Module 2

Representation of Signals in Frequency Domain

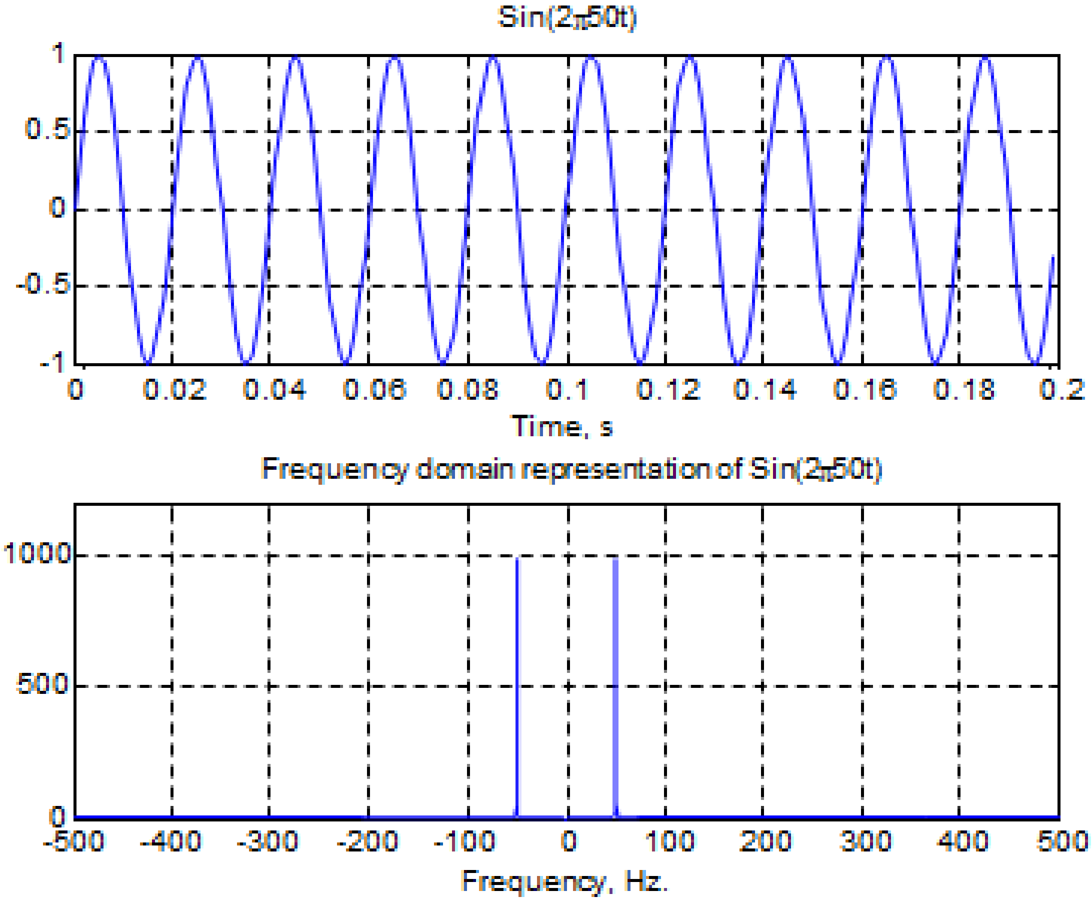
Intro

- Time domain operation are often not very informative and/or efficient in signal processing

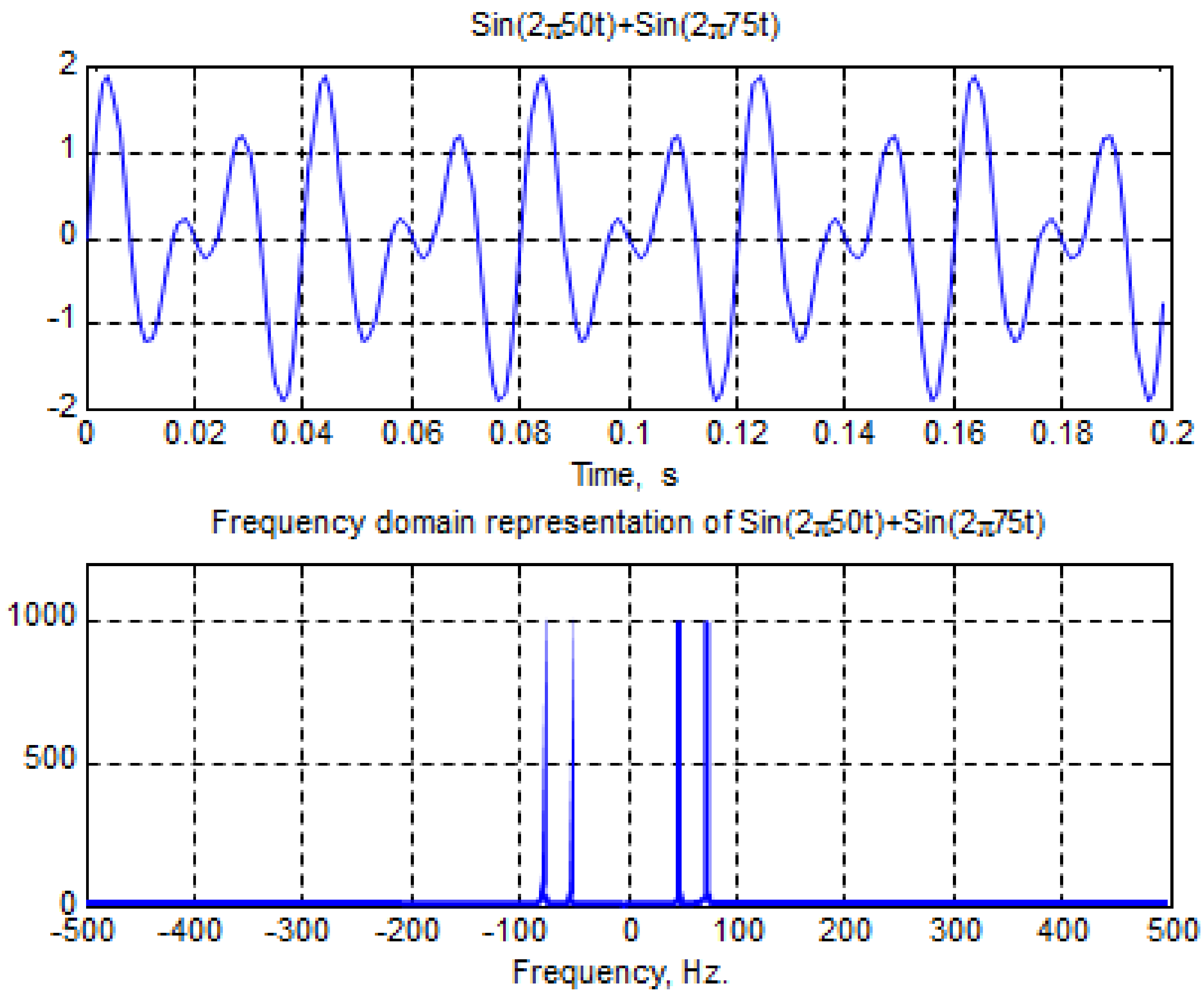
- An alternative representation and characterization of signals and systems can be made in transform domain

- more information can be extracted from a signal in the transform / frequency domain.

- Many operations that are complicated in time domain become rather simple algebraic expressions in transform domain.



Sin(2π50t)
1
0.5
0
-0.5
-1
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
Time, s
Frequency domain representation of Sin(2π50t)
1000
500
0
-500
-400
-300
-200
-100
0
100
200
300
400
500
Frequency, Hz.



Sin(2π50t)+Sin(2π75t)
2
1
0
-1
-2
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
Time, s
Frequency domain representation of Sin(2π50t)+Sin(2π75t)
1000
500
0
-500
-400
-300
-200
-100
0
100
200
300
400
500
Frequency, Hz.

Jean B. Joseph Fourier
(1768-1830)

*"An **arbitrary** function, continuous or with discontinuities, defined in a finite interval by an arbitrarily capricious graph can always be expressed as a sum of sinusoids"*

J.B.J. Fourier

December 21, 1807

Fourier Transforms

Fourier Series (FS) ← CTFS ⊃ DTFS

- A periodic function can be represented as a finite, weighted sum of sinusoids that are integer multiples of the fundamental frequency ω_0 of the signal.
- These frequencies are harmonically related, or simply **harmonics**.

Continuous Time Fourier Transform (FT)

- Extension of Fourier series to non-periodic functions: Any continuous aperiodic function can be represented as an infinite sum (integral) of sinusoids.

Discrete Time Fourier Transform (DTFT)

- Extension of FT to discrete sequences. Any discrete function can also be represented as an infinite sum (integral) of sinusoids.

Discrete Fourier Transform (DFT)

- An extension to DTFT is DFT, where the frequency variable is also discretized.

Fast Fourier Transform (FFT)

- Mathematically identical to DFT, however a significantly more efficient implementation.

Dirichlet's Conditions

Dirichlet put the final period to the discussion on the feasibility of Fourier transform by proving the necessary conditions for the existence of Fourier representations of signals

- signal must have finite number of discontinuities
- signal must have finite number of extremum points within its period
- signal must be absolutely integrable within its period

$$\int_{t_0}^{t_0+T} |x(t)| dt < \infty$$

CT

Fourier Series

Periodic

Time ⇒ Continuous Signal

Freq ⇒ Discrete

***Synthesis equation*:**

$$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t}=\sum_{k=-\infty}^{\infty}a_k e^{jk(2\pi/T)t}$$

***Analysis equation*:**

$$a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt=\frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

The Fourier Transform

$T \to \infty$

$\omega_0 \to$ infinitely small $\to 0$.

The continuous Fourier transform (FT), can be obtained from the FS representation of a signal, by assuming that the non-periodic signal is periodic with an infinite period.

Time → Contin

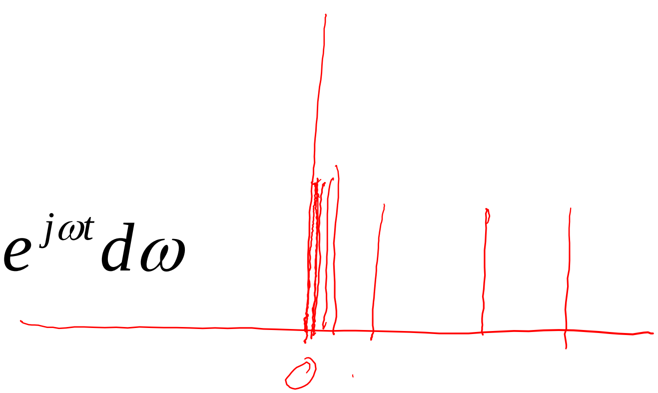
freq → continuous

***Analysis equation*:**

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

***Synthesis equation*:**

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$



Property	Signal	Fourier transform
	$x(t)$	$X(\omega)$
	$x_1(t)$	$X_1(\omega)$
	$x_2(t)$	$X_2(\omega)$
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{\omega}{a}\right)$
Time reversal	$x(-t)$	$X(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time differentiation	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
Frequency differentiation	$(-jt)x(t)$	$\frac{dX(\omega)}{d\omega}$

Integration	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\pi X(0)\delta(\omega) + \frac{1}{j\omega} X(\omega)$
Convolution	$x_1(t) * x_2(t)$	$X_1(\omega) X_2(\omega)$
Multiplication	$x_1(t) x_2(t)$	$\frac{1}{2\pi} X_1(\omega) * X_2(\omega)$
Real signal	$x(t) = x_e(t) + x_o(t)$	$X(\omega) = A(\omega) + jB(\omega)$
		$X(-\omega) = X^*(\omega)$
Even component	$x_e(t)$	$\operatorname{Re}\{X(\omega)\} = A(\omega)$
Odd component	$x_o(t)$	$j\operatorname{Im}\{X(\omega)\} = jB(\omega)$

Parseval's relations

$$\int_{-\infty}^{\infty} x_1(\lambda) X_2(\lambda)\, d\lambda = \int_{-\infty}^{\infty} X_1(\lambda) x_2(\lambda)\, d\lambda$$

$$\int_{-\infty}^{\infty} x_1(t) x_2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) X_2(-\omega)\, d\omega$$

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

Discrete Time Fourier Transform

- Similar to Continuous Time (CT) signals, the Discrete Time (DT) signals or sequences can also be periodic or non-periodic
- **Periodic or non-periodic DT sequences** result in **DTFS** or **DTFT** respectively
- DTFT is important since most of the signals in engineering applications are non-periodic
- DTFT is denoted as $X(e^{j\omega})$ or simply $X(\omega)$

$z =$

$X(re^{j\omega})$

	Time	Freq
DTFS	Discrete	Discrete
DTFT	Discrete	Continuous.

- **DTFT Analysis equation**

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

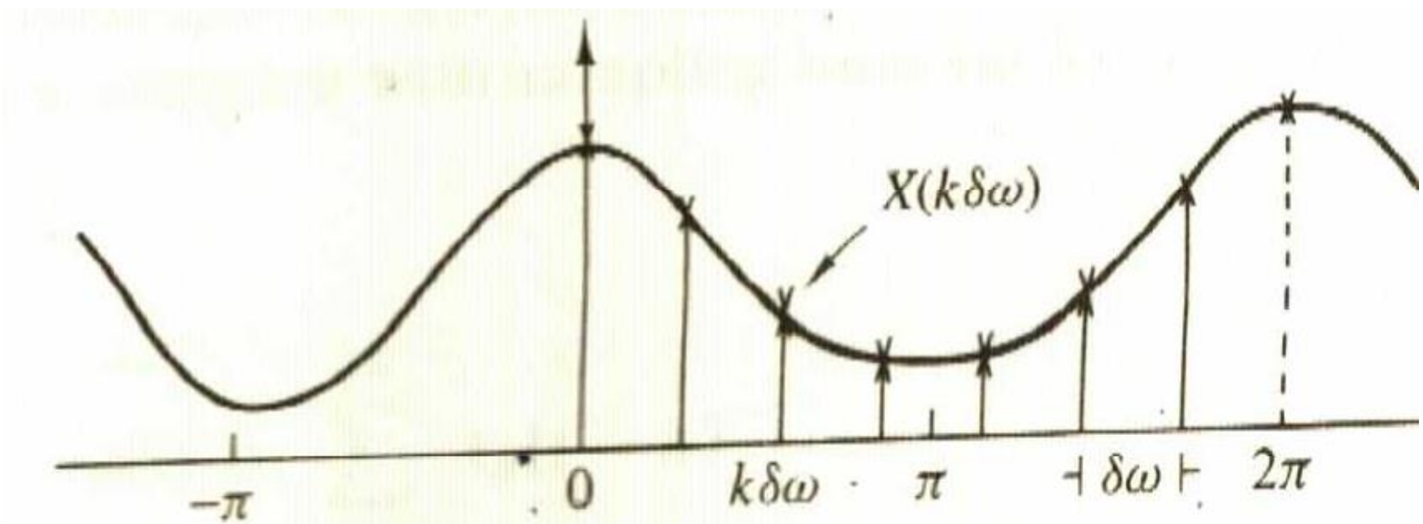
- **DTFT Synthesis equation**

$$x[n] = \frac{1}{2\pi}\int_{0}^{2\pi} X(e^{j\omega})\, e^{j\omega n}\, \mathrm{d}\omega$$

DTFT

- Fourier transform of discrete time signal

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$



Discrete Fourier Transform (DFT)

- **The sampled DTFT of a finite length DT signal is known as DFT**
- DFT is representation of a sequence x[n] by samples of its spectrum X(ω)

Definition - The simplest relation between a length-N sequence $x[n]$, defined for $0 \leq n \leq N-1$, and its DTFT X(ω) is obtained by uniformly sampling X(ω) i.e. frequency domain sampling, on the ω-axis

$$0 \leq \omega \leq 2\pi \text{ at } \omega_k = 2\pi k/N \qquad 0 \leq k \leq N-1$$

DFT evolving from DTFT

- Consider a DT signal x[n] having a finite duration, in the range $0 \leq n \leq$ N-1, DTFT of this signal is

  $X(e^{j\omega})$ =$\sum_{n=0}^{\mathrm{N}-1} x[n]e^{-j\omega n}$ ....(1)

- X(ω) is periodic with period 2π

- Sample X(ω) using a total of N equally spaced samples in: $\omega \in (0, 2\pi)$.

- The sampling interval is 2π/N

DFT

$$\omega = \frac{2\pi}{N} k$$

$$X\left(\frac{2\pi}{N} k\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\left(\frac{2\pi}{N}\right)nk} \qquad k = 0,1,\ldots N-1$$

$$X\left(\frac{2\pi}{N} k\right) = \ldots + \sum_{n=-N}^{-1} x[n] e^{-j\left(\frac{2\pi}{N}\right)nk} + \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2\pi}{N}\right)nk}$$

$$+ \sum_{n=N}^{2N-1} x[n] e^{-j\left(\frac{2\pi}{N}\right)nk} + \ldots$$

$$X\left(\frac{2\pi}{N} k\right) = \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x[n] e^{-j\left(\frac{2\pi}{N}\right)nk}$$

Changing inner summation n to (n-lN), and interchange the order

$$X\left(\frac{2\pi}{N}k\right)=\sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty}x[n-lN]\right]e^{-j\left(\frac{2\pi}{N}\right)nk} \quad k=0,1,\ldots N-1$$

$$x_p[n]=\sum_{l=-\infty}^{\infty}x[n-lN]$$

$$X\left(\frac{2\pi}{N}k\right)=\sum_{n=0}^{N-1}x_p[n]e^{-j\left(\frac{2\pi}{N}\right)nk} \quad k=0,1,\ldots N-1$$

$x_p(n)$ is periodic repetition of x(n) after N samples, so it $x_p(n)$ is periodic with fundamental period 'N'

So $x_p(n)$ can be expressed using Fourier series

$$x_p[n]=\sum_{k=0}^{N-1}a_k e^{j\left(\frac{2\pi}{N}\right)nk} \quad n=0,1,\ldots N-1$$

$$x_p[n] = \sum_{k=0}^{N-1} a_k e^{j\left(\frac{2\pi}{N}\right)nk} \qquad n = 0,1,\ldots N-1$$

FS coeff.

$$a_k = \frac{1}{N}\sum_{k=0}^{N-1} x_p[n] e^{-j\left(\frac{2\pi}{N}\right)nk} \qquad k = 0,1,\ldots N-1$$

Comparing above equation with X(2π/N nk)

$$a_k = \frac{1}{N} X\left(\frac{2\pi}{N}k\right) \qquad k = 0,1,\ldots N-1$$

IDFT.

$$x_p[n] = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j\left(\frac{2\pi}{N}\right)nk} \qquad n = 0,1,\ldots N-1$$

$x(n) = \{ x_p(n) \quad N$

DFT:

$$X\left(\frac{2\pi}{N}k\right) \equiv X(k) = \sum_{n=0}^{N-1} x[n]e^{-j\left(\frac{2\pi}{N}\right)nk} \quad k = 0,1,\ldots N-1$$

IDFT:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j\left(\frac{2\pi}{N}\right)nk} \quad n = 0,1,\ldots N-1$$

$$x[n] = \{1, 2, 1, 2\} \qquad X(k) = \sum_{n=0}^{3} x[n] e^{-j\frac{2\pi}{4}nk}$$

$$k = 0, 1, 2, 3$$

$$X(0) = \sum_{n=0}^{3} x[n] = 1 + 2 + 1 + 2 = 6$$

$$X(1) = \sum_{n=0}^{3} x[n] e^{-j\frac{\pi}{2}n} = x[0] + x[1] e^{-j\pi/2} + x[2] e^{-j\pi} + x[3] e^{-j3\pi/2}$$

$$= 1 + 2\left(\cos\pi/2 - j\sin\pi/2\right) + 1(-1) + 2(+j)$$

$$= 1 - 2j - 1 + 2j = 0$$

$$X(2) = \sum_{n=0}^{3} x[n] e^{-jn\pi} = -2$$

$$X(3) = \sum_{n=0}^{3} x[n] e^{-j\frac{3\pi}{2}n} =$$

$$X(3) = \sum_{n=0}^{3} x[n] e^{-j\frac{3\pi}{2}n}$$

$$= x(0) + x(1)\left(\cos 3\pi/2 - j\sin 3\pi/2\right)$$

$$+ x(2)\left[\cos 3\pi - j\sin 3\pi\right)$$

$$+ x(3)\left[\cos 9\pi/2 - j\sin 9\pi/2\right)$$

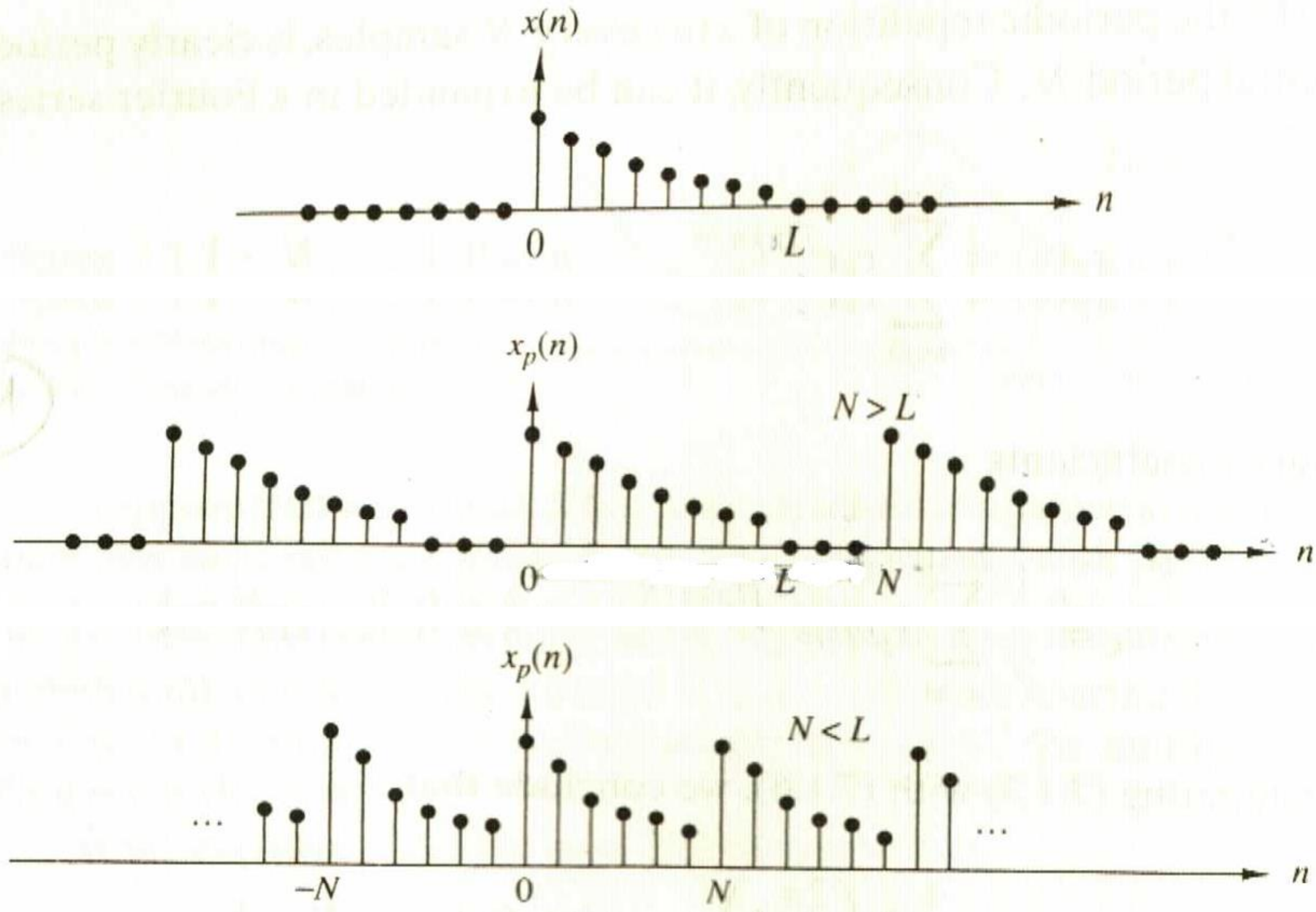
$$= 1 + 2(j) + 1(-1) + 2\left(-j\sin\left(\pi + \pi/2\right)\right)$$

$$= 1 + 2j - 1 - 2j = 0$$

$$X(k) = \{6, 0, -2, 0\}$$

Relation between DTFT and DFT

- DTFT is a continuous transform. Sampling the DTFT at regularly spaced intervals around the unit circle gives the DFT
- The sampling operation in one domain causes the transform in the other domain to be periodic
- Similar to reconstructing a CT signal from its samples, DTFT can also be synthesized from its DFT samples
- This reconstruction is possible, as long as the no. of points N at which DFT is computed is equal to or larger than the samples of original signal


x(n)
n
0
L
x_p(n)
N > L
n
0
L
N
x_p(n)
N < L
...
...
n
−N
0
N

DFT as a linear Transformation

- DFT : $$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2\pi}{N}\right)nk} \quad k = 0,1,\ldots N-1$$
- IDFT : $$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\left(\frac{2\pi}{N}\right)nk} \quad n = 0,1,\ldots N-1$$
- Twiddle factor is periodic with period N $$W_N = e^{-j\left(\frac{2\pi}{N}\right)}$$
- Then,
- DFT: $$X(k) = \sum_{n=0}^{N-1} x[n] W_N^{nk} \quad k = 0,1,\ldots N-1$$
- IDFT : $$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-nk} \quad n = 0,1,\ldots N-1$$

- Computation of each point of DFT requires
  - N complex multiplications
  - (N-1) complex additions
- N-point DFT have
  - N^2 Complex multiplication
  - N(N-1) complex addition
- Let us define an N point Vector x_N for the signal sequence x[n]
- DFT X(k) as N point vector X_N

DFT in Matrix form

$$X(k) = DFT\ \{x[n]\} = \sum_{n=0}^{N-1} x[n]{W_N}^{kn} \qquad 0 \le k \le N-1$$

$X(0) = {W_N}^0$ x(0) + ${W_N}^0$ x(1) + ………+ ${W_N}^0$ x(N-1)

$X(1) = {W_N}^0$ x(0) + ${W_N}^1$ x(1) +………+ ${W_N}^{(N-1)}$x(N−1)

$X(2) = W_N^0 x(0) + W_N^2 x(1) + W_N^4 x(2) + \ldots W_N^{2(N-1)} x(N-1)$

..

..

..

X(N−1) = ${W_N}^0$ x(0) + ${W_N}^{N-1}$x(1) +………+ ${W_N}^{(N-1)(N-1)}$x(N-1)

$$\mathrm{x} = \begin{bmatrix} \mathrm{x}[0] \\ \mathrm{x}[1] \\ \vdots \\ \mathrm{x}[N-1] \end{bmatrix} \qquad \mathrm{X} = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\begin{array}{ccccc} 0 & 1 & k \quad 2 & \cdots & N-1 \end{array}$$

$$W_N{}^{0}$$

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \\ N-1 \end{array} = \begin{bmatrix} W_N{}^{0} & W_N{}^{0} & W_N{}^{0} & .. & W_N{}^{0} \\ W_N{}^{0} & W_N{}^{1} & W_N{}^{2} & .. & W_N{}^{N-1} \\ W_N{}^{0} & W_N{}^{2} & W_N{}^{4} & .. & W_N{}^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ W_N{}^{0} & W_N{}^{N-1} & W_N{}^{2(N-1)} & .. & W_N{}^{(N-1)^2} \end{bmatrix}$$

DFT can be expressed in matrix form as $[X] = [W_N][x]$

W_N is a symmetric matrix

IDFT

$X = W_N x$

$$W_N^{-1} X = W_N^{-1} W_N \, x$$

$$x = W_N^{-1} X$$

$N = 4$

- From previous IDFT equation we express in matrix form

$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} x(k) e^{j\frac{2\pi}{N}nk}$

IDFT

$$\boxed{[x_N] = \frac{1}{N} [W_N^*] [X_N]}$$

$W_N = e^{-j2\pi/N}$

W^*_N denotes the complex conjugate of W_N

$$where \;\; W_N^{-1} = \frac{1}{N} W_N^*$$

$$W_N W_N^* = N I_N$$

Orthonormality property

$$[X] = [W_N][x] \qquad x[n] = \{1, 2, 1, 2\}$$

$$N = 4$$

$$[W_N] = \begin{array}{c} \\ 0 \\ n=1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & W_N^3 \\ W_N^0 & W_N^2 & W_N^4 & W_N^6 \\ W_N^0 & W_N^3 & W_N^6 & W_N^9 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-j\pi/2} & e^{-j\pi} & e^{-j3\pi/2} \\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} \\ 1 & e^{-j3\pi/2} & e^{-j3\pi} & e^{-j9\pi/2} \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$[X] = [W][x]$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 1-2j-1+2j \\ 1-2+1-2 \\ 1+2j-1-2j \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

IDFT

$$x = \frac{1}{N} W_N^{*} X$$

$$W_N^{*} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} & & W_N^{*} & & \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$