2.6 Problems

Module:2 Linear and Planar Arrays

Course: BECE305L – Antenna and Microwave Engineering

-Dr Richards Joe Stanislaus

Assistant Professor - SENSE

Email: richards.stanislaus@vit.ac.in

Vellore Institute of Technology

(Deemed to be University under section 3 of UGC Act, 1956)

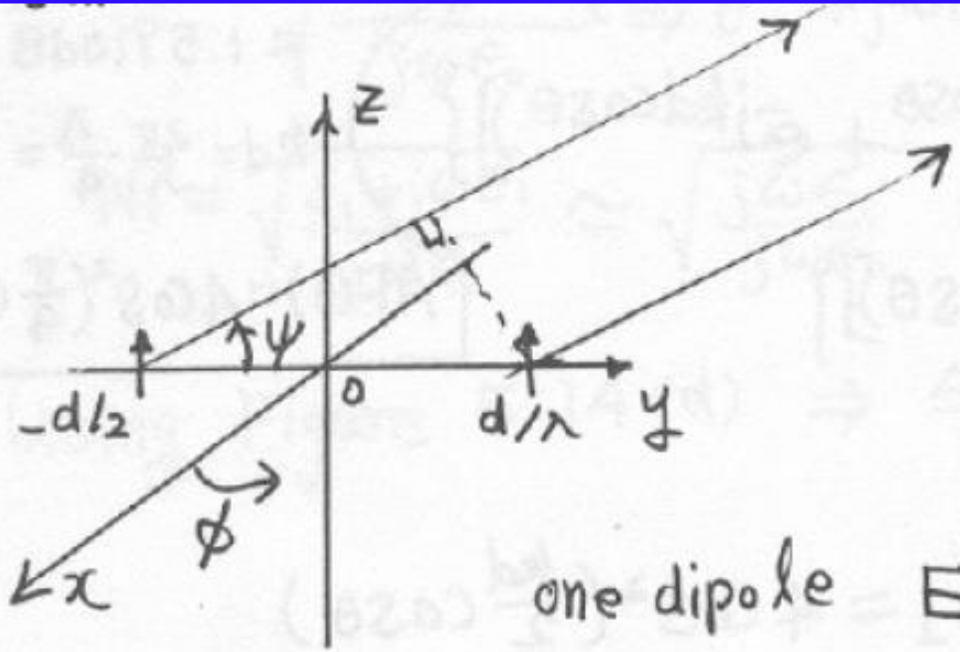
CHENNAI

Two very short dipoles ("infinitesimal") of equal length are equidistant from the origin with their centers lying on the y-axis, and oriented parallel to the z-axis. They are excited with currents of equal amplitude. The current in dipole 1 (at $y = -d/2$) leads the current in dipole 2 (at $y = +d/2$) by 90° in phase. The spacing between dipoles is one quarter wavelength. To simplify the notation, let E_0 equal the maximum magnitude of the far field at distance r due to either source alone.

(a) Derive expressions for the following six principal-plane patterns:

1. $|E_\theta(\theta)|$ for $\phi = 0°$
2. $|E_\theta(\theta)|$ for $\phi = 90°$
3. $|E_\theta(\phi)|$ for $\theta = 90°$
4. $|E_\phi(\theta)|$ for $\phi = 0°$
5. $|E_\phi(\theta)|$ for $\phi = 90°$
6. $|E_\phi(\phi)|$ for $\theta = 90°$

(b) Sketch the six field patterns.



one dipole $E_\theta \simeq j\eta \frac{k I_0 l e^{-jkr}}{4\pi r} \cdot \sin\theta$

Array Factor:

$$(AF)_2 = E_0\left[e^{j\frac{\pi}{2}} \cdot e^{-j\frac{\lambda}{8}\cdot\frac{2\pi}{\lambda}\cos\psi} + e^{+j\frac{\lambda}{8}\frac{2\pi}{\lambda}\cos\psi}\right]$$

$$= E_0 e^{j\frac{\pi}{4}}\left[e^{-j\frac{\pi}{4}(\cos\psi - 1)} + e^{j\frac{\pi}{4}(\cos\psi - 1)}\right]$$

$$= E_0 e^{j\frac{\pi}{4}} \cdot 2 \cdot \cos\left(\frac{\pi}{4}(\cos\psi - 1)\right) = E_0 e^{j\frac{\pi}{4}} \, 2\cos\left(\frac{\pi}{4}(\sin\theta\sin\phi - 1)\right)$$

$\left(\hat{a}_y \cdot \hat{a}_r = \sin\theta \cdot \sin\phi = \cos\psi\right) \overset{At\ y,z\ plane,\ \phi = 90^\circ}{=} \sin\theta$

(1) $|E_\theta(\theta)| \propto \left|\sin\theta \cdot \cos\left(\frac{\pi}{4}\right)\right|$, (x-z plane)

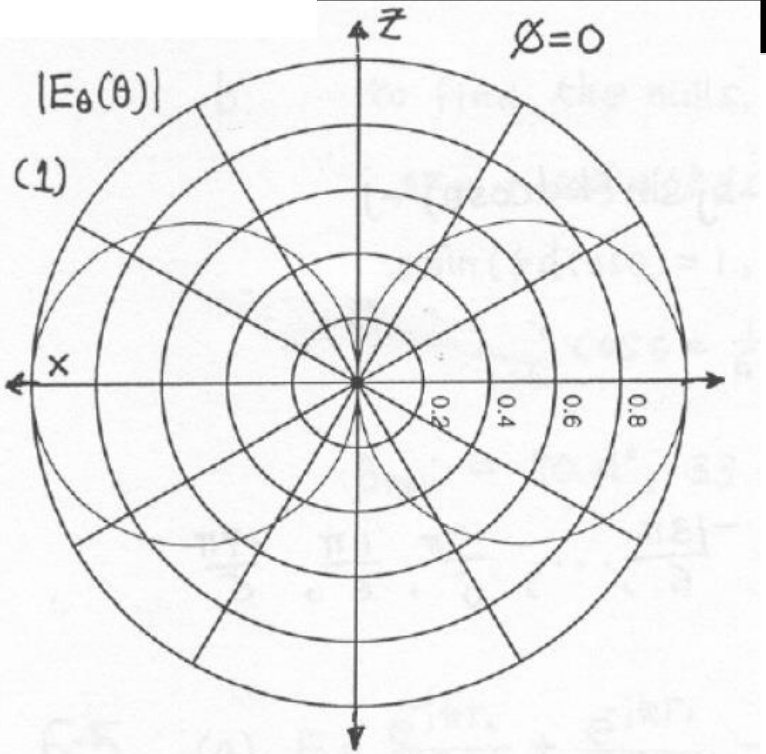
$\phi = 0°$;

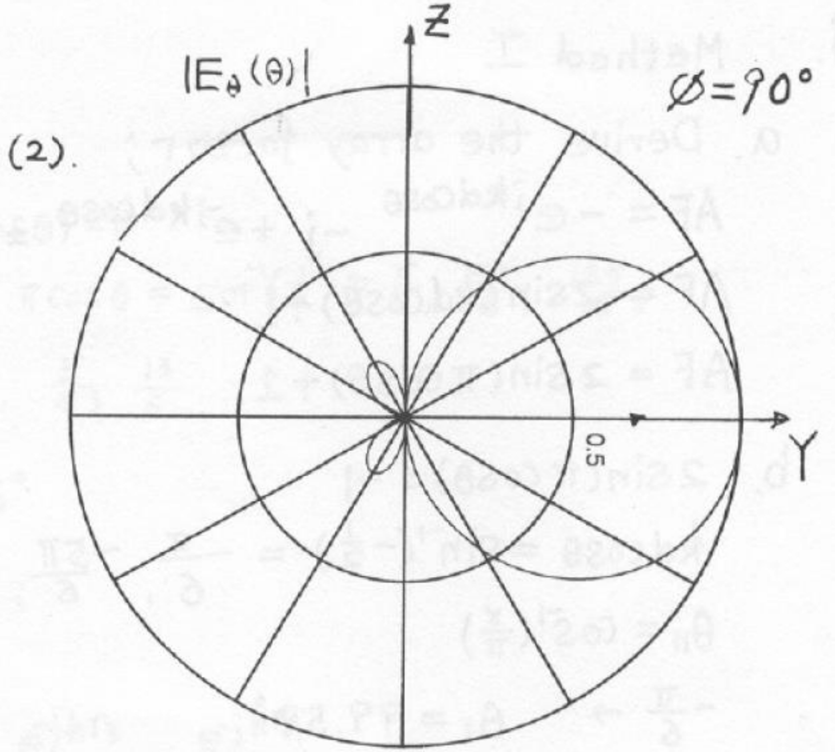
(2) $|E_\theta(\theta)| \propto \left|\underset{0<\theta<\pi}{\sin\theta} \cdot \cos\left(\frac{\pi}{4}(\underset{-\pi<\theta<\pi}{\sin\theta} - 1)\right)\right|$, (y-z plane)

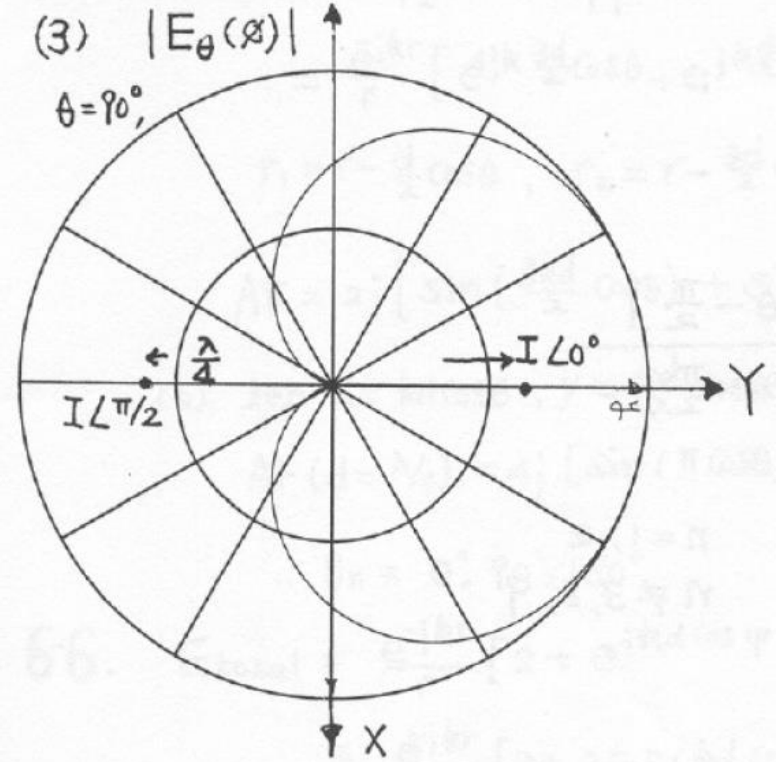
$\phi = 90°$

(3) $|E_\theta(\phi)| \propto \left|\cos\left(\frac{\pi}{4}(\sin\phi - 1)\right)\right|$, (x, y. plane)

$\theta = 90°$



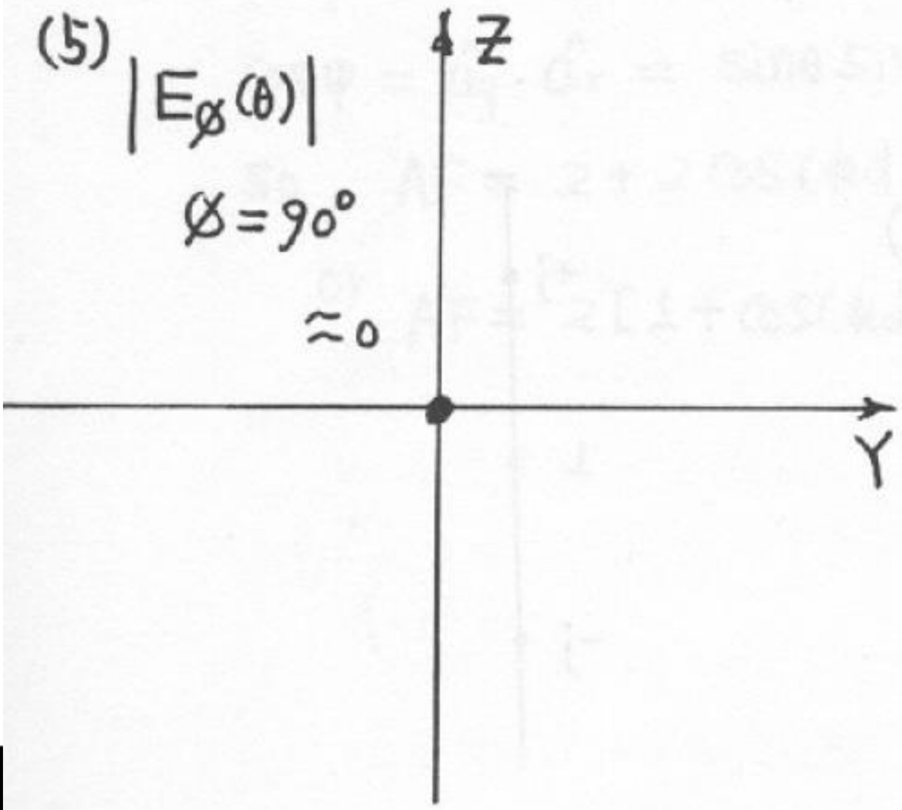


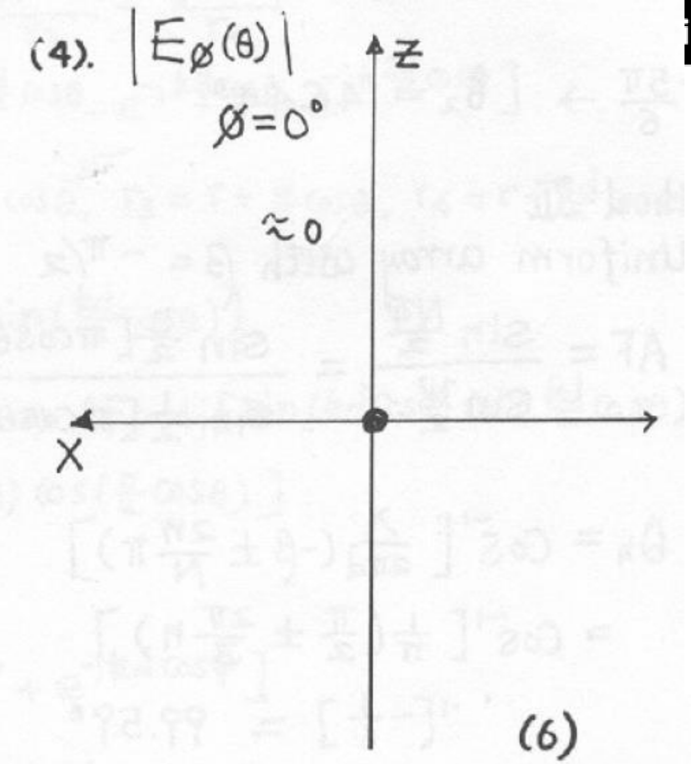


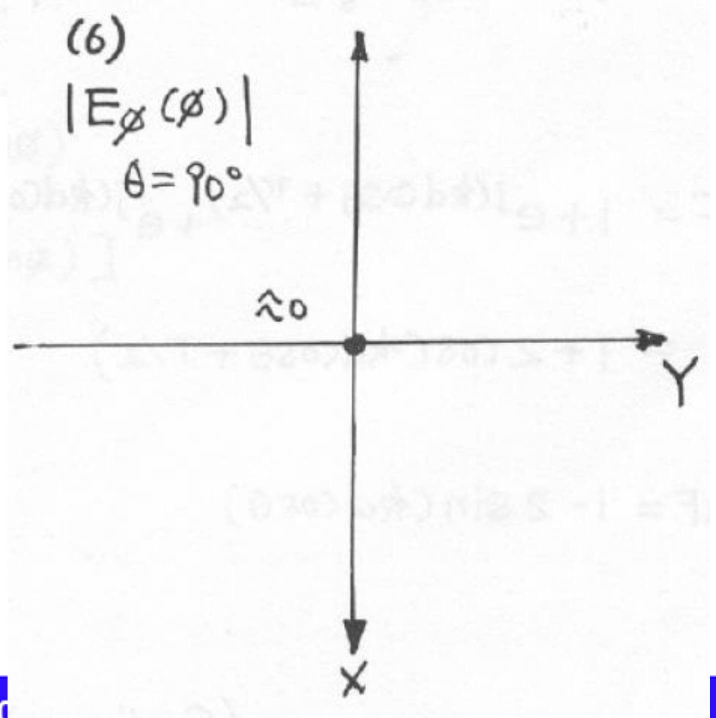
(4) $|E_\phi(\theta)|_{\phi=0^\circ} \propto 0$

(5) $|E_\phi(\theta)|_{\phi=90^\circ} \propto 0$

(6) $|E_\phi(\theta)|_{\theta=90^\circ} \propto 0$







A three-element array of isotropic sources has the phase and magnitude relationships shown. The spacing between the elements is $d = \lambda/2$.

(a) Find the array factor.

(b) Find all the nulls.

Uniform array with $\beta = -\pi/2$

a. $$AF = \frac{\sin \frac{N\psi}{2}}{\sin \frac{\psi}{2}} = \frac{\sin \frac{3}{2}\left[\pi\cos\theta - \frac{\pi}{2}\right]}{\sin \frac{1}{2}\left[\pi\cos\theta - \frac{\pi}{2}\right]}$$

b. $$\theta_n = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2n}{N}\pi\right)\right] \qquad n = 1, 2$$

$$= \cos^{-1}\left[\frac{1}{\pi}\left(\frac{\pi}{2} \pm \frac{2\pi}{3}n\right)\right] \qquad n \neq 3, 6, 9$$

$n=1$; $\cos^{-1}\left[-\frac{1}{6}\right] = 99.59°$

$n=2$; $\cos^{-1}\left[\frac{-5}{6}\right] = 146.44°$

Design a two-element uniform array of isotropic sources, positioned along the z-axis a distance $\lambda/4$ apart, so that its only maximum occurs along $\theta_0 = 0°$. Assuming ordinary end-fire conditions, find the

(a) relative phase excitation of each element (b) array factor of the array

(c) directivity using the computer program **Directivity** of Chapter 2. Compare it with Kraus' approximate formula.

Placing one element at the origin and the other at d distance above it, the array factor is equal to

$$AF(\theta) = 1 + e^{j(kd\cos\theta+\beta)} = 2e^{j\frac{1}{2}(kd\cos\theta+\beta)}\left[\frac{e^{-j\frac{1}{2}(kd\cos\theta+\beta)} + e^{+j\frac{1}{2}(kd\cos\theta+\beta)}}{2}\right]$$

$$AF(\theta) = 2e^{j\frac{1}{2}(kd\cos\theta+\beta)}\cos\left[\tfrac{1}{2}(kd\cos\theta+\beta)\right]$$

which in normalized form can be written as

$$(AF)_n = \cos\left[\tfrac{1}{2}(kd\cos\theta+\beta)\right]$$

Design a two-element uniform array of isotropic sources, positioned along the z-axis a distance $\lambda/4$ apart, so that its only maximum occurs along $\theta_0 = 0°$. Assuming ordinary end-fire conditions, find the

(a) relative phase excitation of each element (b) array factor of the array

(c) directivity using the computer program **Directivity** of Chapter 2. Compare it with Kraus' approximate formula.

$$(AF)_n = \cos\left[\tfrac{1}{2}(kd\cos\theta + \beta)\right]$$

a. $\beta = -kd = -\frac{2\pi}{\lambda}\left(\frac{\lambda}{4}\right) = -\frac{\pi}{2}$.

b. For $d = \lambda/4$, $(AF)_n = \cos\left[\frac{\pi}{4}(\cos\theta - 1)\right]$

c. $(AF)_n\big|_{max} = 1 = \cos\left[\frac{\pi}{4}(\cos\theta_m - 1)\right] \Rightarrow \theta_m = 0°$

$$(AF)_n = 0.707 = \cos\left[\tfrac{\pi}{4}(\cos\theta_h - 1)\right] \Rightarrow \tfrac{\pi}{4}(\cos\theta_h - 1) = \cos^{-1}(0.707) = \begin{cases} +\frac{\pi}{4} \\ -\frac{\pi}{4} \end{cases}$$

For $+\pi/4 \Rightarrow \cos\theta_h - 1 = 1 \Rightarrow \cos\theta_h = 2 \Rightarrow \theta_h = \cos^{-1}(2) \Rightarrow$ Does not exist

For $-\pi/4 \Rightarrow \cos\theta_h - 1 = -1 \Rightarrow \cos\theta_h = 0 = \theta_h = \cos^{-1}(0) = 90° = \frac{\pi}{2}$ radians

Therefore $\Theta_{1r} = \Theta_{2r} = 2\left(\frac{\pi}{2} - \theta\right) = \pi$

and $D_0 \simeq \dfrac{4\pi}{\Theta_{1r}\Theta_{2r}} = \dfrac{4\pi}{(\pi)^2} = \dfrac{4}{\pi} = 1.273 = 1.049\,\text{dB}$

Computer Program $\left(U = \cos^2\left[\frac{\pi}{4}(\cos\theta - 1)\right]\right)$

$D_0 = 1.42451 = 1.5367\,\text{dB}$

Design a two-element uniform array of isotropic sources, positioned along the z-axis a distance $\lambda/4$ apart, so that its only maximum occurs along $\boxed{\theta = 180°}$. Assuming ordinary end-fire conditions, find the

(a) relative phase excitation of each element (b) array factor of the array

(c) directivity using the computer program **Directivity** of Chapter 2. Compare it with Kraus' approximate formula.

a. $\beta = +kd = +\frac{\pi}{2}$

b. $(AF)_n = \cos\left[\frac{\pi}{4}(\cos\theta + 1)\right]$

$(AF)_n\big|_{max} = 1 = \cos\left[\frac{\pi}{4}(\cos\theta_m + 1)\right] \Rightarrow \theta_m = 180° = \pi$ radians

$(AF)_n = 0.707 = \cos\left(\frac{\pi}{4}(\cos\theta_h + 1)\right) \Rightarrow \theta_h = 90° = \frac{\pi}{2}$ radians

$\Theta_{1r} = \Theta_{2r} = 2\left(\pi - \frac{\pi}{2}\right) = \pi$

and

$D_0 \simeq \frac{4\pi}{\pi^2} = \frac{4}{\pi} = 1.273 = 1.049$ dB

Computer Program result.

$U = \cos^2\left[\frac{\pi}{4}(\cos\theta + 1)\right]$

$D_0 = 1.42451 = 1.5367$ dB

An array of 10 isotropic elements are placed along the z-axis a distance d apart. Assuming uniform distribution, find the progressive phase (in degrees), half-power beamwidth (in degrees), first-null beamwidth (in degrees), first side lobe level maximum beamwidth (in degrees), relative side lobe level maximum (in dB), and directivity (in dB) (using equations

(a) broadside (b) ordinary end-fire

arrays when the spacing between the elements is $d = \lambda/4$.

$N=10, \quad d = \lambda/4$

a. Broadside (Table 6.1 and 6.2) $\Rightarrow \beta = 0$

$HPBW = 2[90° - \cos^{-1}(\frac{1.391 \times 4}{10\pi})] = 2(90° - 79.80°) = 20.4°$

$FNBW = 2[90° - \cos^{-1}(\frac{4}{10})] = 2(90° - 66.42°) = 47.16°$

$FSLBW = 2[90° - \cos^{-1}(\frac{6}{10})] = 2(90° - 53.13°) = 73.74°$

From (6-17a) $\Rightarrow$ Relative sidelobe maximum $= -13.46$dB

From Table 6.7 $\Rightarrow D_0 = 2N(\frac{d}{\lambda}) = 2 \cdot 10 \cdot \frac{1}{4} = 5 = 6.99$dB

An array of 10 isotropic elements are placed along the z-axis a distance d apart. Assuming uniform distribution, find the progressive phase (in degrees), half-power beamwidth (in degrees), first-null beamwidth (in degrees), first side lobe level maximum beamwidth (in degrees), relative side lobe level maximum (in dB), and directivity (in dB) (using equations

(a) broadside (b) ordinary end-fire

arrays when the spacing between the elements is $d = \lambda/4$.

b. Ordinary End-Fire (Tables 6.3 and 6.4) $\Rightarrow \beta = \pm kd = \pm \pi/2 = \pm 90^\circ$

$$HPBW = 2\cos^{-1}\left[1 - \frac{1.391(4)}{10\pi}\right] = 2\,(34.62^\circ) = 69.25^\circ$$

$$FNBW = 2\cos^{-1}\left[1 - \frac{4}{10}\right] = 2\cos^{-1}(0.6) = 2\,(53.13^\circ) = 106.26^\circ$$

$$FSLBW = 2\cos^{-1}\left[1 - \frac{3(4)}{20}\right] = 2\,(66.42) = 132.84^\circ$$

From (6-17a) $\Rightarrow$ Relative side lobe maximum $= -13.46$ dB

From Table 6.7 $\Rightarrow D_0 = 4N\left(\frac{d}{\lambda}\right) = 4\,(10)\,\frac{1}{4} = 10 = 10\,dB$

A uniform array of 20 isotropic elements is placed along the z-axis a distance $\lambda/4$ apart with a progressive phase shift of β rad. Calculate β (give the answer in radians) for the following array designs:

(a) broadside

(b) end-fire with maximum at $\theta_0 = 0°$

(c) end-fire with maximum at $\theta_0 = 180°$

(d) phased array with maximum aimed at $\theta_0 = 30°$

$$kd = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$$

a. $\beta = 0$ radians

b. $\beta = -\pi/2$

c. $\beta = +\pi/2$

d. $\beta = -1.36 = -\frac{\sqrt{3}}{4}\pi = -0.433\pi$

Design a 19-element uniform linear scanning array with a spacing of $\lambda/4$ between the elements.

(a) What is the progressive phase excitation between the elements so that the maximum of the array factor is 30° from the line where the elements are placed?

(b) What is the half-power beamwidth (in degrees) of the array factor of part a?

(c) What is the value (in dB) of the maximum of the first minor lobe?

$N = 19, \quad d = \lambda/4$

a. $\beta = -kd\cos\theta_0 \Big|_{\substack{\theta_0 = 30^\circ \\ d = \lambda/4}} = -\frac{2\pi}{\lambda}\left(\frac{\lambda}{4}\right)\cos(30^\circ) = -\frac{\pi}{2}\frac{\sqrt{3}}{2} = -\frac{\pi\sqrt{3}}{4} = -1.3603$

$\beta = -\frac{\pi\sqrt{3}}{4} = -1.3603 \text{ (rad)} = -77.942^\circ$

b. $\theta_h = \cos^{-1}\left[\cos\theta_0 - 0.443\frac{\lambda}{L+d}\right]_{\theta_0=30^\circ} - \cos^{-1}\left[\cos\theta_0 + 0.443\frac{\lambda}{L+d}\right]_{\theta_0=30^\circ}$

$= \cos^{-1}\left[0.866 - \frac{0.443}{5}\right] - \cos^{-1}\left[0.866 + \frac{0.443}{5}\right]$

$= \cos^{-1}(0.7774) - \cos^{-1}(0.9546) = 38.9769^\circ - 17.3309^\circ = 21.6459^\circ$

$\theta_h = 21.6459^\circ$

c. $\boxed{-13.5 \text{ dB}}$

For a uniform broadside linear array of 10 isotropic elements, determine the approximate directivity (in dB) when the spacing between the elements is

(a) $\lambda/4$ (b) $\lambda/2$ (c) $3\lambda/4$ (d) λ

$$D_0 \simeq 2N(d/\lambda)$$

a. $d = \frac{\lambda}{4}$, $D_0 \simeq 2 \cdot 10 \cdot \frac{1}{4} = 5 = 6.99\,dB$

b. $d = \frac{\lambda}{2}$, $D_0 \simeq 2 \cdot 10 \cdot \frac{1}{2} = 10 = 10\,dB$

c. $d = \frac{3\lambda}{4}$, $D_0 \simeq 2 \cdot 10 \cdot (0.75) = 15 = 11.76\,dB$

d. $d = \lambda$, $D_0 \simeq 2 \cdot 10 \cdot (1) = 20 = 13.0\,dB$

Design a three-element binomial array of isotropic elements positioned along the z-axis a distance d apart. Find the

(a) normalized excitation coefficients

(b) array factor

(c) nulls of the array factor for $d = \lambda$

(d) maxima of the array factor for $d = \lambda$

a. The excitation coefficients for a 3-element array are 1, 2, 1.

Placing one element at the origin, one above it, and the other below it, the problem is identical to that of Problem 6.1. Thus the array factors are identical and equal to

b. $(AF)_n = 1 + \cos(kd\cos\theta) = 2\cos^2\left(\frac{kd}{2}\cos\theta\right)$

Design a three-element binomial array of isotropic elements positioned along the z-axis a distance d apart. Find the

(a) normalized excitation coefficients (b) array factor

(c) nulls of the array factor for $d = \lambda$ (d) maxima of the array factor for $d = \lambda$

c. The nulls of the pattern can be found using either of the above forms, as it was demonstrated in Problem 6.1. Using either one

$d=\lambda \Rightarrow \theta_n = \cos^{-1}(n\lambda/2d) = \cos^{-1}(n/2)$, $n = \pm1, \pm3, \pm5, \cdots$

$n=\pm1$: $\theta_1 = \cos^{-1}(\pm 1/2) = \cos^{-1}(\pm 0.5) = 60°, 120°$

$n=\pm3$: $\theta_3 = \cos^{-1}(\pm 3/2) = \cos^{-1}(\pm 1.5) =$ Does not exist.

$n=\pm5$: $\theta_5 = \cos^{-1}(\pm 5/2) = \cos^{-1}(2.5) =$ Does not exist. The same holds for $|n| \geq 7$.

d. The maxima of the pattern can also be found either of the forms. Using the results of Problem 6.1.

$d=\lambda \Rightarrow \theta_m = \cos^{-1}(m\lambda/d) = \cos^{-1}(m)$, $m = 0, \pm1, \pm2, \pm3, \cdots$

$m=0$: $\theta_0 = \cos^{-1}(0) = 90°$

$m=\pm1$: $\theta_1 = \cos^{-1}(\pm1) = 0°, 180°$

$m=\pm2$: $\theta_2 = \cos^{-1}(\pm2) =$ Does not exist. The same holds for $n \geq 3$.

Four isotropic sources are placed symmetrically along the z-axis a distance d apart. Design a binomial array. Find the

(a) normalized excitation coefficients (b) array factor

(c) angles (in degrees) where the array factor nulls occur when $d = 3\lambda/4$

The excitation coefficients of a 4-element binomial array are 1,3,3,1

a. $\left.\begin{array}{l} a_1 = 3 \\ a_2 = 1 \end{array}\right\}$ $N = 2M = 4 \Rightarrow M = 2$

b. $(AF)_4 = \sum_{n=1}^{M=2} a_n \cos[(2n-1)u]$, $u = \frac{\pi d}{\lambda}\cos\theta$, using (6-61a) and (6-61c).

Thus

$$(AF)_4 = a_1 \cos(u) + a_2 \cos(3u) = 3\cos\left(\frac{\pi d}{\lambda}\cos\theta\right) + \cos\left(\frac{3\pi d}{\lambda}\cos\theta\right)$$

which can also be written, using (6-66) for $m=3$, as

$$(AF)_4 = 3\cos\left(\frac{\pi d}{\lambda}\right) + 4\cos^3\left(\frac{\pi d}{\lambda}\cos\theta\right) - 3\cos\left(\frac{\pi d}{\lambda}\cos\theta\right) = 4\cos^3\left(\frac{\pi d}{\lambda}\cos\theta\right)$$

$$(AF)_4 = 4\cos^3\left(\frac{\pi d}{\lambda}\cos\theta\right)$$

Four isotropic sources are placed symmetrically along the z-axis a distance d apart. Design a binomial array. Find the

(a) normalized excitation coefficients (b) array factor

(c) angles (in degrees) where the array factor nulls occur when $d = 3\lambda/4$

C. The nulls occur when

$$(AF)_4 = 4\cos^3\left(\frac{\pi d}{\lambda}\cos\theta_n\right) = 0 \Rightarrow \frac{\pi d}{\lambda}\cos\theta_n = \cos^{-1}(0) = \pm\frac{(2n+1)\pi}{2},\ n=0,1,2,\cdots$$

$$\text{or}\quad \theta_n = \cos^{-1}\left[\pm\frac{(2n+1)\lambda}{2d}\right] \overset{d=3\lambda/4}{=} \cos^{-1}\left[\pm\frac{(2n+1)2}{3}\right],\ n=0,1,2,\cdots$$

$n=0$: $\theta_0 = \cos^{-1}\left(\pm\frac{2}{3}\right) = 48.19^\circ,\ 131.81^\circ$

$n=1$: $\theta_1 = \cos^{-1}(\pm 2) =$ Does not exist. The same holds for $n \geq 2$.

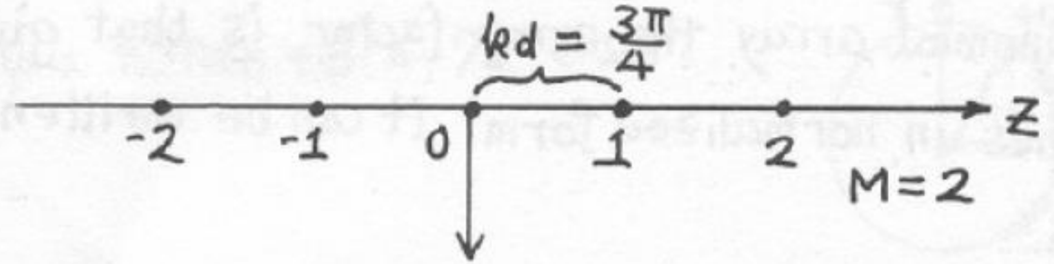
Five isotropic sources are placed symmetrically along the z-axis, each separated from its neighbor by an electrical distance $kd = 5\pi/4$. For a binomial array, find the

(a) excitation coefficients

(b) array factor

(c) normalized power pattern

(d) angles (in degrees) where the nulls (if any) occur



a. Use Pascal's triangle to find excitation coefficients

$2a_0 = 6 \rightarrow a_3 = 3$

$a_1 = 4$

$a_2 = 1$

b. $AF = 2\sum_{n=0}^{2} a_n \cos(nkd\cos\theta)$

$= 2\{3 + 4\cos(kd\cos\theta) + \cos(2kd\cos\theta)\}$

$\leftarrow \cos(2kd\cos\theta) = 2\cos^2(kd\cos\theta) - 1$

$= 4\{1 + 2\cos(kd\cos\theta) + \cos^2(kd\cos\theta)\}$

$= 4\{1 + \cos(kd\cos\theta)\}^2 = 16\cos^4\left(\frac{kd}{2}\cos\theta\right)$

c. $U(\theta,\phi) = |AF|^2 = 256\cos^8\left(\frac{kd}{2}\cos\theta\right)$

$U_{max} = U(\theta = \frac{\pi}{2}, \phi) = 256$

$P_n(\theta,\phi) = \frac{U(\theta,\phi)}{U_{max}} = \cos^8\left(\frac{kd}{2}\cos\theta\right)$

d. nulls occurs when $\cos\left(\frac{kd}{2}\cos\theta_n\right) = 0$

$kd\cos\theta_n = (2n+1)\pi \quad , \quad n = 0, \pm 1, \pm 2, \cdots$

$\theta_n = \cos^{-1}\left\{(2n+1)\frac{\pi}{kd}\right\} = \cos^{-1}\left\{(2n+1)\frac{4}{5}\right\}$

$n = 0: \quad \theta_0 = \cos^{-1}\left(\frac{4}{5}\right) = 36.9^\circ$

$n = -1: \quad \theta_0 = \cos^{-1}\left(-\frac{4}{5}\right) = 143.1^\circ$

Design a broadside binomial array of six elements placed along the z-axis separated by a distance $d = \lambda/2$.

(a) Find the amplitude excitation coefficients (a_n's).

(b) What is the progressive phase excitation between the elements?

(c) Write the array factor.

(d) Now assume that the elements are $\lambda/4$ dipoles oriented in the z-direction. Write the expression for the electric field *vector* in the far field.

a. From (6-63), $a_1 = 10$, $a_2 = 5$, $a_3 = 1$, ← Verified with computer program

b. Since the array is broadside, the progressive phase shift between the elements as required by (6-18a) is zero ($\beta = 0$.)

c. $(AF)_6 = 2\sum_{n=1}^{3} a_n \cos[(2n-1)u]$, $u = \frac{\pi d}{\lambda}\cos\theta = \frac{\pi}{2}\cos\theta$, Computer Program $\Rightarrow D_0 = 6.089$ dB At $d = \lambda/2$

d. $\underline{E} = \hat{a}_\theta\, j\eta \frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos(\frac{\pi}{4}\cos\theta) - \cos(\frac{\pi}{4})}{\sin\theta}\right]\left\{10\cos(\frac{\pi}{2}\cos\theta) + 5\cos(\frac{3\pi}{2}\cos\theta) + \cos(\frac{5\pi}{2}\cos\theta)\right\}$

Design a broadside binomial array of 7 elements placed along the z-axis separated by a distance $d = \lambda/2$.

(a) Find the amplitude excitation coefficients (a_n's).

(b) What is the progressive phase excitation between the elements?

(c) Write the array factor.

(d) Now assume that the elements are $\lambda/4$ dipoles oriented in the z-direction. Write the expression for the electric field *vector* in the far field.

a. From (6-63), $a_1=10$, $a_2=15$, $a_3=6$, $a_4=1$ ← Verified with computer program

b. Same answer like (b) in Problem 6-30. ($D_o = 6.467\text{dB}$)

At $d=\frac{\lambda}{2}$

c. $$AF=\sum_{n=1}^{4} a_n \cos[2(n-1)u] = 10 + 15\cos 2u + 6\cos 4u + \cos 6u$$

$$\left(\leftarrow u = \frac{\pi d}{\lambda}\cos\theta = \frac{\pi}{2}\cos\theta\right)$$

d. Field of E_θ at origin : From (4-62a)

$$E_\theta \simeq j\frac{\eta I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{\pi}{4}\cos\theta\right)-\cos\left(\frac{\pi}{4}\right)}{\sin\theta}\right]$$ ← one dipole of $\frac{\lambda}{4}$ length

Array.

$$E_\theta \simeq j\eta\frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{\pi}{4}\cos\theta\right)-0.707}{\sin\theta}\right]\left[10+15\cos 2u+6\cos 4u+\cos 6u\right]$$

$$\left(\leftarrow u = \frac{\pi d}{\lambda}\cos\theta\right)$$

Design a three-element, −40 dB side lobe level Dolph-Tschebyscheff array of isotropic elements placed symmetrically along the z-axis. Find the

(a) amplitude excitation coefficients
(b) array factor
(c) angles where the nulls occur for $d = 3\lambda/4 (0° \le \theta \le 180°)$
(d) directivity for $d = 3\lambda/4$
(e) half-power beamwidth for $d = 3\lambda/4$

34. The procedure for this problem is identically the same as that of Problem 6.33 except that the side lobe level for this one is −40 dB instead of −26 dB.

a. $(AF)_3 = \sum_{n=1}^{2} a_n \cos[2(n-1)u] = a_1 + a_2 \cos(2u) = (a_1 - a_2) + 2a_2 \cos^2 u$

$R_0 = 40\text{ dB} \Rightarrow R_0(\text{voltage ratio}) = 100$

$z_0 = \frac{1}{2}\left[(100+\sqrt{100^2-1})^{1/2} + (100-\sqrt{100^2-1})^{1/2}\right] = 7.1063$

Thus

$(a_1-a_2) + 2a_2\cos^2 u = (a_1-a_2) + 2a_2\left(\frac{z}{z_0}\right)^2 = 2z^2-1 = T_2(z)$

$2a_2\left(\frac{z}{z_0}\right)^2 = 2z^2 \Rightarrow a_2 = z_0^2 = (7.1063)^2 = 50.5$

$a_1 - a_2 = -1 \Rightarrow a_2 = z_0^2 = (7.1063)^2$

$\Rightarrow a_1 = -1 + a_2 = 49.5$

$\left.\begin{array}{l} a_1 = 49.5 \\ a_2 = 50.5 \end{array}\right\}$ normalized $\begin{array}{l} a_{1n} = a_1/a_2 = 49.5/50.5 = 0.9802 \\ a_{2n} = a_2/a_2 = 50.5/50.5 = 1.0000 \end{array}$

b. $(AF)_3 = a_{1n} + a_{2n}\cos(2u) = 0.9802 + \cos\left(\frac{2\pi d}{\lambda}\cos\theta\right)$

or $(AF)_3 = (a_{1n} - a_{2n}) + 2a_{2n}\cos^2 u = -0.0198 + 2\cos^2\left(\frac{\pi d}{\lambda}\cos\theta\right)$

c. For $d = 3\lambda/4 \quad (0° < \theta \leq 180°)$

$$(AF)_3 = 0.9802 + \cos\left[\frac{2\pi(3)}{4}\cos\theta_n\right] = 0.9802 + \cos\left(\frac{3\pi}{2}\cos\theta_n\right) = 0$$

$$\frac{3\pi}{2}\cos\theta_n = \cos^{-1}(-0.9802) = \begin{cases} \pm 168.58°\ (\pm 2.9423\ \text{rad}) \\ \pm 191.42°\ (\pm 3.3409\ \text{rad}) \end{cases}$$

Therefore $\theta_n = \cos^{-1}\left(\pm\frac{2(2.9423)}{3\pi}\right) = 51.36°,\ 128.64°$

$$\theta_n = \cos^{-1}\left(\pm\frac{2(3.3409)}{3\pi}\right) = 44.85°,\ 135.15°$$

d. $D_0 = \dfrac{2R_0^2}{1 + (R_0^2 - 1) f\left(\frac{\lambda}{L+d}\right)}, \quad f = 1 + 0.636\left\{\frac{2}{R_0}\cosh\left[\sqrt{(\cosh^{-1}R_0)^2 - \pi^2}\right]\right\}^2$

$$f = 1 + 0.636\left\{\frac{2}{100}\cosh\left[\sqrt{(\cosh^{-1}(100))^2 - \pi^2}\right]\right\} = 1.323$$

$$D_0 = \frac{2(100)^2}{1 + (100^2 - 1) + 1.323\left(\frac{4}{9}\right)} = 3.4 = 5.32\text{dB}$$

Using the computer program at the end of Chapter 2, $D_0 = 3.76 = 5.75\text{dB}$

$$\text{HPBW} = \left\{\cos^{-1}\left[\cos\theta_0 - 0.443\left(\frac{\lambda}{L+d}\right)\right] - \cos^{-1}\left[\cos\theta_0 - 0.443\left(\frac{\lambda}{L+d}\right)\right]\right\} f \Big|_{\theta_0 = 90°}$$

$$= \left\{\cos^{-1}\left[0 - 0.443\left(\frac{4}{9}\right)\right] - \cos^{-1}\left[0 - 0.443\left(\frac{4}{9}\right)\right]\right\} 1.323 = 30.05°$$

- Alternately

The excitation coefficients can also be found using (6-77b) or

$$a_n = \sum_{\xi=n}^{M+1} (-1)^{M-\xi+1} \cdot (z_0)^{2(\xi-1)} \cdot \frac{(\xi+M-2)!\,(2M)}{\epsilon_n\,(\xi-n)!\,(\xi+n-2)!\,(M-\xi+1)!} \quad \text{with } M=1$$

Thus

$$a_1 = \sum_{\xi=1}^{2} (-1)^{2-\xi} \cdot (7.1063)^{2(\xi-1)} \cdot \frac{(\xi-1)!\;2}{2(\xi-1)!\,(\xi-1)!\,(2-\xi)!} = -2+(7.1063)^2 = 49.5$$

$$a_2 = \sum_{\xi=2}^{2} (-1)^{2-\xi} \cdot (7.1063)^{2(\xi-1)} \cdot \frac{(\xi-1)!\;2}{(\xi-2)!\,\xi!\,(2-\xi)!} = (7.1063)^2 = 50.5$$

Design a four-element, −40 dB side lobe level Dolph-Tschebyscheff array of isotropic elements placed symmetrically about the z-axis. Find the

(a) amplitude excitation coefficients (b) array factor

(c) angles where the nulls occur for $d = 3\lambda/4$.

5.

a. $(AF)_4 = \sum_{n=1}^{2} a_n \cos[(2n-1)u]$

$= a_1 \cos u + a_2 \cos 3u = (a_1 - 3a_2)\cos u + 4a_2 \cos^3 u$

$R_0 = 40\text{dB} \Rightarrow R_0 = 100$

$z_0 = \cosh[\frac{1}{3}\cosh^{-1}(100)] = 3.0095$

Therefore

$(AF)_4 = (a_1 - 3a_2)\frac{z}{z_0} + 4a_2\left(\frac{z}{z_0}\right)^3 = -3z + 4z^3 = T_3(z)$

$\frac{4a_2}{(3.0095)^3} = 4 \Rightarrow a_2 = 27.257$

$\frac{a_1 - 3(27.257)}{(3.0095)} = -3 \Rightarrow a_1 = 72.742$

$a_1 = 2.668$

$a_2 = 1$

b. $AF = 2.668 \cos u + \cos 3u \quad , \; u = \frac{\pi d}{\lambda} \cos\theta$

c. $d = \frac{3\lambda}{4}, \qquad u = \frac{3}{4}\pi \cos\theta$

$$AF = 2.668 \cos u + \cos 3u = 2.668 \cos u - 3\cos u + 4\cos^3 u$$

$$= -0.332 \cos u + 4\cos^3 u = \cos\left(\frac{3\pi}{4}\cos\theta\right)\left[-0.332 + 4\cos^2\left(\frac{3\pi}{4}\cos\theta\right)\right]$$

$$= \cos\left(\frac{3\pi}{4}\cos\theta\right)\left[1.668 + 2\cos\left(\frac{3\pi}{2}\cos\theta\right)\right] = 0$$

$$\therefore \cos\left(\frac{3\pi}{4}\cos\theta_n\right) = 0 \quad \text{or} \quad \frac{3\pi}{2}\cos\theta_n = \cos^{-1}(-0.834) = \begin{cases} 2.5571 \\ 3.7261 \end{cases}$$

$$\theta_n = \cos^{-1}\left(\frac{\pi}{2}\cdot\frac{4}{3\pi}\right) = 48.19^\circ,$$

$$\theta_n = \cos^{-1}\left[\frac{2}{3\pi}(2.5571)\right] = 57.137^\circ$$

$$\theta_n = \cos^{-1}\left[\frac{2}{3\pi}(3.7261)\right] = 37.7487^\circ$$

Computer Result.

Directivity

$D_0 = 6.859\,dB$

Design a five element, −40 dB side lobe level Dolph-Tschebyscheff array of isotropic elements placed symmetrically about the z-axis. Find the

(a) amplitude excitation coefficients (b) array factor

(c) angles where the nulls occur for $d = 3\lambda/4$.

$$(AF)_5 = \sum_{n=1}^{3} a_n \cos[2(n-1)u] = a_1 \cos(0u) + a_2 \cos(2u) + a_3 \cos(4u)$$

$$= a_1 + a_2(2\cos^2 u - 1) + a_3(8\cos^4 u - 8\cos^2 u + 1)$$

$$= (2a_2 - 8a_3)\cos^2 u + (8\cos^4 u)a_3 + (a_1 - a_2 + a_3)$$

$R_0 = 10$, $z_0 = \cosh\left(\frac{1}{4}\cosh^{-1}(10)\right) = 1.2933$, $T_4(z) = 8z^4 - 8z^2 + 1$

$$\therefore (2a_2 - 8a_3)\left(\frac{z}{z_0}\right)^2 + 8a_3\left(\frac{z}{z_0}\right)^4 + (a_1 - a_2 + a_3) = -8z^2 + 8z^4 + 1$$

$$\frac{8z^4 \cdot a_3}{(1.2933)^4} = 8z^4, \rightarrow a_3 = 2.7976$$

$$\frac{z^2[2a_2 - 8(2.7976)]}{(1.2933)^2} = -8z^2 \rightarrow a_2 = 4.49992$$

$$a_1 - a_2 + a_3 = 1 \rightarrow a_1 = 2.7023$$

a. $a_3 = 1$, $a_2 = 1.6085$, $a_1 = 0.966$

b. array factor.

$$(AF) = 0.966 + 1.6085\cos(2u) + \cos(4u)$$

$$u = \frac{\pi d}{\lambda}\cos\theta$$

c. $d = \frac{3\lambda}{4}$, $u = \frac{3\pi}{4}\cos\theta$.

$$0 = 0.966 + 1.6085(2\cos^2 u - 1) + 8\cos^4 u - 8\cos^2 u + 1$$

$$= 8\cos^4 u - 4.783\cdot\cos^2 u + 0.3575$$

$$\therefore \cos\left(\frac{3\pi}{4}\cos\theta\right) = \pm 0.7143562028, \pm 0.2959226515$$

$$\therefore \cos(\theta_n) = \frac{4}{3\pi}\cos^{-1}(0.7143562028)$$

$$\theta_n = 70.79°,$$

$$\cos(\theta_n) = \frac{4}{3\pi}\cos^{-1}(\pm 0.2959226515)$$

$$\theta_n = 57.37°, \quad 37.423°.$$

Null degree

$$\theta_n = 37.423°, 57.37°, 70.79°$$

Design a six element, −40 dB side lobe level Dolph-Tschebyscheff array of isotropic elements placed symmetrically about the z-axis. Find the
(a) amplitude excitation coefficients (b) array factor
(c) angles where the nulls occur for $d = 3\lambda/4$.

$$(AF)_6 = \sum_{n=1}^{3} a_n \cos[(2n-1)u] = a_1 \cos u + a_2 \cos 3u + a_3 \cos 5u$$

$$R_o(\text{dB}) = 20 = 20\log_{10}(R_o) \quad \therefore R_o = 10$$

$$z_o = \cosh\left(\tfrac{1}{4}\cosh^{-1}(10)\right) = 1.2933.$$

$$(AF)_6 = a_1 \cos u + a_2(4\cos^3 u - 3\cos u) + a_3(16\cos^5 u - 20\cos^3 u + 5\cos u)$$

$$= a_3 (16)\cos^5 u + (4a_2 - 20a_3)\cos^3 u + (a_1 - 3a_2 + 5a_3)\cos u$$

$$= 16z^5 - 20z^3 + 5z$$

$$\frac{a_3(16)}{(1.2933)^5} = 16, \quad \rightarrow a_3 = 3.618$$

$$\frac{4a_2 - 20(3.618)}{(1.2933)^3} = -20, \quad \rightarrow a_2 = 7.275$$

$$\frac{a_1 - 3(7.275) + 5(3.618)}{(1.2933)} = 5 \rightarrow a_1 = 10.2015$$

a. $a_1 = 2.81965, \quad a_2 = 2.011, \quad a_3 = 1$

b. $(AF)_6 = 2.81965 \cdot \cos(u) + 2.011 \cdot \cos(3u) + \cos(5u)$

c. Null point, $u = \frac{3\pi}{4}\cos\theta$

$(AF)_6 = 2.81965(\cos u) + 2.011(4\cos^3 u - 3\cos u) + 16\cos^5 u - 20\cos^3 u + 5\cos u$

$= 16\cos^5 u - 11.956\cos^3 u + 1.78665\cos u = 0.$

$\therefore \cos u = 0, \pm 0.7353555305, \pm 0.4544251796.$

$\cos(\frac{3\pi}{4}\cos\theta) = 0, \quad \theta_n = \cos^{-1}(\frac{\pi}{2}\cdot\frac{4}{3\pi}) = 48.19°$

$\cos(\frac{3\pi}{4}\cos\theta) = +0.73535553, \quad \theta_n = 71.57°$

$\cos(\frac{3\pi}{4}\cos\theta) = \pm 0.4544251796, \quad \theta_n = 62.19°, 29.90°$

$\therefore$ Null degree.

$\theta_n = 29.90°, 62.19°, 48.19°, 71.57°$

Design a five-element, −40 dB side lobe level Dolph-Tschebyscheff array of isotropic elements. The elements are placed along the x-axis with a spacing of $\lambda/4$ between them. Determine the:

(a) normalized amplitude coefficients (b) array factor

(c) directivity (d) half-power beamwidth

-40. $(AF)_5 = \sum_{n=1}^{3} a_n \cos[2(n-1)u] = a_1 + a_2\cos(2u) + a_3\cos(4u)$

$= a_1 + a_2(2\cos^2 u - 1) + a_3(8\cos^4 u - 8\cos^2 u + 1)$

a. $R_0 = 40\text{dB} \Rightarrow R_0 (\text{Voltage ratio}) = 100$

$z_0 = \frac{1}{2}\left\{\left[100 + \sqrt{100^2 - 1}\right]^{1/4} + \left[100 - \sqrt{100^2 - 1}\right]^{1/4}\right\} = 2.013248$

letting $\cos u = z/z_0$

$(AF)_5 = (a_1 - a_2 + a_3) + (2a_2 - 8a_3)\cos^2 u + 8a_3\cos^4 u$

$= (a_1 - a_2 + a_3) + (2a_2 - 8a_3)\left(\frac{z}{z_0}\right)^2 + 8a_3\left(\frac{z}{z_0}\right)^4 = 1 - 8z^2 + 8z^4$

Equating alike terms yields $a_3 = 16.429$, $a_2 = 49.503$, $a_1 = 34.074$

or in normalized form $a_{3n} = a_3/a_3 = 1.0$, $a_{2n} = a_2/a_3 = 3.013$, $a_{1n} = a_1/a_3 = 2.074$

b. $(AF)_5 = 2.074 + 3.013\cos\left(\frac{\pi}{2}\sin\theta\cos\phi\right) + \cos(\pi\sin\theta\cos\phi)$, $u = \frac{\pi d}{\lambda}\sin\theta\cos\phi \Big|_{d=\lambda/4}$

c. $f = 1 + 0.636\left\{\frac{2}{100}\cosh\left[\sqrt{(\cosh^{-1}100)^2 - \pi^2}\right]\right\}^2 = 1 + 0.636\left[\frac{2}{100}(35.64)\right]^2 = 1.323$

$$D_0 = \frac{2R_0^2}{1 + (R_0^2 - 1)f\left(\frac{\lambda}{L+d}\right)} = \frac{2(100)^2}{1 + (100^2 - 1)1.323\frac{1}{1.25}} = 1.889 = 2.76\text{dB}$$

D_0 (Computer Program) $= 1.998 = 3.01$dB
at the end of Chapter 2

d. $\text{HPBW} = f\left\{\cos^{-1}\left(\cos\theta_0 - 0.443\frac{\lambda}{L+d}\right) - \cos^{-1}\left(\cos\theta_0 + 0.443\frac{\lambda}{L+d}\right)\right\}_{\theta_0 = 90^\circ}$

$= 1.323\left\{\cos^{-1}\left[-0.443\left(\frac{1}{1.25}\right)\right] - \cos^{-1}\left[0.443\left(\frac{1}{1.25}\right)\right]\right\} = 1.323(41.513^\circ)$

$\text{HPBW} = 54.9^\circ$

The total length of a discrete-element array is 4λ. For a -30 dB side lobe level Dolph-Tschebyscheff design and a spacing of $\lambda/2$ between the elements along the z-axis, find the:

(a) number of elements

(b) excitation coefficients

(c) directivity

(d) half-power beamwidth

a. $N = 2M+1 = 9 \Rightarrow M = 4$, $R_o = 30\text{dB} \Rightarrow R_o(\text{voltage Ratio}) = 10^{1.5} = 31.662$

$$z_o = \frac{1}{2}\left\{\left[31.662 + \sqrt{(31.662)^2 - 1}\right]^{1/8} + \left[31.662 - \sqrt{(31.662)^2 - 1}\right]^{1/8}\right\}$$

$$z_o = 1.679244 + 0.595506 = 1.137375$$

b. $(AF)_9 = \sum_{n=1}^{5} a_n \cos[2(n-1)u] = a_1 + a_2\cos(2u) + a_3\cos(4u) + a_4\cos(6u) + a_5\cos(8u)$

$= a_1 + a_2(2\cos^2 u - 1) + a_3(8\cos^4 u - 8\cos^2 u + 1) + a_4(32\cos^6 u - 48\cos^4 u + 18\cos^2 u - 1) + a_5(128\cos^8 u - 256\cos^6 u + 160\cos^4 u - 32\cos^2 u + 1)$

$(AF)_9 = (a_1 - a_2 + a_3 - a_4 + a_5) + (2a_2 - 8a_3 + 18a_4 - 32a_5)\cos^2 u + (8a_3 - 48a_4 + 160a_5)\cos^4 u + (32a_4 - 256a_5)\cos^6 u + (128a_5)\cos^8 u$

letting $z/z_0 = \cos u$ and equating $(AF)_9 = T_8(z) = 1 - 32z^2 + 160z^4 - 256z^6 + 128z^8$

leads to $a_5 = z_0^8 = (1.13737)^8 = 2.8004$

$-256 = \dfrac{32a_4 - 256a_5}{z_0^6} \Rightarrow a_4 = (-256z_0^6 + 256a_5)/32 = 5.085$

$160 = \dfrac{8a_3 - 48a_4 + 160a_5}{z_0^4} \Rightarrow a_3 = (160z_0^4 + 48a_4 - 160a_5)/8 = 7.970$

$-32 = \dfrac{2a_2 - 8a_3 + 18a_4 - 32a_5}{z_0^2} \Rightarrow a_2 = (-32z_0^2 + 8a_3 - 18a_4 + 32a_5)/2 = 10.226$

$1 = a_1 - a_2 + a_3 - a_4 + a_5 \Rightarrow a_1 = 1 + a_2 - a_3 + a_4 - a_5 = 5.540$

Thus
$a_1 = 5.540 \Rightarrow 2a_1 = 11.080$
$a_2 = 10.226$
$a_3 = 7.970$
$a_4 = 5.085$
$a_5 = 2.8004$

In Normalized Form $\Rightarrow$
$a_{1n} = a_1/a_5 = 1.978, 2a_{1n} = 3.956$
$a_{2n} = a_2/a_5 = 3.652$
$a_{3n} = a_3/a_5 = 2.846$
$a_{4n} = a_4/a_5 = 1.815$
$a_{5n} = a_5/a_5 = 1.0$

c. $f = 1 + 0.636\left\{\frac{2}{31.662}\cosh\left[\sqrt{(\cosh^{-1}31.662)^2-\pi^2}\right]\right\}^2 = 1+0.636\left[\frac{2}{31.662}(7.5373)\right]^2 = 1.144$

$$D_0 = 2R_0^2\Big/\left\{1+(R_0^2-1)f\left(\frac{\lambda}{L+d}\right)\right\} = 2(31.662)^2\Big/\left\{1+(31.662^2-1)\,1.144\left(\frac{1}{4.5}\right)\right\}$$

$$= 7.844 = 8.945\ \text{dB}$$

d. $$\text{HPBW} = f\left\{\cos^{-1}\left(\cos\theta_0 - 0.443\frac{\lambda}{L+d}\right) - \cos^{-1}\left(\cos\theta_0 + 0.443\frac{\lambda}{L+d}\right)\right\}_{\theta_0=90^\circ}$$

$$= 1.144\left\{\cos^{-1}\left(-0.443\frac{1}{4.5}\right) - \cos^{-1}\left(0.443\frac{1}{4.5}\right)\right\} = 1.144(95.65 - 84.35)$$

$$\text{HPBW} = 1.144\,(11.30) = 12.93^\circ$$