Module-2
Problems on
Channel Models and Capacity

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Question 1

For the channel shown in Figure, write the channel matrix.

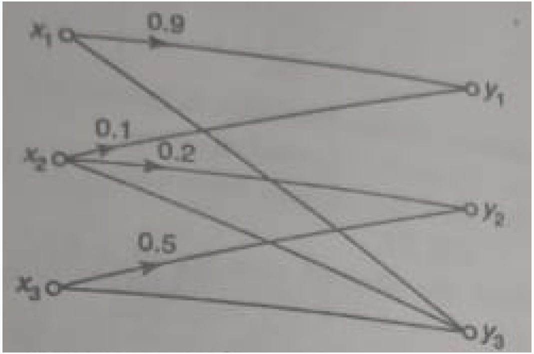

Figure 1: Channel diagram.

Solution to Question 1

The channel matrix for the given channel diagram is:

$$P(Y|X) = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.1 & 0.9 \end{bmatrix}$$

Question 2

For the given channel matrix:

$$P(Y|X) = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

Find the probability distribution of the output symbols.

Solution to Question 2

Assume input symbols X are equally likely, so $P(X=0)=P(X=1)=0.5$. The probability distribution of the output symbols is given by:

$$
\begin{aligned}
P(Y=0) &= P(Y=0|X=0)P(X=0)+P(Y=0|X=1)P(X=1) \\
&= 0.7\cdot 0.5+0.4\cdot 0.5 \\
&= 0.35+0.2=0.55 \\
P(Y=1) &= P(Y=1|X=0)P(X=0)+P(Y=1|X=1)P(X=1) \\
&= 0.3\cdot 0.5+0.6\cdot 0.5 \\
&= 0.15+0.3=0.45
\end{aligned}
$$

Question 3

Show that

$$H(X, Y) = H(X|Y) + H(Y) = H(Y|X) + H(X)$$

Solution to Question 3

By definition of joint entropy and conditional entropy:

$$H(X,Y) = -\sum_{x\in X}\sum_{y\in Y} P(x,y)\log P(x,y)$$

$$H(X|Y) = -\sum_{x\in X}\sum_{y\in Y} P(x,y)\log P(x|y)$$

$$H(Y|X) = -\sum_{x\in X}\sum_{y\in Y} P(x,y)\log P(y|x)$$

$$H(Y) = -\sum_{y\in Y} P(y)\log P(y)$$

$$H(X) = -\sum_{x\in X} P(x)\log P(x)$$

Therefore,

$$H(X,Y) = H(X|Y) + H(Y)$$
$$H(X,Y) = H(Y|X) + H(X)$$

Question 4

For the JPM matrix given, find

1. $H(X)$
2. $H(Y)$
3. $H(X, Y)$
4. $H(Y|X)$
5. $H(X|Y)$

$$\text{JPM} = P(X, Y) = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.1 & 0.3 & 0.2 \\ 0.1 & 0.2 & 0.2 \end{bmatrix}$$

Solution to Question 4 - Part 1

Finding $H(X)$:

$$H(X) = -\sum_{x \in X} P(x) \log_2 P(x)$$

Calculate $P(x)$:

$$P(x_1) = 0.2 + 0.1 + 0.1 = 0.4$$
$$P(x_2) = 0.1 + 0.3 + 0.2 = 0.6$$
$$P(x_3) = 0.1 + 0.2 + 0.2 = 0.5$$

Now,

$$\begin{aligned} H(X) &= -[0.4 \log_2 0.4 + 0.6 \log_2 0.6 + 0.5 \log_2 0.5] \\ &= -[0.4 \cdot (-1.322) + 0.6 \cdot (-0.737) + 0.5 \cdot (-1)] \\ &= 0.529 + 0.442 + 0.5 \\ &= 1.471 \text{ bits} \end{aligned}$$

Finding $H(Y)$:

$$H(Y) = -\sum_{y \in Y} P(y) \log_2 P(y)$$

Calculate $P(y)$:

$$P(y_1) = 0.2 + 0.1 + 0.1 = 0.4$$
$$P(y_2) = 0.1 + 0.3 + 0.2 = 0.6$$
$$P(y_3) = 0.1 + 0.2 + 0.2 = 0.5$$

Now,

$$\begin{aligned} H(Y) &= -[0.4 \log_2 0.4 + 0.6 \log_2 0.6 + 0.5 \log_2 0.5] \\ &= -[0.4 \cdot (-1.322) + 0.6 \cdot (-0.737) + 0.5 \cdot (-1)] \\ &= 0.529 + 0.442 + 0.5 \\ &= 1.471 \text{ bits} \end{aligned}$$

Solution to Question 4 - Part 3

Finding $H(X, Y)$:

$$H(X,Y) = -\sum_{x \in X} \sum_{y \in Y} P(x,y) \log_2 P(x,y)$$

Now,

$$\begin{aligned}
H(X,Y) &= -[0.2 \log_2 0.2 + 0.1 \log_2 0.1 + 0.1 \log_2 0.1 \\
&\quad + 0.1 \log_2 0.1 + 0.3 \log_2 0.3 + 0.2 \log_2 0.2 \\
&\quad + 0.1 \log_2 0.1 + 0.2 \log_2 0.2 + 0.2 \log_2 0.2] \\
&= 0.464 + 0.332 + 0.332 + 0.332 + 0.521 + 0.464 \\
&\quad + 0.332 + 0.464 + 0.464 \\
&= 2.445 \text{ bits}
\end{aligned}$$

Solution to Question 4 - Part 4

Finding $H(Y|X)$:

$$H(Y|X) = H(X, Y) - H(X)$$

Now,

$$\begin{aligned} H(Y|X) &= 2.445 - 1.471 \\ &= 0.974 \text{ bits} \end{aligned}$$

Solution to Question 4 - Part 5

Finding $H(X|Y)$:

$$H(X|Y) = H(X,Y) - H(Y)$$

Now,

$$\begin{aligned} H(X|Y) &= 2.445 - 1.471 \\ &= 0.974 \text{ bits} \end{aligned}$$

Question 5

Consider a BSC with a transition probability of $P(0|0) = P(1|1) = 0.8$. The inputs are equally likely to occur. Draw the channel diagram. Also find:

1. $H(X)$
2. $H(Y)$
3. $H(X, Y)$
4. $H(Y|X)$
5. $H(X|Y)$

Solution to Question 5 - Part 1

Finding $H(X)$:

$$H(X) = 1 \text{ bit}$$

Solution to Question 5 - Part 2

Finding $H(Y)$:

$$H(Y) = -\sum_{y\in\{0,1\}} P(y)\log P(y)$$

Since inputs are equally likely, $P(0) = P(1) = 0.5$. Therefore,

$$\begin{aligned} H(Y) &= -[0.5\log 0.5 + 0.5\log 0.5] \\ &= 1 \text{ bit} \end{aligned}$$

Solution to Question 5 - Part 3

Finding $H(X, Y)$:

$$H(X, Y) = -\sum_{x,y} P(x, y) \log P(x, y)$$

The joint probabilities $P(x, y)$ are:

$$P(0, 0) = 0.5 \times 0.8 = 0.4$$
$$P(1, 1) = 0.5 \times 0.8 = 0.4$$
$$P(0, 1) = 0.5 \times 0.2 = 0.1$$
$$P(1, 0) = 0.5 \times 0.2 = 0.1$$

Therefore,

$$H(X, Y) = -[0.4 \log 0.4 + 0.4 \log 0.4 + 0.1 \log 0.1 + 0.1 \log 0.1]$$
$$= 1.685 \text{ bits}$$

Solution to Question 5 - Part 4

Finding $H(Y|X)$:

$$H(Y|X) = H(X,Y) - H(X)$$

$$\begin{aligned} H(Y|X) &= 1.685 - 1 \\ &= 0.685 \text{ bits} \end{aligned}$$

Solution to Question 5 - Part 5

Finding $H(X|Y)$:

$$H(X|Y) = H(X,Y) - H(Y)$$

$$\begin{aligned} H(X|Y) &= 1.685 - 1 \\ &= 0.685 \text{ bits} \end{aligned}$$

Question 6

Find the rate of information transmission for the JPM given. Assume $r_s = 1000$ symbols/s:

$$\text{JPM} = P(X, Y) = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.1 & 0.3 & 0.1 \\ 0.1 & 0.2 & 0.2 \end{bmatrix}$$

Solution to Question 6

The rate of information transmission is given by:

$$R = r_s \times I(X;Y)$$

where $I(X;Y)$ is the mutual information. Calculating $I(X;Y)$:

$$I(X;Y) = \sum_{x,y} P(x,y) \log \frac{P(x,y)}{P(x)P(y)}$$

With the given JPM, we calculate $I(X;Y)$ and then R.

Question 7

Consider the cascaded channel with the following channel matrices:

$$P(Y|X) = \begin{bmatrix} 0.7 & 0 & 0.3 \\ 0.8 & 0.2 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

Solution to Question 7

The cascaded channel matrix is given by the product of the individual channel matrices:

$$P(Z|X) = P(Z|Y) \cdot P(Y|X)$$

Assume the second channel matrix as:

$$P(Z|Y) = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.3 & 0.7 \end{bmatrix}$$

Solution to Question 7 - Part 2

Calculate the resulting channel matrix $P(Z|X)$:

$$P(Z|X) = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.7 & 0 & 0.3 \\ 0.8 & 0.2 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

Performing the matrix multiplication:

$$P(Z|X) = \begin{bmatrix} (0.6 \cdot 0.7 + 0.4 \cdot 0.8) & (0.6 \cdot 0 + 0.4 \cdot 0.2) & (0.6 \cdot 0.3 + 0.4 \cdot 0) \\ (0.5 \cdot 0.7 + 0.5 \cdot 0.8) & (0.5 \cdot 0 + 0.5 \cdot 0.2) & (0.5 \cdot 0.3 + 0.5 \cdot 0) \\ (0 \cdot 0.7 + 0.3 \cdot 0.8) & (0 \cdot 0 + 0.3 \cdot 0.2) & (0 \cdot 0.3 + 0.3 \cdot 0.5 + 0.7 \cdot 0.5) \end{bmatrix}$$

Solution to Question 7 - Part 3

Simplifying the matrix:

$$P(Z|X) = \begin{bmatrix} 0.42 + 0.32 & 0 + 0.08 & 0.18 + 0 \\ 0.35 + 0.40 & 0 + 0.10 & 0.15 + 0 \\ 0 + 0.24 & 0 + 0.06 & 0 + 0.15 + 0.35 \end{bmatrix}$$

$$P(Z|X) = \begin{bmatrix} 0.74 & 0.08 & 0.18 \\ 0.75 & 0.10 & 0.15 \\ 0.24 & 0.06 & 0.50 \end{bmatrix}$$

Question 8

Assume $P(X) = [0.3, 0.5, 0.2]$.

1. Find the mutual information $I(X; Z)$.

$$P(Z|Y) = \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \\ 0.1 & 0.9 \end{bmatrix}$$

Solution to Question 8 - Part 1

Finding Mutual Information $I(X; Z)$:

$$I(X; Z) = \sum_{x,z} P(x, z) \log \frac{P(x, z)}{P(x)P(z)}$$

Calculate the joint probabilities $P(x, z)$:

$$P(x_1, z_1) = 0.3 \times 0.7 = 0.21$$

$$P(x_1, z_2) = 0.3 \times 0.3 = 0.09$$

$$P(x_2, z_1) = 0.5 \times 0.5 = 0.25$$

$$P(x_2, z_2) = 0.5 \times 0.5 = 0.25$$

$$P(x_3, z_1) = 0.2 \times 0.1 = 0.02$$

$$P(x_3, z_2) = 0.2 \times 0.9 = 0.18$$

Solution to Question 8 - Part 2

Calculate the marginal probabilities $P(z)$:

$$\begin{aligned} P(z_1) &= P(x_1, z_1) + P(x_2, z_1) + P(x_3, z_1) \\ &= 0.21 + 0.25 + 0.02 = 0.48 \\ P(z_2) &= P(x_1, z_2) + P(x_2, z_2) + P(x_3, z_2) \\ &= 0.09 + 0.25 + 0.18 = 0.52 \end{aligned}$$

Solution to Question 8 - Part 3

Now calculate $I(X;Z)$:

$$
\begin{aligned}
I(X;Z) &= 0.21 \log \frac{0.21}{0.3 \times 0.48} + 0.09 \log \frac{0.09}{0.3 \times 0.52} \\
&\quad + 0.25 \log \frac{0.25}{0.5 \times 0.48} + 0.25 \log \frac{0.25}{0.5 \times 0.52} \\
&\quad + 0.02 \log \frac{0.02}{0.2 \times 0.48} + 0.18 \log \frac{0.18}{0.2 \times 0.52} \\
&= 0.21 \log \frac{0.21}{0.144} + 0.09 \log \frac{0.09}{0.156} + 0.25 \log \frac{0.25}{0.24} \\
&\quad + 0.25 \log \frac{0.25}{0.26} + 0.02 \log \frac{0.02}{0.096} + 0.18 \log \frac{0.18}{0.104} \\
&= 0.21 \log 1.458 + 0.09 \log 0.577 + 0.25 \log 1.042 \\
&\quad + 0.25 \log 0.962 + 0.02 \log 0.208 + 0.18 \log 1.731 \\
&= 0.21 \times 0.164 + 0.09 \times -0.239 + 0.25 \times 0.018 \\
&\quad + 0.25 \times -0.017 + 0.02 \times -0.683 + 0.18 \times 0.238 \\
&= 0.0344 - 0.0215 + 0.0045 - 0.0043 - 0.0137 + 0.0428 \\
&= 0.0422 \text{ bits}
\end{aligned}
$$

Question 9

Find the channel capacity of a BEC with transition probability of 0.7.

Solution to Question 9

The channel capacity C of a Binary Erasure Channel (BEC) is given by:

$$C = 1 - p$$

where p is the erasure probability. Here, $p = 0.7$.

$$C = 1 - 0.7 = 0.3$$

Question 10

Find the capacity of the channel for the following channel matrix:

$$P(Y|X) = \begin{bmatrix} 0.2 & 0.5 & 0.3 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$$

Solution to Question 10 - Part 1

The channel capacity C is found by maximizing the mutual information $I(X;Y)$ over all possible input distributions $P(X)$.

$$C = \max_{P(X)} I(X;Y)$$

Start by calculating the joint probabilities $P(x, y)$ for a given distribution $P(X)$.

Solution to Question 10 - Part 2

Assume $P(X) = [p_1, p_2, p_3]$. The joint probabilities are:

$$P(x_1, y_1) = p_1 \times 0.2$$
$$P(x_1, y_2) = p_1 \times 0.5$$
$$P(x_1, y_3) = p_1 \times 0.3$$
$$P(x_2, y_1) = p_2 \times 0.2$$
$$P(x_2, y_2) = p_2 \times 0.6$$
$$P(x_2, y_3) = p_2 \times 0.2$$
$$P(x_3, y_1) = p_3 \times 0.1$$
$$P(x_3, y_2) = p_3 \times 0.1$$
$$P(x_3, y_3) = p_3 \times 0.8$$

Solution to Question 10 - Part 3

Calculate the marginal probabilities $P(y)$:

$$
\begin{aligned}
P(y_1) &= p_1 \times 0.2 + p_2 \times 0.2 + p_3 \times 0.1 \\
&= 0.2p_1 + 0.2p_2 + 0.1p_3 \\
P(y_2) &= p_1 \times 0.5 + p_2 \times 0.6 + p_3 \times 0.1 \\
&= 0.5p_1 + 0.6p_2 + 0.1p_3 \\
P(y_3) &= p_1 \times 0.3 + p_2 \times 0.2 + p_3 \times 0.8 \\
&= 0.3p_1 + 0.2p_2 + 0.8p_3
\end{aligned}
$$

Solution to Question 10 - Part 4

Now calculate $I(X;Y)$:

$$I(X;Y) = \sum_{x,y} P(x,y) \log \frac{P(x,y)}{P(x)P(y)}$$

Substitute the joint and marginal probabilities and solve for $I(X;Y)$. The goal is to maximize this value over the input distribution $P(X)$.

Example 11

The noise characteristics of a channel are as shown in Fig. Find the capacity of the channel using Muroga's method.

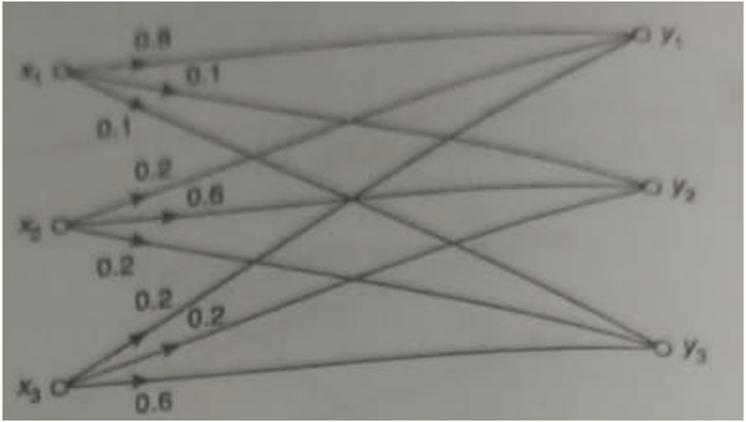

Figure 2: Communication Channel

Solution to Example 11 : Part -1

$$P\left(\frac{Y}{X}\right) = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$$

Using Muroga's method:

$$\begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} \cdot \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} 0.8 \log_2 0.8 + 0.1 \log_2 0.1 + 0.1 \log_2 0.1 \\ 0.2 \log_2 0.2 + 0.6 \log_2 0.6 + 0.2 \log_2 0.2 \\ 0.2 \log_2 0.2 + 0.2 \log_2 0.2 + 0.6 \log_2 0.6 \end{bmatrix} = \begin{bmatrix} -0.743 \\ -0.971 \\ -0.971 \end{bmatrix}$$

Solution to Example 11 : Part -2

$$0.8Q_1 + 0.1Q_2 + 0.1Q_3 = -0.922$$
$$0.2Q_1 + 0.6Q_2 + 0.2Q_3 = -1.371$$
$$0.2Q_1 + 0.2Q_2 + 0.6Q_3 = -1.371$$

Solving the above equations, we get:

$$Q_1 = -1.186$$
$$Q_2 = -1.471$$
$$Q_3 = -1.471$$

$$C = \log_2\left(\sum_{i=1}^{3} 2^{-Q_i}\right) = \log_2(2^{1.186} + 2^{1.471} + 2^{1.471}) = \log_2(0.437 + 0.358 + 0.358)$$
$$= \log_2(1.153) \approx 0.202 \text{ bit/sym}$$