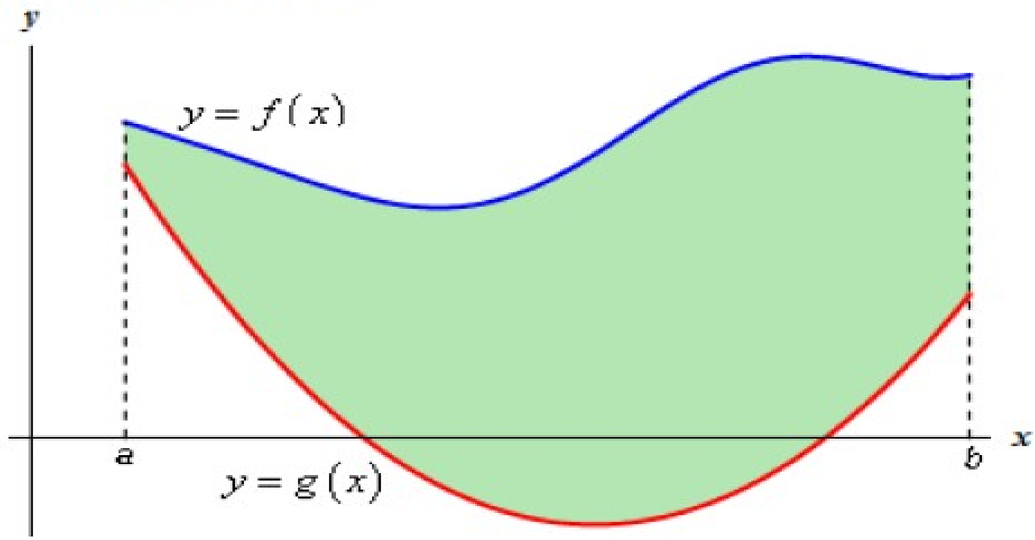
AREA BETWEEN CURVES

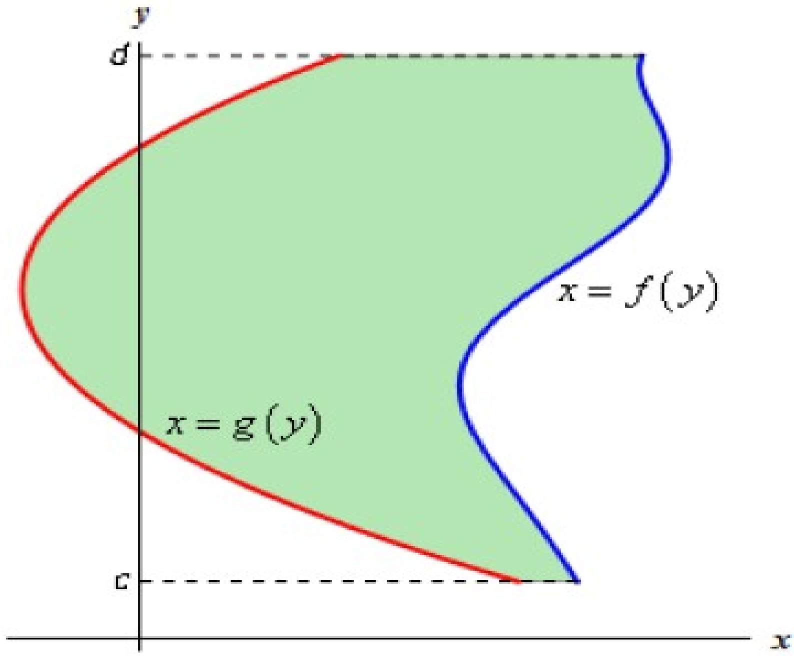
In the first case we want to determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a,b]$. We are also going to assume that $f(x) \geq g(x)$. Take a look at the following sketch to get an idea of what we're initially going to look at.



$$A = \int_a^b f(x) - g(x)\,dx$$

$$A = \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx, \qquad a \leq x \leq b$$

The second case is almost identical to the first case. Here we are going to determine the area between $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$ with $f(y) \ge g(y)$.



In this case the formula is,

$$A = \int_c^d f(y) - g(y)\, dy$$

$$A = \int_c^d \begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix} dy, \qquad c \le y \le d$$

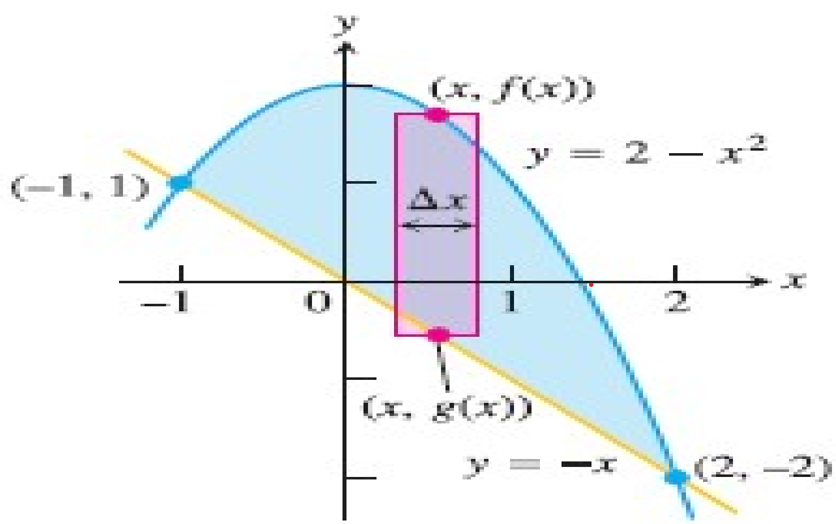
$f(x)$

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$. -2 1

$g(x)$

Solution First we sketch the two curves (Figure). The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.

$$
\begin{aligned}
2 - x^2 &= -x \\
x^2 - x - 2 &= 0 \\
(x + 1)(x - 2) &= 0 \\
x = -1, \qquad x &= 2.
\end{aligned}
$$



The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1, b = 2$.

The area between the curves is

$$A = \int_a^b [f(x) - g(x)]\,dx = \int_{-1}^{2} [(2 - x^2) - (-x)]\,dx$$

$$= \int_{-1}^{2} (2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}$$

2. Determine the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$.

Sol:-

Given $y = x^2$ — ① & $y = \sqrt{x}$ — ②

Using ② in ① $x^2 = \sqrt{x}$

Squaring both sides

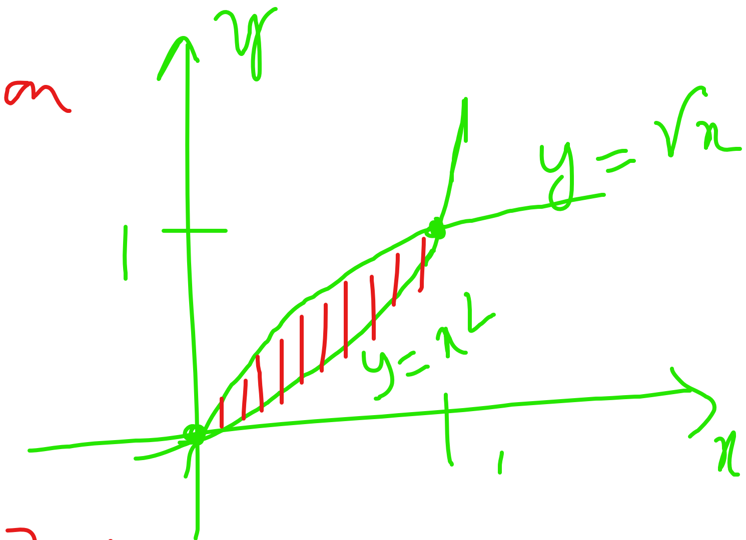
$$x^4 = x \Rightarrow x^4 - x = 0$$

$$x(x^3 - 1) = 0$$

$$x = 0 \ \& \ 1$$

The pts of intersection are $(0,0)$ & $(1,1)$

The limits of the integration are $x = 0$ to $x = 1$



$$\text{Area}(A) = \int_0^1 \left[\sqrt{x} - x^2\right] dx$$

$$= \int_0^1 \left(x^{1/2} - x^2\right) dx = \left(\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right)_0^1$$

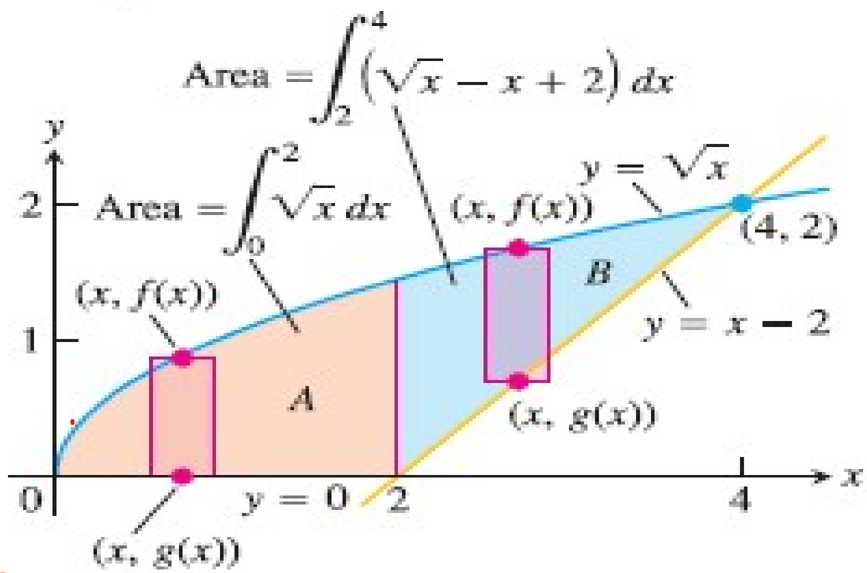
$$= 1/3$$

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution The sketch (Figure) shows that the region's upper boundary is the graph of $f(x) = \sqrt{x}$. The lower boundary changes from $g(x) = 0$ for $0 \le x \le 2$ to $g(x) = x - 2$ for $2 \le x \le 4$ (both formulas agree at $x = 2$). We subdivide the region at $x = 2$ into subregions A and B, shown in Figure

$x = y^2 = g(y)$

$x = y + 2 = f(y)$



$y = f(x)$

$y = g(x)$

$y + 2 = y^2$

$y^2 - y - 2 = 0$

$(y+1)(y-2) = 0$

$y = -1,\ y = 2$

$A = \int_0^2 (y + 2 - y^2)\,dy = 10/3$

The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:

$$
\begin{aligned}
\sqrt{x} &= x - 2 \\
x &= (x - 2)^2 = x^2 - 4x + 4 \\
x^2 - 5x + 4 &= 0 \\
(x - 1)(x - 4) &= 0 \\
x &= 1, \qquad x = 4.
\end{aligned}
$$

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

For $0 \leq x \leq 2$: $\quad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$

For $2 \leq x \leq 4$: $\quad f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2$

We add the areas of subregions A and B to find the total area:

$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\,dx}_{\text{area of } B}$$

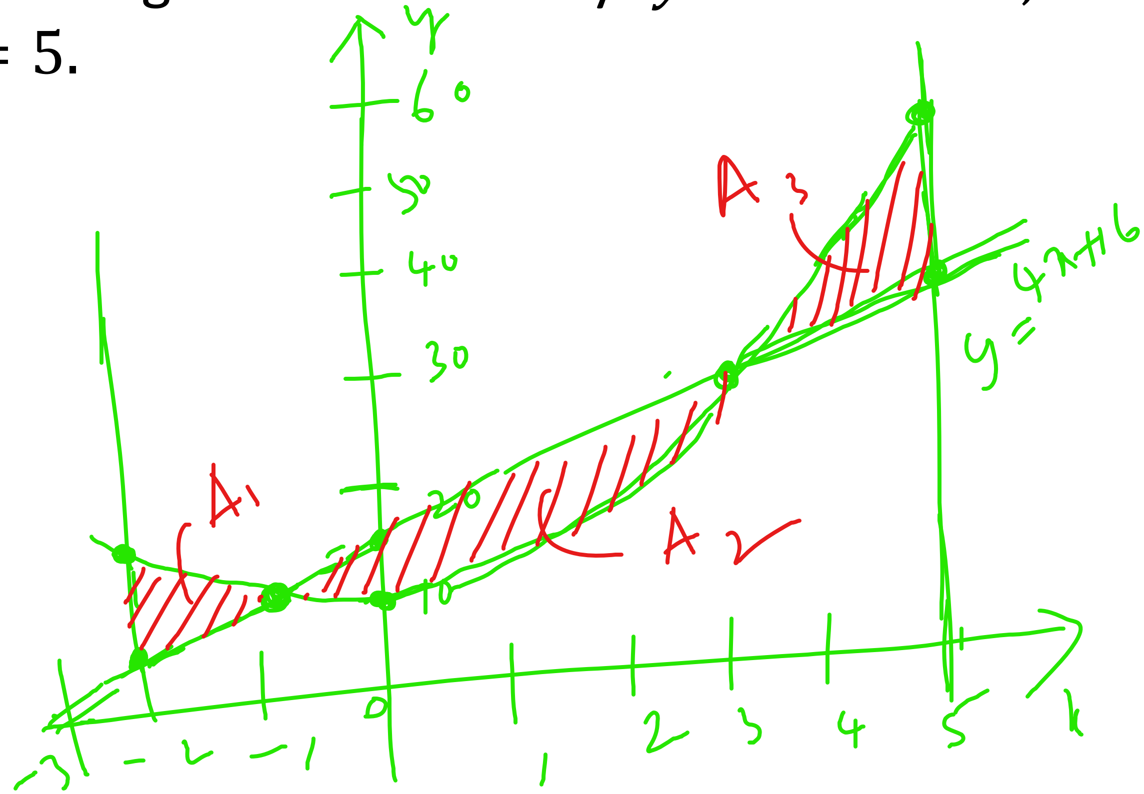
$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

$$= \frac{2}{3}(8) - 2 = \frac{10}{3}.$$

4. Determine the area of the region bounded by $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$ and $x = 5$.

Sol:-



$$A = \int_{-2}^{-1} \left[2x^2+10-(4x+16) \right] dx + \int_{-1}^{3} \left[(4x+16)-(2x^2+10) \right] dx + \int_{3}^{5} \left[(2x^2+10)-(4x+16) \right] dx$$

$$A = A_1 + A_2 + A_3$$

where $$A_1 = \int_{-2}^{-1} \left[2x^2 + 10 - (4x+16) \right] dx$$

$$= \int_{-2}^{-1} (2x^2 + 10 - 4x - 16)\, dx = \int_{-2}^{-1} (2x^2 - 4x - 6)\, dx$$

$$= \left(2\frac{x^3}{3} - \cancel{4}^{2}\frac{x^2}{\cancel{2}} - 6x \right)_{-2}^{-1}$$

$$= \left(-\frac{2}{3} - 2 + 6 \right) - \left(-16/3 - 8 + 12 \right) = 14/3$$

$$A_2 = \int_{-1}^{3} \left[(4x+16) - (2x^2+10) \right] dx$$

$$= \int_{-1}^{3} \left(-2x^2 + 4x + 6 \right) dx$$

$$= \left(-\frac{2x^3}{3} + \frac{\overset{2}{\cancel{4}}x^2}{\cancel{2}} + 6x \right)_{-1}^{3}$$

$$= \left(-\cancel{18} + \cancel{18} + 18 \right) - \left(\frac{2}{3} + 2 - 6 \right) = 64/3$$

$$A_3 = \int_{3}^{5} \left[(2x^2+10) - (4x+16) \right] dx = 64/3$$

$$\therefore A = \frac{14}{3} + \frac{64}{3} + \frac{64}{3} = \frac{142}{3}$$