z - transform → To analyse linear time invariant discrete time system in the frequency domain. ⑥

Properties of z-transform:

Linearity:

If $X_1(z) = z\{x_1(n)\}$

$X_2(z) = z\{x_2(n)\}$ then

$$z\{a x_1(n) + b x_2(n)\} = a X_1(z) + b X_2(z).$$

Proof:

$$z\{a x_1(n) + b x_2(n)\} = \sum_{n=-\infty}^{\infty} [a x_1(n) + b x_2(n)] z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}.$$

$$= a X_1(z) + b X_2(z).$$

Time shift or translation:

If $X(z) = z\{x(n)\}$ and the intial conditions for $x(n)$ are zero, then

$$z\{x(n-m)\} = z^{-m} X(z).$$

Proof:

$$z\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m) z^{-n}$$

$$= z^{-m} \sum_{n=-\infty}^{\infty} x(n-m) z^{-(n-m)}.$$

Let $(n-m) = l$,

$$z\{x(n-m)\} = z^{-m} \sum_{l=-\infty}^{\infty} x(l) z^{-l}$$

$$= z^{-m} X(z)$$

b) If $X(z) = Z\{x(n)\}$ then

i) $Z\{x(n-m)\} = z^{-m}\left\{X(z) + \sum_{k=1}^{m} x(-k) z^{k}\right\}$

'm' is +ve integer.

Proof:

$$Z\{x(n-m)\} = \sum_{n=0}^{\infty} x(n-m) z^{-n}$$

$$= z^{-m} \sum_{n=0}^{\infty} x(n-m) z^{-(n-m)}$$

$$= z^{-m} \sum_{l=-m}^{\infty} x(l) z^{-l} \quad [\because l = n-m]$$

$$= z^{-m}\left\{ \sum_{l=0}^{\infty} x(l) z^{-l} + \sum_{l=-m}^{-1} x(l) z^{-l} \right\}$$

$$= z^{-m}\left\{ X(z) + \sum_{k=1}^{m} x(-k) z^{k} \right\} \quad [\because l = -k]$$

ii) $Z\{x(n+m)\} = z^{m}\left\{X(z) - \sum_{k=0}^{m-1} x(k) z^{-k}\right\}$

$$Z\{x(n+m)\} = \sum_{n=0}^{\infty} x(n+m) z^{-n}$$

$$= z^{m}\left\{ \sum_{n=0}^{\infty} x(n+m) z^{-(n+m)} \right\}$$

$$= z^{m}\left\{ \sum_{l=m}^{\infty} x(l) z^{-l} \right\} \quad [\because l = n+m]$$

$$= z^{m}\left\{ \sum_{l=0}^{\infty} x(l) z^{-l} - \sum_{l=0}^{m-1} x(l) z^{-l} \right\}$$

$$= z^{m}\left\{ X(z) - \sum_{k=0}^{m-1} x(k) z^{-k} \right\} \quad [\because l = k]$$

Multiplication by an exponential sequence (8)

If $X(z) = Z\{x(n)\}$ then $Z\{a^n x(n)\} = X(a^{-1}z)$

Proof:

$$Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1}z)^{-n}$$

$$= X(a^{-1}z)$$

ROC is $|a| R_1 < |z| < |a| R_2$

Time Reversal:

If $X(z) = Z\{x(n)\}$, then $Z\{x(-n)\} = X(z^{-1})$

Proof:

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

$$= \sum_{l=-\infty}^{\infty} x(l) z^{l} \; ; \quad l = -n$$

$$= \sum_{l=-\infty}^{\infty} x(l) (z^{-1})^{-l}$$

$$= X(z^{-1})$$

ROC : $\frac{1}{R_2} < |z| < \frac{1}{R_1}$.

b) If $X(z) = Z\{x(n)\}$ then

i) $Z\{x(n-m)\} = z^{-m}\left\{X(z) + \sum_{k=1}^{m} x(-k) z^{k}\right\}$

'm' is +ve integer.

Proof:

$$Z\{x(n-m)\} = \sum_{n=0}^{\infty} x(n-m) z^{-n}$$

$$= z^{-m} \sum_{n=0}^{\infty} x(n-m) z^{-(n-m)}$$

$$= z^{-m} \sum_{l=-m}^{\infty} x(l) z^{-l} \quad [\because l = n-m]$$

$$= z^{-m}\left\{ \sum_{l=0}^{\infty} x(l) z^{-l} + \sum_{l=-m}^{-1} x(l) z^{-l} \right\}$$

$$= z^{-m}\left\{ X(z) + \sum_{k=1}^{m} x(-k) z^{k} \right\} \quad [\because l = -k]$$

ii) $Z\{x(n+m)\} = z^{m}\left\{X(z) - \sum_{k=0}^{m-1} x(k) z^{-k}\right\}$

$$Z\{x(n+m)\} = \sum_{n=0}^{\infty} x(n+m) z^{-n}$$

$$= z^{m}\left\{ \sum_{n=0}^{\infty} x(n+m) z^{-(n+m)} \right\}$$

$$= z^{m}\left\{ \sum_{l=m}^{\infty} x(l) z^{-l} \right\} \quad [\because l = n+m]$$

$$= z^{m}\left\{ \sum_{l=0}^{\infty} x(l) z^{-l} - \sum_{l=0}^{m-1} x(l) z^{-l} \right\}$$

$$= z^{m}\left\{ X(z) - \sum_{k=0}^{m-1} x(k) z^{-k} \right\} \quad [\because l = k]$$

Differentiation of X(z)

If $X(z) = Z\{x(n)\}$ then $Z\{n x(n)\} = -z \frac{d}{dz} X(z)$.

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Differentiating the z-transform

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} x(n) \frac{d}{dz}(z^{-n})$$

$$= \sum_{n=-\infty}^{\infty} x(n)(-n) z^{-n-1}$$

$$= -\frac{1}{z} \sum_{n=-\infty}^{\infty} n x(n) z^{-n}$$

Multiply $-z$,

$$-z \frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} n x(n) z^{-n}$$

$$= Z\{n x(n)\}.$$

Parseval's theorem:

Let us consider two complex sequences $x_1(n)$ & $x_2(n)$. Parseval's theorem states that.

$$\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi j} \oint_c X_1(v) X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv.$$

Parseval's relation in terms of Fourier transform.

$$\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\omega}) X_2^*(e^{j\omega}) d\omega$$

Convolution theorem:

If $X(z) = Z\{x(n)\}$ and $H(z) = Z\{h(n)\}$
then $Z\{x(n) * h(n)\} = X(z) \cdot H(z)$.

Let $y(n) = x(n) * h(n)$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$Y(z) = Z\{y(n)\} = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \right] z^{-n}.$$

$$Y(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k} \sum_{n=-\infty}^{\infty} h(n-k) z^{-(n-k)}.$$

[Interchange order of Σ]

Replace $(n-k)$ by l, in second Σ

$$Y(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k} \sum_{l=-\infty}^{\infty} h(l) z^{-l}.$$

$$= X(z)\, H(z).$$

Determine the pole-zero plot for the system described by difference equation,

$$y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n) - x(n-1).$$

Taking z-transform.

$$Y(z) - \frac{3}{4} z^{-1} Y(z) + \frac{1}{8} z^{-2} Y(z) = X(z) - z^{-1} X(z)$$

$$\frac{Y(z)}{} = \frac{1 - z^{-1}}{} \qquad \frac{1 - z^{-1}}{}$$

$x(n) = u(n-2)$.

WKT, $Z\{u(n)\} = \dfrac{z}{z-1}$

using time shifting property,

$$Z[x(n-m)] = z^{-m} X(z)$$

$$Z[u(n-2)] = z^{-2} \cdot \frac{z}{z-1} = \frac{z^{-1}}{z-1} \; : \; \text{ROC}: |z| > 1$$

$x(n) = (n+0.5)(1/3)^n u(n)$.

$$x(n) = n(1/3)^n u(n) + 0.5(1/3)^n u(n)$$

$$Z\left[(1/3)^n u(n)\right] = \frac{1}{1-\frac{1}{3}z^{-1}}$$

using differentiation property,

$$Z(n\,x(n)) = -z \frac{d}{dz} X(z)$$

$$Z\left[n\,(1/3)^n u(n)\right] = -z \frac{d}{dz}\left(\frac{1}{1-\frac{1}{3}z^{-1}}\right) = \frac{\frac{1}{3}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)^2}$$

$$\therefore X(z) = \frac{\frac{1}{3}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)^2} + \frac{0.5}{1-\frac{1}{3}z^{-1}}$$

ROC : $|z| > \frac{1}{3}$