Gram-Schmidt Orthogonalization Procedure (GSOP)

Any signal $s_i(t)$ in a set of M energy signals $\{s_i(t) \mid 1 \le i \le M\}$ can be represented by a linear combination of a set of N orthonormal functions $\{\phi_j(t) \mid 1 \le j \le N\}$ where $N \le M$ as

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t) \quad , \quad i = 1, 2, \ldots, M \, , \quad 0 \le t \le T \qquad (1)$$

where $s_{ij} = \langle s_i(t), \phi_j(t) \rangle = \int_0^T s_i(t)\phi_j(t)\,dt$

Note : $\int_0^T s_i(t)\phi_j(t)\,dt$ gives s_{ij} which is the projection of $s_i(t)$ on $\phi_j(t)$.

Gram-Schmidt Orthogonalization Procedure (GSOP)

The representation (1) has the following matrix form:

$$s_i(t) = (s_{i1}, s_{i2}, \ldots, s_{iN}) \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \\ \vdots \\ \phi_N(t) \end{pmatrix}$$

Thus, using the set of the basis functions, each signal $s_i(t)$ maps to a set of N real numbers, which is a N dimensional real-valued vector

$$\mathbf{s}_i = (s_{i1}, s_{i2}, \ldots, s_{iN})$$

This is a 1-1 correspondence between the signal set (or equivalently, the message symbol set) and the N dimensional vector space.

Gram-Schmidt Orthogonalization Procedure (GSOP)

Input: M signals $\{s_i(t)\}$ for $0 \le t < T$ and $i = 1, \ldots, M$

Output:

$\{\phi_j(t)\} : N \le M$ orthonormal basis functions for $0 \le t < T$ and $j = 1, \ldots, N$ such that

$$s_i(t) = \sum_{j=1}^{N} s_{ij} \phi_j(t)$$

where $s_{ij} = < s_i(t), \phi_j(t) > = \int_0^T s_i(t)\phi_j(t)dt$

Gram-Schmidt Orthogonalization Procedure (GSOP)

Step 1.

$$g_1(t) = s_1(t) \qquad \text{(direction)}$$

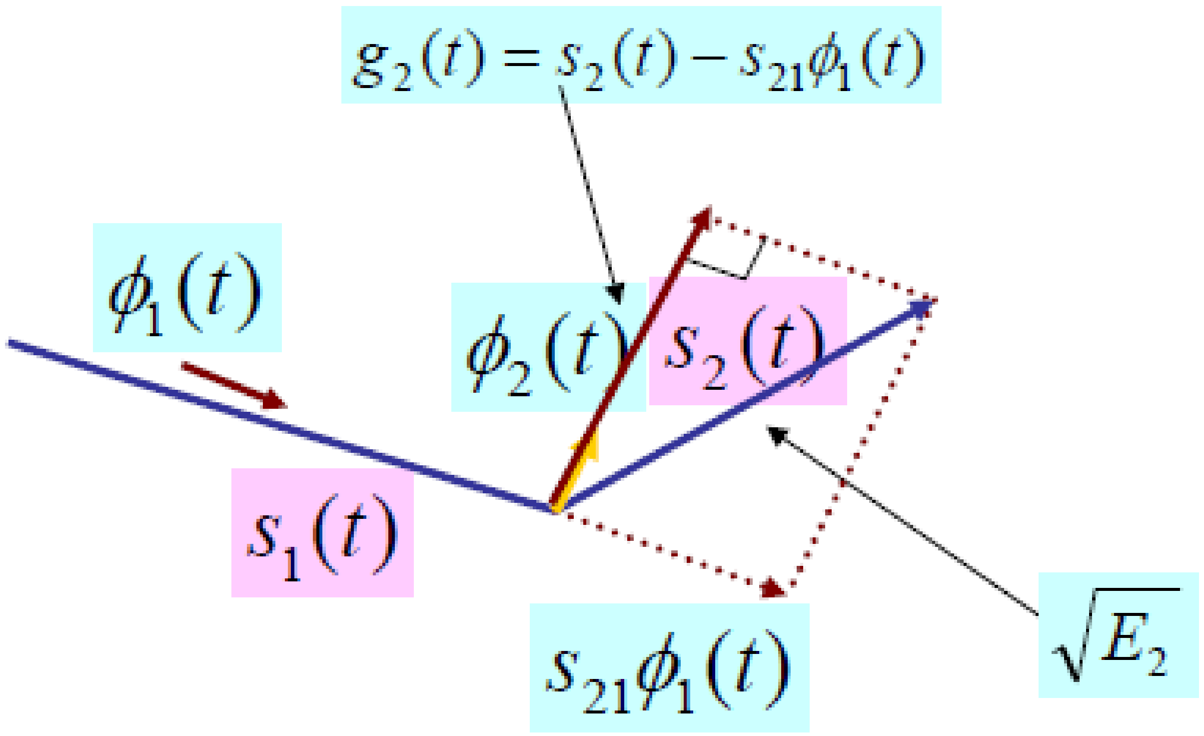
$$\phi_1(t) = \frac{g_1(t)}{\|g_1(t)\|} \qquad \text{(unit length)}$$

$$= \frac{s_1(t)}{\sqrt{E_1}} \qquad E_1 = \int_0^T s_1^2(t)\,dt$$

$$\Rightarrow s_1(t) = \sqrt{E_1}\,\phi_1(t) = s_{11}\phi_1(t)$$

where $s_{11} = \sqrt{E_1}$

and $\phi_1(t)$ has unit energy.



Gram-Schmidt Orthogonalization Procedure (GSOP)

Step 2. Compute:

$$s_{21} = \int_0^T s_2(t)\phi_1(t)dt$$

Set

$$g_2(t) = s_2(t) - s_{21}\phi_1(t) \qquad (g_2(t) \perp \phi_1(t))$$

$$\Rightarrow < g_2(t), \phi_1(t) > \; = 0 \qquad \text{(direction)}$$

($\phi_2(t)$ is the normalized version of $g_2(t)$)

Compute the norm of $g_2(t)$:

$$E_2 = \int_0^T s_2^2(t)dt$$

$$\|g_2(t)\| = \sqrt{\int_0^T g_2^2(t)dt} = \sqrt{E_2 - 2s_{21}^2 + s_{21}^2} = \sqrt{E_2 - s_{21}^2}$$

Set

$$\phi_2(t) = \frac{g_2(t)}{\|g_2(t)\|} = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}$$

Gram-Schmidt Orthogonalization Procedure (GSOP)

$$\text{Now, } \int_0^T g_2(t)\,\phi_1(t)\,dt = \int_0^T [s_2(t) - s_{21}\,\phi_1(t)]\,\phi_1(t)\,dt$$

$$= \int_0^T s_2(t)\,\phi_1(t)\,dt - s_{21}\int_0^T \phi_1(t)\,\phi_1(t)\,dt$$

$$= s_{21} - s_{21} = 0$$

$\therefore\ g_2(t)$ is orthogonal to $\phi_1(t)$

Gram-Schmidt Orthogonalization Procedure (GSOP)

$$g_2(t) = \int_0^T [s_2(t) - s_{21}\,\phi_1(t)]^2\,dt$$

$$g_2(t) = \int_0^T [s_2(t) - s_{21}\,\phi_1(t)]^2\,dt$$

$$g_2(t) = \int_0^T s_2^2(t)\,dt - 2.s_{21}\int_0^T s_2(t)\,\phi_1(t) + s_{21}^2\int_0^T \phi_1^2(t)\,dt$$

$$\int_0^T g_2^2(t)\,dt = E_2 - 2.s_{21}s_{21} + s_{21}^2$$

$$= E_2 - s_{21}^2 \qquad \ldots($$

Gram-Schmidt Orthogonalization Procedure (GSOP)

Similarly, third basis function for $i = 3$,

$$g_3(t) = s_3(t) - \sum_{j=1}^{2} s_{3j}\,\phi_j(t) \quad 0 \le t < T$$

$$= s_3(t) - [s_{31}\,\phi_1(t) + s_{32}\,\phi_2(t)]$$

where $s_{31} = \int_0^T s_3(t)\,\phi_1(t)\,dt$ and $s_{32} = \int_0^T s_3(t)\,\phi_2(t)\,dt$

Now $\phi_3(t) = \dfrac{g_3(t)}{\text{Energy of } g_3(t)} = \dfrac{g_3(t)}{\sqrt{\int_0^T g_3^2(t)\,dt}}$

Gram-Schmidt Orthogonalization Procedure (GSOP)

Step *n*. Compute: $s_{nj} = \langle s_n(t), \phi_j(t) \rangle, \; j = 1, \ldots, n-1$

$$g_n(t) = s_n(t) - \sum_{j=1}^{n-1} s_{nj}\phi_j(t) \qquad \text{(direction)}$$

$$\| g_n(t) \| = \sqrt{E_n - \sum_{j=1}^{n-1} s_{nj}^2}$$

$$\left(E_n = \|s_n(t)\|^2 = \int_0^T s_n^2(t)\,dt\right)$$

$$\phi_n(t) = \frac{g_n(t)}{\|g_n(t)\|} \qquad \text{(unit length)}$$

$$= \frac{s_n(t) - \sum_{j=1}^{n-1} s_{nj}\phi_j(t)}{\sqrt{E_n - \sum_{j=1}^{n-1} s_{nj}^2}}$$

components of $s_n(t)$ already accounted for by $\phi_1(t), \ldots, \phi_{n-1}(t)$

Gram-Schmidt Orthogonalization Procedure (GSOP)

- The dimension N is less than or equal to the number of given signals, M depending on two possibilities:

i. For $N = M$, the signals $s_1(t), s_2(t), \dots, s_M(t)$ are *linearly independent*

i. For $N < M$, the signals $s_1(t), s_2(t), \dots, s_M(t)$ are *not linearly independent*

Gram-Schmidt Orthogonalization Procedure (GSOP)

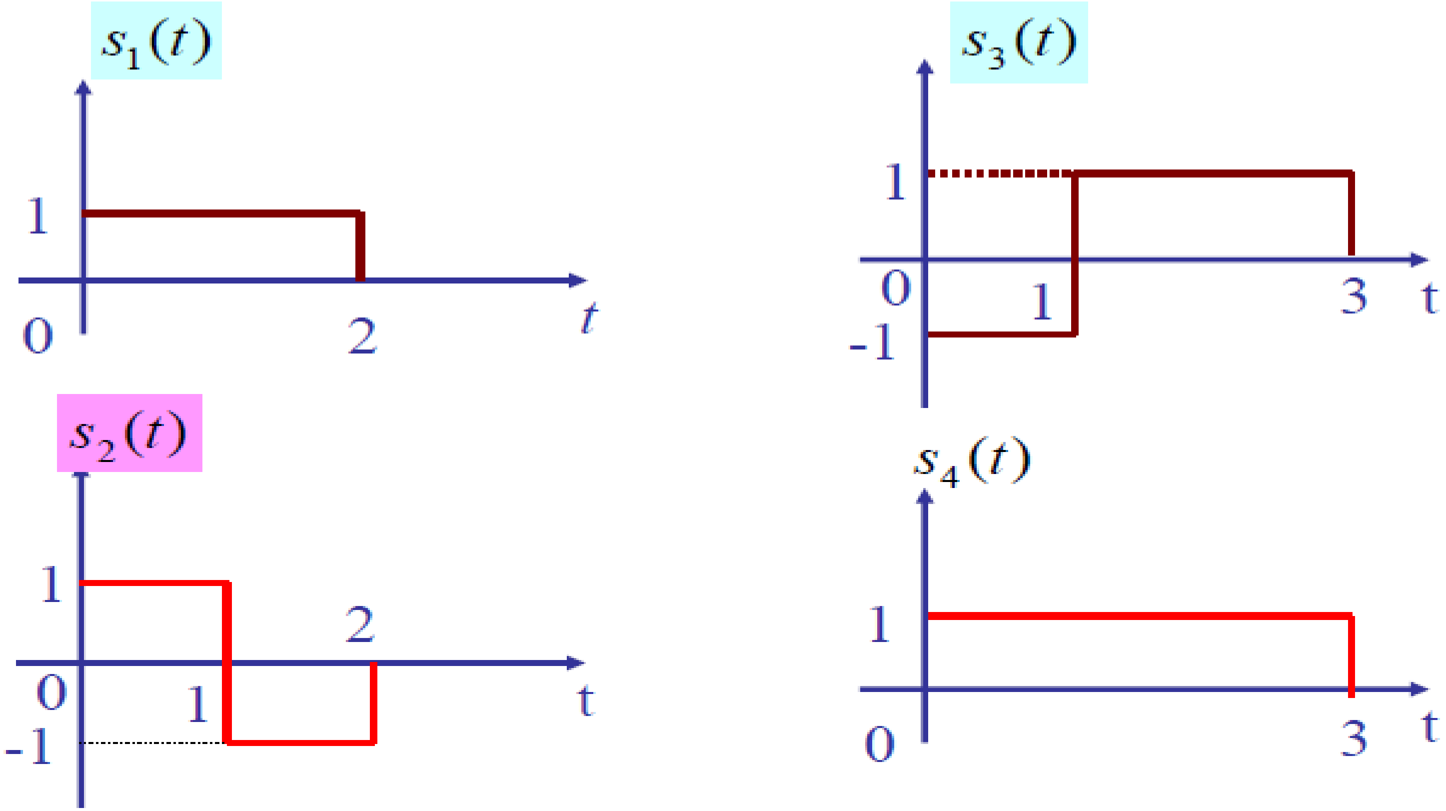
This procedure is proceeded until all signals $s_1(t), \ldots, s_M(t)$ are processed.

It may happen that $s_n(t) - \sum_{j=1}^{n-1} s_{nj}\phi_j(t) = 0$

So, $\phi_n(t) = 0$ because $s_n(t)$ has no component not already accounted for by a combination of $\phi_1(t), \ldots, \phi_{n-1}(t)$. In this case, just skip any signal $s_n(t)$ that gives $\phi_n(t) = 0$.

GSOP -Numerical

Example. A set of four waveform is illustrated as below. Find an orthonormal set for this set of signals by applying the Gram-Schmidt procedure.

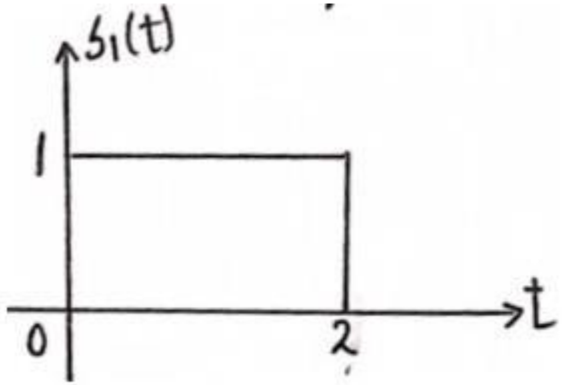


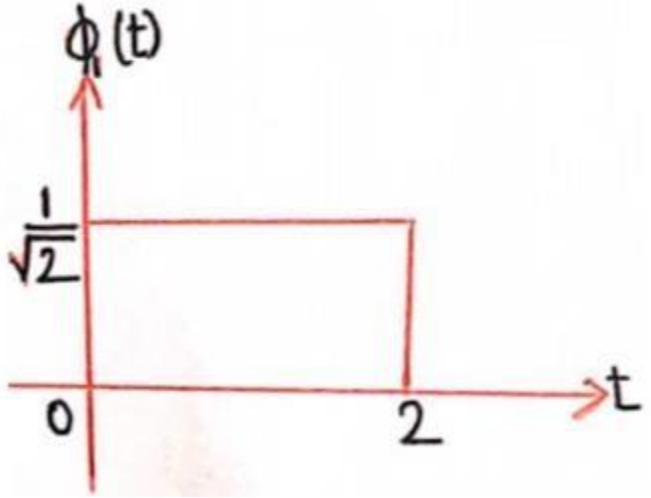
GSOP -Numerical

1) To find $\phi_1(t)$

Energy of $s_1(t)$, $E_1 = \int_0^T s_1^2(t)\,dt = \int_0^2 1^2\,dt = \underline{2}$

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{\sqrt{2}}$$





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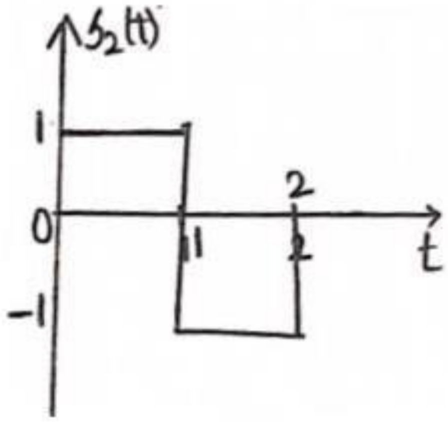
GSOP -Numerical

2) To find $\phi_2(t)$

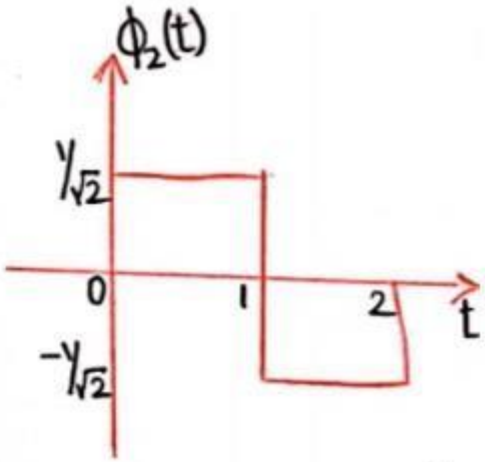
$$s_{21} = \int_0^T s_2(t)\,\phi_1(t)\,dt = \int_0^1 \frac{1}{\sqrt{2}}\,dt + \int_1^2 -\frac{1}{\sqrt{2}}\,dt$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \underline{0}$$



$$g_2(t) = s_2(t) - s_{21}\,\phi_1(t) = s_2(t)$$

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)\,dt}} = \frac{s_2(t)}{\sqrt{\int_0^T s_2^2(t)\,dt}}$$



Energy of $s_2(t)$, $E_2 = \int_0^T s_2^2(t)\,dt = \int_0^1 1^2\,dt + \int_1^2 1^2\,dt = 2$

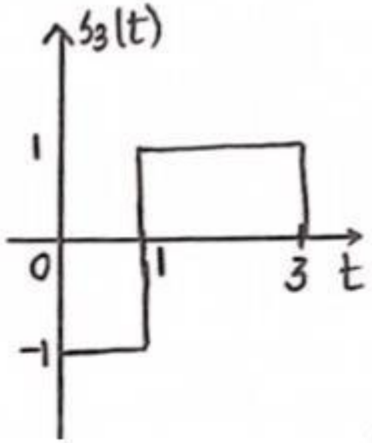
$$\phi_2(t) = \underline{\underline{\frac{s_2(t)}{\sqrt{2}}}}$$

GSOP -Numerical

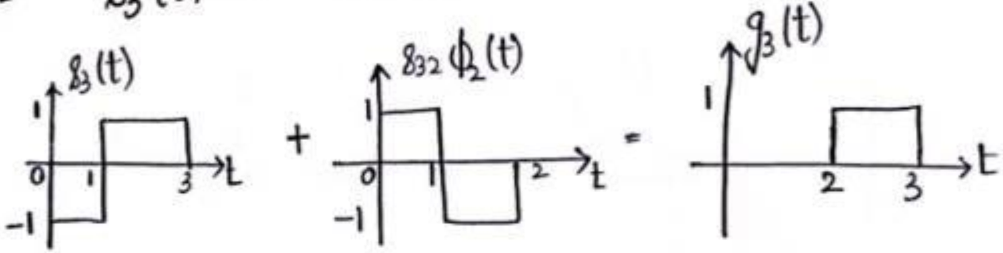
3) To find $\phi_3(t)$

$$s_{31} = \int_0^T s_3(t)\,\phi_1(t)\,dt = \int_0^1 -1/\sqrt{2}\,dt + \int_1^2 \frac{1}{\sqrt{2}}\,dt = 0$$

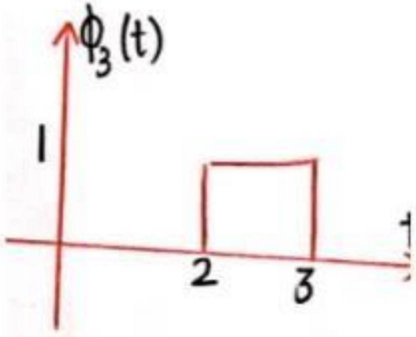
$$s_{32} = \int_0^T s_3(t)\,\phi_2(t)\,dt = \int_0^1 -\frac{1}{\sqrt{2}}\,dt + \int_1^2 -\frac{1}{\sqrt{2}}\,dt = -\sqrt{2}$$



$$g_3(t) = s_3(t) - s_{31}\,\phi_1(t) - s_{32}\,\phi_2(t)$$



$$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t)\,dt}} = \frac{g_3(t)}{\sqrt{\int_2^3 1^2\,dt}} = g_3(t)$$



GSOP -Numerical

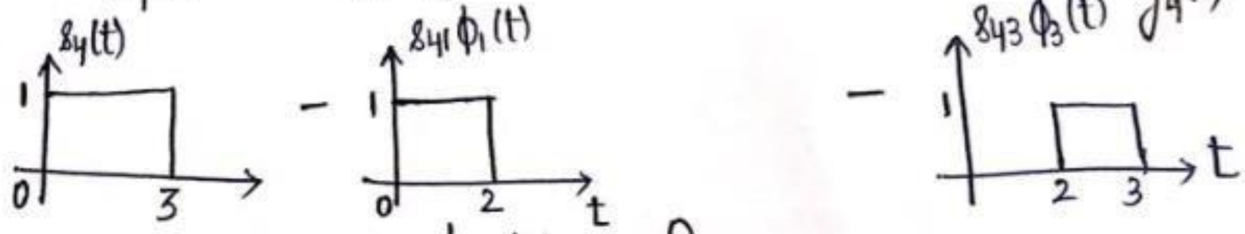
4) To find $\phi_4(t)$

$$s_{41} = \int_0^T s_4(t)\,\phi_1(t)\,dt = \int_0^2 1/\sqrt{2}\,dt = \underline{\sqrt{2}}$$

$$s_{42} = \int_0^T s_4(t)\,\phi_2(t)\,dt = \int_0^1 \frac{1}{\sqrt{2}}\,dt - \int_1^2 \frac{1}{\sqrt{2}}\,dt = \underline{0}$$

$$s_{43} = \int_0^T s_4(t)\,\phi_3(t)\,dt = \int_2^3 1\,dt = \underline{1}$$

$$g_4(t) = s_4(t) - s_{41}\,\phi_1(t) - s_{42}\,\phi_2(t) - s_{43}\,\phi_3(t) = \underline{0}$$



$g_4(t) = \underline{0}$

$\therefore\ \phi_4(t) = \underline{0}$

GSOP -Numerical

- Orthogonalization continues till N≤M
- If $g_i(t) = 0$ at any step, then there is no further basis function.
- Here, No. of orthonormal basis functions

  N = 3< 4 (No. of signals)

- This implies $s_4(t)$ can be expressed as the linear combination of other basis functions:

$$s_4(t) = \sqrt{2}\emptyset_1(t) + \emptyset_3(t)$$