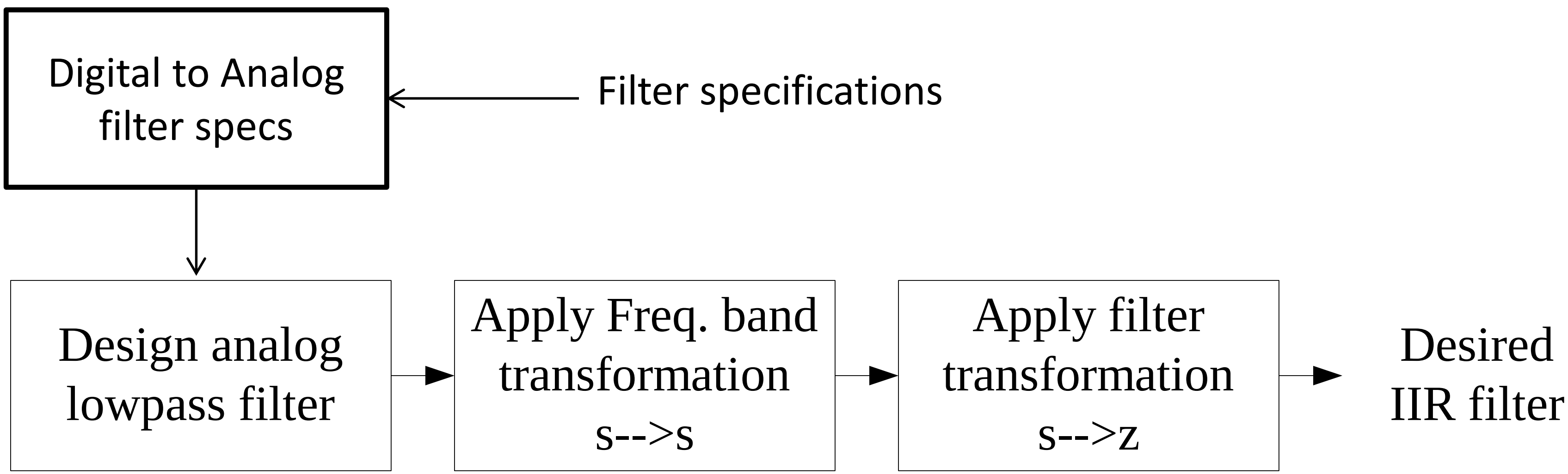
Analog IIR Filter Design

IIR Filter

- IIR filters are one of two primary types of digital filters used in Digital Signal Processing (DSP) applications (the other type being FIR). “IIR” means “Infinite Impulse Response.”

- The impulse response is “infinite” because there is feedback in the filter; if you put in an impulse (a single “1” sample followed by many “0” samples), an infinite number of non-zero values will come out (theoretically.)


Digital to Analog filter specs
Filter specifications
Design analog lowpass filter
Apply Freq. band transformation s-->s
Apply filter transformation s-->z
Desired IIR filter

Analog IIR Filter Types

- Butterworth Filters
- Chebyshev I Filters
- Chebyshev II Filters
- Elliptic Filters

- Butterworth filters have a very smooth passband, relatively wide transmission region.

- Chebyshev–I filter is equiripple in the passband and monotonic in the stopband.

- Chebyshev-II filter is equiripple in the stopband and monotonic in the passband.

- Elliptic filter is equiripple in the both stopband and passband.

Butterworth Filters

- The magnitude-square response of an N^{th} order analog

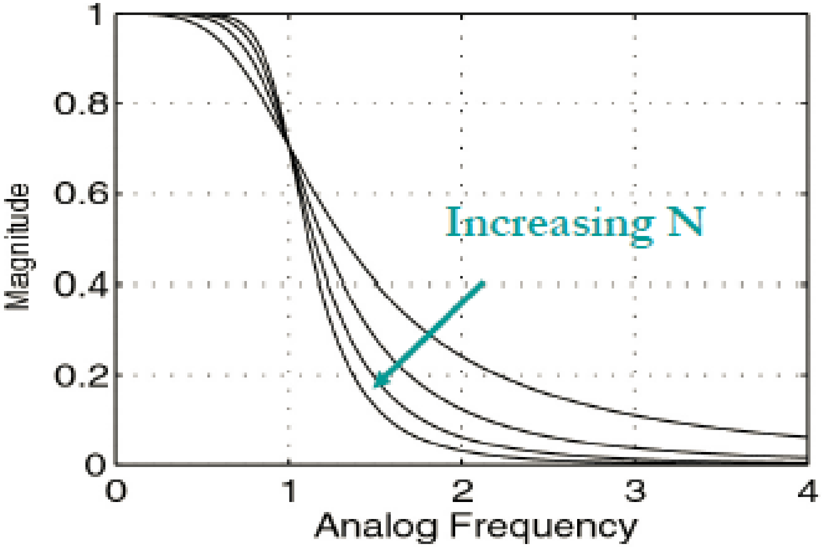
lowpass Butterworth filter:

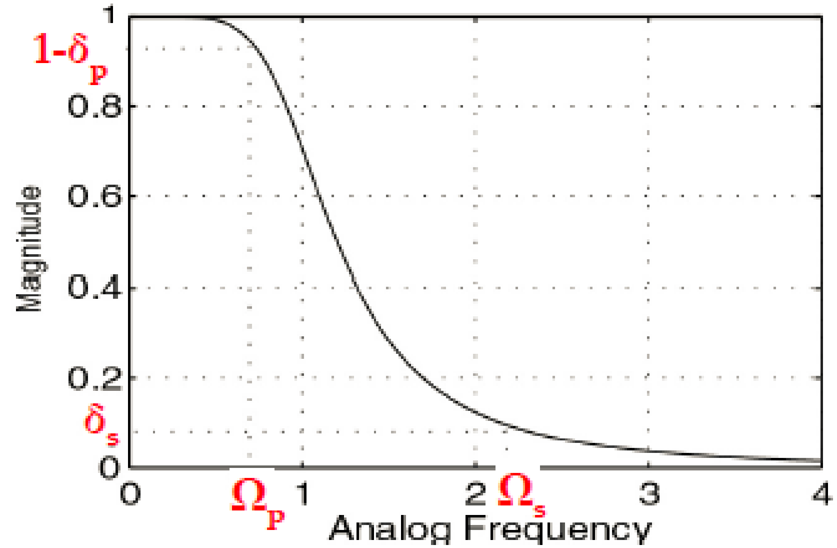
$$|H(j\Omega)| = \frac{1}{\left[1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]^{\frac{1}{2}}}$$

- Where N is the order of the filter.

- Ω_c is defined as the cutoff frequency.

- where the filter magnitude is $\frac{1}{\sqrt{2}}$ times the dc gain (Ω=0).





- $|H(j\Omega)|$=1 for all N
- $|H(j\Omega)|$=$\frac{1}{\sqrt{2}}$ $at\ \Omega = \Omega_c$, implies $20log|H(j\Omega_c)| = -3.01\ dB$.
- $|H(j\Omega)|$=0 as $\Omega \rightarrow \infty$
- The magnitude characteristic is said to be maximally flat because $\left.\frac{d^n|H(j\Omega)|}{d\Omega^n}\right|_{\Omega=0} = 0\ for\ n = 1, 2, \ldots \ldots 2N - 1$
- $|H(j\Omega)|$ is a monotonically decreasing function of frequency $|H(j\Omega_2)|$< $|H(j\Omega_1)|$ for any values of Ω_1 and Ω_2.

Normalized Butterworth Polynomial

- The magnitude-squared frequency response of the normalized (Ω_c=1) low pass butterworth filter is

$$|H_N(j\Omega)|^2 = \frac{1}{1+\Omega^{2N}} \text{ -------------------(1)}$$

$$H_N(j\Omega)\ H_N(-j\Omega)=\frac{1}{1+\Omega^{2N}} \text{ ----------------(2)}$$

Replacing $j\Omega$ by s and hence Ω by $\frac{s}{j}$ in the above equation

$$H_N(s)\ H_N(-s)= \frac{1}{1+(\frac{s}{j})^{2N}} \text{ --------------------(3)}$$

- The transfer function $H_N(s)\ H_N(-s)$ has no finite zeros.
- The poles of the product $H_N(s)\ H_N(-s)$ are determined by equating the denominator of equation (3)to zero.

$$1+\left(\frac{s}{j}\right)^{2N} = 0 \Rightarrow s = (-1)^{\frac{1}{2N}} j \text{ -------(4)}$$

- $-1 = e^{j\pi(2k+1)}, k = 0, 1, \ldots \ldots 2N-1$ Cos $\pi(2k)$

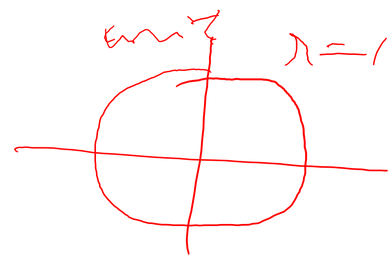
$$j = e^{j\frac{\pi}{2}},$$

- The poles are given by

$$s_k = e^{j\pi\frac{(2k+1)}{2N}} e^{j\frac{\pi}{2}}. \text{-----------------------} (5)$$

- The poles are the product $H_N(s)\ H_N(-s)$ and it is distributed on the unit circle in the s-plane;.

$$\theta_k = \frac{\pi}{N}k + \frac{\pi}{2N} + \frac{\pi}{2} \qquad\qquad k = 0, 1, \ldots \ldots 2N-1$$



For N=1

$$\theta_k = \frac{\pi}{N}k + \frac{\pi}{2N} + \pi/2$$

$k = 0, 1$

$$\theta_k = \pi k + \pi$$

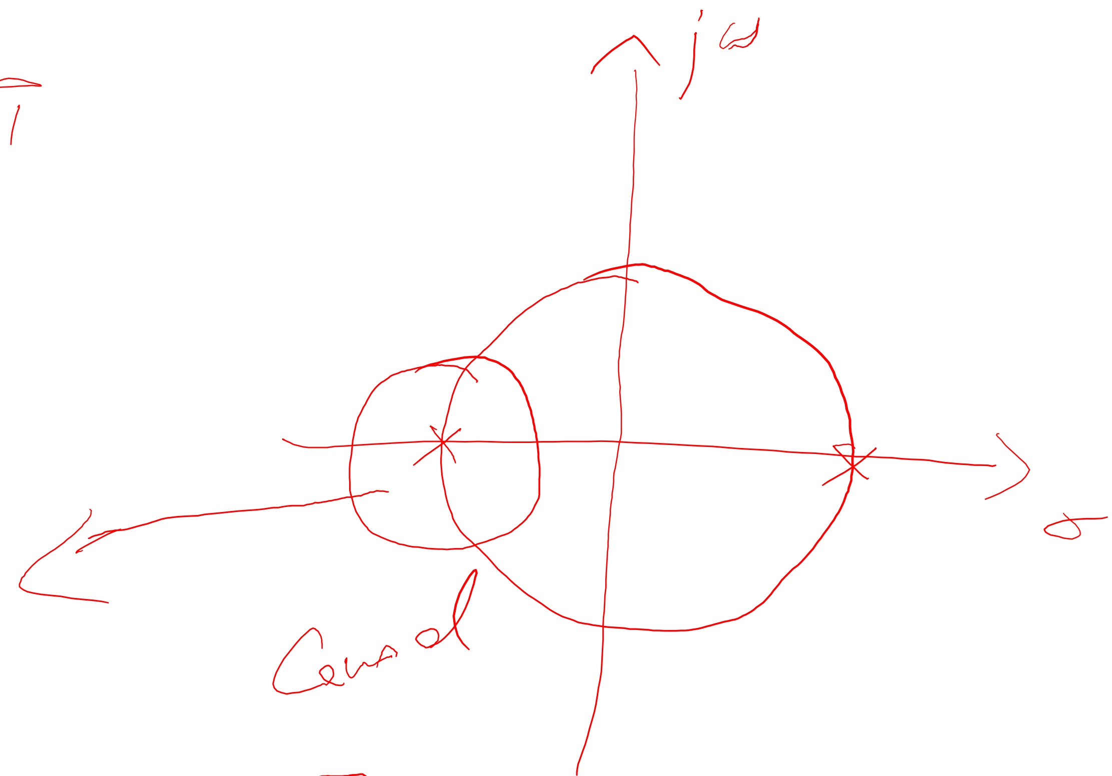
$k = 0 \quad \theta_k = \pi$

$k = 1 \quad \theta_k = 2\pi$

$$s = e^{j\pi}$$

$$s = -1$$

$$\boxed{H(s) = \frac{1}{s+1}}$$

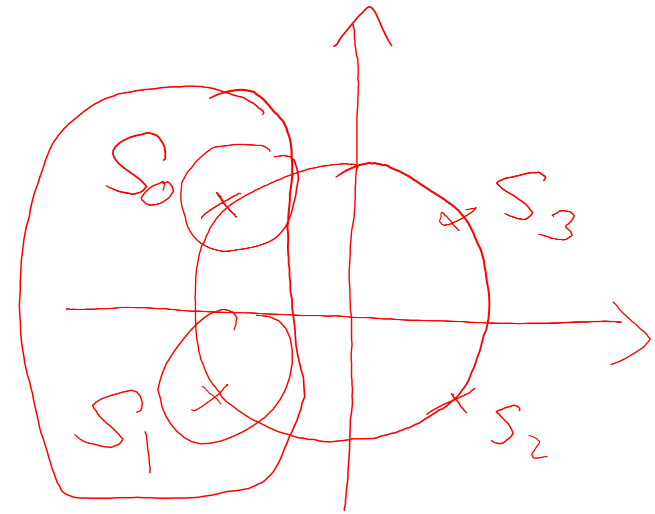


For N=2

$$\theta_k = \frac{\pi}{N}k + \frac{\pi}{2N} + \frac{\pi}{2} \qquad k = 0, 1, 2, 3$$

$$\theta_0 = \frac{3\pi}{4} \qquad \theta_1 = \frac{5\pi}{4}$$

$$\theta_2 = \frac{7\pi}{4} \qquad \theta_3 = \frac{9\pi}{4}$$



$$S_0 = \cos\frac{3\pi}{4} + j\sin\frac{3\pi}{4}$$

$$S_0 = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \qquad S_1 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$H(s) = \frac{1}{\prod_k (s - s_k)} = \frac{1}{\left(s + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)\left(s + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)} = \frac{1}{s^2 + \sqrt{2}s + 1}$$

- Half ($H_N(s)$)on the left-half plane and ($H_N(-s)$)half on the right-half plane.

Thus, $H_{N(s)} = \frac{1}{\prod_{LHP}(s-s_k)} = \frac{1}{B_N(s)}$ ....(6)

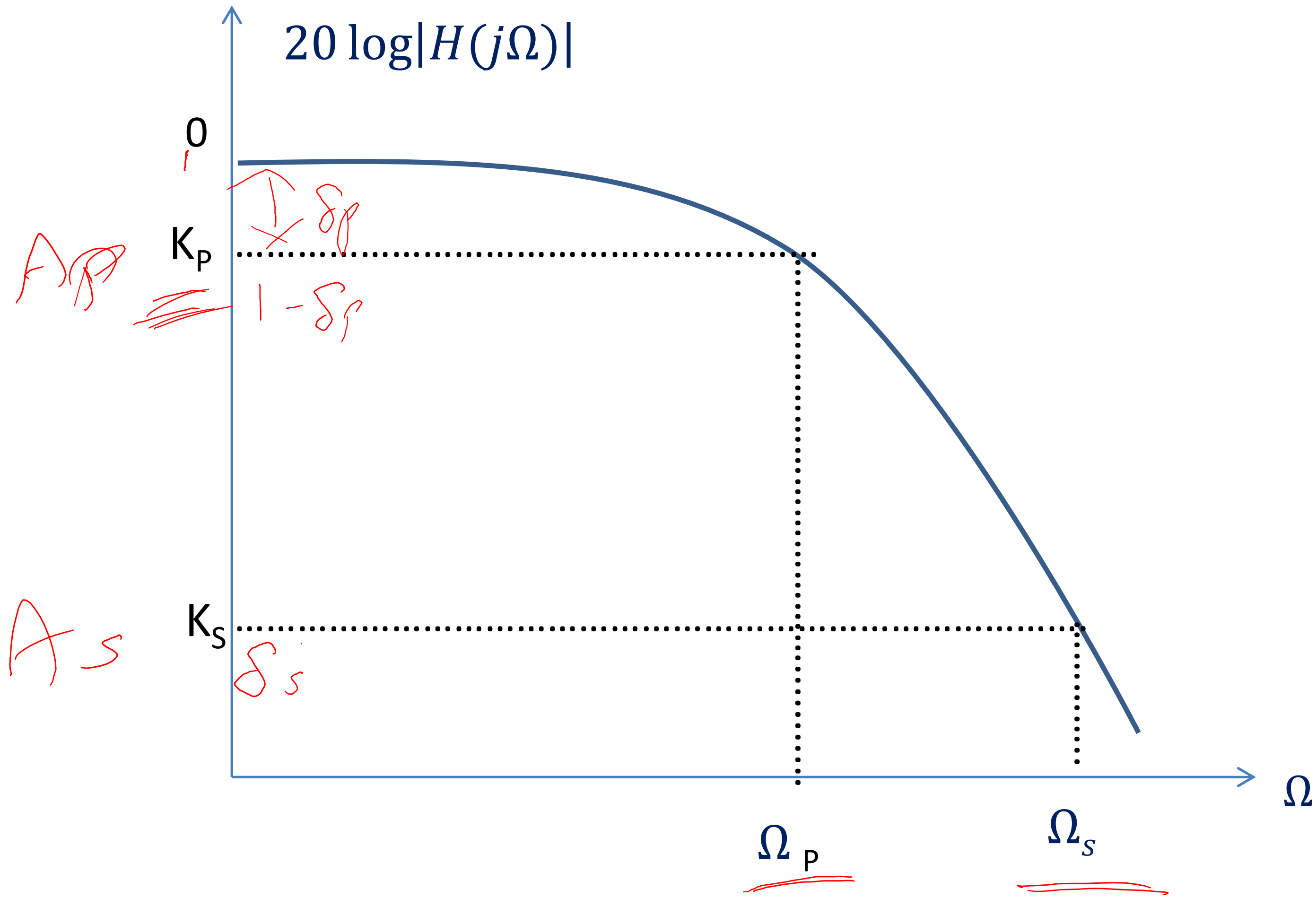
s_k are all the left-half poles

$B_N(s)$ is the Butterworth polynomial of order N.

$H(s) = \frac{1}{B_N(s)}$

Order N	Butterworth polynomial $B_N(s)$
1	$s + 1$
2	$s^2 + \sqrt{2}s + 1$
3	$(s^2 + s + 1)(s + 1)$
4	$(s^2 + 0.76536s + 1)(s^2 + 1.8477s + 1)$
5	$(s + 1)((s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)$

Design of Lowpass Butterworth filter



Determine Order N

- The design of a lowpass filter amounts to the determination of its transfer function. This necessitates the value of the filter order N and cutoff frequency Ω_c.
- The magnitude frequency response of a lowpass Butterworth filter is given by

$$|H(j\Omega)| = \frac{1}{\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]^{\frac{1}{2}}} \text{..................................(7)}$$

Taking 20 log on both the sides of equation (7)

$$20log|H(j\Omega)| = A = -20log\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]^{\frac{1}{2}}$$

$$= -10log\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right] \text{....................(8)}$$

For Passband sub $\Omega = \Omega_P$, $A = A_P$ *in eq. (8)*

$$A_P = -10log\left[1 + \left(\frac{\Omega_P}{\Omega_c}\right)^{2N}\right] \text{.................(9)}$$

$$\left(\frac{\Omega_P}{\Omega_c}\right)^{2N} = 10^{\frac{-A_P}{10}} - 1 \text{.....................(10)}$$

Similarly, for stopband $\Omega = \Omega_S$, $A = A_S$ *in eq.(8)*

$$A_S = -10log\left[1 + \left(\frac{\Omega_S}{\Omega_c}\right)^{2N}\right] \text{................(11)}$$

$$\left(\frac{\Omega_S}{\Omega_c}\right)^{2N} = 10^{\frac{-A_S}{10}} - 1 \text{.....................(12)}$$

- Divide equation (10) by equation (12), we get

$$\left(\frac{\Omega_P}{\Omega_S}\right)^{2N} = \frac{\left(10^{\frac{-A_P}{10}}-1\right)}{\left(10^{\frac{-A_S}{10}}-1\right)} \text{......................(13)}$$

Take log of equation (13)

$$2N\log\left(\frac{\Omega_P}{\Omega_S}\right) = \log\left[\left(10^{\frac{-A_P}{10}}-1\right) \Big/ \left(10^{\frac{-A_S}{10}}-1\right)\right]$$

$$N = \frac{\log\left[\left(10^{\frac{-A_P}{10}}-1\right) \Big/ \left(10^{\frac{-A_S}{10}}-1\right)\right]}{2\log\left(\frac{\Omega_P}{\Omega_S}\right)} \text{................(14)}$$

The filter order must be rounded up to the next larger integer value. Example: N=1.2 then order is 2

- To meet the passband requirement exactly the cutoff frequency is selected from equation (10).

$$\Omega_c = \frac{\Omega_P}{\left(10^{\frac{-A_P}{10}} - 1\right)^{\frac{1}{2N}}}$$

- To meet the stopband requirement exactly the cutoff frequency is selected from equation (12).

$$\Omega_c = \frac{\Omega_S}{\left(10^{\frac{-A_S}{10}} - 1\right)^{\frac{1}{2N}}}$$

- The third option is to take the cutoff frequency as the arithmetic mean of the two cutoff frequencies found above

$$A_P \leq 20log|H(j\Omega)| \leq 0 \qquad for\ all\ \Omega \leq \Omega_P$$

$$20log|H(j\Omega)| \leq A_S \qquad for\ all\ \Omega \ \geq \ \Omega_S$$

Pass band gain at $\Omega = \Omega p$, $Ap = 20 \log(1- \delta p)$

Stop band gain at $\Omega = \Omega s$, $As = 20 \log(\delta s)$

$$0.8 \le |H(\omega)| \le 1 \qquad 0 < \omega < 0.2\pi$$

$$|H(\omega)| \le 0.2 \qquad 0.32\pi < \omega < \pi$$

Find N

$$K_p = 20\log(0.8) = -1.93$$

$$K_s = 20\log(0.2) = -13.979$$

$$\Omega_p = 0.2\pi, \quad \Omega_s = 0.32\pi$$

$$N = \frac{\log_{10}\left[\left(10^{\frac{1.93}{10}} - 1\right) \Big/ 10^{\frac{13.979}{10}} - 1\right]}{2\log_{10}\frac{0.2\pi}{0.32\pi}}$$

$$= 3.986 \simeq 4$$

Frequency Transformations

- If the specified filter is low pass then apply lowpass to lowpass transformation on the normalized low pass filter by replacing $s \rightarrow . \frac{s}{\Omega c}$

- If the specified filter is high pass then apply lowpass to highpass transformation on the normalized low pass filter by replacing $s \rightarrow \frac{\Omega_c}{s}.$

- If the specified filter is Bandpass then apply lowpass to Bandpass transformation on the normalized low pass filter by replacing

$$\mathrm{s} \rightarrow \frac{s^2+\Omega_u\Omega_l}{s(\Omega_u-\Omega_l)}.$$

- If the specified filter is Bandstop then apply lowpass to Bandstop transformation on the normalized low pass filter by replacing

$$\mathrm{s} \rightarrow \frac{s(\Omega_u-\Omega_l)}{s^2+\Omega_u\Omega_l}.$$

- If the given filter is high pass then the backward design equation to find the stopband edge frequency $\boldsymbol{\Omega_s}$ is $\frac{\boldsymbol{\Omega_P}}{\boldsymbol{\Omega_S}}$.

- If the given filter is Bandpass then the backward design equation to find the stopband edge frequency $\boldsymbol{\Omega_s} =$ $Min\{|A|, |B|\}$. Where $A = \frac{-{\Omega_1}^2 + \Omega_l \Omega_u}{\Omega_1 \ (\Omega_u - \Omega_l)}$, $B = \frac{{\Omega_2}^2 - \Omega_l \Omega_u}{\Omega_2 \ (\Omega_u - \Omega_l)}$

- If the given filter is Bandstop then the backward design equation to find the stopband edge frequency $\boldsymbol{\Omega_s} =$ $Min\{|A|, |B|\}$. Where $A = \frac{\Omega_1 \ (\Omega_u - \Omega_l)}{-{\Omega_1}^2 + \Omega_l \Omega_u}$, $B = \frac{\Omega_2 \ (\Omega_u - \Omega_l)}{{\Omega_2}^2 - \Omega_l \Omega_u}$

- The normalized passband edge frequency $\boldsymbol{\Omega_P}$ is always equal to 1 rad/sec irrespective of the given filter.

Problem 1

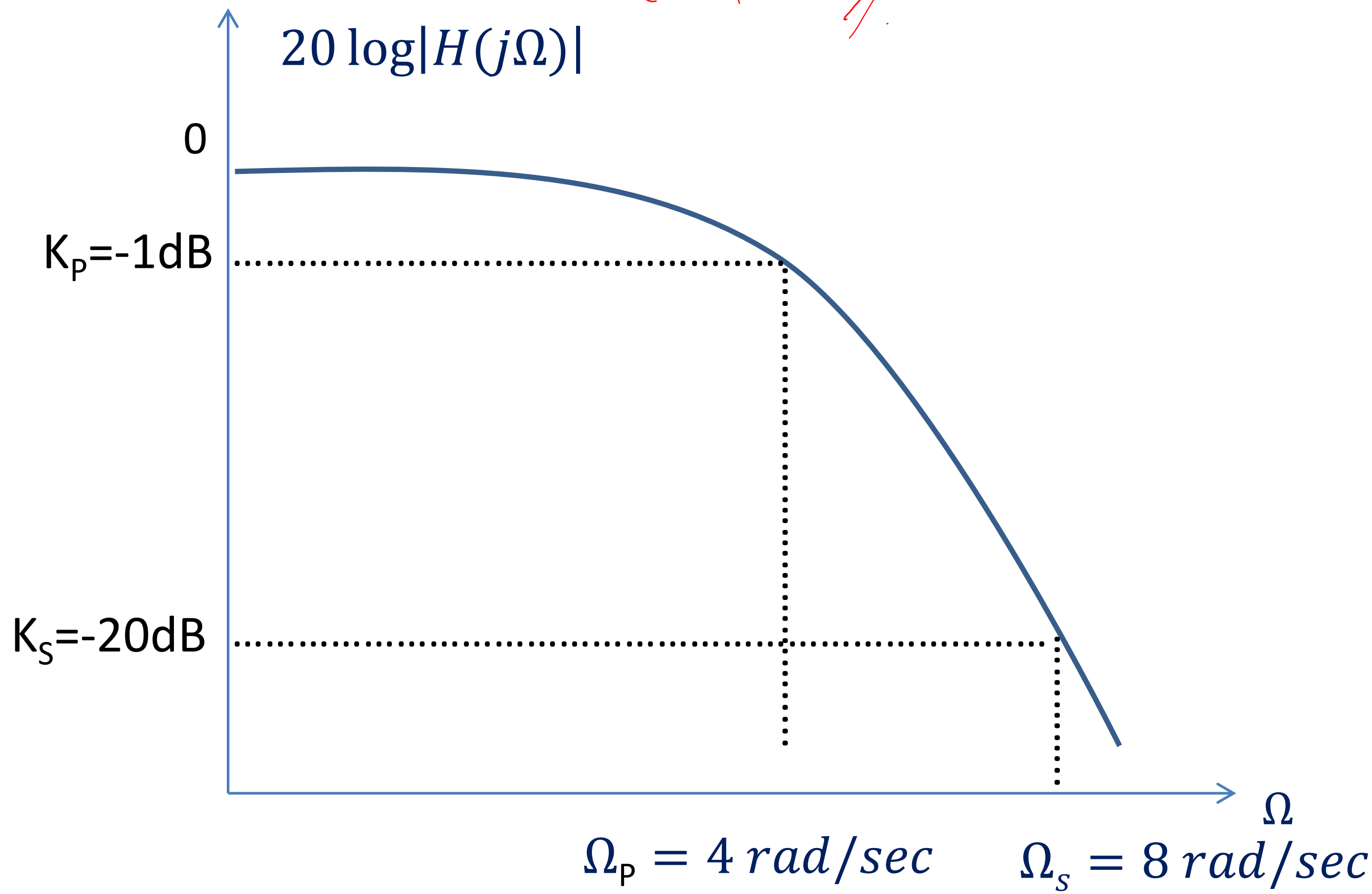
- A Butterworth lowpass filter has to meet the following specifications.

Passband gain, $K_P = -1\text{dB}$ at $\Omega_P = 4\, rad/sec$.

Stopband attenuation greater than or equal to 20dB at $\Omega_S = 8\, rad/sec$.

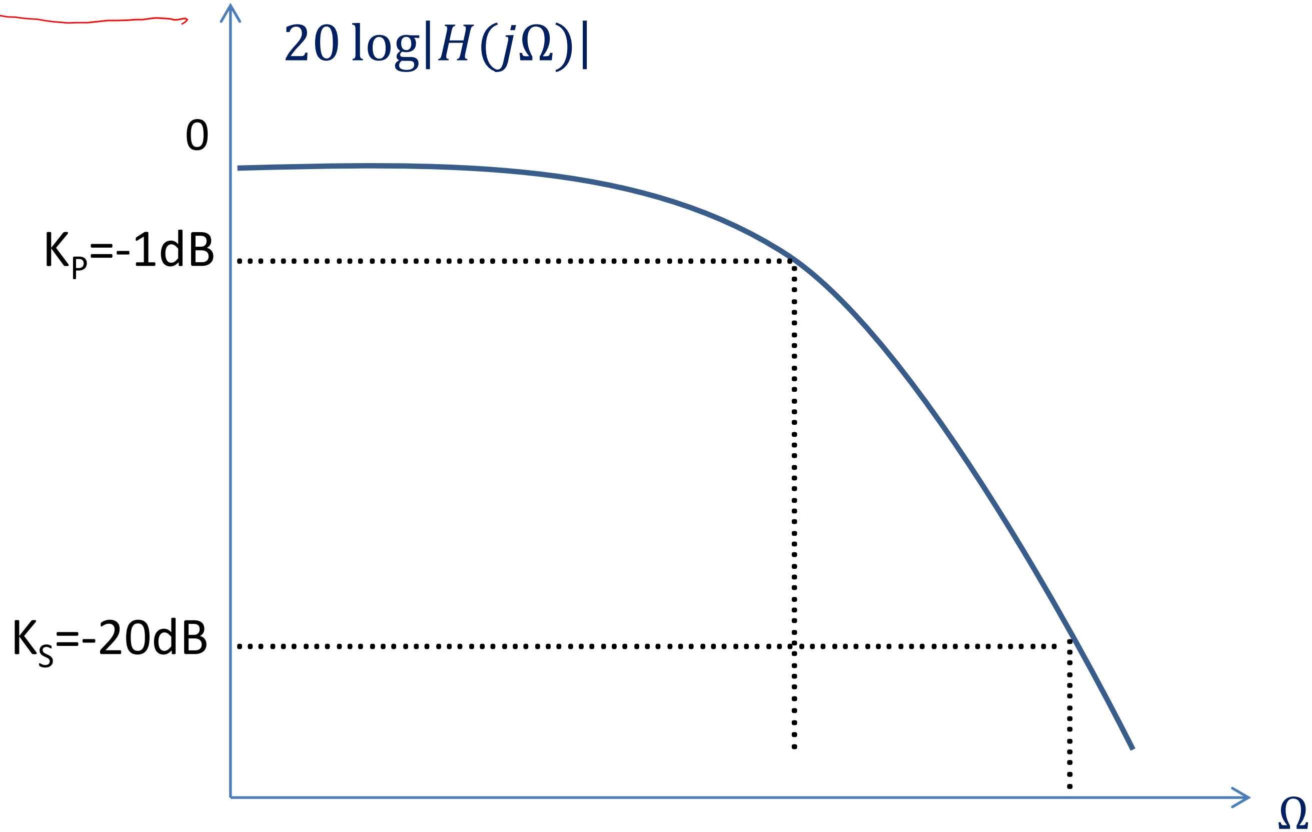
Determine the transfer function $H_a(s)$ of the lowest-order Butterworth filter to meet the above specification.

- Solution: Step 1 Digital Filter Specs



Specified magnitude frequency response of the lowpass Butterworth filter

- **Step 2**: Magnitude frequency response of the normalized lowpass filter



$\boldsymbol{\Omega_P = 1\, rad/sec}$ $\boldsymbol{\Omega_s = 2\, rad/sec}$

The pass band edge frequency $\boldsymbol{\Omega_P}$ of the normalized low pass filter is 1 rad/sec
Let us the backward design equation to find the stopband edge frequency $\boldsymbol{\Omega_s}$ of the normalized low pass filter $\boldsymbol{\Omega_s} = \frac{\boldsymbol{\Omega_s}}{\boldsymbol{\Omega_P}} = \frac{8}{4} = 2\, rad/sec$.

This backward equation is for Low pass to Lowpass transformation

Step 3: Find the order N of the filter using equation (14).

Sub K_P=-1 dB, K_S=-20 dB, Ω_P=1 rad/sec, Ω_S=2 rad/sec.

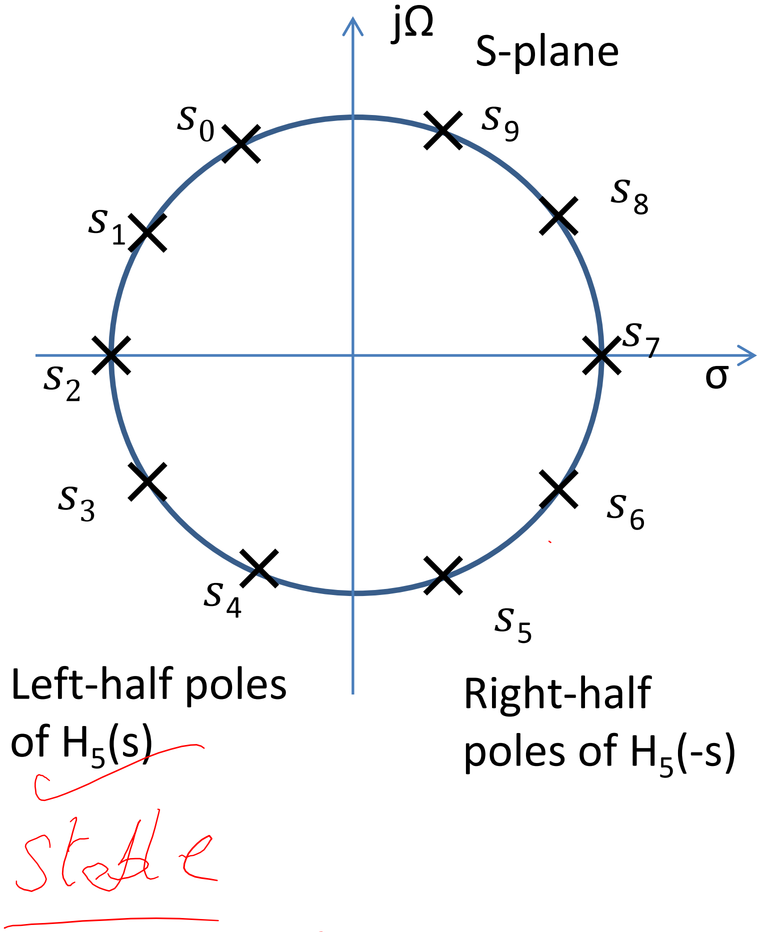
$$N = \frac{log\left[\left(10^{\frac{-K_P}{10}}-1\right)\Big/\left(10^{\frac{-K_S}{10}}-1\right)\right]}{2log\left(\frac{\Omega_P}{\Omega_S}\right)} = 4.289 = 5$$

Step 4: Now proceed to find the transfer function of the 5th order normalized lowpass filter. Find the poles of the 5th order normalized low pass filter using equation (5)

$$s_k = e^{j\pi\frac{(2k+1)}{2N}} e^{j\frac{\pi}{2}} \qquad k = 0,1,2, \ldots \ldots 2N-1$$

N=5, so, $K = 0, 1, 2,3,4,5,6,7,8,9$

Total 10 poles from S_0 to S_9



Poles	σ+jΩ
s_0	-0.309+j0.951
s_1	-0.809+j0.588
s_2	-1
s_3	-0.809-j0.588
s_4	-0.309-j0.951
s_5	0.309-j0.951
s_6	0.809-j0.588
s_7	1
s_8	0.809+j0.588
s_9	0.309+j0.951

Step 5:

Hence, the transfer function of the 5th order normalized lowpass Butterworth filter is

$$H_{N(s)} = \frac{1}{\prod_{LHP}(s - s_k)}$$

- $H_5(s) = \dfrac{1}{(s-s_0)(s-s_1)(s-s_2)(s-s_3)(s-s_4)}$

$$= \frac{1}{(s + 0.309 - j0.951)(s + 0.809 - j0.588)(s + 1)(s + 0.809 + j0.588)(S + 0.309 + j0.951)}$$

$$H_5(s) = \frac{1}{(s + 1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)}$$

STEP 6: Find the Cutoff frequency Ω_c to exactly meet the pass band requirement.

$$\Omega_c = \frac{\Omega_P}{\left(10^{\frac{-K_P}{10}} - 1\right)^{\frac{1}{2N}}} = 4.5787$$

$s \to \dfrac{s}{\Omega_c}$

Sub $\Omega_P = 4\, rad/\sec\;\; N = 5\;\; K_P = -1\text{dB}$

- Step 7: The specified Lowpass filter is obtained by applying lowpass to lowpass transformation on the normalized low pass filter.
- $H_a(s) = H_5(s)|s \to \frac{s}{\Omega_c = 4.5787}$
- $= \frac{2012.4}{s^5 + 14.82s^4 + 109.8s^3 + 502.6s^2 + 14222.3s + 2012.4}$

Step 8 S -> Z domain BLT ->

IIV ->

3) $H(s) = \dfrac{1}{S^2+S+1}$ LPF with $\Omega_p = 1\text{rad/sec}$

Convert $\Rightarrow$ LPF with $\Omega_p = 10\text{rad/sec}$

$\Rightarrow$ HPF with $\Omega_c = 1\text{rad/sec}$

$\Rightarrow$ HPF with $\Omega_c = 10\text{rad/sec}$.