, e^{-jn\omega_s t}\, dt.$$

$$= \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \sum_m \delta(t-mT_s)\, e^{-jn\omega_s t}\, dt$$

$= \frac{1}{T_s}$, all n, because only $\delta(t)$ falls in the range of integration. Thus,

$$\sum_{n=-\infty}^{\infty} \delta(t-nT_s) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

The Fourier transform

$$P_{x_s x_s}(\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} P_{xx}(\omega - n\omega_s)$$

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Proof

$$R_{x_s x_s}(\tau) = R_{xx}(\tau) \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{jn\omega_s \tau}$$

$$P_{x_s x_s}(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} R_{xx}(\tau) e^{jn\omega_s \tau} e^{-j\omega\tau} d\tau}$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} P_{xx}(\omega - n\omega_s)$$

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Now $P_{xx}(\omega)$ is the central term, where $n=0$

If we write a rectangular function $\text{rect}(\omega/\omega_s)$ on the both sides to select out $P_{xx}(\omega)$

$$P_{xx}(\omega) = T_s \, \text{rect}(\omega/\omega_s) \, P_{x_s x_s}(\omega)$$

It shows that the process power spectrum is T_s times the central period portion of the periodic power spectrum of the DT process.

We can obtain $P_{xx}(\omega)$ alternative using sampling theory.

$$R_{xx}(\tau) = \sum_{n=-\infty}^{\infty} R_{xx}(nT_s) \, S_a\left[\omega_s(\tau - nT_s)/2\right] \quad (6)$$

The direct Fourier transformation of eqn (6) produces.

$$P_{xx}(\omega) = T_s \, \text{rect}(\omega/\omega_s) \sum_{n=-\infty}^{\infty} R_{xx}(nT_s) e^{-jn\omega T_s}$$

$$= T_s \, \text{rect}(\omega/\omega_s) \, P_{x_s x_s}(\omega)$$

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Discrete time sequences

Ω is called discrete frequency

$$\Omega = \omega T_s$$

Z-transforms are extensively used in DSP.

The auto correlation function of τ

Transform of sequence of samples

$$P_{x_s x_s}(z) = \sum_{n=-\infty}^{\infty} R_{xx}[n]\, z^{-n} \quad \text{--- (1)}$$

$$P_{x_s x_s}(z) = \sum_{n=-\infty}^{\infty} R_{xx}[nT_s]\, e^{-jn\omega T_s}$$

$$= \sum_{n=-\infty}^{\infty} R_{xx}[n]\, e^{-jn\Omega}$$

$$= P_{x_s x_s}(z)\Big|_{z=e^{j\Omega}} = P_{x_s x_s}(e^{j\Omega}) \quad \text{--- (2)}$$

However eqn (2) shows that the power density spectrum is the Z transform evaluated on the unit circle

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$\mathbb{P}_{x_s x_s}(e^{j\Omega})$ is the DTFT of $R_{xx}(m)$

Inverse DTFT

$$\mathbb{P}_{x_s x_s}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} R_{xx}[n]\, e^{-jn\Omega} \quad \text{(DTFT)} \quad \text{—(3)}$$

$$R_{xx}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbb{P}_{x_s x_s}(e^{j\Omega})\, e^{jn\Omega}\, d\Omega \quad (\text{I DTFT}) \quad \text{—(4)}$$

(3) & (4) involves sequences of finite length N.

Let us define the extent of the samples as NT_s and the discrete values ω_k and Ω_k of ω and Ω

$$\omega_k = 2\pi k / NT_s \qquad k = 0, 1, \ldots N-1$$

$$\Omega_k = \omega T_s \big|_{\omega=\omega_k} = \frac{2\pi k}{N} \qquad k = 0, 1, \ldots N-1$$

the discrete values of the power spectrum

$$\mathbb{P}_{x_s x_s}(\omega_k) = \mathbb{P}_{x_s x_s}(e^{j\Omega_k}) = \mathbb{P}_{x_s x_s}(e^{j\omega_k T_s}) \quad \text{—(5)}$$

$$\mathbb{P}_{x_s x_s}[k] = \mathbb{P}_{x_s x_s}(e^{j\Omega_k}) = \mathbb{P}_{x_s x_s}(e^{j2\pi k/N}) \quad \text{—(6)}$$

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truncated version of (3) & discretized version of (4)

$$P_{x_s x_s}[k] = \sum_{n=0}^{N-1} R_{xx}[n]\, e^{-j2\pi nk/N}$$

$$k = 0, 1, \ldots (N-1) \quad (7)$$

$$R_{xx}[n] = \frac{1}{N} \sum_{k=0}^{N-1} P_{x_s x_s}[k]\, e^{j2\pi nk/N}$$

$$n = 0, 1, \ldots (N-1) \quad (8)$$

(7) is discrete Fourier transform of the finite autocorrelation sequence.

(8) is IDFT of the discretized power spectrum.

<u>Alternate form</u>

$$P_{x_s x_s}(\omega_k) = \sum_{n=0}^{N-1} R_{xx}(nT_s)\, e^{-jn\omega_k T_s}$$

$$k = 0, 1, \ldots N-1$$

$$R_{xx}(nT_s) = \frac{1}{N} \sum_{k=0}^{N-1} P_{x_s x_s}(\omega_k)\, e^{jn\omega_k T_s}$$

$$n = 0, 1, \ldots (N-1)$$

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Power spectrums of complex processes.

If power spectrum is defined for complex processes. we should consider at least WSS.

$R_{ZZ}(\tau)$ is the auto correlation function of a complex random process $Z(t)$.

The power spectrum is defined as its FT

$$P_{ZZ}(\omega) = \int_{-\infty}^{\infty} R_{ZZ}(\tau)\, e^{-j\omega\tau}\, d\tau$$

Inverse form.

$$R_{ZZ}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_{ZZ}(\omega)\, e^{j\omega\tau}\, d\omega.$$

For two jointly WSS complex processes $Z_m(t)$ and $Z_n(t)$ their cross power density spectrum and cross correlation function

$$P_{Z_mZ_n}(\omega) = \int_{-\infty}^{\infty} R_{Z_mZ_n}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R_{Z_mZ_n}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_{Z_mZ_n}(\omega)\, e^{j\omega\tau}\, d\omega.$$

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① Given random process

$$x(t) = A_0 \cos(\omega_0 t + \theta)$$

where A_0 and ω_0 are constant and θ is a random variable uniformly distributed on the interval $(0, \pi)$

(a) Is $x(t)$ WSS

(b) Find the power in $x(t)$

(a) $E[x(t)] = \int_{-\infty}^{\infty} x(t) f(\theta) \, d\theta$.

uniformly distributed $(0, \pi)$.

$$f(\theta) = \frac{1}{\pi}$$

$$= \int_{0}^{\pi} \frac{1}{\pi} A_0 \cos(\omega_0 t + \theta) \, d\theta$$

$$= \frac{A_0}{\pi} \left[\sin(\omega_0 t + \pi) - \sin(\omega_0 t) \right]$$

$$= -\frac{2 A_0}{\pi} \sin(\omega_0 t).$$

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The mean value is time dependent. So $x(t)$ is not WSS.

(B) $E[x^2(t)] = E[A_0^2 \cos^2(\omega t + \theta)]$

$$= \frac{A_0^2}{2}\left[E\left\{1 + \cos(2\omega_0 t + 2\theta)\right\}\right]$$

$$= \frac{A^2}{2} + \frac{A^2}{2}\int_0^{\pi} \cos(2\omega_0 t + 2\theta) \cdot \frac{1}{\pi}\, d\theta = A^2/2.$$

$$P_{xx} = \lim_{T \to \infty} \frac{1}{2T}\int_{-T}^{T} \frac{A^2}{2}\, dt = A^2/2.$$

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Assume a random process has a power spectrum.

$$P_{xx}(\omega) = \begin{cases} 4 - \left(\frac{\omega^2}{9}\right) & |\omega| \le 6 \\ 0 & \text{elsewhere} \end{cases}$$

Find (a) the average power, (b) RMS B.W

$$P_{xx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_{xx}(\omega)\, d\omega = \frac{1}{2\pi}\int_{-6}^{6}\left[4 - \frac{\omega^2}{9}\right] d\omega$$

$$= \frac{16}{\pi}$$

(b) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 P_{xx}(\omega)\, d\omega = \frac{1}{2\pi}\int_{-6}^{6}\left[4\omega^2 - \frac{\omega^4}{9}\right] d\omega$$

$$= \frac{16 \times 36}{5\pi}$$

$$W_{rm}^2 = \frac{\frac{36 \times 16}{5\pi}}{16/\pi} = 36/5$$

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(1) Compute the variance of the random process $x(t)$ whose autocorrelation function is given by

$$R_{xx}(\tau) = 25 + \frac{4}{1+6\tau^2}$$

<u>Mean</u>

$$(E[x(t)])^2 = \mu_x^2 = \lim_{|\tau| \to \infty} R_{xx}(\tau)$$

$$= \lim_{|\tau| \to \infty} 25 + \frac{4}{1+6\tau^2} = 25 + \frac{4}{\infty}$$

$$= 25$$

$$\mu_x^2 = 25 \Rightarrow \mu_x = \pm 5$$

$$E[x^2(t)] = R_{xx}(0) = 25+4 = 29$$

<u>Variance</u>

$$\sigma^2 = E[x^2(t)] - \mu_x^2 = 29-25$$

$$= \underline{4}$$

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(2) Determine the autocorrelation function of the random process with the power spectral density given by

$$S_{xx}(\omega) = \begin{cases} S_0 & |\omega| < \omega_0 \\ 0 & \end{cases}$$

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} S_0\, e^{j\omega\tau}\, d\omega$$

$$= \frac{S_0}{2\pi} \left[\frac{e^{j\omega\tau}}{j\tau} \right]_{-\omega_0}^{\omega_0}$$

$$= \frac{S_0}{2\pi j\tau} \left[e^{j\omega_0\tau} - e^{-j\omega_0\tau} \right]$$

$$= \frac{S_0}{\pi\tau} \left[\frac{e^{j\omega_0\tau} - e^{-j\omega_0\tau}}{2j} \right]$$

$$R_{xx}(\tau) = \frac{S_0}{\pi\tau} \times \sin \omega_0\tau$$

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(3) A stationary random process $x(t)$ has the power spectral density

$$S_{xx}(\omega) = \frac{24}{\omega^2+16}$$

Find the mean-square value of the process

$$E[x^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{24}{\omega^2+16}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{24}{16\left[1+\left(\frac{\omega}{4}\right)^2\right]}\,d\omega$$

let $\omega/4 = \tan\theta$

$d\omega = 4\sec^2\theta\,d\theta$

$1+\left(\frac{\omega}{4}\right)^2 = 1+\tan^2\theta = \sec^2\theta$

put $\omega = -\infty$, $\theta = -\pi/2$

$\omega = \infty$, $\theta = \pi/2$

$$E[x^2(t)] = \frac{24}{32\pi}\int_{-\pi/2}^{\pi/2} \frac{4\cancel{\sec^2\theta}\,d\theta}{\cancel{\sec^2\theta}} = \frac{3}{\pi}\int_{-\pi/2}^{\pi/2} d\theta$$

$$= \frac{3}{\pi}[\theta]_{-\pi/2}^{\pi/2} = \frac{3}{\pi}\left[\frac{\pi}{2} - \frac{-\pi}{2}\right] = 3$$

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Alternate method

$$e^{-a|\tau|} \longleftrightarrow \frac{2a}{a^2+\omega^2}.$$

$$S_{xx}(\omega) = \frac{24}{\omega^2+16} = \frac{24}{\omega^2+4^2} = 3\left\{\frac{2(4)}{(\omega^2+4^2)}\right\}$$

Here $a=4$ and the auto correlation function.

$$R_{xx}(\tau) = 3e^{-4|\tau|}$$

$\therefore$ the mean-square of the process is

$$E[x^2(t)] = R_{xx}(0) = 3$$

(4) Let $y(t) = x(t) + N(t)$ be a WSS process where $x(t)$ is the actual signal and $N(t)$ is a zero mean noise process with variance σ_N^2 and independent of $x(t)$. Find the power spectral density of $y(t)$

$x(t)$ & $N(t)$ are independent random process.

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$$R_{YY}(\tau) = E[Y(t)\,Y(t+\tau)]$$

$$= E[\{x(t)+N(t)\}\{x(t+\tau)+N(t+\tau)\}]$$

$$= E[x(t)\,x(t+\tau)] + \overset{0}{\cancel{E[x(t)\,N(t+\tau)]}} + \overset{0}{\cancel{E[N(t)\,x(t+\tau)]}}$$

$$+ E[N(t)\,N(t+\tau)]$$

$$= R_{XX}(\tau) + R_{NN}(\tau)$$

$$= R_{XX}(\tau) + \sigma_N^2\,\delta(\tau)$$

Thus power spectral density of $Y(t)$ is given by

$$S_{YY}(\omega) = S_{XX}(\omega) + \sigma_N^2$$

~~If~~ The random variable is uncorrelated, If the white noise is a Gaussian WSS process.

then $X[n]$ consist of sequence of independent and identically distributed random variables with variance σ^2 and mean 0

~~Ca~~ auto correlation function is given by

$$R_{XX}(m) = \sigma^2\,\delta(m) \qquad \delta(m) = \begin{cases} 1 & m=0 \\ 0 & m\neq 0 \end{cases}$$

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Find the power spectral density of a random sequence $X[n]$ whose auto correlation function is given by $R_{XX}[m] = a^{|m|}$

The power spectral density

$$S_{XX}(\Omega) = \sum_{m=-\infty}^{\infty} R_{XX}(m)\, e^{-j\Omega m}$$

$$= \sum_{m=-\infty}^{\infty} a^{|m|}\, e^{-j\Omega m}.$$

$$= \sum_{m=-\infty}^{-1} e^{-j\Omega m} a^{-m} + \sum_{m=0}^{\infty} a^{m} e^{-j\Omega m}$$

$$= \sum_{m=1}^{\infty} e^{j\Omega m} a^{m} + \sum_{m=0}^{\infty} a^{m} e^{-j\Omega m}$$

$$= \sum_{m=0}^{\infty} e^{j\Omega m} a^{m} - 1 + \sum_{m=0}^{\infty} a^{m} e^{-j\Omega m}$$

$$= \sum_{m=0}^{\infty} (a e^{j\Omega})^{m} - 1 + \sum_{m=0}^{\infty} (a e^{-j\Omega})^{m}$$

$$= \frac{1}{1 - a e^{j\Omega}} - 1 + \frac{1}{1 - a e^{-j\Omega}}$$

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$$= \frac{(1-ae^{-j\Omega}) - (1-ae^{j\Omega})(1-ae^{-j\Omega}) + a(1-ae^{j\Omega})}{(1-ae^{j\Omega})(1-ae^{-j\Omega})}$$

$$= \frac{1-ae^{-j\Omega} - (1+a^2-ae^{-j\Omega}-ae^{j\Omega}) + 1-ae^{j\Omega})}{1+a^2-ae^{j\Omega}-ae^{-j\Omega}}$$

$$= \frac{\cancel{1} - \cancel{ae^{-j\Omega}} - \cancel{1} - a^2 + \cancel{ae^{-j\Omega}} + \cancel{ae^{j\Omega}} + \cancel{1} - \cancel{ae^{j\Omega}}}{1+a^2-2a\cos\Omega}$$

$$= \frac{1-a^2}{1+a^2-2a\cos\Omega}$$

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A random process is given by

$$W(t) = Ax(t) + By(t)$$

where A and B are real constants and $x(t)$, $y(t)$ are jointly WSS process

(a) Find the power spectrum $S_{ww}(\omega)$ of $W(t)$

(b) If $x(t)$ and $y(t)$ are uncorrelated

(b) Find the cross power spectrum $S_{xw}(\omega)$

For uncorrelated $x(t)$ and $y(t)$ =

$$R_{ww}(t, t+\tau) = E\left[\{Ax(t) + By(t)\}\{Ax(t+\tau) + By(t+\tau)\}\right]$$

$$= A^2 E\{x(t)x(t+\tau)\} + AB\, E[x(t)y(t+\tau)] + AB\, E[y(t)x(t+\tau)] + B^2 E[y(t)y(t+\tau)]$$

$$= A^2 R_{xx}(\tau) + AB\, R_{yx}(\tau) + AB\, R_{xy}(\tau) + B^2 R_{yy}(\tau)$$

$x(t)$ and $y(t)$ are uncorrelated

$$\therefore R_{yx}(\tau) = R_{xy}(\tau) = \bar{x}\,\bar{y}$$

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∴ Power spectrum

$$S_{WW}(\omega) = A^2 S_{XX}(\omega) + B^2 S_{YY}(\omega) + 2\pi \bar{X}\bar{Y} AB\,\delta(\omega) + 2\pi \bar{X}\bar{Y} AB\,\delta(\omega)$$

$$\boxed{S_{WW}(\omega) = A^2 S_{XX}(\omega) + B^2 S_{YY}(\omega) + 4\pi \bar{X}\bar{Y} AB\,\delta(\omega)}$$

(2) $S_{XW}(\omega)$

$$R_{XW}(t, t+\tau) = E\left[x(t)\{A\,x(t+\tau) + B\,y(t+\tau)\}\right]$$

$$= A\,E[x(t)\,x(t+\tau)] + B\,E[x(t)\,y(t+\tau)]$$

$$= A\,R_{XX}(\tau) + B\,R_{XY}(\tau)$$

$$\boxed{S_{XW}(\omega) = A\,S_{XX}(\omega) + B\,S_{XY}(\omega)}$$

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Let $x(t)$ and $y(t)$ be statiscally independent processes with power spectrums

$$S_{xx}(\omega) = 2\delta(\omega) + \frac{1}{1+(\omega/10)^2}$$

$$S_{yy}(\omega) = \frac{4}{1+\left(\frac{\omega}{2}\right)^2}$$

A complex process

$$z(t) = [x(t) + j\,y(t)]\exp(j\omega_0 t)$$

is formed where ω_0 is a constant. Determine the auto correlation function of $z(t)$.

$$\frac{\alpha}{2\pi} \longleftrightarrow \alpha\,\delta(\omega)$$

$$e^{-\alpha|t|} \longleftrightarrow \frac{2\alpha}{\alpha^2+\omega^2}$$

$$2\delta(\omega) \longleftrightarrow \frac{1}{\pi}, \quad \frac{1}{1+(\omega/10)^2} = \frac{100}{\omega^2+100^2} = \frac{5\times(2\times10)}{\omega^2+100^2}$$

$$\frac{5\times(2\times10)}{\omega^2+100^2} \longleftrightarrow 5e^{-10|\tau|}$$

$$\frac{4\times(2\times2)}{\omega^2+2^2} \longleftrightarrow 4e^{-2|\tau|}$$

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$$R_{zz}(t, t+\tau) = E\left[z^*(t)\, z(t+\tau)\right]$$

$$= E\left\{\left[x(t) - j\,y(t)\right] e^{-j\omega_0 t} \cdot \left[x(t+\tau) + j\,y(t+\tau)\right] e^{j\omega_0 t} \cdot e^{j\omega_0 \tau}\right]$$

$$= e^{j\omega_0 \tau}\left\{E\left[x(t)\, x(t+\tau)\right] + j\,E\left[x(t) \cdot y(t+\tau)\right] - j\,E\left[y(t)\, x(t+\tau)\right] + E\left[y(t)\, y(t+\tau)\right]\right\}$$

$$= e^{j\omega_0 \tau}\left[R_{xx}(\tau) + R_{yy}(\tau) + j\bar{x}\bar{y} - j\bar{x}\bar{y}\right]$$

$$R_{zz}(\tau) = e^{j\omega_0 \tau}\left[\frac{1}{\pi} + 5e^{-10|\tau|} + 4e^{-2|\tau|}\right]$$

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A complex random process is given by

$$Z(t) = Ae^{j\Omega t}$$

where Ω is a random variable with probability density function $f_\Omega(\cdot)$ and A is a complex constant. Show that the power spectrum of $Z(t)$ is

$$P_{ZZ}(\omega) = 2\pi |A^2| f_\Omega(\omega)$$

$$R_{ZZ}(t, t+\tau) = E[z^*(t)\, z(t+\tau)]$$

$$= E\left[A^* e^{-j\Omega t} \cdot A e^{j\Omega(t+\tau)}\right]$$

$$= |A|^2 E\left[\cancel{e^{-j\Omega t}} \cdot \cancel{e^{j\Omega t}} \cdot e^{j\Omega\tau}\right]$$

$$= |A|^2 E\left[e^{j\Omega\tau}\right] = A^2 \int_{-\infty}^{\infty} f_\Omega(\omega)\, e^{j\omega\tau}\, d\tau$$

The integral is 2π times the inverse fourier transform of $f_\Omega(\omega)$. $2\pi |A^2| f_\Omega(\omega)$ must be the power spectrum of $Z(t)$.

$$\therefore \quad R_{ZZ}(\tau) \longleftrightarrow P_{ZZ}(\omega) = 2\pi |A^2| f_\Omega(\omega)$$

$$\boxed{\begin{aligned} \Phi_X(\omega) &= E[e^{j\omega X}] = \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx. \\ f_X(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_X(\omega)\, e^{-j\omega x}\, d\omega. \end{aligned}}$$

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A zero mean gaussian random process has an auto correlation function

$$R_{xx}(\tau) = \begin{cases} 13\left[1-(|\tau|/6\right] & |\tau| \le 6 \\ 0 & \text{elsewhere} \end{cases}$$

Find the covariance function necessary to specify the jointly density random variables defined at times $t_i = 2(i-1)$, $i = 1, 2 \ldots 5$ Give the covariance matrix for $x_i = x_i(t)$

$$t_k - t_i = (2k-2) - (2i-2) = 2(k-i)$$

$$R_{xx}(t_k - t_i) = \begin{cases} 13\left[1 - |k-i|/3\right] & |k-i| \le 3 \\ 0 & |k-i| > 3 \end{cases}$$

$$C_{xx}(t_i - t_k) = R_{xx}(t_i, t_k) - \bar{x}_i \bar{x}_k$$

$$= \begin{cases} 13\left[1 - |k-i|/3\right] & |k-i| \le 3 \\ 0 & |k-i| > 3 \end{cases}$$

$$[C_x] = \begin{bmatrix} 13 & \left(\frac{26}{3}-0\right) & \left(\frac{13}{3}-0\right) & 0 & 0 \\ \left(\frac{26}{3}-0\right) & 13 & \left(\frac{26}{3}-0\right) & \left(\frac{13}{3}-0\right) & 0 \\ \left(\frac{13}{3}-0\right) & \left(\frac{26}{3}-0\right) & 13 & \left(\frac{26}{3}-0\right) & \left(\frac{13}{3}-0\right) \\ 0 & \left(\frac{13}{3}-0\right) & \left(\frac{26}{3}-0\right) & 13 & \left(\frac{26}{3}-0\right) \\ 0 & 0 & \left(\frac{13}{3}-0\right) & \left(\frac{26}{3}-0\right) & 13 \end{bmatrix}$$

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Normal distribution

the CDF of x is given by

$$F_x(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\mu_x}{\sigma_x}} e^{-y^2/2}\, dy = \phi\left(\frac{x-\mu_x}{\sigma_x}\right)$$

If $x = N(3;9)$ which means that x is a normal random variable with $\mu_x = 3$ and $\sigma_x^2 = 9$

find the probability that x lies between 2 and 5,

$$P[2 < x < 5] = F_x(5) - F_x(2)$$

$$= \phi\left(\frac{5-\mu_x}{\sigma_x}\right) - \phi\left(\frac{2-\mu_x}{\sigma_x}\right)$$

$$= \phi\left(\frac{5-3}{3}\right) - \phi\left(\frac{2-3}{3}\right) = \phi\left(\frac{2}{3}\right) - \phi\left(-\frac{1}{3}\right)$$

$$\boxed{\phi(-x) = 1 - \phi(x)}$$

$$= \phi(2/3) - (1 - \phi(1/3))$$

$$= \phi(2/3) + \phi(1/3) - 1 = 0.7486 + 0.6293 - 1$$

$$= 0.3779$$

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