2.3 Non-uniform excitation - Binomial, Chebyshev distribution

Module:2 Linear and Planar Arrays

Course: BECE305L – Antenna and Microwave Engineering

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CHENNAI

Module:2 Linear and Planar Arrays

- Two element array, N-element linear array - broadside array, End fire array - Directivity, radiation pattern, pattern multiplication. Non-uniform excitation - Binomial, Chebyshev distribution, Arrays: Planar array, circular array, Phased Array antenna (Qualitative study).
- Source of the contents: Constantine A. Balanis - Antenna theory analysis and design (2016)

1. *N*-ELEMENT LINEAR ARRAY: UNIFORM SPACING, NONUNIFORM AMPLITUDE

- broadside arrays with uniform spacing but nonuniform amplitude distribution
- binomial and
  Dolph-Tschebyscheff broadside arrays (also spelled Tchebyscheff or Chebyshev)
- *(uniform, binomial, and Tschebyscheff),*
  a uniform amplitude array yields the smallest half-power beamwidth.
  followed by the Dolph-Tschebyscheff and
  binomial arrays.
- *In contrast, binomial arrays usually possess the smallest side lobes*
  followed by the Dolph-Tschebyscheff and
  uniform arrays

1. *N*-ELEMENT LINEAR ARRAY: UNIFORM SPACING, NONUNIFORM AMPLITUDE

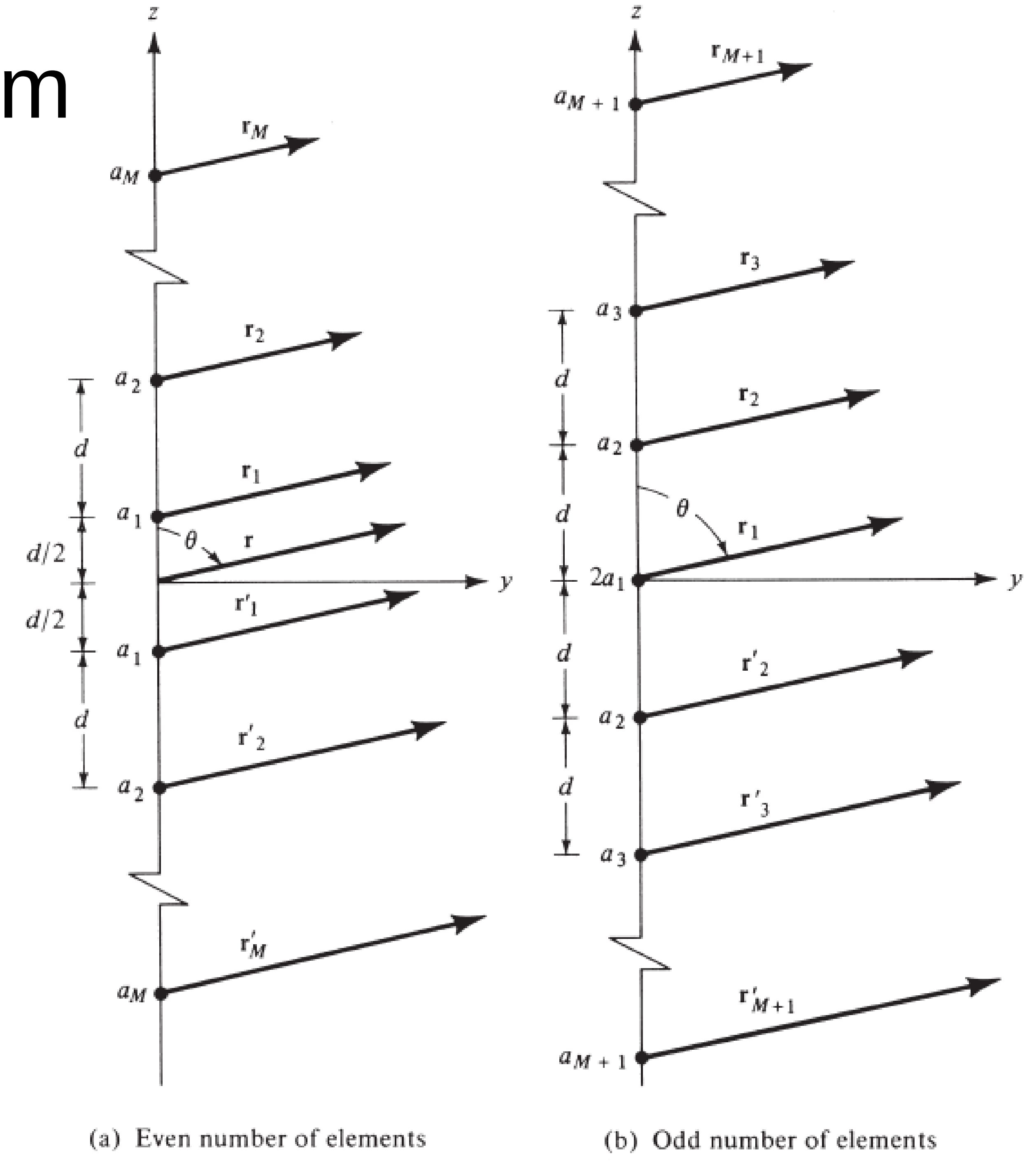
- Binomial with element spacing equal or less than $\lambda/2$ have no side lobes.
- designer must compromise between side lobe level and beamwidth.
- *shown analytically that*
- *for a given side lobe level*
  the Dolph-Tschebyscheff array produces the smallest beamwidth between the first nulls.
- *Conversely, for a given beamwidth between the first nulls, the Dolph-Tschebyscheff design leads to the smallest possible side lobe level.*

1. *N*-ELEMENT LINEAR ARRAY: UNIFORM SPACING, NONUNIFORM AMPLITUDE

- Uniform arrays usually possess the largest directivity.
- However, superdirective (or super gain as most people refer to them) antennas possess directivities higher than those of a uniform array
- superdirective arrays usually require very large currents with opposite phases between adjacent elements.
- Net current and efficiency of each array are very small compared to the corresponding values of an individual element.

2. Array factor – nonuniform amplitude distributions

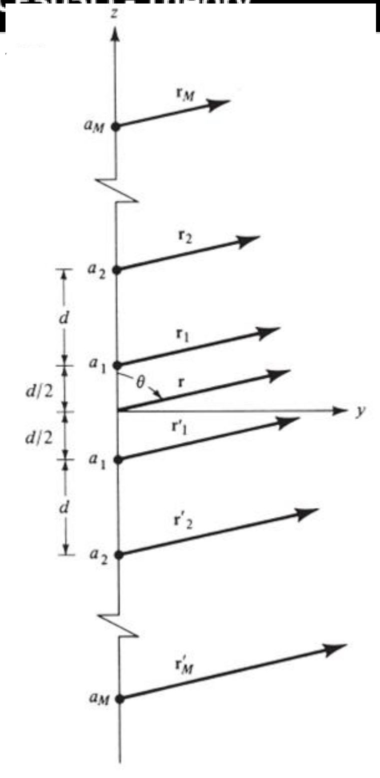
- An array of an even number of isotropic elements 2*M* (where *M* is an integer) is positioned symmetrically along the *z*-axis.
- separation between the elements is *d*, and *M* elements are placed on each side of the origin.
- amplitude excitation is symmetrical about the origin



2. Array factor – nonuniform amplitude distributions

- array factor for a nonuniform amplitude broadside array

$$
\begin{aligned}
(\mathrm{AF})_{2M} = {} & a_1 e^{+j(1/2)kd\cos\theta} + a_2 e^{+j(3/2)kd\cos\theta} + \cdots \\
& + a_M e^{+j[(2M-1)/2]kd\cos\theta} \\
& + a_1 e^{-j(1/2)kd\cos\theta} + a_2 e^{-j(3/2)kd\cos\theta} + \cdots \\
& + a_M e^{-j[(2M-1)/2]kd\cos\theta}
\end{aligned}
$$



(a) Even number of elements

$$
(\mathrm{AF})_{2M} = 2\sum_{n=1}^{M} a_n \cos\left[\frac{(2n-1)}{2}kd\cos\theta\right]
$$

M elements are placed on each side of the origin.

normalized form reduces to

$$
(\mathrm{AF})_{2M} = \sum_{n=1}^{M} a_n \cos\left[\frac{(2n-1)}{2}kd\cos\theta\right]
$$

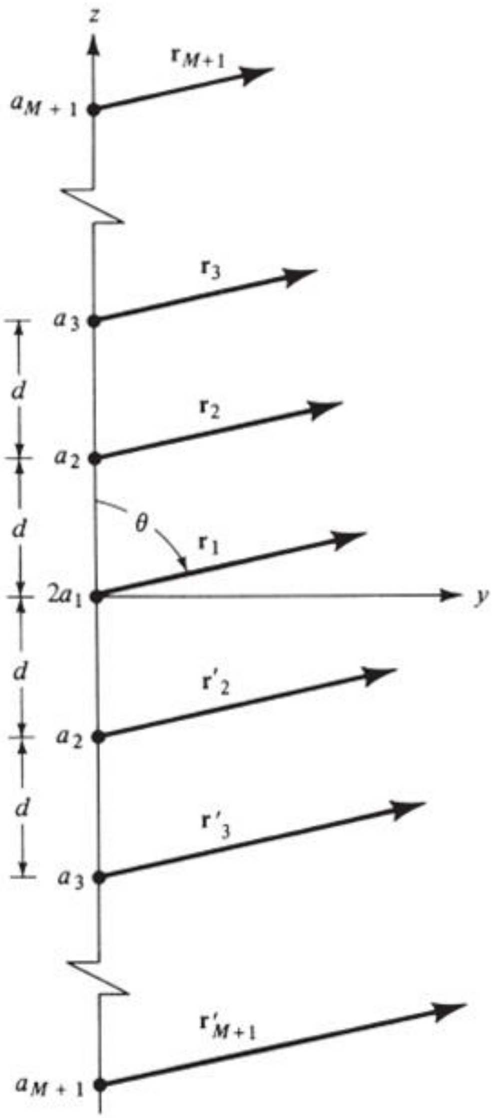
2. Array factor – nonuniform amplitude distributions

- total number of isotropic elements of the array is odd $2M + 1$ (where M is an integer),

$$(\mathrm{AF})_{2M+1} = 2a_1 + a_2 e^{+jkd\cos\theta} + a_3 e^{j2kd\cos\theta} + \cdots + a_{M+1} e^{jMkd\cos\theta}$$
$$+ a_2 e^{-jkd\cos\theta} + a_3 e^{-j2kd\cos\theta} + \cdots + a_{M+1} e^{-jMkd\cos\theta}$$

$$(\mathrm{AF})_{2M+1} = 2\sum_{n=1}^{M+1} a_n \cos[(n-1)kd\cos\theta]$$

The amplitude excitation of the center element is 2a1.



(b) Odd number of elements

normalized form reduces to

$$(\mathrm{AF})_{2M+1} = \sum_{n=1}^{M+1} a_n \cos[(n-1)kd\cos\theta]$$

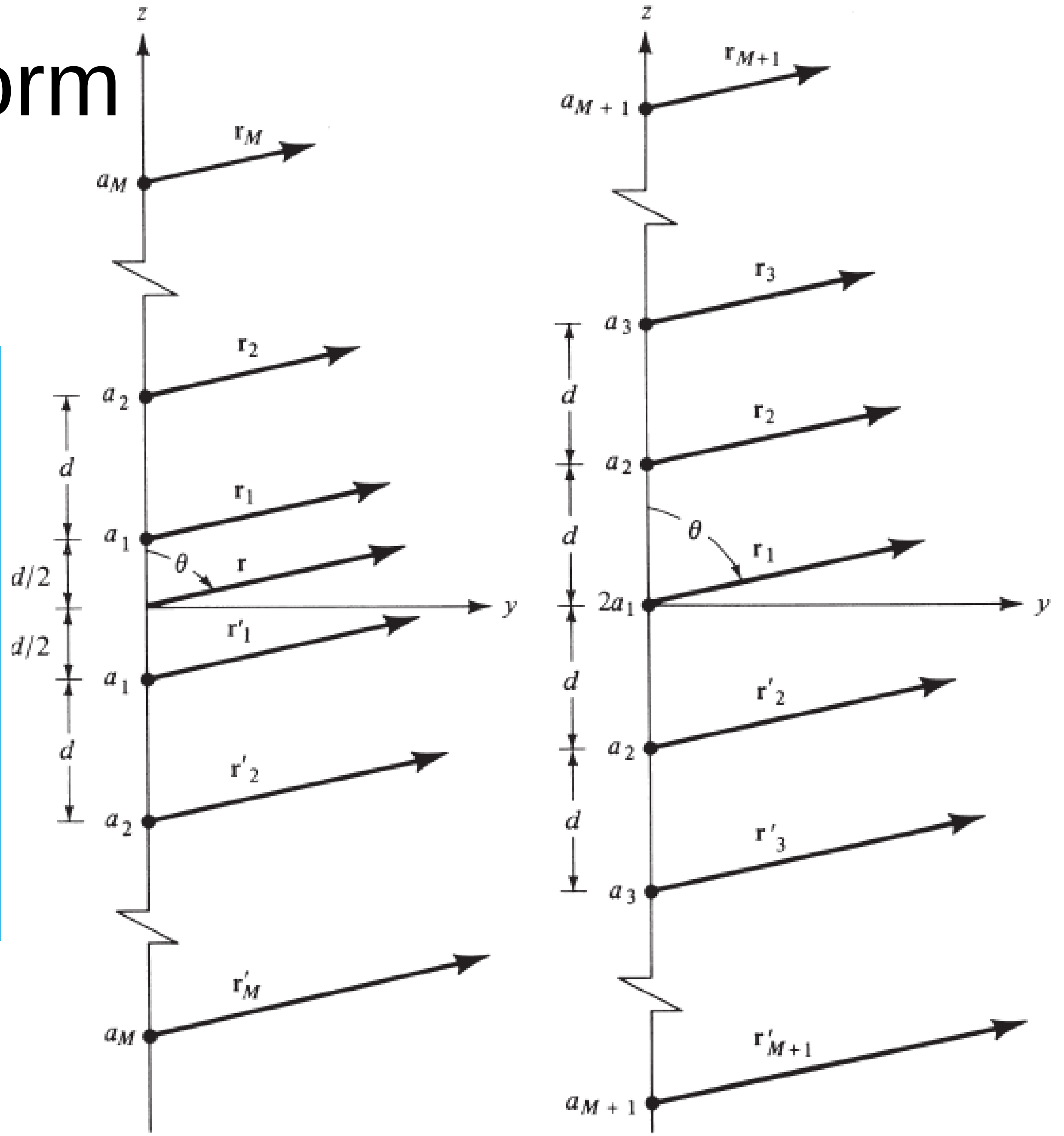
2. Array factor – nonuniform amplitude distributions

$$(\mathrm{AF})_{2M}(\mathrm{even}) = \sum_{n=1}^{M} a_n \cos[(2n-1)u]$$

$$(\mathrm{AF})_{2M+1}(\mathrm{odd}) = \sum_{n=1}^{M+1} a_n \cos[2(n-1)u]$$

where

$$u = \frac{\pi d}{\lambda}\cos\theta$$



(a) Even number of elements (b) Odd number of elements

3. Binomial Array: A. Excitation coefficients

- The array factor for the binomial array is represented where the a_n's are the excitation coefficients
- To determine the excitation coefficients of a binomial array

$$(1+x)^{m-1} = 1 + (m-1)x + \frac{(m-1)(m-2)}{2!}x^2 + \frac{(m-1)(m-2)(m-3)}{3!}x^3 + \cdots$$

- **coefficients are determined from a binomial series expansion**, the array is known as a ***binomial array***.
- Pascal's triangle. If the values of *m* are used to represent the number of elements of the array, then the coefficients of the expansion represent the relative amplitudes of the elements

3. Binomial Array: A. Excitation coefficients

$$
\begin{array}{lccccccccccccccccccc}
m=1 & & & & & & & & & & 1 & & & & & & & & & \\
m=2 & & & & & & & & & 1 & & 1 & & & & & & & & \\
m=3 & & & & & & & & 1 & & 2 & & 1 & & & & & & & \\
m=4 & & & & & & & 1 & & 3 & & 3 & & 1 & & & & & & \\
m=5 & & & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & & \\
m=6 & & & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & & \\
m=7 & & & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & & \\
m=8 & & & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & & \\
m=9 & & 1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1 & \\
m=10 & 1 & & 9 & & 36 & & 84 & & 126 & & 126 & & 84 & & 36 & & 9 & & 1
\end{array}
$$

3. Binomial Array: A. Excitation coefficients

1. Two elements ($2M = 2$)

$$a_1 = 1$$

2. Three elements ($2M + 1 = 3$)

$$2a_1 = 2 \Rightarrow a_1 = 1$$

$$a_2 = 1$$

3. Four elements ($2M = 4$)

$$a_1 = 3$$

$$a_2 = 1$$

4. Five elements ($2M + 1 = 5$)

$$2a_1 = 6 \Rightarrow a_1 = 3$$

$$a_2 = 4$$

$$a_3 = 1$$

3. Binomial Array: *B. Design Procedure*

- amplitude excitation coefficients : accomplished using either expansion of the series or the Pascal triangle.
- *binomial arrays do not exhibit any minor lobes provided the spacing between the elements is equal or less than one-half of a wavelength*

$$\mathrm{HPBW}(d = \lambda/2) \simeq \frac{1.06}{\sqrt{N-1}} = \frac{1.06}{\sqrt{2L/\lambda}} = \frac{0.75}{\sqrt{L/\lambda}}$$

$$D_0 = \frac{2}{\int_0^{\pi} \left[\cos\left(\frac{\pi}{2}\cos\theta\right)\right]^{2(N-1)} \sin\theta \, d\theta}$$

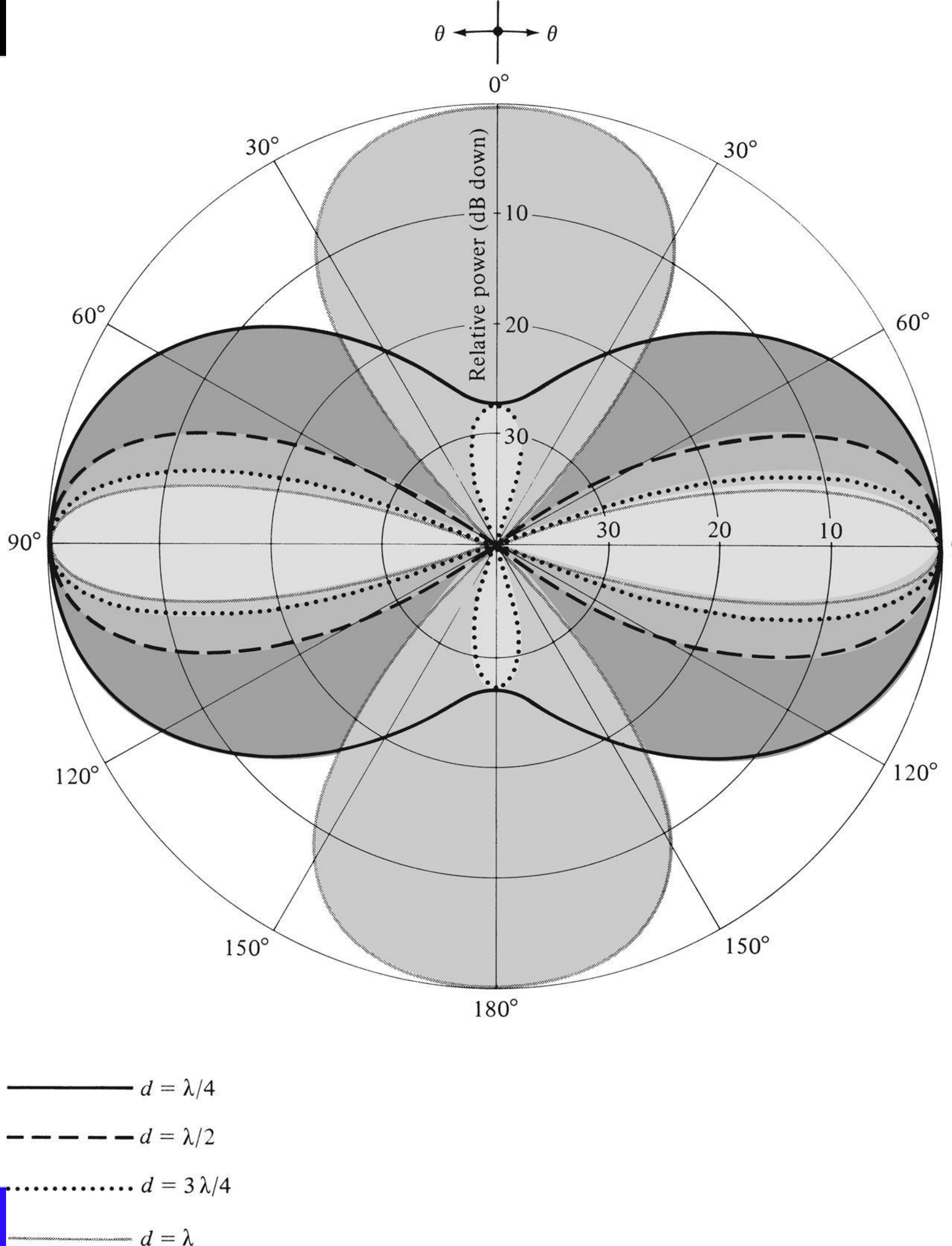
$$D_0 = \frac{(2N-2)(2N-4)\cdots 2}{(2N-3)(2N-5)\cdots 1}$$

$$D_0 \simeq 1.77\sqrt{N} = 1.77\sqrt{1 + 2L/\lambda}$$

3. Binomial Array: *B. Design Procedure*

- amplitude excitation coefficients : accomplished using either expansion of the series or the Pascal triangle.
- 10-element binomial array ($2M = 10$) with spacings between the elements of $\lambda/4$, $\lambda/2$, $3\lambda/4$, and λ, respectively.
- The coefficients of ***a*1 = 126**, $a2 = 84$, $a3 = 36$, $a4 = 9$, and $a5 = 1$.
- no minor lobes for the arrays with spacings of $\lambda/4$ and $\lambda/2$ between the elements.
- very low level minor lobes, they exhibit larger beamwidths
- **Disadvantage: wide variations between the amplitudes** of the different elements of an array, especially for an array with a large number of elements. Hence have very low efficiencies for the feed network

Array factor power patterns for a 10-element broadside binomial array with $N = 10$ and $d = \lambda/4$, $\lambda/2$, $3\lambda/4$, and λ.



Problem: For a 10-element binomial array with a spacing of λ/2 between the elements, whose amplitude pattern is displayed in previous Figure 6.20, determine the half-power beamwidth (in degrees) and the maximum directivity (in dB). Compare the answers with other available data

$$\text{HPBW} \simeq \frac{1.06}{\sqrt{10-1}} = \frac{1.06}{3} = 0.353 \text{ radians} = 20.23^\circ$$

$$\text{HPBW}(d = \lambda/2) \simeq \frac{1.06}{\sqrt{N-1}} = \frac{1.06}{\sqrt{2L/\lambda}} = \frac{0.75}{\sqrt{L/\lambda}}$$

- the value of the directivity is equal for $N = 10$

$$D_0 = 5.392 = 7.32 \text{ dB}$$

$$D_0 = \frac{(2N-2)(2N-4)\cdots 2}{(2N-3)(2N-5)\cdots 1}$$

$$D_0 = 5.392 = 7.32 \text{ dB}$$

$$D_0 \simeq 1.77\sqrt{N} = 1.77\sqrt{1 + 2L/\lambda}$$

$$D_0 = 1.77\sqrt{10} = 5.597 = 7.48 \text{ dB}$$

4. Dolph-Tschebyscheff Array: Broadside

- **also spelled Tchebyscheff or Chebyshev**
- many pratical applications
- It is primarily a compromise between uniform and binomial arrays
- excitation coefficients are related to Tschebyscheff polynomials
- A Dolph-Tschebyscheff array with no side lobes (or side lobes of $-\infty$ dB) reduces to the binomial design.

4.1 Dolph-Tschebyscheff Array: Array Factor

- Array factor of an array of even or odd number of elements with symmetric amplitude excitation is a summation of M or $M + 1$ cosine terms
- largest harmonic of the cosine terms is one less than the total number of elements.

$$(AF)_{2M}(\text{even}) = \sum_{n=1}^{M} a_n \cos[(2n-1)u]$$

$$(AF)_{2M+1}(\text{odd}) = \sum_{n=1}^{M+1} a_n \cos[2(n-1)u]$$

where

$$u = \frac{\pi d}{\lambda} \cos\theta$$

- Each cosine term, whose argument is an integer times a fundamental frequency, can be rewritten as a series of cosine functions with the fundamental frequency as the argument.

$$m = 0 \quad \cos(mu) = 1$$

$$m = 1 \quad \cos(mu) = \cos u$$

$$m = 2 \quad \cos(mu) = \cos(2u) = 2\cos^2 u - 1$$

4.1 Dolph-Tschebyscheff Array: Array Factor

$$m = 0 \quad \cos(mu) = 1$$
$$m = 1 \quad \cos(mu) = \cos u$$
$$m = 2 \quad \cos(mu) = \cos(2u) = 2\cos^2 u - 1$$
$$m = 3 \quad \cos(mu) = \cos(3u) = 4\cos^3 u - 3\cos u$$
$$m = 4 \quad \cos(mu) = \cos(4u) = 8\cos^4 u - 8\cos^2 u + 1$$
$$m = 5 \quad \cos(mu) = \cos(5u) = 16\cos^5 u - 20\cos^3 u + 5\cos u$$
$$m = 6 \quad \cos(mu) = \cos(6u) = 32\cos^6 u - 48\cos^4 u + 18\cos^2 u - 1$$
$$m = 7 \quad \cos(mu) = \cos(7u) = 64\cos^7 u - 112\cos^5 u + 56\cos^3 u - 7\cos u$$
$$m = 8 \quad \cos(mu) = \cos(8u) = 128\cos^8 u - 256\cos^6 u + 160\cos^4 u - 32\cos^2 u + 1$$
$$m = 9 \quad \cos(mu) = \cos(9u) = 256\cos^9 u - 576\cos^7 u + 432\cos^5 u - 120\cos^3 u + 9\cos u$$

$$(\text{AF})_{2M}(\text{even}) = \sum_{n=1}^{M} a_n \cos[(2n-1)u]$$

$$(\text{AF})_{2M+1}(\text{odd}) = \sum_{n=1}^{M+1} a_n \cos[2(n-1)u]$$

where

$$u = \frac{\pi d}{\lambda} \cos\theta$$

- Above are obtained using $\sin^2 u = 1 - \cos^2 u$. and

$$[e^{ju}]^m = (\cos u + j\sin u)^m = e^{jmu} = \cos(mu) + j\sin(mu)$$

4.1 Dolph-Tschebyscheff Array: Array Factor

$$m = 0 \quad \cos(mu) = 1 = T_0(z)$$
$$m = 1 \quad \cos(mu) = z = T_1(z)$$
$$m = 2 \quad \cos(mu) = 2z^2 - 1 = T_2(z)$$
$$m = 3 \quad \cos(mu) = 4z^3 - 3z = T_3(z)$$
$$m = 4 \quad \cos(mu) = 8z^4 - 8z^2 + 1 = T_4(z)$$
$$m = 5 \quad \cos(mu) = 16z^5 - 20z^3 + 5z = T_5(z)$$
$$m = 6 \quad \cos(mu) = 32z^6 - 48z^4 + 18z^2 - 1 = T_6(z)$$
$$m = 7 \quad \cos(mu) = 64z^7 - 112z^5 + 56z^3 - 7z = T_7(z)$$
$$m = 8 \quad \cos(mu) = 128z^8 - 256z^6 + 160z^4 - 32z^2 + 1 = T_8(z)$$
$$m = 9 \quad \cos(mu) = 256z^9 - 576z^7 + 432z^5 - 120z^3 + 9z = T_9(z)$$

$$\boxed{z = \cos u}$$

$$(\mathrm{AF})_{2M}(\text{even}) = \sum_{n=1}^{M} a_n \cos[(2n-1)u]$$

$$(\mathrm{AF})_{2M+1}(\text{odd}) = \sum_{n=1}^{M+1} a_n \cos[2(n-1)u]$$

where

$$u = \frac{\pi d}{\lambda}\cos\theta$$

- each is related to a Tschebyscheff (Chebyshev) polynomial $T_m(z)$.

4.1 Dolph-Tschebyscheff Array: Array Factor

- These relations between the cosine functions and the Tschebyscheff polynomials are valid only in $-1 \le z \le +1$ range because $z = \cos u$
- Note: for |z| > 1, the Tschebyscheff polynomials are related to the hyperbolic cosine functions.
- Recursion formula for Tschebyscheff polynomial:

$$\boxed{T_m(z) = 2zT_{m-1}(z) - T_{m-2}(z)}$$

- used to find one Tschebyscheff polynomial if the polynomials of the previous two orders are known
- Tschebyscheff polynomials can also be found using

$$T_m(z) = \cos[m\cos^{-1}(z)] \qquad -1 \le z \le +1$$

$$T_m(z) = \cosh[m\cosh^{-1}(z)]^{\dagger} \qquad z < -1, z > +1$$

$$(\mathrm{AF})_{2M}(\text{even}) = \sum_{n=1}^{M} a_n \cos[(2n-1)u]$$

$$(\mathrm{AF})_{2M+1}(\text{odd}) = \sum_{n=1}^{M+1} a_n \cos[2(n-1)u]$$

where

$$u = \frac{\pi d}{\lambda}\cos\theta$$

$m = 0 \quad \cos(mu) = 1 = T_0(z)$
$m = 1 \quad \cos(mu) = z = T_1(z)$
$m = 2 \quad \cos(mu) = 2z^2 - 1 = T_2(z)$
$m = 3 \quad \cos(mu) = 4z^3 - 3z = T_3(z)$
$m = 4 \quad \cos(mu) = 8z^4 - 8z^2 + 1 = T_4(z)$
$m = 5 \quad \cos(mu) = 16z^5 - 20z^3 + 5z = T_5(z)$
$m = 6 \quad \cos(mu) = 32z^6 - 48z^4 + 18z^2 - 1 = T_6(z)$
$m = 7 \quad \cos(mu) = 64z^7 - 112z^5 + 56z^3 - 7z = T_7(z)$
$m = 8 \quad \cos(mu) = 128z^8 - 256z^6 + 160z^4 - 32z^2 + 1 = T_8(z)$
$m = 9 \quad \cos(mu) = 256z^9 - 576z^7 + 432z^5 - 120z^3 + 9z = T_9(z)$

$T_m(z)$

1. All polynomials, of any order, pass through the point (1, 1).

2. Within the range $-1 \leq z \leq 1$, the polynomials have values within -1 to $+1$.

3. All roots occur within $-1 \leq z \leq 1$, and all maxima and minima have values of $+1$ and -1, respectively

- Number of maximas = m-1 between $-1 \leq z \leq 1$

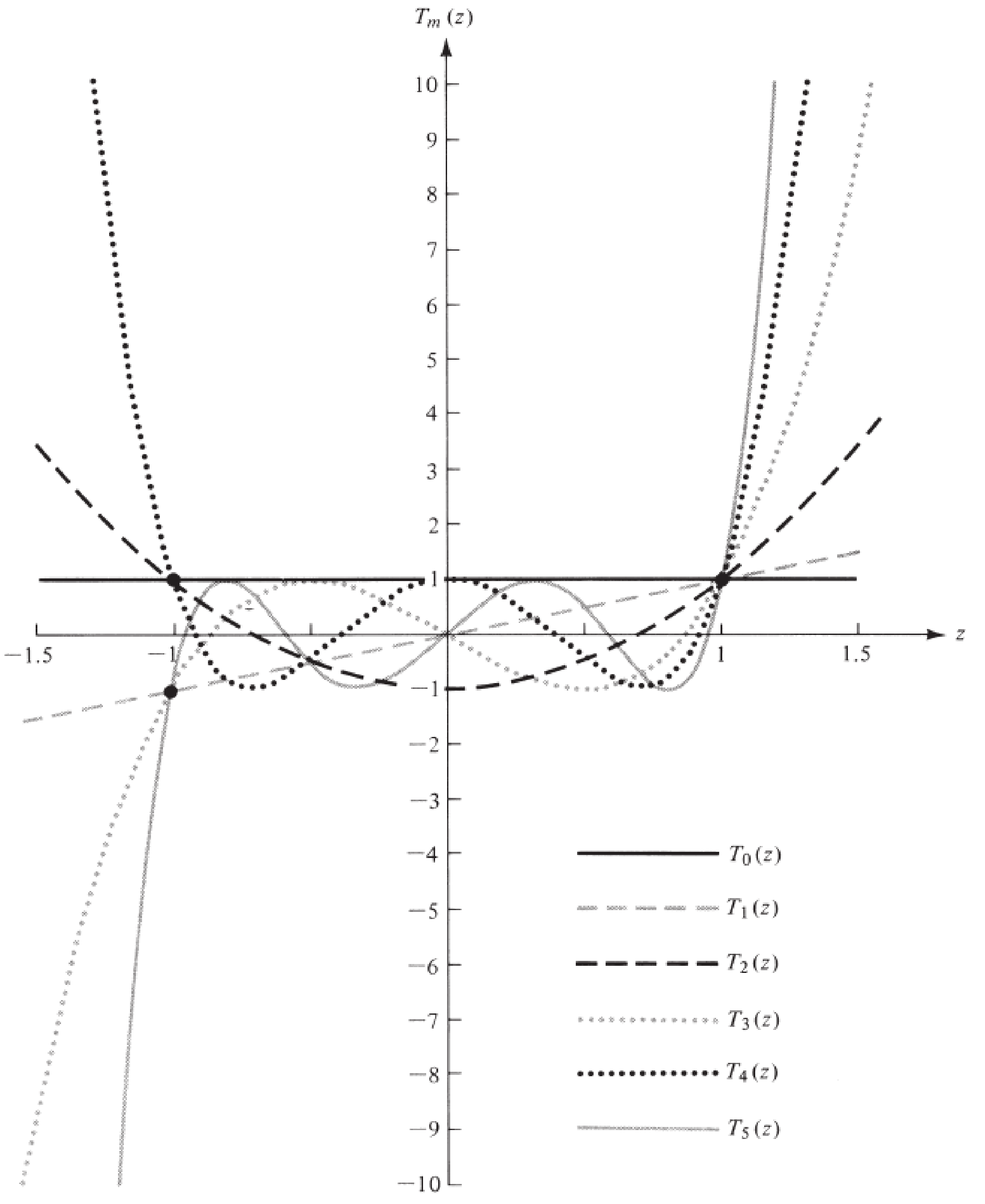


Figure 6.21 Tschebyscheff polynomials of orders zero through five.

4.1 Dolph-Tschebyscheff Array

$$(\text{AF})_{2M}(\text{even}) = \sum_{n=1}^{M} a_n \cos[(2n-1)u]$$

$$(\text{AF})_{2M+1}(\text{odd}) = \sum_{n=1}^{M+1} a_n \cos[2(n-1)u]$$

where

$$u = \frac{\pi d}{\lambda}\cos\theta$$

- Since the **array factor** of an even or odd number of elements is a **summation of cosine terms** whose form is the **same as the Tschebyscheff polynomia** the unknown coefficients of the array factor can be determined by equating the series representing the cosine terms of the array factor to the appropriate Tschebyscheff polynomial.
- *The order of the polynomial should be one less than the total number of elements of the array*.

$$z = \cos u$$

$$T_m(z) = \cos[m\cos^{-1}(z)] \qquad -1 \le z \le +1$$

$$T_m(z) = \cosh[m\cosh^{-1}(z)]^{\dagger} \qquad z < -1, z > +1$$

$$\boxed{T_m(z) = 2zT_{m-1}(z) - T_{m-2}(z)}$$

4.2 Dolph-Tschebyscheff Array: Array Design

Design statement: Design a broadside Dolph-Tschebyscheff array of 2*M* or 2*M* + 1 elements with spacing *d* between the elements. The side lobes are R_0 dB below the maximum of the major lobe. Find the excitation coefficients and form the array factor.

$$(\mathrm{AF})_{2M}(\text{even}) = \sum_{n=1}^{M} a_n \cos[(2n-1)u]$$

$$(\mathrm{AF})_{2M+1}(\text{odd}) = \sum_{n=1}^{M+1} a_n \cos[2(n-1)u]$$

where

$$u = \frac{\pi d}{\lambda}\cos\theta$$

Procedure:

1. **Select the appropriate array factor using**:

2. **Expand the array factor**. Replace each cos(*mu*) function (*m* = 0, 1, 2, 3,…) by its appropriate series expansion found in

$$\begin{aligned}
m &= 0 & \cos(mu) &= 1\\
m &= 1 & \cos(mu) &= \cos u\\
m &= 2 & \cos(mu) &= \cos(2u) = 2\cos^2 u - 1\\
m &= 3 & \cos(mu) &= \cos(3u) = 4\cos^3 u - 3\cos u\\
m &= 4 & \cos(mu) &= \cos(4u) = 8\cos^4 u - 8\cos^2 u + 1\\
m &= 5 & \cos(mu) &= \cos(5u) = 16\cos^5 u - 20\cos^3 u + 5\cos u\\
m &= 6 & \cos(mu) &= \cos(6u) = 32\cos^6 u - 48\cos^4 u + 18\cos^2 u - 1\\
m &= 7 & \cos(mu) &= \cos(7u) = 64\cos^7 u - 112\cos^5 u + 56\cos^3 u - 7\cos u\\
m &= 8 & \cos(mu) &= \cos(8u) = 128\cos^8 u - 256\cos^6 u + 160\cos^4 u - 32\cos^2 u + 1\\
m &= 9 & \cos(mu) &= \cos(9u) = 256\cos^9 u - 576\cos^7 u + 432\cos^5 u - 120\cos^3 u + 9\cos
\end{aligned}$$

4.2 Dolph-Tschebyscheff Array: Array Design

3. **Determine the point z = z0 such that** $T_m(z_0)$ = R_0 (voltage ratio). ***The order m of the Tschebyscheff polynomial is always one less than the total number of elements.*** The design procedure requires that the Tschebyscheff polynomial in the −1 ≤ *z* ≤ *z*1, where *z*1 is the null nearest to *z* = +1, be used to represent the minor lobes of the array. The major lobe of the pattern is formed from the remaining part of the polynomial up to point *z*0 (*z*1 < *z* ≤ *z*0).

$$T_m(z) = \cos[m\cos^{-1}(z)] \qquad -1 \le z \le +1$$

$$T_m(z) = \cosh[m\cosh^{-1}(z)]^{\dagger} \qquad z < -1, z > +1$$

4. **Substitute** **cos(*u*) = *z*/*z*0** **in the array factor** of step 2.
Note: The cos(*u*) is replaced by *z*/*z*0, for keeping it under 1.

$z = \cos u$

$T_m(z) = 2zT_{m-1}(z) - T_{m-2}(z)$

5. Equate the array factor from step 2, after substitution of **$\cos(u) = \frac{z}{z_0}$** , to a $T_m(z)$

m=2M-1 for even 2M elements)

m=2M for odd number of 2M+1 elements.

- This will allow the determination of the excitation coefficients a_n.

6. Write array factor using coefficients found in step 5.

$$
\begin{aligned}
m = 0 &\quad \cos(mu) = 1 = T_0(z)\\
m = 1 &\quad \cos(mu) = z = T_1(z)\\
m = 2 &\quad \cos(mu) = 2z^2 - 1 = T_2(z)\\
m = 3 &\quad \cos(mu) = 4z^3 - 3z = T_3(z)\\
m = 4 &\quad \cos(mu) = 8z^4 - 8z^2 + 1 = T_4(z)\\
m = 5 &\quad \cos(mu) = 16z^5 - 20z^3 + 5z = T_5(z)\\
m = 6 &\quad \cos(mu) = 32z^6 - 48z^4 + 18z^2 - 1 = T_6(z)\\
m = 7 &\quad \cos(mu) = 64z^7 - 112z^5 + 56z^3 - 7z = T_7(z)\\
m = 8 &\quad \cos(mu) = 128z^8 - 256z^6 + 160z^4 - 32z^2 + 1 = T_8(z)\\
m = 9 &\quad \cos(mu) = 256z^9 - 576z^7 + 432z^5 - 120z^3 + 9z = T_9(
\end{aligned}
$$

$$(\mathrm{AF})_{2M}(\text{even}) = \sum_{n=1}^{M} a_n \cos[(2n-1)u]$$

$$(\mathrm{AF})_{2M+1}(\text{odd}) = \sum_{n=1}^{M+1} a_n \cos[2(n-1)u]$$

where

$$u = \frac{\pi d}{\lambda} \cos\theta$$

Problem: Design a broadside Dolph-Tschebyscheff array of 10 elements with spacing *d* between the elements and with a major-to-minor lobe ratio of 26 dB. Find the excitation coefficients and form the array factor

1. **For 10 elements: $Even: 2M = 10$, M=5**

$$(AF)_{2M} = \sum_{n=1}^{M=5} a_n \cos[(2n-1)u]$$

$$u = \frac{\pi d}{\lambda} \cos\theta$$

2. **Expanding array factor:**

$$(AF)_{10} = a_1 \cos(u) + a_2 \cos(3u) + a_3 \cos(5u) + a_4 \cos(7u) + a_5 \cos(9u)$$

Replace cos(*u*), cos(3*u*), cos(5*u*), cos(7*u*), and cos(9*u*) by their series expansions found in

$m = 0 \quad \cos(mu) = 1$

$m = 1 \quad \cos(mu) = \cos u$

$m = 2 \quad \cos(mu) = \cos(2u) = 2\cos^2 u - 1$

$m = 3 \quad \cos(mu) = \cos(3u) = 4\cos^3 u - 3\cos u$

$m = 4 \quad \cos(mu) = \cos(4u) = 8\cos^4 u - 8\cos^2 u + 1$

$m = 5 \quad \cos(mu) = \cos(5u) = 16\cos^5 u - 20\cos^3 u + 5\cos u$

$m = 6 \quad \cos(mu) = \cos(6u) = 32\cos^6 u - 48\cos^4 u + 18\cos^2 u - 1$

$m = 7 \quad \cos(mu) = \cos(7u) = 64\cos^7 u - 112\cos^5 u + 56\cos^3 u - 7\cos u$

$m = 8 \quad \cos(mu) = \cos(8u) = 128\cos^8 u - 256\cos^6 u + 160\cos^4 u - 32\cos^2 u + 1$

$m = 9 \quad \cos(mu) = \cos(9u) = 256\cos^9 u - 576\cos^7 u + 432\cos^5 u - 120\cos^3 u + 9\cos$

Problem: Design a broadside Dolph-Tschebyscheff array of 10 elements with spacing *d* between the elements and with a major-to-minor lobe ratio of 26 dB. Find the excitation coefficients and form the array factor

- 3. *R*0 (dB) = 26 = 20 log10(*R*0) or *R*0 (voltage ratio) = 20.
  $T_m(z_0) = R_0$ ***(The order m of the Tschebyscheff polynomial is always one less than the total number of elements)***

$$T_m(z) = \cos[m\cos^{-1}(z)] \qquad -1 \le z \le +1$$

$$T_m(z) = \cosh[m\cosh^{-1}(z)]^{\dagger} \qquad z < -1, z > +1$$

- Here m=10-1=9
- Determine *z*0 by equating *R*0 to T_9(*z*0). As Tm(z0) is above 1,

$$R_0 = 20 = T_9(z_0) = \cosh[9\cosh^{-1}(z_0)]$$

$$z_0 = \cosh[\tfrac{1}{9}\cosh^{-1}(20)] = 1.0851$$

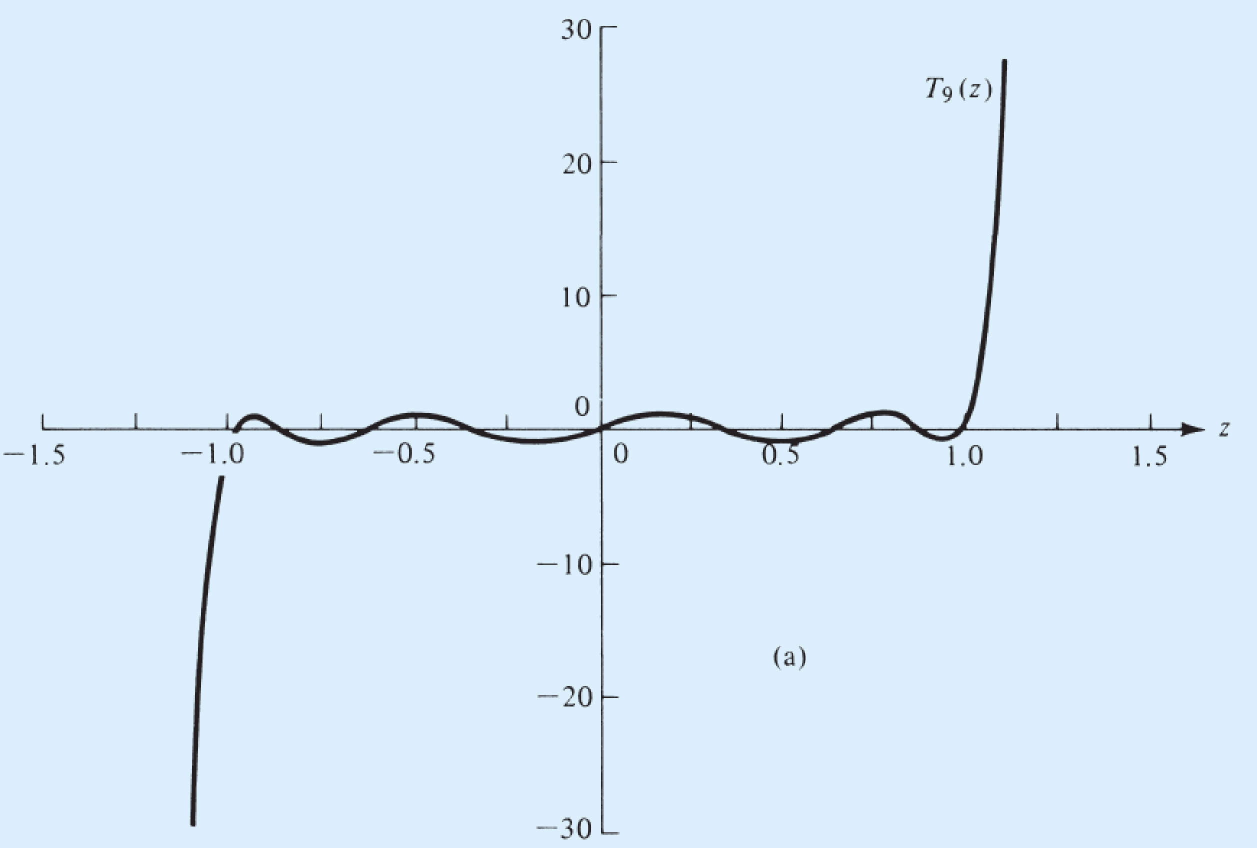
- Another way without hyperbolic method: Based on plot:

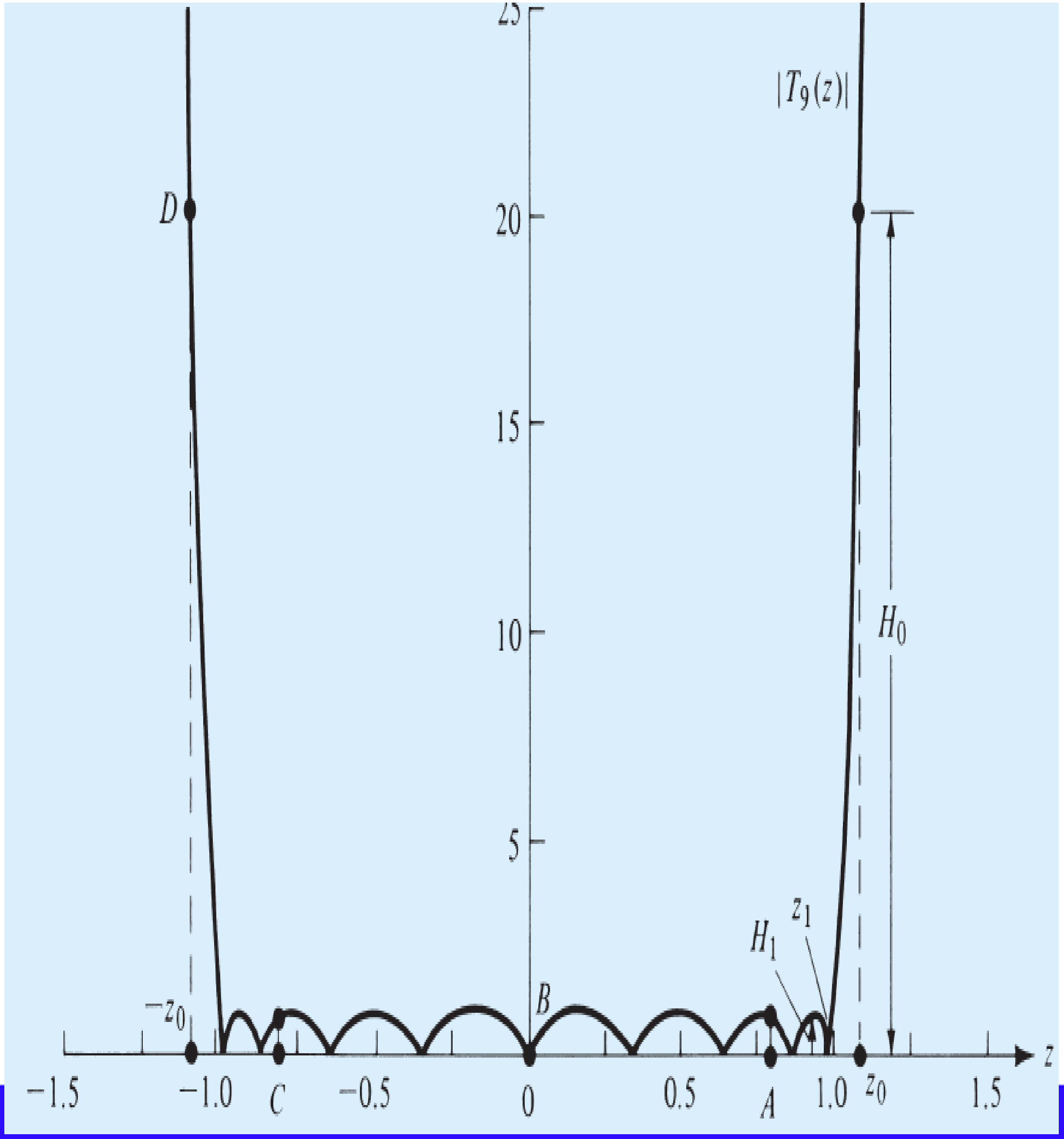
$$z_0 = \frac{1}{2}\left[\left(R_0 + \sqrt{R_0^2 - 1}\right)^{1/P} + \left(R_0 - \sqrt{R_0^2 - 1}\right)^{1/P}\right]$$

P is an integer equal to one less than the number of array elements (in this case *P* = 9). *R*0 = *H*0⁄*H*1 and *z*0 are identified

Problem: Design a broadside Dolph-Tschebyscheff array of 10 elements with spacing *d* between the elements and with a major-to-minor lobe ratio of 26 dB. Find the excitation coefficients and form the array factor

- Tsebychev polynomial of order 9: and its magnitude





Problem: Design a broadside Dolph-Tschebyscheff array of 10 elements with spacing *d* between the elements and with a major-to-minor lobe ratio of 26 dB. Find the excitation coefficients and form the array factor

4. after expanding the array factor of

$$(\mathrm{AF})_{10} = a_1 \cos(u) + a_2 \cos(3u) + a_3 \cos(5u) + a_4 \cos(7u) + a_5 \cos(9u)$$

$$\begin{aligned}
m &= 0 \quad \cos(mu) = 1 \\
m &= 1 \quad \cos(mu) = \cos u \\
m &= 2 \quad \cos(mu) = \cos(2u) = 2\cos^2 u - 1 \\
m &= 3 \quad \cos(mu) = \cos(3u) = 4\cos^3 u - 3\cos u \\
m &= 4 \quad \cos(mu) = \cos(4u) = 8\cos^4 u - 8\cos^2 u + 1 \\
m &= 5 \quad \cos(mu) = \cos(5u) = 16\cos^5 u - 20\cos^3 u + 5\cos u \\
m &= 6 \quad \cos(mu) = \cos(6u) = 32\cos^6 u - 48\cos^4 u + 18\cos^2 u - 1 \\
m &= 7 \quad \cos(mu) = \cos(7u) = 64\cos^7 u - 112\cos^5 u + 56\cos^3 u - 7\cos u \\
m &= 8 \quad \cos(mu) = \cos(8u) = 128\cos^8 u - 256\cos^6 u + 160\cos^4 u - 32\cos^2 u + 1 \\
m &= 9 \quad \cos(mu) = \cos(9u) = 256\cos^9 u - 576\cos^7 u + 432\cos^5 u - 120\cos^3 u + 9\cos u
\end{aligned}$$

Substitute $$\cos(u) = \frac{z}{z_0} = \frac{z}{1.0851}$$ in array factor

Problem: Design a broadside Dolph-Tschebyscheff array of 10 elements with spacing *d* between the elements and with a major-to-minor lobe ratio of 26 dB. Find the excitation coefficients and form the array factor

5. Equate $(AF)_{10} = T_9(z)$

$$m = 0 \quad \cos(mu) = 1 = T_0(z)$$
$$m = 1 \quad \cos(mu) = z = T_1(z)$$
$$m = 2 \quad \cos(mu) = 2z^2 - 1 = T_2(z)$$
$$m = 3 \quad \cos(mu) = 4z^3 - 3z = T_3(z)$$
$$m = 4 \quad \cos(mu) = 8z^4 - 8z^2 + 1 = T_4(z)$$
$$m = 5 \quad \cos(mu) = 16z^5 - 20z^3 + 5z = T_5(z)$$
$$m = 6 \quad \cos(mu) = 32z^6 - 48z^4 + 18z^2 - 1 = T_6(z)$$
$$m = 7 \quad \cos(mu) = 64z^7 - 112z^5 + 56z^3 - 7z = T_7(z)$$
$$m = 8 \quad \cos(mu) = 128z^8 - 256z^6 + 160z^4 - 32z^2 + 1 = T_8(z)$$
$$m = 9 \quad \cos(mu) = 256z^9 - 576z^7 + 432z^5 - 120z^3 + 9z = T_9(z)$$

$$\begin{aligned}
(\mathrm{AF})_{10} = {} & z[(a_1 - 3a_2 + 5a_3 - 7a_4 + 9a_5)/z_0] \\
& + z^3[(4a_2 - 20a_3 + 56a_4 - 120a_5)/z_0{}^3] \\
& + z^5[(16a_3 - 112a_4 + 432a_5)/z_0{}^5] \\
& + z^7[(64a_4 - 576a_5)/z_0{}^7] \\
& + z^9[(256a_5)/z_0{}^9] = 9z - 120z^3 + 432z^5 - 576z^7 + 256z^9
\end{aligned}$$

Matching similar terms allows the determination of the a_n's.

Problem: Design a broadside Dolph-Tschebyscheff array of 10 elements with spacing *d* between the elements and with a major-to-minor lobe ratio of 26 dB. Find the excitation coefficients and form the array factor

$$
\begin{aligned}
(\mathrm{AF})_{10} = {} & z[(a_1 - 3a_2 + 5a_3 - 7a_4 + 9a_5)/z_0] \\
& + z^3[(4a_2 - 20a_3 + 56a_4 - 120a_5)/z_0{}^3] \\
& + z^5[(16a_3 - 112a_4 + 432a_5)/z_0{}^5] \\
& + z^7[(64a_4 - 576a_5)/z_0{}^7] \\
& + z^9[(256a_5)/z_0{}^9] = 9z - 120z^3 + 432z^5 - 576z^7 + 256z^9
\end{aligned}
$$

- Coefficient of z^9: $256a_5/z_0^9 = 256 \Rightarrow a_5 = 2.0860$
- Coefficient of z^7: $(64a_4 - 576a_5)/z_0{}^7 = -576 \Rightarrow a_4 = 2.8308$
- Coefficient of z^5: $(16a_3 - 112a_4 + 432a_5)/z_0{}^5 = 432 \Rightarrow a_3 = 4.1184$
- Coefficient of z^1: $(4a_2 - 20a_3 + 56a_4 - 120a_5)/z_0{}^3 = -120 \Rightarrow a_2 = 5.2073$
- Coefficient of z^9: $(a_1 - 3a_2 + 5a_3 - 7a_4 + 9a_5)/z_0 = 9 \Rightarrow a_1 = 5.8377$

Problem: Design a broadside Dolph-Tschebyscheff array of 10 elements with spacing *d* between the elements and with a major-to-minor lobe ratio of 26 dB. Find the excitation coefficients and form the array factor

$a_5 = 2.0860$

$a_4 = 2.8308$

$a_3 = 4.1184$

$a_2 = 5.2073$

$a_1 = 5.8377$

- normalized form, the a_n coefficients
- w.r.t amplitude of the elements at the edge

$a_5 = 1$

$a_4 = 1.357$

$a_3 = 1.974$

$a_2 = 2.496$

$a_1 = 2.798$

- normalized form, the a_n coefficients
- w.r.t amplitude of the center element

$a_5 = 0.357$

$a_4 = 0.485$

$a_3 = 0.706$

$a_2 = 0.890$

$a_1 = 1$

Problem: Design a broadside Dolph-Tschebyscheff array of 10 elements with spacing *d* between the elements and with a major-to-minor lobe ratio of 26 dB. Find the excitation coefficients and form the array factor

6. Using the first (left) set of normalized coefficients, the array factor (from step 2)

$$(AF)_{10} = a_1 \cos(u) + a_2 \cos(3u) + a_3 \cos(5u) + a_4 \cos(7u) + a_5 \cos(9u)$$

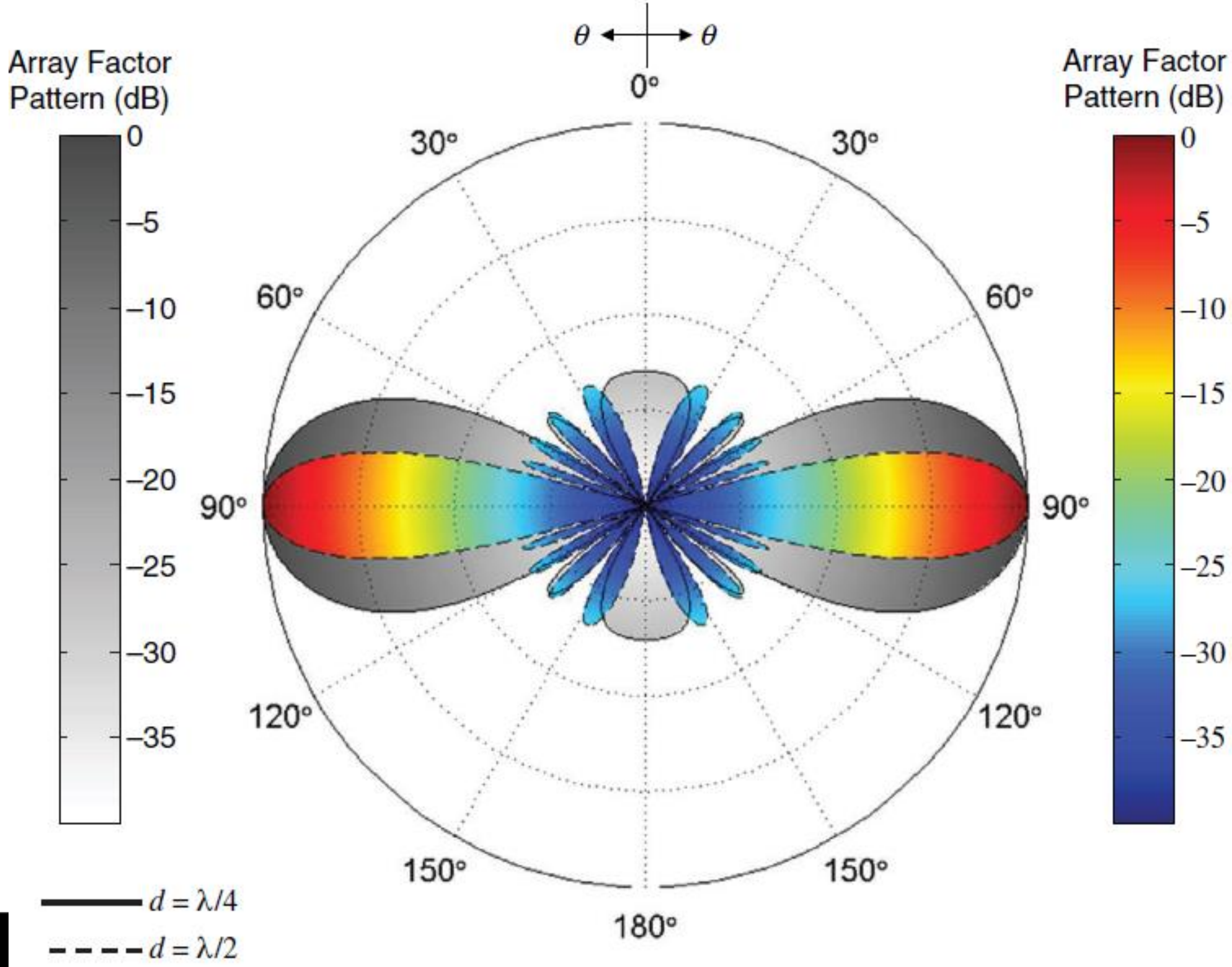
$$(AF)_{10} = 2.798 \cos(u) + 2.496 \cos(3u) + 1.974 \cos(5u) + 1.357 \cos(7u) + \cos(9u)$$

$$u = [(\pi d/\lambda) \cos\theta].$$

for $d = \lambda/4$ and $\lambda/2$

- spacing is less than λ
  $d < \lambda$,
  maxima exist only at broadside ($\theta_0 = 90^\circ$).
- However when the spacing is equal to λ ($d = \lambda$), two more maxima appear (one toward $\theta_0 = 0^\circ$ and the other toward $\theta_0 = 180^\circ$).

For $d = \lambda$ the array has four maxima, and it acts as an *end-fire* as well as a *broadside* array.



Eq: 6.74:

$$z = z_0 \cos u = z_0 \cos\left(\frac{\pi d}{\lambda}\cos\theta\right) = 1.0851 \cos\left(\frac{\pi d}{\lambda}\cos\theta\right)$$

- $N = 10$, $R0 = 20$

	$d = \lambda/4$	$d = \lambda/2$	$d = 3\lambda/4$	$d = \lambda$
θ	z (Eq. 6-74)	z (Eq. 6-74)	z (Eq. 6-74)	z (Eq. 6-74)
0°	0.7673	0.0	−0.7673	−1.0851
10°	0.7764	0.0259	−0.7394	−1.0839
20°	0.8028	0.1026	−0.6509	−1.0657
30°	0.8436	0.2267	−0.4912	−0.9904
40°	0.8945	0.3899	−0.2518	−0.8049
50°	0.9497	0.5774	0.0610	−0.4706
60°	1.0025	0.7673	0.4153	0.0
70°	1.0462	0.9323	0.7514	0.5167
80°	1.0750	1.0450	0.9956	0.9276
90°	1.0851	1.0851	1.0851	1.0851
100°	1.0750	1.0450	0.9956	0.9276
110°	1.0462	0.9323	0.7514	0.5167
120°	1.0025	0.7673	0.4153	0.0
130°	0.9497	0.5774	0.0610	−0.4706
140°	0.8945	0.3899	−0.2518	−0.8049
150°	0.8436	0.2267	−0.4912	−0.9904
160°	0.8028	0.1026	−0.6509	−1.0657
170°	0.7764	0.0259	−0.7394	−1.0839
180°	0.7673	0.0	−0.7673	−1.0851

Beam width and Directivity of Chebyshev array

- Beam broadening factor:

$$f = 1 + 0.636 \left\{ \frac{2}{R_0} \cosh \left[\sqrt{(\cosh^{-1} R_0)^2 - \pi^2} \right] \right\}^2$$

- Where major-to-minor lobe ratio is R_0
- half-power beamwidth of a Dolph-Tschebyscheff array can be determined by
  (1) calculating the beamwidth of a uniform array (of the same number of elements and spacing)
  (2) multiplying the beamwidth of part (1) by the appropriate beam broadening factor *f*
- Directivity

Beam width and Directivity of Chebyshev array

- Directivity:

$$D_0 = \frac{2R_0^2}{1 + (R_0^2 - 1) f \dfrac{\lambda}{(L+d)}}$$

- The directivity of a Dolph-Tschebyscheff array, with a given side lobe level, increases as the array size or number of elements increases
- a given number of elements in the array, the directivity does not necessarily increase as the side lobe level decreases. As a matter of fact, a −15 dB side lobe array has smaller directivity than a −20 dB side lobe array

Beam width and Directivity of Chebyshev array

- The beamwidth and the directivity of an array depend linearly, but not necessarily at the same rate, on the overall length or total number of elements of the array. Therefore, the beamwidth and directivity must be related to each other.
- For a uniform broadside array this relation is $$D_0 = \frac{101.5}{\Theta_d}$$
- Θ*d* is the 3-dB beamwidth (in degrees).

Calculate the half-power beamwidth and the directivity for the Dolph-Tschebyscheff array of Example 6.9 for a spacing of $\lambda/2$ between the elements.

$$R_0 = 26 \text{ dB} \Rightarrow R_0 = 20 \quad \text{(voltage ratio)}$$

- beam broadening factor *f*

$$f = 1.079$$

- the beamwidth of a uniform broadside array with *L* + *d* = 5λ is equal to

$$\Theta_h = \cos^{-1}\left[\cos\theta_0 - 0.443\frac{\lambda}{(L+d)}\right] - \cos^{-1}\left[\cos\theta_0 + 0.443\frac{\lambda}{(L+d)}\right]$$

$$\Theta_h = 10.17^\circ$$

- the beamwidth of a Dolph-Tschebyscheff array is equal to

$$\Theta_h = 10.17^\circ f = 10.17^\circ(1.079) = 10.97^\circ$$

- Directivity:

$$D_0 = \frac{2R_0^2}{1 + (R_0^2 - 1)f\dfrac{\lambda}{(L+d)}}$$

$$D_0 = \frac{2(20)^2}{1 + [(20)^2 - 1]\dfrac{1.079}{5}} = 9.18 \text{ (dimensionless)} = 9.63 \text{ dB}$$