MOD - 2

Z - Transform

$x(n) \rightarrow$ Transform $X(z)$

→ unilateral or one-sided Z-transform.

→ Bilateral or double-sided Z-transform.

$$Z[x(n)] = X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n}$$

$$Z[x(n)] = X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}.$$

$$x(n) \underset{Z^{-1}}{\overset{Z}{\rightleftarrows}} X(Z)$$

Z^{-1} → inverse Z-transform

$$Z^{-1}[X(Z)] = x(n) = \frac{1}{2\pi j} \oint_c X(Z) Z^{n-1} dZ.$$

Methods to find Inverse

1. Long division
2. Partial fraction. (preferred method)
3. Residue method.

Q. Find the Z-transform for the following discrete time signals:

1. $x(n) = \{1, 2, -1, -3\}$
2. $x(n) = \delta(n)$
3. $u(n)$
4. $\delta(n-k)$
5. $\delta(n+k)$
6. $u(n-k)$

① $$X(Z) = \sum_{n=0}^{3} x(n) Z^{-n}$$

$$= x(0) Z^{-0} + x(1) Z^{-1} + x(2) Z^{-2} + x(3) Z^{-3}.$$

$$= 1 \cdot 1 + 2 \cdot \frac{1}{Z} + (-1)\frac{1}{Z^2} + (-3)\frac{1}{Z^3}$$

$$= 1 + \frac{2}{Z} - \frac{1}{Z^2} - \frac{3}{Z^3}.$$

② $Z[\delta(n)]$

$$= X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

At $n = 0$, $X(Z) = \delta(n) Z^0$

$= 1.$

$\therefore X(Z) = 1$ (0 for others).

③ $Z[u(n)] = X(Z)$

$$= \sum_{n=-\infty}^{\infty} u(n) Z^{-n}.$$

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$X(Z) = \sum_{n=0}^{\infty} u(n) Z^{-n}$$

$$= 1 \cdot Z^0 + 1 \cdot Z^{-1} + \ldots$$

$$= (Z^0 + Z^{-1} + Z^{-2} + \ldots)$$

$$= 1 + Z^{-1} + Z^{-2} + \ldots$$

④ $\delta(n-K)$

$Z[\delta(n-K)]$

$= X(Z) = \sum_{n=-\infty}^{\infty} \delta(n-K) Z^{-n} \Big|_{n=K}$

$= 1 \cdot Z^{-K}$

$= Z^{-K}$

⑤ $\delta(n+K) = X(Z) = \sum_{n=-\infty}^{\infty} \delta(n+K) Z^{-n} \Big|_{n=-K}$

$= 1 \cdot Z^{K}$

$= Z^{K}$

⑥ $Z[u(n-K)] = X(Z) = \sum_{n=K}^{\infty} u(n) Z^{-n}$

$= 1(Z^{-K} + Z^{-K+1} + \ldots)$

$$\boxed{\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad |a| < 1}$$

* $x(n) = a^n u(n)$

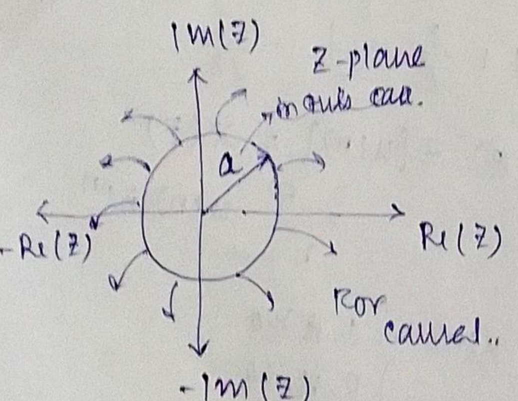
$X(Z) = \sum_{n=0}^{\infty} a^n Z^{-n}$

$= \sum_{n=0}^{\infty} (aZ^{-1})^n$

$= \frac{1}{1-aZ^{-1}} = \frac{Z}{Z-a}, \quad |aZ^{-1}| < 1.$

$\left|\frac{a}{Z}\right| < 1.$

$|Z| > |a|.$ $\cancel{|a| > |Z|}$.



ROC

Region of convergence.

Exterior to circle.

(4) $x(n) = -6^n u(-n-1)$

$$X(Z) = \sum_{n=-\infty}^{\infty} -6^n u(-n-1) \cdot Z^{-n}$$

$$u(-n) = \begin{cases} 1, & n \le 0 \\ 0, & \text{otherwise} \end{cases}$$

$$u(-n-1) = \begin{cases} 1, & n \le -1 \\ 0, & \text{otherwise} \end{cases}$$

$$= -\sum_{n=-\infty}^{-1} 6^n Z^{-n}$$

$$= -\sum_{n=\infty}^{1} 6^{-n} Z^n \quad [- \to +]$$

$$= -\sum_{n=1}^{\infty} 6^{-n} Z^n \quad [\text{Reverse}]$$

$$= -\sum_{n=0}^{\infty} 6^{-n} Z^n - 1 \quad [\text{Include } 0]$$

$$= -\sum_{n=0}^{\infty} (6^{-1} Z)^n - 1$$

$$= \frac{-1}{1 - 6^{-1} Z} - 1 \qquad |6^{-1} Z| < 1.$$

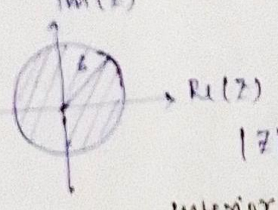
$$|Z/6| < 1, \quad |Z| < 6.$$

$$= \frac{-1}{1 - \frac{Z}{6}} - 1$$

ROC

$$= \frac{-6}{6 - Z} + 1 = \cancel{\frac{-6 - 6 + Z}{6 - Z}} = \cancel{\frac{Z}{6 - Z}}.$$

$$= \frac{-6 + 6 - Z}{6 - Z} = \frac{-Z}{6 - Z} = \frac{Z}{Z - 6}$$



$|Z| < 6$.

Interior to circle (Although same ans as previous).

(5) Find the Z-transform and plot the ROC for the signal

$x(n) = a^n + b^n \cdot u(-n-1)$.

$$X(Z) = \sum_{n=-\infty}^{-1} (a^n + b^n) \cdot Z^{-n}$$

$$= \cancel{\sum_{n=\infty}^{1} (a^{-n} + b^{-n}) \cdot Z^n}$$

$$= \cancel{\sum_{n=1}^{\infty} (a^{-n} + b^{-n}) \cdot Z^{-n}}$$

$$= \cancel{\sum_{n=0}^{\infty} (a^{-n} + b^{-n}) \cdot Z^n - 2}$$

$$X(Z) = \frac{Z}{Z - a} - \frac{Z}{Z - b}.$$

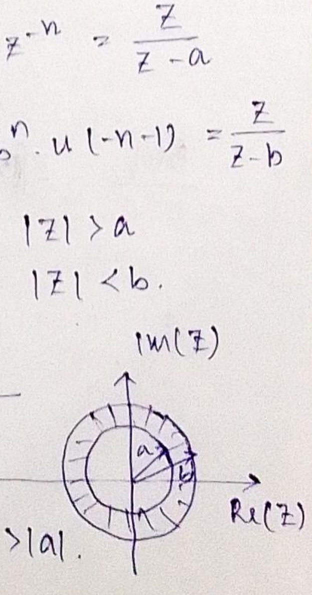
$$\sum_{n=0}^{\infty} a^n Z^{-n} = \frac{Z}{Z - a}$$

$$\sum_{n=0}^{\infty} b^n \cdot u(-n-1) = \frac{Z}{Z - b}$$

$|Z| > a$

$|Z| < b$.

ROC



$|b| > |Z| > |a|$.

Properties of Z-transform

① Linearity

if $x_1(n) \xrightarrow{Z} X_1(Z)$

if $x_2(n) \xrightarrow{Z} X_2(Z)$.

$a x_1(n) + b x_2(n) \xrightarrow{Z} a X_1(Z) + b X_2(Z)$.

Proof $\quad Z[a x_1(n) + b x_2(n)] = \sum_{n=-\infty}^{\infty} (a x_1(n) + b x_2(n)) \cdot Z^{-n}$

$$= a \sum_{n=-\infty}^{\infty} x_1 n Z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n) Z^{-n}$$

$$= a X_1(Z) + b X_2(Z).$$

② Time shifting.

if $x_1(n) \xrightarrow{Z} X_1(Z)$.

$x(n-k) \xrightarrow{Z} Z^{-k} X(Z)$. $[-k \to -k, \ +k \to +k]$

Proof $\quad Z[x(n-k)] = \sum_{n=-\infty}^{\infty} x(n-k) \cdot Z^{-n}$

let, $n-k = l$. $\therefore n = l+k$.

$$\therefore Z[x(l)] = \sum_{l=-\infty}^{\infty} x(l) \cdot Z^{-(l+k)}$$

$$= Z^{-k} \sum_{l=-\infty}^{\infty} x(l) \cdot Z^{-l}$$

$$= Z^{-k} \cdot X(Z)$$

③ Time Reversal.

if $x(n) \xrightarrow{Z} X(Z)$

$x(-n) \xrightarrow{Z} X(Z^{-1})$.

Proof $\quad Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) \cdot Z^{-n}$.

let, $-n = l$; $n = -l$.

$$\therefore Z[x(l)] = \sum_{l=\infty}^{-\infty} x(l) Z^{-l}$$

$$= \sum_{l=-\infty}^{\infty} x(l) Z^{+l} = X(Z^{-1}).$$

④ Time scaling

if $x(n) \xrightarrow{Z} X(Z)$.

$a^n x(n) \longrightarrow X(a^{-1} Z)$

Proof $Z[a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n) \cdot Z^{-n}$

$= \sum_{n=-\infty}^{\infty} (a^{-1} z^{-1})^n \cdot x(n)$

$= \sum_{n=-\infty}^{\infty} x(n) \cdot (a^{-1} Z)^{-n}$

$= X(a^{-1} Z)$.

⑤ Convolution

if $x_1(n) \xrightarrow{Z} X_1(Z)$

& $x_2(n) \xrightarrow{Z} X_2(Z)$.

then, $\underbrace{x(n) = x_1(n) * x_2(n)}$, $\xrightarrow{Z} X_1(Z) \cdot X_2(Z)$.

$\sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k)$.

Proof $Z[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) \right] \cdot Z^{-n}$

let, $n-k = l$; $n = l+k$.

$= \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k) x_2(l) \cdot Z^{-(l+k)}$

$= \left[\sum_{l=-\infty}^{\infty} x_2(l) Z^{-l} \right] \cdot \left[\sum_{k=-\infty}^{\infty} x_1(k) \cdot Z^{-k} \right]$.

$= X_2(Z) \cdot X_1(Z)$.

$= X_1(Z) \cdot X_2(Z)$.

⑥ Differentiation

if $x(n) \xrightarrow{Z} X(Z)$

then $n x(n) \xrightarrow{Z} -Z \frac{d}{dZ} X(Z)$.

System Analysis using Z-transform :

(OR)

Solution to difference equation using Z-transform :

$x(n) \rightarrow$ DT system $h(n)$ $\rightarrow y(n) = x(n) * h(n)$

$$= \sum_{k=-\infty}^{\infty} x(n)\, h(n-k)$$

$h(n) \rightarrow$ impulse response of the response system.

Apply Z-transform, $Y(Z) = Z[x(n) * h(n)]$

$$= X(Z) \cdot H(Z).$$

System function (or) Transfer function

$$\boxed{H(Z) = \frac{Y(Z)}{X(Z)}} \quad Y(Z) \rightarrow \text{O/P}, \quad X(Z) \rightarrow \text{I/P}$$

$$\boxed{\begin{aligned} H(Z) &= Z[h(n)] \\ h(n) &= Z^{-1}[H(Z)] \end{aligned}}$$ impulse response.

Q. Given, $x(n) = \{-1, 2, 3, 4\}$

$h(n) = \{1, 2, 3, 4\}$

find $y(n)$.

* when sequence length less, add 0 at end.

$\rightarrow Z[x(n)] = X(Z) = \sum_{n=0}^{3} x(n) Z^{-n}$

$$= x(0) Z^{-0} + x(1) Z^{-1} + x(2) Z^{-2} + x(3) Z^{-3}$$

$$= -1 + 2Z^{-1} + 3Z^{-2} + 4Z^{-3}$$

$Z[h(n)] = H(Z) = \sum_{n=0}^{3} x(n) Z^{-n}$

$$= 1 + 2Z^{-1} + 3Z^{-2} + 4Z^{-3}.$$

$\therefore Y(Z) = X(Z) \cdot H(Z)$

$$= (-1 + x)(1 + x) \quad [x = 2Z^{-1} + 3Z^{-2} + 4Z^{-3}].$$

$$= (x-1)(x+1)$$

$$= x^2 - 1 = (2Z^{-1} + 3Z^{-2} + 4Z^{-3})^2 - 1.$$

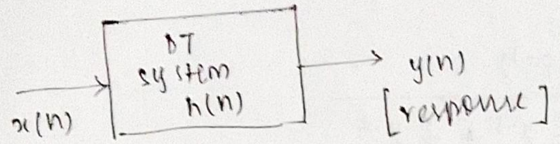
$\therefore y(n) = Z^{-1}[Y(Z)].$

$$= Z^{-1}[(2Z^{-1} + 3Z^{-2} + 4Z^{-3})^2 - 1].$$

$$= Z^{-1}[4Z^{-2} + 9Z^{-4} + 16Z^{-6} + 12Z^{-3} + 24Z^{-5} + 16Z^{-4} - 1]$$

$= Z^{-1}[-1 + 4Z^{-2} + 12Z^{-3} + 25Z^{-4} + 24Z^{-5} + 16Z^{-6}]$

$\therefore y(n) = \{-1, 0, 4, 12, 25, 24, 16\}.$



* i/p and o/p can be related by difference equation.
[differential].

Q. Find the transfer function, impulse response, output / response of the system. Also, plot the pole-zero pattern of the ~~diffe~~ transfer function and determine whether the system is stable or not.

$$y(n) + \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + x(n-1).$$

→ Apply Z-transform,

$$Y(Z) + \frac{3}{4}Z^{-1}Y(Z) + \frac{1}{8}Z^{-2}Y(Z) = X(Z) + Z^{-1}X(Z).$$

$$\Rightarrow Y(Z)\left[1 + \frac{3}{4}Z^{-1} + \frac{1}{8}Z^{-2}\right] = X(Z)\left[1 + Z^{-1}\right].$$

Tr. func, $H(Z) = \dfrac{Y(Z)}{X(Z)} = \dfrac{(1+Z^{-1})}{(1+\frac{3}{4}Z^{-1}+\frac{1}{8}Z^{-2})} = \dfrac{Z(Z+1)}{Z^2+\frac{3}{4}Z+\frac{1}{8}}$

$$= \frac{Z(Z+1)}{(Z+\frac{1}{2})(Z+\frac{1}{4})}$$

→ zeroes

→ poles

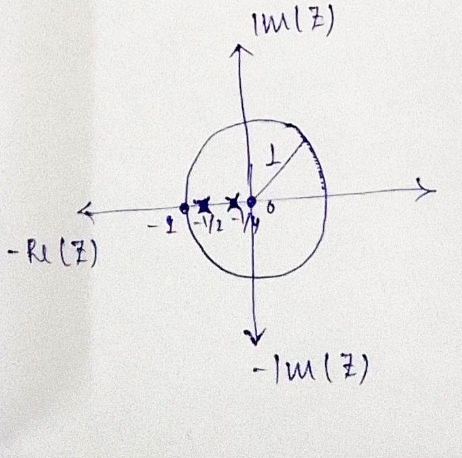
impulse response, $h(n) = Z^{-1}[H(Z)]$

$$= Z^{-1}\left[\frac{1+Z^{-1}}{1+\frac{3}{4}Z^{-1}+\frac{1}{8}Z^{-2}}\right]$$

$=$

o/p or response, where $Y(Z) = X(Z) \cdot H(Z)$

$\therefore y(n) = Z^{-1}[Y(Z)].$



Zeros:	Poles:
$Z = 0$	$Z = -1/2$
$Z = -1$	$Z = -1/4$.

All poles lies inside unit circle.
So stable system.

↳ only req. when stability asked.

Impulse response,

$x(n) = \delta(n)$

$X(Z) = Z[\delta(n)]$

$= 1.$

$$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{(1+Z^{-1})}{1+\frac{3}{4}Z^{-1}+\frac{1}{8}Z^{-2}}$$

Taking $X(Z) = 1$,

$$H(Z) = \frac{Y(Z)}{1} = \frac{Z(Z+1)}{(Z+1/2)(Z+1/4)}$$

Taking inverse, (Partial fraction)

$$h(n) = y(n) = \frac{A}{Z+1/2} + \frac{B}{Z+1/4}.$$

$$\cancel{A(Z+1/4) + B(Z+1/2) = Z(Z+1)}$$

$$\cancel{AZ + \frac{A}{4} + ZB + \frac{B}{2} = Z^2+1.}$$

$$\frac{y(Z)}{Z} = \left(\frac{Z}{Z+1/2}\right) - \left(\frac{3/4\,Z}{Z+1/4}\right)$$

inverse,

$$h(n) = y(n) = \left(\frac{-1}{2}\right)^n u(n) - \frac{3}{4}\cdot\left(\frac{-1}{4}\right)^n u(n)$$

$Z(A+B)$

$\frac{A}{4} + \frac{B}{2}$

$A+4$

$A+$

$Z[a^n u($

$Z^{-1}\left(\frac{Z}{Z}\right.$

$= (a)^n$

$\left(\frac{-1}{2}\right)$

① $y(n) - 2y(n-1) + 3y(n-2) = x(n) + x(n-1)$

② $y(n) + 6y(n-1) + 12y(n-2) = x(n-1) + 2x(n-2)$

③ $y(n) = y(n-1) - 0.5\,y(n-2) + x(n) + x(n-1)$

④ $y(n) = 1.8\,y(n-1) - 0.72\,y(n-2) + x(n) + 0.5\,x(n-1)$