Signals

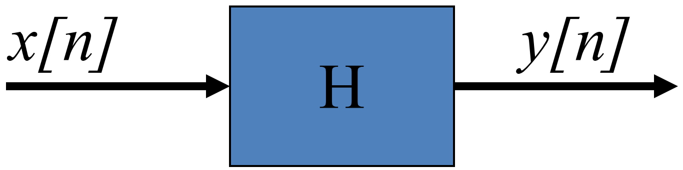
Fundamentals >> Signals

- Broad definition: Functions of independent variables.
  - Examples: music, velocity of some car, your cash, voltage or current in a circuit, your body temperature, your heart’s blood pumping rate..
- Signals are functions of single independent variable, time t.

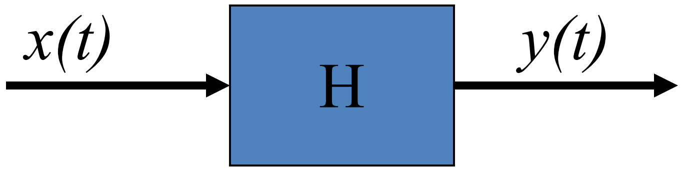
- Signals can be:
  - Discrete *x[n]*, *n* is integer.
  - Continuous *x(t)*, *t* is real.
- Signals can be represented in mathematical form:
  - $x(t) = e^t$, $x[n] = n/2$
  - $y(t) = \begin{cases} 0 & ,t < 0 \\ t^2 & ,t \geq 0 \end{cases}$
- Discrete signals can also be represented as sequences:
  - $\{y[n]\} = \{\ldots,1,0,1,0,\underline{1},0,1,0,1,0,\ldots\}$

Fundamentals >> Systems

- System is a black box that transforms input signals to output signals.
  - *Discrete-Time System*: Input and output signals are discrete.



  - *Continuous-Time System*: Input and output signals are continuous.



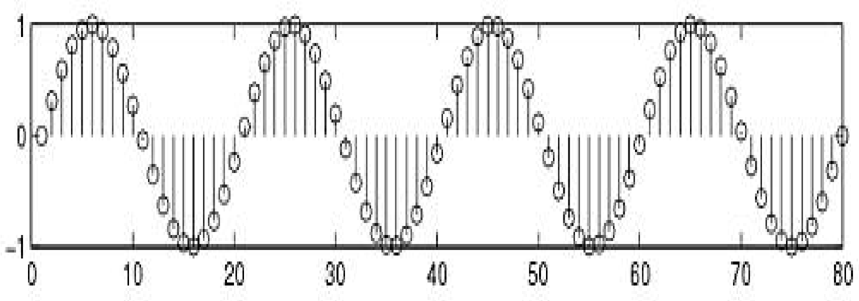
  - Combination is also possible, e.g. analog-to-digital or digital-to-analog converters.

Classification of signals

- Continuous time & Discrete time
- Deterministic & Random
- Periodic & Non periodic
- Causal & Non causal
- Even & Odd
- Energy & Power

Deterministic signal

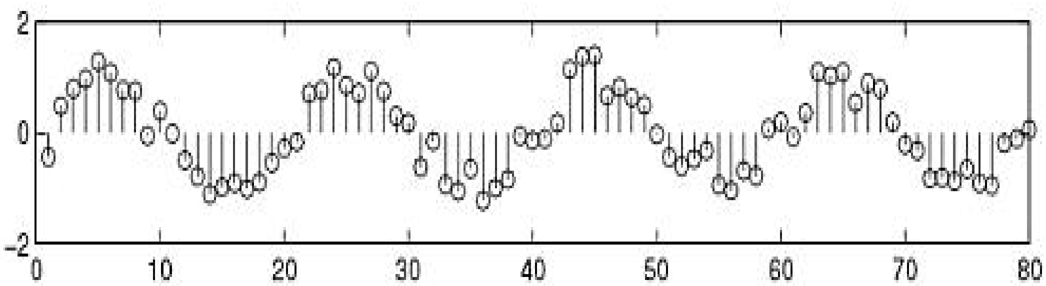
- A deterministic signal is a signal in which each value of the signal is fixed and can be determined by a mathematical expression
- Because of this the future values of the signal can be calculated from past values with complete confidence



Deterministic signal

Random Signal

- A random signal has a lot of uncertainty about its behavior
- The future values of a random signal cannot be accurately predicted and can usually only be guessed based on the averages of sets of signals



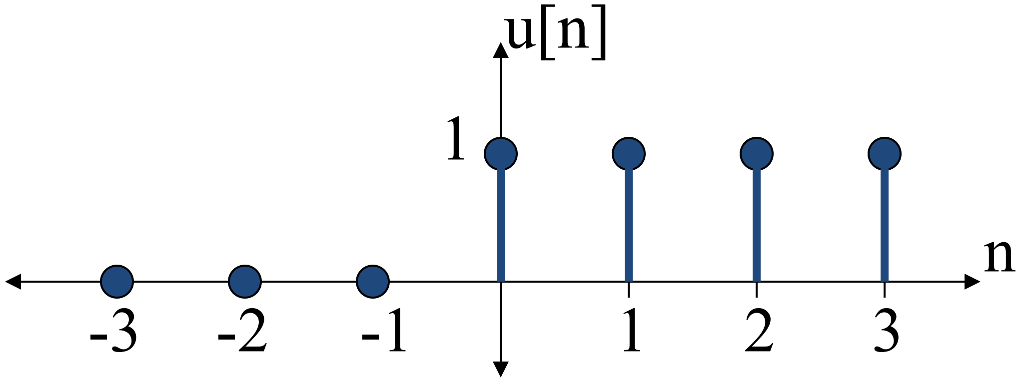
Random signal

Elementary Signals

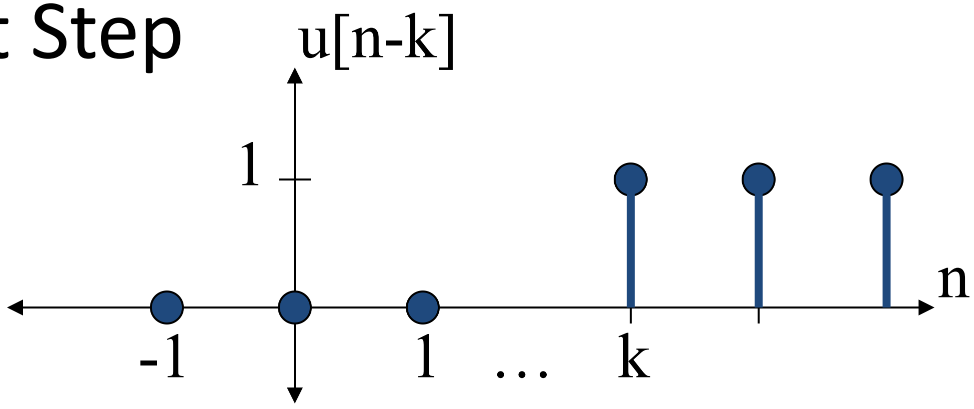
- Unit impulse or impulse signal
- Unit step or step signal
- Ramp
- Sinusoidal
- Exponential

Unit Step

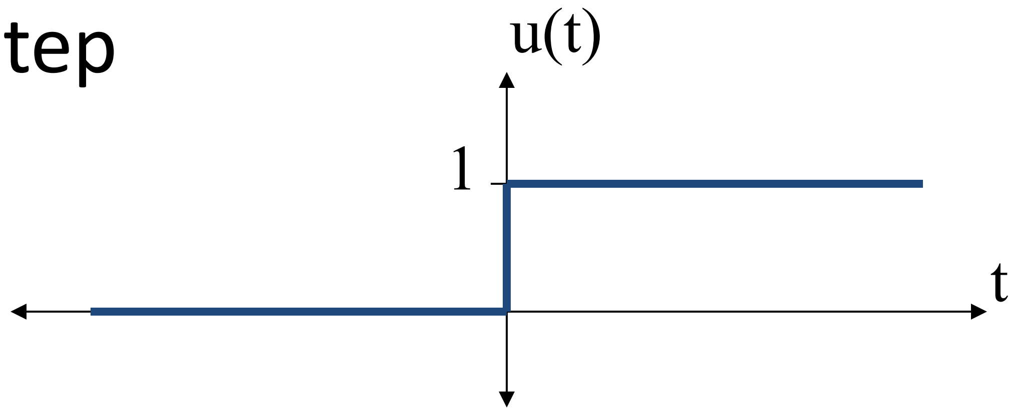
- Discrete Unit Step
  - u[n]= $\begin{cases} 1 & ,n \geq 0 \\ 0 & ,n < 0 \end{cases}$



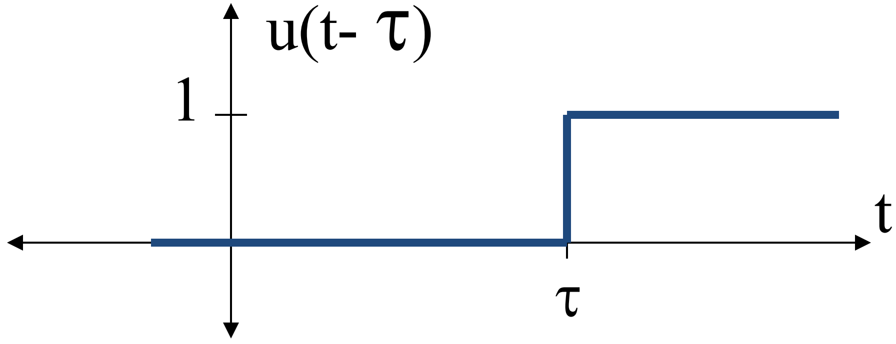
- Discrete Shifted Unit Step
  - u[n-k]= $\begin{cases} 1 & ,n \geq k \\ 0 & ,n < k \end{cases}$



- Continuous Unit Step
  - u(t)= $\begin{cases} 1 & ,t > 0 \\ 0 & ,t < 0 \end{cases}$

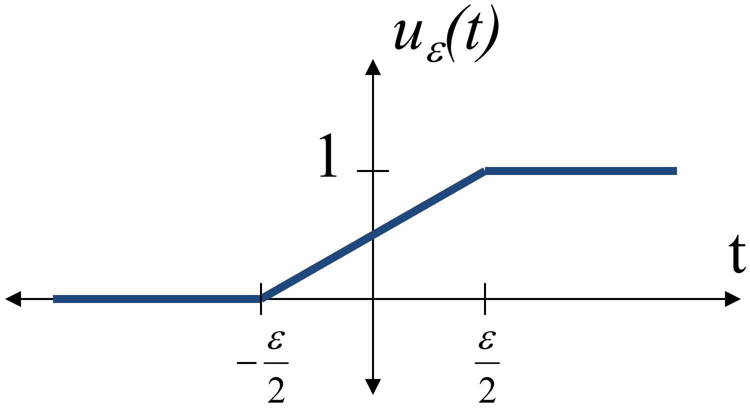


- Continuous Shifted Unit Step
  - u(t-τ)= $\begin{cases} 1 & ,t > \tau \\ 0 & ,t < \tau \end{cases}$



- Continuous Unit Step is discontinuous at t=0, so is not differentiable!
- Define delayed unit step:

$$u_\varepsilon(t) = \begin{cases} 1 & ,t > \varepsilon/2 \\ 0 & ,t < -\varepsilon/2 \\ \dfrac{t}{\varepsilon} + \dfrac{1}{2} & ,otherwise \end{cases}$$



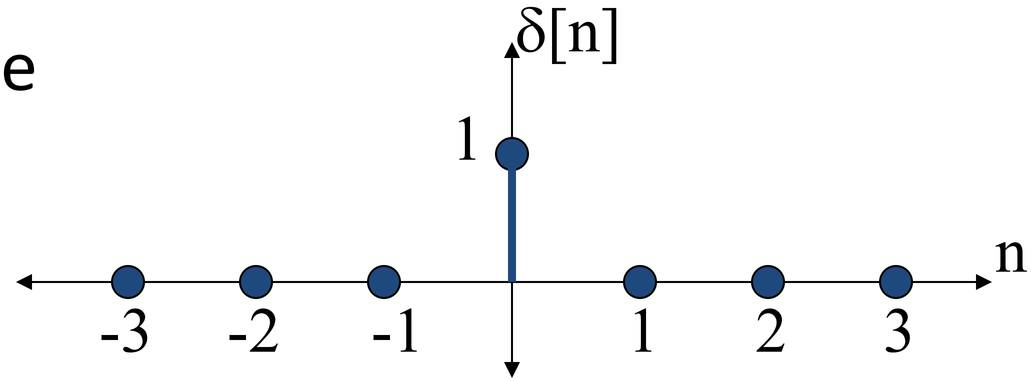
- $u_\varepsilon(t)$ is continuous and differentiable.

$$u(t) = \lim_{\varepsilon \to 0} u_\varepsilon(t) \qquad \frac{du_\varepsilon(t)}{dt} = \begin{cases} \dfrac{1}{\varepsilon} & ,-\varepsilon/2 < t < \varepsilon/2 \\ 0 & ,otherwise \end{cases}$$

Unit Impulse

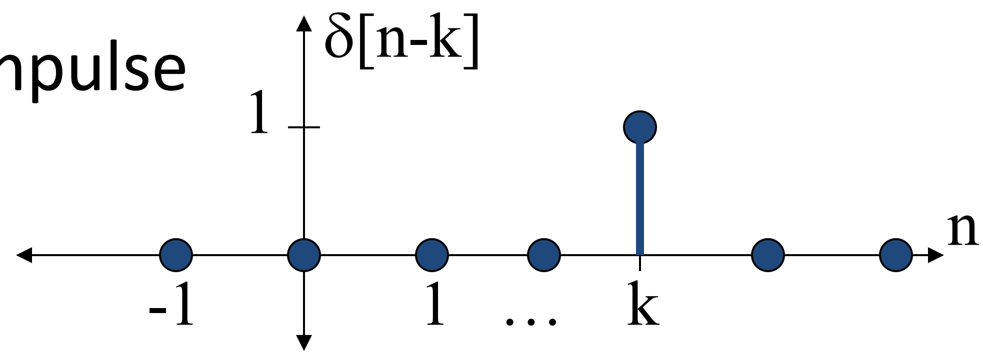
- Discrete Unit Impulse

$$\delta[n] = \begin{cases} 1 & ,n = 0 \\ 0 & ,n \neq 0 \end{cases}$$



- Discrete Shifted Unit Impulse

$$\delta[n-k] = \begin{cases} 1 & ,n = k \\ 0 & ,n \neq k \end{cases}$$



- Properties of discrete Unit Impulse functions:

$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

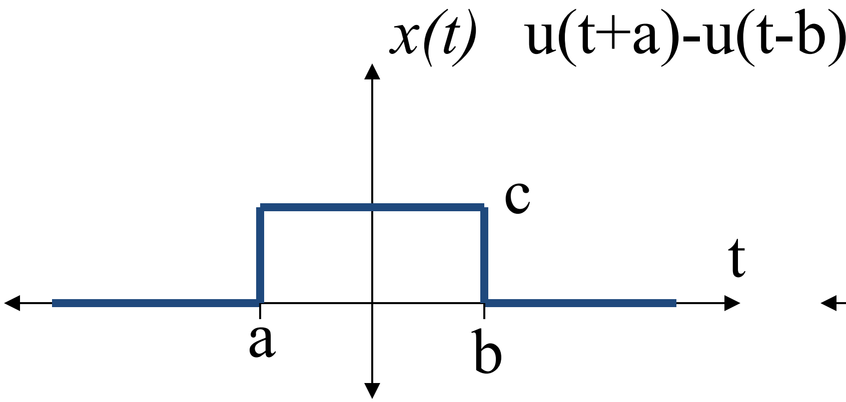
$$\mathrm{r}(t) = \int_0^t u(t)dt$$

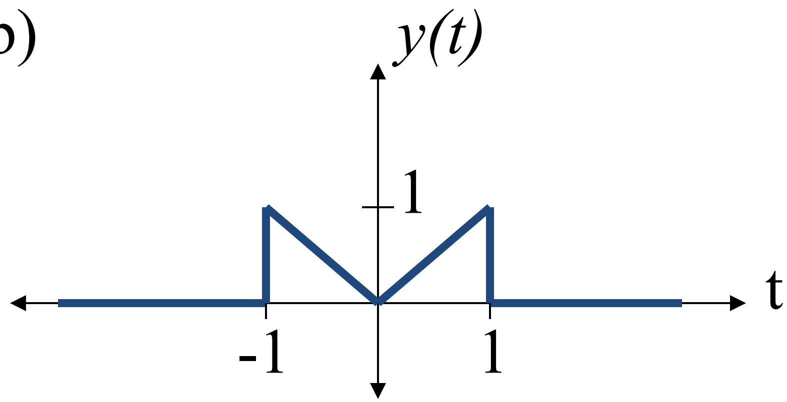
$$x[n]\delta[n] = x[0]\delta[n]$$

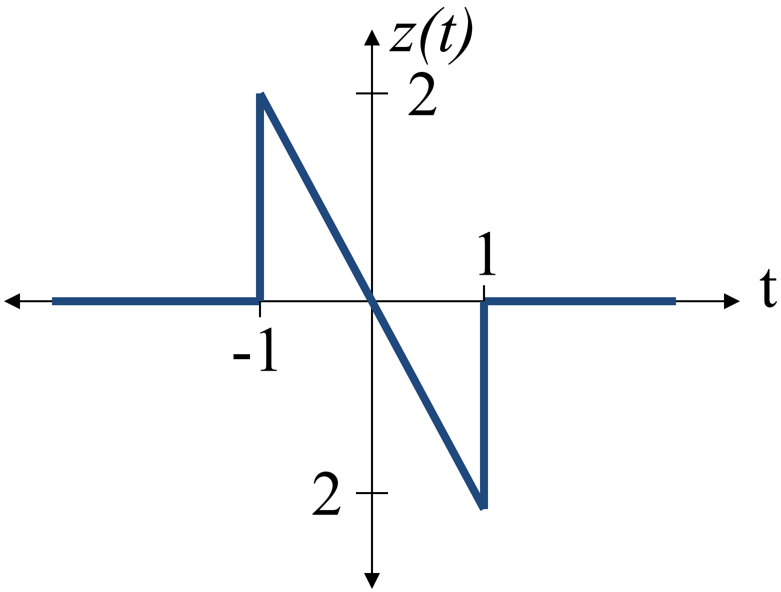
$$x[n]\delta[n-k] = x[k]\delta[n-k]$$

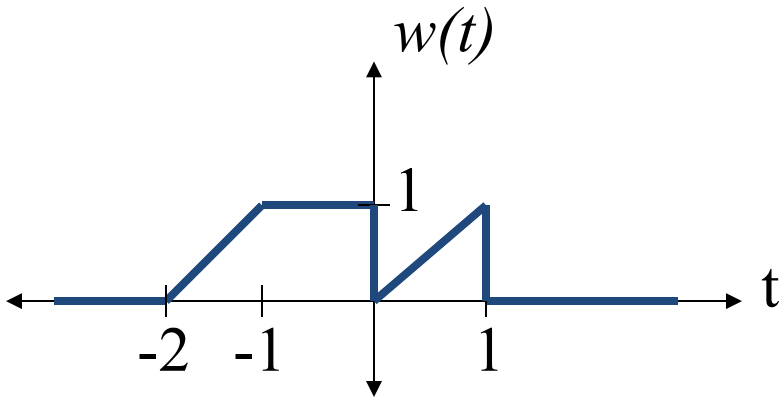
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Write Signal Functions

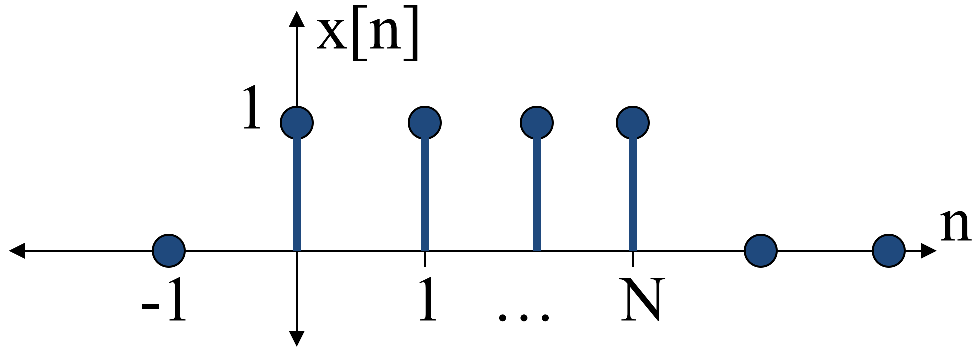


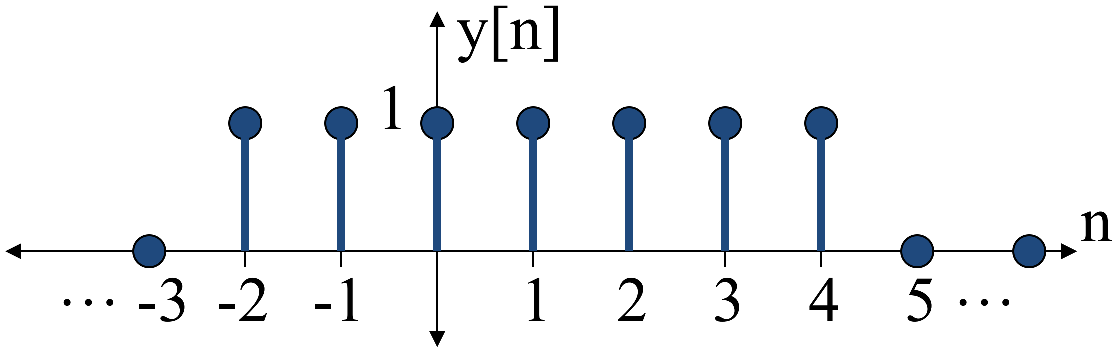






Write Signal Functions





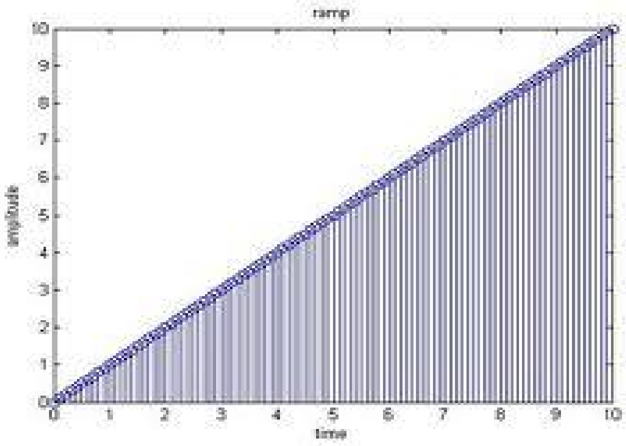
Ramp signal

- Denoted as r(t) or r[n]

- Defined as

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$r[n] = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$$



- Otherwise, r(t) = tu(t) or r[n] = nu[n]

Sinusoidal signal

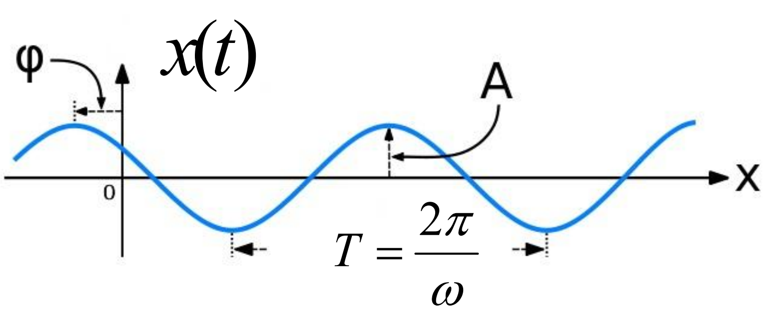
- General form is

$$x(t) = A\sin(\omega t + \phi)$$

where A = amplitude

ω = angular frequency in radians

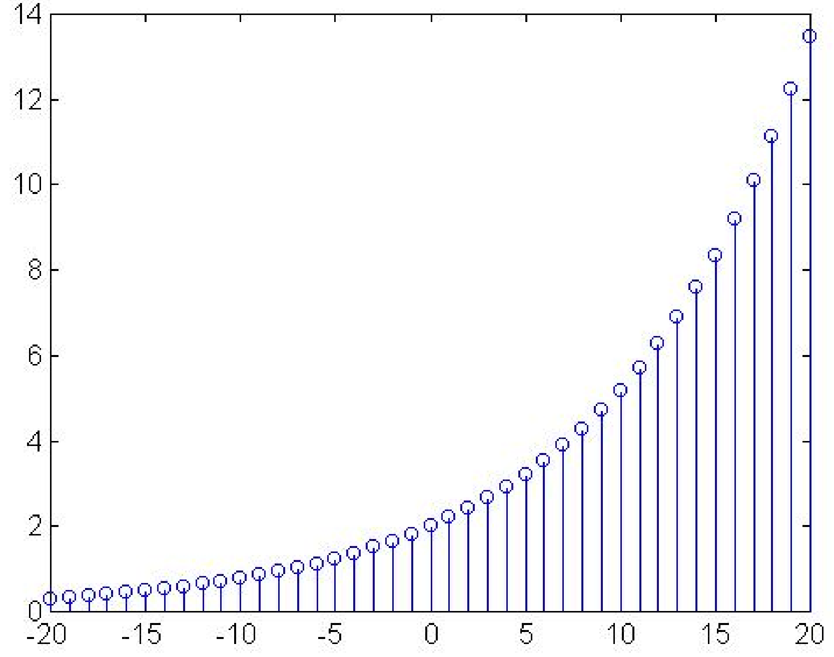
ϕ = phase angle in radians



Exponential signal $C*\alpha^n$

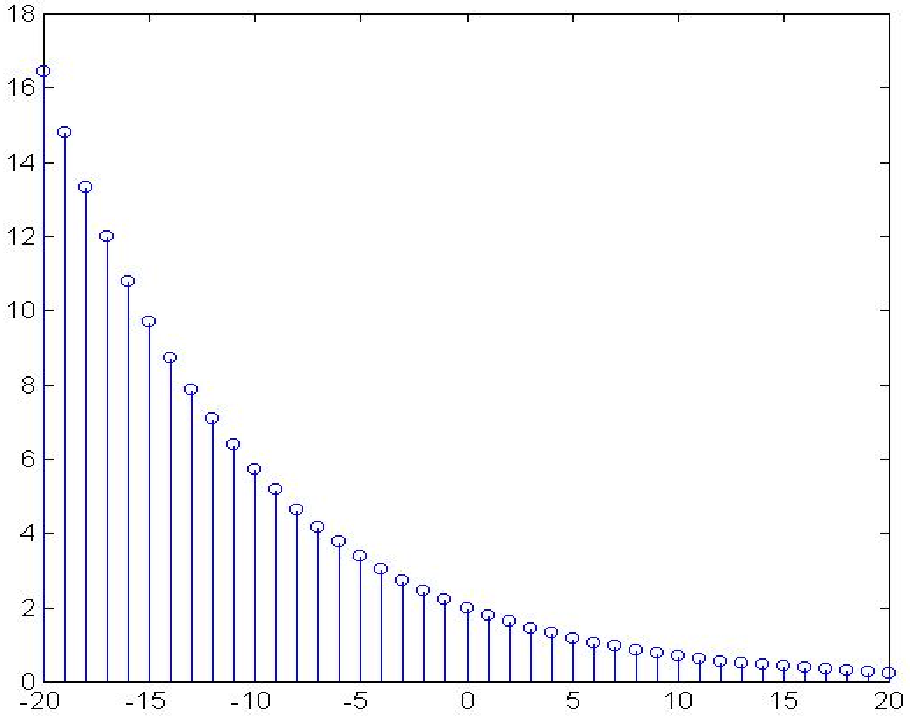
- If C and ‘α’ are real then it will be real exponential

$$x[n] = C*\alpha^n \text{ where } \alpha > 1 \qquad e.g.\quad x[n] = 2*1.1^n$$



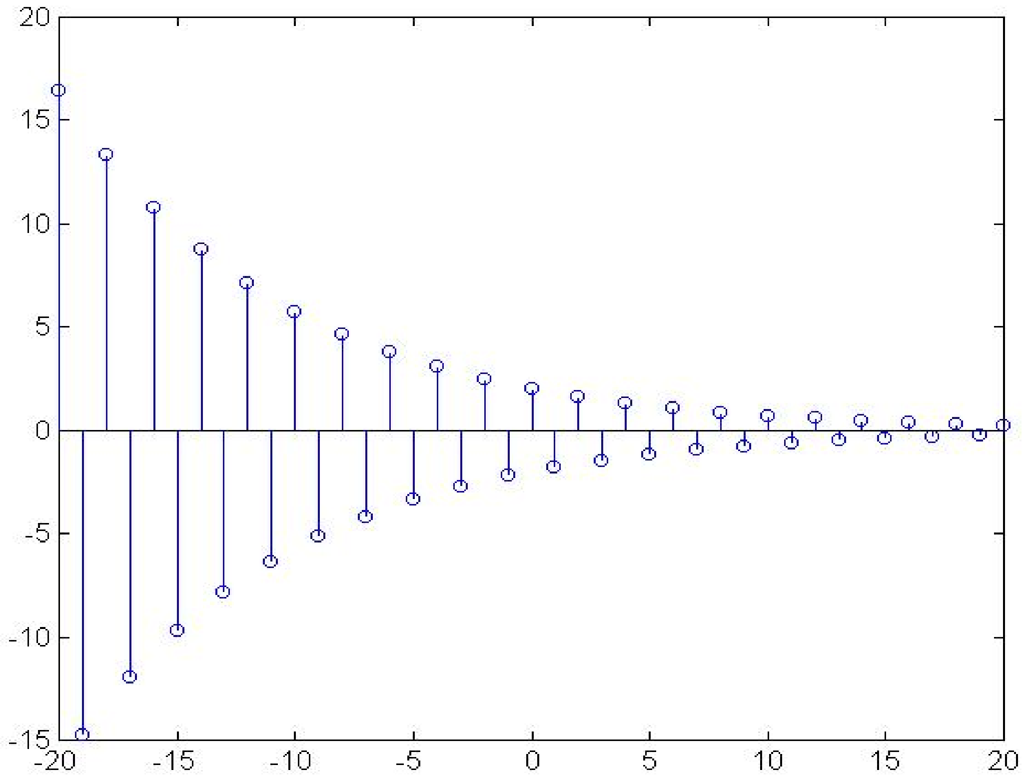
Real exponential signal - examples

$x[n] = C * \alpha^n$ where $0 < \alpha < 1$ $\qquad x[n] = 2 * 0.9^n$



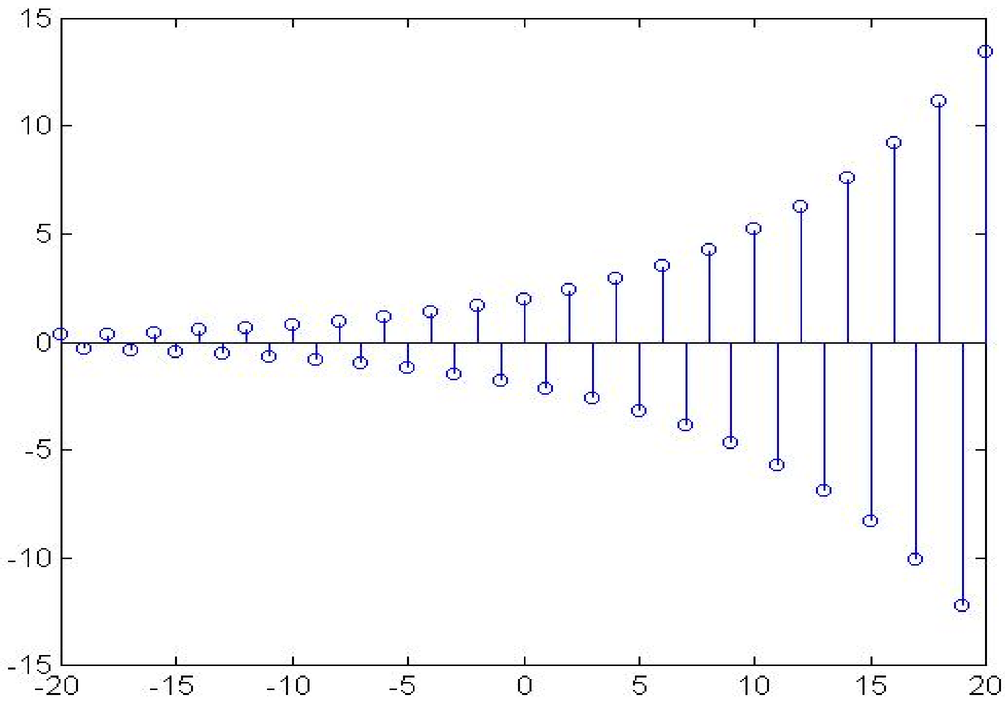
Real exponential signal - examples

$x[n] = C * \alpha^n$ where $-1 < \alpha < 0$ $\quad x[n] = 2*(-0.9)^n$



Real exponential signal - examples

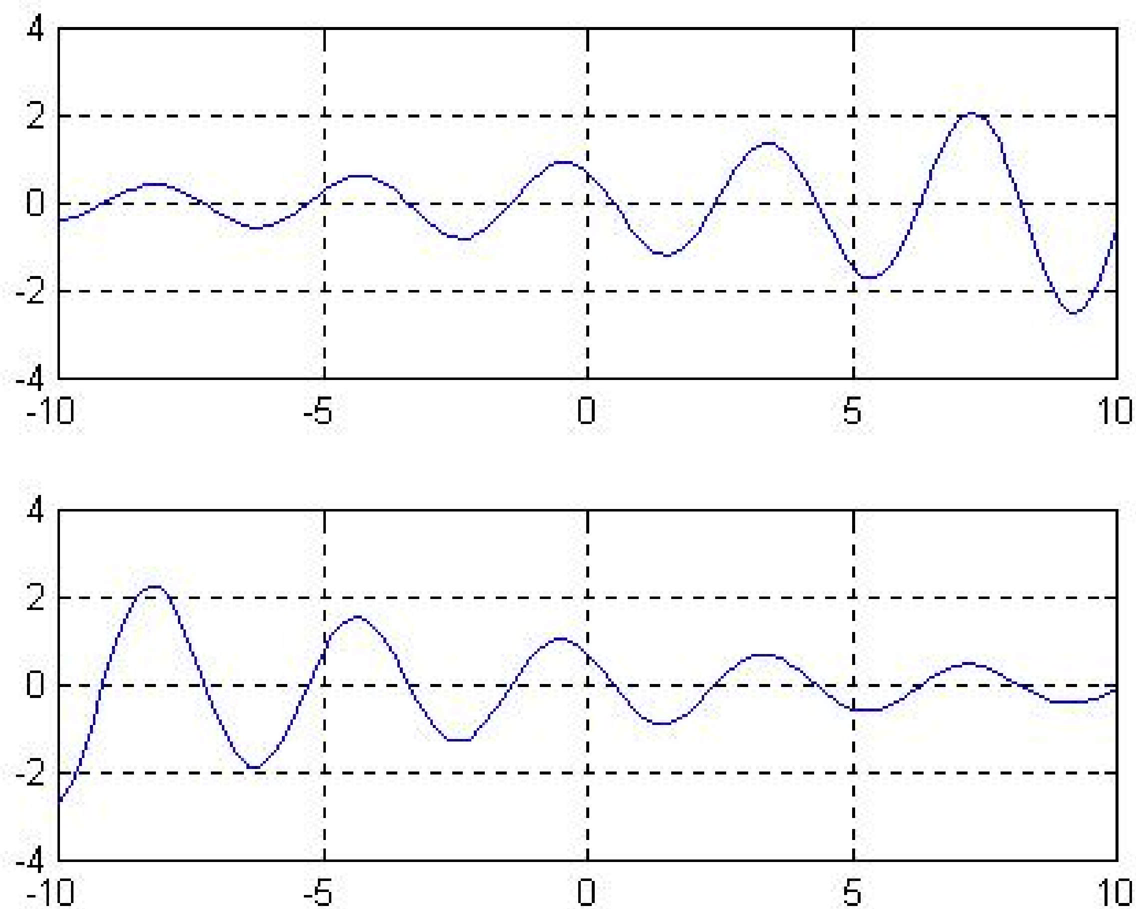
$x[n] = C * \alpha^n$ where $\alpha < -1$ $\quad$ $x[n] = 2*(-1.1)^n$



Growing & Decaying Sinusoidal Signals

r>0

r<0



Relationship between real sinusoids and complex exponentials

- A real sinusoid can be considered as the real part of a complex exponential

$$A\cos(\omega_0 t + \phi) = \mathrm{Re}\left[Ae^{j(\omega_0 t + \phi)}\right] = \mathrm{Re}\left[Ae^{j\phi}e^{j\omega_0 t}\right]$$

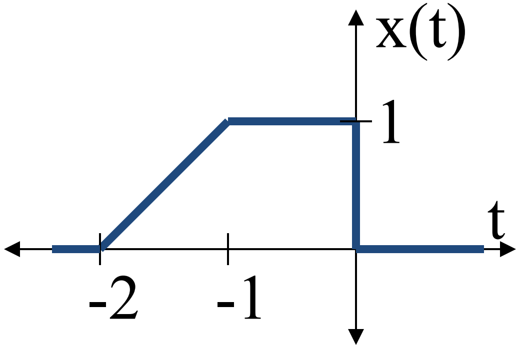
$Ae^{j\phi}$ is called the phasor of the sinusoid, which contains the amplitude and initial phase of the signal

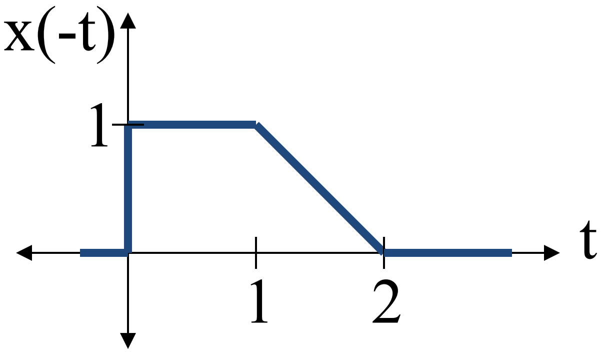
Transformations of Time Variable

- Three possible ***time*** transformations:
  - *Time Flip (or reverse)*: x(-t), x[-n]
    - Flips the signal over the vertical axis.
  - *Time Shift*: x(t+a), x[n+a]
    - On horizontal axis, shifts to the right when a<0, shifts to the left when a>0.
  - *Time Scale*: x(at), x[an] for a>0.
    - On horizontal axis, scales the signal length down when a>1, scales it up when a<1.

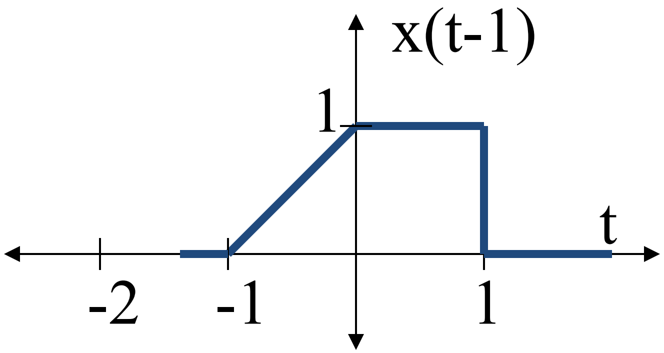
Transformations of Time Variable (cont'd)

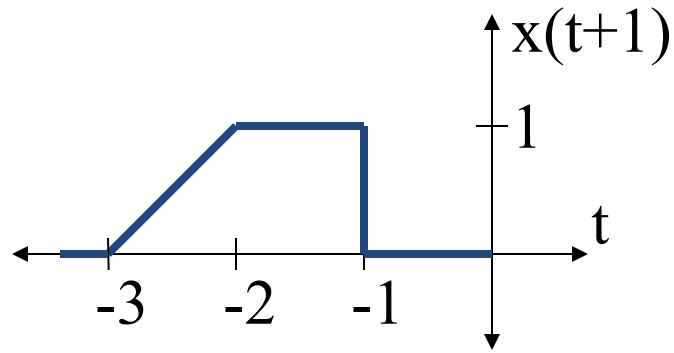
- Time-flip example:





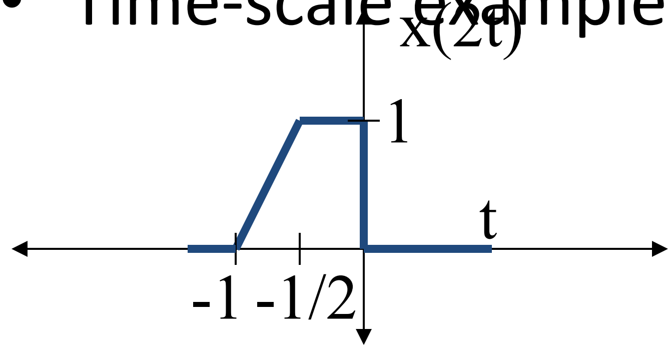
- Time-shift example:

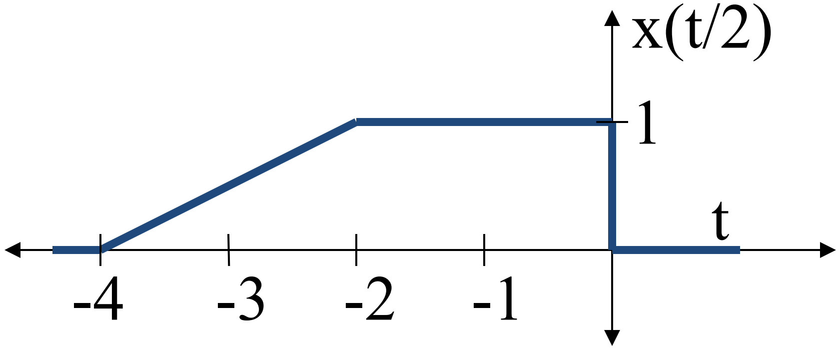




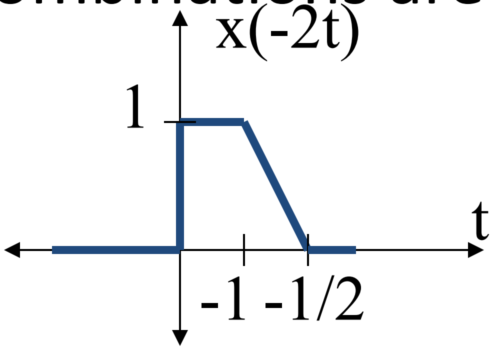
Transformations of Time Variable (cont'd)

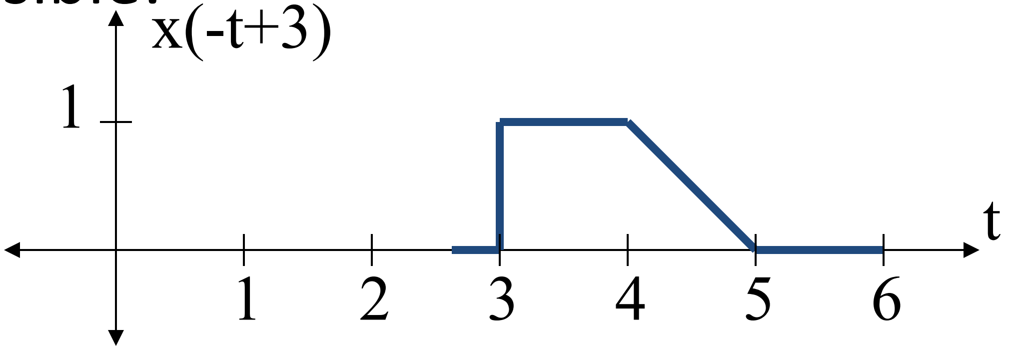
- Time-scale example:





- Combinations are possible:

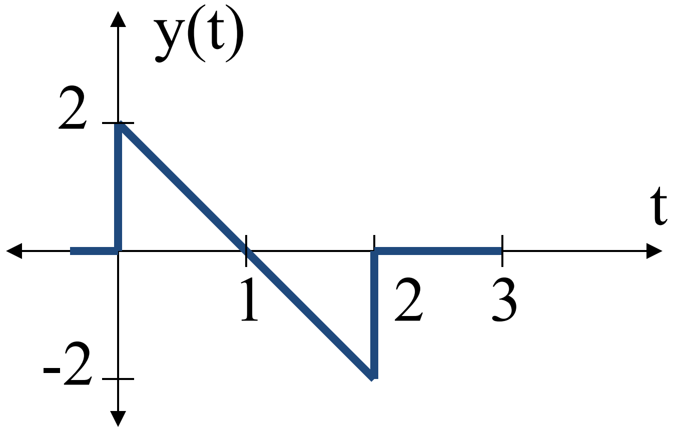




Transformations of Time Variable (cont'd)

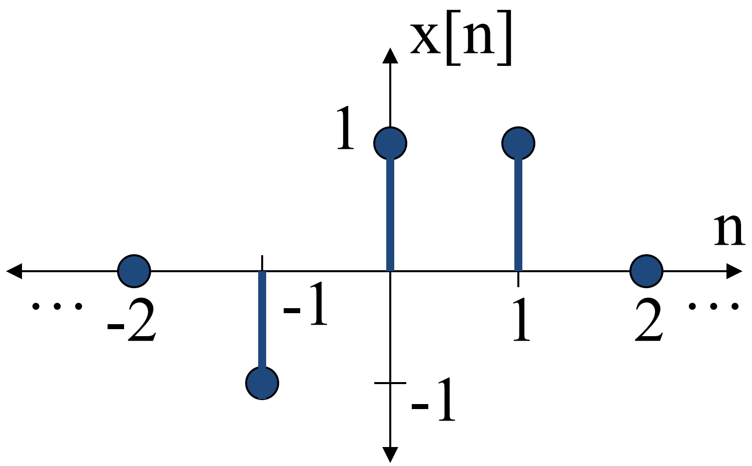
- Be careful when combining the transformations.
  - e.g. $x(-t+3) = x_1(t-3)$ where $x_1(t)= x(-t)$ *or*

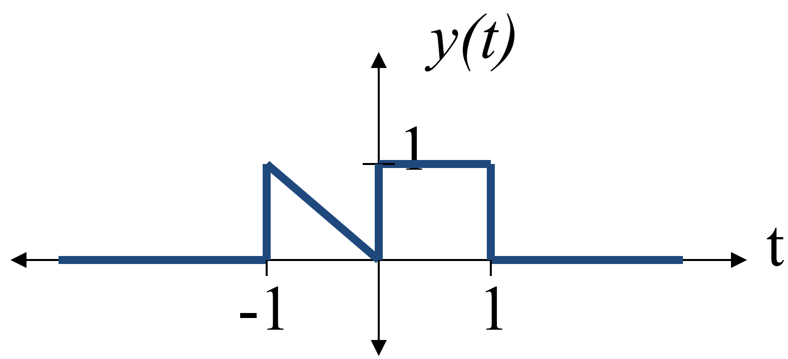
    $x(-t+3) = x_2(-t)$ where $x_2(t)= x(t-3)$
- Given y(t) below, find y(-3t+6) in different orders: flip/shift/scale flip/scale/shift, shift/flip/scale.



Even and Odd Signals

- $x[n]$ is even, if $x[n]=x[-n]$
- $x[n]$ is odd, if $x[-n]=-x[n]$
- Any signal x[n] can be divided into two parts:
  - Ev{x[n]} = (x[n]+x[-n])/2
  - Od{x[n]} = (x[n]-x[-n])/2
- The arguments above are also valid for continuous signals.
- Exercise: Divide the following signals into even and odd parts:





Periodic and Non Periodic signal

- A periodic signal $x(t)$ is a function that satisfies the condition,

$$\underline{x(t) = x(t+T)} \text{ for all } t$$

  - T that satisfied the above equation is called fundamental period of $x(t)$
  - The reciprocal of fundamental period is called fundamental frequency $f = {}^{1}\!/_{T}$

Periodic and Non Periodic signal

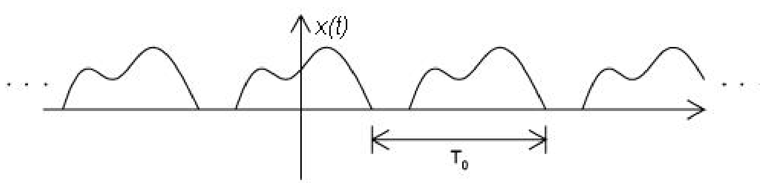
- The frequency *f* is measured in hertz (Hz) or cycles per second
- The angular frequency is measured in radians per second

$$\omega = {}^{2\pi}\!/_{T}$$

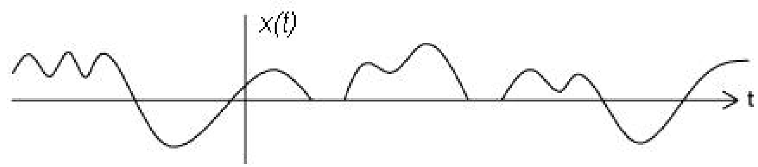
- Any signal *x(t)* for which there is no value of *T* to satisfy the previous slide equation is called aperiodic or non periodic signal

Periodic and Non Periodic signal - examples

Periodic Signal with period T_0



Non periodic signal



Periodicity for combination signals

- Let $x_1(t)$ and $x_2(t)$ be periodic signals with fundamental periods T_1 and T_2 respectively.

- Under what conditions is the sum

  $x(t) = x_1(t) + x_2(t)$ periodic ?

- If periodic, What will be the fundamental period of $x(t)$?

Periodicity for combination signals

- $x_1(t) = x_1(t+T_1) = x_1(t+mT_1)$,   m = +ve integer
- $x_2(t) = x_2(t+T_2) = x_2(t+kT_2)$,   k = +ve integer

$x(t) = x_1(t) + x_2(t)$

- $x(t) = x_1(t+mT_1) + x_2(t+kT_2)$
- x(t) is periodic with period T
- $x(t) = x(t+T) = x_1(t+T) + x_2(t+T)$
- mT1 = kT2 = T → $$\frac{T_1}{T_2} = \frac{k}{m}$$

Important Points

- Sum of two periodic signals is periodic only if the ratio (T1/ T2) is a rational number or ratio of two integers

- The fundamental period is the LCM of T1 and T2

- If T1/T2 is irrational no, then x1(t) and x2(t) do not have common period and hence x(t) is not periodic

Are these signals are periodic ? If so, find its fundamental period

- $x(t) = 3u(t)+2\sin(2t)$
- Aperiodic

$x(t) = 2\cos(10t + 1) - \sin(4t-1)$

- $T1/T2 = 2/5$

Periodicity in discrete signals

- Condition: $x[n] = x[n+N]$
- $X[n] = A \sin(\omega_0 n+\theta)$
- $X[n+N] = A \sin[\omega_0(n+N)+\theta] = A \sin[\omega_0 n+\omega_0 N + \theta]$

- So, $\omega_0 N = 2\pi * m$ where m = integer

$$N = 2\pi\left[\frac{m}{\omega_0}\right]$$

Are these signals are periodic ? If so, find its fundamental period

- $X[n] = 12 \cos (20n)$
  - Aperiodic

$$x[n] = e^{j\frac{3\pi}{5}\left(n+\frac{1}{2}\right)}$$

– When m = 3, then N will be an integer = 10

Power and Energy of Signals

- Energy: accumulation of absolute of the signal

$$E_{\infty} \overset{\Delta}{=} \lim_{T\to\infty} \int_{-T}^{T} \left|x(t)\right|^2 dt = \int_{-\infty}^{\infty} \left|x(t)\right|^2 dt$$

$$E_{\infty} \overset{\Delta}{=} \lim_{N\to\infty} \sum_{n=-N}^{N} \left|x[n]\right|^2 = \sum_{-\infty}^{\infty} \left|x[n]\right|^2$$

- Power: average of absolute of the signal

$$P_{\infty} \overset{\Delta}{=} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|x(t)\right|^2 dt = \lim_{T\to\infty} \frac{E_{\infty}}{2T}$$

$$P_{\infty} \overset{\Delta}{=} \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|x[n]\right|^2 = \lim_{N\to\infty} \frac{E_{\infty}}{2N+1}$$

Power and Energy of Signals (cont'd)

- *Energy* signal iff $0<E<\infty$, and so $P=0$.
  - e.g: $$x(t) = \begin{cases} 0, & t < 0 \\ e^{-t}, & t \geq 0 \end{cases}$$
- *Power* signal iff $0<P<\infty$, and so $E=\infty$.
  - e.g: $\{x[n]\} = \{\ldots -1, 1, \underline{-1}, 1, -1, 1 \ldots\}$
- Neither energy nor power, when both E and P are infinite.
  - e.g: $x(t) = t$

.

Is these energy or power signal ?

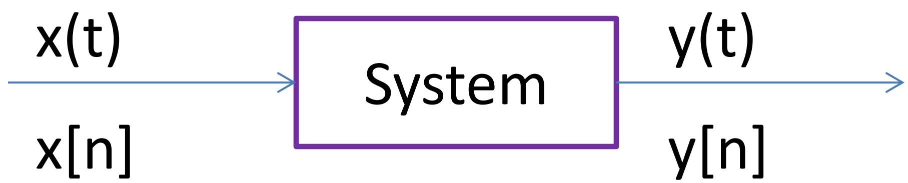
- $X(t) = e^{-at}\, u(t)$, where a>0 Energy signal
- $x(t) = u(t)$ Power signal
- $X(t) = tu(t)$ Neither energy nor Power signal
- $X(t) = \sin^2\omega_0 t$ Power signal
- $X[n] = \left(\frac{1}{2}\right)^n u[n]$ Energy signal

- Signal → not periodic , may be energy signal
- Signal → periodic, may be power signal

SYSTEMS

Introduction to systems

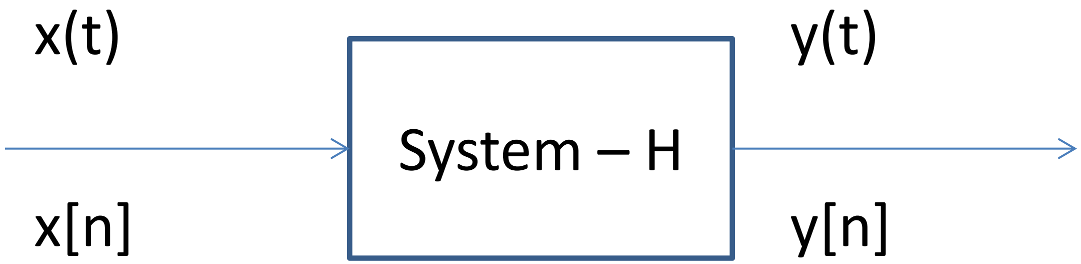
- A system is a physical device that generates a response or output signal for an input signal



- Example:
  - Speaker recognition system
  - Communication system
  - Aircraft landing system etc.,

A system is formally defined as an entity that manipulates one or more signal to accomplish a function there by yielding new signals

System representation



- H denotes the action of the system
- y(t) = H { x(t) }
- y[n] = H { x[n] }

Classification – based on independent variable

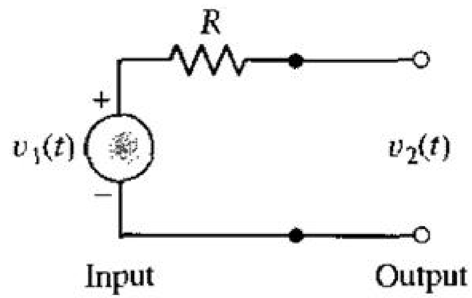
- Continuous time(CT) system
  - A CT system is one which operates on a CT signal and produces CT output signal
- Discrete time (DT) system
  - A DT system is one which operates on DT signal and produces DT output signal
- Mixed system
  - A mixed system is one which operates on CT / DT signal and produces DT / CT signal respectively

Classification – based on character

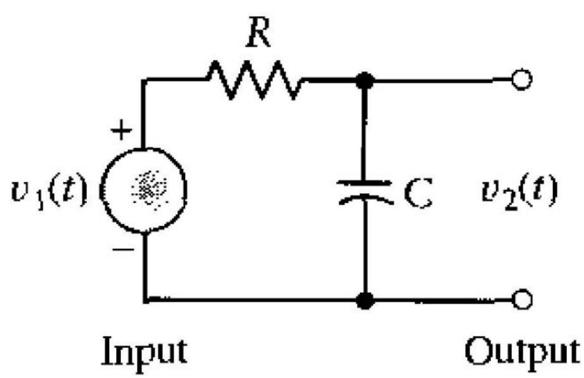
- Memory
  - Static & dynamic
- Causality
  - Causal & non causal
- Linearity
  - Linear & non linear
- Time variance
  - Time variant & time invariant
- Stability
  - Stable & unstable

Memory – static & dynamic

- Static or memory less
  - Output at any instant depends on the input at that instant but not on the past or future values of input



- Dynamic or memory
  - Output signal depends on past or future values of the input



Example

- $y(t) = x(t+1) + 5$
  - Dynamic
- $y[n] = x[n] + x[n-1]$
  - Dynamic
- $y(t) = x(t) \sin(2t)$
  - Static
- $y[n] = u[n]$
  - Static

Causality

- Causal system or anticipative system
  - System output at any time depends on present and /or past inputs not future inputs

- Non Causal system
  - System output at any time depends on future inputs

Example

- y(t) = x(t) + x(t-1)
  - Causal system

- y[n] = x[n+3] + x^2[n]
  - non causal system

$$y(t) = \int_{-\infty}^{2t} x(t)\,dt$$

$$t = 0,\ y(0) = \int_{-\infty}^{0} x(t)\,dt = P(0) - P(-\infty)$$

$$t = 1,\ y(1) = \int_{-\infty}^{2} x(t)\,dt = P(2) - P(-\infty)$$

Non causal system

Linearity

- $x_1(t) \Rightarrow y_1(t)$
- $x_2(t) \Rightarrow y_2(t)$

- $x_1(t) + x_2(t) \Rightarrow y_1(t) + y_2(t)^{also}$
- Should satisfy scaling property (i.e)
- $a_1x_1(t) \Rightarrow a_1y_1(t)$
- $a_2x_2(t) \Rightarrow a_2y_2(t)$

- $a_1x_1(t) + a_2x_2(t) \Rightarrow a_1y_1(t) + a_2y_2(t)$

Linearity *(continued)*

- System is said to be linear if, the weighted sum of several inputs produces the weight sum of outputs

- If not, the system is non linear

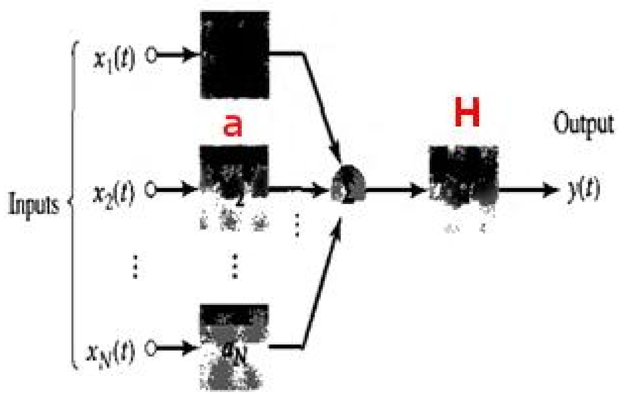
$$H\left\{ax_1(t)+bx_2(t)\right\}=aH\left\{x_1(t)\right\}+bH\left\{x_2(t)\right\}$$

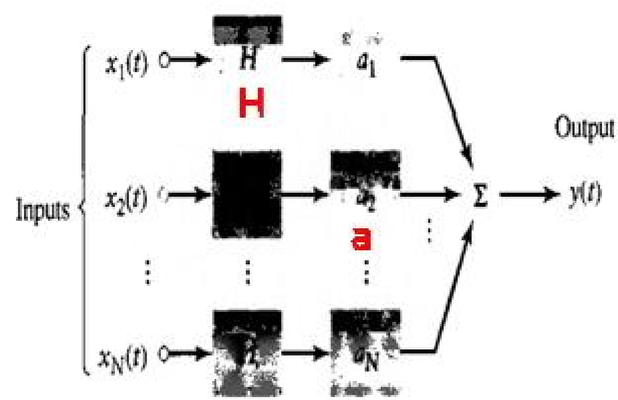
$$H\left\{ax_1[n]+bx_2[n]\right\}=aH\left\{x_1[n]\right\}+bH\left\{x_2[n]\right\}$$

Linearity *(continued)*

$$H\left\{a_1 x_1(t) + a_2 x_2(t)\right\} = a_1 H\left\{x_1(t)\right\} + a_2 H\left\{x_2(t)\right\}$$

$$H\left\{a_1 x_1[n] + a_2 x_2[n]\right\} = a_1 H\left\{x_1[n]\right\} + a_2 H\left\{x_2[n]\right\}$$





Testing the linearity

- Step 1:
  - $y1(t) = H \{ x1(t) \}$; $y2(t) = H \{ x2(t) \}$
  - Find weighted sum
  - $y3(t) = a1\ y1(t) + a2\ y2(t)$
  - $y3(t) = a1\ H \{ x1(t) \} + a2\ H \{ x2(t) \}$
- Step 2
  - For the linear combination of i/p [a1 x1(t) + a2 x2(t)], find the o/p for weighted sum
  - $y4(t) = H \{ a1\ x1(t) + a2\ x2(t) \}$
- If y3 = y4, system is linear

Example

$$y(t) = \{2x(t)\}^2 \qquad y_1(t) = 4x_1^{\,2}(t) \qquad y_2(t) = 4x_2^{\,2}(t)$$

$$y_3(t) = a_1 y_1(t) + a_2 y_2(t)$$

$$y_3(t) = 4a_1 x_1^{\,2}(t) + 4a_2 x_2^{\,2}(t)$$

$$y_4(t) = H\{a_1 x_1(t) + a_2 x_2(t)\} \qquad y_4(t) = 4\{a_1 x_1(t) + a_2 x_2(t)\}^2$$

$$y_4(t) = 4a_1 x_1^{\,2}(t) + 4a_2 x_2^{\,2}(t) + 8a_1 a_2 x_1(t) x_2(t)$$

$$y_3(t) \neq y_4(t)$$

Non Linear

Example

$$y(t) = x(t^2) \qquad y_1(t) = x_1(t^2) \qquad y_2(t) = x_2(t^2)$$

$$y_3(t) = a_1 y_1(t) + a_2 y_2(t)$$

$$y_3(t) = a_1 x_1(t^2) + a_2 x_2(t^2)$$

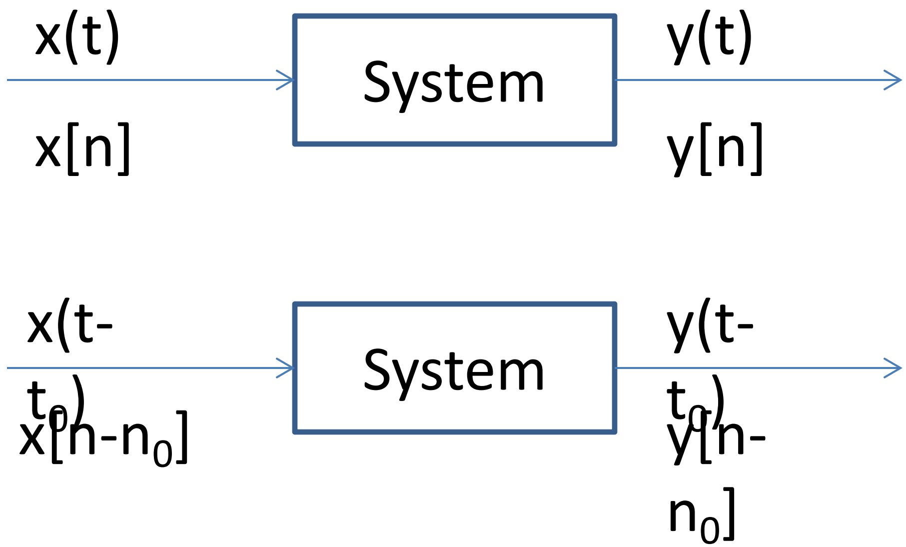
$$y_4(t) = H\left\{a_1 x_1(t) + a_2 x_2(t)\right\}$$

$$y_4(t) = a_1 x_1(t^2) + a_2 x_2(t^2)$$

$$y_3(t) = y_4(t)$$ Linear

Time invariant & time variant system

- A system is said to be time invariant if its input and output characteristics do not change with time



- If i/p is delayed by t_0 second, then o/p is also delayed by t_0 second ➔ time invariant system

Testing time variance

- $x(t)$ = input and $x(t-t_0)$ delayed input
- $y(t) = H\{ x(t) \}$ be the output for the input $x(t)$
- $y(t, t_0) = H\{ x(t-t_0) \} = y(t)|_{x(t)=x(t-t0)}$
- $y(t-t_0) = y(t)|_{t = t - t0}$
- $y(t, t_0) = y(t-t_0)$ ➔ time invariant system

Example

Time variant system

- $y(t) = t\,x(t)$
- $y(t, t_0) = H\{\, x(t-t_0) \,\}$
  $= y(t)|_{x(t)=x(t-t0)}$

- $y(t, t_0) = t\,x(t-t_0)$

- $y(t-t_0) = y(t)|_{t\,=\,t\,-\,t0}$
- $y(t-t_0) = (t-t_0)\,x(t-t_0)$

- $y(t,t_0) \neq y(t-t_0)$

Time Invariant system

- $y(t) = e^{x(t)}$
- $y(t, t_0) = e^{x(t-t0)}$

- $y(t-t_0) = y(t)|_{t\,=\,t\,-\,t0}$
- $y(t-t_0) = e^{x(t-t0)}$

- $y(t,t_0) = y(t-t_0)$

Stable & Unstable CT system

- A CT system is said to be stable system, if the **bounded input** to the system produces **bounded output**

- if x(t) is bounded, y(t) should also be bounded for the system to be stable

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

Example

$$h(t) = e^{4t}u(t-3)$$

$$\int_{-\infty}^{\infty} |h(t)| dt$$

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{3}^{\infty} e^{4t} dt$$

$$= \left[\frac{e^{4t}}{4} \right]_3^{\infty} = \frac{1}{4}\left[\infty - e^{12} \right] = \infty$$

Unstable system

$$x(t) = e^{-2|t|}$$

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{0} e^{2t} dt + \int_{0}^{\infty} e^{-2t} dt$$

$$= \frac{1}{2}\left[e^{2t} \right]_{-\infty}^{0} - \frac{1}{2}\left[e^{-2t} \right]_{0}^{\infty}$$

$$= \frac{1}{2}\left[1 + 1 \right] = 1 < \infty$$

Stable system

Stable & Unstable DT system

- A DT system is said to be stable if bounded input produces bounded output (i.e.) **absolutely summable**

$$x[n] \leq M_x < \infty \qquad y[n] \leq M_x < \infty$$

$$\sum_{-\infty}^{\infty} \left| h[n] \right| < \infty$$

Example

$$h[n] = nu[n]$$

$$y[n] = \sum_{0}^{\infty} n$$

$$= 1 + 2 + \ldots + \propto$$

$$= \propto$$

Unstable system

$$h[n] = 2^n u[-n-1]$$

$$y[n] = \sum_{-\infty}^{-1} 2^n$$

Changing
n= -n

$$y[n] = \sum_{1}^{\infty} 2^{-n}$$

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots$$

$$= \frac{1}{2}\left\{1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \ldots\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{1 - \left(\frac{1}{2}\right)}\right\} = 1 \quad < \infty$$