Module-2
Channel Models and Capacity

Dr. Markkandan S

School of Electronics Engineering (SENSE)
Vellore Institute of Technology
Chennai

Outline

Channel Models and Its Types

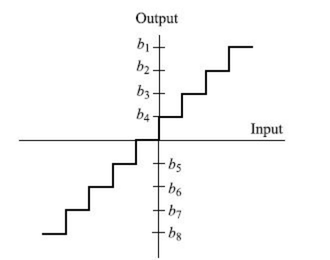
INTRODUCTION

The three parameters taken into account by a digital communications engineer:

- The Transmitted Signal Power
- The Channel Bandwidth
- The Reliability of Communication System (in terms of bit error rate)

Noise Effects and Correction

- The detector, which follows the demodulator, may decide on whether the transmitted bit is a 0 or a 1. This is called a hard decision decoding. This decision process at the decoder is like a binary quantization with two levels.
- If there are more than 2 levels of quantization, the detector is said to perform soft decision decoding. In the extreme case, no quantization is performed for soft decision decoding.
- The use of hard decision decoding causes an irreversible loss of information at the receiver. Suppose that the modulator sends only binary symbols, but the demodulator has an alphabet with Q symbols. Assuming the use of the quantizer as depicted in Fig. Binary input Q-ary output Discrete Memoryless Channel.



Introduction to Channel Models

- Channel models are essential for understanding and mitigating the effects of noise and interference in communication systems.
- Common types include:
  - Binary Symmetric Channel (BSC)
  - Binary Erasure Channel (BEC)
  - Additive White Gaussian Noise (AWGN) Channel
  - Rayleigh Fading Channel

Importance of Channel Models

- Channel models help in designing efficient communication systems.
- They provide insights into the performance limits of communication systems.
- Enable the development of error correction and detection algorithms.

TYPES OF CHANNELS - BINARY SYMMETRIC CHANNEL (BSC)

- If the modulator employs binary waveforms and the detector makes hard decisions, then the channel may be viewed as one in which a binary bit stream enters at the transmitting end and another bit stream comes out at the receiving end.
- This binary Discrete-input, Discrete-output channel is characterized by the set X = 0,1 of possible inputs, the set Y = 0,1 of possible outputs, and a set of conditional probabilities that relate the possible outputs to the possible inputs.

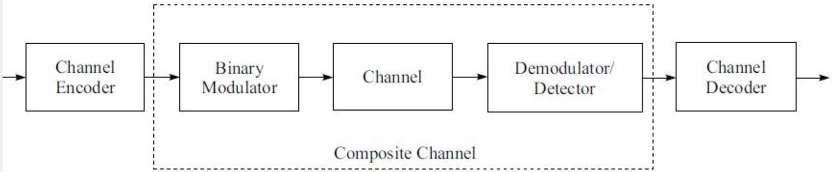
(a) A binary symmetric channel (BSC). (b) How an image might look after transmission through a BSC.

- A binary symmetric channel has two inputs (x_0, x_1) and two outputs (y_0, y_1).
- The channel transition probabilities are given by:

$$P(y_0|x_0) = 1 - p, \quad P(y_1|x_0) = p$$

$$P(y_0|x_1) = p, \quad P(y_1|x_1) = 1 - p$$

Types of Channels – Binary Symmetric Channel (BSC)



A composite discrete-input, discrete-output channel formed by including the modulator and demodulator/detector.

Example: Binary Symmetric Channel (BSC)

- A binary channel can be visualized as one that transports 1's and 0's from the transmitter (T_x) to the receiver (R_x).
- It makes an error occasionally with probability p.
- A BSC channel flips 1 to 0 and vice versa.
- The channel transition probabilities are as follows:

Joint Probability Matrix

We already know that if x_i are the inputs and y_j are the outputs, then the joint probability $P(x_i, y_j)$ is:

$$P(x_i, y_j) = P(y_j|x_i)P(x_i) = P(x_i|y_j)P(y_j)$$

and also we defined the PTM as:

$$P(Y|X) = \begin{bmatrix} P(y_1|x_1) & P(y_1|x_2) & \cdots & P(y_1|x_n) \\ P(y_2|x_1) & P(y_2|x_2) & \cdots & P(y_2|x_n) \\ \vdots & \vdots & \ddots & \vdots \\ P(y_m|x_1) & P(y_m|x_2) & \cdots & P(y_m|x_n) \end{bmatrix}$$

$$P(X|Y) = \begin{bmatrix} P(x_1|y_1) & P(x_1|y_2) & \cdots & P(x_1|y_m) \\ P(x_2|y_1) & P(x_2|y_2) & \cdots & P(x_2|y_m) \\ \vdots & \vdots & \ddots & \vdots \\ P(x_n|y_1) & P(x_n|y_2) & \cdots & P(x_n|y_m) \end{bmatrix}$$

Joint Probability Matrix

Multiplying first row by $P(x_1)$, second row by $P(x_2)$, and so on, we get:

$$\begin{bmatrix} P(x_1)P(y_1|x_1) & P(x_1)P(y_1|x_2) & \cdots & P(x_1)P(y_1|x_n) \\ P(x_2)P(y_2|x_1) & P(x_2)P(y_2|x_2) & \cdots & P(x_2)P(y_2|x_n) \\ \vdots & \vdots & \ddots & \vdots \\ P(x_n)P(y_m|x_1) & P(x_n)P(y_m|x_2) & \cdots & P(x_n)P(y_m|x_n) \end{bmatrix} =$$

$$\begin{bmatrix} P(x_1, y_1) & P(x_1, y_2) & \cdots & P(x_1, y_m) \\ P(x_2, y_1) & P(x_2, y_2) & \cdots & P(x_2, y_m) \\ \vdots & \vdots & \ddots & \vdots \\ P(x_n, y_1) & P(x_n, y_2) & \cdots & P(x_n, y_m) \end{bmatrix}$$

Properties of Joint Probability Matrix (JPM)

Since $P(x_i, y_j) = P(y_j|x_i)P(x_i)$, $P(x_i, y_j)$ is the joint probability matrix (JPM), which has the following properties:

1. $\sum_{j=1}^{m} P(x_i, y_j) = P(x_i)$ (i.e. probability of the input can be obtained by adding all the elements in the row of JPM - probability of input symbols).
2. $\sum_{i=1}^{n} P(x_i, y_j) = P(y_j)$ (i.e. probability of the output can be obtained by adding all the elements in the column of JPM - probability of output symbols).
3. $\sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) = 1$, that is, the sum of all the elements of JPM is equal to unity.

Output Symbol Probabilities

- The output probabilities are given by:

$$P(y_0) = P(x_0)P(y_0|x_0) + P(x_1)P(y_0|x_1) = \frac{1}{2}(1-p) + \frac{1}{2}p = \frac{1}{2}$$

$$P(y_1) = P(x_0)P(y_1|x_0) + P(x_1)P(y_1|x_1) = \frac{1}{2}p + \frac{1}{2}(1-p) = \frac{1}{2}$$

Joint Probabilities

- The joint probabilities are given by:

$$P(x_0, y_0) = P(x_0)P(y_0|x_0) = \frac{1}{2}(1-p)$$

$$P(x_0, y_1) = P(x_0)P(y_1|x_0) = \frac{1}{2}p$$

$$P(x_1, y_0) = P(x_1)P(y_0|x_1) = \frac{1}{2}p$$

$$P(x_1, y_1) = P(x_1)P(y_1|x_1) = \frac{1}{2}(1-p)$$

Conditional Probabilities

- The conditional probabilities are:

$$P(x_0|y_0) = \frac{P(x_0)P(y_0|x_0)}{P(y_0)} = \frac{\frac{1}{2}(1-p)}{\frac{1}{2}} = 1-p$$

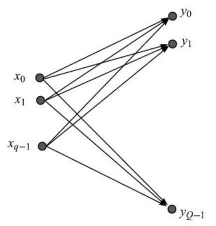
$$P(x_1|y_0) = \frac{P(x_1)P(y_0|x_1)}{P(y_0)} = \frac{\frac{1}{2}p}{\frac{1}{2}} = p$$

$$P(x_0|y_1) = \frac{P(x_0)P(y_1|x_0)}{P(y_1)} = \frac{\frac{1}{2}p}{\frac{1}{2}} = p$$

$$P(x_1|y_1) = \frac{P(x_1)P(y_1|x_1)}{P(y_1)} = \frac{\frac{1}{2}(1-p)}{\frac{1}{2}} = 1-p$$

Types of Channels - Composite Discrete-input, Discrete-output Channel

- Let the input to the channel be q-ary symbols, i.e., $X = \{x_0, x_1, \ldots x_{q-1}\}$.
- The output of the detector at the receiving end of the channel consists of Q-ary symbols, i.e., $Y = \{y_0, y_1, \ldots y_{Q-1}\}$.
- The inputs and the outputs can then be related by a set of qQ conditional probabilities $P(Y = y_i | X = x_j) = P(y_i | x_j)$.
- We assume that the channel and the modulation are both memoryless. This channel is known as Discrete Memoryless Channel (DMC).



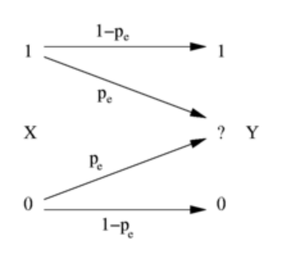
Types of Channels - Binary Erasure Channel (BEC)

A transmitter sends a bit (a zero or a one), and the receiver either receives the bit correctly, or with some probability P_e receives a message that the bit was not received ("erased").

$$P(Y=0|X=0) = P(Y=1|X=1) = 1 - p_e$$

$$P(Y=0|X=1) = P(Y=1|X=0) = 0$$

$$P(Y=e|X=0) = P(Y=e|X=1) = P_e$$



Example 1

Example 1 : For the channel shown in Fig. , write the channel matrix.

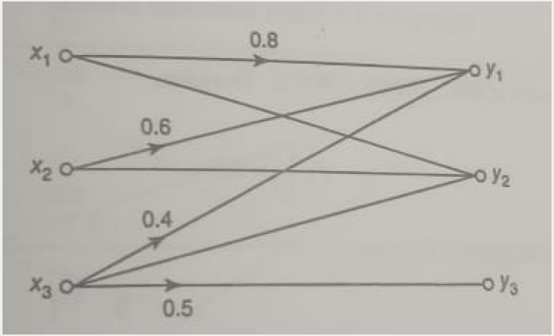


Figure: A communication channel.

Solution: For a communication channel, the sum of all outgoing probabilities from any input must be 1.

Example 1 (contd.)

Thus, the missing probabilities can be calculated as follows:

$$P\left(\frac{y_2}{x_1}\right) = 1 - 0.8 = 0.2$$

$$P\left(\frac{y_2}{x_2}\right) = 1 - 0.6 = 0.4$$

$$P\left(\frac{y_2}{x_3}\right) = 1 - (0.4 + 0.5) = 0.1$$

Thus, the channel matrix is given as follows:

$$\text{Channel matrix} = P\left(\frac{Y}{X}\right) = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.6 & 0.4 & 0 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}$$

Example 2

Example 2: For the BSC shown in Fig. , write the channel matrix.

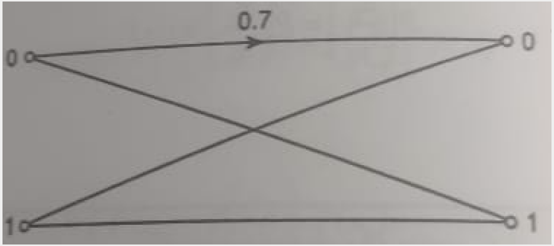


Figure: A binary symmetric channel.

Solution: As the channel is BSC, $P(0|0) = P(1|1)$. Therefore,

$$P\left(\frac{1}{1}\right) = 0.7$$

Thus, the channel matrix is given as follows:

$$\text{Channel matrix} = P\left(\frac{Y}{X}\right) = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

Example 3

For the channel matrix given, find the missing entries. Also, draw the corresponding channel diagram.

$$\text{Channel matrix} = P\left(\frac{Y}{X}\right) = \begin{bmatrix} 0.8 & * & 0.2 \\ 0.6 & 0.2 & * \\ 0.2 & 0.3 & * \end{bmatrix}$$

Example 3 Continued

Solution: We know that the sum of all the elements of a row of a channel matrix is equal to unity. Therefore,

$$P\left(\frac{y_2}{x_1}\right) + P\left(\frac{y_3}{x_1}\right) + P\left(\frac{y_4}{x_1}\right) = 1$$

$$\Rightarrow P\left(\frac{y_2}{x_1}\right) = 1 - 0.8 - 0.2 = 0$$

Similarly,

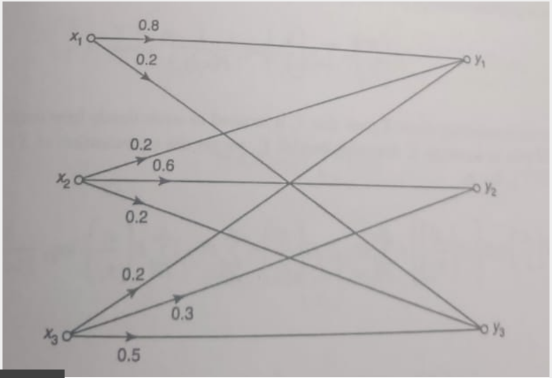
$$P\left(\frac{y_3}{x_2}\right) = 1 - 0.6 - 0.2 = 0.2$$

$$P\left(\frac{y_3}{x_3}\right) = 1 - 0.2 - 0.3 = 0.5$$

EXAMPLE 3 CONTINUED

Thus, the complete channel matrix is:

$$\text{Channel matrix} = P\left(\frac{Y}{X}\right) = \begin{bmatrix} 0.8 & 0 & 0.2 \\ 0.6 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$



Entropies for a Binary Symmetric Channel (BSC)

- Channel Matrix:

$$P\left(\frac{Y}{X}\right) = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$$

- Marginal Probabilities:

$$P(x_0) = P(x_1) = \frac{1}{2}$$

$$P(y_0) = P(y_1) = \frac{1}{2}$$

- Joint Probabilities:

$$P(x_0, y_0) = \frac{1}{2}(1-p), \quad P(x_0, y_1) = \frac{1}{2}p$$

$$P(x_1, y_0) = \frac{1}{2}p, \quad P(x_1, y_1) = \frac{1}{2}(1-p)$$

Entropies for a Binary Symmetric Channel (BSC)

- Conditional Probabilities:

$$P(x_0|y_0) = 1 - p, \quad P(x_1|y_0) = p$$
$$P(x_0|y_1) = p, \quad P(x_1|y_1) = 1 - p$$

- Entropy $H(X)$:

$$H(X) = 1 \text{ bit}$$

- Entropy $H(Y)$:

$$H(Y) = 1 \text{ bit}$$

- Joint Entropy $H(X,Y)$:

$$H(X,Y) = 1 + H(p) \text{ bits}$$

- Conditional Entropy $H(Y|X)$:

$$H(Y|X) = H(p) \text{ bits}$$

- Conditional Entropy $H(X|Y)$:

$$H(X|Y) = H(p) \text{ bits}$$

Example : Entropies for a Binary Symmetric Channel (BSC)

- Channel Matrix:

$$P\left(\frac{Y}{X}\right) = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$$

- Marginal Probabilities:

$$P(x_0) = \frac{1}{2}, \quad P(x_1) = \frac{1}{2}$$
$$P(y_0) = 0.5, \quad P(y_1) = 0.5$$

- Joint Probabilities:

$$P(x_0, y_0) = 0.45, \quad P(x_0, y_1) = 0.05$$
$$P(x_1, y_0) = 0.05, \quad P(x_1, y_1) = 0.45$$

- Conditional Probabilities:

$$P(x_0|y_0) = 0.9, \quad P(x_1|y_0) = 0.1$$
$$P(x_0|y_1) = 0.1, \quad P(x_1|y_1) = 0.9$$

Example : Entropies for a Binary Symmetric Channel (BSC)

- Entropy $H(X)$:

$$H(X) = 1 \text{ bit}$$

- Entropy $H(Y)$:

$$H(Y) = 1 \text{ bit}$$

- Joint Entropy $H(X, Y)$:

$$H(X, Y) = 1.4688 \text{ bits}$$

- Conditional Entropy $H(Y|X)$:

$$H(Y|X) = 0.469 \text{ bits}$$

- Conditional Entropy $H(X|Y)$:

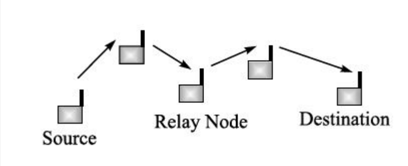
$$H(X|Y) = 0.4688 \text{ bits}$$

Types of Channels - Waveform Channels

- Waveform Channels: Such channels are typically associated with a given bandwidth, W.
- Suppose, $x(t)$ is a bandlimited input to a waveform channel and $y(t)$ is the output of the channel.
- Then, $y(t) = x(t) + n(t)$
- where $n(t)$ is the additive noise.

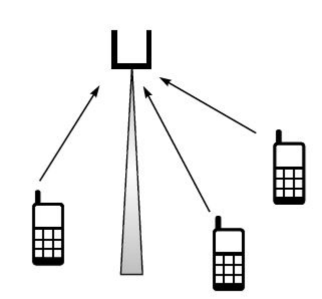
Types of Channels - Relay Channels

- Relay Channels: There is a source, a destination, and intermediate relay nodes. These relay nodes facilitate the communication between the source and the destination.
- Amplify-and-Forward (AF): Each relay node simply receives the signal and forwards it to the next relay node.
- Decode-and-Forward (DF): Relay node first decodes the received signal and then re-encodes the signal before forwarding it to the next relay node.



Types of Channels - Multiple Access Channels

- M transmitters (say, mobile phone users) want to communicate with a single receiver (say, the base station) over a common channel. This scenario is known as a Multiple Access Channel.

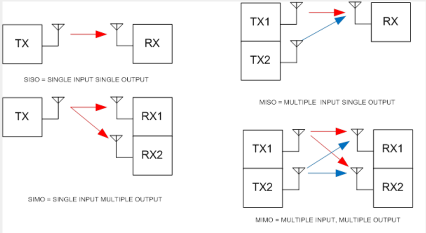


TYPES OF CHANNELS - BROADCAST CHANNELS

- A single transmitter (say, a low earth orbit satellite) wants to communicate with M receivers (say, the home dish antennas) over a common channel. This is an example of a Broadcast Channel.

Types of Channels - Multiple Input Multiple Output (MIMO) Channel

- **Single Input Single Output (SISO)**: Single antenna at transmitter and receiver.
- **Single Input Multiple Output (SIMO)**: Single antenna at transmitter and multiple antennas at the receiver.
- **Multiple Input Single Output (MISO)**: Multiple antennas at transmitter but only a single antenna at the receiver.
- **Multiple Input Multiple Output (MIMO)**: Multiple antennas both at the transmitter and receiver.



Types of Channels - Multiple Input Multiple Output (MIMO) Channel

- Consider a MIMO system with M_T transmit antennas and M_R receive antennas.
- The impulse response between j^{th} (j=1,2,... M_T) transmit antenna and i^{th} (i=1,2,... M_R) receive antennas is denoted by $h_{i,j}(\tau, t)$.
- The MIMO channel can be represented as follows:

$$H(\tau, t) = \begin{bmatrix} h_{1,1}(\tau,t) & h_{1,2}(\tau,t) & \dots & h_{1,M_T}(\tau,t) \\ h_{2,1}(\tau,t) & h_{2,2}(\tau,t) & \dots & h_{2,M_T}(\tau,t) \\ \vdots & \vdots & \dots & \vdots \\ h_{M_R,1}(\tau,t) & h_{M_R,2}(\tau,t) & \dots & h_{M_R,M_T}(\tau,t) \end{bmatrix}$$

- The variable τ is used to capture the time-varying nature of the channel.

Types of Channels - Multiple Input Multiple Output (MIMO) Channel

- If a signal $s_j(t)$ is transmitted from the j^{th} transmit antenna, the received signal at i^{th} receive antenna is

$$y_i(t) = \sum_{j=1}^{M_T} h_{i,j}(\tau, t) s_j(t), \quad i = 1, 2, \dots M_R$$

- The input-output relation of the MIMO channel is expressed as

$$y(t) = H(\tau, t) s(t)$$

- Where, $s(t) = [s_1(t) \; s_2(t) \dots s_{M_T}(t)]^T$ and $y(t) = [y_1(t) \; y_2(t) \dots y_{M_T}(t)]^T$.

Channel Capacity Calculations

Channel Capacity

Consider a DMC having an input alphabet $X = \{x_0, x_1, ..., x_{q-1}\}$ and an output alphabet $Y = \{y_0, y_1, ..., y_{r-1}\}$. The set of channel transition probabilities are $P(y_i|x_j)$. The average mutual information provided by the output Y for the input X is,

$$I(X;Y) = \sum_{j=0}^{q-1} \sum_{i=0}^{r-1} P(x_j) P(y_i|x_j) \log \frac{P(y_i|x_j)}{P(y_i)}$$

The Capacity of a DMC is defined as the maximum average mutual information in any single use of the channel. Where the maximum is over all possible input probabilities.

$$C = \max_{P(x_j)} (I(X;Y))$$

Hence,

$$C = \max_{P(x_j)} \sum_{j=0}^{q-1} \sum_{i=0}^{r-1} P(x_j) P(y_i|x_j) \log \frac{P(y_i|x_j)}{P(y_i)}$$

Channel Capacity - Properties

The units of channel capacity are bits per channel use (provided the base of the logarithm is 2). **Properties**

1. $C \geq 0$, since $I(X;Y) \geq 0$.
2. $C \leq \log|X|$, since $C = \max I(X;Y) \leq \max H(X) = \log|X|$.
3. $C \leq \log|Y|$, since $C = \max I(X;Y) \leq \max H(Y) = \log|Y|$.

Shannon's Channel Capacity Theorem

- For a given channel with capacity C, any transmission rate $R < C$ can be achieved with an arbitrarily small error probability using proper coding schemes.
- Ensures reliable communication over noisy channels.
- Consider a channel with bandwidth W and signal-to-noise ratio $\frac{P}{N_0}$.
- The maximum data rate C is given by:

$$C = W \log_2 \left(1 + \frac{P}{N_0}\right)$$

Shannon's Channel Coding Theorem

- States that reliable communication is possible if the transmission rate R is less than the channel capacity C.
- For rates $R > C$, reliable communication is not possible.
- Introduces the concept of error-correcting codes to achieve this reliable transmission.

Implications of Shannon's Theorem

- The theorem sets a fundamental limit on the efficiency of communication systems.
- Drives the design of practical coding schemes.
- Helps in understanding the trade-offs between bandwidth, power, and error rates.

Example: Shannon's Theorem in Practice

- Consider a communication system with:
  - Bandwidth $W = 1$ MHz
  - SNR $= 10$ dB
- Calculate the maximum data rate:

$$\text{SNR} = 10^{\frac{10}{10}} = 10$$

$$C = 10^6 \log_2(1 + 10) \approx 10^6 \log_2(11) \approx 3.46 \times 10^6 \text{ bps}$$

Shannon's Limit and Practical Systems

- Shannon's limit serves as a benchmark for the performance of practical systems.
- Engineers strive to design systems that approach this limit.
- Coding techniques like Turbo codes and LDPC codes are examples of practical implementations.

Advanced Example: Shannon's Limit

- Calculate the channel capacity for a system with:
  - Bandwidth $W = 5$ MHz
  - SNR $= 20$ dB
- Solution:

$$\text{SNR} = 10^{\frac{20}{10}} = 100$$

$$C = 5 \times 10^6 \log_2(1 + 100) \approx 5 \times 10^6 \log_2(101) \approx 33.2 \times 10^6 \text{ bps}$$

Example : Channel Capacity of Binary Symmetric Channel (BSC)

- Consider a BSC with channel transition probabilities

$$P(0|1) = p = P(1|0)$$

- The capacity of BSC is

$$C = 1 + plog_2p + (1-p)log_2(1-p)$$

- The entropy function

$$H(p) = -plog_2p - (1-p)log_2(1-p)$$

- The Capacity of Binary Symmetric Channel is

$$C = 1 - H(p)$$

Binary Symmetric Channel (BSC)

- Each bit transmitted has a probability p of being flipped.
- Transition probability matrix:

$$\begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$$

- Capacity C is given by:

$$C = 1 - H(p)$$

- Where $H(p)$ is the binary entropy function:

$$H(p) = -p\log_2(p) - (1-p)\log_2(1-p)$$

Example Calculation for BSC

- For $p = 0.1$:
$$H(0.1) = -0.1 \log_2(0.1) - 0.9 \log_2(0.9) \approx 0.469$$
- Capacity C is:
$$C = 1 - H(0.1) \approx 0.531 \text{ bits per channel use}$$

Exercise: BSC Capacity Calculation

- Calculate the capacity of a BSC with $p = 0.2$.
- Solution:

$$H(0.2) = -0.2\log_2(0.2) - 0.8\log_2(0.8) \approx 0.722$$

$$C = 1 - H(0.2) \approx 0.278 \text{ bits per channel use}$$

Binary Erasure Channel (BEC)

- Each bit transmitted can be erased with a probability e.
- Transition probability matrix:

$$\begin{bmatrix} 1-e & e \\ e & 1-e \end{bmatrix}$$

- Capacity C is given by:

$$C = 1 - e$$

- For $e = 0.1$:

$$C = 1 - 0.1 = 0.9 \text{ bits per channel use}$$

Exercise: BEC Capacity Calculation

- Calculate the capacity of a BEC with $e = 0.3$.
- Solution:

$$C = 1 - 0.3 = 0.7 \text{ bits per channel use}$$

Channel Capacity of Discrete Memoryless Channel

- A discrete memoryless channel is said to be Weakly Symmetric if the rows of the channel transition probability matrix are permutations of each other and the column sums $\sum_{x_i} p(y|x_i)$ are equal.
- For weakly symmetric channels, the capacity is given by

$$C = \log |Y| - H(\text{row of transition matrix})$$

Example: Discrete Memoryless Channel

- Consider the transition probability matrix given by

$$P_1 = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.5 & 0.3 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

- This represents the symmetric channel. The rows and columns are permutations of each other.
- Now consider another probability transition matrix

$$P_2 = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/6 & 1/2 & 1/3 \end{bmatrix}$$

- This is a weakly symmetric channel where rows are permutations but columns are not. The capacity is,

$$C = \log(3) - H\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right) = 0.1258 \text{ bits/use}$$

Muroga's Theorem

For the channel with noise characteristic $P(Y|X)$ is square and non-singular, the capacity can be calculated using the following equation:

$$C = \log\left(\sum_{i=1}^{n} 2^{-q_i}\right)$$

Solution for Muroga's Theorem

The matrix equation $P(Y|X) \cdot Q = h$ is solved where $h = \{h_1, h_2, \ldots, h_n\}$ are the entropies of $P(Y|X)$.

$$\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \cdot \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}$$

$$C = \log \left(\sum_{i=1}^{n} 2^{-q_i} \right)$$

$$p'_i = 2^{-q_i - C}$$

$$p'_i = p_i \cdot \begin{bmatrix} p_{1i} & p_{2i} & \cdots & p_{ni} \end{bmatrix}$$

Example: Transition Probability Matrix

Consider a discrete memoryless channel with the following transition probability matrix:

$$P(Y|X) = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$$

Step 1: Calculate Entropies

Calculate the entropies h_i of each row:

$$h_1 = -(0.6 \log 0.6 + 0.2 \log 0.2 + 0.2 \log 0.2) \approx 0.971$$
$$h_2 = -(0.2 \log 0.2 + 0.6 \log 0.6 + 0.2 \log 0.2) \approx 0.971$$
$$h_3 = -(0.2 \log 0.2 + 0.2 \log 0.2 + 0.6 \log 0.6) \approx 0.971$$

Step 2: Set Up the Matrix Equation

Solve the matrix equation $P(Y|X) \cdot Q = h$:

$$\begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} \cdot \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} 0.971 \\ 0.971 \\ 0.971 \end{bmatrix}$$

Step 3: Solve the Matrix Equation

Solving this matrix equation, we get:

$$Q_1 = Q_2 = Q_3 = 0.971$$

Step 4: Calculate Channel Capacity

Calculate the channel capacity C:

$$C = \log\left(\sum_{i=1}^{3} 2^{-q_i}\right)$$

Substituting $q_i = 0.971$:

$$C = \log\left(2^{-0.971} + 2^{-0.971} + 2^{-0.971}\right)$$

$$C = \log\left(3 \times 2^{-0.971}\right)$$

$$C = \log 3 + \log 2^{-0.971}$$

$$C = \log 3 - 0.971$$

$$C = 1.585 - 0.971 = 0.614 \text{ bits/use}$$

Example: Transition Probability Matrix (Non-Symmetric)

Consider a discrete memoryless channel with the following transition probability matrix:

$$P(Y|X) = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.4 & 0.2 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

Step 1: Calculate Entropies

Calculate the entropies h_i of each row:

$$
\begin{aligned}
h_1 &= -(0.5\log 0.5 + 0.3\log 0.3 + 0.2\log 0.2) \approx 1.029 \\
h_2 &= -(0.4\log 0.4 + 0.4\log 0.4 + 0.2\log 0.2) \approx 0.971 \\
h_3 &= -(0.3\log 0.3 + 0.2\log 0.2 + 0.5\log 0.5) \approx 1.029
\end{aligned}
$$

Step 2: Set Up the Matrix Equation

Solve the matrix equation $P(Y|X) \cdot Q = h$:

$$\begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.4 & 0.2 \\ 0.3 & 0.2 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} 1.029 \\ 0.971 \\ 1.029 \end{bmatrix}$$

Step 3: Solve the Matrix Equation

Solving this matrix equation, we get:

$$Q_1 \approx 0.913, \quad Q_2 \approx 1.114, \quad Q_3 \approx 1.029$$

Step 4: Calculate Channel Capacity

Calculate the channel capacity C:

$$C = \log\left(\sum_{i=1}^{3} 2^{-q_i}\right)$$

Substituting $q_1 \approx 0.913, q_2 \approx 1.114, q_3 \approx 1.029$:

$$C = \log\left(2^{-0.913} + 2^{-1.114} + 2^{-1.029}\right)$$

$$C = \log(0.518 + 0.470 + 0.494)$$

$$C = \log(1.482)$$

$$C = 0.170 \text{ bits/use}$$

Channel Capacities Summary

- **Binary Symmetric Channel (BSC)**

$$C = 1 - H(p)$$

where $H(p) = -p\log_2 p - (1-p)\log_2(1-p)$

- **Binary Erasure Channel (BEC)**

$$C = 1 - e$$

where e is the erasure probability.

- **Discrete Memoryless Channel (DMC)**

$$C = \max_{P(X)} I(X;Y)$$

where $I(X;Y) = \sum_{x,y} P(x,y)\log_2 \frac{P(x,y)}{P(x)P(y)}$

- **Symmetric Channel**

$$C = \log_2 M - H(\mathbf{p})$$

where M is the number of output symbols and $H(\mathbf{p})$ is the entropy of the output distribution.

- **Muroga's Theorem for Channels with Noise**

$$C = \log_2\left(\sum_{i=1}^{n} 2^{-q_i}\right)$$

where q_i are the solutions to the matrix equation $P(Y|X) \cdot Q = h$.

Channel Capacity of MIMO Systems

- The MIMO channel of dimension $M_R \times M_T$ can be represented as follows:

$$H(\tau,t) = \begin{bmatrix} h_{1,1}(\tau,t) & h_{1,2}(\tau,t) & \dots & h_{1,M_T}(\tau,t) \\ h_{2,1}(\tau,t) & h_{2,2}(\tau,t) & \dots & h_{2,M_T}(\tau,t) \\ \vdots & \vdots & \dots & \vdots \\ h_{M_R,1}(\tau,t) & h_{M_R,2}(\tau,t) & \dots & h_{M_R,M_T}(\tau,t) \end{bmatrix}$$

- Assuming transmit symbol energy is E_s, the sampled signal model for frequency flat fading channels is

$$y[k] = \sqrt{\frac{E_s}{M_T}} H_s[k] + n[k]$$

- The covariance matrix is

$$R_{ss} = E\{ss^H\} \quad H - \text{ Hermitian Operation}$$

CHANNEL CAPACITY OF MIMO SYSTEMS (CONTD.)

- If CSI is known, The channel capacity of MIMO system is

$$C = \max_{T_r(R_{ss})=M_T} W \log_2 \det\left(I_{MR} + \frac{E_s}{M_T N_0} H R_{ss} H^H\right) \text{ bits per second}$$

  Here, $T_r(R_{ss}) = M_T$ is the total average energy transmitted over a symbol period.

- If CSI is unknown, The capacity of MIMO channel is

$$C = W \sum_{i=1}^{r} \log_2\left(1 + \frac{E_s}{M_T N_0} \lambda_i\right) \text{ bits per second}$$

- For full rank MIMO channel $M_R = M_T$

$$C = W M \log_2\left(1 + \frac{E_s}{N_0}\right)$$

Channel Capacity and Coding Theorems

Channel Coding

- All real-life channels are affected by noise. Noise causes discrepancies (errors) between the input and the output data sequences of a digital communication system.
- For a typical noisy channel, the probability of bit error may be as high as 10^{-2}.
- This means that, on an average, 1 bit out of every 100 bits that are transmitted has an error. This level of reliability is far from adequate.
- Different applications require different levels of reliability (which is a component of the quality of service).

Application	*Probability of Error*
Speech telephony	10^{-4}
Voice band data	10^{-6}
Electronic mail, Electronic newspaper	10^{-6}
Internet access	10^{-6}
Streaming services	10^{-7}
Video telephony, High speed computing	10^{-7}

- An Error Control Code for a channel, represented by the channel transition probability matrix $P(y|x)$, consists of:
  - A message set {1, 2, … , M}.
  - An encoding function, X^n, which maps each message to a unique codeword, i.e., $1 \rightarrow X^n(1), 2 \rightarrow X^n(2), \ldots M \rightarrow X^n(M)$. The set of codewords is called a (codebook).
  - A decoding function, $D \rightarrow \{1, 2, \ldots, M\}$, which makes a guess based on a decoding strategy in order to map back the received vector to one of the possible messages.

Code Rate - Block Codes

- In this class of codes, the incoming message sequence is first divided into sequential blocks, each of length k bits.
- The cardinality of the message set is $M = 2^k$.
- Each k-bit long information block is mapped to an n-bit block by the channel coder.
- Where $n > k$. For every k-bit information, $n - k$ redundant bits are added.
- Code rate is $r = \frac{k}{n} = \frac{\log M}{k}$.
- Rate r is said to be achievable if there exists a coding scheme (n, k) such that the maximal probability of error tends to 0 as $n \to \infty$. The (n, k) code may also be expressed as a $(2^{nr}, n)$ code or a (M, n) code, where $M = 2^k = 2^{nr}$.

Noisy Channel Coding Theorem or Channel Coding Theorem

- Let a DMS with an alphabet X have entropy $H(X)$ and produce symbols every T_s seconds. Let a discrete memoryless channel have capacity C and be used once every T_c seconds. Then if

$$\frac{H(X)}{T_s} \leq \frac{C}{T_c}$$

  there exists a coding scheme and signal reception can be achieved with low probability of error.
- If,

$$\frac{H(X)}{T_s} > \frac{C}{T_c}$$

  it is not possible to transmit information and reconstruct it with a low probability of error.
- $\frac{C}{T_c}$ is called the critical rate.

Information Capacity Theorem or Channel Capacity Theorem

- The information capacity of a continuous channel of bandwidth W Hertz, perturbed by additive white Gaussian noise with power spectral density $N_0/2$ and limited in bandwidth W, is given by

$$C = W \log_2 \left(1 + \frac{P}{N_0 W}\right) \text{ bits/second}$$

- This theorem is also called the Channel Capacity Theorem.
- This channel capacity is the fundamental limit on the rate of reliable communication for a power-limited, band-limited Gaussian channel.

Shannon's Limit

- Consider a Gaussian Channel that is limited both in power and bandwidth.
- Let us define an ideal system with transmission rate R_b which is equal to the capacity of the channel C. Suppose the energy per bit is E_b. Then the average transmitted power is,

$$P = E_b R_b = E_b C$$

- Channel Capacity theorem for this ideal channel is

$$\frac{C}{W} = \log_2\left(1 + \frac{E_b}{N_0}\frac{C}{W}\right)$$

- This equation can be rewritten as

$$\frac{E_b}{N_0} = \left(\frac{2^{C/W} - 1}{C/W}\right)$$

Shannon's Limit - Bandwidth Efficiency

- The plot of the bandwidth efficiency $\frac{R_b}{W}$ versus $\frac{E_b}{N_0}$ is called the Bandwidth Efficiency Diagram.
- For infinite bandwidth, the ratio $\frac{E_b}{N_0}$ tends to a limiting value

$$\left.\frac{E_b}{N_0}\right|_{W\to\infty} = \ln 2 = 0.693 = -1.6 \text{ dB}$$

- This value is called the Shannon Limit.
- The channel capacity corresponding to this limiting value is

$$C|_{W\to\infty} = \frac{P}{N_0}\log_2 e$$

- Thus, at infinite bandwidth, the capacity of the channel is determined by the SNR.

Shannon's Limit - Bandwidth Efficiency Diagram

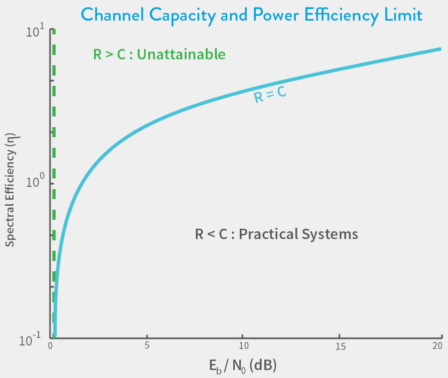


Figure: Shannon's Limit

- The curve for the critical rate $R_b = C$ is known as the Capacity Boundary.
- The trade-offs between the quantities $\frac{R_b}{W}$, $\frac{E_b}{N_0}$, and the probability of error P_e. The bandwidth and the power can be traded one for the other to provide the desired BER.
- Any point on the bandwidth efficiency diagram corresponds to an operating point corresponding to a set of values of SNR, bandwidth efficiency, and BER.