Joint space calculation path planning

* to ensure the smoothness of the joints in the manipulator, the angular velocity & angular acceleration must be conted. (Jerky motion are not preferred).

* Trajectory functions (common techiques)

1. Cubic polynomial (3rd order)
2. Fifth order polynomial (Quintic polyn)
3. Linear Trajectory planning.

I. Cubic Polynomial.

~~Case 1~~ Conditions :→ Initial & final values of Joint angle are known.

→ Angular velocity at begining & end of the cycle are kept Zero.

1^{st} condition: ① At time, $t = 0 = t_i$.

$\theta = \theta_i$. $\dot{\theta} = 0$.

② At time, $t = t_f$.

$\theta = \theta_f$, $\dot{\theta} = 0$.

* the order of polynomial is based on no. of boundary conditions. If 'n' is no. of boundary conditions, order of polynomial is $(n-1)$.

* in general 4 boundary conditions are there $\boxed{t_i = t_f}$ $\theta_i = \theta_f = $ known. $v_i = v_f \Rightarrow \dot{\theta} = 0$.

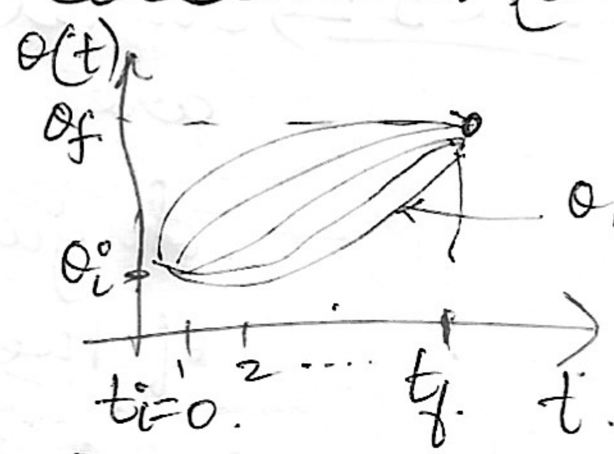
No. of boundary condition $= 4$ so order of polynomial is 3.

$$\theta(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \dots$$

Angular Displacement

$$\boxed{\theta(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3} \quad \text{Eq (1)}$$

is generic trajectory cubic polynomial based on 4 boundary conditions (Constrai



From eq (1),

Angular velocity

$$\boxed{\dot{\theta}(t) = \frac{d\theta}{dt} = c_1 + 2c_2 t + 3c_3 t^2} \quad \text{Eq (2)}$$

at $t=0$; $\theta = \theta_i$.

Boundary Condt (1)

$\therefore$ Eq. (1) $\Rightarrow$ $\theta_i = c_0 + c_1(0) + c_2(0) + c_3(0)$

$\theta_i = c_0$. ie) $\boxed{c_0 = \theta_i}$ — Eq (3)

initial case for displacem

$\cancel{\text{Eq (2)} \Rightarrow \dot{\theta}_i(t) = [c_1 + 2c_2(0) + 3c_3(0)]}$

$\cancel{\dot{\theta}_i = 0}$

boundary condⁿ ②

sub. in ②. $t=0$, $\dot{\theta}=0$ [initial case for velocity]

$$\dot{\theta}(t) = c_1 + 2c_2(t) + 3c_3t^2.$$

$$0 = c_1 + 0 + 0.$$

ie) $\boxed{c_1 = 0}$ → Eq. ④

boundary condition ③ & ④. [final case for displacement & velocity]

$t = t_f$.

$$\theta_f = c_0 + c_1 t_f + 2c_2 t_f^2 + 3c_3 t_f^3 \rightarrow \text{Eq } ⑤$$

$$\dot{\theta}_f = c_1 + 2c_2 t_f + 3c_3 t_f^2.$$

since final velocity is zero,

$$0 = c_1 + 2c_2 t_f + 3c_3 t_f^2.$$

WKT $c_1 = 0$.

$$\therefore 0 = \cancel{c_1} + 2c_2 t_f + 3c_3 t_f^2.$$

ie) $\boxed{c_2 = -\frac{3}{2} c_3 t_f}$ → Eq. ⑥

~~substituting all values in $\dot{\theta}(t)$~~

~~$\theta(t) = \theta i +$~~ ~~$c_3 = c - 2c_2 t_f$~~

~~In Eq ⑤ subs. c_0 & c_1~~

$$\cancel{\theta_f = \theta i + 0 + 2c_2 t_f^2 + 3c_3 t_f^3.}$$

$$\cancel{\theta_f - \theta i = 2c_2 t_f^2 + 3c_3 t_f^3.}$$

$$\cancel{\theta_f - \theta i = -\frac{3}{2} c_3 t_f^3 + 3c_3 t_f^3}$$

④

similarly ... $C_3 = \frac{-2(\theta_f - \theta_i)}{t_f^3}$.

subs. this C_3 in eq. ⑥

$$C_2 = \frac{-3}{\cancel{2}} \cdot \frac{(-\cancel{2}(\theta_f - \theta_i))\cancel{t_f}}{t_f^{\cancel{3}\,2}}$$

$$\boxed{C_2 = \frac{3(\theta_f - \theta_i)}{t_f^2}}$$

subs. all values in $\dot{\theta}(t)$ & $\theta(t)$.

$$\theta(t) = \theta_i + 0 + \frac{3(\theta_f - \theta_i)}{t_f^2} \cdot t^2 - \frac{2(\theta_f - \theta_i)}{t_f^3} \cdot t^3$$

$$\dot{\theta}(t) = 2 \times \frac{3(\theta_f - \theta_i)}{t_f^2} \cdot t - \frac{3 \times 2(\theta_f - \theta_i)}{t_f^3} \cdot t^2$$

$$= \frac{6(\theta_f - \theta_i)t}{t_f^2} - \frac{6(\theta_f - \theta_i)t^2}{t_f^3}$$

$$\dot{\theta}(t) = 6(\theta_f - \theta_i)\left[\frac{t}{t_f^2} - \frac{t^2}{t_f^3}\right]$$

Thus if θ_i, θ_f + t_f are known then $\theta(t)$ & $\dot{\theta}(t)$ ie ~~dis~~ displacement & angular velocity can be found out.

P1) A robot arm has initial & final positions as 10° & 40°. The link connected to the joint star from rest & reaches final position in 3 sec. & becomes motionless. Find the cubic polynomial trajectory function for the sam

Soln: $\theta = \theta_i = 10^\circ$ $\quad t = t_0 = 0$ (rest)

$\theta = \theta_f = 40^\circ$. $\quad t = t_f = 3$ sec (final)

At $t=0$ & $t=t_f$ } $\dot{\theta} = 0$. & $\dot{\theta}_f = 0$.

so $\theta_i = 10^\circ$ $\quad \theta_f = 40^\circ$ $\quad t_f = 3$.

subs. in expression for 3rd order (cubic) polynomial.

$$C_0 = \theta_i = 10 \qquad C_1 = 0$$

$$C_2 = \frac{3(\theta_f - \theta_i)}{t_f^2} = \frac{3(40-10)}{3^2} \Rightarrow C_2 = 10.$$

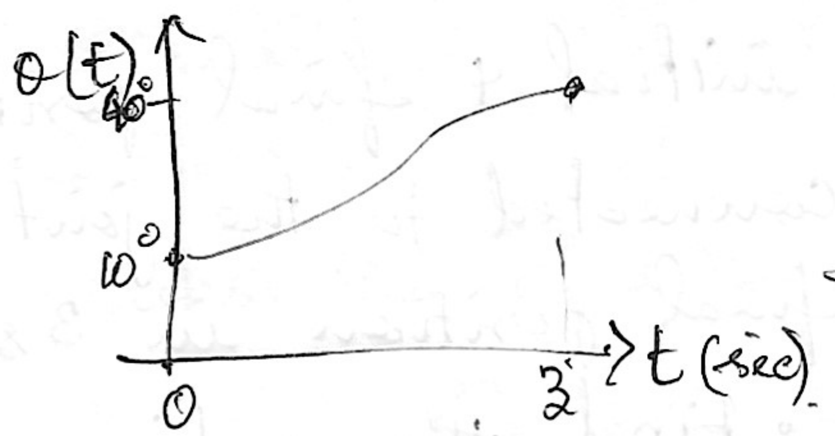
$$C_3 = \frac{-2(\theta_f - \theta_i)}{t_f^3} = \frac{-2(40-10)}{3^3} \Rightarrow C_3 = -2.22$$

Displacement trajectory is

$$\therefore \theta(t) = C_0 + C_1(t) + C_2 t^2 + C_3 t^3.$$

$$\theta(t) = 10 + 0 + 10t^2 - 2.22t^3.$$

$$\boxed{\theta(t) = 10 + 10t^2 - 2.22t^3.}$$



Also velocity trajectory is

$$\boxed{\dot{\theta}(t) = 20t^2 - 6.66t^2.}$$

Case 2: * Initial & final joint angles are known
& Initial & final angular velocities are non zero.

$t = t_i = 0$. $\theta = \theta_i, \ \dot{\theta}_i = \dot{\theta}_i$

$t = t_f$. $\theta = \theta_f, \ \dot{\theta} = \dot{\theta}_f$

4 boundary conditions, so 3^{rd} order polynomial (cubic).

$$\theta(t) = \theta_i + \dot{\theta}_i \{ \quad \} t_f^2 + [\quad] t_f^3.$$

Case 3: θ_i, θ_f → displacement is known.

$\dot{\theta}_i, \dot{\theta}_f$ → angular velocity at begining & final known

$\ddot{\theta}_i, \ddot{\theta}_f$ → angular acceleration at begining & final known.

So 6 boundary conditions,

Order of polynomial is $(6-1) = 5^{th}$ order.

$$\theta(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5$$

The same is called as Quintic Polynomial Trajectory in matrix form

$$\begin{bmatrix} \theta_0 \\ \omega_0 \\ \alpha_0 \\ \theta_f \\ \omega_f \\ \alpha_f \end{bmatrix} = \begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 & t_0^4 & t^5 \\ 0 & 1 & 2t_0 & 3t_0^2 & 4t_0^3 & 5t_0^4 \\ 0 & 0 & 2 & 6t_0^2 & 12t_0^2 & 20t_0^3 \\ 1 & t_f & t_f^2 & t^3 & t_f^4 & t_f^5 \\ 0 & 1 & 2t_f^2 & 3t_f^2 & 4t_f^3 & 5t_f^4 \\ 0 & 0 & 2 & 6t_f & 12t_f^2 & 20t_f^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

If the 't' matrix is know we can find $a_0\, a_1 \ldots a_5$ by computing inverse.

ie).
$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_5 \end{bmatrix} = \begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 & t_0^4 & t^5 \\ \vdots & & & & & \\ 0 & \ldots & & & & 20t_f^3 \end{bmatrix}^{-1} \begin{bmatrix} \theta_0 \\ \omega_0 \\ \alpha_0 \\ \theta_f \\ \omega_f \\ \alpha_f \end{bmatrix}$$

Why can't we use cubic polynomial used for 6 boundary condition.

$$\theta(t) = c_0 + c_1(t) + c_2(t^2) + c_3(t^3)$$

$$\dot{\theta}(t) = c_1 + 2c_2 t + 3c_3 t^2$$

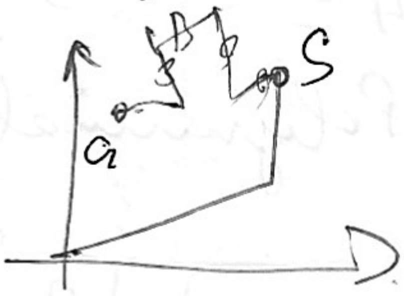
$$\ddot{\theta}(t) = 2c_2 + 6c_3 t^2.$$

At $t=0$. $\ddot{\theta}(t_0) = \text{Const}\ldots$ ie) $\ddot{\theta}_0 = \text{Const}.$

which is impractical.

⑧

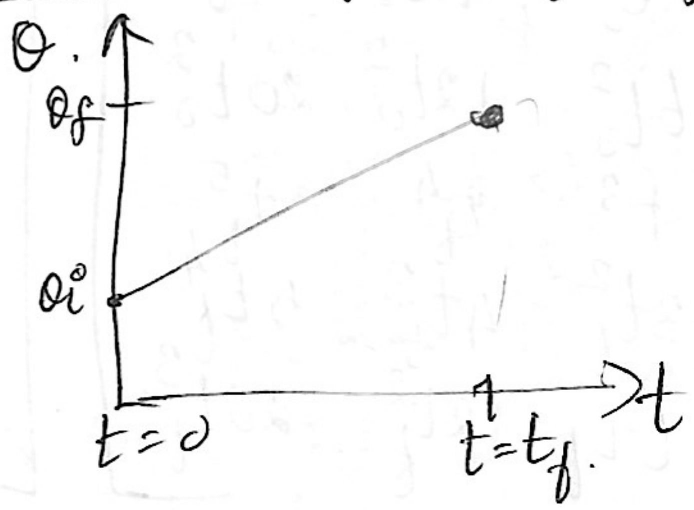
If there are obstacles in the trajectory



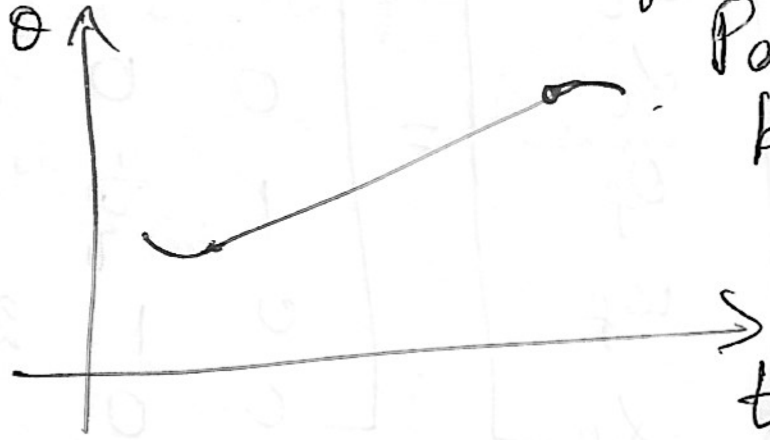
→ we go for single high order polynomial

→ 'm' points → (m+1) cubic polynomial

Linear Trajectory function

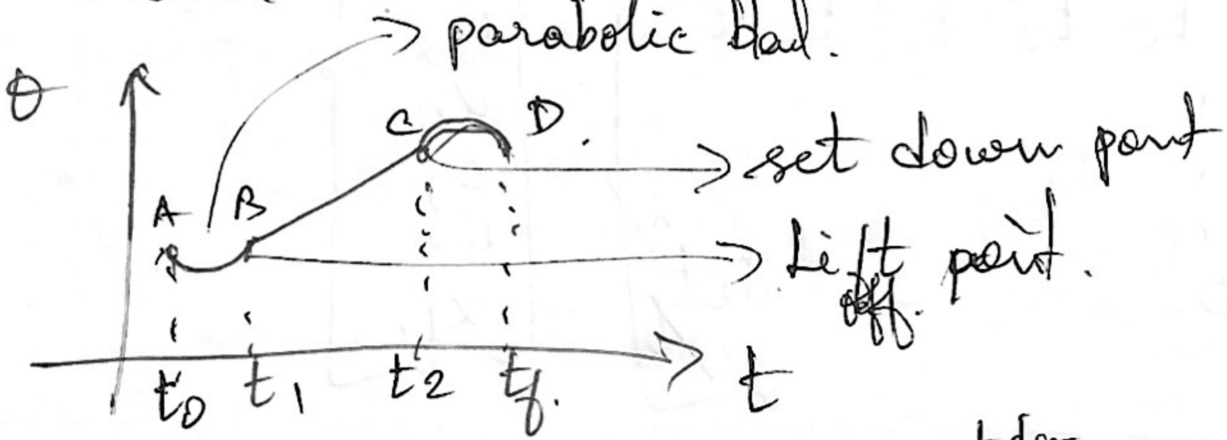


~~modifi.~~

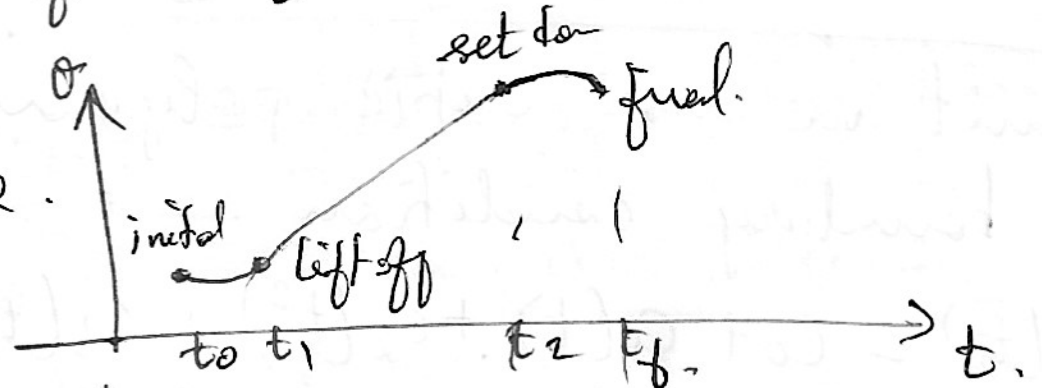


Parabolic blend

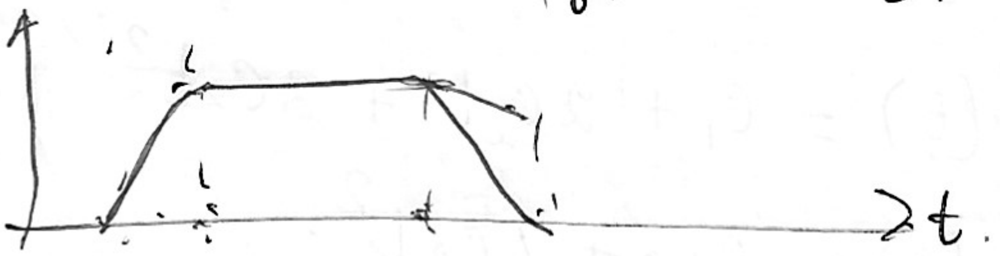
* Parabolic blend is modification in linear trajectory fn. to avoid jerks in beginning & end.



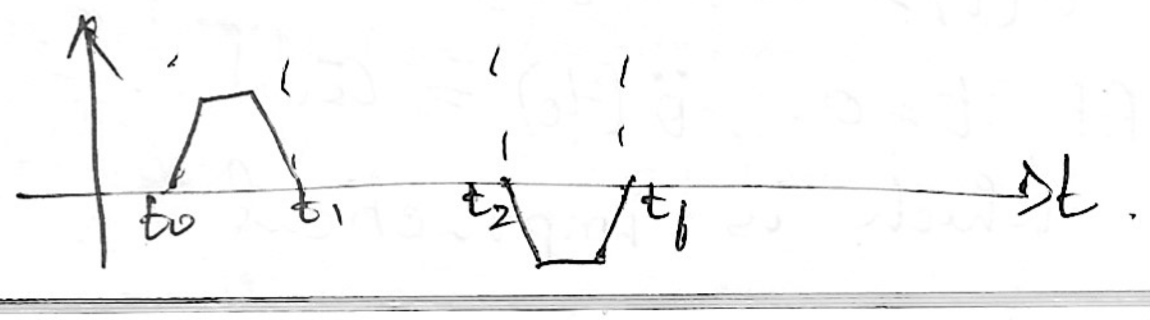
Path profile.



Velocity profile.



Acceleration profile.



parameters are

1) Total cycle time $= t_c = t_f - t_0$.

2) Time duration of each blend. (A to B). (C to D) $\} = \boxed{t_1 - t_0 = t_f - t_2}$

3) Angular displacement & velocity at the 2 junction of parabolic blend should be found out. $\}$ $\boxed{\theta(t_1) = \theta(t_0) - \frac{1}{2}\ddot{\theta}t_b^2}$

[WKT, $S = ut + \frac{1}{2}at^2$ physics.]

here $\ddot{\theta}$ is magnitude of acceleration or deceleration

4) $\dot{\theta}(t_1) = \dot{\theta}(t_0) + \ddot{\theta} \times t_b$. (WKT $v = u + at$)

5). $\theta(t_2)$:-

$\theta_{t_f} - \theta_{t_2} = \theta_{t_1} - \theta_{t_0}$. → due to symmetry.

$\therefore \theta(t_2) = \theta(t_f) - \theta(t_1) - \theta(t_0)$.

$$\dot{\theta}(t_2) = \frac{\theta(t_2) - \theta(t_1)}{t_2 - t_1}$$ (WKT $v = \frac{s}{t}$)

⟹ PTO

Eg: Calculate the angular velocity & angular displacement based on parabolic blend process

$t = t_i = 0$ $\quad \theta_i = 20$ $\quad \dot{\theta} = 0$. $\quad t_b = 3$ sec

$t = t_f = 12$ sec. $\quad \theta_f = 74^0$, $\quad \dot{\theta} = 0$. $\quad \ddot{\theta} = 2^0$ deg/sec

what are angular displacements?

$\theta(t_1) \Rightarrow$ angular displacement at t_1

$= \theta(t_0) + \frac{1}{2}\ddot{\theta}(t_b^2)$

$= 20 + \frac{1}{2} \times 2 \times 3.$

$\theta(t_1) = 29$ deg.

similarly.

$\dot{\theta}(t) = 6$ deg.

$\theta(t_2) = \theta(t_f) - \theta(t_1) + \theta(t_0)$

$= 74 - 29 + 20$

$= 65^0.$

$\dot{\theta}(t) = 6$ deg. $= \dot{\theta}(t)$

So ang. vel at t_1 & t_2 are same, so no jerky movements.

$\Rightarrow$ PTO. For parabolic blends,

$\theta_b = \theta_0 + \frac{1}{2}\ddot{\theta} t_b^2$

$t_b = \frac{t}{2} - \frac{\sqrt{\ddot{\theta}^2 t^2 - 4\ddot{\theta}(\theta_f - \theta_0)}}{2\ddot{\theta}}$

$\ddot{\theta} \geq \frac{4(\theta_f - \theta_0)}{t^2}$

Path Trajectory.

$\theta(t) = \theta_0 + \frac{1}{2}\ddot{\theta}t^2$; $0<t<$ (blen

$\theta(t) = \theta_b + \ddot{\theta} t_b (t - t_b)$; $t_b \leq t < t_f - t_b$ (lin

$\theta(t) = \theta_f - \frac{1}{2}\ddot{\theta}(t - t_f)^2$; $t_f - t_b \leq t \leq t_f$ (blend2

$t_b \Rightarrow$ blend 1 = blend 2

P2) A . B . C .

1 DOF.

	A	B	C
given	$\theta_0 = 2^0$	$\theta_v = 15^0$	$\theta_f = 20^0$
	$\dot{\theta}_0 = 0.$	$\dot{\theta}_v = 0$	$\dot{\theta}_f = 0.$

$t = 2$ $\qquad$ $t = 2.$

Soln: 2 function — A to B
$\qquad\qquad\qquad\qquad$ — B to C.

① From A to B.

at $t=0.$ $\quad \theta(0) = \theta_0 = a_0 + a_1(0) + a_2(0^2) + a_3(0^3)$

$$2 = a_0 + 0 + 0 + 0.$$

$$\boxed{\therefore a_0 = 2}$$

at $t = 2.$ $\quad \theta(2) = 2 + 0 + a_2(2^2) + a_3(2^3) = 15.$

rearranging $\quad 4a_2 + 8a_3 = 13$ ——— ①

$$\dot{\theta}(2) = 0 = 2a_2(2) + 3a_3(2^2)$$

$$\Rightarrow 4a_2 + 12a_3 = 0. \text{ ——— } ②$$

solving ① & ②.

$$a_2 = 9.75 \qquad a_3 = -3.25.$$

Repeating same for B to C.

$$b_0 = 15 \quad b_1 = 0 \quad b_2 = 3.75 \quad b_3 = -1.25$$

Final step is trajectory expression

$\theta(t) = 2 + 9.75t^2 - 3.25t^3 \quad (A \to B)$

$\theta(t) = 15 + 3.75t^2 - 1.25t^3 \quad (B \to C).$

P3) Arm moves from A to B.

$$A \xrightarrow{t=3} B$$

$\theta_0 = 15^\circ$ $\quad$ $\theta_f = 75^\circ$

$\dot{\theta}_0 = 0$ $\quad$ $\dot{\theta}_f = 0$.

Assume blend times of both sections are equal & velocity at end of blend is equal to velocity at linear section

Solu: $\ddot{\theta} \geq \dfrac{4(75-15)}{3^2} = 26.7^\circ$. choose $\ddot{\theta} = 30^\circ$.

$$t_b = \frac{3}{2} - \frac{\sqrt{30^2(3^2) - 4(30)(75-15)}}{2(30)} = 1 \text{ sec}.$$

$$\theta_b = 15 + \frac{1}{2}(30)(1^2) = 30^\circ.$$

subs. these into path trajectory equations.

for blend 1. $\quad \theta(t) = \theta_0 + \frac{1}{2}\ddot{\theta}t^2$
$(0 \leq t < 1)$ $\quad = 15 + 15t^2$

for linear $\quad \theta(t) = 30 + 30(1)(t-1)$
$(1 \leq t < 2)$

for blend 2. $\quad \theta(t) = 75 - \frac{1}{2} 30(t-3)^2$.
$(2 \leq t \leq 3)$ $\quad = 75 - 15(t-3)^2$

→ Run Time → path generator $(\theta, \dot{\theta}, \ddot{\theta})$ → feeds info control system.

→ for cubic polynomial mthd → a_0 a_1 a_2 ....

→ for parabolic blend. →

P4). Given a robot joint moves from $\frac{\pi}{3}$ to $\frac{7\pi}{4}$ rad in 4s, starting from rest & then stopping. Find an appropriate Quintic polynomial trajectory.

Solu: steps → ① specify initial & final condn.

② take $\frac{d}{dt}$, $\frac{d^2}{dt^2}$

③ matrix form $TC = Q$.

④ solve $C = T^{-1}Q$.

	θ (rad)	$\dot{\theta}$ (rad/sec)	$\ddot{\theta}$ (rad/sec²)	t (sec)
initial	$\pi/3$	0	0	0
final.	$7\pi/4$.	0	0	4

$$\theta(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5$$

$$\dot{\theta}(t) = c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4$$

$$\ddot{\theta}(t) = 2c_2 + 6c_3 t + 12c_4 t^2 + 20c_5 t^3.$$

Matrix form $\underset{(6\times6)}{T}\,\underset{(6\times1)}{C} = \underset{(6\times1)}{Q}$

$$\begin{bmatrix} T: \text{Coffs.} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_5 \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \dot{\theta}_0 \\ \ddot{\theta}_0 \\ \theta_f \\ \dot{\theta}_f \\ \ddot{\theta}_f \end{bmatrix}$$

At time $t=0$

$$\theta = c_0 \qquad \theta_f = c_0 + 4c_1 + 16c_2 + 64c_3 + 256c_4 + 1024c_5$$

$$\dot{\theta} = c_1 \qquad \dot{\theta}_f = c_1 + 8c_2 + 48c_3 + 256c_4 + 1280c_5$$

$$\ddot{\theta} = 2c_2. \qquad \ddot{\theta}_f = 2c_2 + 24c_3 + \cancel{\ } 192c_4 + 1280c_5$$

$$\begin{array}{c c}
 & \begin{array}{cccccc} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \end{array} \\
\begin{array}{c} \theta_0 \\ \dot{\theta}_0 \\ \ddot{\theta}_0 \\ \theta_f \\ \dot{\theta}_f \\ \ddot{\theta}_f \end{array} &
\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 \\
\hline
1 & 4 & 16 & 64 & 256 & 1024 \\
0 & 1 & 8 & 48 & 256 & 1280 \\
0 & 0 & 2 & 24 & 192 & 1280
\end{array}\right]
\end{array}
\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \dot{\theta}_0 \\ \ddot{\theta}_0 \\ \theta_f \\ \dot{\theta}_f \\ \ddot{\theta}_f \end{bmatrix} = \begin{bmatrix} \pi/3 \\ 0 \\ 0 \\ 7\pi/4 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}^{-1}_{6\times 6} \begin{bmatrix} \theta_0 \\ \dot{\theta}_0 \\ \ddot{\theta}_0 \\ \theta_f \\ \dot{\theta}_f \\ \ddot{\theta}_f \end{bmatrix}$$

Solving,

gives.

$c_0 = 1.047 \qquad c_1 = 0 \qquad c_2 = 0$

$c_3 = 0.695 \qquad c_4 = -2.608 \qquad c_5 = 0.0261$

$$\boxed{\therefore\ \theta(t) = 1.047 + 0.695t^3 - 2.608t^4 + 0.0261t^5.}$$