**ECE2006- DIGITAL SIGNAL PROCESSING**

**Contents of Module-2 from syllabus**

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| **Module:2** | **Frequency Analysis of Signals and Systems-II** | **5 hours** | **CO: 2** |

Frequency domain sampling- Sampling rate conversion - Aperiodic correlation estimation-Cepstrum processing- Band limited discrete time signals- Phase and group delay- DFT-Properties. Frequency analysis of signals using DFT-FFT Algorithm-Radix-2 FFT algorithms-Applications of FFT.

**Reference Books: (used for Module 2)**

1) J. G. Proakis, and D.G. Manolakis , “Digital Signal Processing Principles, Algorithms and Applications”, 2012, 4th edition, Pearson Education, Noida, India.

2) D.Ganesh Rao and V.P.Gejji, “Digital Signal Processing “, 2017, Cengage Learning India Pvt. Ltd., Delhi, India

**DFT as a linear transformation**

* DFT of a finite length sequence x[n] is

=…..(1), where 0≤ ≤

Range (0, N-1) in above eqn. (1)

x(0) + x(1) + ………+ x(N-1)

………+ x(N-1)

………………………………………………………………………………..

………+ x(N-1)

Eq.(1) of DFT can be expressed in matrix form as

= …….(2)

* From eqn.(2), both X and x are (Nx1) matrices and is an (N x N) square matrix called the DFT matrix. The complete matrix is described by
* The elements of are known as Twiddle Factors

**Problems on DFT:**

1. Using DFT Matrix method, Compute the 4-point DFT of the sequence x[n] = {1,2,1,0}.

Solution: Given N = 4

= 1, = = cos π/2 – j sin π/2 = – j

= ()2 = cos π – j sin π = -1

= ()3 = cos 3π/2 – j sin 3π/2 = –j(-1)=j

=

* =

Ans: DFT X(k) = (, , 0, )

1. Find the DFT of a sequence x[n] = {1, 2, 3, 4} using matrix method?

DFT is =, 0≤ ≤

using matrix method, X4 = W4. x4

Ans: X(k) = { 10, -2+2j, -2, -2-2j }

1. Find the DFT of a sequence x[n] = {0,1, 2, 3}? Also cross check your answer by taking arrived DFT as input and find IDFT x[n]?

using matrix method, X4 = W4. x4

Answer is X(k) =

**Cross Checking Procedure** **using IDFT Formula Method**

DFT X(k) ={6, -2+2j, -2,-2-2j}

IDFT Formula is

Sub n=0 and k=0,1,2,3 we get x[0] =0

Sub n=1 and k=0,1,2,3 we get x[1] = 1

Sub n=2 and k=0,1,2,3 we get x[2] = 2

Sub n=3 and k=0,1,2,3 we get x[3] = 3

Therefore sequence x[n] ={0,1,2,3} (cross verified the answer)

**Applications of DFT**

* DFT plays a key role in applications of DSP which includes linear filtering, correlation analysis and spectrum analysis

**Matrix relation to compute IDFT**

* = …..(1)

Therefore = …….(2)

From eqn.(2), is called IDFT matrix

* The equation to find IDFT of a sequence X(k)
* …….(3), 0≤ ≤
* Expressing IDFT in matrix form ….(4)
* Comparing Eqn.(2) & Eqn.(4)

We get = ……(5)

**IDFT problem:**

* Compute the IDFT of the 4-point sequence X(k) = (, , 0, ) using the DFT?

Solution:

Formula to find IDFT

* Step 1: Find the complex conjugate of X(k)

That is = , , 0, )

* Step 2: Find the DFT of

=

* Step 3:

Find the conjugate of the result and divide it by N=4 to get IDFT IDFT {X(k)} = x[n] = (4,8,4,0)\* = (4,8,4,0)

Ans: x[n] = (1,2,1,0)

**DFT and its Properties**

**Periodicity Property**

* The N-point DFT and N-point IDFT are implicit periodic with period N, although x[n] and X(k) are sequences of length N each.
* Note: DFT X(k) and IDFT x[n] can be shown to be periodic with a period N, since the exponentials in the defining equations of DFT and IDFT are periodic with a period N
* Hence X(k) and x[n] are called implicit periodic sequences. That is

X(k + N) = X(k) and x[n + N] = x[n]

* **Proof**: DFT X(k) = …….(1)
* Now X(k + N) =

= …..(2)

Since, = = = 1 …..(3)

Sub. Eq.(3) into Eq.(2),  **X(k + N) = X(k)** ..…(4) (**Proved**)

…….(5)

Now x[n + N] =

* x[n + N] = ……(6)

Since, = = = 1 …..(7)

Sub. Eq.(7) into Eq.(6), **x[n + N) = x[n]** ..… (8) (**Proved**)

**Linearity Property of DFT**

* **Statement**

DFT=+, where k=0,1,2,……,N-1

Here (k) and (k) are the N-point DFTs of the sequences and respectively.

* **Proof**

=…..(1), where 0≤ ≤

DFT {.}

System

Input x[n] DFT X(k)

Let = …….(2)

Sub. equivalent from Eq.(2) into Eq.(1)

* DFT

=

= (k) + (k) …….(3), where k=0,1,2,……,N-1

Thus proving the linearity property,

} =(k) + (k)

**Problem**:

Use linearity property to find the 4-point DFT of the sequence given by, ) + ) ?

* **Solution:** N=4 , we know = , that is = =

Now = 1

= = cos π/2 – j sin π/2 = – j

= ( )2 = cos π – j sin π = -1

= ()3 = cos 3π/2 – j sin 3π/2 = 0–j(-1)= j

* Let = ) and = ) .The values of and for 0≤n≤3 are tabulated below:

**Table 1**: **Values of and for 0≤n≤3**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| n | 0 | 1 | 2 | 3 |
| = ) | 1 | 1/√2 | 0 | - 1/√2 |
| = ) | 0 | 1/√2 | 1 | 1/√2 |

* Computing 4-point DFTs , (k) and (k)
* (k) = DFT =, for 0≤ ≤ 3
* Applying values of

(k) = 1+ 1/√2 + 0 - 1/√2 …… (1)

* (0) = 1+ 1/√2- 1/√2 = 1
* (1) = 1+ 1/√2 + 0 - 1/√2 = 1 - j1.414
* (2) = 1 + 1/√2 + 0 - 1/√2

(2) = 1 + 1/√2 + 0 - 1/√2 = 1

* (3) = 1 + 1/√2 + 0 -1/√2

(3) = 1 + 1/√2 + 0 -1/√2 = 1 + j1.414

Now (k) = DFT =

Applying values of we get

(k) = 1/√2 + 1/√2 …….(2)

* (0) = 1/√2 + 1 +1/√2 = 2.414

(1) = 1/√2 + + 1/√2 = -1

* (2) = 1/√2 + +1/√2

(2) = 1/√2 + +1/√2 = -0.414

* (3) = 1/√2 + + 1/√2

(3) = 1/√2 + +1/√2 = -1

* Applying linearity property finally, we get

= DFT = (k) + (k)

= {(0) + (0), (1) + (1), (2) + (2), (3) + (3)}

= (3.414, -j1.414, 0.586, j1.414)

**Circular Folding (or) Time-reversal property of DFT**

**Statement**: It states that reversing the N-Point sequence in time-domain is equivalent to reversing the DFT sequence in frequency domain. That is, if DFT {x[n]} = X(k), then

DFT {x[N-n]} = X(N-k)

(OR)

=

**Proof:** DFT {x[N-n]} = ….(1)

Let m = N-n , therefore n=N-m ….(2)

Eq.(1) implies, DFT {x[N-n]} =

=

= ……..(3)

DFT {x[N-n]} = (Since = 1 )

DFT {x[N-n]} = ……(4)

=

=

= ………(5)

DFT {x[N-n]} = X(N-k)

Thus, DFT {x[N-n]} = X(N-k)

(OR)

= ……(6) (hence the proof)

**Problem illustrating Circular Folding property of DFT**

Compute the 4-point DFT of the sequence ={1,2,1,0}. Also, find Y(k) if

, 0 ≤ k ≤ 3 ?

**Solution**

We know that =

Since N=4, we get = =

= 1, = – j, = -1 and = j

= , 0≤ k ≤ 3

= 1 x + 2 x + 1 x + 0 x

1 + + ……(1)

Sub k=0 in eq.(1) , 1+2+1 = 4

Sub k=1 in eq.(1) , 1 + 2 + = –j2

Sub k=2 in eq.(1) , 1 + 2 + = 1 + 2 + =0

Sub k=3 in eq.(1) , 1 + 2 + = 1 + 2 + =j2

Therefore X(k) = {4, –j2, 0, j2}

Since is real, it may be noted that the symmetry property,

(N-k) is observed

Given:

Hence, Y(k) = =(k) , 0 ≤ k ≤ 3 (since is real)

Implies, Y(k) = {4, j2, 0, -j2}

**Circular time shift property of DFT**

**Statement:**

If DFT { x[n] } = X(k), then} = X(k), 0≤ k ≤ N

**Proof:** Consider the DFT viewed as a linear operator

This means that x[n] and X(k) are of same length N

From Inverse DFT definition, we know

, 0≤ n ≤ N-1 ………(1)

Implies, ………(2)

Since, the time shift is circular, we can write eq.(2) as

= ……(3)

= ] ……..(4)

(OR)

} = X(k) ……..(5) (Hence the proof)

**Problem to illustrate Circular Time Shift Property**

**Problem 1**

Find the 4-point DFT of the sequence, (1,-1,1,-1). Also using time shift property, find the DFT of the sequence, ?

**Solution:** Given N=4

= 1, = – j, = -1 and = j

= = , 0≤ k ≤ 3

= 1 x - 1 x + 1 x - 1 x

1 - + - ……(1)

Sub k=0 in eq.(1) , 1-1+1-1 = 0

Sub k=1 in eq.(1) , 1- + - = 1 + j -1 –j = 0

Sub k=2 in eq.(1) , 1- + - = 1- + -

1+1+1+1 =4

Sub k=3 in eq.(1) , 1- + - = 1- + -

1-j-1+j = 0

Therefore DFT (0,0,4,0)

G and we now found (0,0,4,0)

Applying circular time shift property, we get

DFT Y(k) = X(k) , where k=0,1,2,3 ……….(2)

Sub k=0 in eq.(2) , Y X(0) = 1 X 0 = 0

Sub k=1 in eq.(2) , Y X(1) = -1 X 0 = 0  
Sub k=2 in eq.(2) , Y X(2) = X(2) = 1 x 4 = 4

Sub k=3 in eq.(2) , Y X(3) = X(3) = -1 x 0 = 0

Hence, DFT Y (0,0,4,0)

**Problem 2**

Suppose is an 8-point sequence defined as {0,1,2,3,4,5,6,7}

1. Illustrate
2. If DFT { x[n] } = X(k),what is DFT {} ?

**Solution: (a)** Given N=8 and {0,1,2,3,4,5,6,7}

To generate , move the last 2 samples of to the beginning

That is , = {6,7,0,1,2,3,4,5}

It should be noted that is implicitly periodic with a period N=8

(b) Let

Applying circular time shift property, we get

DFT Y(k) = X(k) , where k=0,1,2,3,4,5,6,7

**Circular Convolution Property of DFT**

**Circular Convolution in Time**

Let be two sequences of length N each. Then

That is

(or)

, where n= 0,1,2,………(N-1)

**Circular Convolution in time-domain is equivalent to multiplication in frequency-domain**

DFT { = H(k) X(k), where k = 0,1,2,……….(N-1)

**Proof:** DFT { = DFT {

=

=

=

= H(k) X(k)

= RHS (Proved)

**Problems to illustrate** **Circular Convolution Property of DFT**

1. Given the sequences = {1,1,1} and = {1,-2,2}, compute the circular convolution of with N=3?

**Solution:**

Let = =

Here N=3 , now = {1,1,1} and = {1,-2,2}

= {2,-2,1} {obtained by folding = (1,-2,2) w.r.to origin}  
 = {1,2,-2} {obtained by circularly shifting = (2,-2,1)}

The table below demonstrates the computation of y[n] using above equation:

|  |  |  |  |
| --- | --- | --- | --- |
| **n** |  |  |  |
| 0 | {1,1,1} | {1,2,-2} | (1 x 1) + (1 x 2) + (1 x -2) = 1 |
| 1 | {1,1,1} | {-2,1,2} | (1 x -2) + (1 x 1) + (1 x 2) = 1 |
| 2 | {1,1,1} | {2,-2,1} | (1 x 2) + (1 x -2) + (1 x 1) = 1 |

**Answer**: Circular convolution = {1,1,1}

**Alternate Method**

**Circular convolution by matrix method**

=

=

= {1,1,1}

1. Apply DFT- IDFT formula method corresponding to the sequences = {1,2,0,1} and

= {2,2,1,1} and determine such that Y(k) is defined as Y(k) = X1 (k). X2 (k)?

**Solution**

**DFT calculation for 1st sequence x1[n]**  Given: N =4

=

Sub k=0 and n=0,1,2,3 we get ] = 4

Sub k=1 and n=0,1,2,3 we get ] = 1-j

Sub k=2 and n=0,1,2,3 we get ] = -2

Sub k=3 and n=0,1,2,3 we get ] = 1+j

Therefore

**DFT calculation for 2nd sequence x2[n]**

=

Sub k=0 and n=0,1,2,3 we get ] = 6

Sub k=1 and n=0,1,2,3 we get ] = 1-j

Sub k=2 and n=0,1,2,3 we get ] = 0

Sub k=3 and n=0,1,2,3 we get ] = 1+j

Therefore

**Product of DFTs** Y(k) = X1 (k). X2 (k)

Therefore

**IDFT of**

Sub n=0 and k=0,1,2,3 we get y[0] =6

Sub n=1 and k=0,1,2,3 we get y[1] = 5

Sub n=2 and k=0,1,2,3 we get y[2] = 6

Sub n=3 and k=0,1,2,3 we get y[3] = 7

Therefore sequence = {6,5,6,7}

**Problem to illustrate Linear through Circular Convolution**

3) Given the sequences x[n] = {1, 2, 3, 1} and h[n] = {1, 1, 1}. Obtain the result of linear convolution using Circular convolution?

**Solution**

**Linear convolution y[n] through Circular convolution**

Length of x[n] & h[n] is L=4 and M=3 respectively

Linear convolution output length = L+M-1 =4+3-1=6 samples

Perform zero padding and append with appropriate no. of zeros to make both the sequences of length= L+M-1=6 samples. Hence now x[n] = {1, 2, 3, 1, 0, 0} and h[n] = {1, 1, 1, 0, 0, 0}. Hence now express h[n] as 6 x 6 matrix and express x[n] as 6 x 1 column matrix.

Multiply h[n] with x[n] we get y[n] as 6 x 1 column matrix.

Output sequence = {1,3,6,6,4,1}

**Parseval’s Theorem in DFT**

**Statement:**

**“**For complex valued sequences x(n) and y(n), if DFT {x(n)} is X(k) and DFT {y(n)} is Y(k) then

……………(1)

Parseval’s theorem also states that, “energy of the signal in time domain can be expressed in terms of the frequency components {X(k)} in the frequency domain”. Parseval’s theorem in DFT is stated as ……………(2)

**Proof of Parseval’s Theorem**

From definition of IDFT, we have

Take conjugate on both sides,

LHS =

=

= 1/N …..(1)

= RHS (Proved)

Suppose if y[n] = x[n], then

Taking Magnitude we get,

…….(2) = RHS (Proved)

**Problem to illustrate significance of Parseval’s Theorem**

If the energy of the signal in time domain can be expressed in terms of the frequency components {X(k)} in the frequency domain, then apply this property to this signal x[n] = {1,2,0,3,-2,4,7,5} and evaluate the following: (i) X(4) (ii) (iii) ?

**Solution:**

From given data, N=8 , implies n=0 to 7

Here k=4

=

= =

= 1-2-3-2-4+7-5

1. Therefore
2. By using IDFT definition

Sub n=0 we get

Therefore] = 8 (1) = 8

1. By Parseval’s identity we know

8 {} =

8 {1+4+0+9+4+16+49+25} =

Therefore

**Introduction to fast Fourier Transform (FFT)**

The fast Fourier Transform (FFT) refers to the algorithms that compute the discrete Fourier

Transform (DFT) in a numerically efficient manner. Radix-2 FFT algorithms often present two frequently used algorithms: decimation-in-time (DIT) FFT and decimation-in-frequency (DIF) FFT. Radix-2 means, the number of points is assumed as a power of 2, that is N=2p, where p is some integer. This would require p stages, where p = log2N

**Computational Advantages of FFT algorithm over Direct DFT Evaluation**

Overall both the decimation-in-time (DIT) FFT and decimation-in-frequency (DIF) FFT minimizes the computational complexity over Direct DFT evaluation. Since there are log2N stages and each stage has N complex multiplications, hence the total number of complex multiplications have reduced from N2 (direct evaluation) to N log2N (after decimation process).

The total number of complex multiplications required for calculating N-point DFT using DIT-FFT algorithm based on Cooley-Tukey is. Also, since there are two additions per butterfly,

butterflies per stage and log2N stages, the total number of complex additions required for calculating N-point DFT using DIT-FFT is N log2N

**In-place computations**

Other than speedy computations, another advantage that naturally comes from this algorithm is the reduction in storage requirement. Each stage involves only butterflies per stage and these butterflies operate on a pair of complex numbers and produce another pair of complex numbers. Once the output pair is calculated, we don’t need the input pair anymore. Hence the output pair can be stored in the same locations as input pair. Thus, for an v, we need N complex registers or 2N real registers for the first stage. The same registers will be used for other stages to follow. This is known as in-place computation.

**DIT-FFT Algorithm versus DIF-FFT Algorithm**

The procedure in DIT-FFT Algorithm is that the input data samples appear in bit-reversal order and the final DFT (Output) will be obtained in the natural (normal) order.

The procedure in DIF-FFT Algorithm is that the input data samples appear in the normal order and the final DFT (Output) will be obtained in the bit-reversal order.

**Problem to illustrate Radix-2 Decimation-in-Frequency** (**DIF) FFT algorithm**

Determine the 8-point DFT of the sequence x[n] = {1, 2, 3, 4, 1, 0, 1, 2} using Radix-2 Decimation-in-Frequency FFT algorithm?

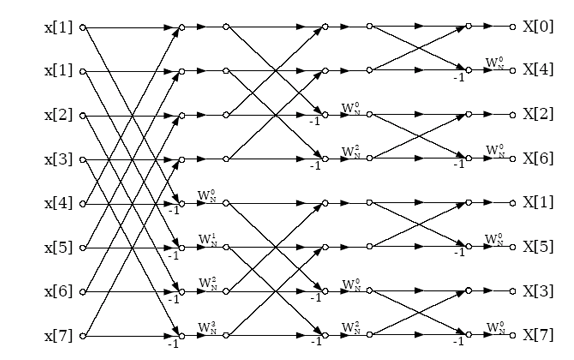
**Solution**

To compute 8-point DFT using DIF-FFT algorithm

No. of stages =log2 8 = 3. Draw the butterfly flowgraph.

The input sequence must be written in normal order. The output will be in bit reversal order. The values of Twiddle factor are

= 1, = - , = -j and = -



**Radix-2 DIF-FFT Butterfly Diagram**

**Calculation of Stage 1 output:**

X1[0] = 1 + 1 = 2 ; X1[4] = 2 + 0 = 2; X1[2] = 3 + 1 =4; X1[6] = 4 + 2 =6;

X1[1] = (1 – 1) W80 = 0; X1[5] = (2 – 0) W81= 2 ( - ) = ;

X1[3] = (3 – 1) W82 = 2(-j) = -2j ;

X1[7] = (4 – 2) W83 = 2 ( - ) = ;

**Calculation of Stage 2 output:**

X2[0] = 2 + 4 = 6 ; X2[4] = 2 + 6 = 8;

X2[2] = (2 – 4) W80 = -2; X2[6] = (2 – 6) W82= -4 (-j) = 4j ;

X2[1] = 0 – j2 = -j2 ; X2[5] = = ;

X2[3] = (0 + j2) W80 = 2j ; X2[7] = ( +) W82 = ;

**Calculation of Stage 3 output:**

X[0] = 6 + 8 = 14 ; X[4] = (6 - 8 ) W80 = -2 ; X[2] = -2 + j4; ; X[6] = -2 - j4

X[1] = -j2 - j ; X[5] = -j2 + j = ;

X[3] = j2 - j = ; X[7] = j2 + j = ;

Rearrange finally and write the output of DFT in normal order

**Answer:**

X(k) = {14, -2 + j4, , -2, , -2 - j4**,**  }

**Problem to illustrate Radix-2 Decimation-in-Time** (**DIT) FFT algorithm**

Determine the 8-point DFT of the sequence x[n] = {1, 1, 1, 1, 0 , 0, 0, 0} using Radix-2 Decimation-in-Time FFT algorithm?

**Solution**

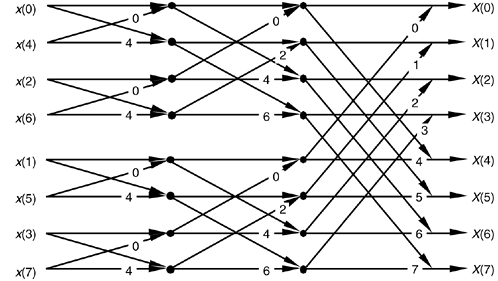
To compute 8-point DFT using DIT-FFT algorithm

No. of stages =log2 8 = 3. Draw the butterfly flowgraph.

The input sequence must be in bit reversal order. The output will be in normal order.

The values of Twiddle factor are

= 1, = - , = -j and = -



**Computation of output during stage 1**

X1[0] = 1 + 0 x = 1

X1[1] = 1 - 0 x = 1

X1[2] = 1 + 0 x = 1

X1[3] = 1 - 0 x = 1

X1[4] = 1 + 0 x = 1

X1[5] = 1 - 0 x = 1

X1[6] = 1 + 0 x = 1

X1[7] = 1 - 0 x = 1

**Computation of output during stage 2**

X2[0] = 1 + x 1 = 1

X2[1] = 1 + 1 x = 1 - j

X2[2] = 1 - x 1 = 0

X2[3] = 1 -1 x = 1 + j

X2[4] = 1 + x 1 = 2

X2[5] = 1 + x 1= 1 - j

X2[6] = 1 - x 1 = 0

X2[7] = 1 - x 1 = 1 + j

**Computation of output during stage 3**

X3[0] =X[0] = 2 + 2= 4

X3[1] =X[1] = 1 – j + (1 – j) = 1 – j 2.414

X3[2] =X[2] = 0 + 0 x = 0

X3[3] =X[3] = (1 + j) + (1 + j) = 1 – j 0.414

X3[4] =X[4] = 2 - 2= 0

X3[5] =X[5] = (1 – j) - (1 – j) = 1 + j 0.414

X3[6] =X[6] = 0 - 0 x = 0

X3[7] =X[7] = (1 + j) - (1 + j) = 1 + j 2.414

**Answer:**

DFTX(k) = {4, 1 – j 2.414, 0, 1 – j 0.414,0, 1 + j 0.414, 0, 1 + j 2.414}

**Applications of Fast Fourier Transform in DSP**

Fast Fourier Transform (FFT) is especially used to efficiently compute the DFT and minimize the computational complexities involved while performing Direct DFT evaluation.

Currently FFT has lots of applications and is extensively used in audio processing, radar, sonar, software defined radio (SDR) etc. In all these applications, a time-domain signal is converted by the FFT into a frequency-domain representation of the signal. The fundamental principle behind the Fourier Transform is that it is possible to decompose a time-varying signal into sine waves. For instance, FFT can be used to calculate the amplitudes and frequencies of all the sine waves that make up an audio signal.

An important application of the FFT algorithm is in FIR linear filtering of long data sequences. There are two methods used for filtering a long data sequence with an FIR filter based on the use of DFT namely, the overlap-add and the overlap-save methods. Even if the FIR filter has linear phase, even then the number of computations per output point is still less with the FFT-based method.

The computation of the cross correlation between two sequences by means of FFT algorithm is similar to linear filtering problem. This indicates that use of FFT algorithm has reduced the cross correlation to an equivalent convolution problem (i.e. a linear FIR filtering problem). Also from the computational point of view, FFT based method is superior when the filter length is relatively large.