2.2 N element Linear array

Module:2 Linear and Planar Arrays

Course: BECE305L – Antenna and Microwave Engineering

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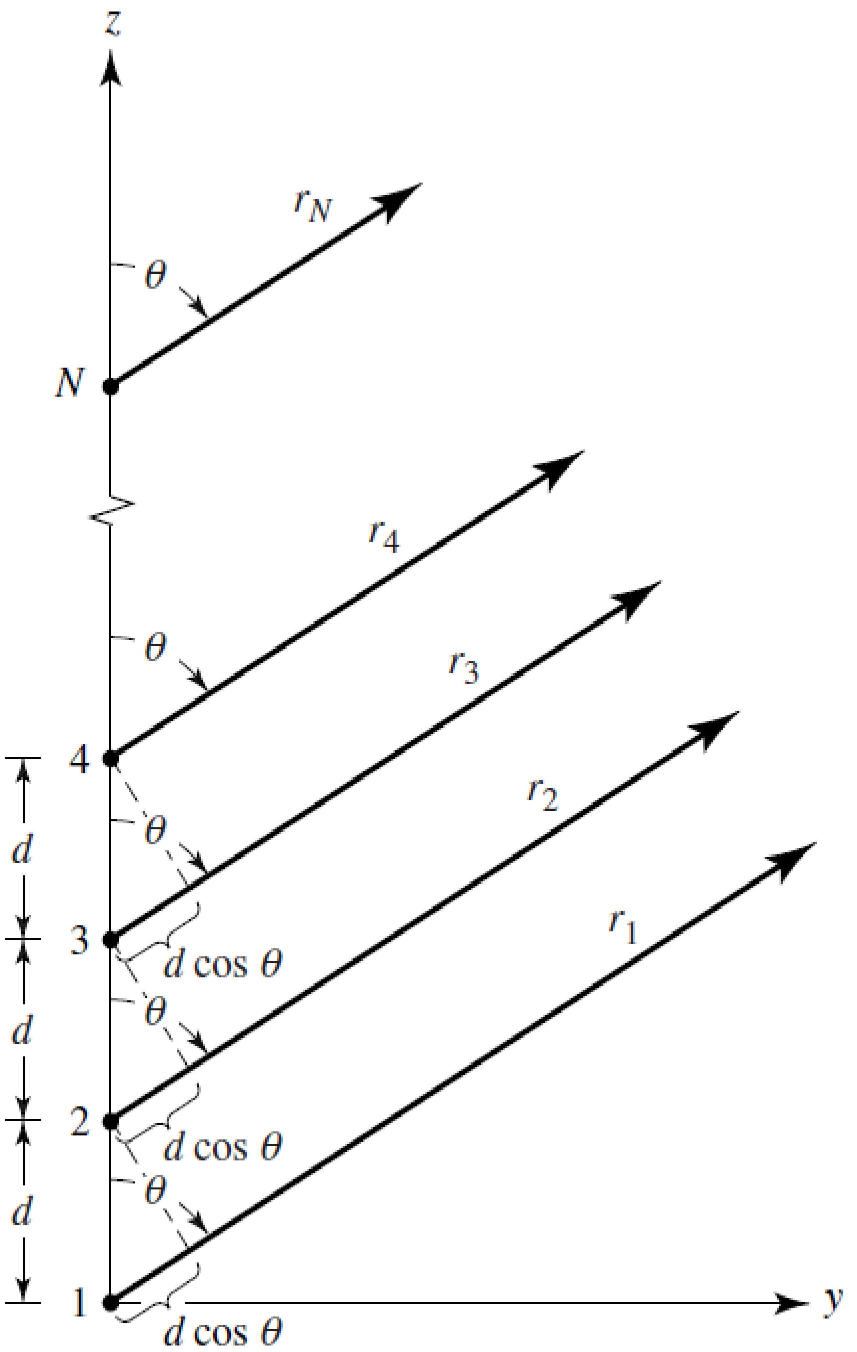
Module:2 Linear and Planar Arrays

- Two element array, N-element linear array - broadside array, End fire array - Directivity, radiation pattern, pattern multiplication.   Non-uniform excitation - Binomial, Chebyshev distribution, Arrays: Planar array, circular array, Phased Array antenna (Qualitative study).

- Source of the contents: Constantine A. Balanis - Antenna theory analysis and design (2016)

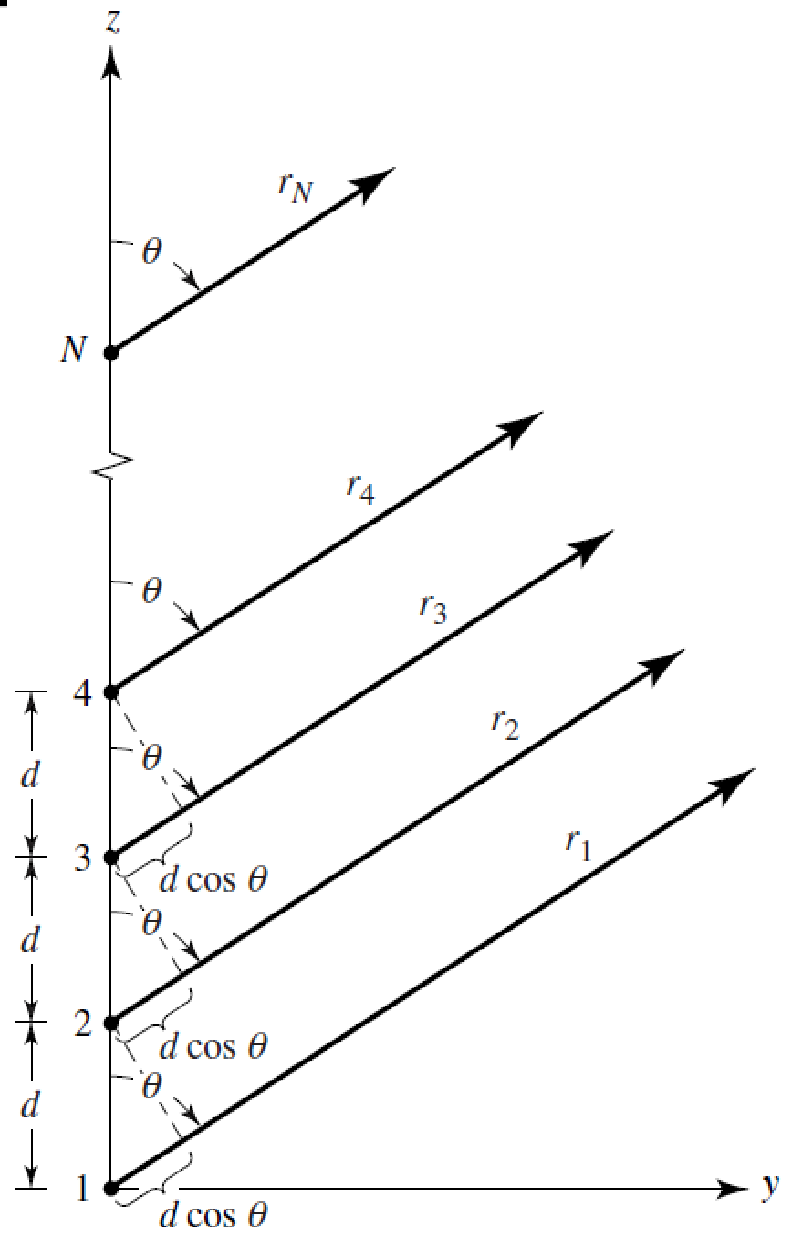
1. N element linear array

- generalize the method to include *N* elements
- assume that all the elements have **identical amplitudes** but each succeeding element has a β progressive phase lead current excitation relative to the preceding one (β represents the phase by which the current in each element leads the current of the preceding element).
- *An array of identical elements all of identical magnitude and each with a progressive phase is referred to as a **uniform array**.*



1. N element linear array

- **array factor** can be obtained by considering the elements to be point sources.
- If the actual elements are not isotropic sources, the **total field** can be formed by multiplying the array factor of the isotropic sources by the field of a single element.
- pattern multiplication rule (it applies only for arrays of identical elements).



$$AF = 1 + e^{+j(kd\cos\theta+\beta)} + e^{+j2(kd\cos\theta+\beta)} + \cdots + e^{j(N-1)(kd\cos\theta+\beta)}$$

$$AF = \sum_{n=1}^{N} e^{j(n-1)(kd\cos\theta+\beta)} \qquad AF = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

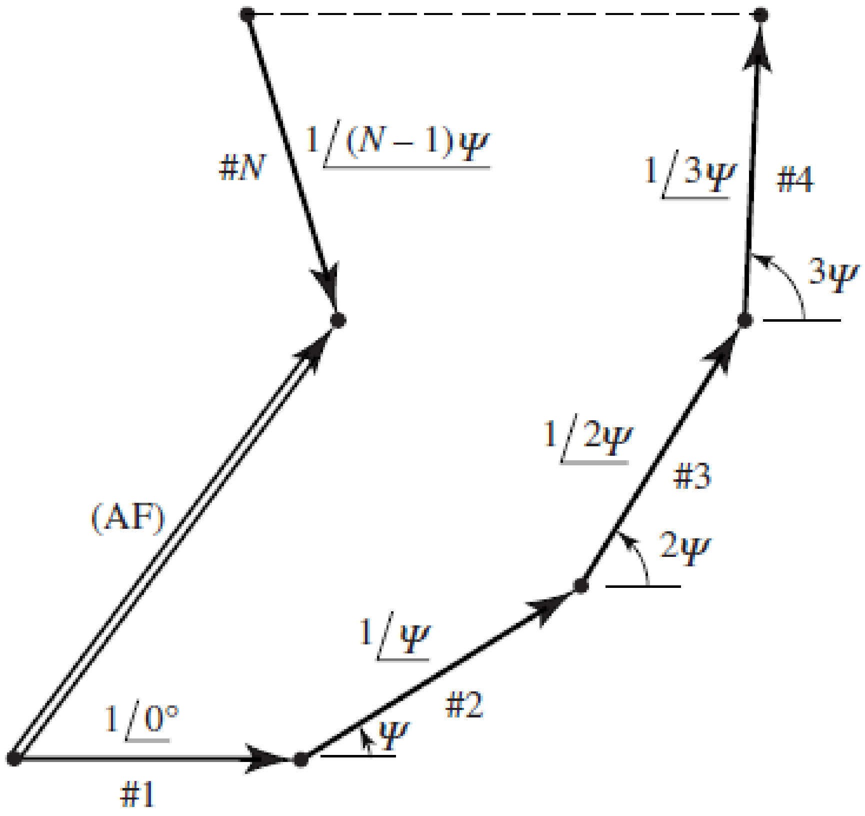
where $\psi = kd\cos\theta + \beta$

1. N element linear array

$$AF = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

- Contribution of every element (with phase shifts) towards observation point
- total array factor for the uniform array is a summation of exponentials, it can be represented by the vector sum of *N* phasors each of unit amplitude and **progressive phase ψ relative to the previous one**
- the amplitude and phase of the AF can be controlled in uniform arrays by properly selecting the **relative phase ψ between the elements**;
- in nonuniform arrays, the amplitude as well as the phase can be used to control the formation and distribution of the total array factor.



(b) Phasor diagram

1. N element linear array

$$\mathrm{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

- Multiplying both sides by $e^{j\psi}$

$$(\mathrm{AF})e^{j\psi} = e^{j\psi} + e^{j2\psi} + e^{j3\psi} + \cdots + e^{j(N-1)\psi} + e^{jN\psi}$$

- Subtracting AF from eqn above

$$\mathrm{AF}(e^{j\psi} - 1) = (-1 + e^{jN\psi})$$

$$\mathrm{AF} = \left[\frac{e^{jN\psi} - 1}{e^{j\psi} - 1}\right] = e^{j[(N-1)/2]\psi}\left[\frac{e^{j(N/2)\psi} - e^{-j(N/2)\psi}}{e^{j(1/2)\psi} - e^{-j(1/2)\psi}}\right]$$

$$= e^{j[(N-1)/2]\psi}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

- If the reference point is the physical center of the array,

$$\mathrm{AF} = \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

$$\mathrm{AF} \simeq \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\frac{\psi}{2}}\right]$$

$$\mathrm{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

1. N element linear array

- $\mathrm{AF} = \left[\dfrac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$ For small values of ψ, $\mathrm{AF} \simeq \left[\dfrac{\sin\left(\frac{N}{2}\psi\right)}{\frac{\psi}{2}}\right]$

- The max value of AF is N, hence, when normalized:

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

for small values of ψ

$$(\mathrm{AF})_n \simeq \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\frac{N}{2}\psi}\right]$$

$$f_N(x) = \left|\frac{\sin(Nx)}{N\sin(x)}\right| \quad N = 1, 3, 5, 10, 20$$

$$f(x) = \frac{\sin(x)}{x}$$

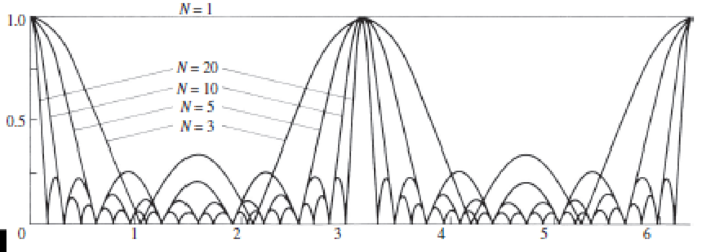
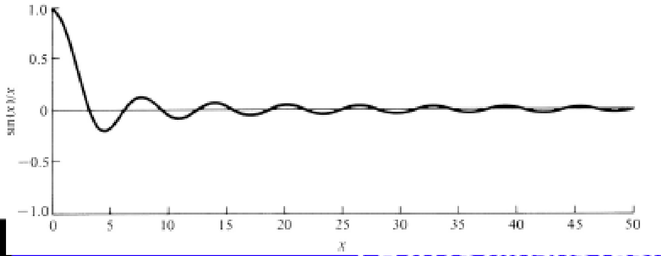


Figure II.1 Curves of $|\sin(Nx)/N\sin(x)|$ function.



$$\mathrm{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

1. N element linear array

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

$$(\mathrm{AF})_n \simeq \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\frac{N}{2}\psi}\right]$$

- **<u>To find null</u>**, equate $(\mathrm{AF})_n = 0$

$$\sin\left(\frac{N}{2}\psi\right) = 0 \Rightarrow \underbrace{\frac{N}{2}\psi|_{\theta=\theta_n} = \pm n\pi}_{} \Rightarrow \theta_n = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2n}{N}\pi\right)\right]$$

$n = 1, 2, 3, \ldots$

$n \neq N, 2N, 3N, \ldots$ with (6-10c)

$$\psi_{\theta_n} = \pm\frac{2n\pi}{N} = kd\cos\theta_n + \beta$$

$$\cos\theta_n = \frac{1}{kd}\left(-\beta \pm \frac{2n\pi}{N}\right) = \frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2n\pi}{N}\right)$$ This value should be ≤ 1

- At n=N, $(AF)_n = 1$ due to $\sin(0)/0$ format. The values of *n* determine the order of the nulls (first, second, etc.). Thus the number of nulls that can exist will be a function of the element separation *d* and the phase excitation difference β

1. N element linear array

$$\mathrm{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

- **<u>To find maximum values</u>**, $(AF)_n = 1$

  when ψ is small, AF has maxima at $m = 0$

$$(\mathrm{AF})_n \simeq \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\frac{N}{2}\psi}\right]$$

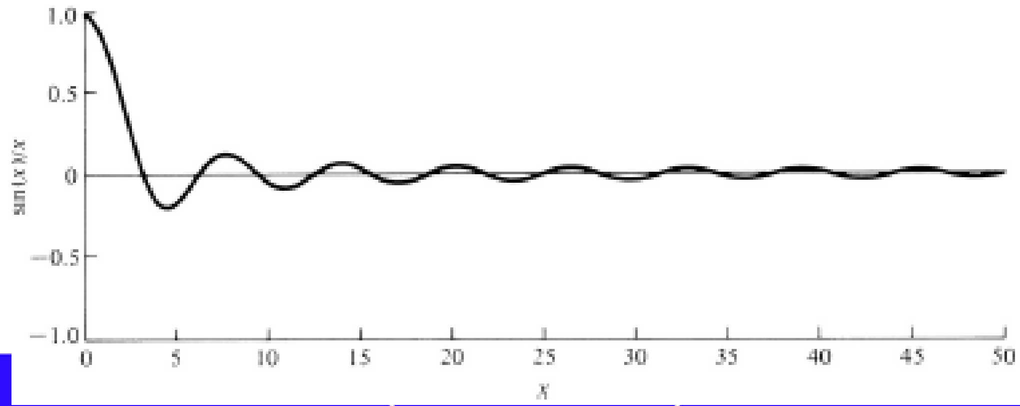
$$\frac{\psi}{2} = \frac{1}{2}(kd\cos\theta + \beta)|_{\theta=\theta_m} = \pm m\pi \Rightarrow \theta_m = \cos^{-1}\left[\frac{\lambda}{2\pi d}(-\beta \pm 2m\pi)\right]$$

$$m = 0, 1, 2, \ldots$$

$$f(x) = \frac{\sin(x)}{x}$$

$$\theta_m = \cos^{-1}\left(\frac{\lambda\beta}{2\pi d}\right)$$

which is the observation angle that makes $\psi = 0$.



$$\mathrm{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

1. N element linear array

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

$$(\mathrm{AF})_n \simeq \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\frac{N}{2}\psi}\right]$$

- **To find 3-dB point for the array factor**,

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

$$\frac{N}{2}\psi = \frac{N}{2}(kd\cos\theta + \beta)|_{\theta=\theta_h} = \pm 1.391$$

$$\Rightarrow \theta_h = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2.782}{N}\right)\right]$$

$$f_N(x) = \left|\frac{\sin(Nx)}{N\sin(x)}\right| \quad N = 1, 3, 5, 10, 20$$

Or $\theta_h = \frac{\pi}{2} - \sin^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2.782}{N}\right)\right]$

large values of $d (d \gg \lambda)$, it reduces to

$$\theta_h \simeq \left[\frac{\pi}{2} - \frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2.782}{N}\right)\right]$$

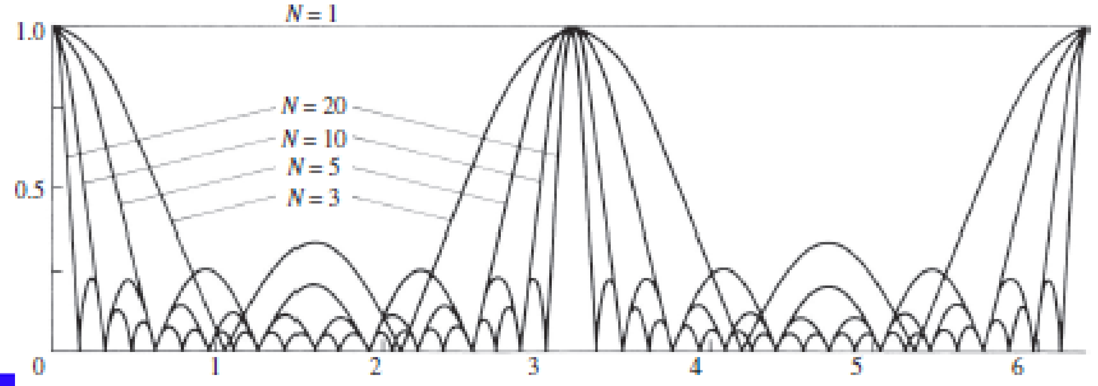


Figure II.1 Curves of $|\sin(Nx)/N\sin(x)|$ function.

$$\mathrm{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

1. N element linear array

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

$$(\mathrm{AF})_n \simeq \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\frac{N}{2}\psi}\right]$$

To find HPBW for the array factor,

- half-power beamwidth Θ_h can be found once the angles of the first maximu $\Theta_h = 2|\theta_m - \theta_h|$ e half-power point (θ_h) are determined.

1. N element linear array

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

$$\mathrm{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

$$(\mathrm{AF})_n \simeq \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\frac{N}{2}\psi}\right]$$

- **<u>To find secondary maxima (maxima of minor lobes),</u>** Numerator appears to be maximum

$$\sin\left(\frac{N}{2}\psi\right) = \sin\left[\frac{N}{2}(kd\cos\theta+\beta)\right]\Big|_{\theta=\theta_s} \simeq \pm 1 \Rightarrow \frac{N}{2}(kd\cos\theta+\beta)\Big|_{\theta=\theta_s}$$

$$\simeq \pm\left(\frac{2s+1}{2}\right)\pi \Rightarrow \theta_s \simeq \cos^{-1}\left\{\frac{\lambda}{2\pi d}\left[-\beta \pm \left(\frac{2s+1}{N}\right)\pi\right]\right\}$$

$$f_N(x) = \left|\frac{\sin(Nx)}{N\sin(x)}\right| \quad N = 1, 3, 5, 10, 20$$

$$\psi = kd\cos\theta + \beta \qquad s = 1, 2, 3, \ldots$$

$$\theta_s \simeq \frac{\pi}{2} - \sin^{-1}\left\{\frac{\lambda}{2\pi d}\left[-\beta \pm \left(\frac{2s+1}{N}\right)\pi\right]\right\}, \quad s = 1, 2, 3, \ldots$$

large values of $d (d \gg \lambda)$

$$\theta_s \simeq \frac{\pi}{2} - \frac{\lambda}{2\pi d}\left[-\beta \pm \left(\frac{2s+1}{N}\right)\pi\right], \quad s = 1, 2, 3, \ldots$$

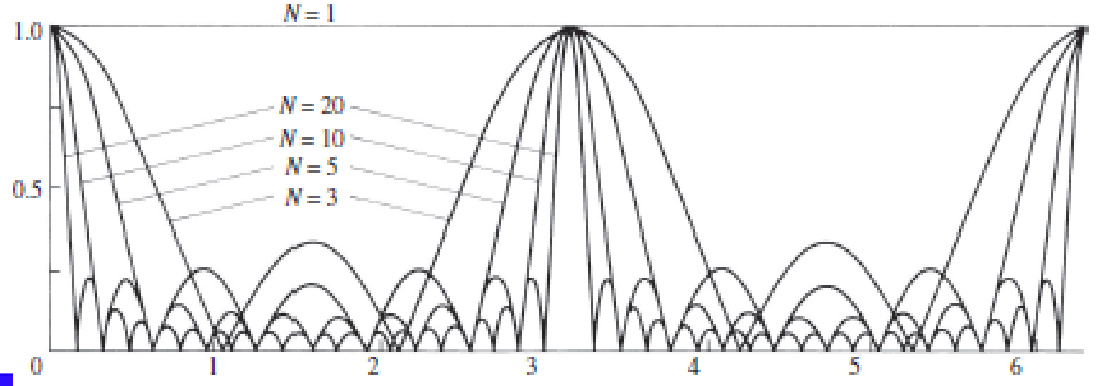


Figure II.1 Curves of $|\sin(Nx)/N\sin(x)|$ function.

1. N element linear array

$$(AF)_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

$$AF = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

$$(AF)_n \simeq \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\frac{N}{2}\psi}\right]$$

- **To find secondary maxima (maxima of minor lobes),** Numerator appears to be maximum

$$\theta_s \simeq \frac{\pi}{2} - \frac{\lambda}{2\pi d}\left[-\beta \pm \left(\frac{2s+1}{N}\right)\pi\right], \quad s = 1, 2, 3, \ldots$$

$$f_N(x) = \left|\frac{\sin(Nx)}{N\sin(x)}\right| \quad N = 1, 3, 5, 10, 20$$

- AF magnitude gets updated as

$$(AF)_n \simeq \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\frac{N}{2}\psi}\right]_{\substack{\theta=\theta_s \\ s=1}} = \frac{2}{3\pi} = 0.212$$

In dB: $$(AF)_n = 20\log_{10}\left(\frac{2}{3\pi}\right) = -13.46 \text{ dB}$$

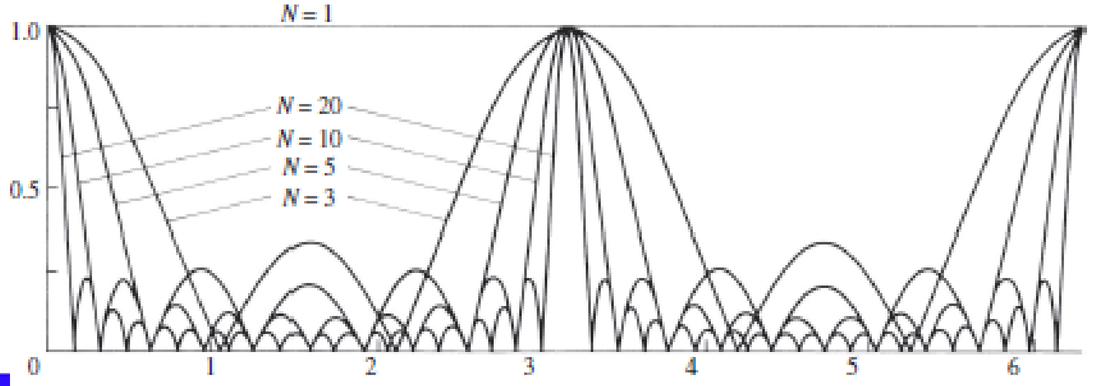


Figure II.1 Curves of $|\sin(Nx)/N\sin(x)|$ function.

$$\mathrm{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

1. N element linear array

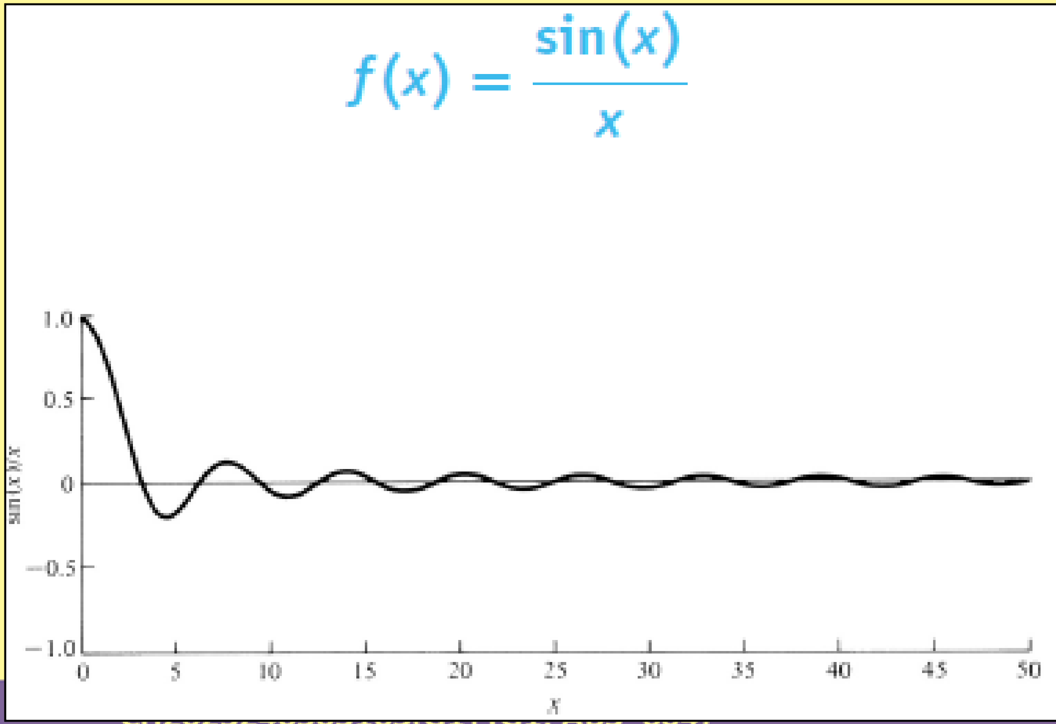
$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

$$(\mathrm{AF})_n \simeq \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\frac{N}{2}\psi}\right]$$

- **<u>To find secondary maxima (maxima of minor lobes),</u>** maximum of the first minor lobe of AF when ψ is small

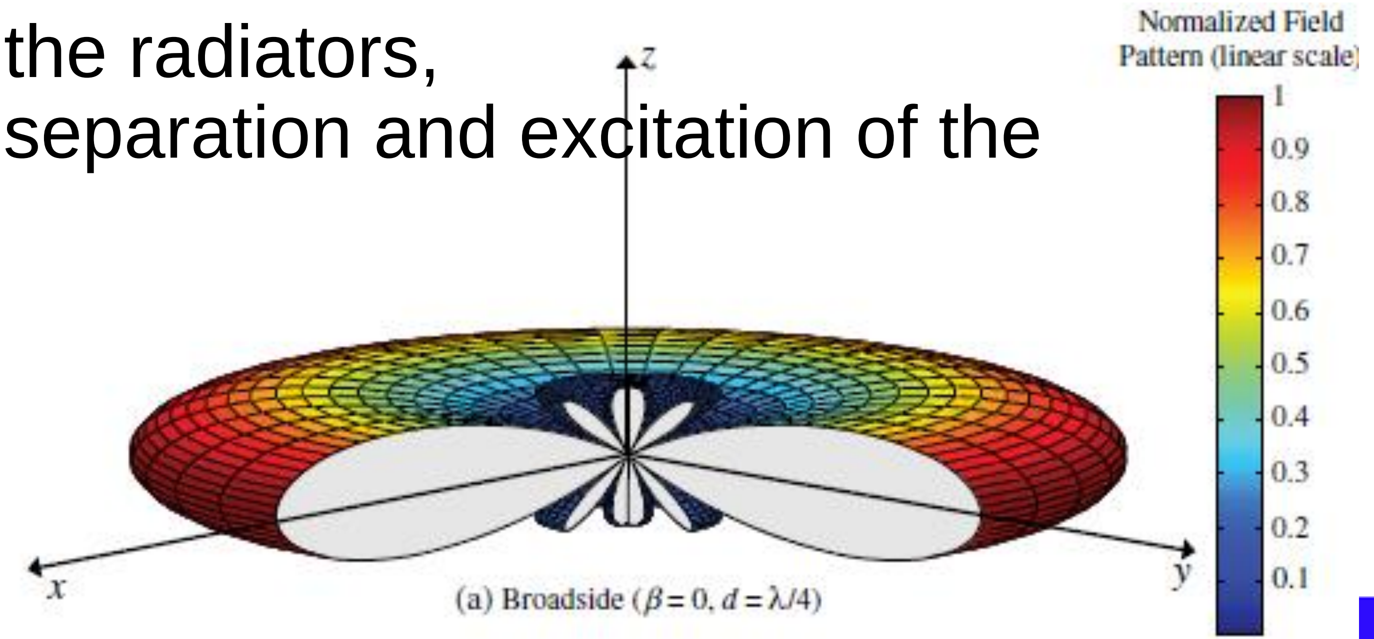
$$\frac{N}{2}\psi = \frac{N}{2}(kd\cos\theta + \beta)|_{\theta=\theta_s} \simeq \pm\left(\frac{3\pi}{2}\right)$$

$$\theta_s = \cos^{-1}\left\{\frac{\lambda}{2\pi d}\left[-\beta \pm \frac{3\pi}{N}\right]\right\}$$



2. Broad side array

- In many applications it is desirable to have the maximum radiation of an array directed normal to the axis of the array [broadside; $\theta_0 = 90°$]
- the maxima of the single element and of the array factor should both be directed toward $\theta_0 = 90°$.
- requirements of
  a) single elements : choice of the radiators,
  b) array factor: by the proper separation and excitation of the individual radiators.
- array factor to "radiate" efficiently broadside



(a) Broadside ($\beta = 0$, $d = \lambda/4$)

$$(AF)_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

$$AF = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

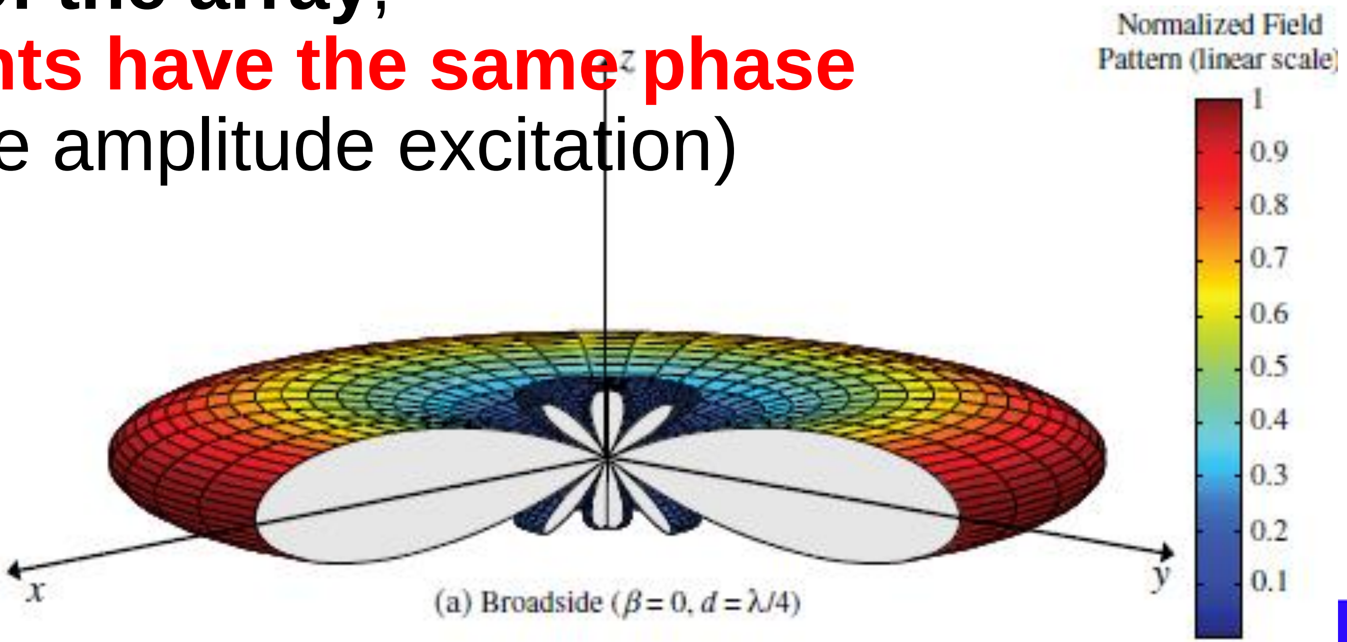
where $\psi = kd\cos\theta + \beta$

2. Broad side array

- **First maximum of the array factor occurs when ψ at first maximum will be** $\psi = kd\cos\theta + \beta = 0$
- We want it to occur at $\theta = 90°$

$$\boxed{\psi = kd\cos\theta + \beta|_{\theta=90°} = \beta = 0}$$

- **to have the maximum** of the array factor of a uniform linear array **directed broadside to the axis of the array**,
  it is necessary that **all the elements have the same phase excitation** (in addition to the same amplitude excitation)



(a) Broadside ($\beta = 0$, $d = \lambda/4$)

2. Broad side array

$$\psi = kd\cos\theta + \beta|_{\theta=90^\circ} = \beta = 0$$

$$(AF)_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

$$(AF)_n \simeq \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\frac{N}{2}\psi}\right]$$

- The separation between the elements can be of any value. To ensure that there are **no principal maxima in other directions**, which are referred to as ***grating lobes***, the **separation between the elements should not be equal to multiples of a wavelength** ($d \neq n\lambda$, $n = 1, 2, 3\ldots$) when $\beta = 0$.
- If $d = n\lambda$, $n = 1, 2, 3, \ldots$ and $\beta = 0$, this makes the array factor attain its maximum value.

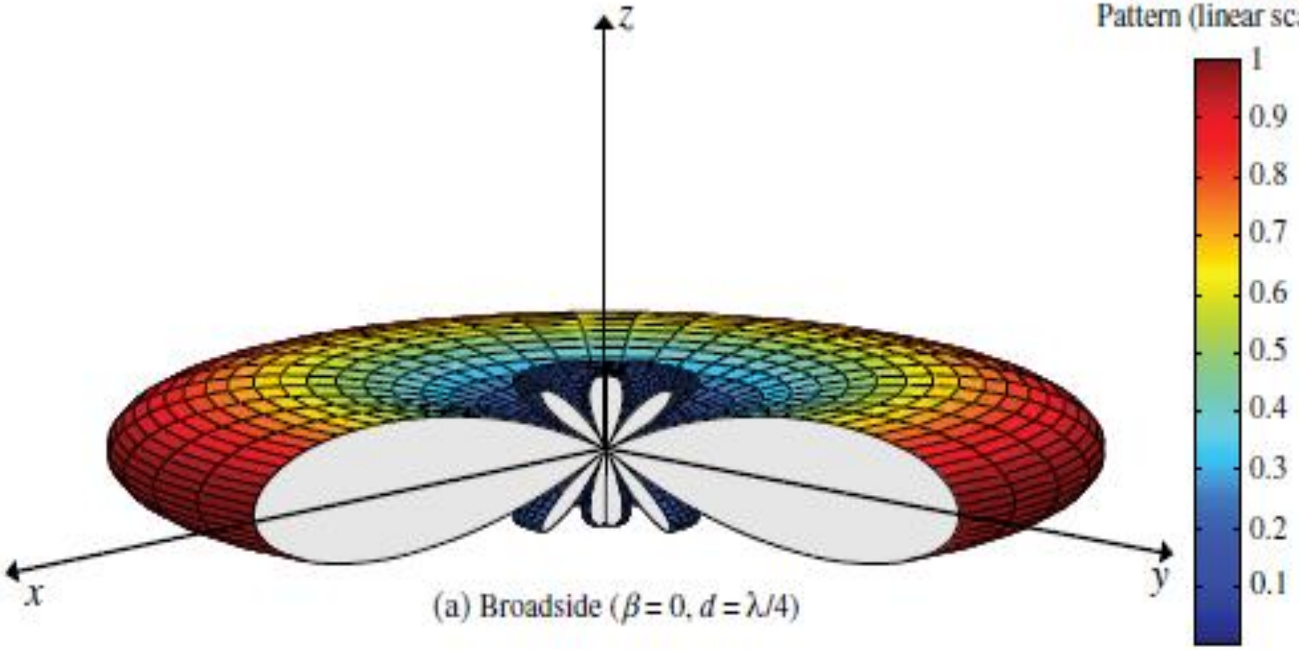
$$\psi = kd\cos\theta + \beta\Big|_{\substack{d=n\lambda \\ \beta=0 \\ n=1,2,3,\ldots}} = 2\pi n\cos\theta\Big|_{\theta=0^\circ,180^\circ} = \pm 2n\pi$$

- Thus for a uniform array with $\beta = 0$ and $d = n\lambda$, in addition to having the maxima of the array factor directed broadside ($\theta_0 = 90^\circ$) to the axis of the array, there are additional maxima directed along the axis ($\theta_0 = 0^\circ, 180^\circ$) of the array (end-fire radiation).

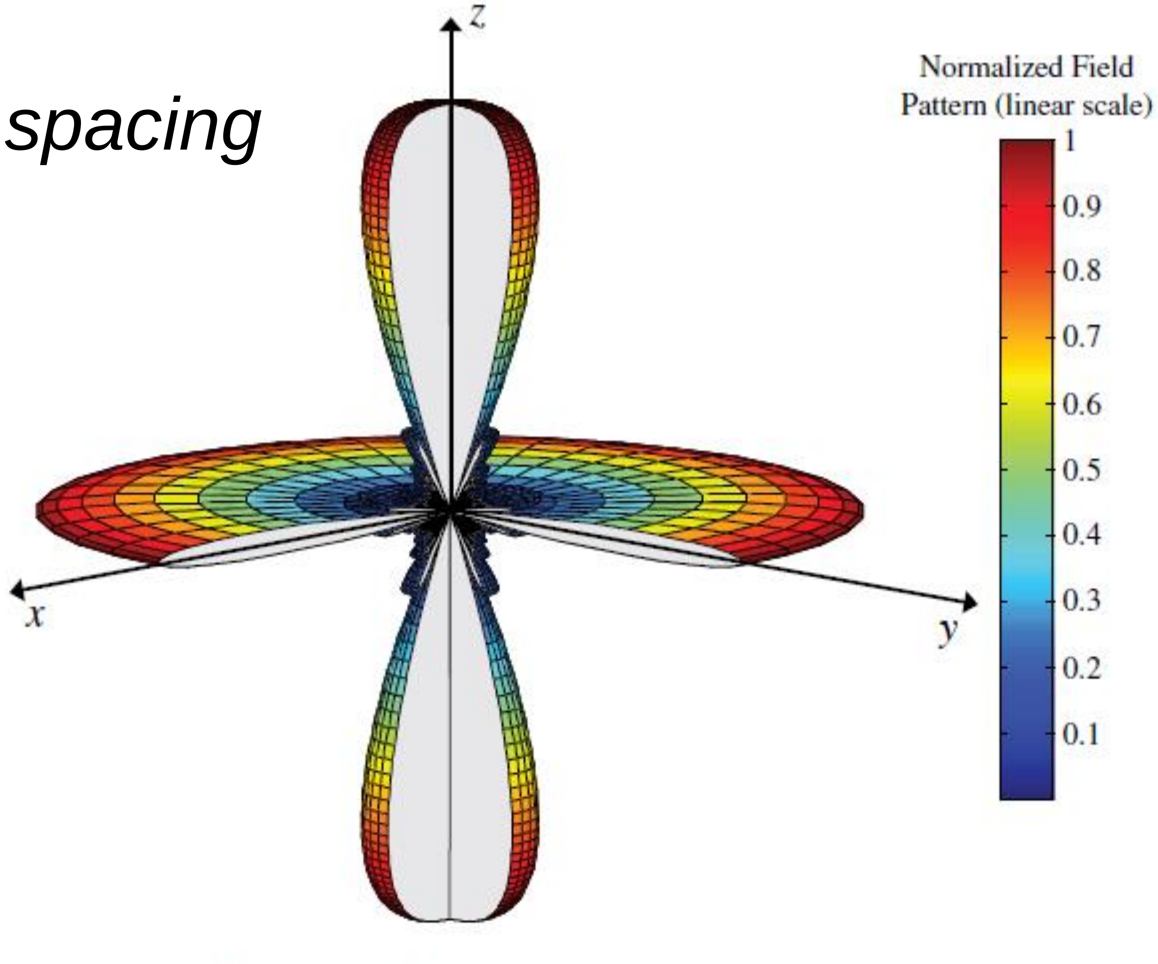
2. Broad side array

$$(AF)_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

- avoid multiple maxima, in addition to the main maximum, which are referred to as *grating lobes*.
- *To avoid any grating lobe, the largest spacing between the elements should be less than one wavelength* (dmax < λ).



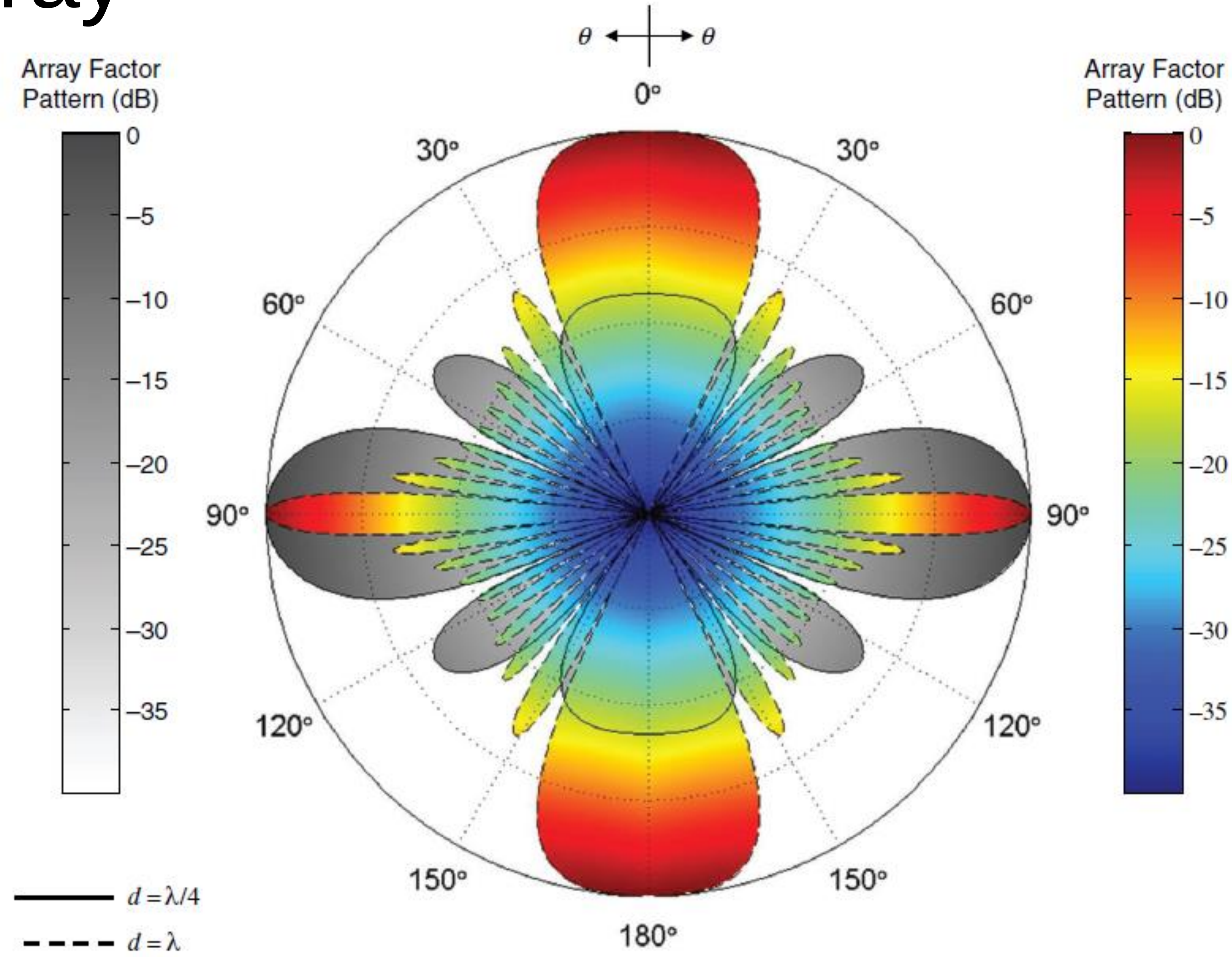
(a) Broadside ($\beta = 0, d = \lambda/4$)



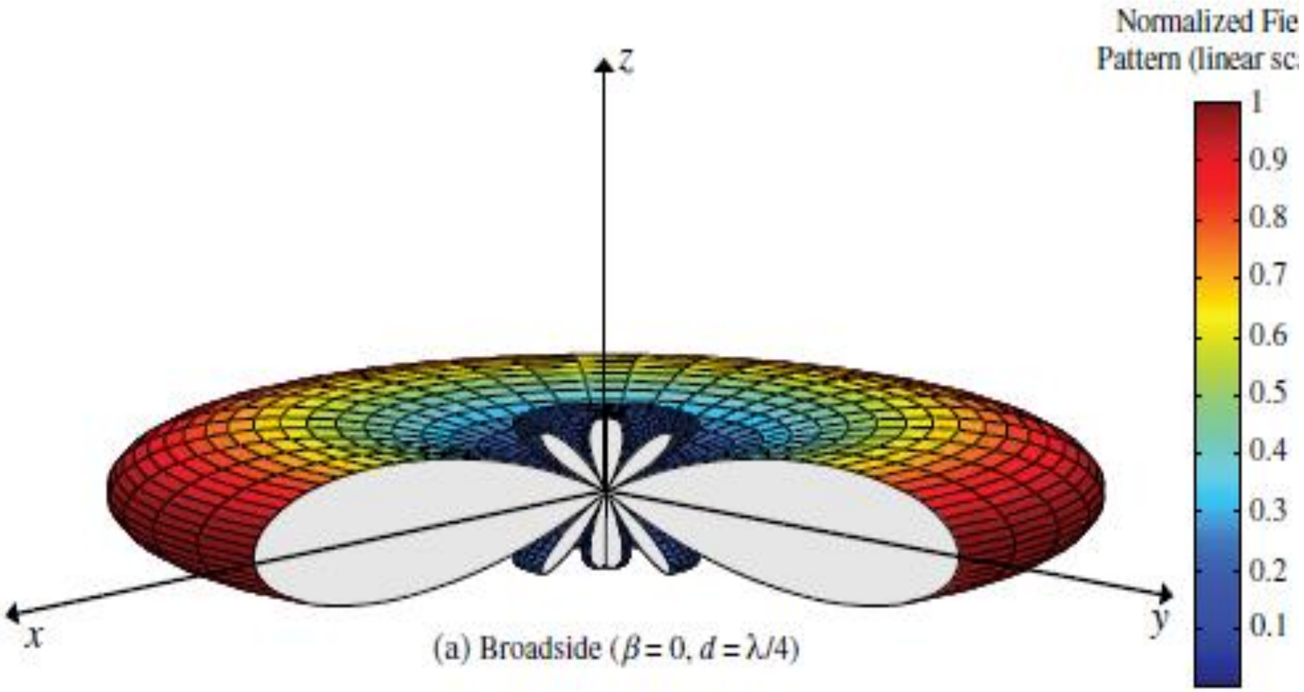
(b) Broadside/end-fire ($\beta = 0, d = \lambda$)

2. Broad side array

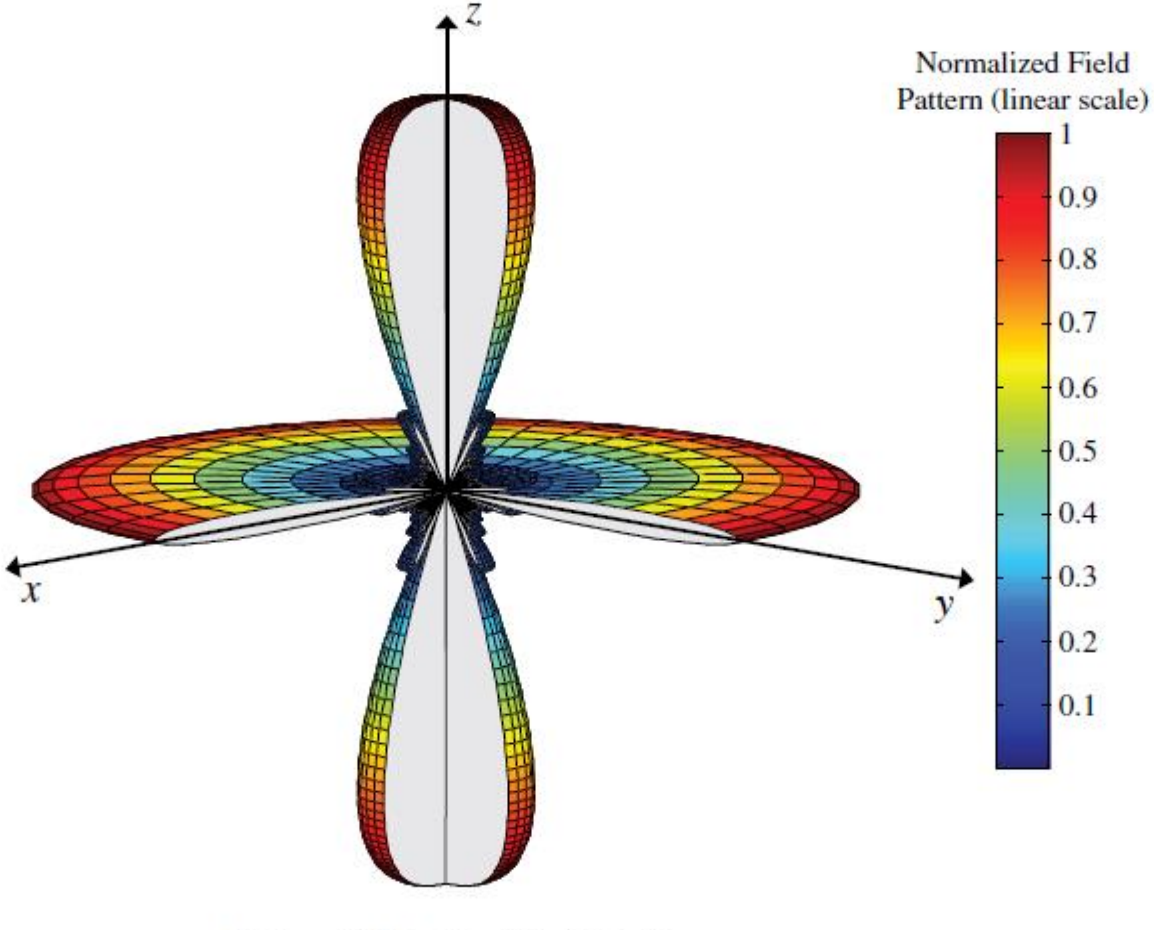
- Comparison



- If the spacing between the elements is chosen between $\lambda < d < 2\lambda$, then the maximum of (a) toward $\theta_0 = 0^\circ$ shifts toward the angular region $0^\circ < \theta_0 < 90^\circ$ while the maximum toward $\theta_0 = 180^\circ$ shifts toward $90^\circ < \theta_0 < 180^\circ$. When $d = 2\lambda$, there are maxima toward 0°, 60°, 90°, 120° and 180°.



(a) Broadside ($\beta = 0$, $d = \lambda/4$)



(b) Broadside/end-fire ($\beta = 0$, $d = \lambda$)

2. Broad side array

- Points of
  Nulls
  Maxima
  Half power
  Minor Lobe maxima

TABLE 6.1 Nulls, Maxima, Half-Power Points, and Minor Lobe Maxima for Uniform Amplitude Broadside Arrays

NULLS	$\theta_n = \cos^{-1}\left(\pm\frac{n}{N}\frac{\lambda}{d}\right)$ $n = 1, 2, 3, \ldots$ $n \neq N, 2N, 3N, \ldots$
MAXIMA	$\theta_m = \cos^{-1}\left(\pm\frac{m\lambda}{d}\right)$ $m = 0, 1, 2, \ldots$
HALF-POWER POINTS	$\theta_h \simeq \cos^{-1}\left(\pm\frac{1.391\lambda}{\pi N d}\right)$ $\pi d/\lambda \ll 1$
MINOR LOBE MAXIMA	$\theta_s \simeq \cos^{-1}\left[\pm\frac{\lambda}{2d}\left(\frac{2s+1}{N}\right)\right]$ $s = 1, 2, 3, \ldots$ $\pi d/\lambda \ll 1$

2. Broad side array

- Beamwidths

TABLE 6.2 Beamwidths for Uniform Amplitude Broadside Arrays

FIRST-NULL BEAMWIDTH (FNBW)	$\Theta_n = 2\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{\lambda}{Nd}\right)\right]$
HALF-POWER BEAMWIDTH (HPBW)	$\Theta_h \simeq 2\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{1.391\lambda}{\pi Nd}\right)\right]$ $\pi d/\lambda \ll 1$
FIRST SIDE LOBE BEAMWIDTH (FSLBW)	$\Theta_s \simeq 2\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{3\lambda}{2dN}\right)\right]$ $\pi d/\lambda \ll 1$

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

2.1 Broad side array: Directivity

- to have the first maximum directed toward $\theta_0 = 90^\circ$, then ***ψ* at first maximum will be**

$$\psi = kd\cos\theta + \beta|_{\theta=90^\circ} = \beta = 0$$

- to have the maximum of the array factor of a uniform linear array **directed broadside to the axis of the array**, it is necessary that all the elements have **the same phase excitation** (in addition to the same amplitude excitation).
- Array factor of Broadside array

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\sin\left(\frac{1}{2}kd\cos\theta\right)}\right]$$

- for a small spacing between the elements ($d \ll \lambda$) can be approximated

$$(\mathrm{AF})_n \simeq \left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\left(\frac{N}{2}kd\cos\theta\right)}\right]$$

$$\psi = kd\cos\theta + \beta|_{\theta=90^\circ} = \beta = 0$$

2.1 Broad side array: Directivity

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\sin\left(\frac{1}{2}kd\cos\theta\right)}\right]$$

- The radiation intensity can be written as

$$U(\theta) = [(\mathrm{AF})_n]^2 = \left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\frac{N}{2}kd\cos\theta}\right]^2 = \left[\frac{\sin(Z)}{Z}\right]^2$$

$$Z = \frac{N}{2}kd\cos\theta$$

$$(\mathrm{AF})_n \simeq \left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\left(\frac{N}{2}kd\cos\theta\right)}\right]$$

- **Directivity** is $D_0 = \frac{4\pi U_{\max}}{P_{\mathrm{rad}}} = \frac{U_{\max}}{U_0}$
- Hence we need U_{max} (max intensity) and U_0 (Average intensity)
- **Max Radiation intensity** is $U_{max} = 1$ at Z=0 (when cosθ=0 with $\theta = 90°$)

$$\psi = kd\cos\theta + \beta|_{\theta=90^\circ} = \beta = 0$$

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\sin\left(\frac{1}{2}kd\cos\theta\right)}\right]$$

2.1 Broad side array: Directivity

- The average radiation intensity can be written as

$$(\mathrm{AF})_n \simeq \left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\left(\frac{N}{2}kd\cos\theta\right)}\right]$$

$$U_0 = \frac{1}{4\pi}P_{\mathrm{rad}} = \frac{1}{2}\int_0^\pi \left[\frac{\sin(Z)}{Z}\right]^2 \sin\theta\, d\theta$$

$$= \frac{1}{2}\int_0^\pi \left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\frac{N}{2}kd\cos\theta}\right]^2 \sin\theta\, d\theta$$

$$Z = \frac{N}{2}kd\cos\theta$$

$$dZ = -\frac{N}{2}kd\sin\theta\, d\theta$$

$$U(\theta) = [(\mathrm{AF})_n]^2 = \left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\frac{N}{2}kd\cos\theta}\right]^2 = \left[\frac{\sin(Z)}{Z}\right]^2$$

$$= -\frac{1}{Nkd}\int_{+Nkd/2}^{-Nkd/2} \left[\frac{\sin Z}{Z}\right]^2 dZ$$

For large array ($Nkd/2 \rightarrow$large),

$$Z = \frac{N}{2}kd\cos\theta$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\mathrm{rad}}} = \frac{U_{\max}}{U_0}$$

$$U_{max} = 1$$

$$\psi = kd\cos\theta + \beta|_{\theta=90^\circ} = \beta = 0$$

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\sin\left(\frac{1}{2}kd\cos\theta\right)}\right]$$

2.1 Broad side array: Directivity

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$$= \frac{1}{2}\int_0^\pi \left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\frac{N}{2}kd\cos\theta}\right]^2 \sin\theta\, d\theta$$

$$Z = \frac{N}{2}kd\cos\theta$$

$$dZ = -\frac{N}{2}kd\sin\theta\, d\theta$$

$$U(\theta) = [(\mathrm{AF})_n]^2 = \left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\frac{N}{2}kd\cos\theta}\right]^2 = \left[\frac{\sin(Z)}{Z}\right]^2$$

$$= -\frac{1}{Nkd}\int_{+Nkd/2}^{-Nkd/2}\left[\frac{\sin Z}{Z}\right]^2 dZ$$

For large array ($Nkd/2 \rightarrow$large),

$$Z = \frac{N}{2}kd\cos\theta$$

$$\boxed{\int_{-\infty}^{+\infty}\left[\frac{\sin(Z)}{Z}\right]^2 dZ = \pi}$$

$$U_0 = \frac{1}{Nkd}\int_{-Nkd/2}^{+Nkd/2}\left[\frac{\sin Z}{Z}\right]^2 dZ \simeq \frac{1}{Nkd}\int_{-\infty}^{+\infty}\left[\frac{\sin Z}{Z}\right]^2 dZ$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{U_{\max}}{U_0}$$

$$U_{max} = 1$$

$$U_0 \simeq \frac{\pi}{Nkd}$$

$$\psi = kd\cos\theta + \beta|_{\theta=90^\circ} = \beta = 0$$

2.1 Broad side array: **Directivity**

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\sin\left(\frac{1}{2}kd\cos\theta\right)}\right]$$

- Directivity: $$D_0 = \frac{U_{\max}}{U_0} \simeq \frac{Nkd}{\pi} = 2N\left(\frac{d}{\lambda}\right)$$

$$(\mathrm{AF})_n \simeq \left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\left(\frac{N}{2}kd\cos\theta\right)}\right]$$

- Total array length: $L = (N-1)d$

$$D_0 \simeq 2N\left(\frac{d}{\lambda}\right) \simeq 2\left(1+\frac{L}{d}\right)\left(\frac{d}{\lambda}\right)$$

$$U(\theta) = [(\mathrm{AF})_n]^2 = \left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\frac{N}{2}kd\cos\theta}\right]^2 = \left[\frac{\sin(Z)}{Z}\right]^2$$

- For very large array ($L \gg d$)

$$Z = \frac{N}{2}kd\cos\theta$$

$$D_0 \simeq 2N\left(\frac{d}{\lambda}\right) = 2\left(1+\frac{L}{d}\right)\left(\frac{d}{\lambda}\right) \overset{L \gg d}{\simeq} 2\left(\frac{L}{\lambda}\right)$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\mathrm{rad}}} = \frac{U_{\max}}{U_0}$$

$$U_0 \simeq \frac{\pi}{Nkd} \qquad U_{max} = 1$$

Given a linear, broadside, uniform array of 10 isotropic elements ($N = 10$) with a separation of $\lambda/4(d = \lambda/4)$ between the elements, find the directivity of the array.

Solution: Using (6-44a)

$$D_0 \simeq 2N\left(\frac{d}{\lambda}\right) = 5 \text{ (dimensionless)} = 10\log_{10}(5) = 6.99 \text{ dB}$$

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

$$\mathrm{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

3. End fire array

- **As usual, for uniform array:**

First maximum of the array factor occurs when

ψ at first maximum will be

$$\psi = kd\cos\theta + \beta = 0$$

$$AF = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

3. End fire array

- Instead of having the maximum radiation broadside to the axis of the array, it may be **desirable to direct it along the axis of the array** (end-fire). As a matter of fact, it may be necessary that it radiates **toward only one direction** (either θ_0 = 0◦ or 180◦)
- To direct the first maximum toward θ_0 = 0◦,

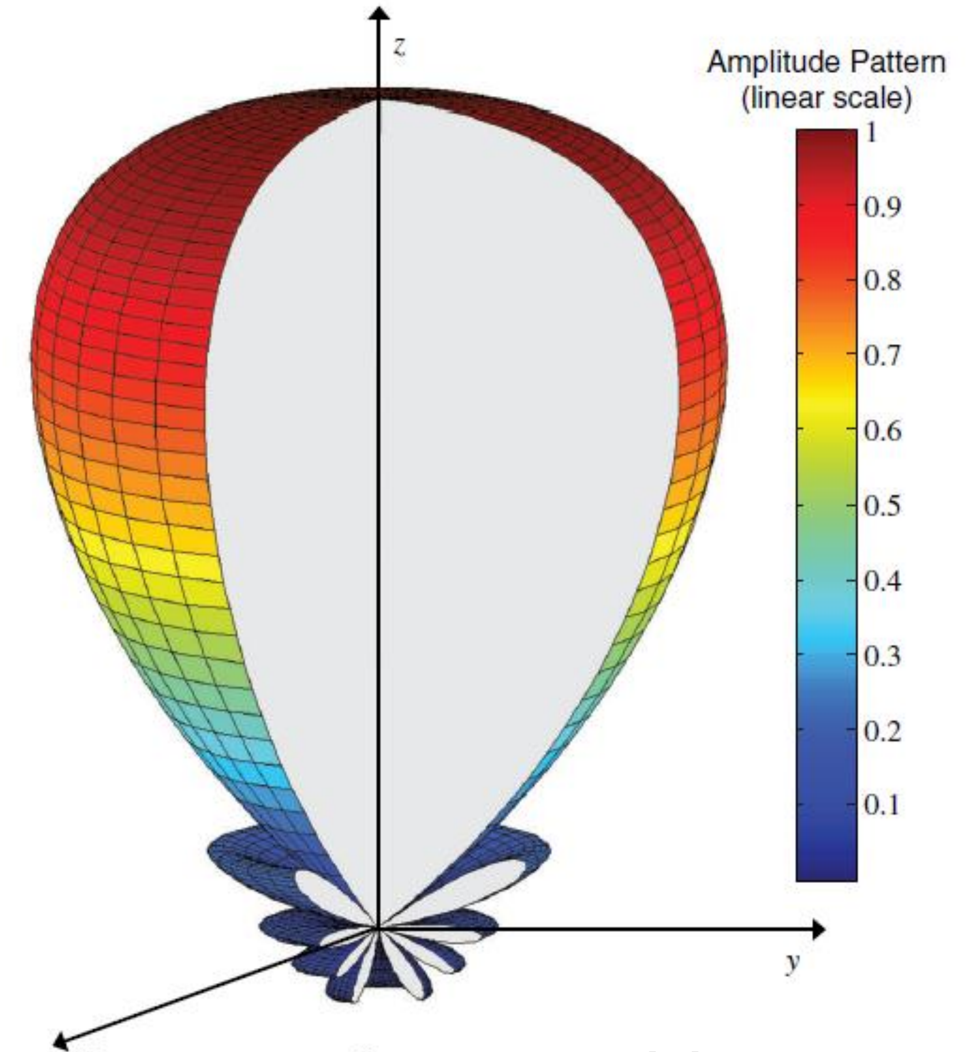
$$\boxed{\psi = kd\cos\theta + \beta|_{\theta=0^\circ} = kd + \beta = 0 \Rightarrow \beta = -kd}$$

- first maximum is desired toward θ_0 = 180◦, then

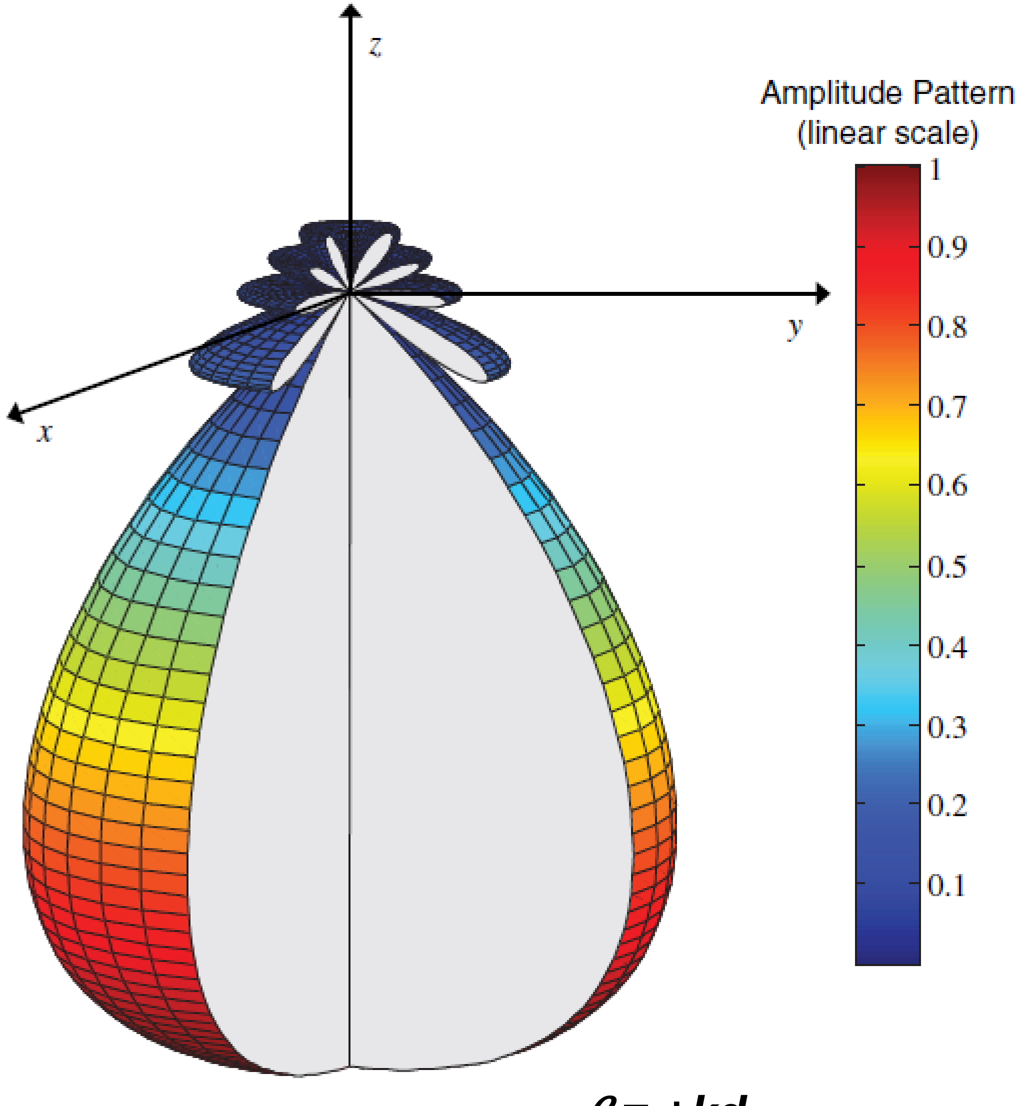
$$\boxed{\psi = kd\cos\theta + \beta|_{\theta=180^\circ} = -kd + \beta = 0 \Rightarrow \beta = kd}$$

- end-fire radiation is accomplished when $\beta = -kd$ (for θ_0 = 0◦) or $\beta = kd$ (for θ_0 = 180◦).

Three-dimensional amplitude patterns for end-fire arrays toward θ_0 = 0◦ and 180◦ (N = 10, d = $\lambda/4$)



(a) $\theta_0 = 0°$ β = -kd



(b) $\theta_0 = 180°$ β = +kd

$$\mathrm{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

3. End fire array

- If the element separation is ***d* = λ/2**, **end-fire radiation exists simultaneously in both directions** ($\theta_0 = 0°$ and $\theta_0 = 180°$).
- If the **element spacing is a multiple of a wavelength** ($d = n\lambda$, $n = 1, 2, 3, \ldots$), then in addition to having **end-fire radiation in both directions**, there also exist **maxima in the broadside directions**.

Thus for ***d* = *n*λ**, $n = 1, 2, 3, \ldots$ **there exist four maxima**; two in the broadside directions and two along the axis of the array.

- *To have **only one end-fire maximum** and **to avoid any grating lobes**, the maximum spacing between the elements should be less than* $d_{max} < \lambda/2$.

3. End fire array

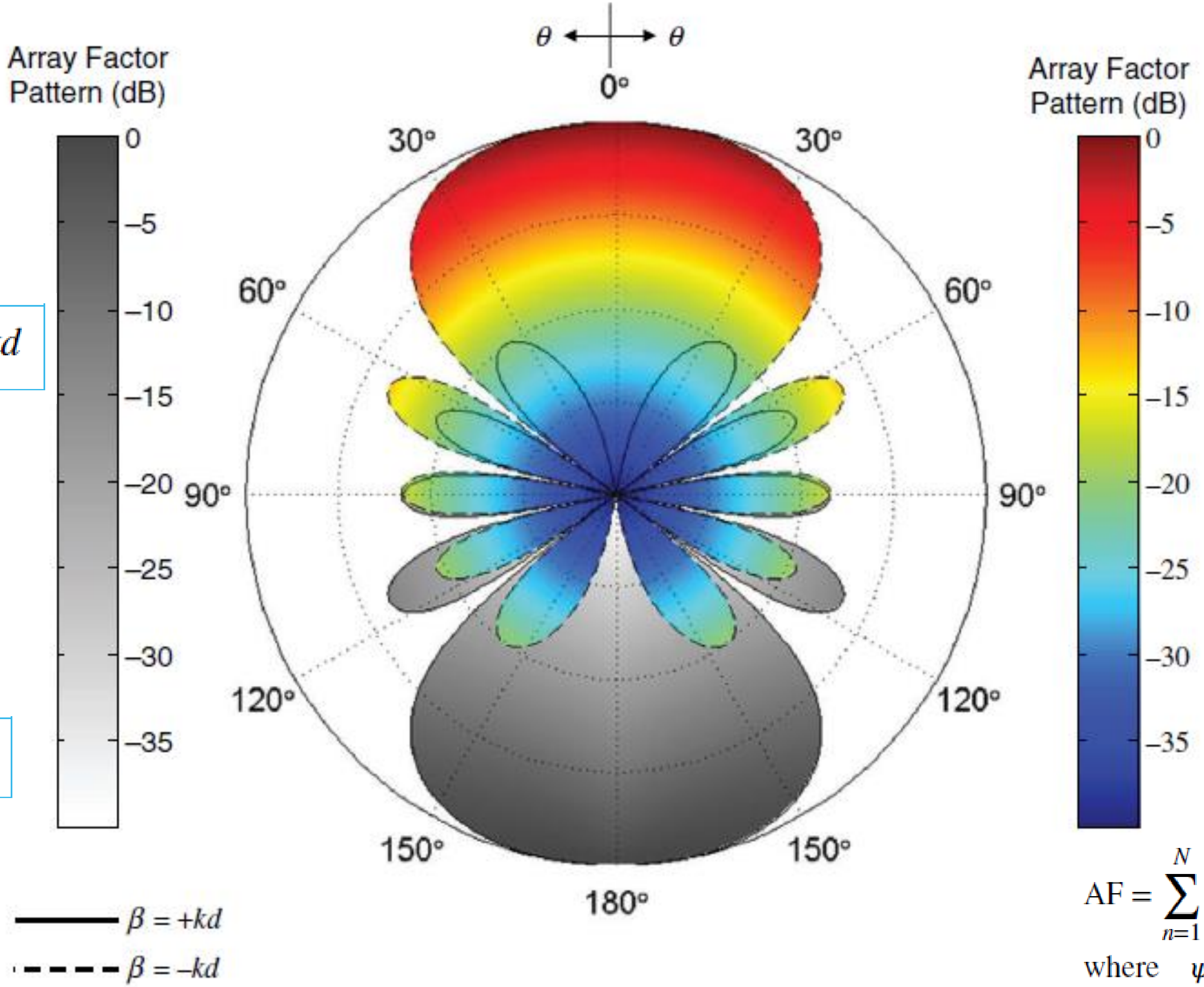
Array factor patterns of a 10-element uniform amplitude end-fire array (N = 10, d = $\lambda/4$).

- To direct the first maximum toward $\theta_0 = 0^\circ$,

$$\psi = kd\cos\theta + \beta|_{\theta=0^\circ} = kd + \beta = 0 \Rightarrow \beta = -kd$$

- first maximum is desired toward $\theta_0 = 180^\circ$, then

$$\psi = kd\cos\theta + \beta|_{\theta=180^\circ} = -kd + \beta = 0 \Rightarrow \beta = kd$$

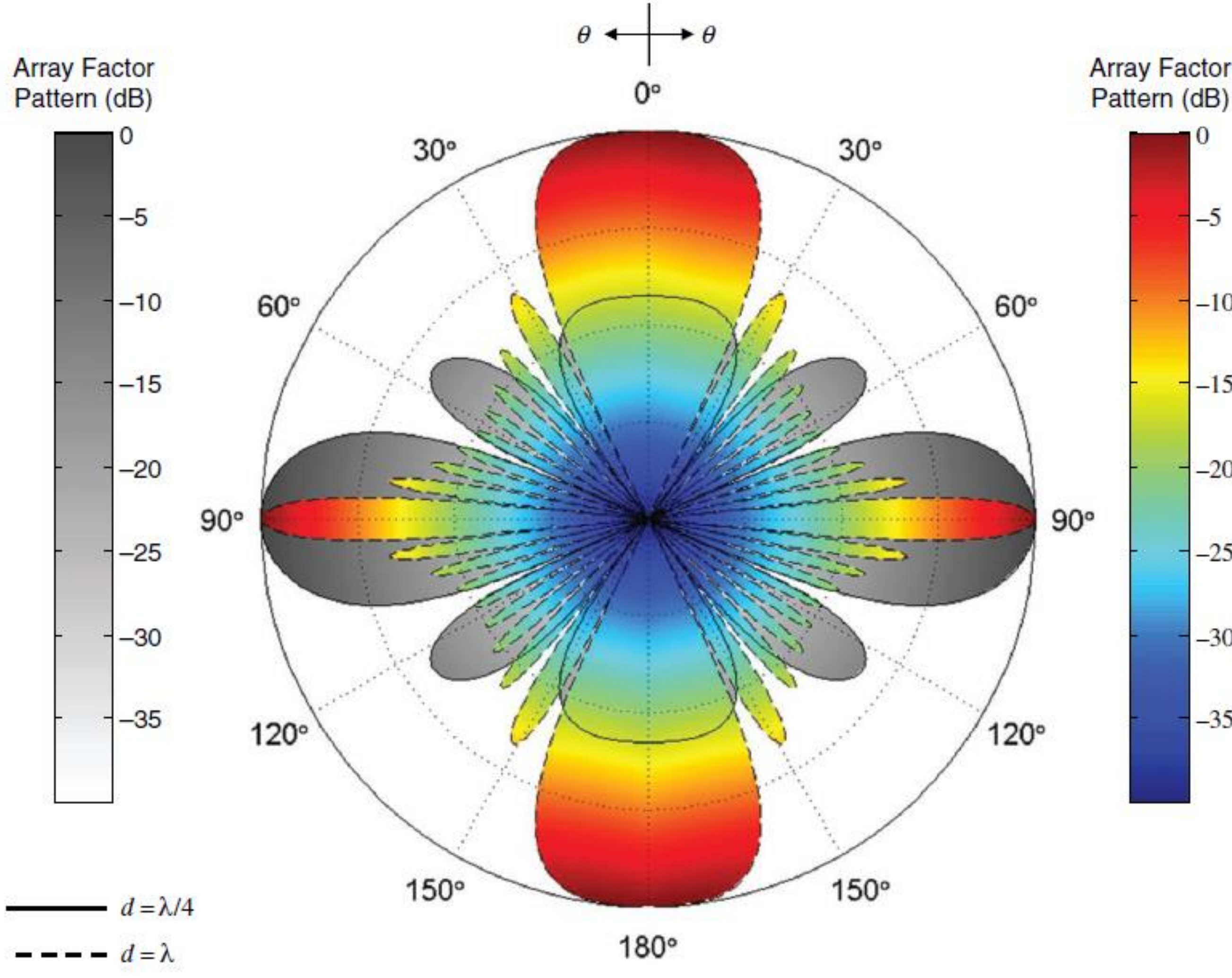


$$\mathrm{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

3. End fire array

Array factor patterns of a 10-element uniform amplitude broadside array (N = 10, β = 0).



3. End fire array

- Points of
  Nulls
  Maxima
  Half power
  Minor Lobe maxima

TABLE 6.3 Nulls, Maxima, Half-Power Points, and Minor Lobe Maxima for Uniform Amplitude Ordinary End-Fire Arrays

NULLS	$\theta_n = \cos^{-1}\left(1 - \frac{n\lambda}{Nd}\right)$ $n = 1, 2, 3, \ldots$ $n \neq N, 2N, 3N, \ldots$
MAXIMA	$\theta_m = \cos^{-1}\left(1 - \frac{m\lambda}{d}\right)$ $m = 0, 1, 2, \ldots$
HALF-POWER POINTS	$\theta_h \simeq \cos^{-1}\left(1 - \frac{1.391\lambda}{\pi d N}\right)$ $\pi d/\lambda \ll 1$
MINOR LOBE MAXIMA	$\theta_s \simeq \cos^{-1}\left[1 - \frac{(2s+1)\lambda}{2Nd}\right]$ $s = 1, 2, 3, \ldots$ $\pi d/\lambda \ll 1$

3. End fire array

- Beamwidths

TABLE 6.4 Beamwidths for Uniform Amplitude Ordinary End-Fire Arrays

FIRST-NULL BEAMWIDTH (FNBW)	$\Theta_n = 2\cos^{-1}\left(1 - \frac{\lambda}{Nd}\right)$
HALF-POWER BEAMWIDTH (HPBW)	$\Theta_h \simeq 2\cos^{-1}\left(1 - \frac{1.391\lambda}{\pi dN}\right)$ $\pi d/\lambda \ll 1$
FIRST SIDE LOBE BEAMWIDTH (FSLBW)	$\Theta_s \simeq 2\cos^{-1}\left(1 - \frac{3\lambda}{2Nd}\right)$ $\pi d/\lambda \ll 1$

TABLE 6.7 **Maximum Element Spacing d_{max} to Maintain Either One or Two Amplitude Maxima of a Linear Array**

Array	Distribution	Type	Direction of Maximum	Element Spacing
Linear	Uniform	Broadside	$\theta_0 = 90°$ *only*	$d_{max} < \lambda$
			$\theta_0 = 0°, 90°, 180°$ *simultaneously*	$d = \lambda$
Linear	Uniform	Ordinary end-fire	$\theta_0 = 0°$ *only*	$d_{max} < \lambda/2$
			$\theta_0 = 180°$ *only*	$d_{max} < \lambda/2$
			$\theta_0 = 0°, 90°, 180°$ *simultaneously*	$d = \lambda$

3.1 End fire array: Directivity

$$\mathrm{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

- For an end-fire array, with the maximum radiation in the $\theta 0 = 0^\circ$ direction, the array factor is given by

$$(\mathrm{AF})_n = \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{N\sin\left[\frac{1}{2}kd(\cos\theta - 1)\right]}\right]$$

- a small spacing between the elements ($d \ll \lambda$), can be approximated by

$$(\mathrm{AF})_n \simeq \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\left[\frac{N}{2}kd(\cos\theta - 1)\right]}\right]$$

- To direct the first maximum toward $\theta_0 = 0^\circ$,

$$\psi = kd\cos\theta + \beta|_{\theta=0^\circ} = kd + \beta = 0 \Rightarrow \beta = -kd$$

- first maximum is desired toward $\theta_0 = 180^\circ$, then

$$\psi = kd\cos\theta + \beta|_{\theta=180^\circ} = -kd + \beta = 0 \Rightarrow \beta = kd$$

3.1 End fire array: Directivity

- Radiation intensity is

$$U(\theta) = [(\mathrm{AF})_n]^2 = \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\frac{N}{2}kd(\cos\theta - 1)}\right]^2 = \left[\frac{\sin(Z)}{Z}\right]^2$$

$$Z = \frac{N}{2}kd(\cos\theta - 1)$$

- maximum value is unity ($U_{max} = 1$) and it occurs at $\theta = 0°$.

$$\mathrm{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

- To direct the first maximum toward $\theta_0 = 0°$,

$$\psi = kd\cos\theta + \beta|_{\theta=0°} = kd + \beta = 0 \Rightarrow \beta = -kd$$

- first maximum is desired toward $\theta_0 = 180°$, then

$$\psi = kd\cos\theta + \beta|_{\theta=180°} = -kd + \beta = 0 \Rightarrow \beta = kd$$

$$(\mathrm{AF})_n = \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{N\sin\left[\frac{1}{2}kd(\cos\theta - 1)\right]}\right]$$

$$(\mathrm{AF})_n \simeq \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\left[\frac{N}{2}kd(\cos\theta - 1)\right]}\right]$$

$$\mathrm{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

3.1 End fire array: Directivity

- The average value of the radiation intensity is given by

$$U_0 = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\frac{N}{2}kd(\cos\theta - 1)}\right]^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{2}\int_0^{\pi}\left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\frac{N}{2}kd(\cos\theta - 1)}\right]^2 \sin\theta\, d\theta$$

$$Z = \frac{N}{2}kd(\cos\theta - 1)$$

$$dZ = -\frac{N}{2}kd\sin\theta\, d\theta$$

- To direct the first maximum toward $\theta_0 = 0°$,

$$\psi = kd\cos\theta + \beta|_{\theta=0°} = kd + \beta = 0 \Rightarrow \beta = -kd$$

- first maximum is desired toward $\theta_0 = 180°$, then

$$\psi = kd\cos\theta + \beta|_{\theta=180°} = -kd + \beta = 0 \Rightarrow \beta = kd$$

$$(\mathrm{AF})_n = \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{N\sin\left[\frac{1}{2}kd(\cos\theta - 1)\right]}\right]$$

$$U_{max} = 1$$

$$(\mathrm{AF})_n \simeq \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\left[\frac{N}{2}kd(\cos\theta - 1)\right]}\right]$$

$$Z = \frac{N}{2}kd(\cos\theta - 1)$$

$$U(\theta) = [(\mathrm{AF})_n]^2 = \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\frac{N}{2}kd(\cos\theta - 1)}\right]^2 = \left[\frac{\sin(Z)}{Z}\right]^2$$

3.1 End fire array: Directivity

- The average value of the radiation intensity is given by

$$U_0 = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\frac{N}{2}kd(\cos\theta - 1)}\right]^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{2}\int_0^{\pi}\left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\frac{N}{2}kd(\cos\theta - 1)}\right]^2 \sin\theta\, d\theta$$

$$Z = \frac{N}{2}kd(\cos\theta - 1)$$

$$dZ = -\frac{N}{2}kd\sin\theta\, d\theta$$

$$= -\frac{1}{Nkd}\int_0^{-Nkd}\left[\frac{\sin(Z)}{Z}\right]^2 dZ = \frac{1}{Nkd}\int_0^{Nkd}\left[\frac{\sin(Z)}{Z}\right]^2 dZ$$

$$\mathrm{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

- To direct the first maximum toward $\theta_0 = 0°$,

$$\psi = kd\cos\theta + \beta|_{\theta=0°} = kd + \beta = 0 \Rightarrow \beta = -kd$$

- first maximum is desired toward $\theta_0 = 180°$, then

$$\psi = kd\cos\theta + \beta|_{\theta=180°} = -kd + \beta = 0 \Rightarrow \beta = kd$$

$$(\mathrm{AF})_n = \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{N\sin\left[\frac{1}{2}kd(\cos\theta - 1)\right]}\right]$$

$$U_{max} = 1$$

$$(\mathrm{AF})_n \simeq \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\left[\frac{N}{2}kd(\cos\theta - 1)\right]}\right]$$

$$Z = \frac{N}{2}kd(\cos\theta - 1)$$

$$U(\theta) = [(\mathrm{AF})_n]^2 = \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\frac{N}{2}kd(\cos\theta - 1)}\right]^2 = \left[\frac{\sin(Z)}{Z}\right]^2$$

$$\mathrm{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

3.1 End fire array: Directivity

- The average value of the radiation intensity is given by

$$U_0 = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\frac{N}{2}kd(\cos\theta - 1)}\right]^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{2}\int_0^{\pi}\left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\frac{N}{2}kd(\cos\theta - 1)}\right]^2 \sin\theta\, d\theta$$

$$Z = \frac{N}{2}kd(\cos\theta - 1)$$

$$dZ = -\frac{N}{2}kd\sin\theta\, d\theta$$

$$= -\frac{1}{Nkd}\int_0^{-Nkd}\left[\frac{\sin(Z)}{Z}\right]^2 dZ = \frac{1}{Nkd}\int_0^{Nkd}\left[\frac{\sin(Z)}{Z}\right]^2 dZ$$

(large array ($Nkd \rightarrow$ large))

$$\simeq \frac{1}{Nkd}\int_0^{\infty}\left[\frac{\sin(Z)}{Z}\right]^2 dZ$$

$$U_0 \simeq \frac{\pi}{2Nkd}$$

- To direct the first maximum toward $\theta_0 = 0^\circ$,

$$\psi = kd\cos\theta + \beta|_{\theta=0^\circ} = kd + \beta = 0 \Rightarrow \beta = -kd$$

- first maximum is desired toward $\theta_0 = 180^\circ$, then

$$\psi = kd\cos\theta + \beta|_{\theta=180^\circ} = -kd + \beta = 0 \Rightarrow \beta = kd$$

$$(\mathrm{AF})_n = \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{N\sin\left[\frac{1}{2}kd(\cos\theta - 1)\right]}\right]$$

$$U_{max} = 1$$

$$(\mathrm{AF})_n \simeq \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\left[\frac{N}{2}kd(\cos\theta - 1)\right]}\right]$$

$$Z = \frac{N}{2}kd(\cos\theta - 1)$$

$$U(\theta) = [(\mathrm{AF})_n]^2 = \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\frac{N}{2}kd(\cos\theta - 1)}\right]^2 = \left[\frac{\sin(Z)}{Z}\right]^2$$

3.1 End fire array: Directivity

- Directivity $D_0 = \frac{U_{\max}}{U_0} \simeq \frac{2Nkd}{\pi} = 4N\left(\frac{d}{\lambda}\right)$

- With **length of array a** $L = (N-1)d$ $: N = \left(1 + \frac{L}{d}\right)$

$$D_0 \simeq 4N\left(\frac{d}{\lambda}\right) = 4\left(1 + \frac{L}{d}\right)\left(\frac{d}{\lambda}\right)$$

$$U_{max} = 1$$

$$U_0 \simeq \frac{\pi}{2Nkd}$$

$$\mathrm{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

- To direct the first maximum toward θ_0 = 0°,

$$\psi = kd\cos\theta + \beta|_{\theta=0^\circ} = kd + \beta = 0 \Rightarrow \beta = -kd$$

- first maximum is desired toward θ_0 = 180°, then

$$\psi = kd\cos\theta + \beta|_{\theta=180^\circ} = -kd + \beta = 0 \Rightarrow \beta = kd$$

$$(\mathrm{AF})_n = \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{N\sin\left[\frac{1}{2}kd(\cos\theta - 1)\right]}\right]$$

$$(\mathrm{AF})_n \simeq \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\left[\frac{N}{2}kd(\cos\theta - 1)\right]}\right]$$

$$Z = \frac{N}{2}kd(\cos\theta - 1)$$

$$U(\theta) = [(\mathrm{AF})_n]^2 = \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\frac{N}{2}kd(\cos\theta - 1)}\right]^2 = \left[\frac{\sin(Z)}{Z}\right]^2$$

$$\mathrm{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$

3.1 End fire array: Directivity

$$U_{max} = 1$$

$$U_0 \simeq \frac{\pi}{2Nkd}$$

$$(\mathrm{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]$$

- Directivity $D_0 = \frac{U_{\max}}{U_0} \simeq \frac{2Nkd}{\pi} = 4N\left(\frac{d}{\lambda}\right)$

- To direct the first maximum toward $\theta_0 = 0°$,

$$\psi = kd\cos\theta + \beta|_{\theta=0°} = kd + \beta = 0 \Rightarrow \beta = -kd$$

- With **length of array a**: $L = (N-1)d$: $N = \left(1 + \frac{L}{d}\right)$

$$D_0 \simeq 4N\left(\frac{d}{\lambda}\right) = 4\left(1 + \frac{L}{d}\right)\left(\frac{d}{\lambda}\right)$$

- first maximum is desired toward $\theta_0 = 180°$, then

$$\psi = kd\cos\theta + \beta|_{\theta=180°} = -kd + \beta = 0 \Rightarrow \beta = kd$$

- for <u>**a large array**</u> ($L \gg d$)

$$(\mathrm{AF})_n = \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{N\sin\left[\frac{1}{2}kd(\cos\theta - 1)\right]}\right]$$

$$D_0 \simeq 4N\left(\frac{d}{\lambda}\right) = 4\left(1 + \frac{L}{d}\right)\left(\frac{d}{\lambda}\right) \overset{L \gg d}{\simeq} 4\left(\frac{L}{\lambda}\right)$$

$$(\mathrm{AF})_n \simeq \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\left[\frac{N}{2}kd(\cos\theta - 1)\right]}\right]$$

$$Z = \frac{N}{2}kd(\cos\theta - 1)$$

- Note: directivity of the end-fire array, is twice that for the broadside array

$$U(\theta) = [(\mathrm{AF})_n]^2 = \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\frac{N}{2}kd(\cos\theta - 1)}\right]^2 = \left[\frac{\sin(Z)}{Z}\right]^2$$

Given a linear, end-fire, uniform array of 10 elements ($N = 10$) with a separation of $\lambda/4(d = \lambda/4)$ between the elements, find the directivity of the array factor. This array is identical to the broadside array of Example 6.3.

Solution: Using (6-49)

$$D_0 \simeq 4N\left(\frac{d}{\lambda}\right) = 10 \text{ (dimensionless)} = 10\log_{10}(10) = 10 \text{ dB}$$

Comparison of Broadside and Endfire array

Array	Directivity	
BROADSIDE	$D_0 = 2N\left(\frac{d}{\lambda}\right) = 2\left(1+\frac{L}{d}\right)\frac{d}{\lambda} \simeq 2\left(\frac{L}{\lambda}\right)$ $N\pi d/\lambda \to \infty, L \gg d$	
END-FIRE (ORDINARY)	$D_0 = 4N\left(\frac{d}{\lambda}\right) = 4\left(1+\frac{L}{d}\right)\frac{d}{\lambda} \simeq 4\left(\frac{L}{\lambda}\right)$ $2N\pi d/\lambda \to \infty, L \gg d$	Only one maximum ($\theta_0 = 0°$ or $180°$)
	$D_0 = 2N\left(\frac{d}{\lambda}\right) = 2\left(1+\frac{L}{d}\right)\frac{d}{\lambda} \simeq 2\left(\frac{L}{\lambda}\right)$	Two maxima ($\theta_0 = 0°$ and $180°$)