IIR Filter Design

IIR Filter

- IIR filters are one of two primary types of digital filters used in Digital Signal Processing (DSP) applications (the other type being FIR). “IIR” means “Infinite Impulse Response.”

- The impulse response is “infinite” because there is feedback in the filter; if you put in an impulse (a single “1” sample followed by many “0” samples), an infinite number of non-zero values will come out (theoretically.)

General Difference Equation for IIR Filter

IIR:

$$Y[n] = b_0 X[n] + b_1 X[n-1] + b_2 X[n-2] + \ldots\ldots\ldots\ldots + b_{M-1} X[n-M]$$
$$-a_1 y[n-1] - a_2 y[n-2] - \ldots\ldots\ldots\ldots\ldots\ldots - a_N y[n-N].$$

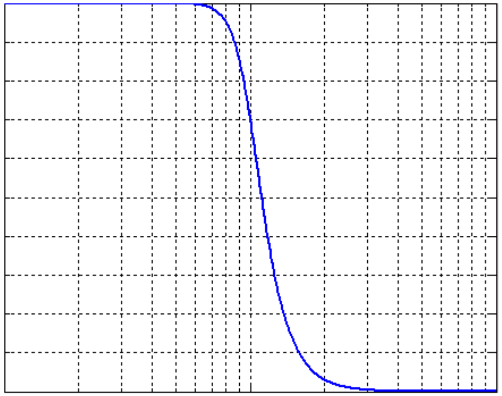
FIR:

$$Y[n] = b_0 X[n] + b_1 X[n-1] + b_2 X[n-2] + \ldots\ldots\ldots\ldots + b_{M-1} X[n-M].$$

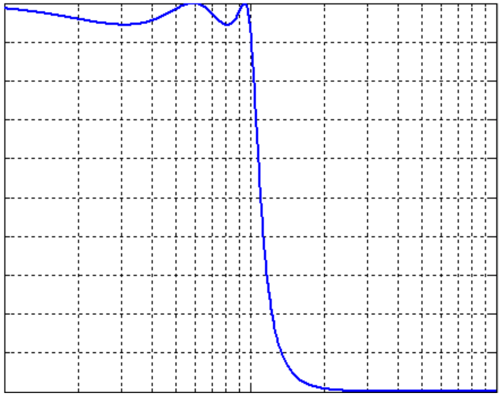
Analog IIR Filter Types

- Butterworth Filters
- Chebyshev I Filters
- Chebyshev II Filters
- Elliptic Filters
- Bessel Filters

Widely used analog filters



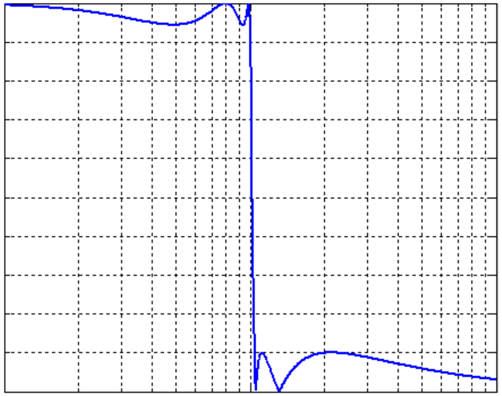
Butterworth

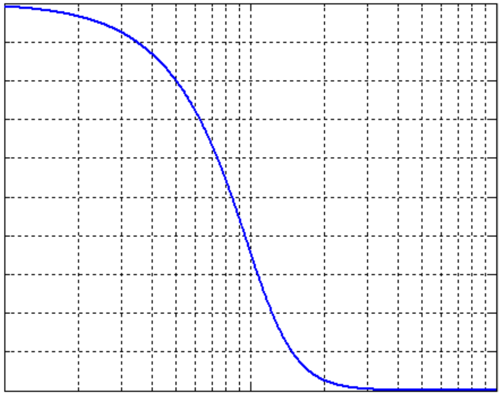


Chebyshev Type I

Chebyshev Type II

Elliptic



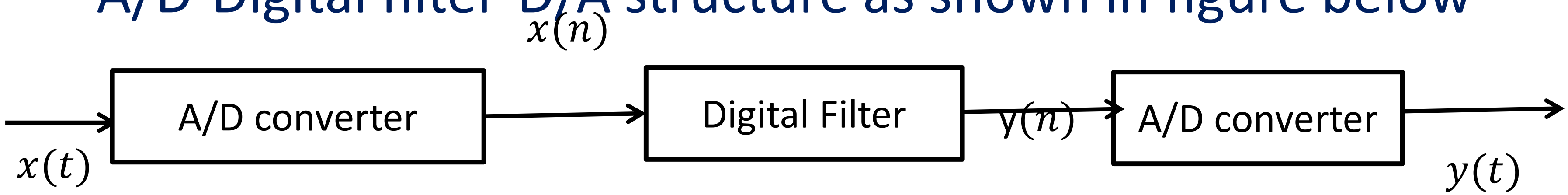


Bessel

Design of IIR Digital Filters

- The design of digital IIR is largely based on analog filter design techniques.

- To simulate an analog filter, the digital filter is issued in A/D-Digital filter-D/A structure as shown in figure below



- The A/D-Digital-D/A structure behaves like an equivalent analog filter having a transfer function, $H_{eq}(s)$.

- A digital filter takes a discrete-time input sequence $x(n)$ and produce a discrete time output sequence y(n).

Steps for typical Design procedure of a digital IIR filter

1. Selecting a method of transformation of a given analog filter to a digital filter having roughly the same frequency response.
2. Mapping the specification of the digital IIR filter to equivalent specifications of an analog IIR filter such that, after mapping from analog to digital is carried out, the digital IIR filter will meet the given specifications.
3. Designing the analog IIR filter according to the mapped specifications.
4. Transforming the analog filter to an equivalent digital filter.

- The different transformation that map a given analog filter to digital filter having roughly the same frequency response. They are
  - Approximation of Derivatives
  - The bilinear transformation
  - The impulse invariant transformation
  - The matched-Z transform

Impulse InVariant (IIV) method

- Suitable analog transfer function H(s) is chosen
- h(t) is obtained by using inverse Laplace tranform
- h(t) is sampled to produce h(nT), where T is the sampling interval
- The desired transfer function is obtained by z transforming h(nT)

H(s) partial fraction h(t)

- When H(s) have distinct poles, it can be expressed through partial fraction as

$$H(s)=\sum_{i=1}^{N}\frac{A_i}{s-p_i}=\frac{A_1}{s-p_1}+\frac{A_2}{s-p_2}+\ldots+\frac{A_N}{s-p_N}$$

$p_1, \ldots, p_N \rightarrow$ Poles

- On taking inverse Laplace transform we get

$$h(t)=\sum_{i=1}^{N}A_i e^{p_i t}=A_1 e^{p_1 t}+A_2 e^{p_2 t}+\ldots+A_N e^{p_N t}$$

$$h(n)=h(t)\big|_{t=nT}=h(nT)=\sum_{i=1}^{N}A_i e^{p_i nT}$$

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

$$z = e^{sT} = e^{\sigma T} e^{j\omega T}$$

$$H(z) = \sum_{n=0}^{\infty} h[n] e^{-sTn}$$

$$e^{-at} \Leftrightarrow \frac{1}{s+a}$$

$$\frac{1}{s-a}$$

$$H(s) = \sum_{k=1}^{N} \frac{A_k}{s-p_k}$$

$\downarrow$ ILT

$$h_a(t) = \sum_{k=1}^{N} A_k e^{p_k t}$$

$$h(n) = h_a(nT)$$

$$h(n) = \sum_{k=} A_k e^{p_k nT}$$

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$= \sum_{k=0} A_k \left[\sum_{k=0}^{\infty} e^{p_k nT} z^{-n} \right]$$

$$= \sum_{k=1}^{N} A_k \frac{1}{1 - e^{p_k T} z^{-1}}.$$

$$\boxed{\frac{1}{s - p_k} \Longleftrightarrow \frac{1}{1 - e^{p_k T} z^{-1}}}$$

$$\sum_{n=\infty}^{\infty} (e^{p_k T} z^{-1})^n.$$

$$\sum_{n=0} a^n$$

$$= \frac{1}{1-a}.$$

IIV method continued

- On taking Z transform we get

$$H(z) = Z\left[h(n)\right] = \frac{A_1}{1-e^{p_1 T}z^{-1}} + \frac{A_2}{1-e^{p_2 T}z^{-1}} + \ldots + \frac{A_N}{1-e^{p_N T}z^{-1}}$$

$$= \sum_{i=1}^{N} \frac{A_1}{1-e^{p_i T}z^{-1}}$$

$$\frac{1}{s-p_i} \xrightarrow[\text{transformed to}]{} \frac{1}{1-e^{p_i T}Z^{-1}}$$

$e^{-2t}u(t) \xleftrightarrow{LT} \frac{1}{s+2}$

$-e^{-2t}u(-t) \xleftrightarrow{LT} \frac{1}{s+2}$

Aliasing

1) Convert into z-domain using IIV

$$H(s) = \frac{2}{(s+1)(s+2)} \quad \text{for } T = 1\,\text{sec.}$$

$$H(s) = \frac{2}{s+1} - \frac{2}{s+2}$$

$$\frac{1}{s - P_k} \longleftrightarrow \frac{1}{1 - e^{P_k T} z^{-1}}$$

$$\frac{2}{s-(-1)} \overset{s-z}{\longleftrightarrow} \frac{2}{1 - e^{-T} z^{-1}}.$$

$$\frac{-2}{s-(-2)} \overset{s-z}{\longleftrightarrow} \frac{-2}{1 - e^{-2T} z^{-1}}$$

$$H(z) = \frac{2}{1-e^{-1}z^{-1}} - \frac{2}{1-e^{-2}z^{-1}}$$

$$= \frac{2}{1-0.367z^{-1}} - \frac{2}{1-0.135z^{-1}} \quad \text{---}$$

$$= \frac{0.466z^{-1}}{1-0.502z^{-1}+0.049z^{-2}}.$$

Tutorial 1) Design given 2^{nd} order LPF filter using IIV method. Assume $T = 1sec$

$$H(s) = \frac{1}{s^2+s+1}.$$

Bilinear Transformation

- The bilinear transformation is used for transforming an analog filter to digital filter.

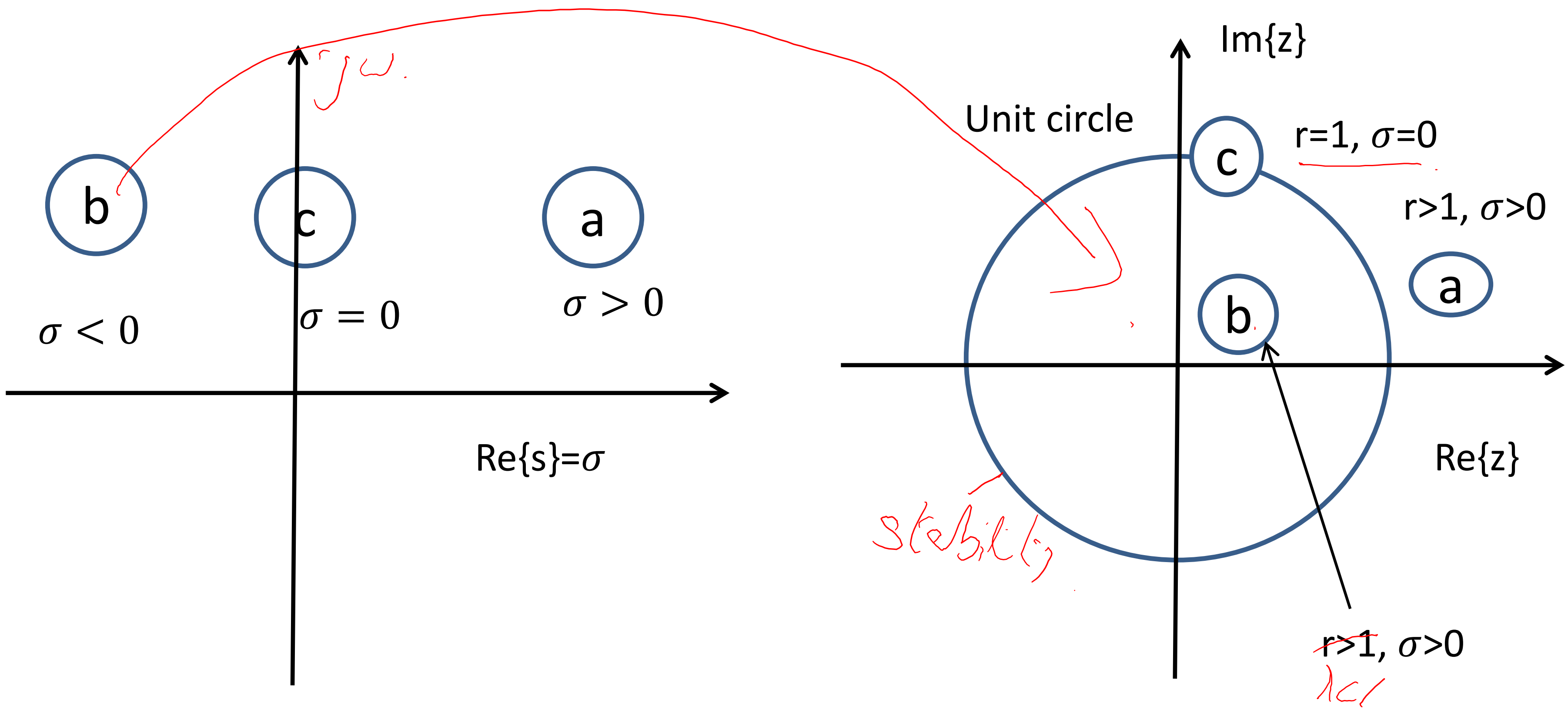
- Bilinear transformation uses trapezoidal rule for integrating a continuous-time function.

- It is defined by the substitution

$$s = \frac{2}{T}\left[\frac{1 - z^{-1}}{1 + z^{-1}}\right]$$

Where T is the sampling period.

Mapping of the s-plane into the z-plane with bilinear transformation.



- The left-half s-plane is now mapped entirely inside the unit circle $|z| = 1$, rather than to a part of it.
- Imaginary axis is mapped on the unit circle.
- Right-half s-plane is mapped outside the unit circle

Bilinear is Frequency to Frequency transformation

$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ or $s = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$

analog freq. digital freq.

Sub $s = \sigma + j\Omega$ and $z = re^{j\omega}$ in the above equation

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{re^{j\omega} - 1}{re^{j\omega} + 1}\right]$$

$$= \frac{2}{T}\left[\frac{(r\,cos\omega - 1) + jr\,sin\omega}{(r\,cos\omega + 1) + jr\,sin\omega}\right]$$

Rationalizing the right-hand side of the above equation

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{\left(r\ cos\omega - 1\right) + jr\ sin\omega}{\left(r\ cos\omega + 1\right) + jr\ sin\omega}\right] \times \frac{\left[\left(r\ cos\omega + 1\right) - jr\ sin\omega\right]}{\left[\left(r\ cos\omega + 1\right) - jr\ sin\omega\right]}$$

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{r^2 - 1}{r^2 + 1 + 2rcos\omega} + j\frac{2r\ sin\omega}{r^2 + 1 + 2rcos\omega}\right]$$

Equating real and imaginary parts on both the sides of the above equation

$$\sigma = \frac{2}{T}\left[\frac{r^2 - 1}{r^2 + 1 + 2rcos\omega}\right]$$

$$\Omega = \frac{2}{T}\left[\frac{2r\ sin\omega}{r^2 + 1 + 2rcos\omega}\right]$$

From the above equation

1. If r<1, we get , $\sigma < 0$. This means that the left-hand side of the s-plane is mapped inside the unit circle $|z| = 1$.
2. If r<1, we get , $\sigma > 0$. This means that the right-hand side of the s-plane is mapped outside the unit circle $|z| = 1$.
3. If r=1, we get , $\sigma = 0$. This means that the Imaginary axis of the s-plane is mapped to the circle of unit radius centered at z=0 in the Z-domain.

Frequency warping

- When r=1, in the previous equation, we get $\sigma = 0$ and

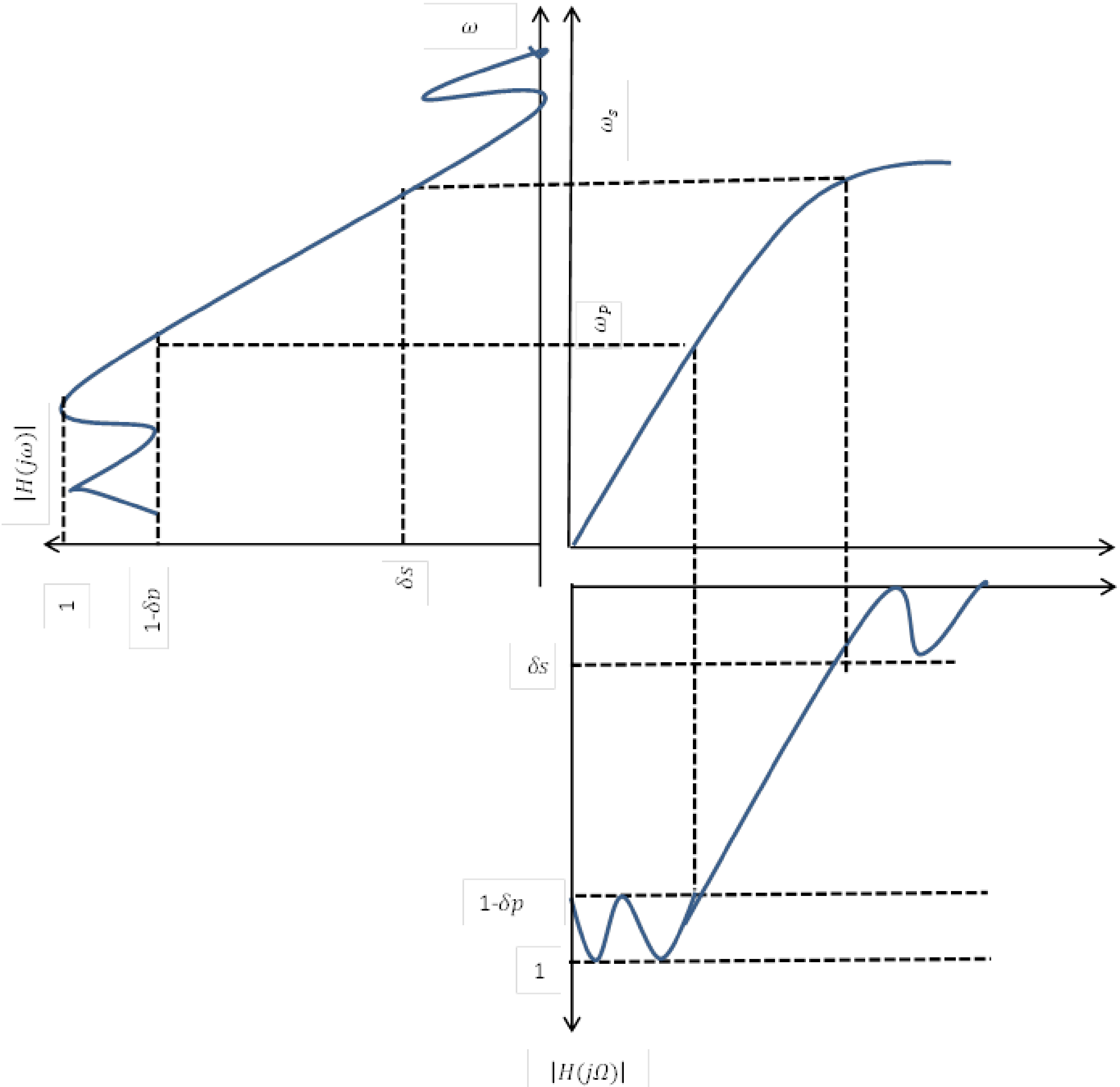
$$\Omega = \frac{2}{T}\left[\frac{2\, sin\omega}{1^2 + 1 + 2cos\omega}\right]$$

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

$$\omega = \frac{2}{T}tan^{-1}\left(\frac{\Omega T}{2}\right)$$

- The digital-domain frequency ω is therefore warped with respect to the analog frequency Ω, using the above the warping function.
- The analog frequencies, $\Omega = \pm\infty$ are mapped to digital frequencies, $\omega = \pm\pi$.
- The frequency mapping is not aliased that is the relationship between ω and Ω is one to one.

Frequency warping introduced by the bilinear transformation



- To design a lowpass filter with passband and stopband edge frequencies ω_P *and* ω_S.
- Convert these frequencies to corresponding analog-domain band edge frequencies $\Omega_P{}'$ and $\Omega_S{}'$ using the following equation.

$$\Omega_P{}' = \frac{2}{T}\tan\left(\frac{\omega_P}{2}\right)$$

Aliasing

$$\Omega_S{}' = \frac{2}{T}\tan\left(\frac{\omega_S}{2}\right)$$

Using above equation we design analog filter. Finally, the required digital low pass filter is obtained by using the bilinear transformation on the analog filter.

BLT

$$S = \frac{2}{T}\left\{\frac{1-z^{-1}}{1+z^{-1}}\right\}$$

Design IIR filter → BLT / IIV

1) Prewarping $\Omega_c, \Omega_p, \Omega_s$

$\left\{ \text{Design of filter} \right\}$

3) $H_a(s) \longrightarrow H(z)$

1) Apply BLT to $H(s) = \dfrac{2}{(s+1)(s+2)}$

with $T=1$ and find $H(z)$

$$s = \frac{2}{T}\left\{\frac{1-z^{-1}}{1+z^{-1}}\right\}$$

$$H(z) = \frac{2}{\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right\}\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right\}}$$

$$H(z) = \frac{1+2z^{-1}+z^{-2}}{6-2z^{-1}}$$

2) Design LPF with 3dB BW of 0.2π using BLT to analog filter $H(s) = \dfrac{\Omega_c}{s+\Omega_c}$.

Digital

$\omega_c = 0.2\pi$.

Analog $\Omega_c = \dfrac{2}{T}\tan\dfrac{\omega_c}{2}$

$= \dfrac{2}{T}\tan 0.1\pi$

$= 0.65/T$

$$H(s) = \frac{0.65/T}{s+0.65/T}\Bigg|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{0.65/T}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + \frac{0.65}{T}}$$

$$= \frac{0.65\left(1+z^{-1}\right)}{2-2z^{-1}+0.65+0.65z^{-1}}$$

$$= \frac{0.65\left(1+z^{-1}\right)}{2.65-1.35z^{-1}}$$

$$= \frac{0.24\left(1+z^{-1}\right)}{1-0.509z^{-1}}$$

$$z = e^{j\omega}.$$

$$H(e^{j\omega}) = H(\omega) = \frac{0.24\left(1+e^{-j\omega}\right)}{1-0.509e^{-j\omega}}.$$

$\omega = 0$ $H(0) = \dfrac{0.24(2)}{1-0.509} = \dfrac{0.48}{0.\ } \approx 0.97$

$= 1$

$\omega = 0.2\pi$

$$\left| H(\omega = 0.2\pi) \right| = 0.707$$