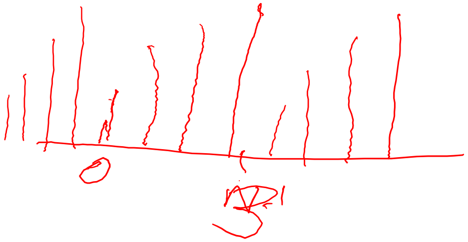
Properties of DFT

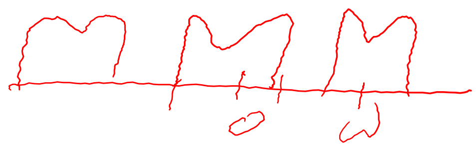


$X(\omega) = \sum x[n] e^{-j\omega n}$

Periodicity

$X(\omega+N) = \sum x(n)\, e^{-j\omega n} e^{-j2\pi n} = \sum x[n] e^{-j\omega n}$

- If x(n) and X(K) are N point DFT pair
  - Time – x(n+N) = x(n) for all n
  - Freq – X(k+N) = X(k) for all k



Sampling in t ⇒ periodic in freq

- Proof:

  (K+N)th coefficient of X(K) is

DFT

Samp in freq ⇒ periodic in time

$$X(k+N) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}(k+N)n} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} e^{-j\frac{2\pi}{N}Nn}$$

$$X(k+N) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = X(k)$$

Symmetry Properties of DFT

DFT of a real sequence

- If x[n] = real 1, 1, 2, 2 {1, 1, 2, 2}
- X(N-k) = X*(k) = X(-k)

X(k) = {6, -1+j, 0, -1-j}

X(4-1) = X(3) = X*(1)

DFT of a Real & even sequence

- If x[n] = real and even (i.e) $2, 1, 2 \{1, 2, 1, 2\}$
  x(n) = x(N - n) , n = 0 to N-1

  $\{6, 0, -2, 0\}$

- DFT will be
$$X(k)=\sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi}{N}\right)nk \qquad 0\le k\le N-1$$

  $e^{-j\frac{2\pi}{N}nk}$

- IDFT also reduces to

  $\cos\frac{2\pi}{N}nk - j\sin\frac{2\pi}{N}nk$

$$x(n)=\frac{1}{N}\sum_{k=0}^{N-1} X(k)\cos\left(\frac{2\pi}{N}\right)nk \qquad 0\le n\le N-1$$

DFT of a Real & Odd sequence

- If x[n] = real and odd (i.e)
  x(n) = - x(N - n) , n = 0 to N-1
- Real part of X(k) = 0, so

$$X(k) = -j\sum_{n=0}^{N-1} x(n)\sin\frac{2\pi kn}{N} \qquad 0 \le k \le N-1$$

- Also IDFT reduces to

$$x(n) = j\frac{1}{N}\sum_{k=0}^{N-1} X(k)\sin\frac{2\pi kn}{N} \qquad 0 \le n \le N-1$$

Complex input

- If N point sequence x(n) and its DFT are complex valued $x(n) = x_r(n) + j\, x_i(n)$, n = 0 to N-1 What will be X(K) ?

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2\pi}{N}\right)nk}$$

$$X(k) = \sum_{n=0}^{N-1} \left(x_r[n] + jx_i[n]\right) e^{-j\left(\frac{2\pi}{N}\right)nk}$$

$$X(k) = \sum_{n=0}^{N-1} \left(x_r[n] + jx_i[n]\right) e^{-j\left(\frac{2\pi}{N}\right)nk}$$

$$X(k) = \sum_{n=0}^{N-1} \left(x_r[n] + jx_i[n]\right) \left\{ \cos\left(\frac{2\pi}{N}\right)nk - j\sin\left(\frac{2\pi}{N}\right)nk \right\}$$

$$X(k) = \sum_{n=0}^{N-1} \left(x_r[n]\cos\left(\frac{2\pi}{N}\right)nk + x_i[n]\sin\left(\frac{2\pi}{N}\right)nk \right)$$

$$- j\sum_{n=0}^{N-1} \left(x_r[n]\sin\left(\frac{2\pi}{N}\right)nk - x_i[n]\cos\left(\frac{2\pi}{N}\right)nk \right)$$

X(k) = X_R(k) + j X_I(k)

$$X_R(k) = \sum_{n=0}^{N-1}\left(x_r[n]\cos\left(\frac{2\pi}{N}\right)nk + x_i[n]\sin\left(\frac{2\pi}{N}\right)nk \right)$$

$$X_I(k) = -\sum_{n=0}^{N-1}\left(x_r[n]\sin\left(\frac{2\pi}{N}\right)nk - x_i[n]\cos\left(\frac{2\pi}{N}\right)nk \right)$$

Purely imaginary input sequence

- If $x(n) = j\, x_I(n)$ then DFT will be $X(k) = X_R(k) + j\, X_I(k)$

$$X_R(k) = \sum_{n=0}^{N-1} x_I(n) \sin \frac{2\pi kn}{N}$$

$X_R(k)$ = Odd
$X_I(k)$ = Even

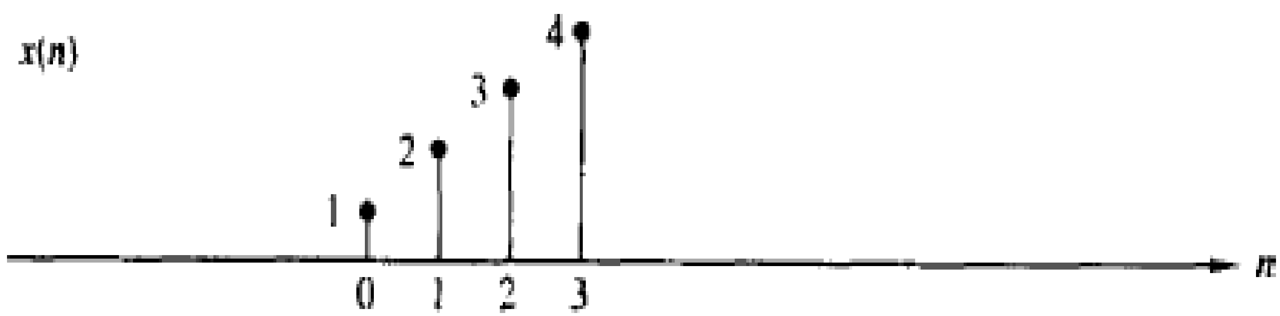
$$X_I(k) = \sum_{n=0}^{N-1} x_I(n) \cos \frac{2\pi kn}{N}$$

- If x(n) = odd → $X_I(k) = 0$ → X(k) = purely real
- If x(n) = even → $X_R(k) = 0$ → X(k) = purely imaginary

Circular symmetries of a sequence

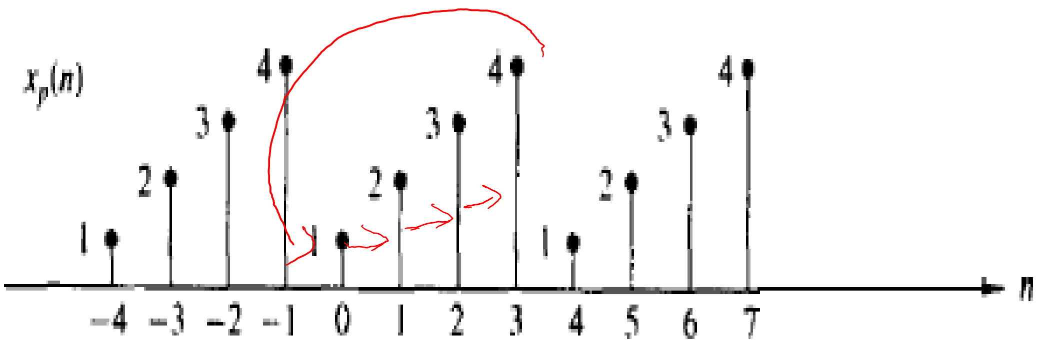
- Consider a finite duration sequence x(n)
- Its periodic extension as $x_p(n)$
  - (i.e) $x_p(n) = x(n+N)$

- Let N = 4, the sequence x(n)

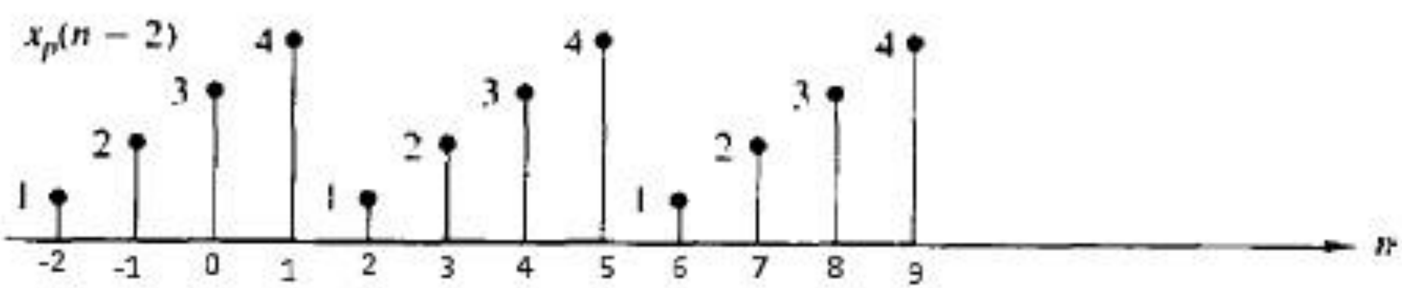


Circular symmetries of a sequence (continued)

- The periodic extension $x_p(n)$

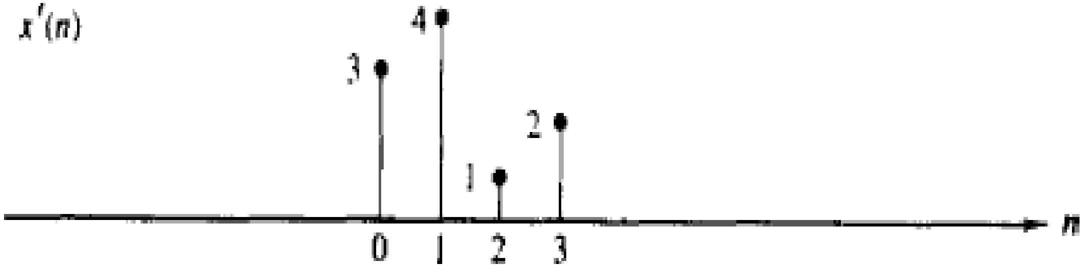


- If $x_p(n)$ is shifted by two units in right



Circular symmetries of a sequence (continued)

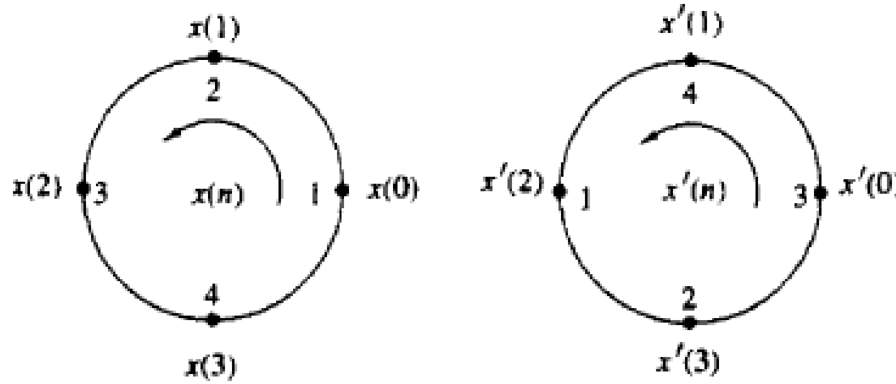
- Now the new sequence will be x'(n)



- From the above signal, x'(n) can be represented as x(n-2, (modulo 4))
  - modulo 4 indicates that sequence repeat after 4 samples

Circular symmetries of a sequence (continued)

- Sequence x(n) and x'(n) can be represented on a circle



- x'(n) = x(n) with circularly shifted by two units

$$x(n\text{-}k, \bmod N) \equiv x((n\text{-}k))_N$$

- <u>Circular shift of a N point sequence is equal to linear shift of its periodic extension and vice-versa</u>

Properties of DFT

Linearity

- If

$$x_1(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)$$

$$x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_2(k)$$

- then for any real valued or complex valued constants a1 and a2

$$a_1 x_1(n) + a_2 x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} a_1 X_1(k) + a_2 X_2(k)$$

Circular time shift

Statement

If DFT { x[n] } = X(k), then $DFT\ \{x((n-m))_N\}$ = $W_N{}^{km}$ X(k), 0≤ k ≤ N

Proof

$IDFT\ x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N{}^{-kn}$, 0≤ n ≤ N-1 ....(1)

Implies, $x[n-m] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N{}^{-k(n-m)}$ ....(2)

Since, the time shift is circular, we can write eq.(2) as

$x((n-m))_N$ = $\frac{1}{N}\sum_{k=0}^{N-1} [X(k)\, W_N{}^{km}]\, W_N{}^{-kn}$ ......(3)

$x((n-m))_N$ = $IDFT\ [\, X(k) W_N{}^{km}\,]$ ....(4)

(OR)

$DFT\ \{x((n-m))_N\}$ = $W_N{}^{km}$ X(k) ....(5) (Hence the proof)

Time shift property

Problem

Find the 4-point DFT of the sequence, $x[n] = $ (1,-1,1,-1). Also using time shift property, find the DFT of the sequence, $y[n] = x((n-2))_4$?

Solution: Given N=4

${W_4}^0$ = 1, ${W_4}^1$ = – j, ${W_4}^2$ = -1 and ${W_4}^3$ = j

$X(k) = DFT\ \{x[n]\}$ = $\sum_{n=0}^{N-1} x[n]{W_N}^{kn}$ = $\sum_{n=0}^{3} x[n]{W_4}^{kn}$, 0≤ k ≤ 3

= 1 x ${W_4}^{0k}$ - 1 x ${W_4}^{k}$ + 1 x ${W_4}^{2k}$ - 1 x ${W_4}^{3k}$

$X(k) =$ 1 - ${W_4}^{k}$ + ${W_4}^{2k}$ - ${W_4}^{3k}$ ......(1)

Sub k=0 in eq.(1) , $X(0) =$ 1-1+1-1 = 0

Sub k=1 in eq.(1) , $X(1) =$ 1- ${W_4}^1$ + ${W_4}^2$ - ${W_4}^3$ = 1 + j -1 –j = 0

Sub k=2 in eq.(1) , $X(2) =$ 1- ${W_4}^2$ + ${W_4}^4$ - ${W_4}^6$ = 1- ${W_4}^2$ + ${W_4}^0$ - ${W_4}^2$

$X(2) =$ 1+1+1+1 =4

Sub k=3 in eq.(1) , $X(3) =$ 1- ${W_4}^3$ + ${W_4}^6$ - ${W_4}^9$ = 1- ${W_4}^3$ + ${W_4}^2$ - ${W_4}^1$

$X(3) =$ 1-j-1+j = 0

Therefore DFT $X(k) =$ (0,0,4,0)

Circular time shift property

Given, $y[n] = x((n-2))_4$ and we now found $X(k) =$ (0,0,4,0)

Applying circular time shift property, we get

DFT Y(k) = $W_4^{\,2k}$ X(k) , where k=0,1,2,3 ……….(2)

Sub k=0 in eq.(2) , Y$(0) = W_4^{\,0}$ X(0) = 1 X 0 = 0

Sub k=1 in eq.(2) , Y$(1) = W_4^{\,2}$ X(1) = -1 X 0 = 0

Sub k=2 in eq.(2) , Y$(2) = W_4^{\,4}$ X(2) = $W_4^{\,0}$ X(2) = 1 x 4 = 4

Sub k=3 in eq.(2) , Y$(3) = W_4^{\,6}$ X(3) = $W_4^{\,2}$ X(3) = -1 x 0 = 0

Hence, DFT Y$(k) =$ (0,0,4,0)

Time-Reversal / Circular Folding Property of DFT

if DFT {x[n]} = X(k), then DFT {x[N-n]} = X(N-k)

Implies, $DFT\ \{x\big((-n)\big)_N\}$ = $X\big((-k)\big)_N$

Proof: DFT {x[N-n]} = $\sum_{n=0}^{N-1} x[N-n]e^{-j2\pi kn/N}$ ....(1)

Let m = N-n , therefore n=N-m ....(2)

Eq.(1) implies, DFT {x[N-n]} = $\sum_{m=0}^{N-1} x[m]e^{-j2\pi k(N-m)/N}$

$= \sum_{m=0}^{N-1} x[m]e^{-j2\pi kN/N}\ e^{j2\pi km/N}$

$= \sum_{m=0}^{N-1} x[m]\ e^{-j2\pi k}\ e^{j2\pi km/N}$ ....(3)

DFT {x[N-n]} $= \sum_{m=0}^{N-1} x[m]\ e^{j2\pi km/N}$ (Since $e^{-j2\pi k}$ = 1)

Circular Folding Property of DFT

DFT {x[N-n]} $= \sum_{m=0}^{N-1} x[m]\ e^{j2\pi km/N}$ ......(4)

$= \sum_{m=0}^{N-1} x[m]\ e^{-j2\pi m}\ e^{j2\pi km/N}$

$= \sum_{m=0}^{N-1} x[m]\ e^{-j2\pi mN/N} e^{j2\pi km/N}$

$= \sum_{m=0}^{N-1} x[m]\ e^{-\frac{j2\pi m(N-k)}{N}}$ ....(5)

= X(N-k)

Implies, DFT {x[N-n]} = X(N-k)

(OR)

$DFT\ \{x\left((-n)\right)_N\} = X\left((-k)\right)_N$ ......(6) (hence the proof)

Circular Folding

Problem

i) Compute the 4-point DFT of the sequence $x[n]$={1,2,1,0}

ii) Also, find DFT Y(k) if $y[n] = x((-n))_4$, 0 ≤ k ≤ 3 ?

Circular Folding

Problem

i) Compute the 4-point DFT of the sequence $x[n]$={1,2,1,0}

ii) Also, find DFT Y(k) if $y[n] = x((-n))_4$, 0 ≤ k ≤ 3 ?

Solution

(I) We know that $Twiddle\ Factor\ \ W_N = e^{-j\frac{2\pi}{N}}$

Since N=4, we get $W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}}$

$Hence,\ \ W_4{}^0 = 1$, $W_4{}^1 = -j$, $W_4{}^2 = -1$ and $W_4{}^3 = j$

$X(k) = DFT\ \{x[n]\}\ = \sum_{n=0}^{3} x[n] W_4{}^{kn}$, 0≤ k ≤ 3

$= 1 \times W_4{}^{0k} + 2 \times W_4{}^{k} + 1 \times W_4{}^{2k} + 0 \times W_4{}^{3k}$

$X(k) = 1 + 2W_4{}^{k} + W_4{}^{2k}$ ……(1)

Now DFT $X(k) = $ 1 + $2W_4{}^k$ + $W_4{}^{2k}$ ……(1)

Sub k=0 in eq.(1) , $X(0) = $ 1+2+1 = 4

Sub k=1 in eq.(1) , $X(1) = $ 1 + $2W_4{}^1$ + $W_4{}^2$ = –j2

Sub k=2 in eq.(1) , $X(2) = $ 1 + $2W_4{}^2$ + $W_4{}^4$ = 1 + $2W_4{}^2$ + $W_4{}^0$ =0

Sub k=3 in eq.(1) , $X(3) = $ 1 + $2W_4{}^3$ + $W_4{}^6$ = 1 + $2W_4{}^3$ + $W_4{}^2$ =j2

Thus, DFT values of X(k) = {4, –j2, 0, j2} …..(2)

ii) Since $x[n]$ is real, it may be noted that the symmetry property,

$$X(k) = X^*(\text{N-k}) \qquad …….(3)$$

Given: $y[n] = x((-n))_4$

Hence, Y(k) = X(4-k) = $X\big((-k)\big)_4$ …….(4)

Circular Folding

$X\big((-k)\big)_4$ = X^*(k) , 0 ≤ k ≤ 3   (since $x[n]$ is real)

Hence, Y(k) = X^*(k)

Implies, Y(k) = {4, j2, 0, -j2}

Therefore, DFT Y(k) = {4, j2, 0, -j2}

Circular Convolution

$x[n]$ $h[n]$ $x(k)$

$$y[n] = \sum x(m)\, h((n-m))_{N} = x[n] \circledast h[n]$$

$x[n] * h(n) \rightarrow N$ Linear

length 4 3+0 4

$x(n) \rightarrow 4$

$h(n) \rightarrow 3$ $3 \rightarrow 4$

$y[n] \rightarrow 4$ $y[n] \rightarrow n_1 + n_2 - 1$

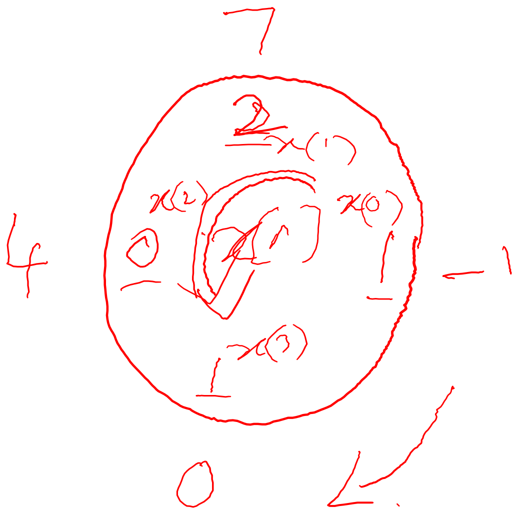
6

$x[n] = \{1, 2, 0, 1\}$ $\quad$ $h[n] = \{-1, 0, 4, 7\}$

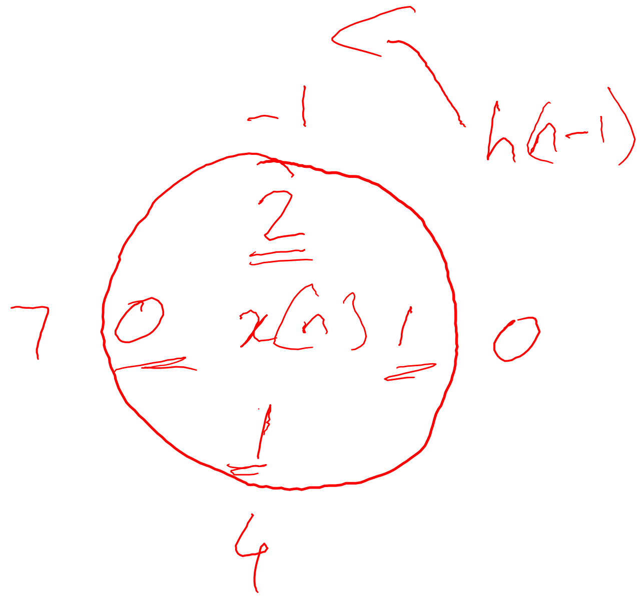
$x(n)$ 1 2 0 1 $\{1\ 2\ 0\ 1\}$...

$h(n)$ $\{7\ 4\ 0\ -1\}\{7\ 4\ 0\ -1$

0 -1 7 4



$y[0] = -1 + 14 = 13$



$y[1] = 0 - 2 + 0 + 4 = 2$

$y[2] = 11$

$y[3] = 14$

$x[n] = [1, 2, 0, 1]$ $h[n] = \{-1, 0, 4, 7\}$

fixed → $x(0)$ $x(1)$ $x(2)$ $x(3)$

$N-1$

$h(3)$ $h(2)$ $h(1)$ $h(0)$ $h(3)$ $h(2)$ $h(1)$

$h(1)$ $h(0)$ $h(3)$ $h(2)$

$h(2)$ $h(1)$ $h(0)$ $h(3)$

$h(3)$ $h(2)$ $h(1)$ $h(0)$

$$\begin{bmatrix} h(0) & h(N-1) & \cdots & h(1) \\ h(1) & h(0) & & h(2) \\ h(2) & h(1) & & h(3) \\ h(3) & h(2) & \cdots & \cancel{h(4)} \end{bmatrix}$$

$h(4) \to h(0)$

$$\begin{bmatrix} -1 & 0 & 4 & 7 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 7 & 4 & 0 \\ 0 & -1 & 7 & 4 \\ 4 & 0 & -1 & 7 \\ 7 & 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 2 \\ 11 \\ 14 \end{bmatrix}$$

Matrix approach

$$\boxed{y[n] = x[n] \circledast_N h[n]}$$

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[N-1] \end{bmatrix} = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \cdots & h[1] \\ h[1] & h[0] & h[N-1] & \cdots & h[2] \\ h[2] & h[1] & h[0] & \cdots & h[3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h[N-1] & h[N-2] & h[N-3] & \cdots & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}$$

Linear Convolution using CConv

N_1+N_2-1.

$\max(N_1, N_2)$

Linear Conv

$x_1(n) = \{1, 2, 3, 0, 0\}$

$x_2(n) = \{2, 4, 1, 0, 0\}$

$\{2, 8, 15, 14, 3\}$

$x_1(n) * x_2(n)$

	1	2	3
2	2	4	6
4	4	8	12
1	1	2	3

$x_1(n) \circledast x_2(n) = \{16, 11, 15\}$

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} =$$

$$x_1(n) \circledast x_2(n) = \begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 15 \\ 14 \\ 3 \end{bmatrix}$$

Circular Convolution Property of DFT

$x(n) * h(n) = X(j\omega) H(j\omega)$

$X(z) H(z)$

<u>Circular Convolution in Time</u>

- Let $x[n] and\ h[n]$ be two sequences of length N each

  Then $y[n] = x[n] *_N \ h[n]$

  $*_N$ indicates $circular\ convolution$ operation with period N

- That is $circular\ convolution\ y[n] = \sum_{m=0}^{N-1} x(m)h((n-m))_N$

  (or)

- $circular\ convolution \qquad y[n] = \sum_{m=0}^{N-1} x((n-m))_N \, h(m)$,

  where n= 0,1,2,………(N-1)

<u>Circular Convolution in time-domain is equivalent to multiplication in frequency-domain</u>

- DFT { $h[n] *_N \ x[n]$ } = H(k) X(k), where k = 0,1,2,……….(N-1)

Circular Convolution Property

Proof: DFT { $h[n] *_N \ x[n]$ } = DFT { $\sum_{l=0}^{N-1} h(l)\ x((n-l))_N$}

= $\sum_{l=0}^{N-1} h(l)\ DFT\ \{x((n-l))_{N\}}$ ....(1)

By circular time shift property,

$DFT\ \{x((n-l))_{N\}}$ = ${W_N}^{kl}\ X(k)$ ....(2)

Sub Eq.(2) in to Eq.(1) we get

Eq.(1) implies, DFT { $h[n] *_N \ x[n]$} = $\sum_{l=0}^{N-1} h(l)\ \ {W_N}^{kl}\ X(k)$

= $X(k) \sum_{l=0}^{N-1} h(l)\ \ {W_N}^{kl}$

= H(k) X(k) ....(3)

= RHS (Proved)

Linear vs. Circular Convolution

All LTI systems are based on the principle of linear convolution, as the output of an LTI system is the linear convolution of the system impulse response and the input to the system, which is equivalent to the product of the respective DTFTs in the frequency domain.

DTFT is based on linear convolution, and DFT is based on circular convolution

For two sequences of length N and M, the linear convolution is of length N+M-1, whereas circular convolution of the same two sequences is of length max(N,M), where the shorter sequence is zero padded to make it the same length as the longer one.

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Circular Convolution Property

$$\left(x[n] * h[n]\right)_N = IDFT\{X[k] \cdot H[k]\}$$

Parseval’s Theorem in DFT

ESD

- Parseval’s theorem also states that, “energy of the signal in time domain can be expressed in terms of the frequency components {X(k)} in the frequency domain”.

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

-

Energy ESD

Parseval's Theorem in DFT

Proof of Parseval's Theorem

From definition of IDFT, we have $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N}$

Take conjugate on both sides,

- $x*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X*(k)e^{-j2\pi kn/N}$
- LHS = $\sum_{n=0}^{N-1} x*(n).x(n) = \sum_{n=0}^{N-1}\left\{\left(\frac{1}{N}\right)\sum_{k=0}^{N-1} X*(k)e^{-\frac{j2\pi kn}{N}}\right\}x(n)$
- $=1/N\sum_{k=0}^{N-1} X*(k)\{\sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi kn}{N}}\}$
- = 1/N $\sum_{k=0}^{N-1} X*(k)X(k)$ .....(1)
- $\sum_{n=0}^{N-1}|x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1}|X(k)|^2$