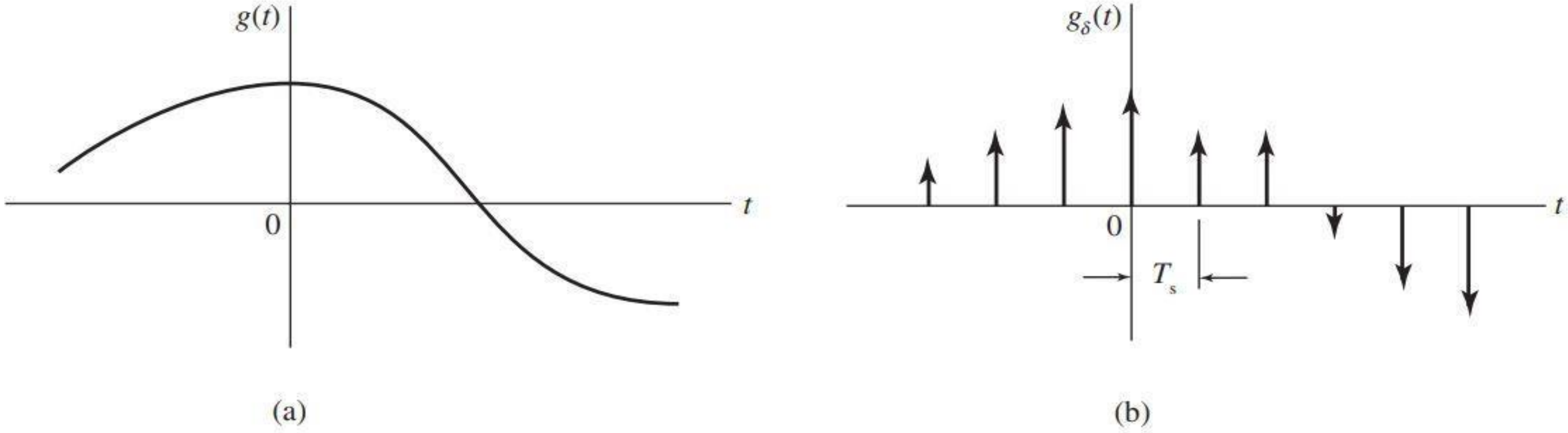
Sampling Process

- **Sampling** is defined as, "The process of measuring the instantaneous values of continuous-time signal in a discrete form."

- **Sample** is a piece of data taken from the whole data which is continuous in the time domain.



The sampling process: (a) Analog signal, (b) Instantaneously sampled version of the analog signal

Sampling Process

Let $g(t)$ be an arbitrary signal of finite energy as shown in figure above, sampled at uniform rate 'T_s' seconds denoted by $g(nT_s)$ where 'n' is an integer

Sampling Rate

- To discretize the signals, the gap between the samples should be fixed.
- That gap can be termed as a **sampling period T_s**.

$$Sampling\ Frequency = \frac{1}{T_s} = f_s$$

Sampling frequency

- **Sampling frequency f_s** is the reciprocal of the sampling period.
- This sampling frequency, can be simply called as **Sampling rate**.
- The sampling rate denotes the number of samples taken per second, or for a finite set of values.

Sampling Process

Representation of $g(t)$ in terms of its samples:

- This is done in 4 steps

1. Define the sampled version $g_\delta(t)$
2. Find the Fourier transform of $g_\delta(t)$ i.e., $G_\delta(f)$
3. Find the relationship between $G(f)$ and $G_\delta(f)$
4. Find the relationship between $g(t)$ and $g(nT_s)$

Sampling Process

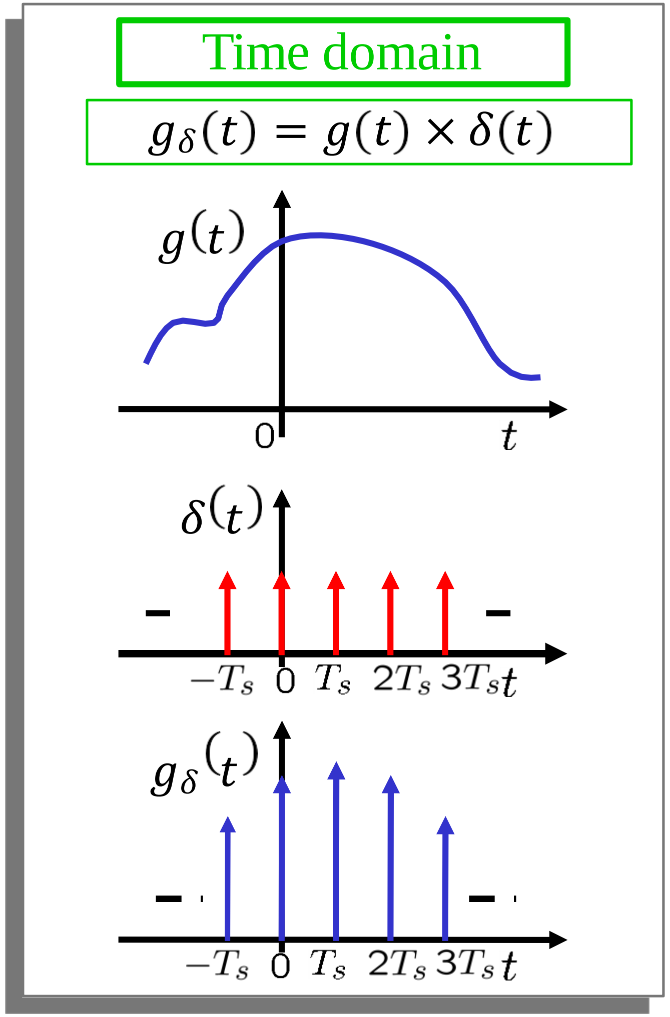
1. **Define the sampled version $g_\delta(t)$**

- Impulse train is represented by $\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$
  - Where $\delta(t - nT_s)$ represents a delta function positioned at time $t = nT_s$
- Signal obtained after sampling

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(t)\delta(t - nT_s)$$

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s)$$

  - $g(nT_s)$ is the instantaneous amplitude of $g(t)$ at instant $t = T_s$



Sampling Process

2. **Find the Fourier transform of $g_\delta(t)$ i.e., $G_\delta(f)$**

- Taking Fourier transform

$$G_\delta(f) = F.T\left[\sum_{n=-\infty}^{\infty} g(t)\delta(t - nT_s)\right]$$

- Product in time domain = Convolution in frequency domain

$$G_\delta(f) = F.T[g(t)] * F.T\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right]$$

$$G_\delta(f) = G(f) * F.T\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right]$$

Sampling Process

- Fourier Transform of the Impulse train $\delta(t - nT_s)$ is,

$$F.T\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right] = f_s \sum_{n=-\infty}^{\infty} G(f - nf_s)$$

- Where $f_s = \frac{1}{T_s}$

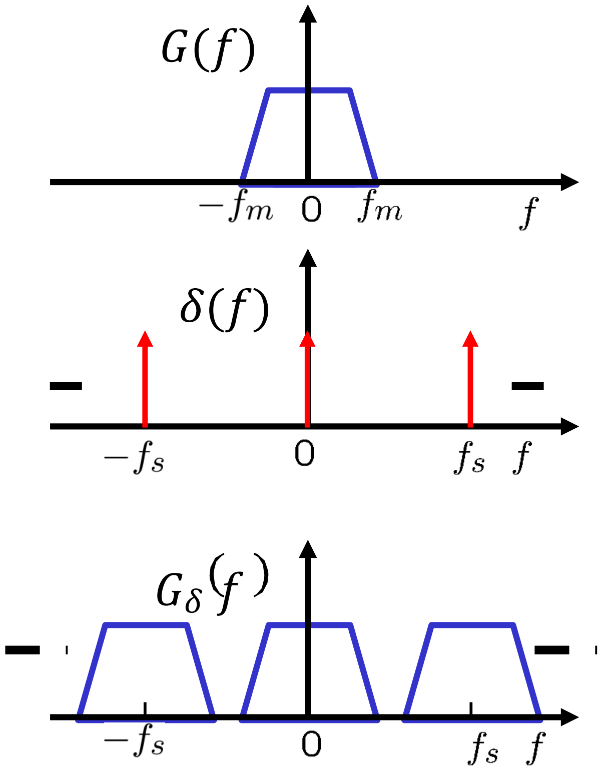
- Therefore, Fourier Transform of the sampled version $g_\delta(t)$ is,

$$G_\delta(f) = G(f) * f_s \sum_{n=-\infty}^{\infty} G(f - nf_s) = f_s \sum_{n=-\infty}^{\infty} G(f) * G(f - nf_s)$$

$$\boldsymbol{G_\delta(f) = f_s \sum_{n=-\infty}^{\infty} G(f - nf_s)}$$

Frequency domain

$$G_\delta(f) = G(f) * \delta(f)$$



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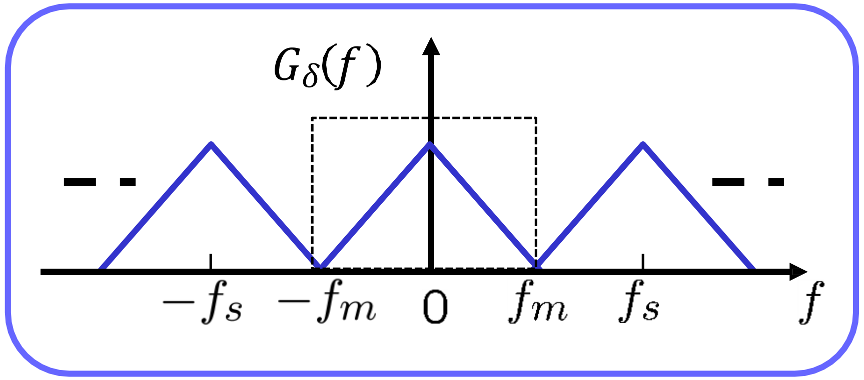
Sampling Process

- $G(f)$ is the Fourier transform of the original signal $g(t)$
- $G(f-nf_s)=G(f)$ at $f=0, \pm f_s, \pm 2f_s, \pm 3f_s \ldots$
- That is same spectrum appears at $f=\ldots, -3f_s, -2f_s, -f_s, 0, +f_s, +2f_s, +3f_s \ldots$
- Which means a periodic spectrum with time period equal to f_s is generated in frequency domain because of sampling $g(t)$ in time domain

$$G_\delta(f) = ..+f_s G(f-2f_s)+f_s G(f-f_s)+\boldsymbol{f_s G(f)}+f_s G(f+f_s)+f_s G(f+2f_s)+f_s G(f+3f_s)+..$$



Sampling Process

- The Fourier transformed version can be written as,

$$G_\delta(f) = f_s G(f) + f_s \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} G(f - nf_s)$$

- The first term $f_s G(f)$ indicates the spectrum without sampling
- The second term $+f_s \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} G(f - nf_s)$ indicates the spectrum repeating at multiple frequencies of sampling frequency f_s

Sampling Process

- Fourier transform of continuous time signal

$$FT[g(t)] = \int_{-\infty}^{\infty} g(t)e^{-2\pi f t}\, dt$$

- Fourier transform of discontinuous time signal

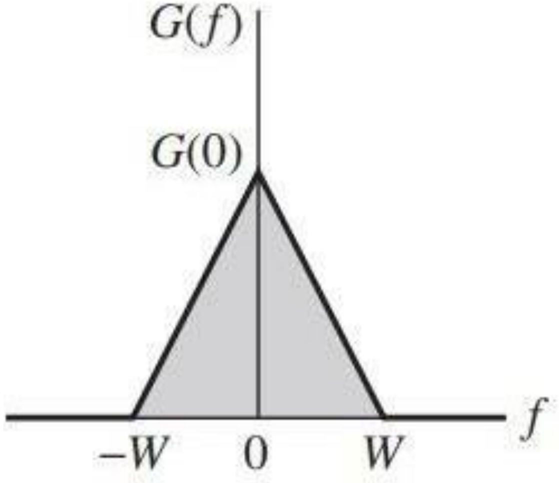
$$F.T\,[g_\delta(t)] = \sum_{n=-\infty}^{\infty} g(nT_s)e^{-j2\pi f nT_s}$$

Sampling Process

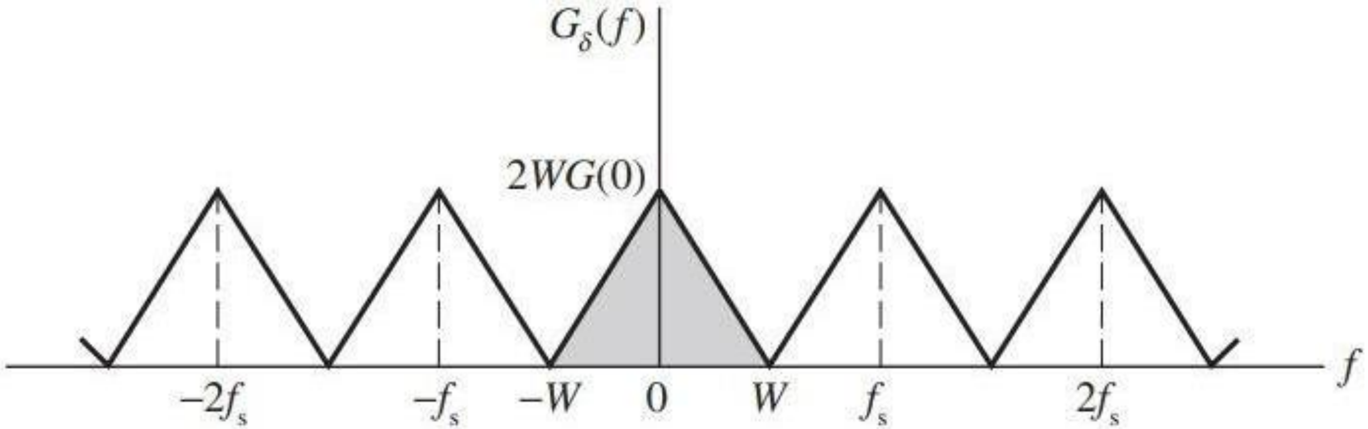
- If the signal g(t) is bandlimited such that there is no frequency components are available greater than the available bandwidth "**W** " Hz then

$$|G(f)| = 0 \; for \; |f| > W$$

- let $f_s = 2W$ or $T_s = \frac{1}{2W}$,where W is the maximum frequency



Spectrum of a strictly band-limited signal $g(t)$



Spectrum of the sampled version of $g(t)$ for a sampling period $T_s = 1/2W$

Sampling Process

3. **Find the relationship between $G(f)$ and $G_\delta(f)$**

- Two assumptions are made: $|G(f)| = 0 \; for \; |f| > W$ and $f_s = 2W$ or $T_s = \dfrac{1}{2W}$
- The equation: $G_\delta(f) = f_s G(f) + f_s \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} G(f - nf_s)$ can be written as

$$f_s G(f) = G_\delta(f) - f_s \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} G(f - nf_s)$$

$$G(f) = \frac{1}{f_s} G_\delta(f) - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} G(f - nf_s) \Rightarrow (f_s = 2W) \Rightarrow \frac{1}{2W} G_\delta(f) - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} G(f - nf_s)$$

$$G(f) = \frac{1}{2W} G_\delta(f) \text{ for } -W \le f \le W$$

Sampling Process

4. **Find the relationship between $g(t)$ and $g(nT_s)$**

- We know that $F.T\,[g_\delta(t)] = G_\delta(f)$

$$G(f) = \frac{1}{2W}\sum_{n=-\infty}^{\infty} g(nT_s)e^{-j2\pi f nT_s} \quad \text{we know that } T_s = \frac{1}{2W}$$

$$G(f) = \frac{1}{2W}\sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)e^{-j2\pi f n/2W} = \frac{1}{2W}\sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)e^{-j\pi f n/W} \quad for -W \le f \le W$$

- $g(t)$ can be obtained by taking Inverse Fourier transform of G (f)

Sampling Process

- We have:

$$g(t) = \sum_{n=-\infty}^{\infty} g(\frac{n}{2W}) \frac{1}{2W} \int_{-W}^{W} \exp\left[j2\pi f\left(t - \frac{n}{2W}\right)\right] df$$

- On further simplifying, we have:

$$g(t) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) sinc(2Wt - n), \qquad -\infty < t < \infty$$

- The above equation is also called as the desired ***reconstruction formula***

The Sampling Theorem

- The sampling theorem for strictly-band limited signals for finite energy that has no frequency components higher than W hertz can be stated in two parts, mentioned as:

1. It can be completely described by specifying the values of the signal instants of time separated by $\frac{1}{2W}$ seconds

$$T_s \leq \frac{1}{2W}$$

2. It can be completely recovered from a knowledge of its samples taken at the rate of $2W$ samples per second

$$f_s \geq 2W$$

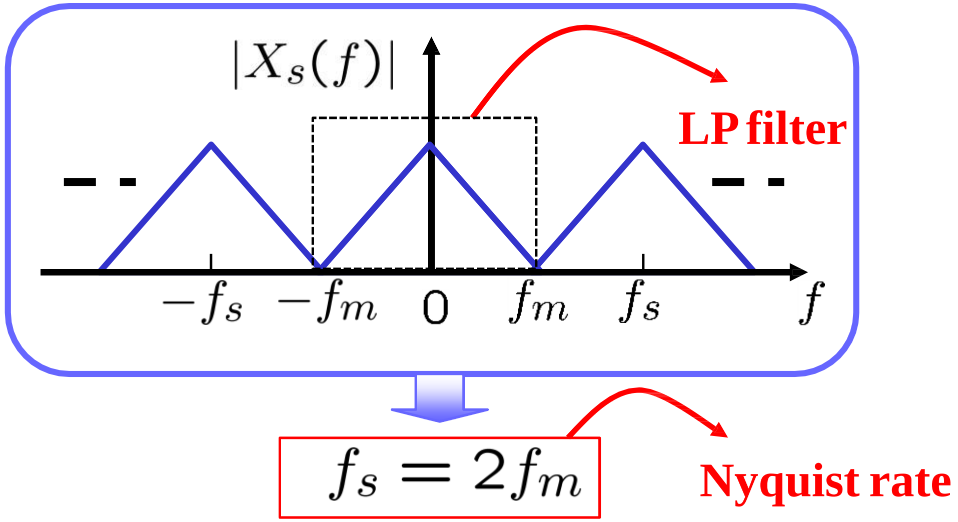
Nyquist Rate

- Suppose that a signal is band-limited with no frequency components higher than **W** Hertz.
- That means, **W** is the highest frequency.
- For such a signal, for effective reproduction of the original signal, the sampling rate should be twice the highest frequency.

$$f_s = 2W$$

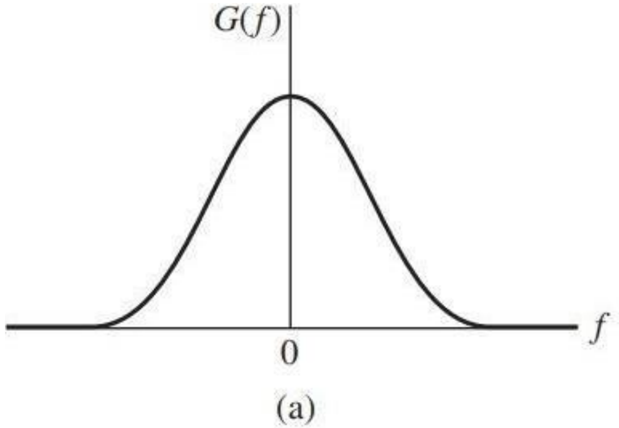
- This rate of sampling is called as **Nyquist rate**
- **Nyquist interval**

$$T_s = \frac{1}{2W}$$

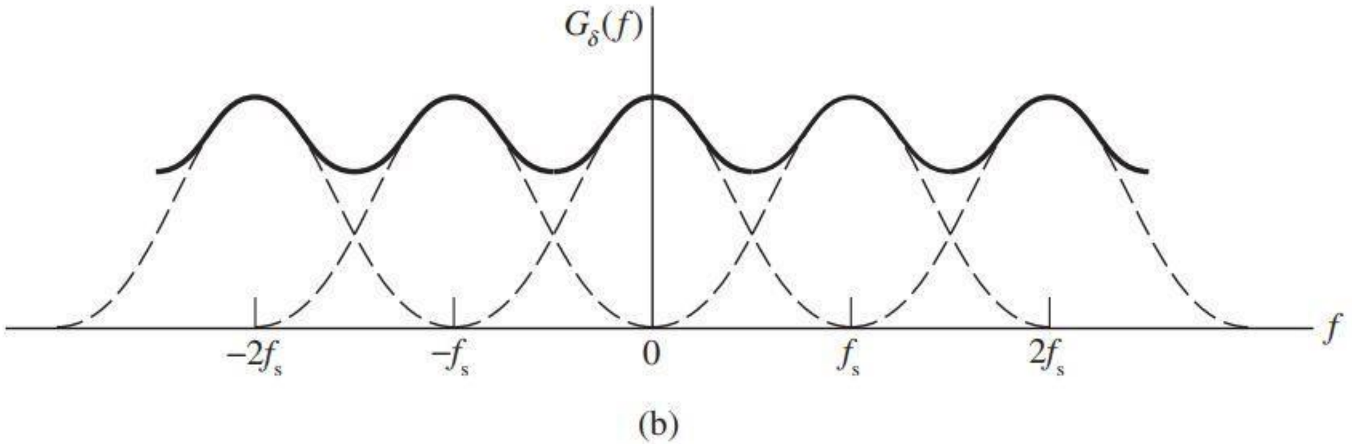


Limitations of Sampling Theorem

- When $f_s < 2W$ in practice $g(t)$ is not strictly bandlimited so aliasing is produced.
- When sampling rate is less than 2W Hz a high frequency component in the spectrum takes the identity of a low frequency component in the spectrum.
- Interference of high frequency components with that of low frequency components in the spectrum is called **Aliasing.**



(a) Spectrum of a signal



(b) Spectrum of an under-sampled version of the signal exhibiting the aliasing phenomenon

Combating Aliasing

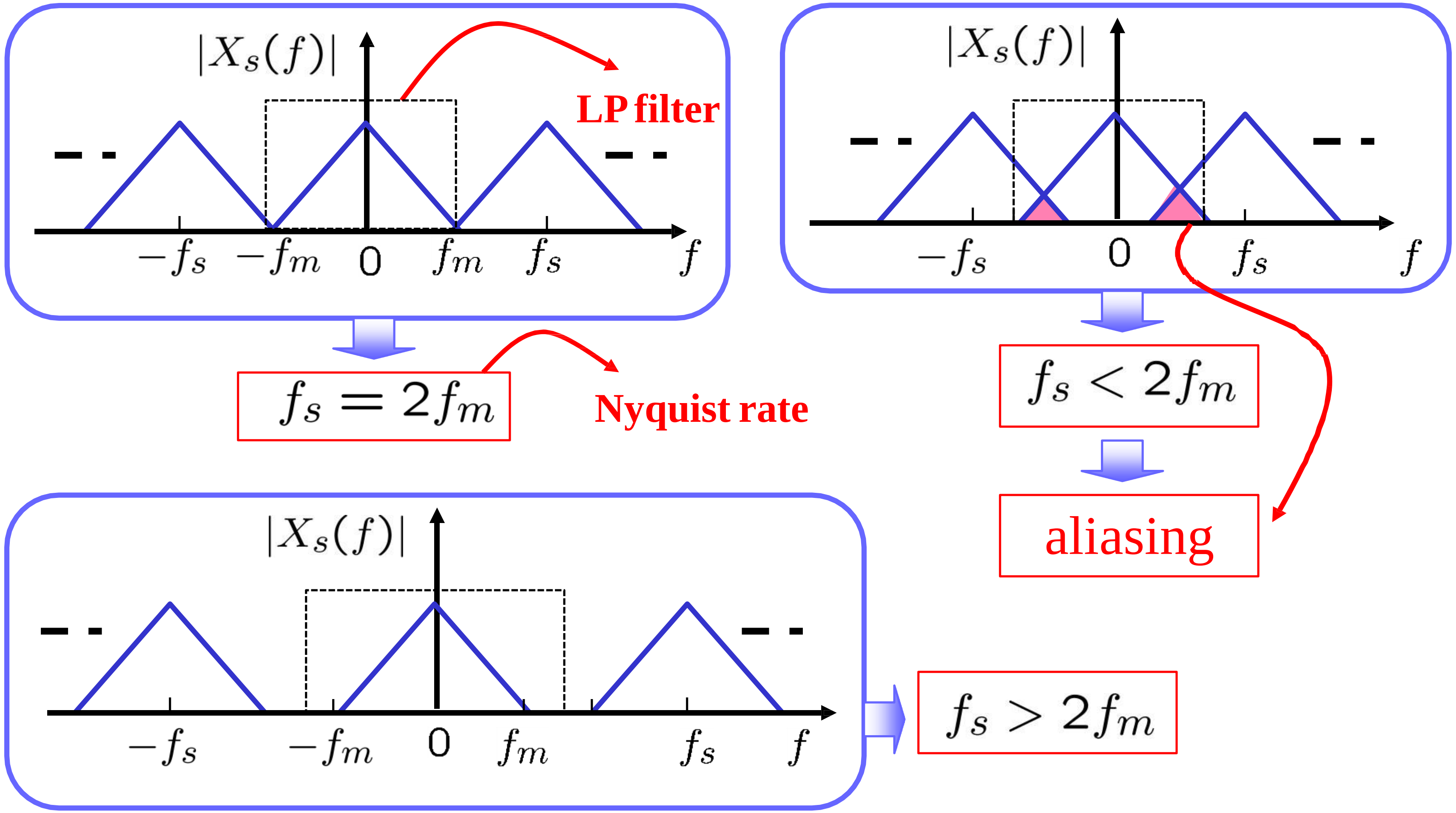
In order to combat aliasing, we may use two corrective measures:

- A *low-pass anti-aliasing filter* can be used prior to sampling in order to attenuate the high-frequency components
- The filtered signal is sampled at a rate slightly higher than the Nyquist rate

Note:

The use of a sampling rate higher than the Nyquist rate also has the beneficial effect of easing the design of the *reconstruction filter* used to recover the original signal from its sampled version

Sampling theorem and Aliasing



Types of Sampling

There are three types of sampling techniques:

1. Ideal /Instantaneous /Impulse Sampling.
2. Natural / Chopper Sampling.
3. Flat Top / Rectangular Sampling.

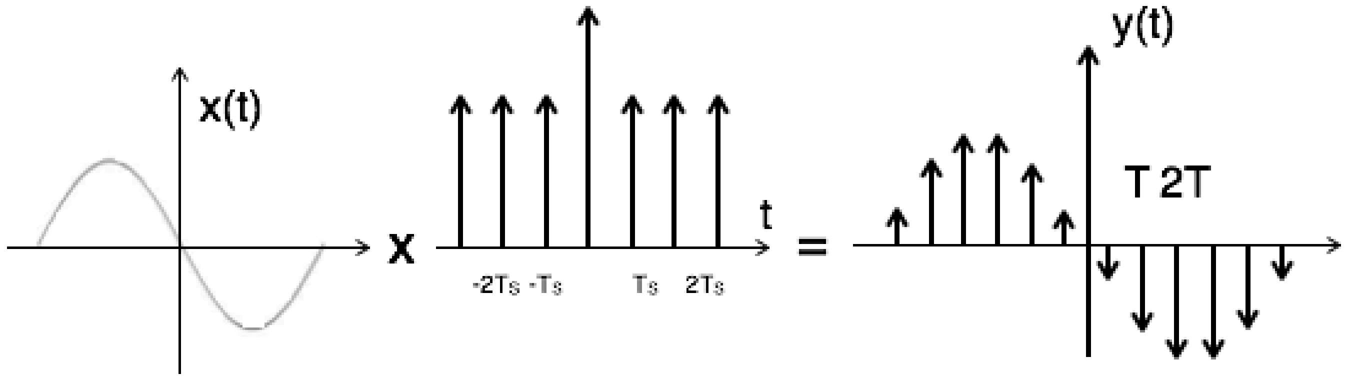
Types of Sampling

Ideal / Instantaneous / Impulse Sampling

- Impulse sampling can be performed by multiplying input signal x(t) with impulse train of period 'T'.

$$\Sigma_{n=-\infty}^{\infty}\delta(t-nT)$$

- Here, the amplitude of impulse changes with respect to amplitude of input signal x(t).
- The output of sampler is given by



Types of Sampling

$$y(t) = x(t) \times \text{ impulse train}$$

$$= x(t) \times \Sigma_{n=-\infty}^{\infty} \delta(t - nT)$$

$$y(t) = y_\delta(t) = \Sigma_{n=-\infty}^{\infty} x(nt)\delta(t - nT) \ldots \ldots 1$$

- To get the spectrum of sampled signal, consider Fourier transform of equation 1 on both sides

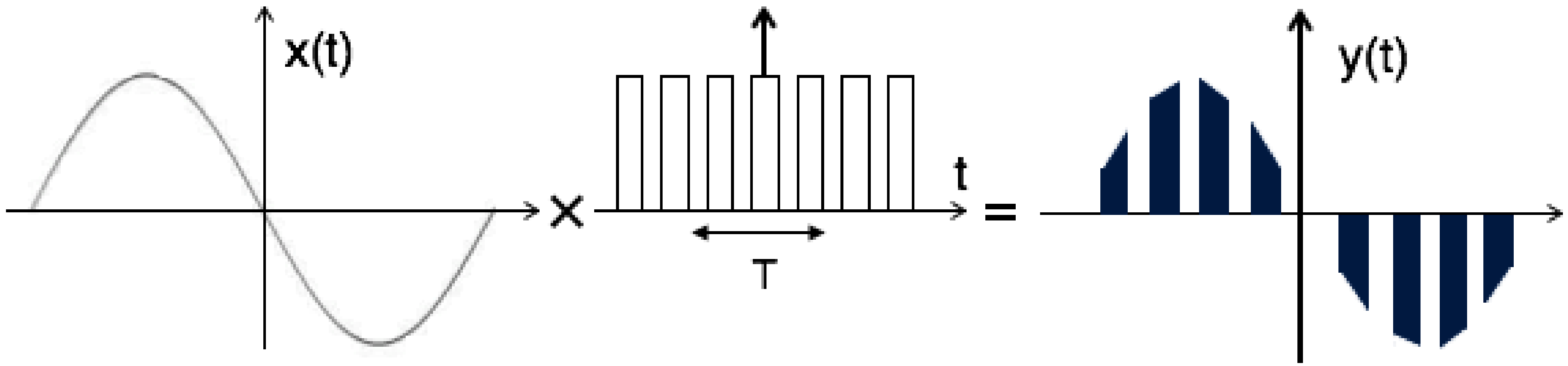
$$Y(\omega) = \frac{1}{T} \Sigma_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

- This is called ideal sampling or impulse sampling. You cannot use this practically because pulse width cannot be zero and the generation of impulse train is not possible practically.

Types of Sampling

Natural / Chopper Sampling

- Natural sampling is similar to impulse sampling,
- except the impulse train is replaced by pulse train of period T.
- i.e. you multiply input signal x(t) to pulse train $\Sigma_{n=-\infty}^{\infty} P(t - nT)$



Types of Sampling

The output of sampler is

$$y(t) = x(t) \times \text{pulse train}$$

$$= x(t) \times p(t)$$

$$= x(t) \times \Sigma_{n=-\infty}^{\infty} P(t - nT) \ldots \ldots (1)$$

The exponential Fourier series representation of p(t) can be given as

$$p(t) = \Sigma_{n=-\infty}^{\infty} F_n e^{jn\omega_s t} \ldots \ldots (2)$$

Where $F_n = \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} p(t) e^{-jn\omega_s t} dt$

$$= \Sigma_{n=-\infty}^{\infty} F_n e^{j2\pi n f_s t}$$

$$= \frac{1}{T} P(n m_s)$$

Types of Sampling

Substitute F_n value in equation 2

$$\therefore p(t) = \Sigma_{n=-\infty}^{\infty} \frac{1}{T} P(n\omega_s) e^{jn\omega_s t}$$

$$= \frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s) e^{jn\omega_s t}$$

Substitute p(t) in equation 1

$$y(t) = x(t) \times p(t)$$

$$= x(t) \times \frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, e^{jn\omega_s t}$$

$$y(t) = \frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, x(t)\, e^{jn\omega_s t}$$

Types of Sampling

- To get the spectrum of sampled signal, consider the Fourier transform on both sides.

$$F.T\,[y(t)] = F.T[\tfrac{1}{T}\Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, x(t)\, e^{jn\omega_s t}]$$

$$= \tfrac{1}{T}\Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, F.T\,[x(t)\, e^{jn\omega_s t}]$$

According to frequency shifting property

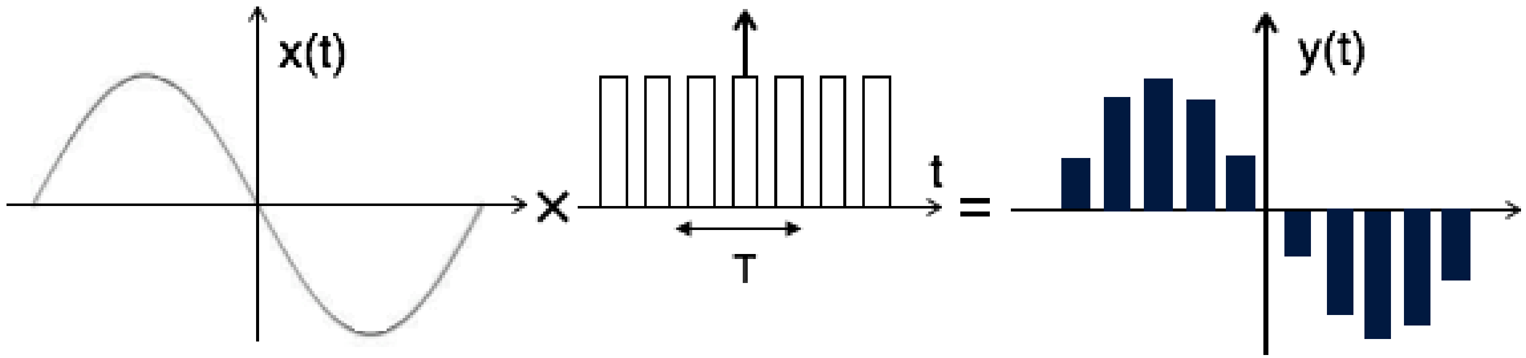
$$F.T\,[x(t)\, e^{jn\omega_s t}] = X[\omega - n\omega_s]$$

$$\therefore\; Y[\omega] = \tfrac{1}{T}\Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, X[\omega - n\omega_s]$$

Types of Sampling

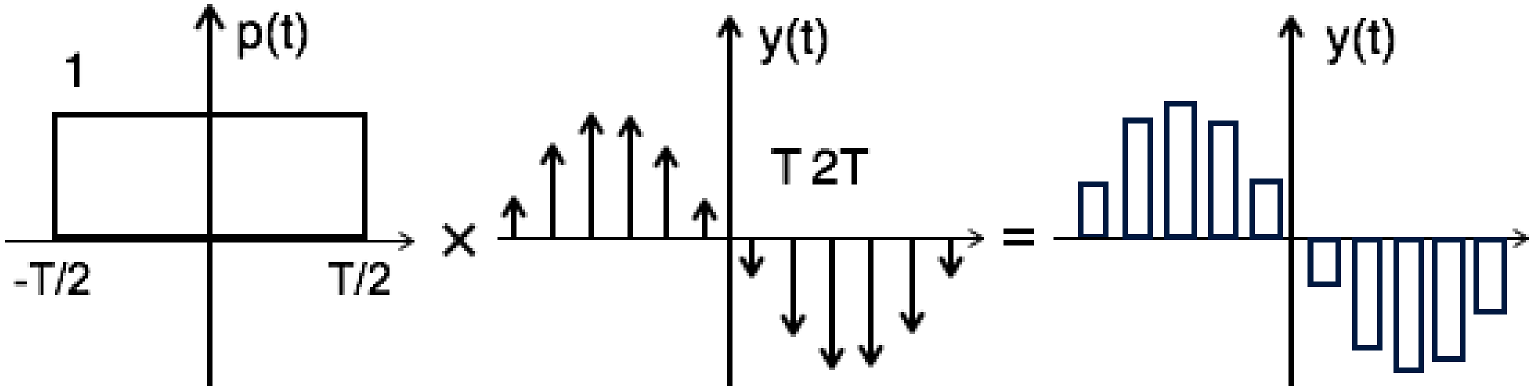
Flat Top / Rectangular Sampling.

- During transmission, noise is introduced at top of the transmission pulse
- Which can be easily removed if the pulse is in the form of flat top.
- Here, the top of the samples are flat i.e. they have constant amplitude.
- Hence, it is called as flat top sampling or practical sampling.
- Flat top sampling makes use of sample and hold circuit.



Types of Sampling

- Theoretically, the sampled signal can be obtained by convolution of rectangular pulse p(t) with ideally sampled signal say $y_\delta(t)$ as shown in the figure below, i.e. $y(t) = p(t) \times y_\delta(t) \ldots\ldots (1)$



- To get the sampled spectrum, consider Fourier transform on both sides for equation 1

$$Y[\omega] = F.T\,[P(t) \times y_\delta(t)]$$

By the knowledge of convolution property,

$$Y[\omega] = P(\omega)\,Y_\delta(\omega)$$

Here $P(\omega) = TSa(\frac{\omega T}{2}) = 2\sin\omega T/\omega$

Numerical

- Consider the analog signal $3cos50\pi t + 10sin300\pi t - cos100\pi t$ what is the Nyquist rate for this signal?

Solution:

Numerical

- Consider the analog signal $3cos50\pi t + 10sin300\pi t - cos100\pi t$ what is the Nyquist rate for this signal?

Solution:

- $x(t) = 3cos50\pi t + 10sin300\pi t - cos100\pi t$
- Nyquist rate = 2W
- $2\pi f_1 t = 50\pi t, f_1 = 25Hz$
- $2\pi f_2 t = 300\pi t, f_2 = 150Hz$ (BW of maximum frequency)
- $2\pi f_3 t = 100\pi t, f_3 = 50Hz$
- Therefore, the Nyquist rate = 2W = $2 \times 150Hz = \mathbf{300Hz}$

Numerical

- $m(t) = sinc200\pi t + sinc^2 200\pi t$. Find the Nyquist rate.

Solution:

Numerical

- $m(t) = sinc200\pi t + sinc^2 200\pi t$.Find the Nyquist rate.

Solution:

$$\frac{sin\pi 200t}{\pi 200t} + \frac{sin^2\pi 200t}{(\pi 200t)^2}$$

- $2\pi f_1 t = 200\pi t, f_1 = 100Hz$
- $2\pi f_2 t = 400\pi t, f_2 = 200Hz$
- Therefore, the Nyquist rate = 2W $= 2 \times 200Hz = \mathbf{400Hz}$

Numerical

- Determine the sampling rate for a signal $\mathrm{v(t)} = 2[cos500\pi t \cdot cos1000\pi t]$

Solution:

Numerical

- Determine the sampling rate for a signal $\mathrm{v(t)} = 2[cos500\pi t \cdot cos1000\pi t]$

Solution:

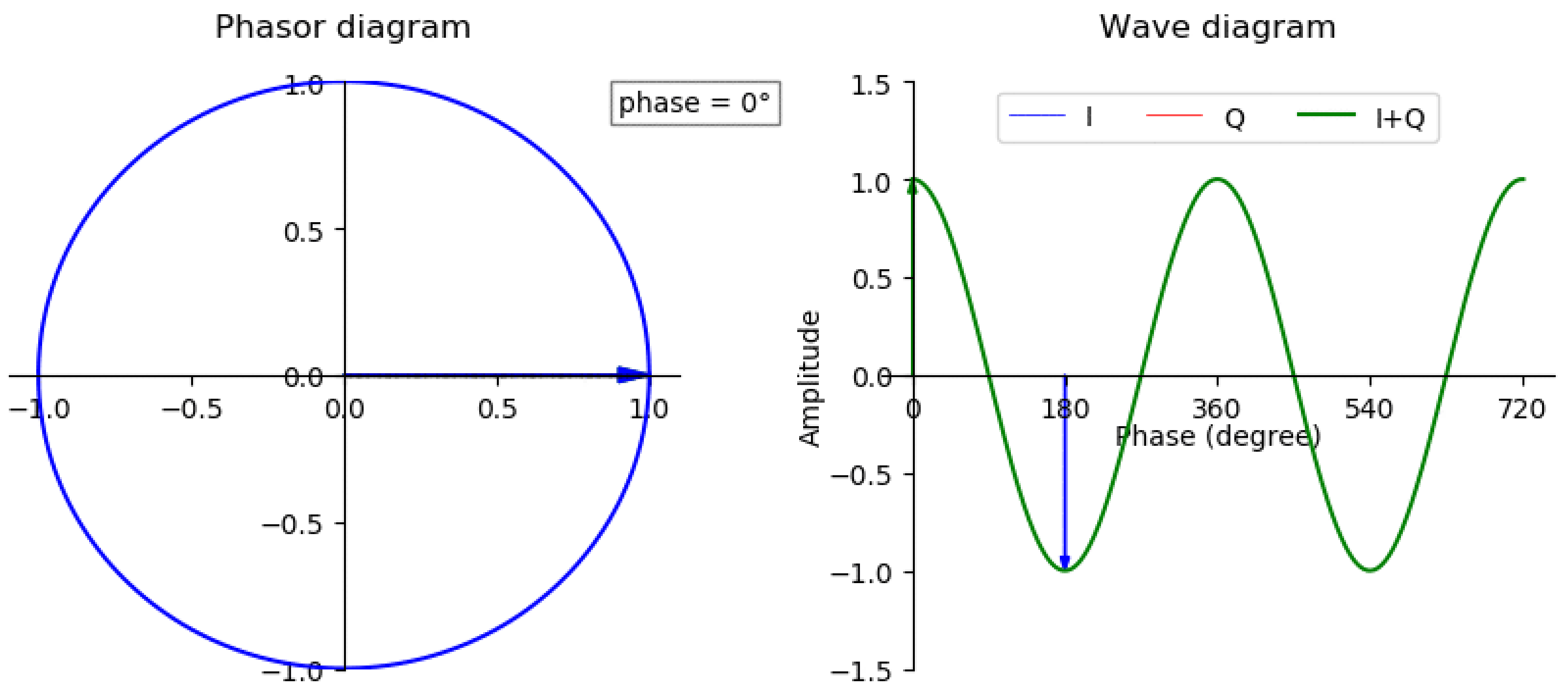
- $\mathrm{v(t)} = 2[cos500\pi t \cdot cos1000\pi t]$

$= 2\ \frac{[\cos(500\pi+1000\pi)t]+[\cos(1000\pi-500\pi)t]}{2} = 2\ \frac{[\cos(1500\pi)t]+[\cos(500\pi)t]}{2} =$

$= \ [\cos(1500\pi)t] + [\cos(500\pi)t]$

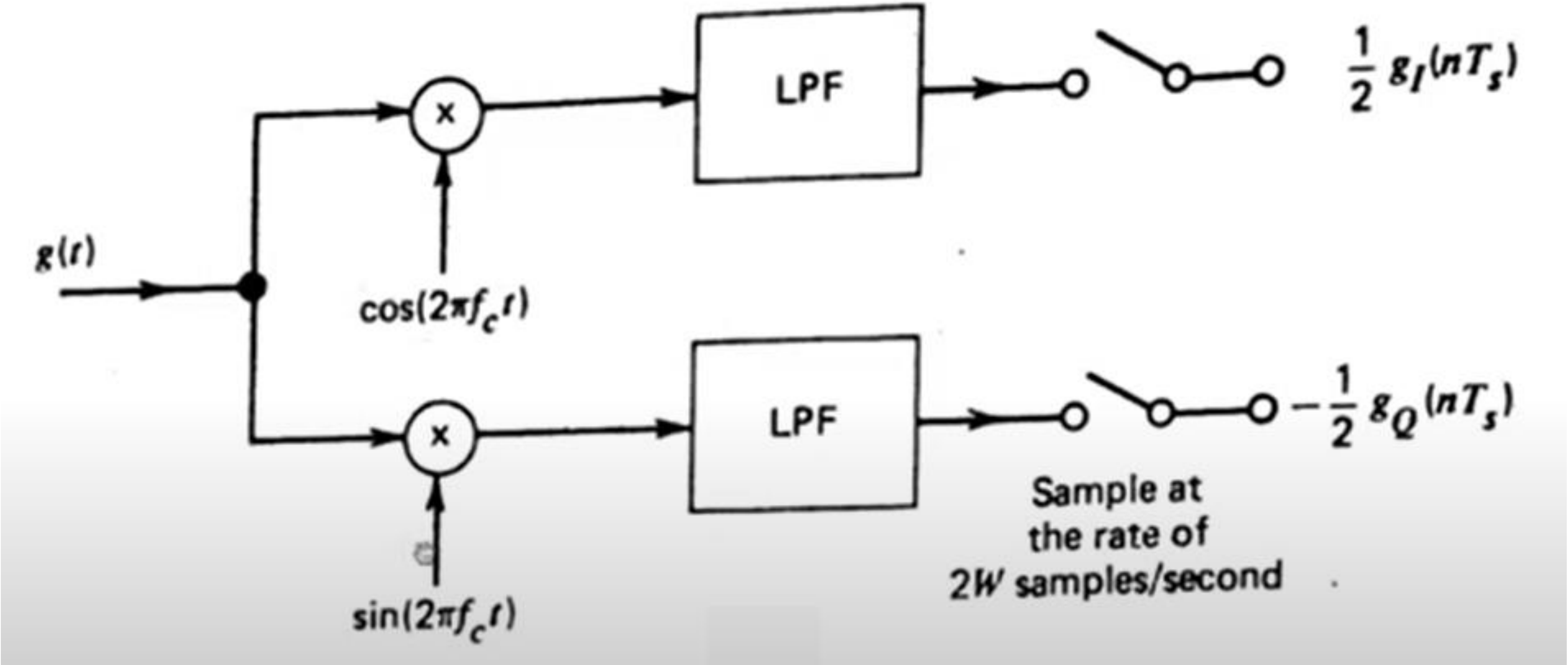
- $f_1 = 750Hz$
- $f_2 = 250Hz$
- Therefore, the sampling rate $f_s \geq 2W \geq 2 \times 750Hz \geq \mathbf{1500}\boldsymbol{Hz}$

Quadrature Sampling of Bandpass Signals



https://pysdr.org/content/sampling.html

Quadrature Sampling of Bandpass Signals



$g(t) = gI(t).\ \cos(2\pi fct) - gQ(t)\ \sin(2\pi fct)$

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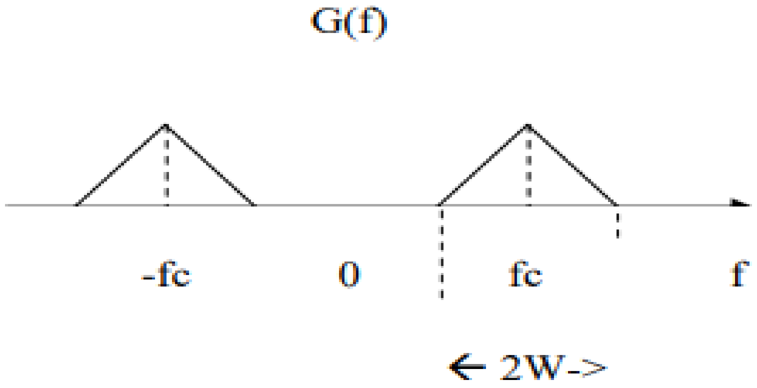
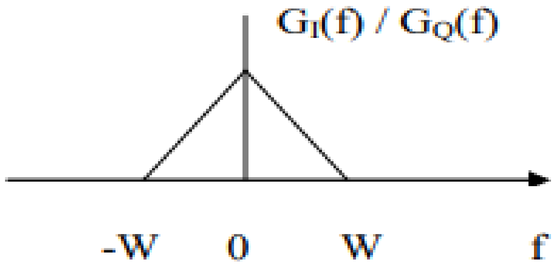
Spectrum

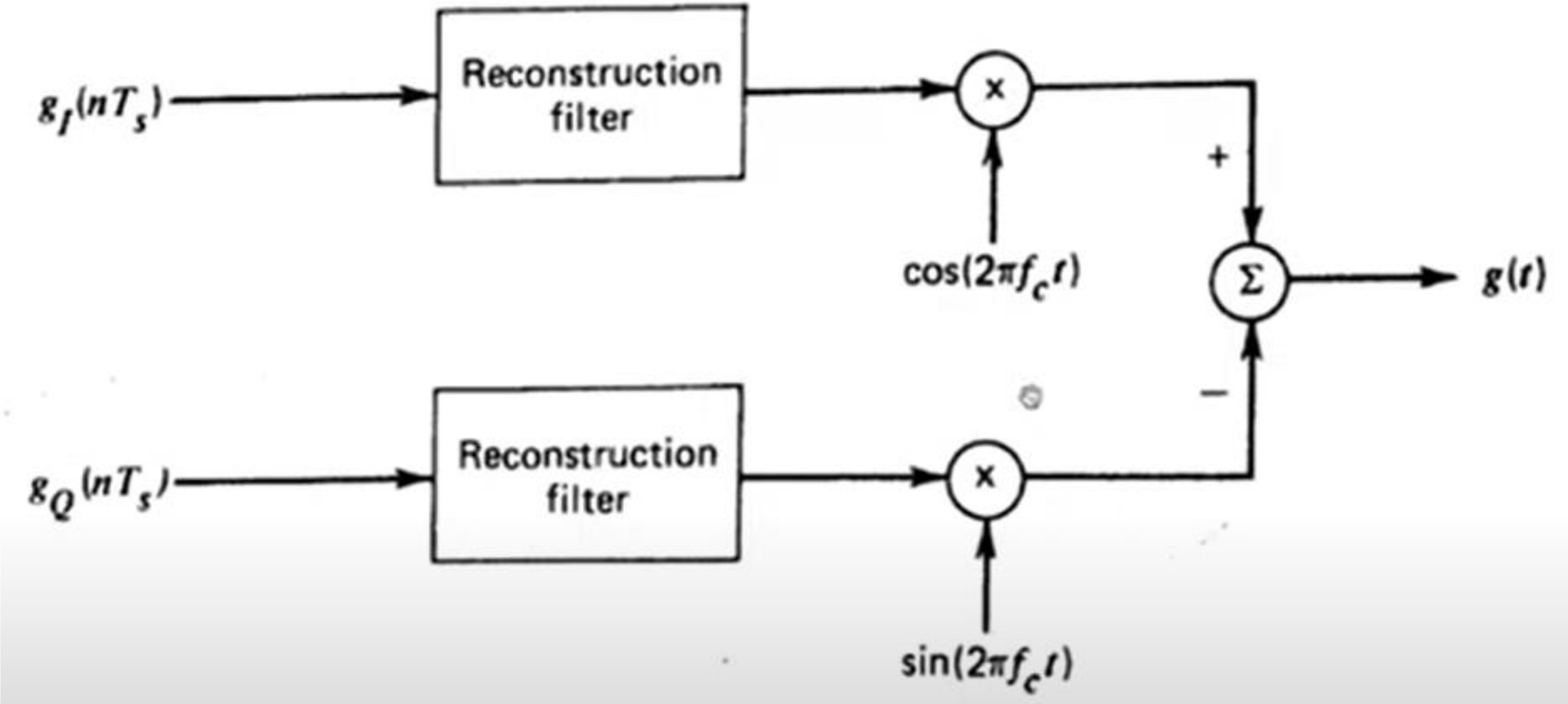


a) Spectrum of a Band pass signal.

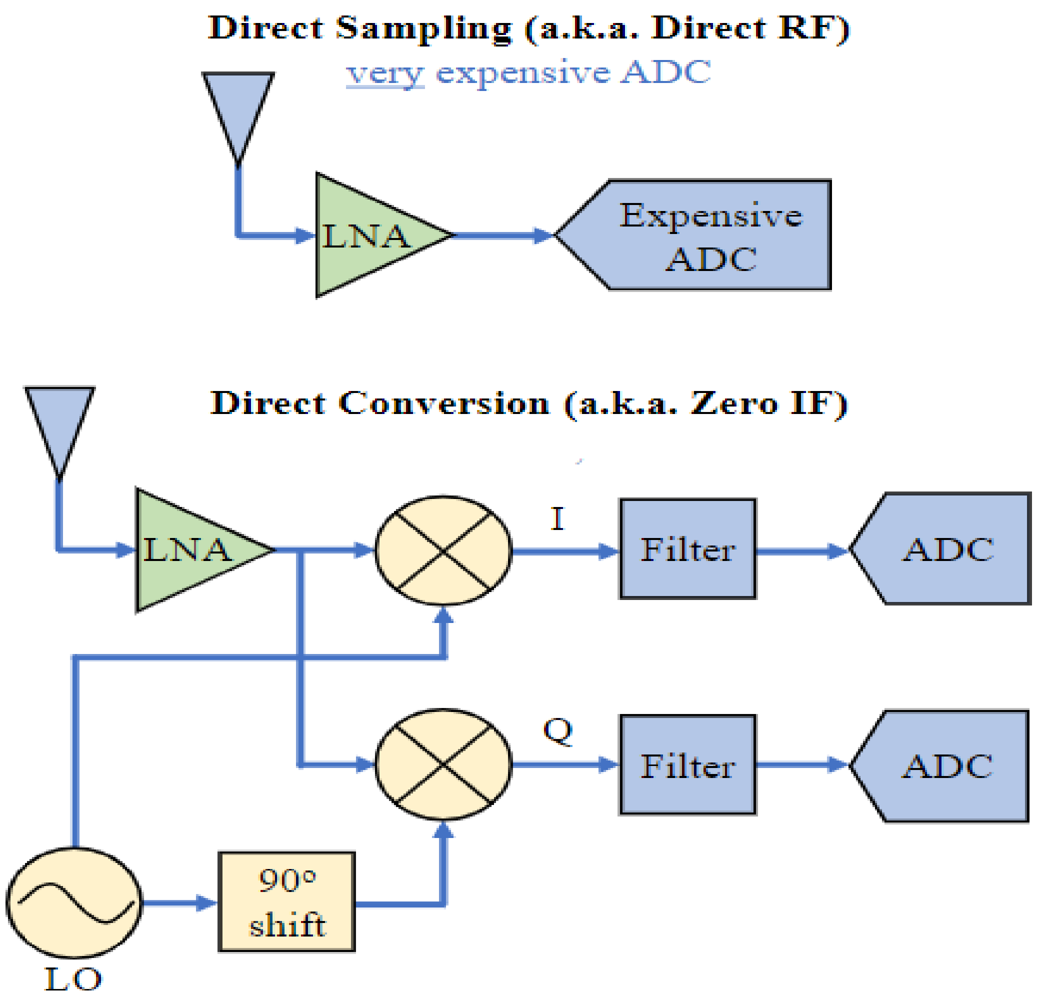


b) Spectrum of $g_I(t)$ and $g_Q(t)$

Reconstruction of g(t)



Direct sampling and Quadrature sampling



Analog to Digital Conversion Blocks in DCS