Module-1
Problems on Information Measures

Dr. Markkandan S

School of Electronics Engineering (SENSE)
Vellore Institute of Technology
Chennai

Problem 1: Discrete Random Variables

Problem: Given a discrete random variable X with possible outcomes $\{a, b, c\}$ and probabilities $P(X = a) = 0.2$, $P(X = b) = 0.5$, $P(X = c) = 0.3$, and another random variable Y with possible outcomes $\{0.3, 0.7\}$ and probabilities $P(Y = 0.3) = 0.4$ and $P(Y = 0.7) = 0.6$:

1. Calculate the self-information $I(x)$ for each outcome.
2. Calculate the average information (entropy) $H(X)$.
3. Given the joint distribution:

$$P(X, Y) = \begin{bmatrix} 0.08 & 0.12 \\ 0.12 & 0.18 \\ 0.20 & 0.30 \end{bmatrix}$$

calculate the mutual information $I(X; Y)$.

Solution 1: Self-information $I(x)$

Solution:

$$
\begin{aligned}
I(a) &= -\log_2 P(X=a) = -\log_2 0.2 = \log_2 5 \approx 2.32 \text{ bits} \\
I(b) &= -\log_2 P(X=b) = -\log_2 0.5 = 1 \text{ bit} \\
I(c) &= -\log_2 P(X=c) = -\log_2 0.3 \approx 1.74 \text{ bits}
\end{aligned}
$$

Solution 1: Entropy $H(X)$

Solution:

$$
\begin{aligned}
H(X) &= -\sum_{x \in \{a,b,c\}} P(X=x)\log_2 P(X=x) \\
&= -(0.2\log_2 0.2 + 0.5\log_2 0.5 + 0.3\log_2 0.3) \\
&\approx 0.2 \times 2.32 + 0.5 \times 1 + 0.3 \times 1.74 \\
&\approx 1.37 \text{ bits}
\end{aligned}
$$

Solution 1: Mutual Information $I(X;Y)$

Solution:

$$P(X,Y) = \begin{bmatrix} 0.08 & 0.12 \\ 0.12 & 0.18 \\ 0.20 & 0.30 \end{bmatrix}$$

$$H(X,Y) = -(0.08 \log_2 0.08 + 0.12 \log_2 0.12 + 0.12 \log_2 0.12 + 0.18 \log_2 0.18$$

$$\approx 1.843 \text{ bits}$$

$$I(X;Y) = H(X) + H(Y) - H(X,Y)$$

$$= 1.37 + 0.971 - 1.843$$

$$= 0.498 \text{ bits}$$

Problem 2: Discrete Random Variables

Problem: Given the joint probability distribution:

$$P(X, Y) = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{bmatrix}$$

Calculate the following:

1. Entropy $H(X)$.
2. Marginal entropy $H(Y)$.
3. Joint entropy $H(X, Y)$.
4. Conditional entropy $H(Y|X)$.

Solution 2: Entropy $H(X)$

Solution:

$$\begin{aligned} P(X=0) &= 0.1 + 0.2 = 0.3 \\ P(X=1) &= 0.3 + 0.4 = 0.7 \\ H(X) &= -(0.3 \log_2 0.3 + 0.7 \log_2 0.7) \\ &= 0.881 \text{ bits} \end{aligned}$$

Solution 2: Marginal Entropy $H(Y)$

Solution:

$$\begin{aligned} P(Y=0) &= 0.1 + 0.3 = 0.4 \\ P(Y=1) &= 0.2 + 0.4 = 0.6 \\ H(Y) &= -(0.4\log_2 0.4 + 0.6\log_2 0.6) \\ &= 0.971 \text{ bits} \end{aligned}$$

Solution 2: Joint Entropy $H(X, Y)$

Solution:

$$
\begin{aligned}
H(X, Y) &= -\sum_{x,y} P(X = x, Y = y) \log_2 P(X = x, Y = y) \\
&= -(0.1 \log_2 0.1 + 0.2 \log_2 0.2 + 0.3 \log_2 0.3 + 0.4 \log_2 0.4) \\
&\approx 1.846 \text{ bits}
\end{aligned}
$$

Solution 2: Conditional Entropy $H(Y|X)$

Solution:

$$\begin{aligned} H(Y|X) &= H(X,Y) - H(X) \\ &= 1.846 - 0.881 \\ &= 0.965 \text{ bits} \end{aligned}$$

Problem 3: Continuous Random Variables

Problem: Given a continuous random variable X with probability density function $f_X(x) = 2e^{-2x}$ for $x \geq 0$:

1. Calculate the differential entropy $h(X)$.
2. Given another continuous random variable Y with $f_Y(y) = 2e^{-2y}$ for $y \geq 0$:
   1. Calculate the differential entropy $h(Y)$.
   2. Calculate the joint entropy $h(X, Y)$ if X and Y are independent.
   3. Calculate the conditional entropy $h(Y|X)$.
   4. Calculate the mutual information $I(X; Y)$.

Solution 3.1: Differential Entropy $h(X)$

Solution:

$$
\begin{aligned}
h(X) &= -\int_0^{\infty} f_X(x) \log f_X(x)\, dx \\
&= -\int_0^{\infty} 2e^{-2x} \log(2e^{-2x})\, dx \\
&= -\int_0^{\infty} 2e^{-2x} (\log 2 - 2x)\, dx \\
&= -(\log 2) \int_0^{\infty} 2e^{-2x}\, dx + 2 \int_0^{\infty} x2e^{-2x}\, dx \\
&= -(\log 2) \cdot 1 + 2 \cdot \frac{1}{2} \\
&= -\log 2 + 1 \\
&\approx -0.693 + 1 \\
&= 0.307 \text{ nats}
\end{aligned}
$$

Solution 3.2: Differential Entropy $h(Y)$

Solution:

$$
\begin{aligned}
h(Y) &= -\int_0^\infty f_Y(y) \log f_Y(y)\, dy \\
&= -\int_0^\infty 2e^{-2y} \log(2e^{-2y})\, dy \\
&= -\int_0^\infty 2e^{-2y} (\log 2 - 2y)\, dy \\
&= -(\log 2) \int_0^\infty 2e^{-2y}\, dy + 2\int_0^\infty y2e^{-2y}\, dy \\
&= -(\log 2) \cdot 1 + 2 \cdot \frac{1}{2} \\
&= -\log 2 + 1 \\
&\approx -0.693 + 1 \\
&= 0.307 \text{ nats}
\end{aligned}
$$

Solution 3.3: Joint Entropy $h(X, Y)$

Solution:

$$\begin{aligned} h(X, Y) &= h(X) + h(Y) \quad \text{(since } X \text{ and } Y \text{ are independent)} \\ &= 0.307 + 0.307 \\ &= 0.614 \text{ nats} \end{aligned}$$

Solution 3.4: Conditional Entropy $h(Y|X)$

Solution:

$$\begin{aligned} h(Y|X) &= h(X, Y) - h(X) \\ &= 0.614 - 0.307 \\ &= 0.307 \text{ nats} \end{aligned}$$

Solution 3.5: Mutual Information $I(X;Y)$

Solution:

$$\begin{aligned} I(X;Y) &= h(X) + h(Y) - h(X,Y) \\ &= 0.307 + 0.307 - 0.614 \\ &= 0 \text{ nats} \end{aligned}$$

Problem 4: Continuous Random Variables

Problem: Given two continuous random variables X and Y with joint probability density function $f_{X,Y}(x, y) = 4e^{-2(x+y)}$ for $x \geq 0$ and $y \geq 0$:

1. Calculate the differential entropy $h(X, Y)$.
2. Calculate the marginal entropies $h(X)$ and $h(Y)$.
3. Calculate the conditional entropy $h(Y|X)$.

Solution:

$$
\begin{aligned}
h(X, Y) &= -\int_0^\infty \int_0^\infty f_{X,Y}(x, y) \log f_{X,Y}(x, y)\, dx\, dy \\
&= -\int_0^\infty \int_0^\infty 4e^{-2(x+y)} \log(4e^{-2(x+y)})\, dx\, dy \\
&= -\int_0^\infty \int_0^\infty 4e^{-2(x+y)}(\log 4 - 2(x + y))\, dx\, dy
\end{aligned}
$$

Solution 4.1: Joint Entropy $h(X, Y)$ (cont.)

Solution:

$$
\begin{aligned}
h(X,Y) &= -(\log 4)\int_0^{\infty}\int_0^{\infty} 4e^{-2(x+y)}\,dx\,dy \\
&\quad + 2\int_0^{\infty}\int_0^{\infty}(x+y)4e^{-2(x+y)}\,dx\,dy \\
&= -(\log 4)\cdot 1 + 2\cdot\frac{1}{4} \\
&= -\log 4 + 0.5 \\
&= -2\log 2 + 0.5 \\
&= -1.386 + 0.5 \\
&= -0.886 \text{ nats}
\end{aligned}
$$

Solution 4.2: Marginal Entropy $h(X)$

Solution: First, find the marginal density function $f_X(x)$:

$$\begin{aligned}
f_X(x) &= \int_0^\infty f_{X,Y}(x,y)\,dy \\
&= \int_0^\infty 4e^{-2(x+y)}\,dy \\
&= 4e^{-2x}\int_0^\infty e^{-2y}\,dy \\
&= 4e^{-2x}\left[-\frac{1}{2}e^{-2y}\right]_0^\infty \\
&= 4e^{-2x}\cdot\frac{1}{2} \\
&= 2e^{-2x}
\end{aligned}$$

Solution:

$$
\begin{aligned}
h(X) &= -\int_0^\infty f_X(x)\log f_X(x)\,dx \\
&= -\int_0^\infty 2e^{-2x}\log(2e^{-2x})\,dx \\
&= -\int_0^\infty 2e^{-2x}(\log 2 - 2x)\,dx \\
&= -(\log 2)\int_0^\infty 2e^{-2x}\,dx + 2\int_0^\infty x2e^{-2x}\,dx \\
&= -(\log 2)\cdot 1 + 2\cdot\frac{1}{2} \\
&= -\log 2 + 1 \\
&\approx -0.693 + 1 \\
&= 0.307 \text{ nats}
\end{aligned}
$$

Solution 4.3: Marginal Entropy $h(Y)$

Solution: First, find the marginal density function $f_Y(y)$:

$$\begin{aligned}
f_Y(y) &= \int_0^{\infty} f_{X,Y}(x,y)\,dx \\
&= \int_0^{\infty} 4e^{-2(x+y)}\,dx \\
&= 4e^{-2y}\int_0^{\infty} e^{-2x}\,dx \\
&= 4e^{-2y}\left[-\frac{1}{2}e^{-2x}\right]_0^{\infty} \\
&= 4e^{-2y}\cdot\frac{1}{2} \\
&= 2e^{-2y}
\end{aligned}$$

Solution:

$$
\begin{aligned}
h(Y) &= -\int_0^\infty f_Y(y) \log f_Y(y)\, dy \\
&= -\int_0^\infty 2e^{-2y} \log(2e^{-2y})\, dy \\
&= -\int_0^\infty 2e^{-2y} (\log 2 - 2y)\, dy \\
&= -(\log 2) \int_0^\infty 2e^{-2y}\, dy + 2\int_0^\infty y2e^{-2y}\, dy \\
&= -(\log 2) \cdot 1 + 2 \cdot \frac{1}{2} \\
&= -\log 2 + 1 \\
&\approx -0.693 + 1 \\
&= 0.307 \text{ nats}
\end{aligned}
$$

Solution 4.4: Conditional Entropy $h(Y|X)$

Solution:

$$\begin{aligned} h(Y|X) &= h(X, Y) - h(X) \\ &= -0.886 - 0.307 \\ &= -1.193 \text{ nats} \end{aligned}$$