SCHOOL OF ELECTRONICS ENGINEERING (SENSE)

BECE313L: Information Theory and Coding

PROBLEM SHEET-1
Advanced Entropy Analysis & Channel Models

Instructions:

1. Total Marks: 20
2. Weightage of marks in grades : 4%
3. Last Date for Submission: 20.08.2024
4. All answers must be handwritten
5. Late submission are not allowed
6. Submission must be through teams

Address each problem with thorough analysis and detailed solutions.

Q.No	Question	Marks
1	A biased die has outcomes $\{1, 2, 3, 4, 5, 6\}$ with probabilities $P(1) = 0.1$, $P(2) = 0.2$, $P(3) = 0.2$, $P(4) = 0.2$, $P(5) = 0.2$, $P(6) = 0.1$. If you roll the die twice, compute the joint entropy of the outcomes. Analyze the impact of bias on the joint entropy. Using the probabilities, derive the entropy $H(X)$ of the biased die and compare it to the entropy of a fair die. Discuss the implications of entropy differences in terms of predictability and information content.	1
2	Consider two discrete random variables X and Y with the joint probability distribution: $$P(X,Y) = \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.4 \end{bmatrix}$$ • Evaluate the mutual information $I(X;Y)$. Discuss how mutual information quantifies the amount of information obtained about one random variable through the other and its relevance in real-world communication systems. • Compute the conditional entropy $H(Y\|X)$ and interpret its significance in the context of information redundancy and data compression. • Determine the joint entropy $H(X,Y)$ and the marginal entropies $H(X)$ and $H(Y)$. Assess how these entropies interrelate and contribute to the overall information content.	1
3	Suppose X is a continuous random variable uniformly distributed over the interval $[0, 1]$. Calculate the differential entropy $h(X)$. Discuss the concept of differential entropy in comparison to discrete entropy and its applications in signal processing.	1

4	Let X be a continuous random variable with probability density function $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. Derive the differential entropy $h(X)$ and analyze its dependency on the parameters μ and σ.	1
5	Consider two continuous random variables X and Y with the joint probability density function: $$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} + \frac{y^2}{\sigma_Y^2} - \frac{2\rho xy}{\sigma_X\sigma_Y}\right)\right)$$ Calculate the mutual information $I(X;Y)$ and evaluate the effect of the correlation coefficient ρ on the mutual information.	1
6	A Markov chain has states $\{S_1, S_2, S_3\}$ with transition matrix: $$P = \begin{bmatrix} 0.2 & 0.5 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$$ Determine the stationary distribution of the Markov chain and discuss its significance in predicting long-term behavior of the system. Using the stationary distribution, compute the entropy rate of the Markov source. Discuss the implications of the entropy rate in the context of data transmission efficiency.	1
7	Consider a Markov source with two states and transition matrix: $$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}$$ With the stationary distribution $\pi = [0.67, 0.33]$, calculate the information rate R. Explain how the information rate influences the design of coding schemes for efficient communication.	1
8	Let P and Q be two probability distributions defined as: $$P = \left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}, \quad Q = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right\}$$ Calculate the Kullback-Leibler divergence $D_{KL}(P\|Q)$ and interpret its meaning in terms of information loss and divergence between distributions.	1
9	A communication channel has a bandwidth of 1 MHz and a signal-to-noise ratio (SNR) of 15 dB. Calculate the channel capacity C using Shannon's formula. Discuss the practical implications of channel capacity in designing communication systems.	1
10	In a sensor network, sensors transmit data to a central server. Each sensor has a probability of failure. Assume the failure probabilities are $P(S_1 \text{ fails}) = 0.1$, $P(S_2 \text{ fails}) = 0.2$, and $P(S_3 \text{ fails}) = 0.3$. Model the entropy of the network and design a coding scheme to ensure reliable data transmission despite sensor failures. Provide numerical calculations for entropy and coding efficiency.	1

11	Consider a channel with the following transition probability matrix: $$P(Y\|X) = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$$ Assume the input probabilities $P(X) = [0.5, 0.3, 0.2]$. • (a) Calculate the joint probability matrix $P(X, Y)$. • (b) Calculate the marginal probabilities $P(Y)$. • (c) Find the mutual information $I(X; Y)$. • (d) Determine the channel capacity.	1
12	Consider a binary symmetric channel with a transition probability $P(1\|0) = P(0\|1) = 0.2$. • (a) Draw the channel diagram. • (b) Calculate the channel capacity. • (c) Compute the mutual information $I(X; Y)$ if the input symbols are equally likely. • (d) Determine the error probability if a simple repetition code (3,1) is used.	1
13	Consider a binary erasure channel with an erasure probability of 0.3. • (a) Draw the channel diagram. • (b) Calculate the channel capacity. • (c) Compute the mutual information $I(X; Y)$ if the input symbols are equally likely. • (d) Determine the error probability if a simple parity check code (3,2) is used.	1

14	For a communication system with a bandwidth of 1 MHz and a signal-to-noise ratio of 15 dB: • (a) Calculate the channel capacity. • (b) Determine the maximum data rate that can be achieved with an arbitrarily small error probability. • (c) If the actual data rate is 8 Mbps, calculate the required bandwidth to achieve this rate with the same SNR. • (d) Discuss the implications of increasing the bandwidth on the channel capacity.	1
15	A Gaussian channel has a bandwidth of 2 MHz and a noise power spectral density of $N_0 = 10^{-9}$ W/Hz. • (a) Calculate the Shannon limit for the channel capacity. • (b) If the transmitted power is 1 W, determine the maximum achievable data rate. • (c) If the actual data rate is 5 Mbps, calculate the required SNR to achieve this rate. • (d) Discuss the trade-off between power and bandwidth for achieving reliable communication.	1
16	A wireless communication system operates over a Rayleigh fading channel with an average SNR of 10 dB. Calculate: 1. The probability density function (PDF) of the received SNR. 2. The outage probability if the system requires a minimum SNR of 5 dB. 3. The average capacity of the channel.	1
17	For a Gaussian channel with bandwidth $W = 2$ MHz and signal power $P = 1$ W. The noise power spectral density is $N_0 = 10^{-9}$ W/Hz. Calculate: 1. The channel capacity. 2. The required signal power to achieve a capacity of 10 Mbps. 3. The change in capacity if the bandwidth is increased to 5 MHz.	1

18	A Multiple Input Multiple Output (MIMO) system with $M_T = 2$ transmit antennas and $M_R = 2$ receive antennas operates over a flat fading channel. The average SNR per receive antenna is 20 dB. Calculate: 1. The capacity of the MIMO channel. 2. The capacity if the number of transmit antennas is increased to 4. 3. The impact on capacity if the SNR is reduced to 10 dB.	1
19	For a communication system with bandwidth $W = 1$ MHz and SNR of 15 dB. Calculate: 1. The channel capacity using Shannon's theorem. 2. The new capacity if the bandwidth is doubled. 3. The required SNR to achieve a capacity of 10 Mbps with the original bandwidth.	1
20	A Binary Erasure Channel (BEC) with erasure probability $e = 0.25$ uses a (7, 4) Hamming code for error correction. Calculate: 1. The probability that a transmitted codeword is correctly decoded. 2. The expected number of erasures per codeword. 3. The probability of correct decoding if the erasure probability is reduced to $e = 0.1$.	1