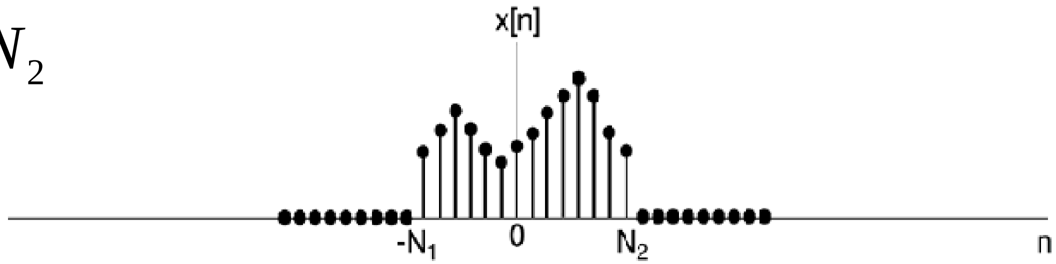
Discrete Time Fourier Transform (DTFT)

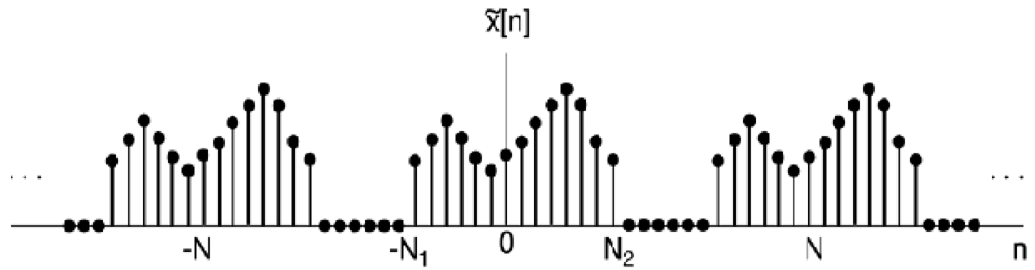
Derivation of the Discrete-time Fourier Transform

$x[n]$ - aperiodic and of finite duration

$$x[n] = 0 \ if \ n < N_1 \ \& \ n > N_2$$

N is large enough





$$\tilde{x}[n] = x[n] \ between \ N_1 \ \& \ N_2 \ and \ periodic \ with \ N$$

$$\tilde{x}[n] = x[n] \text{ for } any \ n \text{ as } N \to \infty$$

Derivation of the Discrete-time Fourier Transform (continued)

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}, \; \omega_0 = \frac{2\pi}{N}$$ DTFS synthesis eq.

$$a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk\omega_0 n}$$ DTFS analysis eq.

$$= \frac{1}{N} \sum_{n=-N_1}^{n=N_2} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=-\infty}^{n=\infty} x[n] e^{-jk\omega_0 n}$$

$$= \frac{1}{N} X(e^{jk\omega_0})$$

where $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

$$\tilde{x}[n] = \sum_{k=<N>} \underbrace{\frac{1}{N} X(e^{jk\omega_0})}_{a_k} e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=<N>} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$

As $N \to \infty$: $\quad \tilde{x}[n] \to x[n]$ for every n $\quad \omega_0 \to 0, \sum \omega_0 \to \int d\omega$

Thus, $x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$

The limit of integration is over any interval of 2π in ω

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Fourier transform pair

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

– Analysis Equation
– FT

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

– Synthesis Equation
– Inverse FT

$x[n] = a^n u[n], |a| < 1$ - Exponentially decaying function

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \underbrace{\left(ae^{-j\omega}\right)^n}_{|ae^{-j\omega}|<1}$$

Infinite sum formula

$$= \frac{1}{1 - ae^{-j\omega}}$$

$$= \frac{1}{(1 - a\cos\omega) + ja\sin\omega}$$

$$= \begin{cases} \dfrac{1}{1-a}, & \omega = 0 \\ \dfrac{1}{1+a}, & \omega = \pi \end{cases}$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}$$

Properties of DTFT

Periodicity: $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$

Linearity: $ax_1[n] + bx_2[n] \longleftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$

Time Shifting: $x[n-n_0] \longleftrightarrow e^{-j\omega n_0}X(e^{j\omega})$

Frequency Shifting: $e^{j\omega_0 n}x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$

Time Reversal: $x[-n] \longleftrightarrow X(e^{-j\omega})$

- First difference

$$x[n] - x[n-1] \leftrightarrow \left(1 - e^{-j\omega}\right)X(\omega)$$

- Summation

$$\sum_{k=-\infty}^{\infty} x(k) \leftrightarrow \pi X(0)\delta(\omega) + \frac{X(\omega)}{(1 - e^{-j\omega})}$$

- Differentiation in Frequency

$$nx[n] \leftrightarrow j\frac{dX(\omega)}{d\omega}$$

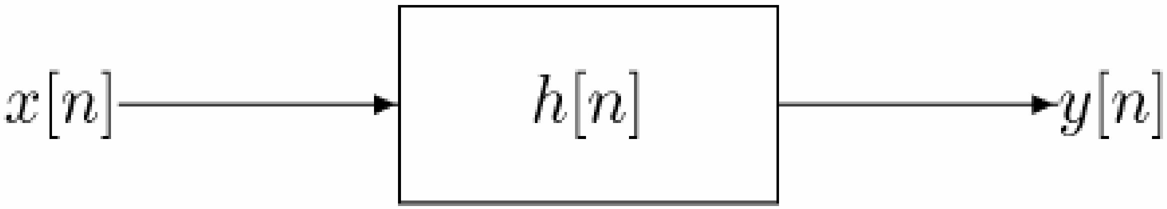
- Parseval's relation

$$\underbrace{\sum_{n=-\infty}^{\infty} |x[n]|^2}_{} = \underbrace{\frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega}_{}$$

- Duality Property

$$X[n] \leftrightarrow 2\pi x(\omega)$$

Convolution property



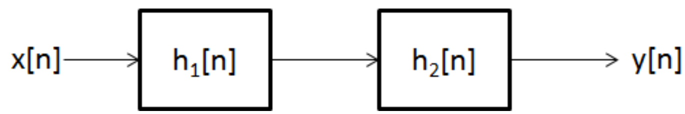
$$y[n] = h[n] * x[n]$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$H(e^{j\omega}) = \text{DTFT of } h[n]$$

Tutorial

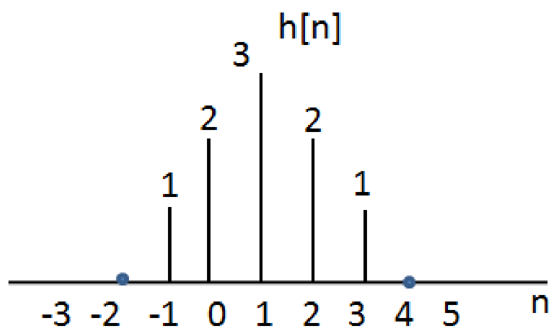
The system is depicted by cascade interconnection of LTI systems as shown below.



The impulse response of $h_2[n]$ is

$$h_2[n] = u[n] - u[n-3]$$

and the overall impulse response, h[n] is given by



i) Find impulse response $h_1[n]$ using DTFT.

ii) Find the output response of the complete system when input is $x[n] = \delta[n] + \delta[n-1] + \delta[n+1]$ using D