Chapter 2 Questions and Solutions

Question 1.

Explain what the terms centrifugal and centripetal mean with regard to a satellite in orbit around the earth.

A satellite is in a circular orbit around the earth. The altitude of the satellite's orbit above the surface of the earth is 1,400 km. (i) What are the centripetal and centrifugal accelerations acting on the satellite in its orbit? Give your answer in m/s^2. (ii) What is the velocity of the satellite in this orbit? Give your answer in km/s. (iii) What is the orbital period of the satellite in this orbit? Give your answer in hours, minutes, and seconds. Note: assume the average radius of the earth is 6,378.137 km and Kepler's constant has the value 3.986004418×10^5 km^3/s^2

Solution to question 1:

In the case of a satellite orbiting the earth, the centrifugal force on the satellite is a force on the satellite that is directly away from the center of gravity of the earth (F_{OUT} in Fig. 2.1) and the centripetal force is one directly towards the center of gravity of the earth (F_{IN} in Fig. 2.1). The centrifugal force on a satellite will therefore try to fling the satellite away from the earth while the centripetal force will try to bring the satellite down towards the earth.

(i) From equation (2.1) centripetal acceleration $a = m/r^2$, where m is Kepler's constant. The value of r = 6,378.137 + 1,400 = 7,778.137 km, thus $a = 3.986004418 \times 10^5 / (7{,}778.137)^2 = 0.0065885$ km/s^2 = 6.5885007 m/s^2. From equation (2.3), the centrifugal acceleration is given by $a = v^2/r$, where v = the velocity of the satellite in a circular orbit.. From equation (2.5) $v = (m/r)^{1/2} = (3.986004418 \times 10^5 / 7{,}778.137)^{1/2} = 7.1586494$ km/s and so $a = 0.0065885007$ km/s^2 = 6.5885007 m/s^2. **NOTE:** since the satellite was in stable orbit, the centrifugal acceleration must be equal to the centripetal acceleration, which we have found to be true here (but we needed only to calculate one of them).

(ii) We have already found out the velocity of the satellite in orbit in part (i) (using equation (2.5)) to be 7.1586494 km/s

(iii) From equation (2.6), the orbital period $T = (2pr^{3/2})/(m^{1/2}) = (2p7{,}778.137^{3/2})/(3.986004418 \times 10^5)^{1/2}$ = (4,310,158.598)/(631.3481146) = 6,826.912916 s = 1 hour 53 minutes 46.92 seconds

Question 2

A satellite is in a 322 km high circular orbit. Determine:
a. The orbital angular velocity in radians per second;
b. The orbital period in minutes; and

c. The orbital velocity in meters per second.
Note: assume the average radius of the earth is 6,378.137 km and Kepler’s constant has the value 3.986004418×10^5 km^3/s^2.

Solution to question 2:

It is actually easier to answer the three parts of this question backwards, beginning with the orbital velocity, then calculating the period, and hence the orbital angular velocity. First we will find the total radius of the orbit r = 322 + 6,378.137 km = 6700.137 km

(c) From eqn. (2.5), the orbital velocity $v = (m/r)^{1/2} = (3.986004418 \times 10^5 / 6700.137)^{1/2}$ = 7.713066 km/s = 7,713.066 m/s.

(b) From eqn. (2.6), $T = (2pr^{3/2})/(m^{1/2}) = (2p6{,}700.137^{3/2})/(3.986004418 \times 10^5)^{1/2}$ = (3,445,921.604)/(631.3481146) = 5,458.037372 seconds = 90.9672895 minutes = 90.97 minutes.

(a) The orbital period from above is 5,458.037372 seconds. One revolution of the earth covers 360^o or 2π radians. Hence 2π radians are covered in 5,458.037372 seconds, giving the orbital angular velocity as 2π/5,458.037372 radians/s = 0.0011512 radians/s. An alternative calculation procedure would calculate the distance traveled in one orbit ($2pr = 2p6700.137$ = 42,098.20236 km). This distance is equivalent to 2π radians and so 1 km is equivalent to 2π/42,098.20236 radians = 0.0001493 radians. From above, the orbital velocity was 7.713066 km/s = $7.713066 \times 0.0001493$ radians/s = 0.0011512 radians/s.

Question 3.

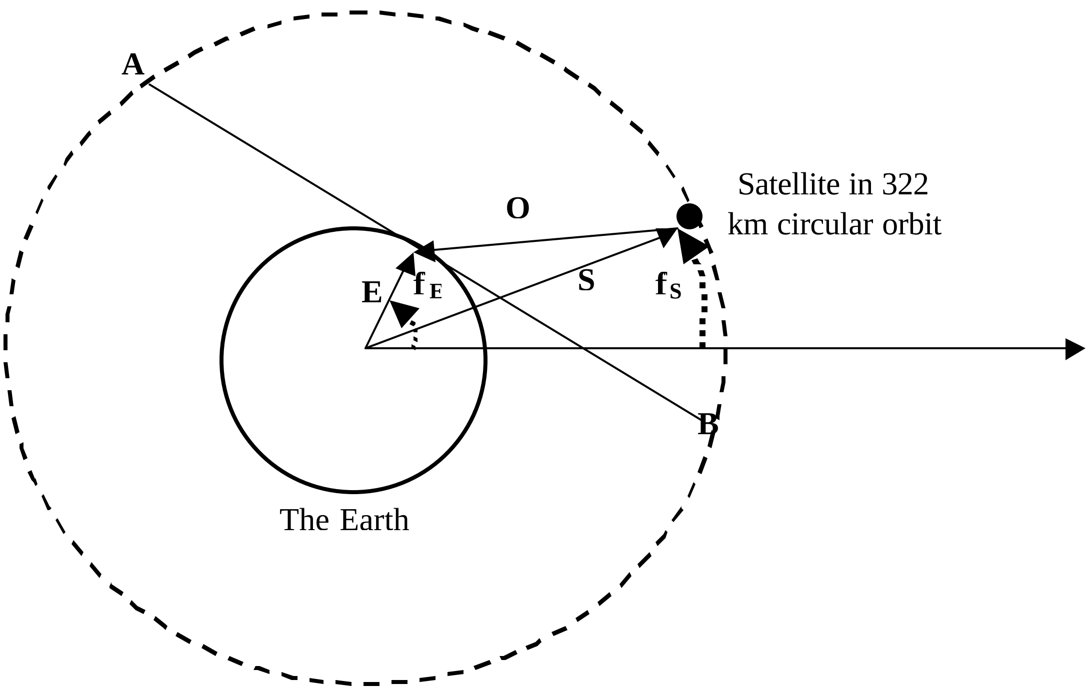
The same satellite in question 2 above (322 km circular orbit) carries a 300 MHz transmitter.
a. Determine the maximum frequency range over which the received signal would shift due to Doppler effects if received by a stationary observer suitably located in space. Note: the frequency can be shifted both up and down, depending on whether the satellite is moving towards or away from the observer. You need to determine the maximum possible change in frequency due to Doppler (i.e. $2\Delta_f$).
b. If an earth station on the surface of the earth at mean sea level, 6,370 km from the center of the earth, can receive the 300 MHz transmissions down to an elevation angle of 0^o, calculate the maximum Doppler shift that this station will observe. Note: Include the earth’s rotation and be sure you consider the *maximum possible* Doppler shift for a 322 km circular orbit.

Solution to question 3

a. The highest Doppler shift would be observed in the plane of the satellite at the orbital height of the satellite: the satellite would be coming directly *at* the observer or directly *away* from the observer. The maximum Doppler shift would therefore be the sum of these two values. The orbital velocity was calculated in question 2 as 7,713.066 m/s. Using equation (2.44a), $Df / f_T = V_T / v_p$, where Df is the Doppler frequency, f_T is the frequency of the transmitter at rest, V_T is the component of the transmitter's velocity directed at the observer, and v_p is the phase velocity of light. Since the observer is at orbital height, the component of the transmitter's velocity towards the observer is the actual velocity of the satellite. Thus $Df = (7{,}713.066 \times 300{,}000{,}000) / 2.9979 \times 10^8 =$ 7,718.468928 Hz. The maximum Doppler shift therefore = 7,718.468928 × 2 = 15,436.93786 Hz = 15,436.94 Hz.

b. It is best to draw a diagram to see what the set up looks like. Below is a view from above the orbit of the satellite (orthogonal to the orbital plane).

The important element in this part of the question is the component of the satellite's velocity towards the earth station.



O = vector from the satellite to the earth station
S = vector from the origin to the satellite
E = vector from the origin to the earth station
f $_S$ = angular coordinate of the satellite
f $_E$ = angular coordinate of the earth station
A and **B** are the points in the satellite's orbit when the elevation angle at the earth station is zero, and hence (if the plane of the orbit takes the satellite directly over the earth station at zenith) the point where the Doppler shift is highest – either positive or negative.

Using the law of cosines,

$\mathbf{O}^2 = \mathbf{E}^2 + \mathbf{S}^2 - 2\mathbf{ES}\cos(f_E - f_S)$

For an orbital height of h, $\mathbf{S} = \mathbf{E} + h$, giving

$\mathbf{O}^2 = \mathbf{E}^2 + (\mathbf{E} + h)^2 - 2\mathbf{E}(\mathbf{E} + \mathrm{h})\cos(f_E - f_S)$

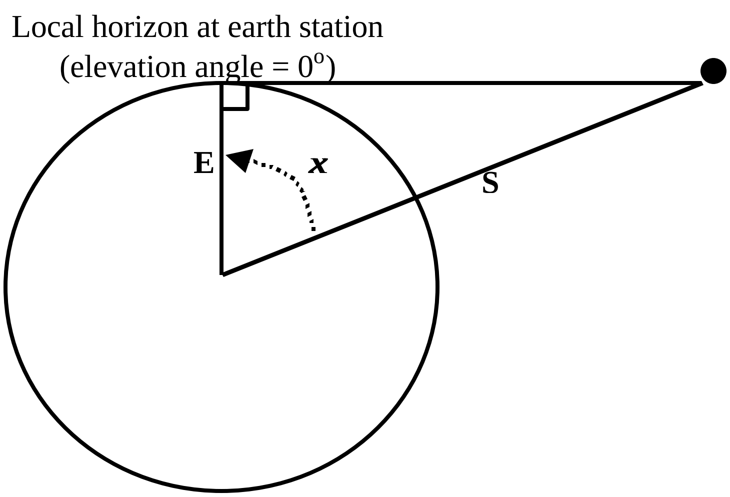
The component of the satellite's velocity towards the earth station is d**O**/dt and we can obtain it by differentiating the above equation (remembering the **E**, h, and **S** may be assumed to be constant values), term by term, thus:

$2\mathbf{O}(d\mathbf{O}/dt) = 2\mathbf{E}(\mathbf{E} + h)\sin(f_E - f_S)(df_E/dt - df_S/dt)$, giving

$(d\mathbf{O}/dt) = (\mathbf{E}(\mathbf{E} + h)\sin(f_E - f_S)(df_E/dt - df_S/dt))/(\mathbf{E}^2 + \mathbf{S}^2 - 2\mathbf{ES}\cos(f_E - f_S))^{1/2}$

df_E/dt is the earth's rotational angular velocity = 2πradians/24hours = 7.2722052×10^{-5} radians/second

$df_S/dt = \pm n$, where n is the orbital angular velocity = 0.0011512 radians/s from question 2 part (a). The sign depends on whether the orbital motion and the earth's rotation are in the same direction. To find the maximum Doppler shift, we need to find $(f_E - f_S)$ when the elevation angle is 0^o. This is drawn below for one of the geometries (the other being the mirror image on the other side).



In this figure $\boldsymbol{x} = (f_E - f_S)$

$\cos \boldsymbol{x} = \mathbf{E}/\mathbf{S} = \mathbf{E}/(\mathbf{E} + h) = (6378.137)/(6378.137 + 322) = 0.9519413$

$(f_E - f_S) = \cos^{-1}(0.9519413) = 17.8352236^o$

Thus

$$\frac{d\mathbf{O}}{dt} = \frac{(6378.137)(6378.137+322)\sin(17.835)(7.272\times10^{-5} \mp 1.15\times10^{-3})}{((6378.137)^2 + (6378.137+322)^2 - 2(6378.137)(6378.137+322)\cos(17.835))^{1/2}}$$

$$= \frac{-14{,}100.03935, +16{,}003.6389}{2052.096993}$$

= -6.8710394 or +7.7986757 km/s

The satellite may be rotating in the same direction as the angular rotation of the earth or against it. Thus, for the same direction rotation,

$$2\Delta f = \frac{2\times7.7986757\times10^3\times300\times10^6}{2.9979\times10^8}$$ = 15,608.27719 = 15.6 kHz

For the opposite direction rotation

$$2\Delta f = \frac{2\times6.8710394\times10^3\times300\times10^6}{2.9979\times10^8}$$ = 13,752.41469 = 13.75 kHz

Question 4

What are Kepler's three laws of planetary motion? Give the mathematical formulation of Kepler's third law of planetary motion. What do the terms perigee and apogee mean when used to describe the orbit of a satellite orbiting the earth?

A satellite in an elliptical orbit around the earth has an apogee of 39,152 km and a perigee of 500 km. What is the orbital period of this satellite? Give your answer in hours. Note: assume the average radius of the earth is 6,378.137 km and Kepler's constant has the value 3.986004418×10^5 km^3/s^2.

Solution to question 4

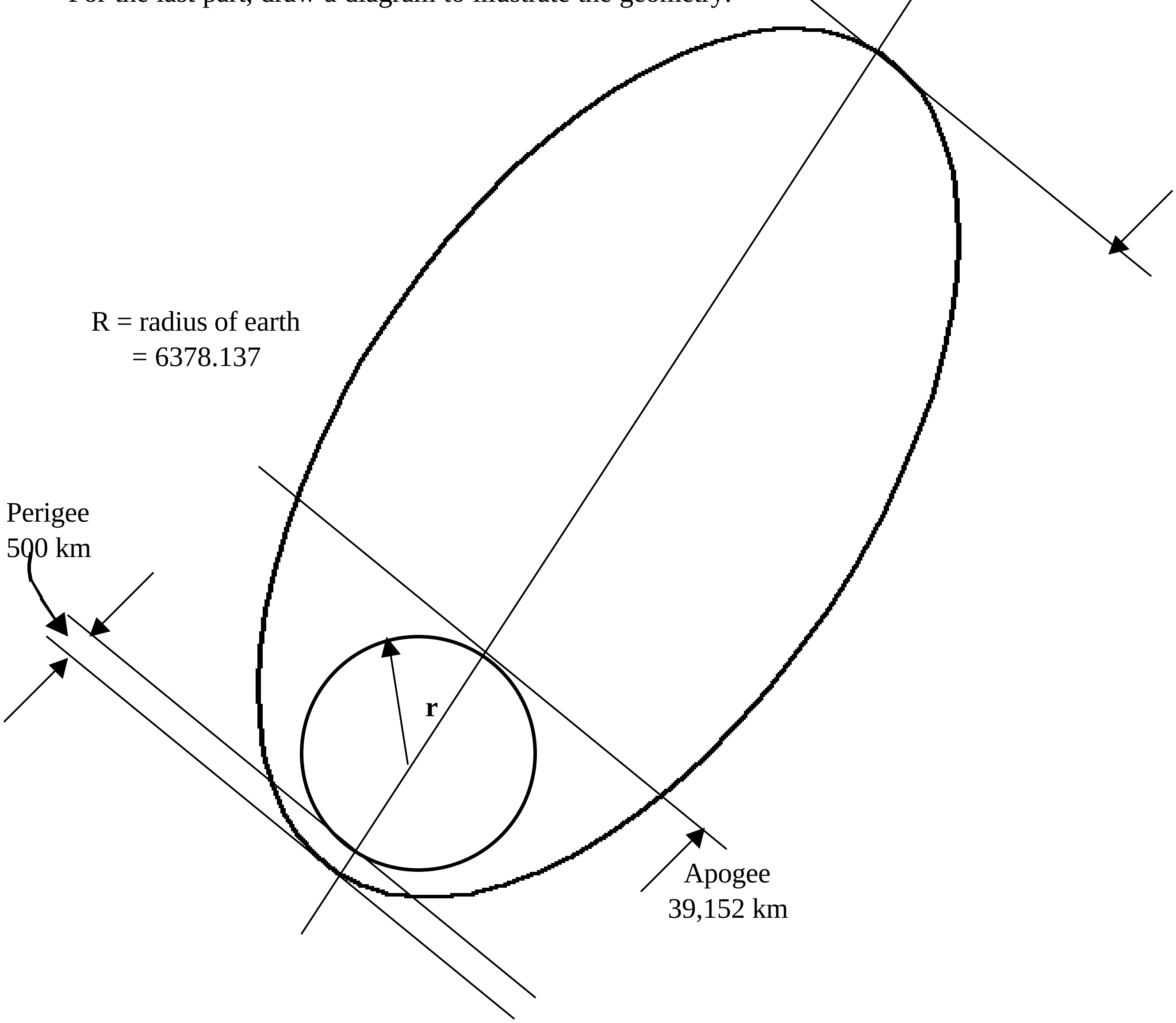
Kepler's three laws of planetary motion are (see page 22)

1. The orbit of any smaller body about a larger body is always an ellipse, with the center of mass of the larger body as one of the two foci.
2. The orbit of the smaller body sweeps out equal areas in time (see Fig. 2.5).
3. The square of the period of revolution of the smaller body about the larger body equals a constant multiplied by the third power of the semimajor axis of the orbital ellipse.

The mathematical formulation of the third law is $T^2 = (4\pi^2 a^3)/\mu$, where T is the orbital period, a is the semimajor axis of the orbital ellipse, and μ is Kepler's constant.

The perigee of a satellite is the closest distance in the orbit to the earth; the apogee of a satellite is the furthest distance in the orbit from the earth.

For the last part, draw a diagram to illustrate the geometry.



The semimajor axis of the ellipse = $(39{,}152 + (2 \times 6378.137) + 500)/2 = 26{,}204.137$ km
The orbital period is

$T^2 = (4\pi^2 a^3)/\mu = (4\pi^2(26{,}204.137)^3)/\ 3.986004418 \times 10^5 = 1{,}782{,}097{,}845.0$

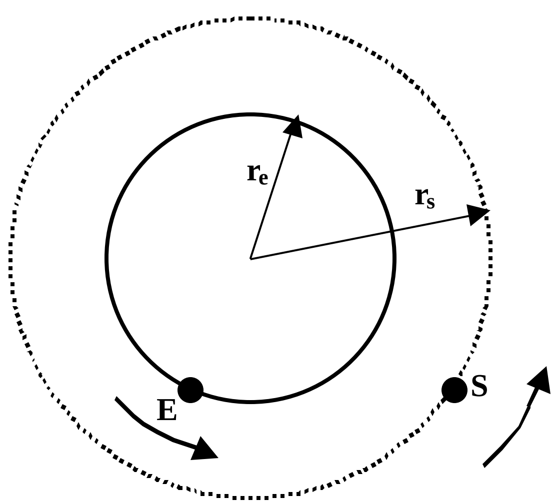
Therefore, $T = 42{,}214.90075$ seconds = 11 hours 43 minutes 34.9 seconds

Question 5

An observation satellite is to be placed into a circular equatorial orbit so that it moves in the same direction as the earth's rotation. Using a synthetic aperture radar system, the satellite will store data on surface barometric pressure, and other weather related parameters, as it flies overhead. These data will later be played back to a controlling earth station after each trip around the world. The orbit is to be designed so that the satellite is directly above the controlling earth station, which is located on the equator, once every 4 hours. The controlling earth station's antenna is unable to operate below an elevation angle of 10^o to the horizontal in any direction. Taking the earth's rotational period to be exactly 24 hours, find the following quantities:
a. The satellite's angular velocity in radians per second.
b. The orbital period in hours.
c. The orbital radius in kilometers.
d. The orbital height in kilometers.
e. The satellite's linear velocity in meters per second.
f. The time interval in minutes for which the controlling earth station can communicate with the satellite on each pass.

Solution to question 5

First, draw a schematic diagram to illustrate the question. The diagram is from the North Pole, above the earth, with the satellite's orbit and the equator in the plane of the paper.



$\mathbf{r_e}$ = earth radius, 6378.137 km
$\mathbf{r_s}$ = orbital radius, km
S = satellite
E = earth station
Arrows above E and S show the direction of motion of the earth station and satellite, respectively

a. Let h_s = satellite angular velocity and h_e = earth's rotational angular velocity. This gives

$$h_e = \frac{2p\,radians}{24\,hours} = \frac{2p}{86{,}400} = 7.2722052 \times 10^{-5}\, radians / s$$

If the satellite was directly over the earth station at time t = 0, their angular separation at time t will be $D_\phi = (h_s - h_e) \times t$. The first overhead pass occurs at $D_\phi = 0$ and the second occurs at $D_\phi = \pm 2\pi$. We want $D_\phi = \pm 2\pi$ to occur at t = 4 hours = 14,400 seconds. Thus $\pm 2\pi = (h_s - h_e) \times 14{,}400$, which gives $(h_s - h_e) = \pm 2\pi/14{,}400 = \pm 0.0004363$. The possible values are (i) $h_s = h_e + 0.0004363 = 7.2722052 \times 10^{-5} + 0.0004363 = 0.0005090221$ radians/s and (ii) $h_e - 0.0004363 = 7.2722052 \times 10^{-5} - 0.0004363 = -0.0003635779$ radians/s.

The first value corresponds to an orbital period of $T = 2\pi/0.0005090221 = 12{,}343.64194$ seconds = 3 hours 25 minutes 43.64 seconds. The second value corresponds to an orbital period of $T = 2\pi/0.0003635779 = 17{,}281.58082$ = 4 hours 48 minutes 1.58 seconds. The second value corresponds to the satellite orbiting in the opposite direction to the earth's rotation. The question stated that the satellite rotated in the same direction as the earth and so the satellite's angular rotation is 0.0005090221 radians/s.

A simple way to check this answer is to note that in 4 hours the earth rotates 60^o or $\pi/3$ radians. A satellite going in the direction of the earth's rotation must cover $(2\pi + \pi/3)$ in 4 hours to catch up with the earth station (that is, one full rotation plus the additional distance the earth has rotated). The angular velocity is therefore $(2\pi + \pi/3)/4$ hours $= (2\pi + \pi/3)/14{,}400 = 0.000509$ radians/s (assuming answer (i) above).

b. The orbital period, T, is given by the number of radians in one orbit (2π) divided by the angular velocity (0.0005090221 radians/s) = 12,343.64194 seconds = 3.43 hours (=3 hours 25 minutes 43.64).

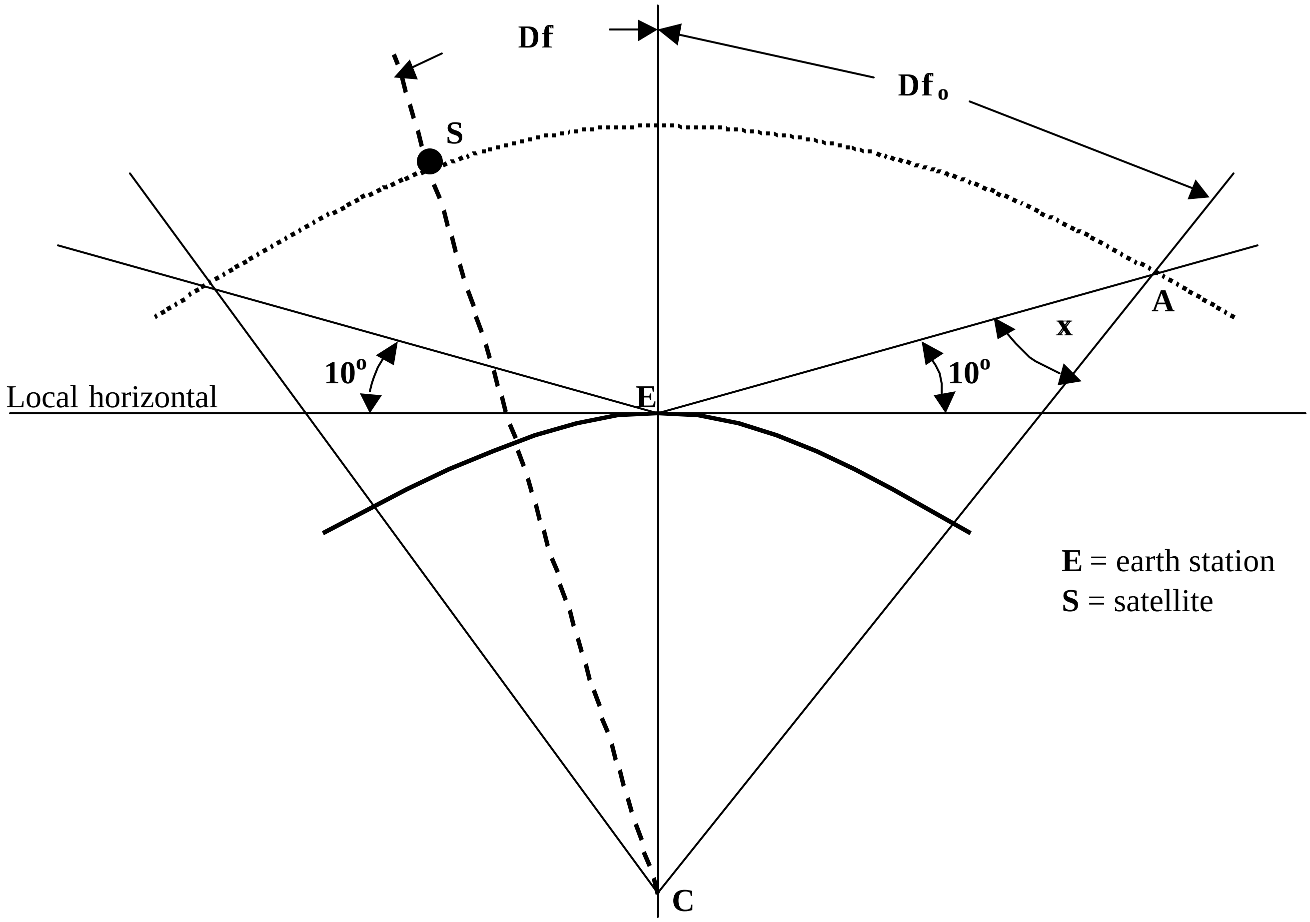
c. From equation (2.25) orbital angular velocity = $(\mu^{1/2})/(a^{3/2})$, where μ is Kepler's constant = 3.986004418×10^5 km^3/s^2 and a is the semimajor axis of the orbit. The orbit is circular in this question and so the radius of the orbit is a. We know μ and the orbital angular velocity, thus
$a = (3.986004418 \times 10^5)^{1/3}/(0.0005090221)^{2/3} = 11{,}543.96203$ km = 11,543.96 km

d. The orbital height is the radius of the orbit (from the earth's center) minus the earth's radius = 11,543.96203 – 6378.137 km = 5,165.825030 = 5,165.83 km

e. The linear velocity of the satellite can be found in two ways. From equation (2.5), the orbital velocity = $(\mu/r)^{1/2} = ((3.986004418 \times 10^5)/(11{,}543.96203))^{1/2}$ = 5.8761306 km/s = 5.876 km/s. Alternatively, orbital velocity = radius of orbit $\times$

orbital angular velocity = 11,543.96203 × 0.0005090221 = 5.8761306 km/s.

f. For this part it is best to draw a diagram illustrating the visibility geometry.



Communications are possible if ½Δφ½ £ $\Delta\phi_o$. We can find $\Delta\phi_o$ by trigonometry. Angle AEC = 100° and, by the law of sines, a/sin(100) = r_e/sinξ, where a is the radius of the orbit. From this, sinξ = r_e sin(100)/a = (6,378.137 × 0.9848078)/11,543.96203 = 0.5441146 giving ξ = 32.9641821 =32.96°.

Given ξ = 32.96°, angle ACE = 180 – 100 – 32.96 = 47.04°. The time the satellite goes from overhead to 10° elevation angle is the angle traveled/relative angular velocity of the satellite = 47.04° in radians/(0.000509 radians/s – 0.000072722052) = 0.8209299/0.0004363 = 1,881.572083 seconds = 31.359347 minutes.

The total time from one extreme to the other of the visible orbit is twice this value = 62.72 minutes.

Question 6

What is the difference, or are the differences, between a *geosynchronous* satellite and a *geostationary* satellite orbit? What is the period of a geostationary satellite? What is the name given to this orbital period? What is the velocity of a geostationary satellite in its orbit? Give your answer in km/s.

A particular shuttle mission released a TDRSS satellite into a circular low orbit, with an orbital height of 270 km. The shuttle orbit was inclined to the earth's equator by approximately 28^{o}. The TDRSS satellite needed to be placed into a geostationary transfer orbit (GTO) once released from the shuttle cargo bay, with the apogee of the GTO at geostationary altitude and the perigee at the height of the shuttle's orbit. (i) What was the eccentricity of the GTO? (ii) What was the period of the GTO? (iii) What was the difference in velocity of the satellite in GTO between when it was at apogee and when it was at perigee? Note: assume the average radius of the earth is 6,378.137 km and Kepler's constant has the value 3.986004418×10^5 km^3/s^2.

Solution to question 6

A *geostationary* satellite orbit is one that has zero inclination to the equatorial plane, is perfectly circular (eccentricity is zero), and is at the correct orbital height to remain apparently stationary in orbit as viewed from the surface of the earth. A *geosynchronous* satellite orbit has most of the attributes of a geostationary orbit, but is either not exactly circular, not in the equatorial plane, or not at exactly the correct orbital height.

From Table 2.1, the orbital period of a geostationary satellite is 23 hours, 56 minutes, and 4.1 seconds.

The orbital period of a geostationary satellite is called a sidereal day.

From Table 2.1, the velocity of a geostationary satellite is 3.0747 km/s.

(i) The Geostationary Transfer Orbit (GTO) will have an apogee of 35,786.03 km (the geostationary altitude) and a perigee of 270 km (the release altitude of the TDRSS).

The semimajor axis $a = (2r_e + h_p + h_a)/2 = (2 \times 6378.137 + 270 + 35{,}786.03)/2 =$ 24,406.152 km

From equation (2.27) and example 2.1.3, $r_o = r_e + h_p$ and the eccentric anomaly $E = 0$ when the satellite is at perigee. From equation (2.27) $r_o = a(1 - e\cos E)$, with $\cos E = 1$. Therefore, $r_e + h_p = a(1 - e)$ and, rearranging the equation, $e = 1 - (r_e + h_p)/a = 1 -$ $(6378.137 + 270)/24{,}406.152 = 0.727604$. The eccentricity of the GTO is 0.728.

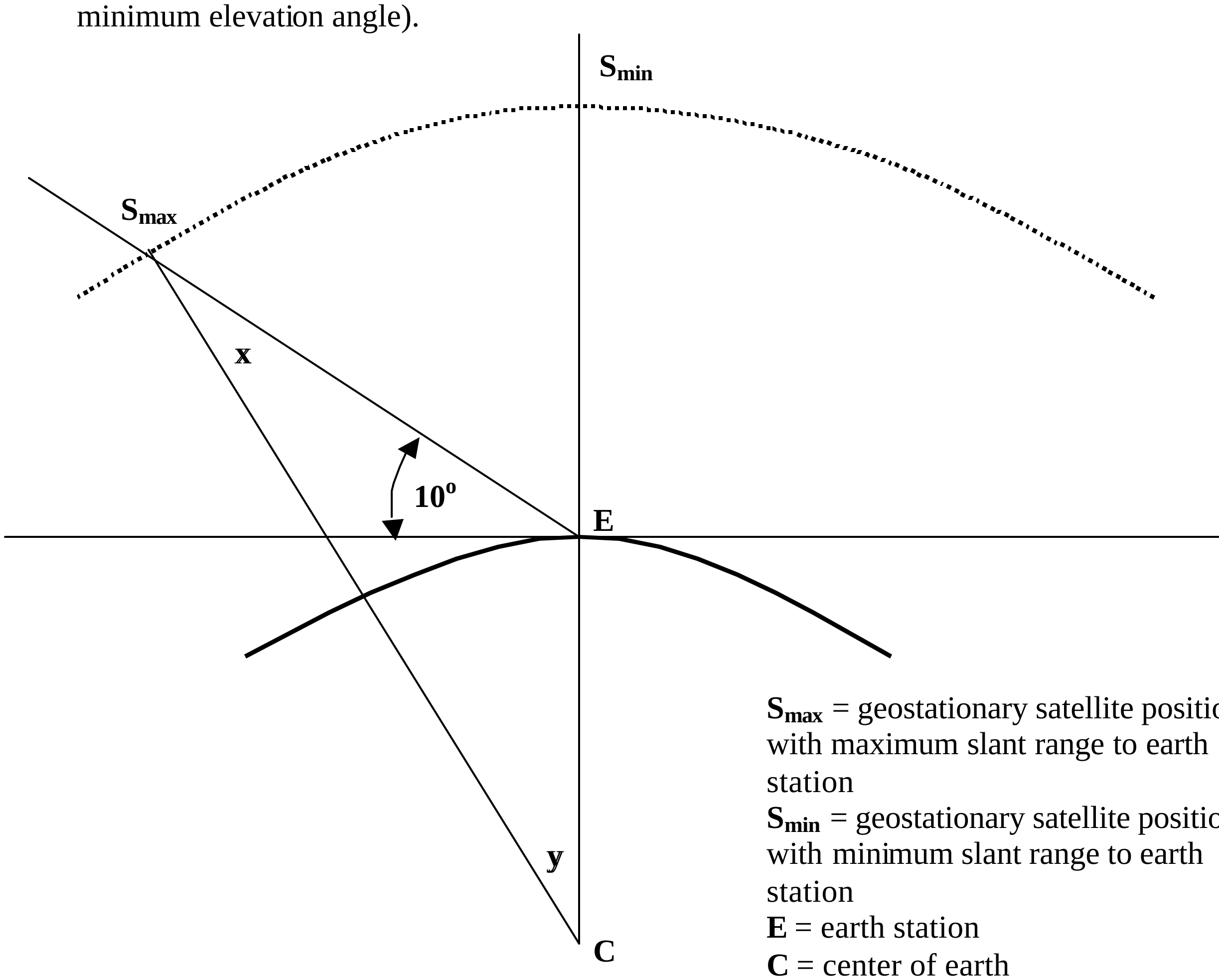
(ii) The orbital period $T = ((4\pi^2 a^3)/\mu)^{1/2} = ((4\pi^2 \times 24{,}406.152^3)/\ 3.986004418 \times 10^5)^{1/2} =$ 37,945.47102 seconds = 10 hours 32 minutes 25.47 seconds.

Question 7

For a variety of reasons, typical minimum elevation angles used by earth stations operating in the commercial Fixed Services using Satellites (FSS) communications bands are as follows:
C-Band 5^o; Ku-Band 10^o; and Ka-Band 20^o.
(i) Determine the maximum and minimum range in kilometers from an earth station to a geostationary satellite in the three bands. (ii) To what round-trip signal propagation times do these ranges correspond? You may assume the signal propagates with the velocity of light in a vacuum even when in the earth's lower atmosphere.

Solution to question 7

Below is a schematic of one of the three frequency situations, in this case Ku-band (10^o minimum elevation angle).



S_{max} = geostationary satellite position with maximum slant range to earth station
S_{min} = geostationary satellite position with minimum slant range to earth station
E = earth station
C = center of earth
r_e = earth radius = 6378.137 km = **CE**
h_s = height of satellite = **ES_{min}** = 35,786.03 km

Part (i)
By the law of sines, $CS_{max}/\sin 100^o = CE/\text{sine}\xi$ giving
$(35{,}786.03 + 6378.137)/\sin 100^o = 6378.137/\sin\xi$ and thus
$\sin\xi = (\sin 100^o \times 6378.137)/42{,}164.167 = 0.1489710$ and so $\xi = 8.5673001$

Angle $\psi = 180 - 100 - \xi = 71.4326999$

Again, by the law of sines, $ES_{max}/\sin\psi = CS_{max}/\sin 100^o$ giving
$ES_{max} = (42{,}164.167 \times \sin 71.4326999)/\sin 100 = 40{,}586.12894$ km

For a minimum elevation angle of 5^o, $\xi = 8.6671185$, $\psi = 180 - 95 - 8.86671185 = 76.3328815^o$, and $ES_{max} = 41.126.78334$ km

For a minimum elevation angle of 20^o, $\xi = 8.1720740$, $\psi = 180 - 110 - 8.1720740 = 61.8279266^o$, and $ES_{max} = 39{,}554.56520$ km

Maximum ranges are 41,126.78 km (C-band), 40,586.13 km (Ku-band), and 39,554.57 km (Ka-band).

The minimum range will be the same for all three frequencies of operation = 35,786.03 km, which assumes the earth station is on the equator and the satellite is directly above the earth station.

Part (ii)
The round trip propagation times, assuming the velocity of light is 2.997×10^8 m/s, are given by distance/velocity = $2 \times 41{,}126.78$ km/2.997×10^8 m/s (C-band), $2 \times 40{,}586.13$ km/2.997×10^8 m/s (Ku-band), and $2 \times 39{,}554.57$ km/2.997×10^8 m/s (Ka-band) = 274.5, 270.8, and 264 milliseconds, respectively.

Question 8

Most commercial geostationary communications satellites must maintain their orbital positions to within $\pm 0.05^o$ of arc. If a geostationary satellite meets this condition (i.e. it has an apparent motion $\pm 0.05^o$ of arc N-S and $\pm 0.05^o$ of arc E-W, as measured from the center of the earth), calculate the maximum range variation to this satellite from an earth station with a mean elevation angle to the center of the satellite's apparent motion of 5^o. You may assume that the equatorial and polar diameters of the earth are the same.

Solution to question 8

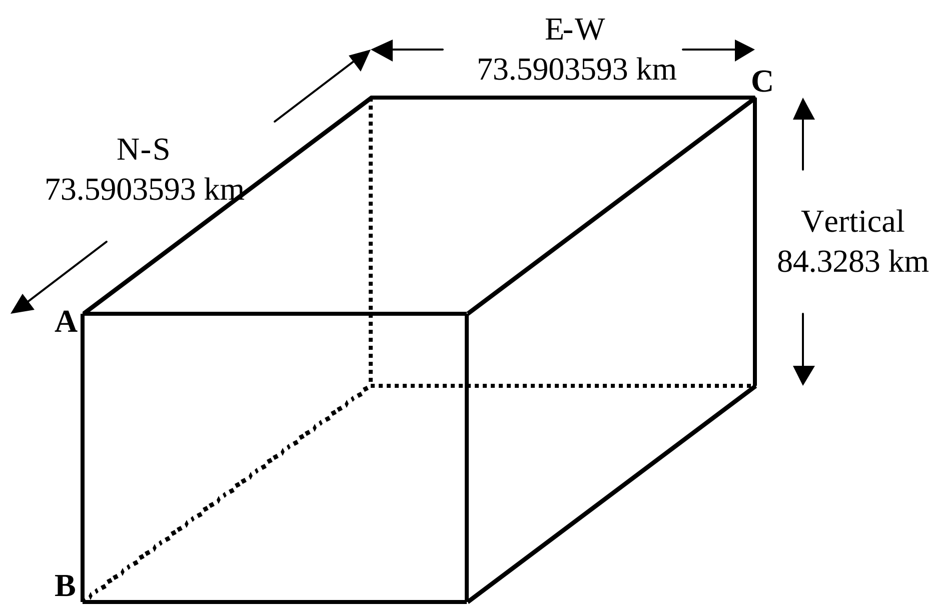
No information was given on eccentricity in this question, so we will assume that it is 0.001.

We can treat the station keeping variation of the satellite as if it were in a box with sides given by (in vertical extent) the eccentricity of the orbit and (in longitudinal and latitudinal extent) the inclination of the orbit.

If Δr is the difference between the maximum orbit radius and the average radius = the difference between the minimum orbit radius and the average radius, then the total vertical movement is $2\Delta r$. The total vertical motion $2\Delta r = r_{max} - r_{min} = a(1 + e) - a(1 - e) = 2ae$, where a is the average orbit radius and e is the eccentricity of the orbit. Thus $2\Delta r = 2 \times 42{,}164.17 \times 0.001 = 84.3283$ km. Alternatively, using equation (10.4), $\Delta r = \pm eR_{av}$, where R_{av} = the average orbit radius, giving $\Delta r = \pm(0.001) \times 42{,}164.17 = \pm 42.16417$, and so the total vertical motion = $2 \times 42.16417 = 84.3283$ km.

The lateral movement of the satellite N-S (longitude) and E-W (latitude) in its orbit is obtained by multiplying the orbit radius by the angular movement in radians. The angular movement is $\pm\ 0.05^o = 0.1^o$ total $\Rightarrow$ 0.0017453 radians and so the physical movement = $0.0017453 \times 42{,}164.17 = 73.5903593$ km.

The movement "box" is shown below.

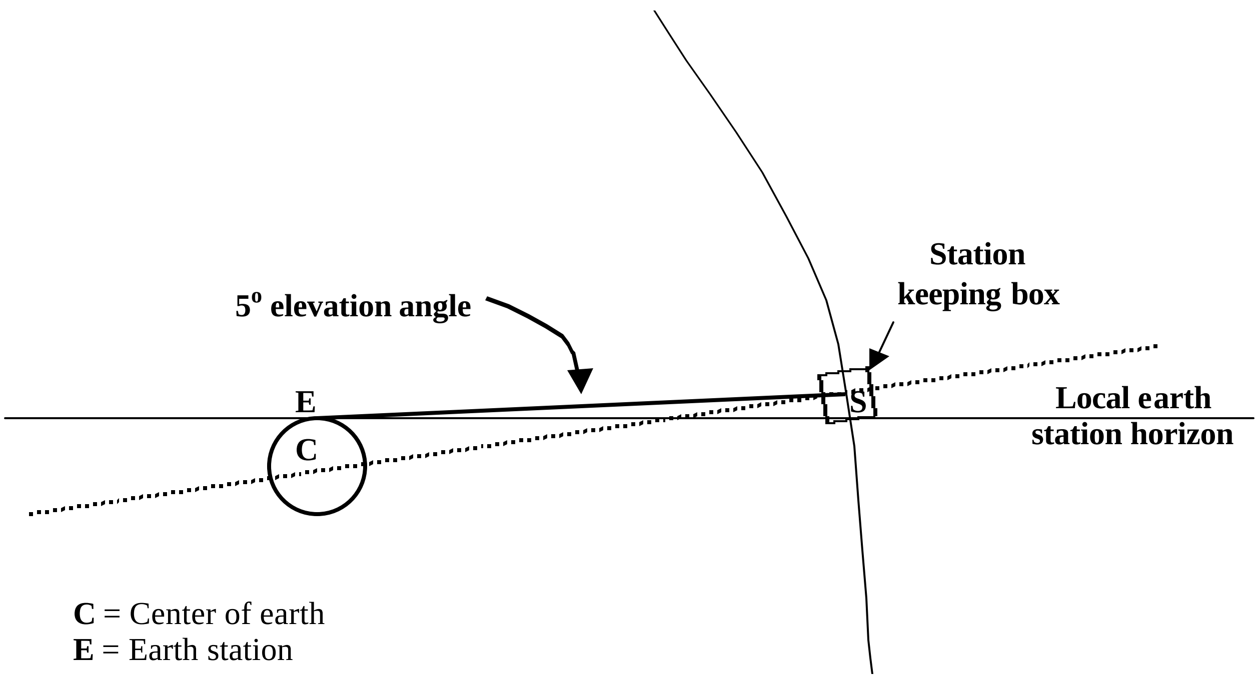


The center of the "box" is the nominal position of the satellite, with excursions up and down given by the eccentricity of the orbit and N-S and E-W given by the inclination of the orbit. Diagonal BC across the box is the maximum movement of the satellite (but note that there are also three other similar lengths within the box, created by joining the relevant opposing corners of the box.). Using Pythagorean geometry, we have

$BC^2 = AC^2 + AB^2$ and $AC^2 = 73.5903593^2 + 73.5903593^2 = 10{,}831.08196$

Thus $BC^2 = 10{,}831.08196 + 84.3283^2 = 10{,}831.08196 + 7{,}111.262181 = 17{,}942.34414$, which gives BC = 133.9490356 km = 133.95 km.

The range variation observed by an earth station will depend on the precise coordinates of the earth station. While the elevation angle (5^o) was given, the earth station coordinates were not. At this elevation angle, the situation looks like that below.



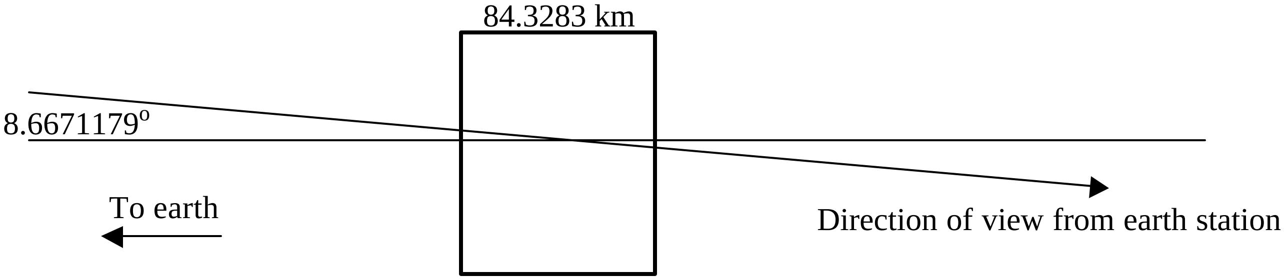
C = Center of earth
E = Earth station
S = Satellite

For an elevation angle of 5^o, Angle CES = 95^o. By the law of sines, we have

$$\sin ESC = \frac{6378.137 \times \sin 95}{42164.17} = 0.1506935$$

Giving angle ESC = 8.6671179^o

The situation inside the station-keeping box is as below



The maximum range variation in this case is 84.3283 × (cos8.6671179°) = 83.3653157 = 83.365 km.

Question 9

An interactive experiment is being set up between the University of York, England (approximately 359.5°E, 53.5°N) and the Technical University of Graz, Austria (approximately 15°E, 47.5°N) that will make use of a geostationary satellite. The earth stations at both universities are constrained to work only above elevation angles of 20° due to buildings, etc., near their locations. The groups at the two universities need to find a geostationary satellite that will be visible to both universities simultaneously, with both earth stations operating at, or above, an elevation angle of 20°. What is the range of sub-satellite points between which the selected geostationary satellite must lie?

Solution to question 9

From equation (2.38), where *El* = elevation angle,

$$\cos(El) = \frac{\sin(g)}{\left[1.02288235 - 0.30253825\cos(g)\right]^{1/2}}$$

Let cos(γ) = X, C = cos (*El*), A = 1.02288235, B = 0.30253825, and so $(1 - X^2)^{1/2} = \sin(\gamma)$

The equation relating *El* to γ can be written as a quadratic in X and solved by the quadratic formula as follows:

$1 - X^2 = C^2(A - BX)$

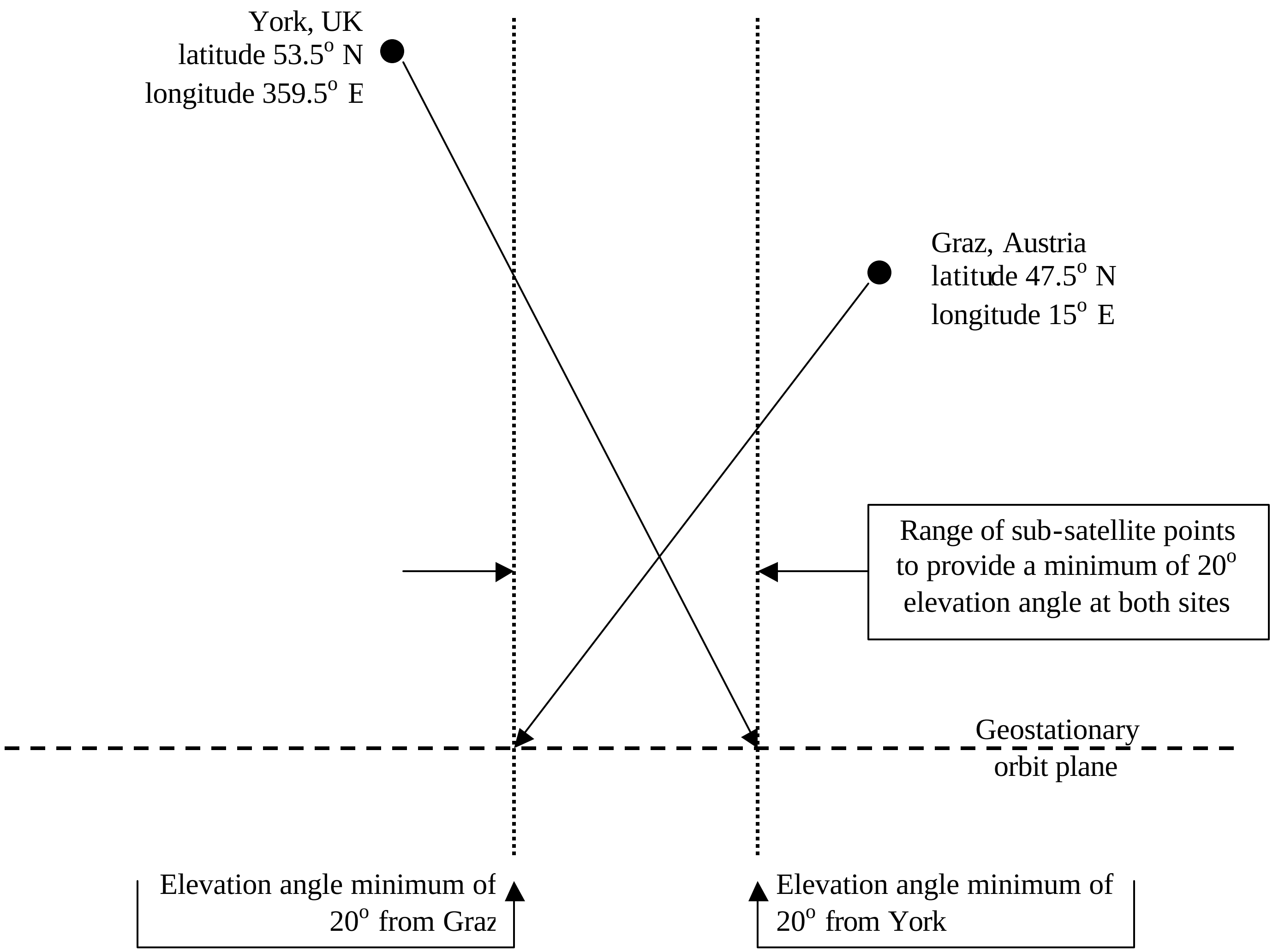
(NOTE: this is the cos (*El*) equation written with the new parameters, and the subject of the equation changed)

Solving gives

$$X = \frac{C^2B \pm (C^4B^2 \ - 4(C^2A - 1))^{1/2}}{2}$$

The correct solution is the positive root. With a minimum elevation angle of 20°, C = 0.9396926 giving X = 0.4721212, and thus γ = $\cos^{-1}$(0.4721212) = 61.8279251°.

The geometry of the linked pair of earth stations is as shown in the schematic below.



From equation (2.36)

$\mathrm{Cos}(\gamma) = \cos(Le)\cos(ls - le)$

For York: cos(61.8279251) = cos(53.5) cos(ls – le) giving cos(ls – le) = 0.7937174. The difference between the York longitude and the sub-satellite point is $\cos^{-1}(0.7937174)$ = 37.4657212°. Since York is at a longitude of 359.5°E, there are two solutions for the placement of the satellite (one east of York and one west of York). Only the placement east of York will be visible to Graz and so the east most placement of the satellite as far as York is concerned is 36.9657212°E = 36.97°E.

For Graz: cos(61.8279251) = cos(47.5) cos(ls – le) giving cos(ls – le) = 0.6988278. The difference between the Graz longitude and the sub-satellite point is $\cos^{-1}(0.6988278)$ = 45.6669685°. Since Graz is at a longitude of 15°E, there are two solutions for the placement of the satellite (one east of Graz and one west of Graz). Only the placement west of Graz will be visible to York and so the west most placement of the satellite as far as Graz is concerned is 329.3330315^{o}E = 329.33^{o}E

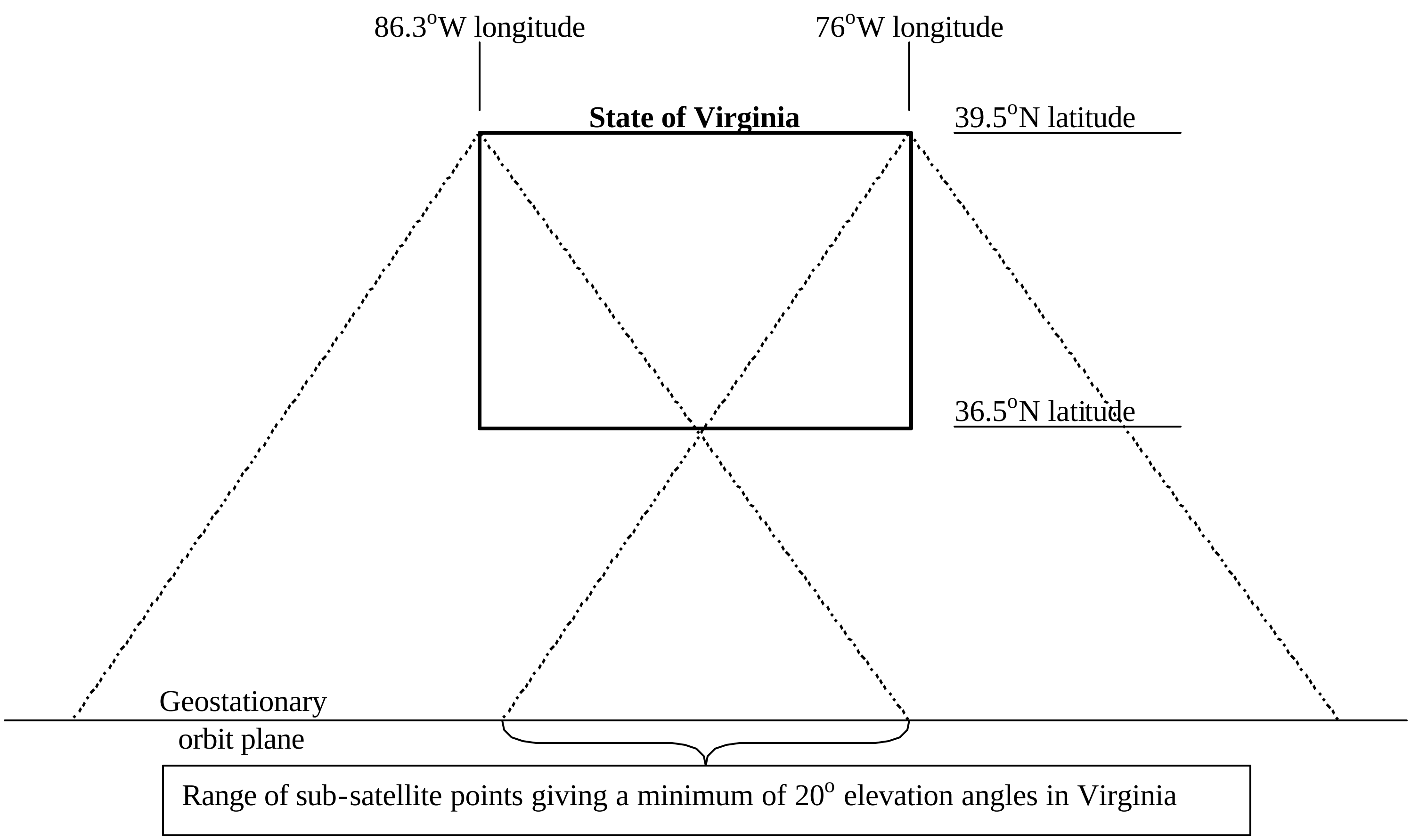
The range of sub-satellite points that will provide an elevation angle of at least 20^{o} to both York and Graz is from 329.33^{o}E to 36.97^{o}E.

Question 10

The state of Virginia may be represented roughly as a rectangle bounded by 39.5^{o} N latitude on the north, 36.5^{o}N latitude on the south, 76.0^{o} W longitude on the east, and 86.3^{o}W longitude on the west. If a geostationary satellite must be visible throughout Virginia at an elevation angle no lower than 20^{o}, what is the range of longitudes within which the sub-satellite point of the satellite must lie?

Solution to question 10

This question is similar in concept to the previous question, and the same quadratic solution may be applied. The schematic below illustrates the geometry of the question.



Using the same quadratic equation in the solution to question 9 gives $\gamma = 61.8279251^o$.

The lowest elevation angles will be at earth stations located at the north most corners of Virginia. That is at latitude 39.5^oN, with one at 76^oW and the other at 86.3^oW. The factor cos(Le) will be the same in both cases, cos(39.5) = 0.7716246, as will cos(ls – le), which will be cos(61.8279251)/(0.7716246) = 0.6118535, which yields a separation between ls and le of 52.2763585^o.

Thus the satellite may be approximately 52.3^o east or west of the earth station to remain at an elevation angle of 20^o. However, to enable the other northern corner of Virginia to still "see" the satellite at an elevation angle of at least 20^o, the satellite must be east of the west most earth station and west of the east most earth station. The east most northern earth station site is at 76^oW and the west most earth station site is at 86.3^oW. The sub-satellite points must therefore be between $76^oW + 52.2763585 = 128.2763585^oW$ and $86.3^oW – 52.2763585 = 34.0236415^oW$

Question 11

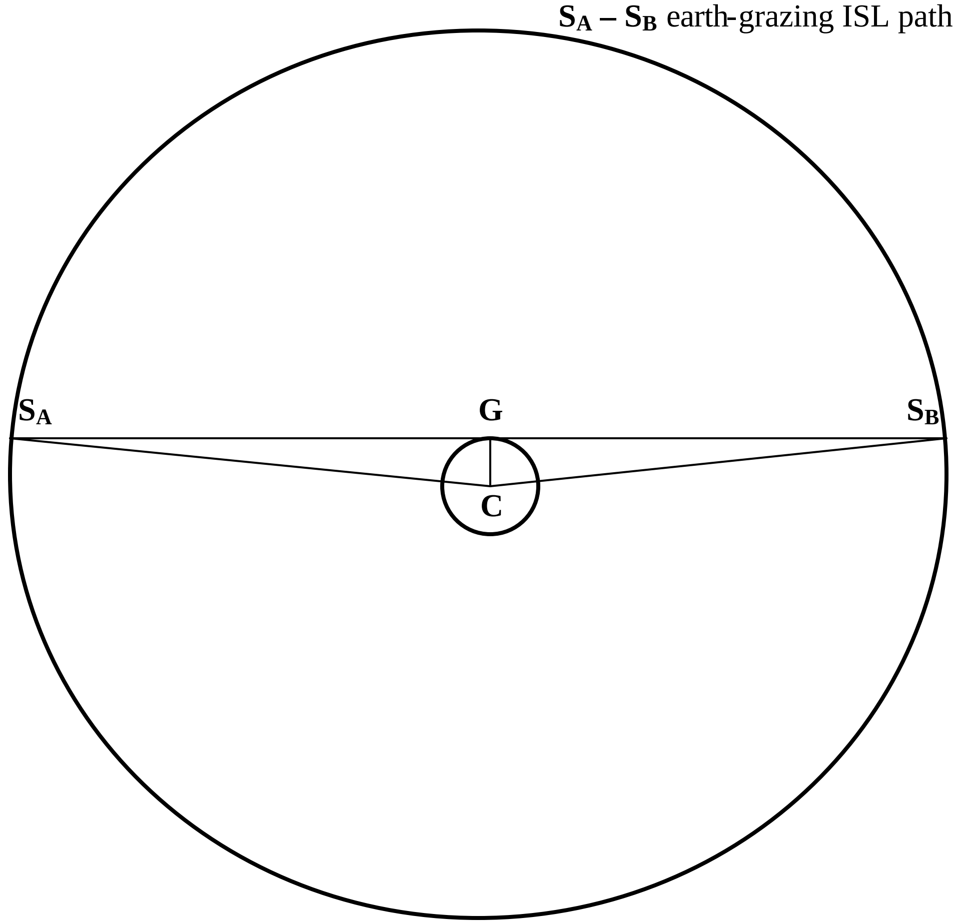
A geostationary satellite system is being built which incorporates inter-satellite links (ISLs) between the satellites. This permits the transfer of information between two earth stations on the surface of the earth, which are not simultaneously visible to any single satellite in the system, by using the ISL equipment to link up the satellites. In this question, the effects of ray bending in the atmosphere may be ignored, processing delays on the satellites may initially be assumed to be zero, the earth may be assumed to be perfectly circular with a flat (i.e. not hilly) surface, and the velocity of the signals in free space (whether in the earth's lower atmosphere or essentially in a vacuum) may be assumed to be the velocity of light in a vacuum.

(i) What is the furthest apart two geostationary satellites may be so that they can still communicate with each other without the path between the two satellites being interrupted by the surface of the earth? Give your answer in degrees longitude between the sub-satellite points. (ii) If the longest, one-way delay permitted by the ITU between two earth stations communicating via a space system is 400 ms, what is the furthest apart two geostationary satellites may be before the transmission delay of the signal from one earth station to the other, when connected through the ISL system of the two satellites, equals 400 ms? The slant path distance between each earth station and the geostationary satellite it is communicating with may be assumed to be 40,000 km. (iii) If the satellites in part (ii) employ on-board processing, which adds an additional delay of 35 ms in each satellite, what is the maximum distance between the ISL-linked geostationary satellites now? (iv) If both of the two earth stations used in parts (ii) and (iii) must additionally now send the signals over a 2,500 km optical fiber line to the end-user on the ground, with an associated transmission delay in the fiber at each end of the link, what is the maximum distance between the ISL-linked geostationary satellites now? You may assume a refractive index of 1.5 for the optical fiber and zero processing delay in the earth station equipment and end-user equipment.

Solution to question 11

A schematic of part (i) is shown below.

S_A = satellite A; **S_B** = satellite B; **C** = Center of earth
G = point where ISL path is tangential to earth
CS_A = GEO radius = 42,164.17 km
CG = radius of earth = 6378.137 km
S_A – S_B earth-grazing ISL path between satellites



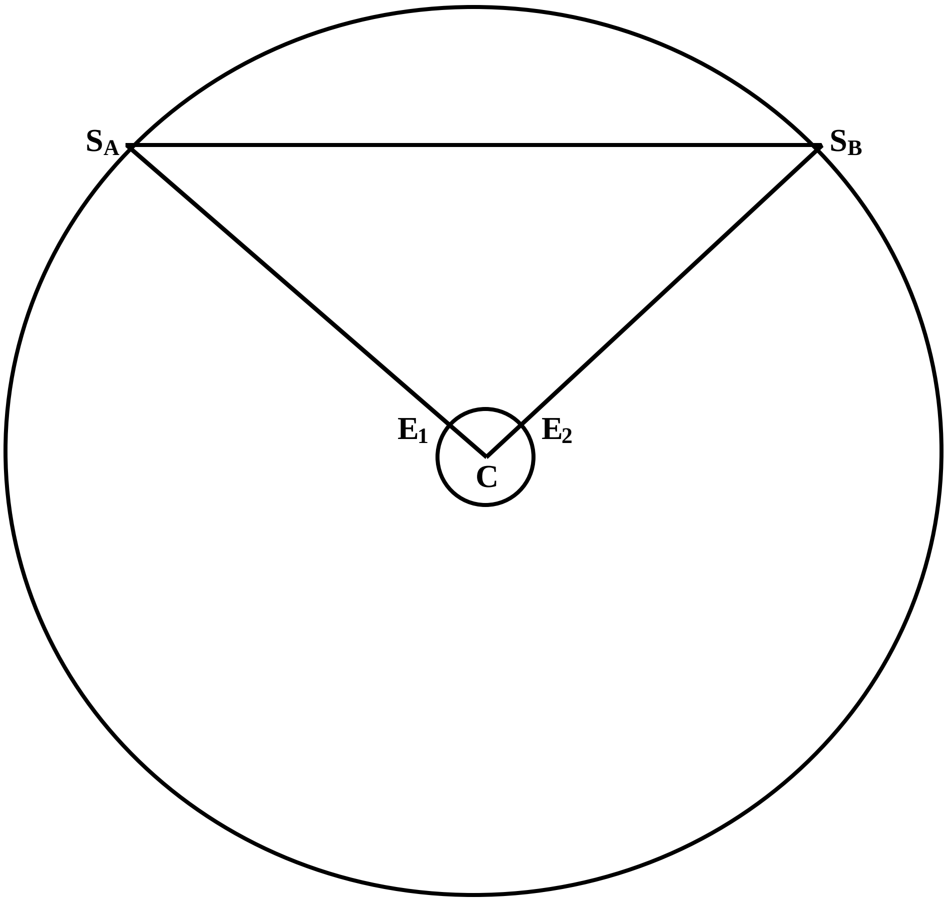
$(GS_A)^2 = (CS_A)^2 – (GC)^2$ = 1,777,817,232 – 40,680,699.86 = 1,737136,562,
Giving GS_A = 41,678.97026 km

By the law of sines, $\text{Sin}(S_ACG) = (S_AG \times \text{Sin}90^o)/(CS_A)$ = 41,678.97026/42,164.17 = 0.9884926, hence angle $S_ACG = 81.2995147^o$.

The difference in sub-satellite point between the two satellites with this geometry is twice angle $S_ACG = 162.6^o$

Part (ii)
A schematic of part (ii) is shown below.

$\mathbf{S_A}$ = satellite A; $\mathbf{S_B}$ = satellite B; **C** = Center of earth
$\mathbf{E_1}$ = Earth station 1 and $\mathbf{E_2}$ = earth station 2
$\mathbf{CS_A} = \mathbf{CS_B}$ = GEO radius = 42,164.17 km
$\mathbf{CE_1} = \mathbf{CE_2}$ = radius of earth = 6378.137 km
$\mathbf{S_A - S_B}$ ISL path between satellites



For a 400 ms delay and a signal velocity of 2.997×10^8 m/s, the total path length traversed must be velocity $\times$ time = $(2.997 \times 10^8 \text{ m/s}) \times (400 \times 10^{-3})$ = 119,880,000 m = 119,880 km.

$E_1S_A = E_1S_B$ = 42,164.17 km – 6378.137 = 35786.033 therefore S_AS_B = 119,880 – (35786.033×2) = 48,307.934 km.

The question, however, stipulates that the slant path distances between the earth stations and their respective satellites (i.e. distances E_1S_A and E_1S_B) are 40,000 km. Thus, S_AS_B = 119,880 – 80,000 = 39,880 km. The maximum physical separation between satellites to keep the end-to-end delay below 400 ms is 39,880 km, which translates into a separation between sub-satellite points of $56.4471538^o = 56.45^o$.

Part (iii)
In this part, an additional total delay of 70 ms occurs in the satellites (35 ms at each satellite), which reduces the total path length for the free space part of the link to $(2.997 \times 10^8$ m/s$) \times ((400 - 70) \times 10^{-3})$ = 98,901,000 m = 98,901 km. With this reduced free space path length available, the maximum physical separation of the two satellites is 98,901 – 80,000 = 18,901 km, which translates into a separation between sub-satellite points of 25.9^o.

Part (iv)
In this part, an additional transmission line exists at both ends of the link, totaling 5,000 km. The velocity of light in the fiber link is $(2.997 \times 10^8$ m/s)/refractive index = 1.998×10^8 m/s, which leads to an additional delay due to the fiber links of $5{,}000{,}000/1.998 \times 10^8$ = 0.0250250 s = 25×10^{-3} s. The available free space path length is therefore reduced to $(2.997 \times 10^8$ m/s$) \times ((400 - 70 - 25) \times 10^{-3})$ = 91,408,500 m = 91,408.5 km. With the on board processing delay and extended fiber optic transmission line at each end, the maximum physical separation of the two satellites is 91,408.5 – 80,000 = 11,408.5 km, which translates into a separation between sub-satellite points of 7.78^o.

Summarizing, we have found the angular separation of satellites connected via an Inter Satellite Link to be as follows:

(i) Maximum separation = 41,678.9702683,357.9 km $\equiv 162.6^o$ between satellites
(ii) With a free space delay of 400 ms, separation = 39,880 km $\equiv 56.45^o$ between satellites
(iii) With a free space delay of 400 ms and 70 ms of total on board processing delay, separation = 18,901 km $\equiv 25.9^o$ between satellites
(iv) With a free space delay of 400 ms, 70 ms of total on board processing delay, and 25 ms total back haul delay at each end of the link, separation = 11,408.5 km $\equiv 7.7^o$.

A 2,500 km back haul link is a typical design requirement, thus on board processing delays need to be kept to a minimum if ISLs are to operate successfully in real time links.