b) Given that, $x(t) = e^{-at} \sin\Omega_0 t$

The discrete time signal x(n) is generated by replacing t by nT, where T is the sampling time period.

$$\therefore\ x(n) = e^{-anT} \sin\Omega_0 nT = e^{-anT} \sin \omega n \ ; \ \text{where } \omega = \Omega_0 T$$

By the definition of one-sided Z-transform we get,

$$X(z) = Z\{x(n)\} = \sum_{n=0}^{\infty} e^{-anT} \sin\omega n\, z^{-n} = \sum_{n=0}^{\infty} e^{-anT}\left(\frac{e^{j\omega n} - e^{-j\omega n}}{2j}\right) z^{-n}$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{1}{2j}\sum_{n=0}^{\infty}\left(e^{-aT}e^{j\omega}z^{-1}\right)^n - \frac{1}{2j}\sum_{n=0}^{\infty}\left(e^{-aT}e^{-j\omega}z^{-1}\right)^n$$

Infinite geometric series sum formula,

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$

$$= \frac{1}{2j}\,\frac{1}{1 - e^{-aT}e^{j\omega}z^{-1}} - \frac{1}{2j}\,\frac{1}{1 - e^{-aT}e^{-j\omega}z^{-1}}$$

$$= \frac{1}{2j}\,\frac{1}{1 - e^{j\omega}/z e^{aT}} - \frac{1}{2j}\,\frac{1}{1 - e^{-j\omega}/z e^{aT}}$$

$$= \frac{1}{2j}\,\frac{z e^{aT}}{z e^{aT} - e^{j\omega}} - \frac{1}{2j}\,\frac{z e^{aT}}{z e^{aT} - e^{-j\omega}}$$

$$= \frac{1}{2j}\left[\frac{z e^{aT}\left(z e^{aT} - e^{-j\omega}\right) - z e^{aT}\left(z e^{aT} - e^{j\omega}\right)}{\left(z e^{aT} - e^{j\omega}\right)\left(z e^{aT} - e^{-j\omega}\right)}\right]$$

$$= \frac{1}{2j}\left[\frac{\left(z e^{aT}\right)[z e^{aT} - e^{-j\omega} - z e^{aT} + e^{j\omega}]}{\left(z e^{aT}\right)^2 - z e^{aT}e^{-j\omega} - z e^{aT}e^{j\omega} + e^{j\omega}e^{-j\omega}}\right]$$

$$= \left[\frac{z e^{aT}\left(e^{j\omega} - e^{-j\omega}\right)/2j}{z^2 e^{2aT} - z e^{aT}\left(e^{j\omega} + e^{-j\omega}\right) + 1}\right]$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}} \qquad \boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= \frac{z e^{aT} \sin\omega}{z^2 e^{2aT} - 2z e^{aT}\cos\omega + 1} \ ; \quad \text{where } \omega = \Omega_0 T$$

Solution

a) **Given that, $x(t) = e^{-at} \cos\Omega_0 t$**

The discrete time signal x(n) is generated by replacing t by nT, where T is the sampling time period.

$$\therefore\ x(n) = e^{-anT} \cos \Omega_0 nT = e^{-anT} \cos \omega n \ ; \text{ where } \omega = \Omega_0 T$$

By the definition of one-sided $\mathcal{Z}$-transform we get,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=0}^{\infty} e^{-anT} \cos \omega n\, z^{-n} = \sum_{n=0}^{\infty} e^{-anT}\left(\frac{e^{j\omega n} + e^{-j\omega n}}{2}\right) z^{-n}$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(e^{-aT}e^{j\omega}z^{-1}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(e^{-aT}e^{-j\omega}z^{-1}\right)^n$$

$$= \frac{1}{2}\,\frac{1}{1 - e^{-aT}e^{j\omega}z^{-1}} + \frac{1}{2}\,\frac{1}{1 - e^{-aT}e^{-j\omega}z^{-1}}$$

Using infinite geometric series sum formula,

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$

$$= \frac{1}{2}\,\frac{1}{1 - e^{j\omega}/z e^{aT}} + \frac{1}{2}\,\frac{1}{1 - e^{-j\omega}/z e^{aT}}$$

$$= \frac{1}{2}\left[\frac{z e^{aT}}{z e^{aT} - e^{j\omega}} + \frac{z e^{aT}}{z e^{aT} - e^{-j\omega}}\right]$$

$$= \frac{1}{2}\left[\frac{z\,e^{aT}\left(z\,e^{aT} - e^{-j\omega}\right) + z\,e^{aT}\left(z\,e^{aT} - e^{j\omega}\right)}{\left(z\,e^{aT} - e^{j\omega}\right)\left(z\,e^{aT} - e^{-j\omega}\right)}\right]$$

$$= \frac{z\,e^{aT}}{2}\left[\frac{z\,e^{aT} - e^{-j\omega} + z\,e^{aT} - e^{j\omega}}{\left(z\,e^{aT}\right)^2 - z\,e^{aT}e^{-j\omega} - z\,e^{aT}e^{j\omega} + e^{j\omega}\,e^{-j\omega}}\right]$$

$$= \frac{z\,e^{aT}}{2}\left[\frac{2z\,e^{aT} - \left(e^{j\omega} + e^{-j\omega}\right)}{z^2\,e^{2aT} - z\,e^{aT}\left(e^{j\omega} + e^{-j\omega}\right) + 1}\right]$$

$$= \left[\frac{z\,e^{aT}\left(z\,e^{aT} - \cos\omega\right)}{z^2\,e^{2aT} - 2z\,e^{aT}\cos\omega + 1}\right] \quad ; \quad \text{where } \omega = \Omega_0 T$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$