Module-1
Problems on Markov Models

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Problem 1: Markov Statistical Model for Information Source

Problem: Consider a first-order Markov source with two states, S_1 and S_2. The transition probability matrix is given by:

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

Calculate the stationary distribution of the states.

Solution 1: Stationary Distribution

Solution: Let the stationary distribution be $\pi = [\pi_1, \pi_2]$. Then,

$$\pi P = \pi$$

This gives us the equations:

$$\pi_1 = 0.7\pi_1 + 0.4\pi_2$$

$$\pi_2 = 0.3\pi_1 + 0.6\pi_2$$

With the normalization condition $\pi_1 + \pi_2 = 1$:

$$\pi_1(1 - 0.7) = \pi_2(0.4)$$

$$0.3\pi_1 = 0.4\pi_2$$

$$\pi_2 = \frac{3}{4}\pi_1$$

Using $\pi_1 + \pi_2 = 1$:

$$\pi_1 + \frac{3}{4}\pi_1 = 1$$

$$\frac{7}{4}\pi_1 = 1$$

Problem 2: Entropy and Information Rate of a Markov Source

Problem: Using the stationary distribution from Problem 1, calculate the entropy of the source and the information rate.

Solution 2.1: Entropy of the Source

Solution: The entropy of the source is given by:

$$H = -\sum_{i,j} \pi_i P_{ij} \log P_{ij}$$

Using the stationary distribution $\pi = \left[\frac{4}{7}, \frac{3}{7}\right]$ and the transition matrix:

$$H = -\left(\frac{4}{7}(0.7\log 0.7 + 0.3\log 0.3) + \frac{3}{7}(0.4\log 0.4 + 0.6\log 0.6)\right)$$

$$H \approx -\left(\frac{4}{7}(0.7 \cdot (-0.514) + 0.3 \cdot (-1.737)) + \frac{3}{7}(0.4 \cdot (-1.322) + 0.6 \cdot (-0.7\right.$$

$$H \approx 0.860 \text{ bits}$$

Solution 2.2: Information Rate

Solution: The information rate R is given by the entropy rate of the Markov source:

$$R = H$$

$$R \approx 0.860 \text{ bits per symbol}$$

Problem 3: Second-Order Markov Model

Problem: Consider a second-order Markov source with three states, S_1, S_2, and S_3. The transition probability matrix is given by:

$$P = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}$$

Calculate the stationary distribution of the states.

Solution 3: Stationary Distribution

Solution: Let the stationary distribution be $\pi = [\pi_1, \pi_2, \pi_3]$. Then,

$$\pi P = \pi$$

This gives us the equations:

$$\pi_1 = 0.6\pi_1 + 0.2\pi_2 + 0.1\pi_3$$

$$\pi_2 = 0.3\pi_1 + 0.7\pi_2 + 0.3\pi_3$$

$$\pi_3 = 0.1\pi_1 + 0.1\pi_2 + 0.6\pi_3$$

With the normalization condition $\pi_1 + \pi_2 + \pi_3 = 1$:

$$\pi_1(1 - 0.6) = \pi_2(0.2) + \pi_3(0.1)$$

$$0.4\pi_1 = 0.2\pi_2 + 0.1\pi_3$$

Using similar steps, solve for π_1, π_2, π_3:

$$\pi = \left[\frac{3}{7}, \frac{2}{7}, \frac{2}{7}\right]$$

Problem 4: Conditional Entropy and Information Rate of a Markov Source

Problem: Using the stationary distribution from Problem 3, calculate the conditional entropy of the source and the information rate.

Solution 4.1: Conditional Entropy of the Source

Solution: The conditional entropy of the source is given by:

$$H(X_{n+1}|X_n) = -\sum_i \pi_i \sum_j P_{ij} \log P_{ij}$$

Using the stationary distribution $\pi = \left[\frac{3}{7}, \frac{2}{7}, \frac{2}{7}\right]$ and the transition matrix:

$$H(X_{n+1}|X_n) = -\left(\frac{3}{7}\sum_j P_{1j} \log P_{1j} + \frac{2}{7}\sum_j P_{2j} \log P_{2j} + \frac{2}{7}\sum_j P_{3j} \log P_{3j}\right)$$

Solution 4.1: Conditional Entropy of the Source (cont.)

Solution:

$$
\begin{aligned}
= -\Bigg(&\frac{3}{7}(0.6\log 0.6 + 0.3\log 0.3 + 0.1\log 0.1) \\
&+ \frac{2}{7}(0.2\log 0.2 + 0.7\log 0.7 + 0.1\log 0.1) \\
&+ \frac{2}{7}(0.1\log 0.1 + 0.3\log 0.3 + 0.6\log 0.6)\Bigg)
\end{aligned}
$$

$$H(X_{n+1}|X_n) \approx 1.356 \text{ bits}$$

Solution 4.2: Information Rate

Solution: The information rate R is given by the conditional entropy rate of the Markov source:

$$R = H(X_{n+1}|X_n)$$

$$R \approx 1.356 \text{ bits per symbol}$$

Problem 5: Information Measures of Continuous Random Variables

Problem: Given a continuous random variable X with probability density function $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$:

1. Calculate the differential entropy $h(X)$.
2. Calculate the mutual information $I(X;Y)$ if $Y = X + N$, where N is a Gaussian noise with zero mean and variance σ_N^2.

Solution 5.1: Differential Entropy $h(X)$

Solution: The differential entropy for a Gaussian random variable X with mean μ and variance σ^2 is given by:

$$h(X) = \frac{1}{2}\log(2\pi e\sigma^2)$$

Solution 5.2: Mutual Information $I(X; Y)$

Solution: Given $Y = X + N$ and $N \sim \mathcal{N}(0, \sigma_N^2)$, the mutual information is:

$$I(X; Y) = h(Y) - h(Y|X)$$

Since Y is also Gaussian with variance $\sigma^2 + \sigma_N^2$:

$$h(Y) = \frac{1}{2}\log(2\pi e(\sigma^2 + \sigma_N^2))$$

$$h(Y|X) = h(N) = \frac{1}{2}\log(2\pi e\sigma_N^2)$$

$$I(X; Y) = \frac{1}{2}\log(2\pi e(\sigma^2 + \sigma_N^2)) - \frac{1}{2}\log(2\pi e\sigma_N^2)$$

$$I(X; Y) = \frac{1}{2}\log\left(\frac{\sigma^2 + \sigma_N^2}{\sigma_N^2}\right)$$

Problem 6: Information Measures of Continuous Random Variables

Problem: Given two continuous random variables X and Y with joint probability density function $f_{X,Y}(x,y) = \frac{1}{\pi}e^{-(x^2+y^2)}$:

1. Calculate the differential entropy $h(X,Y)$.
2. Calculate the marginal entropies $h(X)$ and $h(Y)$.
3. Calculate the mutual information $I(X;Y)$.

Solution 6.1: Joint Entropy $h(X, Y)$

Solution: The joint entropy is given by:

$$h(X, Y) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y) \log f_{X,Y}(x, y)\, dx\, dy$$

Substituting the given $f_{X,Y}(x, y)$:

$$h(X, Y) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\pi} e^{-(x^2+y^2)} \log\left(\frac{1}{\pi} e^{-(x^2+y^2)}\right)\, dx\, dy$$

Solution:

$$
\begin{aligned}
h(X, Y) &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\pi} e^{-(x^2+y^2)} \left(\log \frac{1}{\pi} + \log e^{-(x^2+y^2)} \right) dx\, dy \\
&= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\pi} e^{-(x^2+y^2)} \left(-\log \pi - (x^2 + y^2) \right) dx\, dy \\
&= \log \pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\pi} e^{-(x^2+y^2)}\, dx\, dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\pi} e^{-(x^2+y^2)} (x^2 + y^2
\end{aligned}
$$

Solution 6.1: Joint Entropy $h(X, Y)$ (cont.)

Solution: Since the integral of the Gaussian distribution over all space is 1, we get:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\pi}e^{-(x^2+y^2)}\,dx\,dy = 1$$

The expectation $\mathbb{E}[X^2 + Y^2]$ for the standard normal distribution (since X and Y are independent and identically distributed) is 2, because $\mathbb{E}[X^2] = 1$ and $\mathbb{E}[Y^2] = 1$:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\pi}e^{-(x^2+y^2)}(x^2+y^2)\,dx\,dy = 2$$

So,

$$h(X, Y) = \log\pi + 2$$

Since $\log\pi \approx 1.1447$,

$$h(X, Y) \approx 3.1447 \text{ nats}$$

Solution 6.2: Marginal Entropies $h(X)$ and $h(Y)$

Solution: The marginal density functions are $f_X(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ and $f_Y(y) = \frac{1}{\sqrt{\pi}} e^{-y^2}$.

The marginal entropy for a Gaussian random variable with variance σ^2 is given by:

$$h(X) = \frac{1}{2} \log(2\pi e \sigma^2)$$

Since X and Y are standard normal variables (with $\sigma^2 = 1$):

$$h(X) = \frac{1}{2} \log(2\pi e)$$

Similarly,

$$h(Y) = \frac{1}{2} \log(2\pi e)$$

$$h(X) \approx 1.4189 \text{ nats}$$

$$h(Y) \approx 1.4189 \text{ nats}$$

Solution 6.3: Mutual Information $I(X; Y)$

Solution: The mutual information is given by:

$$I(X; Y) = h(X) + h(Y) - h(X, Y)$$

Substituting the entropies:

$$I(X; Y) = 1.4189 + 1.4189 - 3.1447$$

$$I(X; Y) = 2.8378 - 3.1447$$

$$I(X; Y) = -0.3069 \text{ nats}$$