Design of Linear Phase FIR Filters using Windowing Technique

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Design Procedure of Linear Phase FIR Filters by Windowing Technique

$Given,\ H_d\left(e^{j\omega}\right) \rightarrow$ Desired Frequency Response Function of the Filter

Step1: From the desired frequency response specification $H_d\left(e^{j\omega}\right)$, find the corresponding unit impulse response.

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right)e^{j\omega n}d\omega$$

Step2: In most of the practical cases, the obtained impulse response $h_d[n]$ is of infinite duration and is truncated to get an FIR filter of length M by multiplying $h_d[n]$ with window function $w[n]$.

$$h[n] = h_d[n]w[n]$$

$where,\ w[n] \rightarrow$ Window Function of length M

$h[n] \rightarrow$ Impulse Response of FIR Filter having length M

Step3: The transfer function $H(z)$ is obtained from $h[n]$ by taking its z transform.

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=0}^{M-1} h[n]z^{-n}$$

Step4: The magnitude response $\left|H\left(e^{j\omega}\right)\right|$ is obtained from $H(z)$ by substituting $z = e^{j\omega}$.

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Summary of Symmetric Window Functions

Rectangular Window, $w[n]=\begin{cases}1 & 0\le n\le M-1\\ 0 & otherwise\end{cases}$

Bartlett Window, $w[n]=\begin{cases}1-\dfrac{\left|n-\dfrac{M-1}{2}\right|}{\dfrac{M-1}{2}} & 0\le n\le M-1\\ 0 & otherwise\end{cases}$

Hanning Window, $w[n]=\begin{cases}0.5-0.5\cos\dfrac{2\pi n}{M-1} & 0\le n\le M-1\\ 0 & otherwise\end{cases}$

Hamming Window, $w[n]=\begin{cases}0.54-0.46\cos\dfrac{2\pi n}{M-1} & 0\le n\le M-1\\ 0 & otherwise\end{cases}$

Blackman Window, $w[n]=\begin{cases}0.42-0.5\cos\dfrac{2\pi n}{M-1}+0.08\cos\dfrac{4\pi n}{M-1} & 0\le n\le M-1\\ 0 & otherwise\end{cases}$

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Summary of Frequency Response of Linear Phase FIR Filters

Low Pass Filter

$$H_d\left(e^{j\omega}\right)=\begin{cases} e^{-j\omega\tau} & 0\le|\omega|<\omega_c \\ 0 & \omega_c\le|\omega|\le\pi \end{cases}$$

High Pass Filter

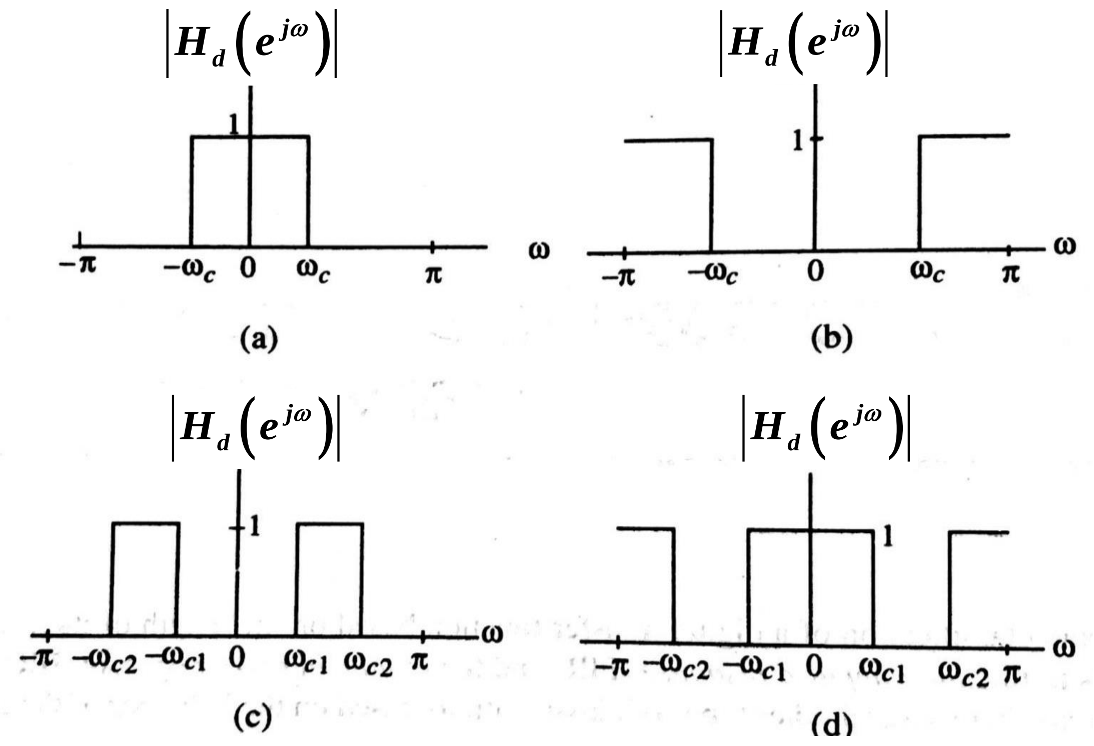
$$H_d\left(e^{j\omega}\right)=\begin{cases} e^{-j\omega\tau} & \omega_c\le|\omega|<\pi \\ 0 & 0\le|\omega|\le\omega_c \end{cases}$$

Band Pass Filter

$$H_d\left(e^{j\omega}\right)=\begin{cases} e^{-j\omega\tau} & \omega_{c1}\le|\omega|<\omega_{c2} \\ 0 & 0\le|\omega|\le\omega_{c1} \text{ and } \omega_{c2}\le|\omega|\le\pi \end{cases}$$

Band Stop Filter

$$H_d\left(e^{j\omega}\right)=\begin{cases} e^{-j\omega\tau} & 0\le|\omega|\le\omega_{c1} \text{ and } \omega_{c2}\le|\omega|\le\pi \\ 0 & \omega_{c1}\le|\omega|<\omega_{c2} \end{cases}$$

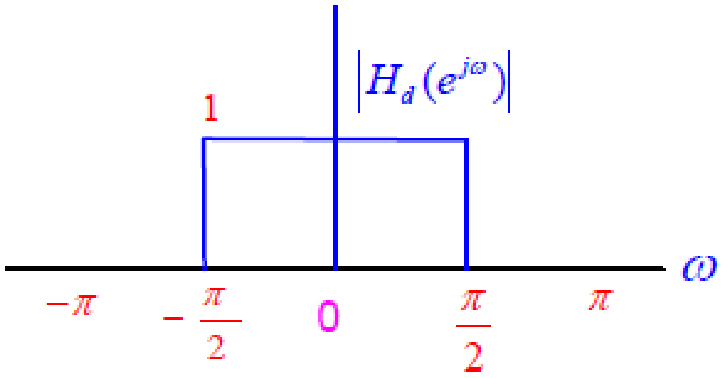


Magnitude Response of Ideal Digital Filters
(a) LPF (b) HPF (c) BPF (d) BSF

Question: For the specifications given below, design a **Symmetric FIR Low Pass Filter** using **Rectangular Window**. Also plot the magnitude response of the designed FIR filter.

Length of the Filter = 11

Cutoff Frequency = $\pi/2$ rad/sample.



Solution:

$$Given\ Specifications;\ M = 11, \omega_c = \frac{\pi}{2} rad/sample, \tau = \frac{M-1}{2} = 5$$

Step 1:

$$H_d\left(e^{j\omega}\right) = \begin{cases} e^{-j\omega\tau} & 0 \le |\omega| < \omega_c \\ 0 & \omega_c \le |\omega| \le \pi \end{cases}$$

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\omega}\right)e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\tau)}d\omega = \frac{1}{2\pi}\left[\left.\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right|_{-\omega_c}^{\omega_c}\right]$$

$$= \frac{1}{(n-\tau)\pi}\left[\frac{e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}}{2j}\right]$$

$$\Rightarrow h_d[n] = \frac{\sin\omega_c(n-\tau)}{\pi(n-\tau)} \quad for\ n \ne \tau$$

$$for\ n = \tau;\ h_d[\tau] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 d\omega = \frac{\omega_c}{\pi}$$

$$h_d[n]=\frac{\sin(0.5\pi(n-5))}{\pi(n-5)} \quad for\ 0\le n\le 10 \text{ and } n\ne 5$$

$$h_d[\tau]=\frac{0.5\pi}{\pi}=0.5 \quad for\ n=\tau=5$$

Step 2: **Rectangular Window**

$$w[n]=1 \quad for\ 0\le n\le 10$$

$$h[n]=h_d[n]w[n]$$

$$h_d[0]=\frac{\sin(0.5\pi(0-5))}{\pi(0-5)}=0.0637 \qquad h_d[10]=\frac{\sin(0.5\pi(10-5))}{\pi(10-5)}=0.0637$$

$$h_d[1]=\frac{\sin(0.5\pi(1-5))}{\pi(1-5)}=0 \qquad h_d[9]=\frac{\sin(0.5\pi(9-5))}{\pi(9-5)}=0$$

$$h_d[2]=\frac{\sin(0.5\pi(2-5))}{\pi(2-5)}=\text{-}0.1061 \qquad h_d[8]=\frac{\sin(0.5\pi(8-5))}{\pi(8-5)}=\text{-}0.1061$$

$$h_d[3]=\frac{\sin(0.5\pi(3-5))}{\pi(3-5)}=0 \qquad h_d[7]=\frac{\sin(0.5\pi(7-5))}{\pi(7-5)}=0$$

$$h_d[4]=\frac{\sin(0.5\pi(4-5))}{\pi(4-5)}=0.3183 \qquad h_d[6]=\frac{\sin(0.5\pi(6-5))}{\pi(6-5)}=0.3183$$

$$h_d[5]=\frac{\omega_c}{\pi}=\frac{0.5\pi}{\pi}=0.5$$

n	$h_d[n]$	$w[n]$	$h[n]$
0	0.0637	1	0.0637
1	0	1	0
2	-0.1061	1	-0.1061
3	0	1	0
4	0.3183	1	0.3183
5	0.5	1	0.5
6	0.3183	1	0.3183
7	0	1	0
8	-0.1061	1	-0.1061
9	0	1	0
10	0.0637	1	0.0637

Step 3:

$$H(z)=\sum_{n=0}^{10} h[n]z^{-n}$$

$$=h[0]+h[1]z^{-1}+h[2]z^{-2}+h[3]z^{-3}+h[4]z^{-4}+h[5]z^{-5}+h[6]z^{-6}+h[7]z^{-7}+h[8]z^{-8}+h[9]z^{-9}+h[10]z^{-1}$$

$$=h[0]\left(1+z^{-10}\right)+h[1]\left(z^{-1}+z^{-9}\right)+h[2]\left(z^{-2}+z^{-8}\right)+h[3]\left(z^{-3}+z^{-7}\right)+h[4]\left(z^{-4}+z^{-6}\right)+h[5]z^{-5}$$

$$=z^{-5}\left[h[0]\left(z^{5}+z^{-5}\right)+h[1]\left(z^{4}+z^{-4}\right)+h[2]\left(z^{3}+z^{-3}\right)+h[3]\left(z^{2}+z^{-2}\right)+h[4]\left(z^{1}+z^{-1}\right)+h[5]\right]$$

$$\Rightarrow H(z)=z^{-5}\left[0.5+0.0637\left(z^{5}+z^{-5}\right)-0.1061\left(z^{3}+z^{-3}\right)+0.3183\left(z^{1}+z^{-1}\right)\right]$$

Step 4:

We have,

$$H(z)=z^{-5}\left[0.5+0.0637\left(z^{5}+z^{-5}\right)-0.1061\left(z^{3}+z^{-3}\right)+0.3183\left(z^{1}+z^{-1}\right)\right]$$

$$H\left(e^{j\omega}\right)=H(z)\Big|_{z=e^{j\omega}}=e^{-j5\omega}\left[0.5+0.0637\left(e^{j5\omega}+e^{-j5\omega}\right)-0.1061\left(e^{j3\omega}+e^{-j3\omega}\right)+0.3183\left(e^{j\omega}+e^{-j\omega}\right)\right]$$

$$\Rightarrow H\left(e^{j\omega}\right)=e^{-j5\omega}\left[0.5+0.0637\times 2\cos 5\omega-0.1061\times 2\cos 3\omega+0.3183\times 2\cos\omega\right]$$

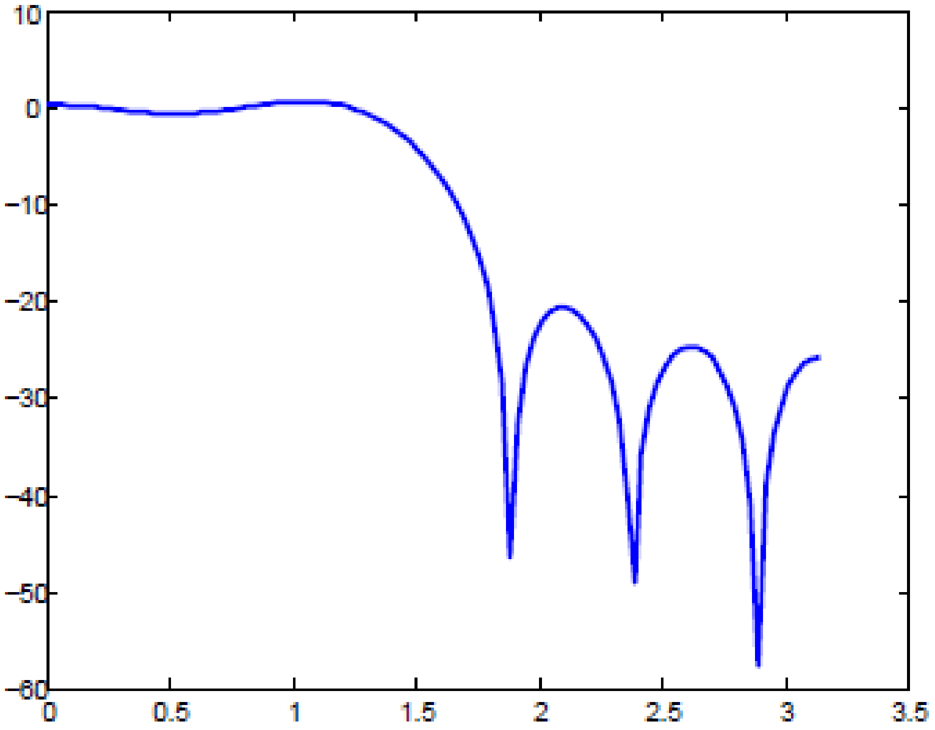
$$\Rightarrow H\left(e^{j\omega}\right)=e^{-j\omega 5}\left[0.5+0.1273\cos 5\omega-0.2122\cos 3\omega+0.6366\cos\omega\right]$$

$$H\left(e^{j\omega}\right)=e^{-j\omega 5}\left[0.5+0.1273\cos 5\omega-0.2122\cos 3\omega+0.6366\cos\omega\right]$$

$$\Rightarrow\left|H\left(e^{j\omega}\right)\right|=\left|0.5+0.1273\cos 5\omega-0.2122\cos 3\omega+0.6366\cos\omega\right|$$

$$\left|H\left(e^{j\omega}\right)\right|_{dB}=20\log_{10}\left|H\left(e^{j\omega}\right)\right|$$

$$\uparrow \quad \left|H\left(e^{j\omega}\right)\right|_{dB}$$



$$\omega\ (rad)\rightarrow$$

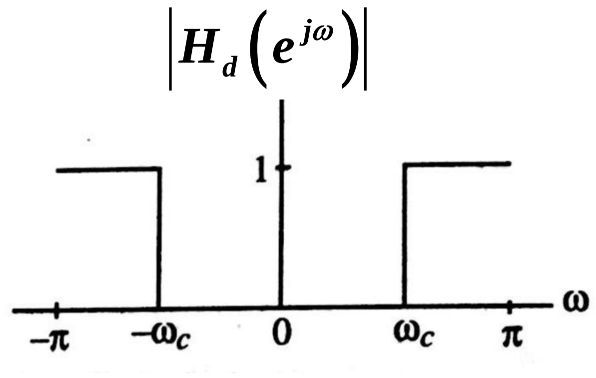
Question: For the given frequency response specifications, design a **Symmetric FIR High Pass Filter** using **Hamming Window**. Also find the magnitude response of the designed FIR filter.

$$H_d\left(e^{j\omega}\right)=\begin{cases} e^{-j3\omega} & 1\le|\omega|<\pi \\ 0 & Otherwise \end{cases}$$

Solution:

$$Given\ Specifications;\ \tau=3,\ \omega_c=1rad/sample,\ M=2\tau+1=7$$

Step 1:



$$h_d[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}H\left(e^{j\omega}\right)e^{j\omega n}d\omega$$

$$=\frac{1}{2\pi}\left[\int_{-\pi}^{-\omega_c}e^{j\omega(n-\tau)}d\omega+\int_{\omega_c}^{\pi}e^{j\omega(n-\tau)}d\omega\right]$$

$$=\frac{1}{2\pi}\left[\left.\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right|_{-\pi}^{-\omega_c}+\left.\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right|_{\omega_c}^{\pi}\right]$$

$$=\frac{1}{(n-\tau)\pi}\left[\frac{e^{-j\omega_c(n-\tau)}-e^{-j\pi(n-\tau)}}{2j}+\frac{e^{j\pi(n-\tau)}-e^{j\omega_c(n-\tau)}}{2j}\right]$$

$$=\frac{1}{(n-\tau)\pi}\left[\frac{\left(e^{j\pi(n-\tau)}-e^{-j\pi(n-\tau)}\right)}{2j}-\frac{\left(e^{j\omega_c(n-\tau)}-e^{-j\omega_c(n-\tau)}\right)}{2j}\right]$$

$$\Rightarrow h_d[n]=\frac{\left[\sin\pi(n-\tau)-\sin\omega_c(n-\tau)\right]}{(n-\tau)\pi}\qquad for\ n\neq\tau$$

$$for\ n=\tau;\ h_d[\tau]=\frac{1}{2\pi}\left[\int_{-\pi}^{-\omega_c}1d\omega+\int_{\omega_c}^{\pi}1d\omega\right]=\frac{-\omega_c+\pi+\pi-\omega_c}{2\pi}$$

$$\Rightarrow h_d[\tau]=1-\frac{\omega_c}{\pi}$$

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$$h_d[n] = \frac{\left[\sin\pi(n-3) - \sin(n-3)\right]}{(n-3)\pi} \quad for\ 0 \le n \le 6 \text{ and } n \ne 3$$

$$h_d[\tau] = 1 - \frac{\omega_c}{\pi} = 1 - \frac{1}{\pi} \quad for\ n = \tau = 3$$

$$h_d[0] = \frac{\left[\sin\pi(0-3) - \sin(0-3)\right]}{(0-3)\pi} = -0.0150$$

$$h_d[1] = \frac{\left[\sin\pi(1-3) - \sin(1-3)\right]}{(1-3)\pi} = -0.1447$$

$$h_d[2] = \frac{\left[\sin\pi(2-3) - \sin(2-3)\right]}{(2-3)\pi} = -0.2678$$

$$h_d[6] = \frac{\left[\sin\pi(6-3) - \sin(6-3)\right]}{(6-3)\pi} = -0.0150$$

$$h_d[5] = \frac{\left[\sin\pi(5-3) - \sin(5-3)\right]}{(5-3)\pi} = -0.1447$$

$$h_d[4] = \frac{\left[\sin\pi(4-3) - \sin(4-3)\right]}{(4-3)\pi} = -0.2678$$

$$h_d[3] = 1 - \frac{1}{\pi} = 0.6817$$

Step 2:

Hamming Window

$$w[n]=\begin{cases}0.54-0.46\cos\dfrac{2\pi n}{M-1} & 0\le n\le M-1\\ 0 & otherwise\end{cases}$$

$$w[0]=0.0800 \qquad w[6]=0.0800$$

$$w[1]=0.3100 \qquad w[5]=0.3100$$

$$w[2]=0.7700 \qquad w[4]=0.7700$$

$$w[3]=1.0000$$

$$h[n]=h_d[n]w[n]$$

n	$h_d[n]$	$w[n]$	$h[n]$
0	−0.0150	0.0800	−0.001197
1	−0.1447	0.3100	−0.044862
2	−0.2678	0.7700	−0.206243
3	0.6817	1	0.6817
4	−0.2678	0.7700	−0.206243
5	−0.1447	0.3100	−0.044862
6	−0.0150	0.0800	−0.001197

Step 3:

$$H(z)=\sum_{n=0}^{6}h[n]z^{-n}$$

$$=h[0]+h[1]z^{-1}+h[2]z^{-2}+h[3]z^{-3}+h[4]z^{-4}+h[5]z^{-5}+h[6]z^{-6}$$

$$=z^{-3}\left[h[3]+h[0]\left(z^{3}+z^{-3}\right)+h[1]\left(z^{2}+z^{-2}\right)+h[2]\left(z^{1}+z^{-1}\right)\right]$$

$$\Rightarrow H(z)=z^{-3}\left[0.6817-0.001197\left(z^{3}+z^{-3}\right)-0.044862\left(z^{2}+z^{-2}\right)-0.206243\left(z^{1}+z^{-1}\right)\right]$$

Step 4:

We have,

$$H(z)=z^{-3}\left[0.6817-0.001197\left(z^{3}+z^{-3}\right)-0.044862\left(z^{2}+z^{-2}\right)-0.206243\left(z^{1}+z^{-1}\right)\right]$$

$$H\left(e^{j\omega}\right)=H(z)\big|_{z=e^{j\omega}}=e^{-j3\omega}\left[0.6817-0.001197\left(e^{j3\omega}+e^{-j3\omega}\right)-0.044862\left(e^{j2\omega}+e^{-j2\omega}\right)-0.206243\left(e^{j\omega}+e^{-j\omega}\right)\right]$$

$$\Rightarrow H\left(e^{j\omega}\right)=e^{-j3\omega}\left[0.6817-0.001197\times 2\cos 3\omega-0.044862\times 2\cos 3\omega-0.044862\times 2\cos\omega\right]$$

$$\Rightarrow H\left(e^{j\omega}\right)=e^{-j3\omega}\left[0.6817-0.0023957\cos 3\omega-0.0897258\cos 2\omega-0.41248674\cos\omega\right]$$

$$\left|H\left(e^{j\omega}\right)\right|=\left|0.6817-0.0023957\cos 3\omega-0.0897258\cos 2\omega-0.41248674\cos\omega\right|$$