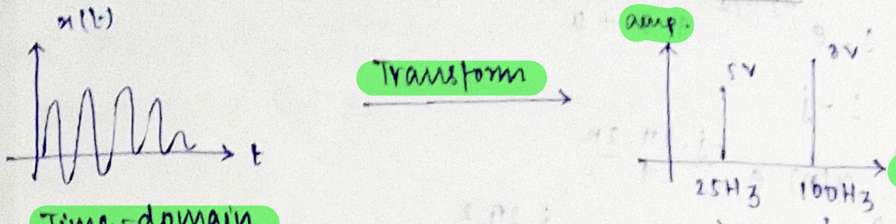
MOD - 3

Discrete Domain Fourier Transform



Time-domain [freq. not well defined]

Bandwidth

freq. domain / spectral domain

Analog (CT)	DT
Fourier Tr.	Z-tr.
Laplace Tr.	DTFT ✓
Fourier Series	DFT (Discrete FT)
	FFT (fast method of DFT)

Discrete Fourier Transform (DFT)

* DFT of a DT signal can be written as:

$$DFT[x(n)] = X(K) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

where, $0 \leq K \leq N-1$

* Inverse DFT

$$IDFT[X(K)] = x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{\frac{j2\pi kn}{N}}$$

where, $0 \leq n \leq N-1$

DFT pairs.

Q. Find the DFT for the given DT sequence, $x(n) = \{1, 1, 0, 0\}$ (indices 0 1 2 3). Take the inverse of $X(K)$ and find $x(n)$. Also plot the magnitude spectrum and phase spectrum.

→ $X(K) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$ → DFT

N = 4 (length)

$K=0$, $X(0) = \sum_{n=0}^{3} x(n)\, \underbrace{e^{\frac{-j2\pi(0)n}{4}}}_{1}$

$= x(0) + x(1) + x(2) + x(3)$

$= 1 + 1 + 0 + 0 = 2.$

$K=1$, $X(1) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi \cdot n}{4}}$

$= x(0).\, e^{-j2\pi.0} + x(1)\, e^{\frac{-j2\pi}{4}} + x(2)\, e^{\frac{-j2\pi.2}{4}} + x(3)\, e^{-j\ldots}$

$= 1 + 1.\, e^{-j\pi/2} + 0 + 0$

$= 1 - j$

$$
\begin{aligned}
e^{-i\pi/2} &= \cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2} \\
&= \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} \\
&= 0 + i\times(-1) \\
&= -i
\end{aligned}
$$

$K=2,\quad X(2) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j.2\pi.2n}{4}}$

$= x(0).\, e^{0} + x(1).\, e^{\frac{-j.2\pi.2}{4}} + 0 + 0$

$= 1 + 1.\, \underbrace{e^{-j.\pi}}_{\cos(\pi) - i\sin(\pi) = -1-0=-1}$

$= 0.$

$\sin(-\pi) + j\cos(-\pi)$

-1

$1 + j.0$

$K=3,\quad X(3) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j.2\pi.3.n}{4}}$

$= x(0).\, e^{0} + x(1).\, e^{\frac{-j.2\pi.3}{4}} + 0 + 0$

$= 1 + 1.\, e^{-j.3\pi/2} \quad (0+j)$

$-1+0$

$= 1 + j$

$\therefore X(K) = \{2,\ 1-j,\ 0,\ 1+j\}$

$n=0,\quad x(0) = \sum_{K=0}^{3} X(K)\, e^{\frac{j2\pi Kn}{N}} \rightarrow$ IDFT

$= X(0).\, e^{0} + X(1).\, e^{\frac{j.2\pi.0}{4}} + X(2).\, e^{\frac{j.2\pi.2.0}{4}} + X(3).\, e^{0}$

$= 2 + 1 - j + 0 + 1 + j$

$= 4.$

$n=1,\quad x(1) = \sum_{K=0}^{3} X(K).\, e^{\frac{j2\pi Kn}{N}}$

$= X(0).\, e^{0} + X(1).\, e^{\frac{j.2\pi.1}{4}} + X(2).\, e^{\frac{j.2\pi.2}{4}} + X(3).\, e^{\ldots}$

$= 2 + (1-j).\, e^{j\pi/2} + 0 + (1+j).\, e^{j3\pi/2}$

$= 2 + (1-j).\, j + (1+j).\, -j$

$= 2 + j + 1 - j + 1$

$= 4$

$n=2,\quad x(2) = \sum_{K=0}^{3} X(K).\, e^{\frac{j2\pi Kn}{N}}$

$= X(0).\, e^{0} + X(1).\, e^{\frac{j.2\pi.2}{4}} + X(2).\, e^{\frac{j.2\pi.2.2}{4}} + X(3).\, e^{\ldots}$

$= 2 + (1-j).\, e^{j\pi} + 0 + (1+j).\, e^{j.3\pi}$

$= 2 + (1-j).(-1) + (1+j)(-1) = 2 + j - 1 - 1 - j = 0$

$n = 3, \; x(3) = 0.$

$\therefore x(n) = \{4, 4, 0, 0\}$

we are getting $4x(n) = N x(n)$.

Dividing by 4, $x(n) = \{1, 1, 0, 0\}$.

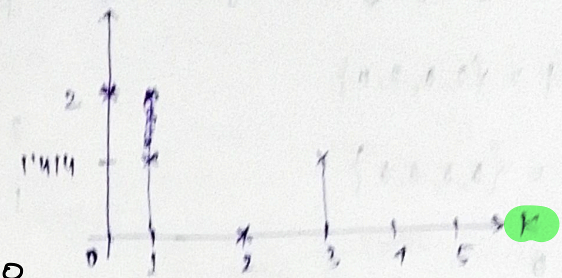
$$\boxed{\begin{array}{c} e^{j\theta} \\ \cos\theta + j\sin\theta \end{array}}$$

Magnitude spectrum : $\sqrt{1^2 + (-1)^2}$

$X(k) = \{2, 1-j, 0, 1+j\}$

$|X(k)| = \{2, \sqrt{2}, 0, \sqrt{2}\}$

$= \{2, 1.414, 0, 1.414\}$



$2 \rightarrow \frac{0}{2} \rightarrow 0$

$0 \rightarrow 0/0 \Rightarrow \infty \rightarrow 90.$

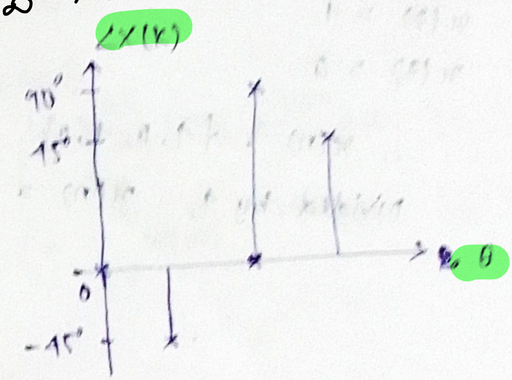
Phase spectrum :

$\angle X(k) = \tan^{-1} \frac{Im}{Re}$

$\rightarrow \infty$

$= \{0, \tan^{-1}\frac{-1}{1}, \frac{0}{0}, \tan^{-1}\frac{1}{1}\}$

$= \{0, -45°, 90°, 45°\}.$



Q. Given, $x(n) = \{1, 0, 1, 0\}$

$\rightarrow \quad X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$

$X(0) = \sum_{n=0}^{3} x(n) \cdot e^{-j2\pi \cdot 0}$

$= x(0) + x(1) + x(2) + x(3) = 2.$

$X(1) = \sum_{n=0}^{3} x(n) \cdot e^{\frac{-j \cdot 2\pi \cdot n}{4\,2}}$

$= x(0) \cdot e^{-j \cdot 0\pi/2} + x(1) \cdot e^{-j\pi/2} + x(2) \cdot e^{-j\pi} + x(3) \cdot e^{-j3\pi/2}$

-1

$= 1 + 0 + (-1) + 0 = 0 \; \cancel{deg} \; 0$

$X(2) = \sum_{n=0}^{3} x(n) \cdot e^{\frac{-j \cdot 2\pi \cdot 2n}{4}}$

$= x(0) \cdot e^0 + \cancel{x(1)}\, 0 + x(2) \cdot e^{\frac{-j \cdot 2\pi \cdot 2 \cdot 2}{4}} + 0$

$= 1 + 1 \cdot e^{\frac{-j2\pi}{1}} = \cancel{1+1} \; 1+1 = 2$

$X(3) = \sum_{n=0}^{3} x(n) \cdot e^{\frac{-j \cdot 2\pi \cdot 3 \cdot n}{4}} = x(0) \cdot e^0 + 0 + x(2) \cdot e^{\frac{-j \cdot 2\pi \cdot 3 \cdot 2}{4}} + 0$

$= 1 + 1 \cdot e^{-j3\pi} = 1 - 1 = 0.$

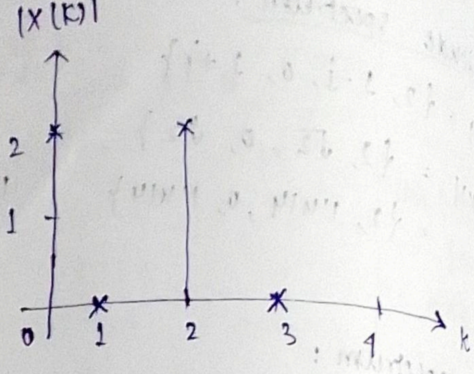
$\frac{180}{4}$ 45, $\frac{3 \times 180}{4}$ 135, $\frac{5 \times 180}{4}$ 225, $\frac{3 \times 180}{2}$ 270

$\therefore X(k) = \{2, 0, 2, 0\}$

$$x(0) = \sum_{k=0}^{3} X(k) \cdot e^{\frac{j2\pi k \cdot 0}{N}}$$

$$= X(0) + X(1) + X(2) + X(3)$$

$$= 2 + 0 + 2 + 0 = 4$$



Magnitude spectrum

$|X(k)| = \{2, 0, 2, 0\}$

$\angle X(k) = \{0, 0, 0, 0\}$

$x(1) = 0$

$x(2) = 4$

$x(3) = 0$

$\therefore x(n) = \{4, 0, 4, 0\}$

Divided by 4, $x(n) = \{1, 0, 1, 0\}$.

H(W).

① $x(n) = \{1, 1, 1, 1, 0, 0\}$ & given $N = 8$.

② $x(n) = \{1, 0, 1, 0, 1, 0, 1, 0\}$, $N = 8$.

① $X(k) = \sum_{n=0}^{7} x(n) \cdot e^{\frac{-j \cdot 2\pi \cdot k \cdot n}{8}}$

$x(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$

$X(0) = \sum_{n=0}^{7} x(n) \cdot e^{\frac{-j \cdot 2\pi \cdot 0}{8}}$

Discrete Time Fourier Transform (DTFT):

* $DTFT[x(n)] = X(e^{jw}) = X(w) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-jwn}$

* $IDTFT[x(n)] = IDTFT[X(e^{jw})] = x(n)$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw}) \, e^{jwn} \, dw.$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(w) \, e^{jwn} \, dw.$

$\boxed{z = e^{jw}}$

Q. Find the discrete time fourier transform (DTFT) of the following:

① $\delta(n)$

$X(w) = \sum_{n=-\infty}^{\infty} \delta(n) \, e^{-jwn}$

$= 1.$

② $u(n)$

$X(w) = \sum_{n=-\infty}^{\infty} u(n) \cdot e^{-jwn}$

$= \sum_{n=0}^{\infty} u(n) \cdot e^{-jwn}$

$= 1 + e^{-jw} + e^{-jw2} + \ldots$ $(1 - e^{-jw})^{-1}$

$= \frac{1}{1 - e^{-jw}}.$

③ $u(n-k)$

④ $\delta(n-k)$

⑤ $a^n u(n)$

⑥ $\sin wn$

⑦ $\cos wn$

⑧ $\delta(n+2) - \delta(n-2)$

⑨ $x(n) = \{-1, 0, \underset{\uparrow}{2}, 1, 2, 3\}.$

③ $n-k \geq 0$

$n \geq k$

$X(w) = \sum_{n=k}^{\infty} u(n) \, e^{-jwn}$

$= e^{-jwk} + e^{-jw(k+1)} + e^{-jw(k+2)} + \ldots$

$= e^{-jwk} (1 + e^{-jw} + e^{-jw2} + \ldots)$

$= e^{-jwk} \cdot \frac{1}{1 - e^{-jw}}$

$= \frac{e^{-jwk}}{1 - e^{-jw}}.$

④ $X(\omega) = \sum_{n=-\infty}^{\infty} \{a^n u(n)\} e^{-j\omega n}$

$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n$

$= \frac{1}{1-ae^{-j\omega}}$

⑧ $X(\omega) = \sum_{n=-\infty}^{\infty} \delta(n+2)\, e^{-j\omega n} - \sum_{n=-\infty}^{\infty} \delta(n-2)\, e^{-j\omega n}$

$= \sum_{n=-2}^{\infty} \delta(n+2)\, e^{-j\omega n} - \sum_{n=2}^{\infty} \delta(n-2)\, e^{-j\omega n}$

$\delta(n) = 1,\ n=0$

$n+2=0$

$n=-2$

$= e^{-j\omega(-2)} - e^{-j\omega 2}$

$= e^{j\omega 2} - e^{-j\omega 2} = e^{2j\omega} - e^{-2j\omega}$

$= 2j\sin(2\omega)$

$\boxed{e^{j\theta} - e^{-j\theta} = 2j\sin\theta}$

⑨ $x(n) = \{-1, 0, \underset{\uparrow}{\overset{0}{2}}, \overset{1}{1}, \overset{2}{2}, \overset{3}{3}\}$

$X(\omega) = \sum_{n=-2}^{3} x(n)\cdot e^{-j\omega n}$

$= (-1)e^{-j\omega(-2)} + 0 + 2e^{-j\omega 0} + 1e^{-j\omega 1} + 2e^{-j\omega 2} + 3e^{-j\omega 3}$

$= -e^{j\omega 2} + 2 + e^{-j\omega} + 2e^{-2j\omega} + 3e^{-3j\omega}$

$= 2 - e^{2j\omega} + e^{-j\omega} + 2e^{-2j\omega} + 3e^{-3j\omega}$

⑥ $X(\omega) = \frac{e^{j\omega n} - e^{-j\omega n}}{2j}$

⑦ $X(\omega) = \frac{e^{j\omega n} + e^{-j\omega n}}{2}$

HW

① $x(n) = a^{|n|}\cdot u(n)$

② $x(n) = a^{|n|}\cdot \cos(\omega n)$

Fast Fourier Transform (FFT)

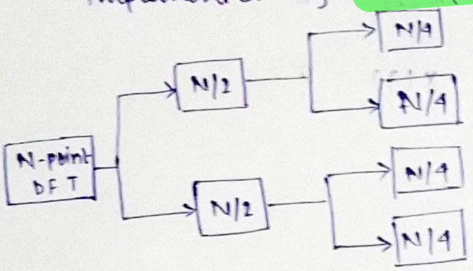
↳ Fast method to compute DFT.

Decimation in time (DIT-FFT) ↳ Reverse.

Decimation in frequency (DIF-FFT)

i/p sequence normal order
o/p sequence bit-reversed order

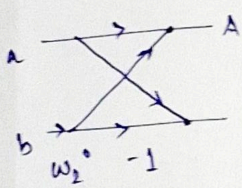
Implemented by butterfly structure



~~Delimit~~ Decimation ↓ 2.

$$\boxed{W_N^k = \text{Twiddle factor} = e^{\frac{-j2\pi k}{N}}}$$

Butterfly structure



$A = a + b W_2^0$
$= a+b$

$B = a + b W_2^0 (-1)$
$= a-b$

2-point (only 2 i/p)

2^n N-point DFT
↳ 2, 4, 8, 16

$a + b\, W_N^k$
$a - b\, W_N^k$

Eg. $x(n) = \{1, 2\}$

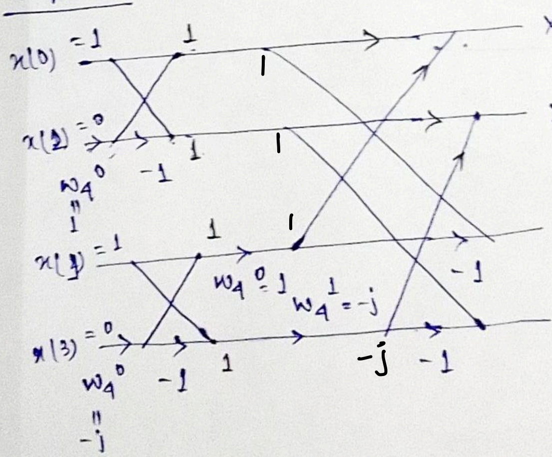
$a = 1, b = 2$

$X(K) = \{3, -1\}$.

$\begin{bmatrix} A = a+b \\ B = a-b \end{bmatrix}$

* Multiply everything on-line
* add at junction of 2 lines.

4-point DIT-FFT (4 i/p).



$X(0) = 1+1 = 2$

$X(1) = 1-j$

$X(2) = 1-1 = 0$

$X(3) = 1+j$

Bit Reversal

index	binary	Bit reversal	Bit reversed index
0	0 0	0 0	0
1	0 1	1 0	2
2	1 0	0 1	1
3	1 1	1 1	3

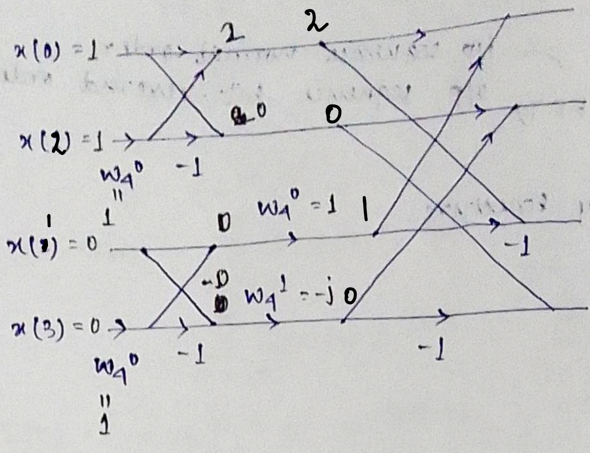
$x(n) = \{1, 1, 0, 0\}$

$X(K) = \{2, 1-j, 0, 1+j\}$

$W_4^0 = 1$

$W_4^1 = e^{\frac{-j2\pi}{4}} = e^{-j\pi/2} = -j$ → line crossing K = 1

$x(n) = \{1, 0, 1, 0\}$



$X(0) = 2 + 0 = 2$

$X(1) = 0$

$X(2) = 2$

$X(3) = 0$

$X(K) = \{2, 0, 2, 0\}$.

8-point

Bit reversal

index	Binary	Bit Rev.	Bit rev. order
0	0 0 0	0 0 0	0
1	0 0 1	1 0 0	4
2	0 1 0	0 1 0	2
3	0 1 1	1 1 0	6
4	1 0 0	0 0 1	1
5	1 0 1	1 0 1	5
6	1 1 0	0 1 1	3
7	1 1 1	1 1 1	7

H/W

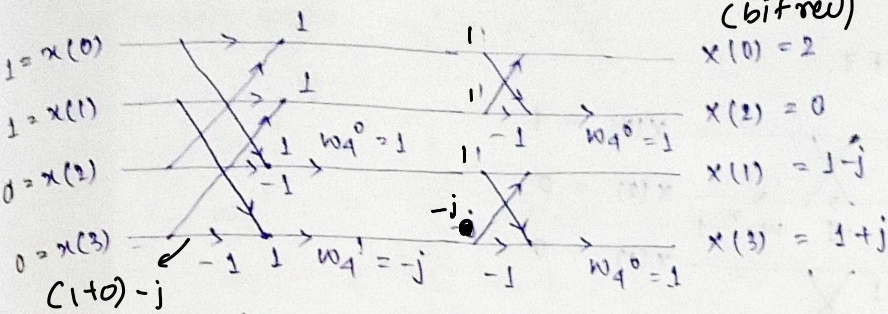
compute : (1) $x(n) = 2^n$, $N = 4$

(2) $x(n) = \cos \pi n$, $N = 4$.

4-point DIF - FFT

$a+b$

$(a-b)\,W^k_F$



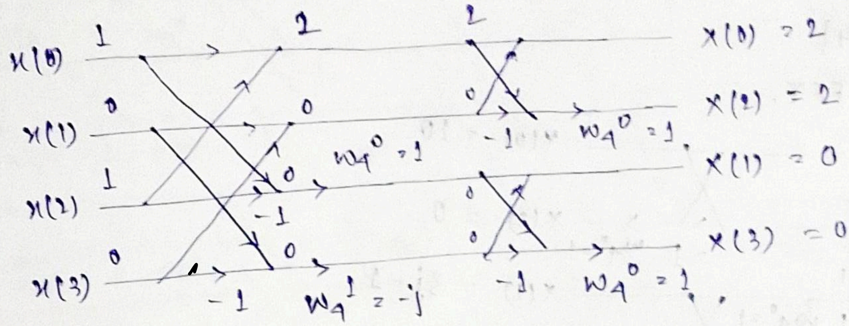
$W_4^0 = 1$

$W_4^1 = -j$

$x(n) = \{1, 1, 0, 0\}$.

$X(K) = \{2,\ 1-j,\ 0,\ 1+j\}$.

Q2. $x(n) = \{1, 0, 1, 0\}$

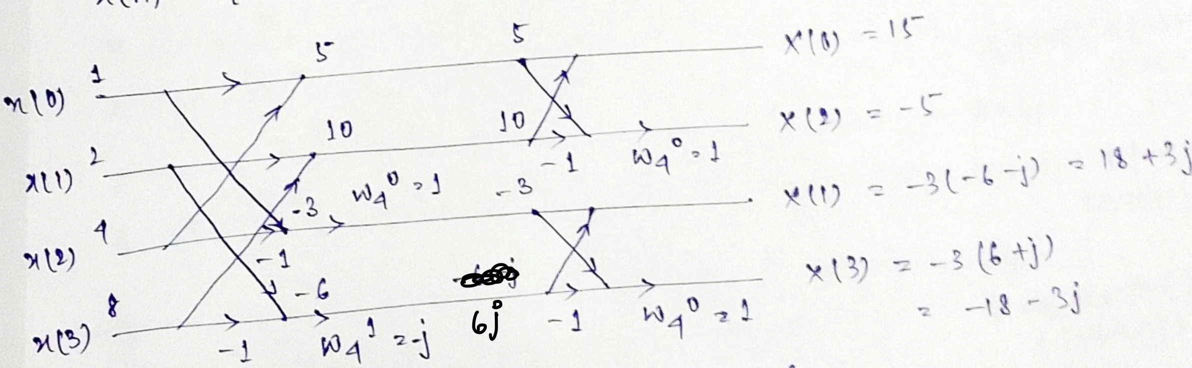


$X(K) = \{2, 0, 2, 0\}$

Q3 HW

① $x(n) = 2^n,\ N = 4. = 0, 1, 2, 3.$

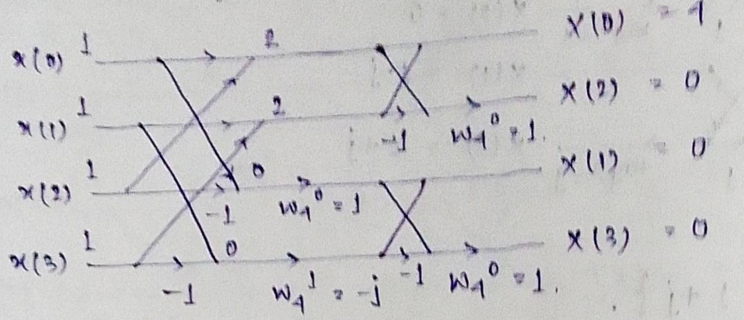
$x(n) = \{1, 2, 4, 8\}$



$\therefore X(K) = \{15,\ 18+3j,\ -5,\ -18-3j\}$.

① $x(n) = \{1, 1, 1, 1\}$

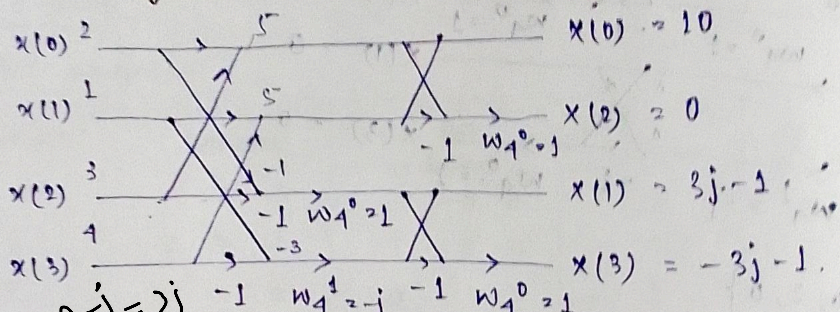
Find DFT using DIF - FFT.



$X(K) = \{4, 0, 0, 0\}$.

② $x(n) = \{2, 1, 3, 4\}$

Using DIF - FFT.



$(-3)(-j) = 3j$

$X(K) = \{10, 3j-1, 0, -3j-1\}$.

8 point DIT - FFT

Twiddle factor $W_N^k = e^{\frac{-j2\pi k}{N}}$

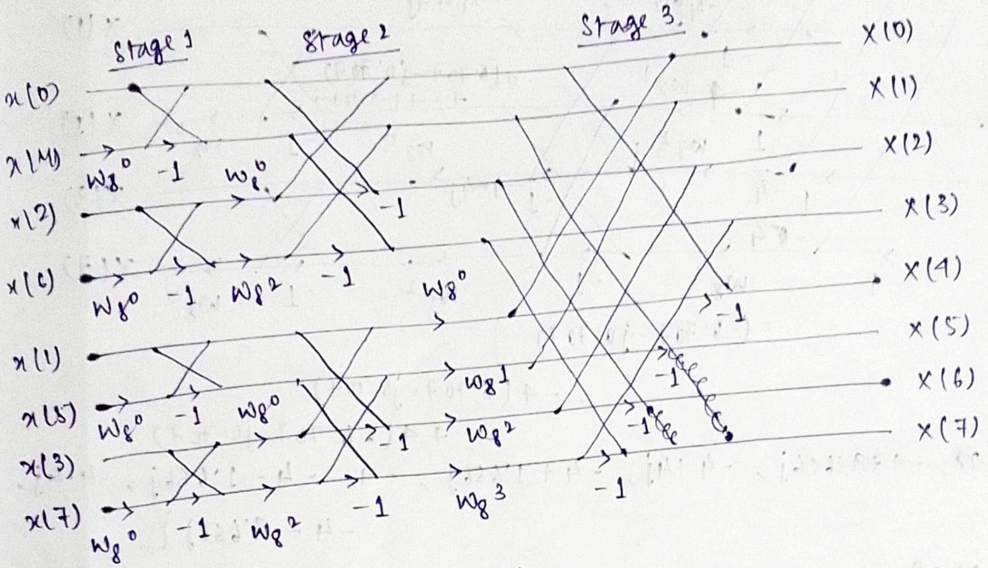
$N = 8$.

$W_8^0 = e^0 = 1$.

$W_8^1 = e^{\frac{-j2\pi}{8}} = 0.707 - j0.707$.

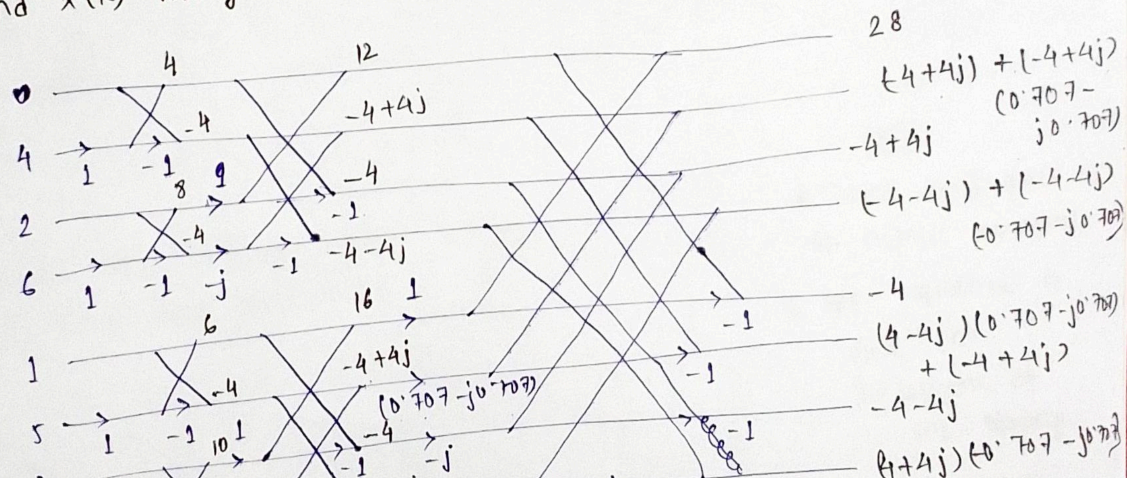
$W_8^2 = e^{\frac{-j2\pi 2}{8}} = -j$

$W_8^3 = e^{\frac{-j2\pi 3}{8}} = -0.707 - j0.707$.



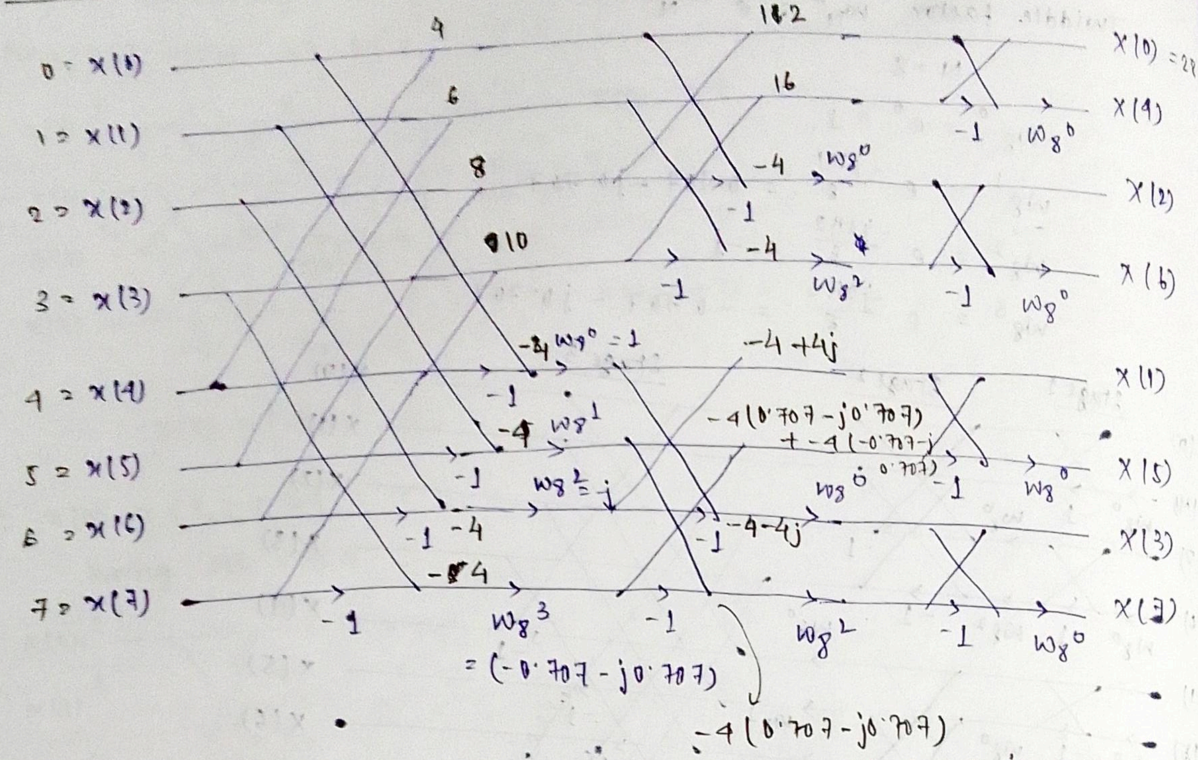
Q. $x(n) = \{0, 1, 2, 3, 4, 5, 6, 7\}$

find X(K) using DIT - FFT butterfly structure



$X(K) = \{28, -4+9.656j, -4+4j, -4+1.656j, -4, -4-1.656j, -4-4j, -4-9.656j\}$.

8-point DIF-FFT



$X(K) = \{28, -4+9.656j, -4+4j, -4+1.656j, -4, -4-1.656j, -4-4j, -4-9.656j\}$

HW

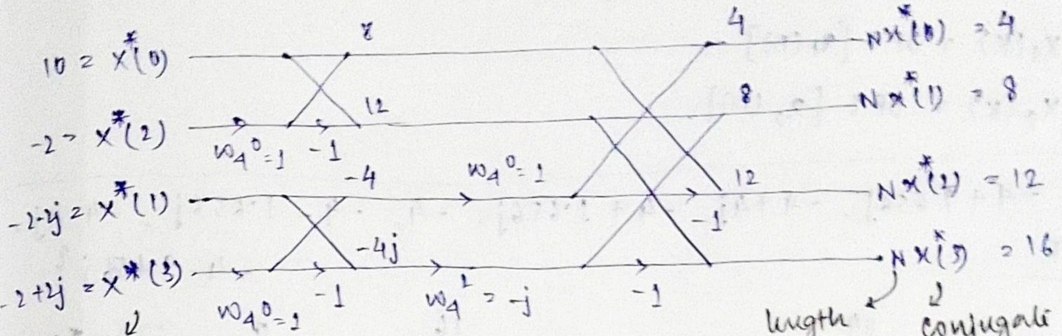
① $x(n) = 2^n$, $N = 8$

② $x(n) = \{1, 1, 1, 1, 2, 2, 2, 2\}$

Inverse DFT (IDFT) using DIT - FFT

Given, $X(k) = \{10, -2+2j, -2, -2-2j\}$.

find $x(n)$.



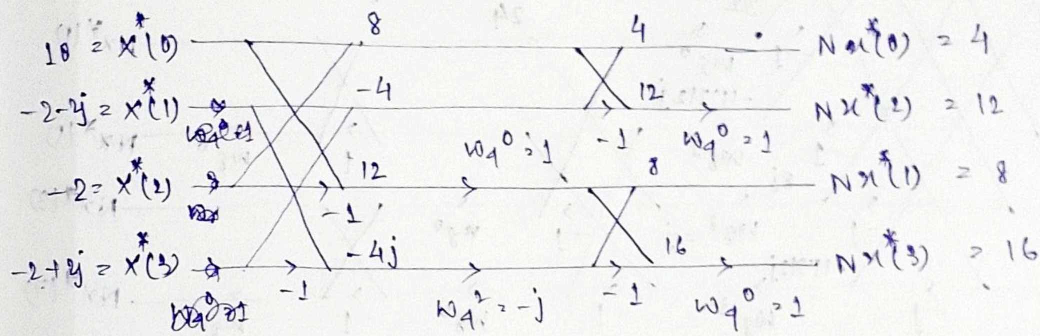
complex conjugate.

length of $X(k)$. conjugate

$Nx^*(n) = \{4, 8, 12, 16\}$

$x^*(n) = \{1, 2, 3, 4\}$.

using DIF - FFT :



$-1-2j+2-2j$

$-2-2j-2+2j$

$x(n) = \{1, 2, 3, 4\}$

(*) 8-point (Prev. Problem) IDFT using DIF - FFT and DIT - FFT.

Note

* Take conjugate of input sequence
* o/p sequence is $Nx^*(n)$
  ↳ length of seq. ~~string~~.

Properties of DFT

(i) Convolution :

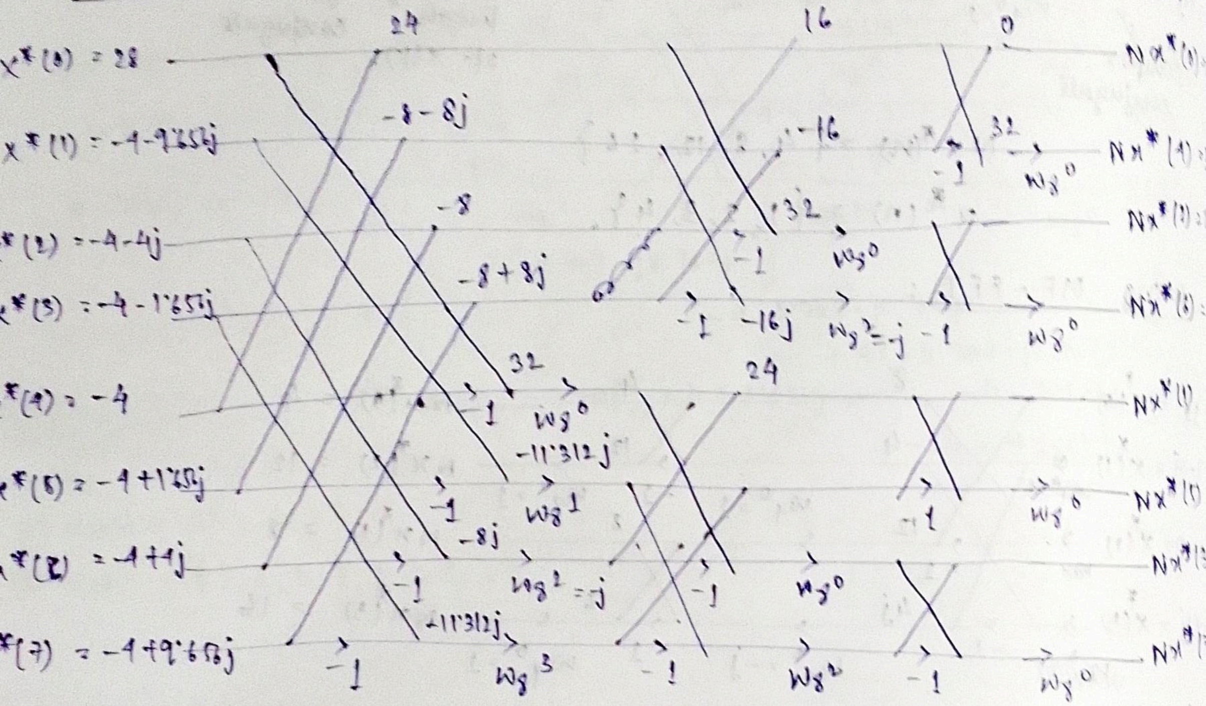
$$x_1(n) * x_2(n) \xrightarrow{T} X_1(k) \cdot X_2(k)$$

$$X_1(k) = DFT[x_1(n)]$$

$$X_2(k) = DFT[x_2(n)].$$

$X(k) = \{28, -4+9.656j, -4+4j, -4+1.656j, -4, -4-1.656j, -4-4j, -4-9.656j\}$

DIF - FFT



$Nx(n) = \{0, 8, 16, 24, 32, 40, 48, 56\}$

$x(n) = \{0, 1, 2, 3, 4, 5, 6, 7\}$

Circular Convolution

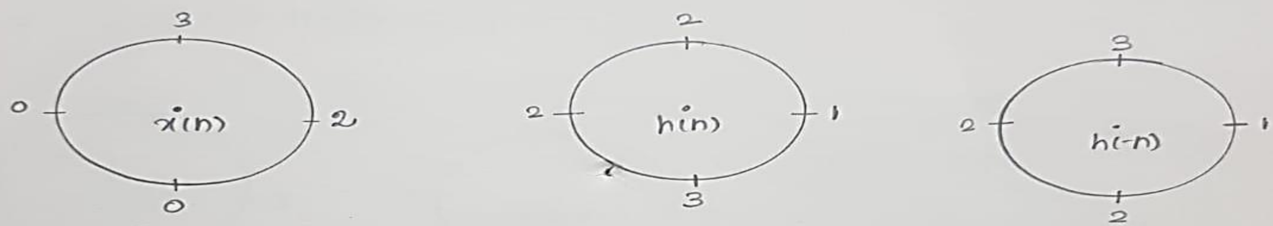
Circular Convolution :-

$x(n) = \{2,3\}$ $\quad h(n) = \{1,2,2,3\}$

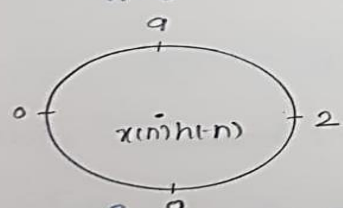
Now,

$x(n) = \{2,3,0,0\}$.

$y(n) = \sum_{n=0}^{3} x(n)\, h((m-n))_N$



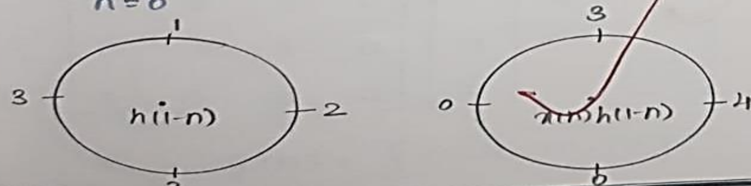
$y(0) = \sum_{n=0}^{3} x(n)\, h((-n))_4$



$= 2+9$

$y(0) = 11.$

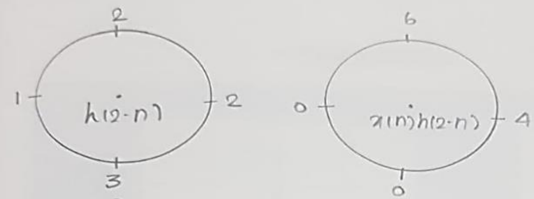
$y(1) = \sum_{n=0}^{3} x(n)\, h((1-n))_4$



$y(1) = 3+4 = 7$

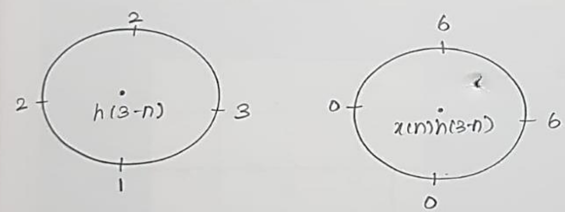
Circular Convolution

$$y(2) = \sum_{n=0}^{3} x(n)\, h((2-n))_N$$



$$y(2) = 6+4 = 10.$$

$$y(3) = \sum_{n=0}^{3} x(n)\, h((3-n))_N$$



$$y(3) = 6+6 = 12.$$

$$y(n) = \{11, 7, 10, 12\}.$$

Matrix method :-

$$\begin{bmatrix} 2 & 0 & 0 & 3 \\ 3 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \\ 10 \\ 12 \end{bmatrix}$$

$$y(n) = \{11, 7, 10, 12\}$$

- **Circular Convolution of discrete time sequences using DFT**

Given $x(n)=\{1,2,1,2\}$, $h(n)=\{1,2,3,4\}$

$DFT\{x(n)\}= \{6,0,-2,0\}$

$DFT\{h(n)\}=\{10,-2+j2,-2,-2-j2\}$

Convolution of two sequence in time domain is equal to multiplication of their transforms in frequency domain→ convolution property

$y(n)=x(n)*h(n)$ → in time domain

$Y(k)= X(k)H(k)$→ in frequency domain

$Y(k)=\{60,0,4,0\}$

Take inverse transform , $y(n)=IDFT\{Y(k)\} =\{16,14,16,14\}$