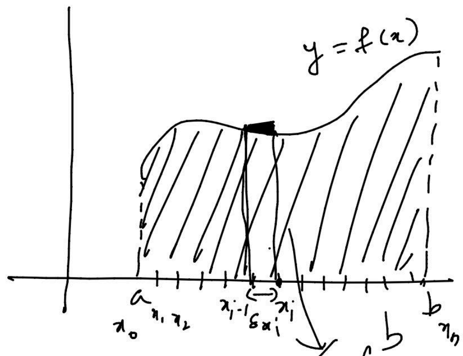
Module : 5

Multiple Integrals

Multiple integrals

01 December 2020 10:58

let $y = f(x)$ in $[a,b]$



Divide $[a,b]$ into n equal sub intervals

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

$$x_i - x_{i-1} = \delta x_i$$

Consider the sum $\sum_{i=1}^{n} f(x_i)\,\delta x_i = f(x_1)\,\delta x_1 + f(x_2)\,\delta x_2 + \dots + f(x_n)\,\delta x_n$.

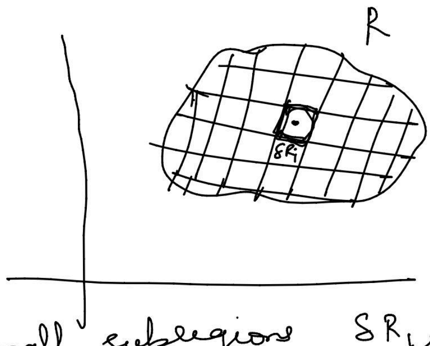
$\int_a^b f(x)\,dx$.

If this sum tends to a finite limit as $n \to \infty$ such that $\delta x_i \to 0$, then this limit is defined to be definite integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\delta x_i$$

→ Double integral:

Consider a region R in xy-plane bounded by one or more curves



let $f(x,y)$ be a function defined in R.

let the region R be divided into small subregions $\delta R_1, \delta R_2, \dots \delta R_n$ which are pairwise non overlapping

let (x_i, y_i) be an arbitrary point in the subregion δR_i

Now Consider the sum

$$f(x_1,y_1)\,\delta R_1 + f(x_2,y_2)\,\delta R_2 + \dots + f(x_n,y_n)\,\delta R_n$$

If this sum tends to a finite limit as $n \to \infty$ such that $\delta R_i \to 0$, then this limit is called as double integral of $f(x,y)$ over the region R and is denoted by $\iint_R f(x,y)\,dR$ (or)

$$\iint_R f(x,y)\,dx\,dy$$

Properties

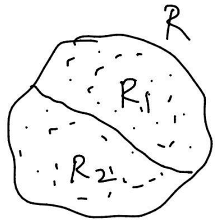
let f & g be two functions of x & y defined in a region R.

Properties

Let f & g be two functions of x & y

1) $\iint_R (f+g)\, dx\,dy = \iint_R f\, dx\,dy + \iint_R g\, dx\,dy$

2) $\iint_R k f\, dx\,dy = k \iint_R f\, dx\,dy$, k is any constant.



3) $\iint_R f\,dx\,dy = \iint_{R_1} f\,dx\,dy + \iint_{R_2} f\,dx\,dy$.

→ Evaluation of Double integrals:

Let $f(x,y)$ be defined over the region R given by

$R: \; a \le x \le b$ & $y_1(x) \le y \le y_2(x)$

Then The double integral

$$\int_{x=a}^{b} \int_{y=y_1(x)}^{y_2(x)} f(x,y)\, dy\, dx = \int_{x=a}^{b} \left[\int_{y=y_1(x)}^{y_2(x)} f(x,y)\, dy \right] dx$$

The integral with in the square brackets is evaluated first. by integrating the integrand w.r.to 'y' partially by keeping 'x' as constant. Then we get the expression in a function of x alone which is then integrated w.r.to x with in the limits $x=a$, $x=b$ gives the required value of double integral.

Ily $R: \; a \le y \le b$, $x_1(y) \le x \le x_2(y)$

$$\iint_R f(x,y)\, dx\,dy = \int_{y=a}^{b} \int_{x=x_1(y)}^{x_2(y)} f(x,y)\, dx\, dy = \int_{y=a}^{b} \left[\int_{x=x_1(y)}^{x_2(y)} f(x,y)\, dx \right] dy$$

If $R: \; a \le x \le b$, $c \le y \le d$

... are constants then we can

If $R: a \le x \le b, \; - - 0$.

i.e both limits of x & y are constants then we can either integrate w.r.to 'x' first or w.r.to 'y' first.

I) Evaluate the following double integrals

1) $\int_0^2 \int_0^3 xy \, dx \, dy$

Sol:
$$\int_{y=0}^{2} \int_{x=0}^{3} xy \, dx \, dy = \int_{y=0}^{2} \left[\int_{x=0}^{3} xy \, dx \right] dy$$

$$= \int_{y=0}^{2} y \left(\frac{x^2}{2}\right)_{x=0}^{3} dy = \frac{1}{2} \int_{y=0}^{2} y \, (9-0) \, dy$$

$$= \frac{9}{2} \int_{y=0}^{2} y \, dy = \frac{9}{2} \left(\frac{y^2}{2}\right)_0^2$$

$$= \frac{9}{4}(4-0) = 9$$

→ 2) $\int_0^2 \int_0^x y \, dy \, dx$

Sol:
$$\int_{x=0}^{2} \int_{y=0}^{x} y \, dy \, dx = \int_{x=0}^{2} \left[\int_{y=0}^{x} y \, dy \right] dx$$

$$= \int_{x=0}^{2} \left(\frac{y^2}{2}\right)_{y=0}^{x} dx = \frac{1}{2} \int_{x=0}^{2} (x^2 - 0) \, dx$$

$$= \frac{1}{2} \int_{x=0}^{2} x^2 \, dx = \frac{1}{2} \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{6} = \frac{4}{3}$$

3) $\int_0^1 \int_{y=x}^{\sqrt{x}} (x^2 + y^2) \, dx \, dy$

Sol:
$$\int^{1} \int^{\sqrt{x}} (x^2 + y^2) \, dx \, dy = \int^{1} \int^{\sqrt{x}} (x^2 + y^2) \, dy \, dx$$

Sol:
$$\int_{x=0}^{1}\int_{y=x}^{\sqrt{x}}(x^2+y^2)\,dx\,dy = \int_{x=0}\int_{y=x}(x^2+y^2)\,dy\,dx$$

$$= \int_{x=0}^{1}\left(x^2y+\frac{y^3}{3}\right)_{y=x}^{\sqrt{x}}dx = \int_{x=0}^{1}\left(x^2\sqrt{x}+\frac{(\sqrt{x})^3}{3}-x^2x-\frac{x^3}{3}\right)dx$$

$$= \int_{x=0}^{1}\left(x^{5/2}+\frac{x^{3/2}}{3}-\frac{4x^3}{3}\right)dx = \left(\frac{x^{7/2}}{7/2}+\frac{1}{3}\frac{x^{5/2}}{5/2}-\frac{4}{3}\frac{x^4}{4}\right)_0^1$$

$$= \left(\frac{2}{7}x^{7/2}+\frac{2}{15}x^{5/2}-\frac{1}{3}x^4\right)_0^1$$

$$\left(\frac{2}{7}+\frac{2}{15}-\frac{1}{3}\right) = \frac{3}{35}$$

$\rightarrow$
$$\int_0^a\int_0^{\sqrt{a^2-y^2}}\sqrt{a^2-x^2-y^2}\,dx\,dy$$

Sol
$$\int_{y=0}^{a}\int_{x=0}^{\sqrt{a^2-y^2}}\sqrt{a^2-x^2-y^2}\,dx\,dy = \int_{y=0}^{a}\int_{x=0}^{\sqrt{a^2-y^2}}\sqrt{(a^2-y^2)-x^2}\,dx\,dy$$

let $a^2-y^2=p^2$
$$\int_{y=0}^{a}\int_{x=0}^{p}\sqrt{p^2-x^2}\,dx\,dy$$

$$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$= \int_{y=0}^{a}\left[\frac{x}{2}\sqrt{p^2-x^2}+\frac{p^2}{2}\sin^{-1}\left(\frac{x}{p}\right)\right]_{x=0}^{p}dy$$

$$= \int_{y=0}^{a}\left[0+\frac{p^2}{2}\sin^{-1}(1)-0-0\right]dy$$

a

$$y=0$$

$$= \frac{1}{2}\int_{y=0}^{a} p^2 \cdot \frac{\pi}{2}\, dy$$

$$= \frac{\pi}{4}\int_{0}^{a} (a^2-y^2)\, dy$$

$$= \frac{\pi}{4}\left(a^2y - \frac{y^3}{3}\right)_0^a$$

$$= \frac{\pi}{4}\left(a^2\cdot a - \frac{a^3}{3} - 0 - 0\right)$$

$$= \frac{\pi}{4}\cdot\frac{2a^3}{3} = \frac{\pi a^3}{6}$$

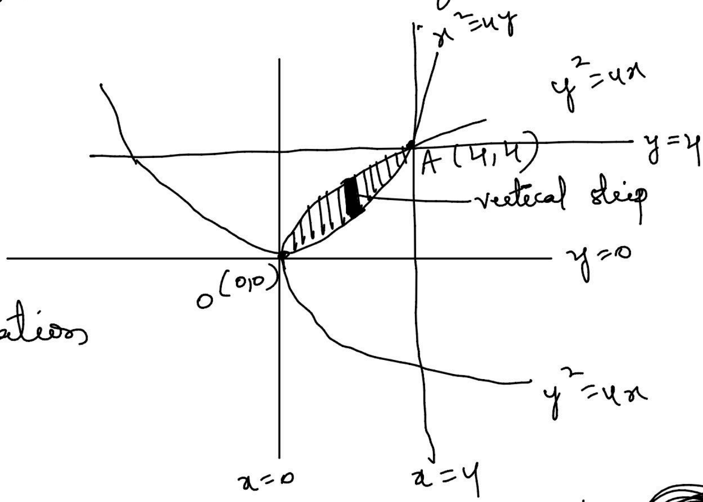
→ Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$ Ans $= \frac{\pi}{4}\log(1+\sqrt{2})$.

$\frac{\pi}{4}\sinh^{-1}(1)$.

→ $\displaystyle\int_0^2\int_0^x e^{x+y}\, dy\, dx$ Ans : $\frac{1}{2}(e^2-1)^2$

→ $\displaystyle\int_0^5\int_0^{x^2} x(x^2+y^2)\, dx\, dy$ Ans : $\frac{29}{24}(5^6)$.

→ Evaluate $\iint_R y\, dx\, dy$ where R is the region bounded by $y^2=4x$ & $x^2=4y$



Sol: The shaded region is 'R' which is the region of integration

$y^2=4x$, $x^2=4y$

$x=\frac{y^2}{4}$ in $x^2=4y$

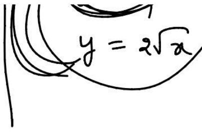
$$\left(\frac{y^2}{4}\right)^2 = 4y = y^4 = 64y$$

$y=0$ & $y=4$

$y=\frac{x^2}{4}$

$y=2\sqrt{x}$

$$\left(\frac{y}{4}\right) = 4y$$



$$\Rightarrow y(y^3 - 64) = 0 \Rightarrow y = 0 \ \& \ y = 4$$

$y = 0 \Rightarrow x = 0$ $\qquad$ $(0,0)$ & $(4,4)$ are points of intersection of parabolas

$y = 4 \Rightarrow x = 4$

Consider a vertical strip in the shaded region. The vertical strip $y = \frac{x^2}{4}$ to $y = 2\sqrt{x}$ slides from $x=0$ to $x=4$ to cover the entire region (shaded region) of integration

$$\iint_R y\,dx\,dy = \int_{x=0}^{4} \int_{y=\frac{x^2}{4}}^{2\sqrt{x}} y\,dy\,dx$$

$$= \int_{x=0}^{4} \left(\frac{y^2}{2}\right)_{y=\frac{x^2}{4}}^{2\sqrt{x}} dx = \frac{1}{2}\int_{x=0}^{4} \left(4x - \frac{x^4}{16}\right) dx$$

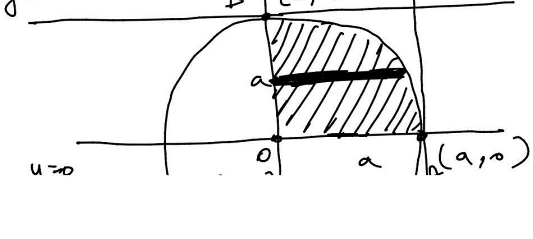
$$= \frac{1}{2}\left[4\frac{x^2}{2} - \frac{x^5}{5\times 16}\right]_0^4$$

$$= \frac{1}{2}\left[2(16-0) - \frac{1}{16\times 5}(4^5 - 0)\right]$$

$$= \frac{1}{2}\left[32 - \frac{4^5}{16\times 5}\right] = \frac{1}{2}\left[32 - \frac{4^3}{5}\right]$$

$$= \frac{1}{2}\left[32 - \frac{64}{5}\right] = \frac{48}{5}$$

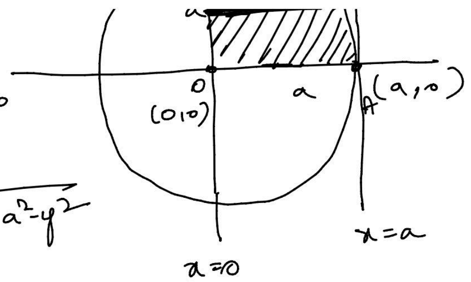
→ Evaluate $\iint xy\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2 = a^2$.



Sol:

Consider a horizontal strip in

sol: Consider a horizontal strip in the shaded region



The horizontal strip $x=0$ to $x=\sqrt{a^2-y^2}$ slides from $y=0$ to $y=a$

$x^2+y^2=a^2$

$x=\sqrt{a^2-y^2}$

$$\iint xy\,dx\,dy = \int_{y=0}^{a}\int_{x=0}^{\sqrt{a^2-y^2}} xy\,dx\,dy$$

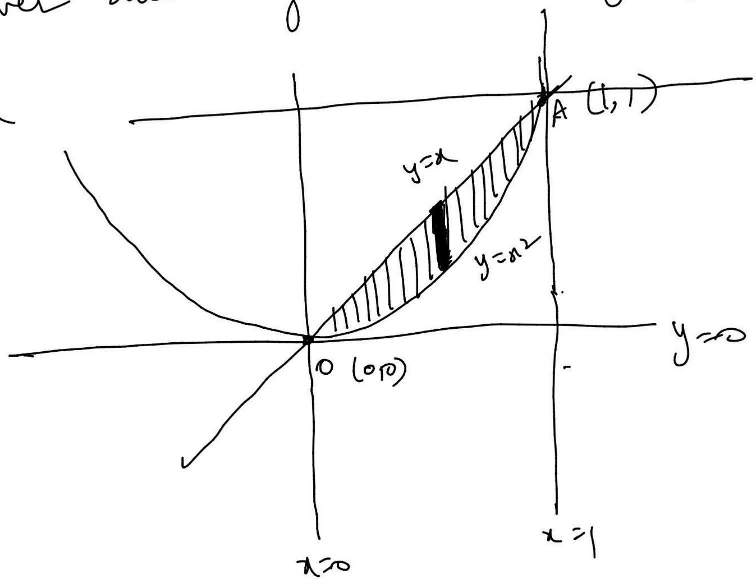
$$= \int_{y=0}^{a} y\left(\frac{x^2}{2}\right)_{x=0}^{\sqrt{a^2-y^2}} dy = \frac{1}{2}\int_{y=0}^{a} y(a^2-y^2)\,dy$$

$$= \frac{1}{2}\int_{y=0}^{a}(a^2y-y^3)\,dy = \frac{1}{2}\left[\frac{a^2y^2}{2}-\frac{y^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right] = \frac{a^4}{8}.$$

→ $\iint xy(x+y)\,dx\,dy$ over the region bounded by $y=x^2$ & $y=x$

$y=x^2$, $y=x$



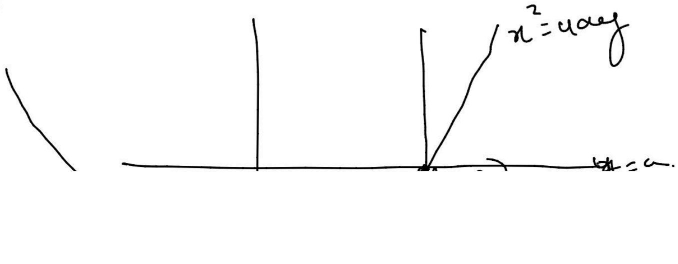
$\underline{y=x^2}$, $\underline{y=x}$

$\underline{x=0,\ x=1}$

$\underline{\text{Ans}} = \frac{3}{56}$

→ $\iint xy\,dx\,dy$, R: x-axis, ordinate $\underline{x=2a}$ & $\underline{x^2=4ay}$

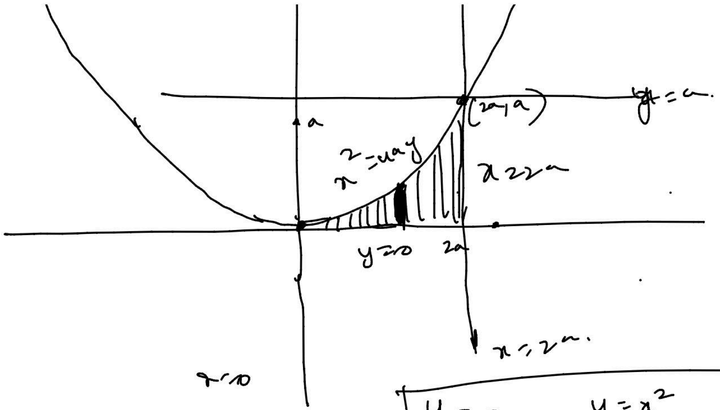
sol



$x^2=4ay$

$x=2a$

Sol



$x = 2a$

$4a^2 = 4ay$

$y = a$

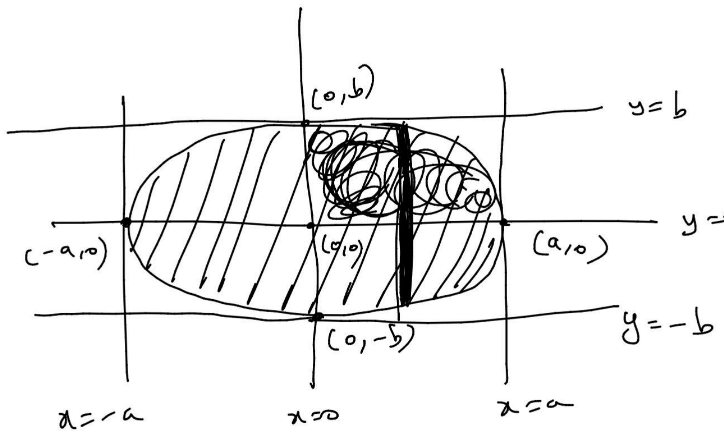
$(2a, a)$

$\boxed{y = 0 \;,\; y = \frac{x^2}{4a}}$ $x = 0 \;,\; x = 2a$

Ans : $\frac{a^4}{3}$

→ $\iint (x+y)^2 \, dx\, dy$ over the area bounded by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol :

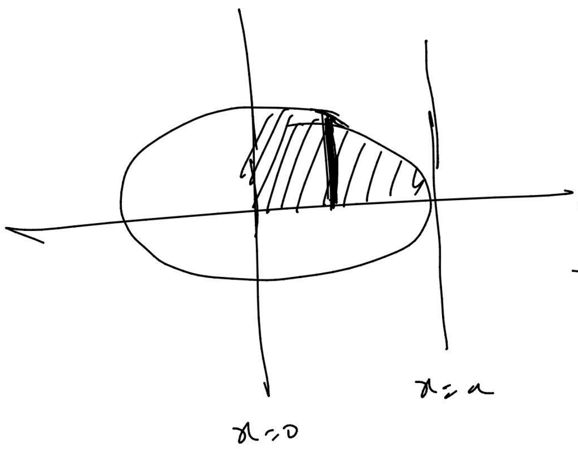


$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$

$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$

vertical strip $y = -b\sqrt{1 - \frac{x^2}{a^2}}$ $y = b\sqrt{1 - \frac{x^2}{a^2}}$ $x = -a$ to $x = a$



$y = 0$ to $y = b\sqrt{1 - \frac{x^2}{a}}$

$x = 0$ to $x = a$

$\frac{\pi}{4} ab(a^2 + b^2)$

$\int^{a} \int^{b\sqrt{1-\frac{x^2}{a^2}}}$

$\bar{u}$

$$\iint_R (x+y)^2\,dx\,dy = 4\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} (x^2+y^2+2xy)\,dy\,dx$$

$$= 4\int_{x=0}^{a}\left(x^2y+\frac{y^3}{3}+\frac{\cancel{2}xy^2}{\cancel{2}}\right)_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}}dx$$

$$= 4\int_{x=0}^{a}\left[x^2b\sqrt{1-\frac{x^2}{a^2}}+\frac{b^3}{3}\left(1-\frac{x^2}{a^2}\right)^{3/2}+xb^2\left(1-\frac{x^2}{a^2}\right)\right]dx$$

put $x = a\sin\theta$ $\quad dx = a\cos\theta\,d\theta$

$x \to 0 \Rightarrow \theta = 0$

$x \to a \Rightarrow \theta = \frac{\pi}{2}$

$$= 4\int_{\theta=0}^{\pi/2}\left[a^2\sin^2\theta\, b\sqrt{1-\sin^2\theta}+\frac{b^3}{3}\left(1-\sin^2\theta\right)^{3/2}+a\sin\theta\, b^2\left(1-\sin^2\theta\right)\right]a\cos\theta\,d\theta$$

$$= 4\int_{0}^{\pi/2}\left[a^2b\sin^2\theta\cos\theta+\frac{b^3}{3}\cos^3\theta+ab^2\sin\theta\cos^2\theta\right]a\cos\theta\,d\theta$$

$$= 4\int_{0}^{\pi/2}\left(a^3b\sin^2\theta\cos^2\theta+\frac{ab^3}{3}\cos^4\theta+a^2b^2\sin\theta\cos^3\theta\right)d\theta$$

$$\boxed{\int_{0}^{\pi/2}\sin^m\theta\cos^n\theta\,d\theta}$$

Change of order of integration and Change of varaibles

15 December 2020 10:30

In some of the problems, we have considered $\iint f(x,y)\,dx\,dy$ & in some we have considered $\iint f(x,y)\,dy\,dx$.

$$\int_{x=a}^{b}\int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx \qquad dx\,dy$$

Whenever these integrals exist, they are equal. Then why not we stick to one form & evaluate?

In some problems by taking a particular form, the evaluation becomes comparitively simple.

In some cases unless we take one of the two forms only, we cannot evaluate the integral.

In some cases, both forms may pose the same amount of ease or difficulty.

→ Change of order of integration:

Change of order of integration implies change of limits of integration.

If the region of integration contains a vertical strip sliding along x-axis then after change of order of integration, a horizontal strip sliding along y-axis is to be considered & vice versa.

Working rule:

Consider $$\int_{x=a}^{b}\int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx.$$

→ Draw the region of integration by drawing the curves $y=y_1(x)$ $y=y_2(x)$, $x=a$ & $x=b$.

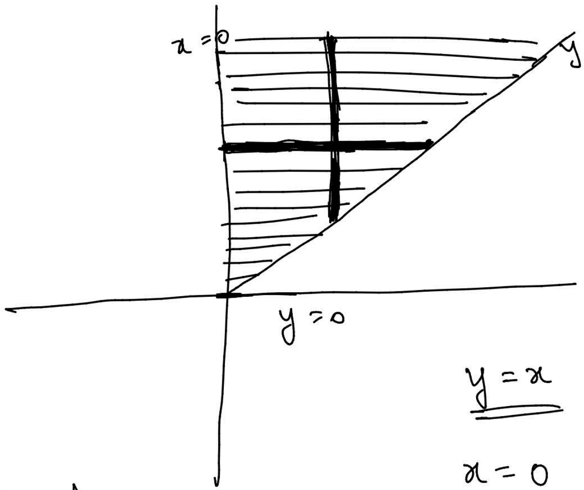
→ If these curves and lines intersect, then draw lines parallel to x-axis to get various sub regions.

→ In each of these sub regions draw elementary strips parallel to x-axis and obtain the limits for x in terms of y & then limits of y as constants.

limits of y as constants.

→ Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$.

Sol: $\int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y}\, dy\, dx$



$x = 0$, $x = \infty$

$y = x$, $y = \infty$

$y = x$ $\quad$ $y = \infty$

$x = 0$ $\quad$ $x = \infty$

From the given limits of integration, a vertical strip $y = x$ to $y = \infty$ is sliding from $x = 0$ to $x = \infty$.

Now after the change of order of integration, we take a horizontal strip in the region.

This horizontal strip, $x = 0$ to $x = y$ sliding from $y = 0$ to $y = \infty$ to cover the region of integration

$$\int_{x=0}^{\infty}\int_{y=x}^{\infty} \frac{e^{-y}}{y}\, dy\, dx = \int_{y=0}^{\infty}\int_{x=0}^{y} \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_{y=0}^{\infty} \left(\frac{e^{-y}}{y}\right)(x)_{x=0}^{y}\, dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{\cancel{y}}\, \cancel{y}\, dy$$

$$= \int_{y=0}^{\infty} e^{-y}\, dy = \left(-e^{-y}\right)_0^{\infty} = 1$$

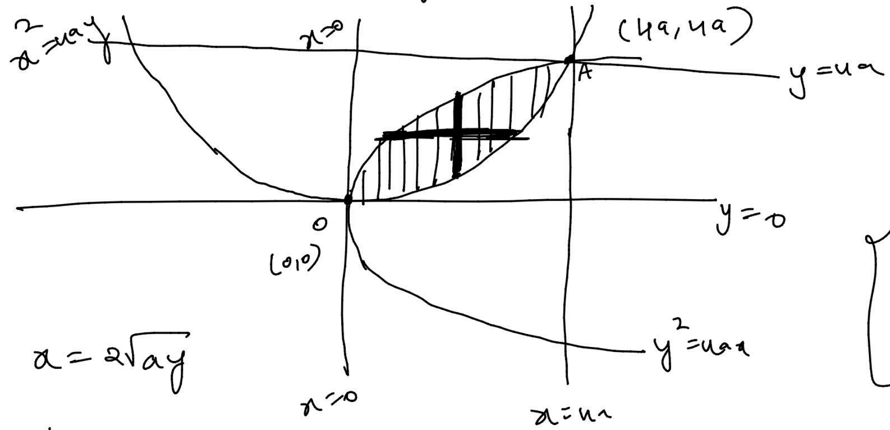
→ Evaluate $\int_{0}^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\, dx$ by changing the order of integration

Sol:

$x = 0 \quad y = \frac{x^2}{4a}$

$x = 0, \quad x = 4a, \quad y = \frac{x^2}{4a}, \quad y = 2\sqrt{ax}$

$x^2 = 4ay, \quad y^2 = 4ax.$



$\begin{cases} y = \frac{x^2}{4a} \\ y = 2\sqrt{ax} \\ x = 0 \\ x = 4a \end{cases}$

$\begin{cases} x = \frac{y^2}{4a} \text{ to } x = 2\sqrt{ay} \\ y = 0 \text{ to } y = 4a \end{cases}$

$$\int_{x=0}^{4a}\int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx = \int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy$$

$$= \int_{y=0}^{4a} \left(x\right)_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$\int_{y=0}^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$\left[2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{1}{4a}\frac{y^3}{3}\right]_0^{4a} = \frac{16a^2}{3}$$

→ Evaluate the following integrals by changing the order of integration

1) $\displaystyle\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dy\,dx$

2) $\displaystyle\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy.$

$1 \quad 2-x$

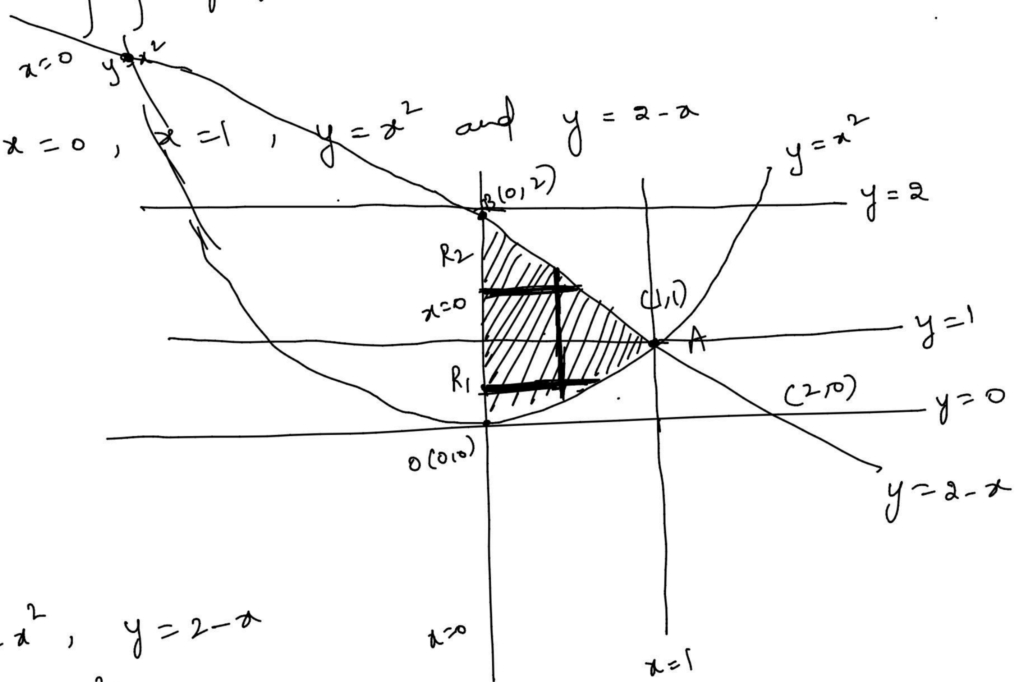
$\int_{0} \int_{x/a}$ $\qquad$ $\int_{0} \int_{1}$

→ change the order of integration in the integral $\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$ & hence evaluate.

Sol: $\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$

$x=0 \quad y=x^2$

$x=0, \; x=1, \; y=x^2$ and $y=2-x$



$y=x^2, \quad y=2-x$

$\Rightarrow x^2 = 2-x$

$x^2+x-2=0$

$x=-2, 1$ $\qquad$ for $x=-2, \; y=4$

$\qquad\qquad\qquad\qquad x=1, \; y=1$

Points of intersection of $y=x^2$ & $y=2-x$ are $(1,1)$ & $(-2,4)$

After change of order of integration, a horizontal strip is taken in the region.

When this horizontal strip is slided from $y=0$ to $y=2$ the left edge remains on y-axis throughout the sliding but the right edge of the strip donot remain on a single curve the right edge of the strip lies on the parabola i,e the right edge of the horizontal strip lies on the straight line. up to A & after A it shifts onto the straight line.

Hence at this point 'A' we divide the region into two sub regions & take two different horizontal strips in these two sub regions.

sub regions ...

two sub regions.

$$\iint_R xy\,dy\,dx = \iint_{R_1} xy\,dy\,dx + \iint_{R_2} xy\,dy\,dx$$

In the region R_1, the horizontal strip $x=0$ to $x=\sqrt{y}$ slides from $y=0$ to $y=1$

In the region R_2, the horizontal strip $x=0$ to $x=2-y$ slides from $y=1$ to $y=2$

$$\iint_R xy\,dy\,dx = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy$$

$$= \int_{y=0}^{1} y\left(\frac{x^2}{2}\right)_{x=0}^{\sqrt{y}} dy + \int_{y=1}^{2} y\left(\frac{x^2}{2}\right)_{x=0}^{2-y} dy$$

$$= \frac{1}{2}\int_{y=0}^{1} y(y-0)\,dy + \frac{1}{2}\int_{y=1}^{2} y(2-y)^2\,dy$$

$$= \frac{1}{2}\left(\frac{y^3}{3}\right)_0^1 + \frac{1}{2}\int_{y=1}^{2} y(y^2+4-4y)\,dy$$

$$= \frac{1}{6} + \frac{1}{2}\left(\frac{y^4}{4} + \frac{4y^2}{2} - \frac{4y^3}{3}\right)_1^2$$

$$\frac{1}{6} + \frac{1}{2}\left(4 + 8 - \frac{32}{3} - \frac{1}{4} - 2 + \frac{4}{3}\right)$$

$$= \frac{3}{8}$$

$\rightarrow$ $$\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy^2\,dy\,dx. \qquad \text{Ans: } \frac{47a^5}{120}$$

... double integrals in polar coordinates:

→ Evaluation of double integrals in polar coordinates:

$$\iint_R f(r,\theta)\, dr\, d\theta$$

$$\int_{\theta=\theta_1}^{\theta_2}\int_{r=r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, dr\, d\theta = \int_{\theta=\theta_1}^{\theta_2}\left[\int_{r=r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, dr\right] d\theta .$$

→ Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\, dr\, d\theta$

Sol:

$$\int_{\theta=0}^{\pi}\int_{r=0}^{a\sin\theta} r\, dr\, d\theta = \int_{\theta=0}^{\pi}\left(\frac{r^2}{2}\right)_{r=0}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi} a^2\sin^2\theta\, d\theta = \frac{a^2}{2}\int_0^{\pi}\frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{a^2}{4}\left[\pi - 0 - 0 - 0\right] = \frac{\pi a^2}{4}$$

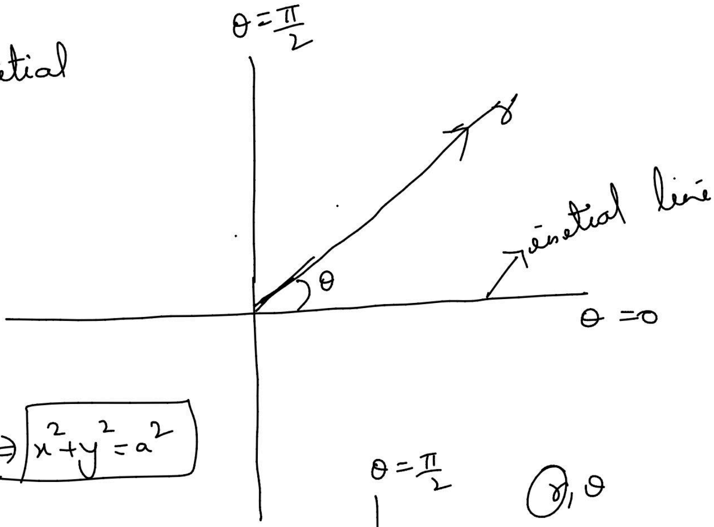
→ $\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2}\, r\, d\theta\, dr$ Ans: $\frac{\pi}{4}$

→ $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\, dr\, d\theta$

→ $\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\, dr\, d\theta}{\sqrt{a^2-r^2}}$

→ Evaluate $\iint_R r^2\sin\theta\, dr\, d\theta$, where R is the semicircle above the initial $\theta=\frac{\pi}{2}$

→ Evaluate $\iint_R r \sin\theta \, dr\, d\theta$ over the cardioid ... $r = 2a\cos\theta$ above the initial



Sol:

$r = a$

$x = r\cos\theta \qquad y = r\sin\theta$

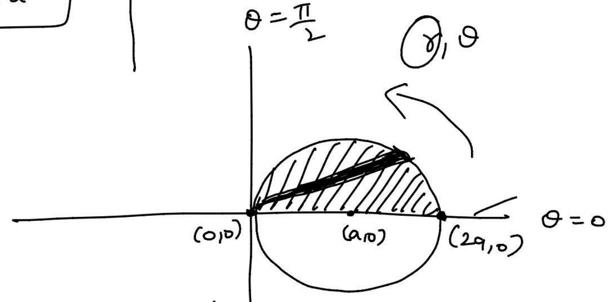
$r = \sqrt{x^2+y^2}$

$\sqrt{x^2+y^2} = a \Rightarrow \boxed{x^2+y^2 = a^2}$

$x^2+y^2 - 2ax = 0$

$r^2 - 2ar\cos\theta = 0$

$\Rightarrow r = 2a\cos\theta$ is a circle with centre on x-axis & passing through origin.



Consider a radial strip in the region $r=0$ to $r = 2a\cos\theta$ which slides from $\theta = 0$ to $\theta = \frac{\pi}{2}$

$$\iint_R r^2 \sin\theta \, dr\, d\theta = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} r^2 \sin\theta \, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \sin\theta \left(\frac{r^3}{3}\right)_{r=0}^{2a\cos\theta} d\theta$$

$$= \frac{8a^3}{3} \int_0^{\pi/2} \sin\theta \left(\cos^3\theta - 0\right) d\theta$$

$$= \frac{8a^3}{3} \int_0^{\pi/2} \sin\theta \cos^3\theta \, d\theta$$

$\cos\theta = t$

$-\sin\theta \, d\theta = dt$

$\sin\theta \, d\theta = -dt$

$\theta = 0 \Rightarrow t = 1$

$\theta = \frac{\pi}{2} \Rightarrow t = 0$

$$= \frac{8a^3}{3}\int_1^0 t^3(-dt)$$

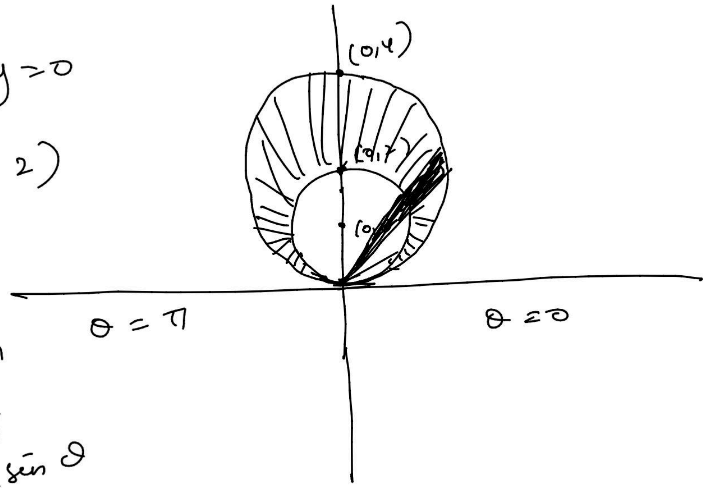
$$= \frac{8a^3}{3}\int_0^1 t^3\,dt = \frac{8a^3}{3}\left(\frac{t^4}{4}\right)_0^1 = \frac{2a^3}{3}$$

→ Evaluate $\iint r^3\,dr\,d\theta$ over the area included b/w the circles $r = 2\sin\theta$ & $r = 4\sin\theta$.

Sol: $x^2+y^2 = 2y$ $\qquad$ $x^2+y^2 = 4y$

$x^2+y^2-2y=0$ $\qquad$ $x^2+y^2-4y=0$

$(0,1)$ $\qquad$ $(0,2)$



The radial strip $r = 2\sin\theta$ to $r = 4\sin\theta$ slides from $\theta = 0$ to $\theta = \pi$

$$\iint r^3\,dr\,d\theta = \int_{\theta=0}^{\pi}\int_{r=2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi}\left(\frac{r^4}{4}\right)_{r=2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi}(256\sin^4\theta - 16\sin^4\theta)\,d\theta$$

$$= 60\int_0^{\pi}\sin^4\theta\,d\theta$$

$$= 60 \times 2\int_0^{\pi/2}\sin^4\theta\,d\theta$$

$$= 60 \times \cancel{2} \times \frac{3}{4} \times \frac{1}{\cancel{2}} \times \frac{\pi}{2} = \frac{45\pi}{2}$$

$$\boxed{\int_0^{\pi/2}\sin^m\theta\,d\theta \qquad \int_0^{\pi/2}\sin^m\theta\cos^n\theta\,d\theta}$$

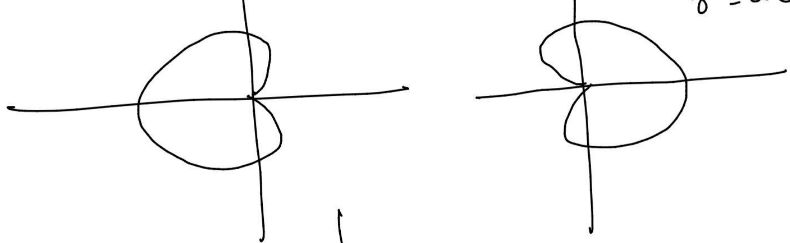
$$\boxed{\begin{aligned}&\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta \\ &\int_0^{\pi/2} \cos^n\theta \, d\theta\end{aligned}}$$

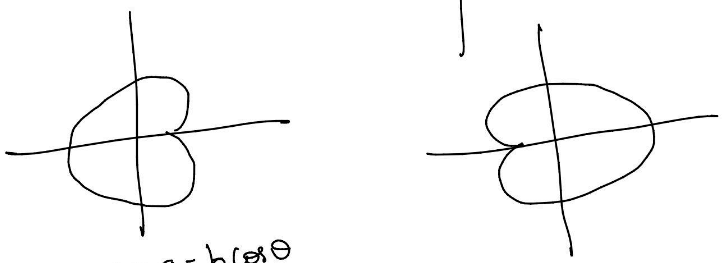
$$= 60 \times \cancel{2} \times \frac{3}{4} \times \frac{1}{\cancel{2}} \times \frac{\pi}{2} = \frac{45\pi}{2}.$$

→ $\iint r^3 \, dr \, d\theta$ over the area b/w the circles $r = 2\cos\theta$ & $r = 4\cos\theta$.

→ Cardiod →

$r = a(1-\cos\theta)$ $\qquad$ $r = a(1+\cos\theta)$

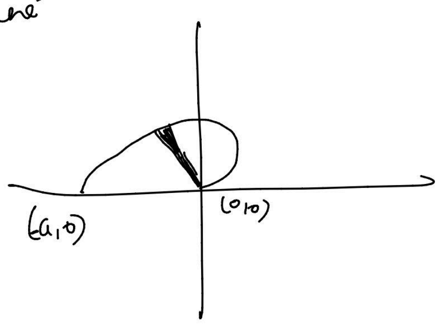




$r = a - b\cos\theta$ $\qquad$ $r = a + b\cos\theta$

→ Evaluate $\iint r \sin\theta \, dr \, d\theta$ over the cardiod $r = a(1-\cos\theta)$ above the initial line

Ans: $\frac{4a^2}{3}$



$r = 0$ to $r = a(1-\cos\theta)$

$\theta = 0$ to $\theta = \pi$

→ Change of variables:-

let $x = f(u,v)$ & $y = g(u,v)$

The above relation represents the transformation of the variables x, y to u, v

thus $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is the Jacobian of transformation

$$\left| \begin{matrix} \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{matrix} \right|$$

Now the double integral $\boxed{\iint_R f(x,y)\,dx\,dy = \iint_{R'} F(u,v)\,|J|\,du\,dv}$

Change of variables to polar coordinates:

$x = r\cos\theta$ & $y = r\sin\theta$ is the relation b/w cartesian (x,y) & polar coordinates (r,θ)

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$J = r$$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} F(r\cos\theta, r\sin\theta)\,|J|\,dr\,d\theta$$

$$= \iint_{R'} F(r\cos\theta, \sin\theta)\, r\,dr\,d\theta.$$

→ Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ changing to polar coordinates.

sol:

$x = 0$, $x = \infty$

$y = 0$, $y = \infty$

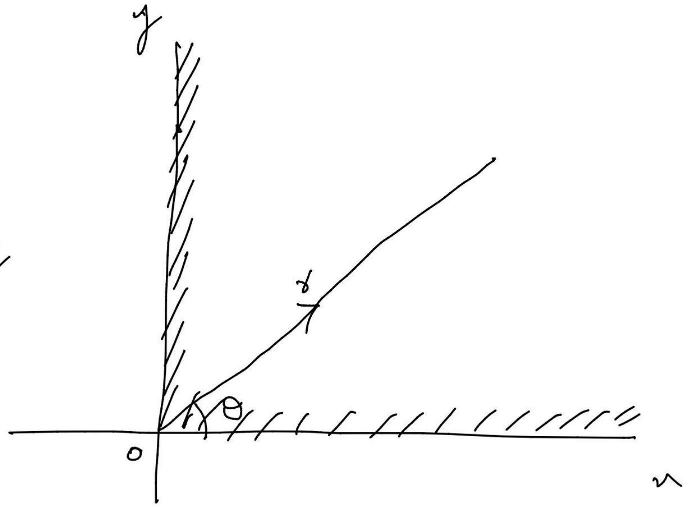
we change x, y to polar by

$x = r\cos\theta$, $y = r\sin\theta$

then $J = r$

$dx\,dy = |J|\,dr\,d\theta$

$dx\,dy = r\,dr\,d\theta$



Region of integration is first quadrant

$r = 0$ to $r = \infty$, $x^2+y^2 = r^2$

$\theta = 0$ to $\theta = \frac{\pi}{2}$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx\,dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} \cdot r\,dr\,d\theta$$

$r^2 = t$

$r\,dr = \frac{dt}{2}$

$r \to 0 \Rightarrow t \to 0$

$r \to \infty \Rightarrow t \to \infty$

$$= \int_{\theta=0}^{\pi/2} \left[\int_{t=0}^{\infty} \frac{e^{-t}}{2} \, dt \right] d\theta$$

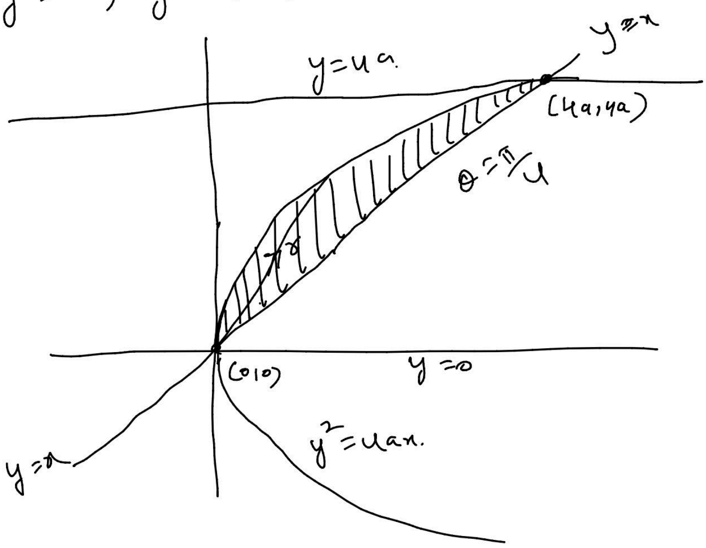
$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} \left(-e^{-t} \right)_{t=0}^{\infty} d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{\pi}{4}$$

→ Evaluate $\displaystyle\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2} \, dx\,dy$ changing to polar coordinates.

Sol: $x = \frac{y^2}{4a}$, $x = y$, $y = 0$, $y = 4a$.

$y^2 = 4ax$



$x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = r^2$

$J = r$

$dx\,dy = |J|\,dr\,d\theta = r\,dr\,d\theta$

$y^2 = 4ax$

$r^2 \sin^2\theta = 4ar\cos\theta$

$r = \frac{4a\cos\theta}{\sin^2\theta}$

$r = 0$ to $r = \frac{4a\cos\theta}{\sin^2\theta}$.

$\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$

$$\int_0^{4a} \int^{y} \frac{x^2-y^2}{\ldots} \, dx\,dy = \int^{\pi/2} \int^{\frac{4a\cos\theta}{\sin^2\theta}} \left(\frac{r^2\cos^2\theta - r^2\sin^2\theta}{\ldots} \right) r\,dr\,d\theta$$

$$\int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=\frac{\pi}{4}}\int_{r=0} \left(\frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\right) r\,dr\,d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} (\cos^2\theta - \sin^2\theta)\, r\,dr\,d\theta.$$

$$= \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos^2\theta - \sin^2\theta)\left(\frac{r^2}{2}\right)_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} d\theta$$

$$= \frac{1}{2}\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos^2\theta - \sin^2\theta)\, 16a^2\frac{\cos^2\theta}{\sin^4\theta}\, d\theta.$$

$$= 8a^2\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} (\cot^4\theta - \cot^2\theta)\, d\theta.$$

$$\boxed{\int \cot^m\theta\, d\theta.}$$

Ans: $8a^2\left(\frac{\pi}{2} - \frac{5}{3}\right).$

→ Apply change of variables to polar coordinates to evaluate $\int_0^2\int_0^{\sqrt{2a-x^2}} (x^2+y^2)\,dy\,dx.$ Ans: $\frac{3\pi}{2}.$

→ By using the transformations $x+y=u$ & $y=uv$

Evaluate $\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\,dy\,dx.$

Sol: $x+y=u \quad , y=uv$

$$x + uv = u$$

$$\Rightarrow x = u - uv$$

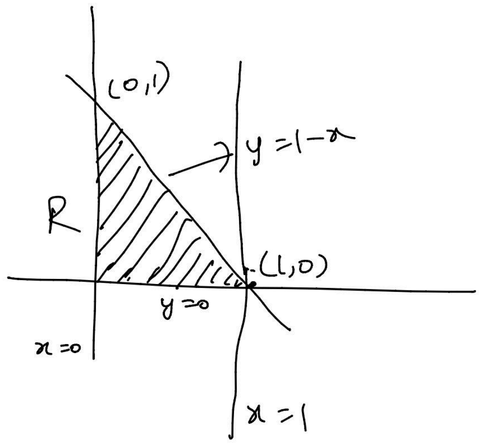
$$y = uv$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u$$

$$J = u$$

$$dx\,dy = |J|\,du\,dv = u\,du\,dv$$

$x = 0,\ x = 1,\ y = 0,\ y = 1-x$



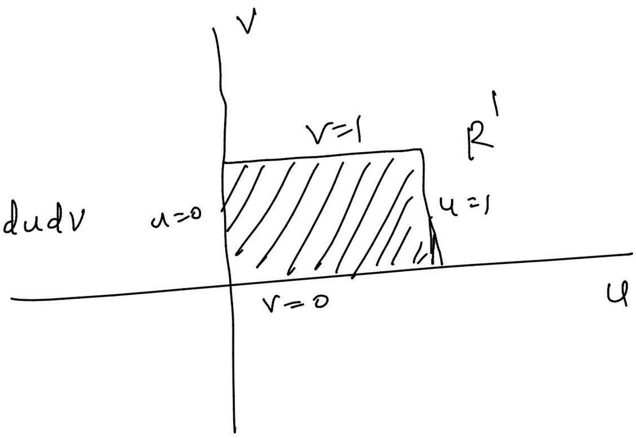
$x + y = u, \quad y = uv$

$y = 0 \Rightarrow uv = 0 \Rightarrow u = 0$ (or) $v = 0$

$y = 1-x \Rightarrow x+y = 1 \Rightarrow u = 1$

$x = 0 \Rightarrow u(1-v) = 0 \Rightarrow u = 0,\ v = 1$

$u = 0,\ u = 1,\ v = 0,\ v = 1$



$$\int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}}\,dy\,dx = \int_{\cancel{u}=0}^{1} \int_{u=0}^{1} e^{\frac{\cancel{u}v}{\cancel{u}}}\,u\,du\,dv$$

$$= \int_{v=0}^{1} \int_{u=0}^{1} e^{v}\,u\,du\,dv$$

$$= \int_{v=0}^{1} e^{v}\left(\frac{u^2}{2}\right)_{u=0}^{1} dv = \frac{1}{2}\int_{v=0}^{1} e^{v}\,dv = \frac{1}{2}\left(e^{v}\right)_0^1$$

$$= \frac{1}{2}(e-1).$$

→ Evaluate $\iint_R \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dx\,dy$ over the first quadrant of the

Ans: $\frac{\pi ab}{8}$

R Ans: $\frac{\pi ab}{8}$

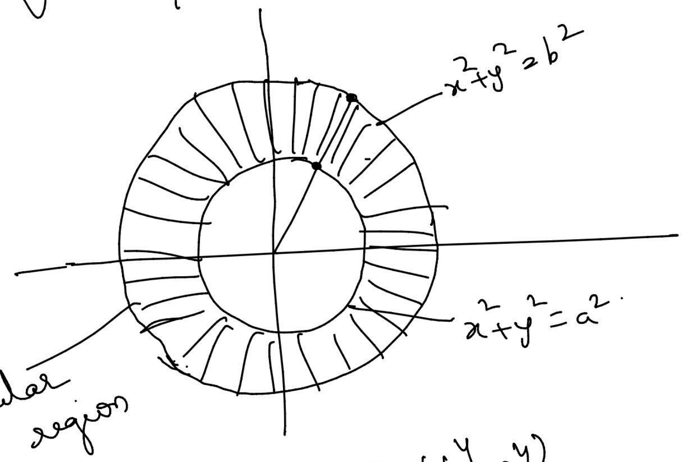
ellipse using $x = au$, $y = bv$.

→ Evaluate $\iint \frac{x^2 y^2}{x^2+y^2}\,dx\,dy$ over the annular region, b/w $x^2+y^2=a^2$ & $x^2+y^2=b^2$, $b>a$, by changing to polar coordinates.

Sol: $x^2+y^2=a^2$ $\quad x^2+y^2=b^2$

$r=a \quad r=b$

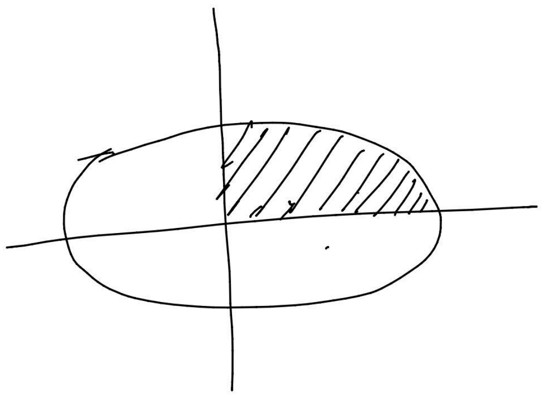
$\theta=0$ to $\theta=2\pi$



Ans: $\frac{\pi}{16}(b^4-a^4)$.

→ $\iint\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dx\,dy$ $\quad x=au,\ y=bv$

Sol $J=\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$



$x = 0$ to a

$y = 0$ to $b\sqrt{1-\frac{x^2}{a^2}}$

$u=0$ to 1

$v=0$ to $\sqrt{1-u^2}$

$\iint_{R'}\left(1-\frac{a^2u^2}{a^2}-\frac{b^2v^2}{b^2}\right)ab\,du\,dv$

$\iint_{R'}(1-u^2-v^2)\,ab\,du\,dv.$

$\frac{4ab}{15}$ ✓

Triple integrals

19 December 2020 10:06

Let $f(x,y,z)$ be defined over the volume V.

Now divide the volume in to sub volumes $\delta V_1, \delta V_2, \delta V_3, \ldots \delta V_n$

Let (x_i, y_i, z_i) be any point in the volume δV_i

Consider the sum

$$f(x_1,y_1,z_1)\,\delta V_1 + f(x_2,y_2,z_2)\,\delta V_2 + \cdots\cdots + f(x_n,y_n,z_n)\,\delta V_n$$

$$= \sum_{i=1}^{n} f(x_i,y_i,z_i)\,\delta V_i$$

If the above sum tends to a finite limit as $n \to \infty$, such that $\delta V_i \to 0$, then that limit is called as triple integral & denoted by $\iiint_V f(x,y,z)\,dV$ (or) $\iiint_V f(x,y,z)\,dx\,dy\,dz$.

→ If $f(x,y,z) = 1$, then $\iiint f(x,y,z)\,dx\,dy\,dz$ represents volume

→ If $f(x,y,z) \neq$ constant, then $\iiint f(x,y,z)\,dx\,dy\,dz$ represents hyper volume (a volume in n-dimensions).

→ Evaluation of triple integrals :

1) $\iiint_V f(x,y,z)\,dx\,dy\,dz$, $V: a_1 \le x \le a_2,\ b_1 \le y \le b_2,\ c_1 \le z \le c_2$.

then in this case, we can either integrate w.r.to 'x', or 'y' or 'z' first by treating the remaining variables constant.

2) If $V: a_1 \le x \le a_2,\ y_1(x) \le y_2 \le y_2(x),\ z_1(x,y) \le z \le z_2(x,y)$

then we first integrate w.r.to z keeping x, y constant & then w.r.to y, keeping x constant & later w.r.to 'x'.

→ $\displaystyle\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\,dz\,dx\,dy$.

$\int uv = u\int v - \int u' \int v$

$\int \log z\,dz = z\log z - \int \frac{1}{z}\cdot z\,dz$

$\int_1 \int_0^{\log y} \int^{e^x}$

$$\int_{1}\int_{1}\int_{1}$$

Sol: $$\int_{y=1}^{e}\int_{x=1}^{\log y}\int_{z=1}^{e^x} \log z \cdot 1\, dz\, dx\, dy$$

$$\int \log z\, dz = z\log z - \int \frac{1}{z}\cdot z\, dz$$
$$= z\log z - z$$

$$= \int_{y=1}^{e}\int_{x=1}^{\log y} \left(z\log z - z\right)_{z=1}^{e^x} dx\, dy$$

$$= \int_{y=1}^{e}\int_{x=1}^{\log y} \left[\left(e^x x - e^x\right) - (0-1)\right] dx\, dy$$

$$= \int_{y=1}^{e} \left(x e^x - e^x - e^x + x\right)_{x=1}^{\log y} dy$$

$$= \int_{y=1}^{e} \left[\left(y\log y - y - y + \log y\right) - \left(\cancel{e} - \cancel{e} - e + 1\right)\right] dy$$

$$= \int_{y=1}^{e} \left(y\log y - 2y + \log y + e - 1\right) dy$$

$$= \left[\log y\, \frac{y^2}{2} - \int \frac{1}{\cancel{y}}\cdot \frac{y^{\cancel{2}}}{2} dy - y^2 + y\log y - y + ey - y\right]_1^e$$

$$= \left[\frac{y^2}{2}\log y - \frac{y^2}{4} - y^2 + y\log y - 2y + ey\right]_1^e$$

$$= \frac{1}{4}\left[e^2 - 8e + 13\right].$$

$\rightarrow$ $$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx.$$ Ans: $\frac{1}{48}$

$\rightarrow$ $$\int_{0}^{1}\int_{0}^{1-x}\int_{0}^{1-x-y} dx\, dy\, dz.$$

$\boxed{\text{Geogebra 3D calculator}}$

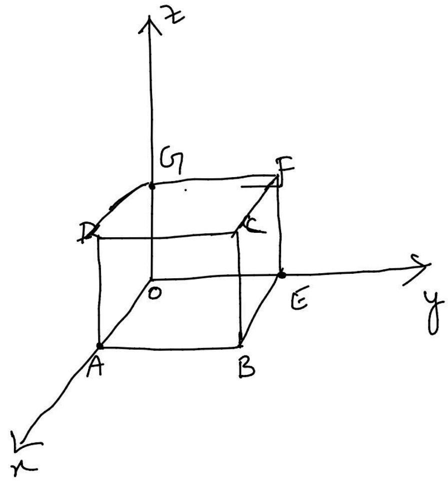
1 octant

$$\int_{x=0}\int_{y=0}\int_{z=0} dx\,dy\,dz$$

$$Ans = \frac{1}{6}.$$

$\boxed{octant}$

→ Evaluate $\iiint_V (xy+yz+zx)\,dx\,dy\,dz$ where V is the region of the space bounded by $\underbrace{x=0,\ y=0,\ z=0}_{\text{coordinate planes}}$ & $x=1, y=1,$ & $z=1$



Sol: Eq of xy-plane is $z=0$ (AOEB)

" yz-plane is $x=0$ (GOEF)

" xz plane is $y=0$ (DGOA

ABCD —— $x=1$

BEFC —— $y=1$

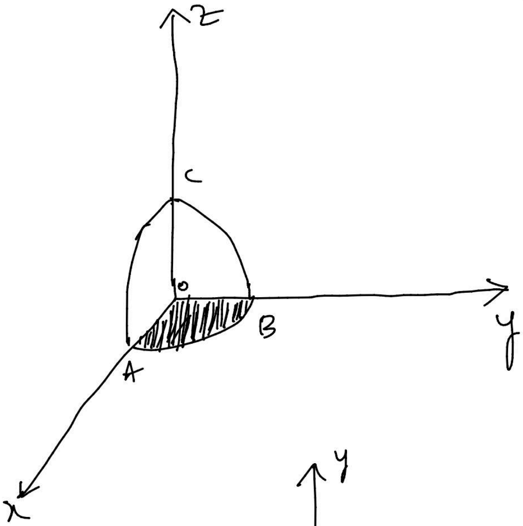
DGFC —— $z=1$

$$\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (xy+yz+zx)\,dz\,dy\,dx.$$

Ans:

→ Evaluate $\iiint xyz\,dx\,dy\,dz$ over the positive octant of the sphere. $x^2+y^2+z^2=a^2$

Sol:

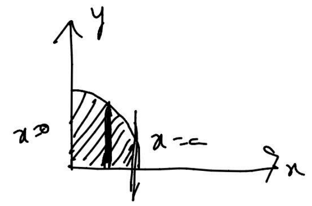


$x=0$ to $x=a$

$y=0$ to $y=\sqrt{a^2-x^2}$

$z=0$ to $z=\sqrt{a^2-x^2-y^2}$

$$\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int^{\sqrt{a^2-x^2-y^2}} xyz\,dz\,dy\,dx$$



on xy-plane $z=0$

$\iiint xyz\, dz\, dy\, dx$

on xy-plane $z=0$

$x^2+y^2+z^2=a^2$

$\Rightarrow x^2+y^2=a^2$

$x=0$, $y=0$, $z=0$

$$\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\left(\frac{z^2}{2}\right)_{z=0}^{\sqrt{a^2-x^2-y^2}} dy\, dx$$

$$= \frac{1}{2}\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\,(a^2-x^2-y^2)\, dy\, dx = \frac{1}{2}\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} (a^2xy - x^3y - xy^3)\, dy\, dx$$

$$= \frac{1}{2}\int_{x=0}^{a}\left(a^2x\frac{y^2}{2} - x^3\frac{y^2}{2} - x\frac{y^4}{2}\right)_{y=0}^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_{x=0}^{a}\left[\underline{\frac{a^2x}{2}(a^2-x^2)} - \frac{x^3}{2}\underline{(a^2-x^2)} - \frac{x}{2}\underline{(a^2-x^2)^2}\right] dx$$

$$= \frac{1}{4}\int_{x=0}^{a} a^2x(a^2-x^2)\, dx - \frac{1}{4}\int_0^a x^3(a^2-x^2)\, dx - \frac{1}{4}\int_0^a x(a^2-x^2)^2\, dx$$

$a^2-x^2=t^2$

$x=a\sin\theta$.

Ans: $\dfrac{a^6}{48}$.

→ $\displaystyle\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ over the volume bounded by the tetrahedron $x=0$, $y=0$, $z=0$ & $x+y+z=1$.

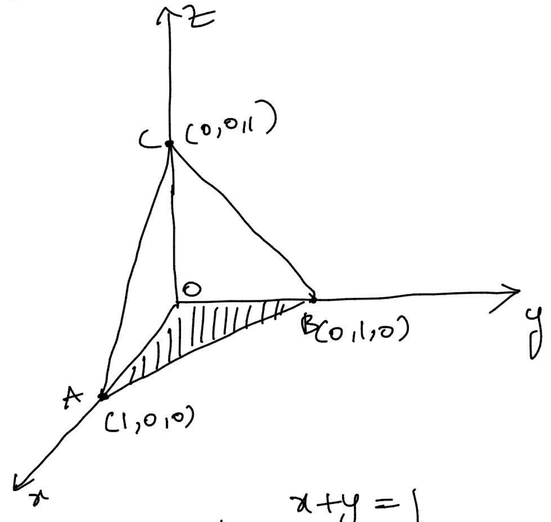
Sol:

OAB — $z=0$

OBC → $x=0$

OAC → $y=0$

ABC — $x+y+z=1$

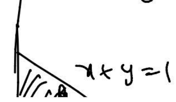


$x=0$ to 1

$y=0$ to $1-x$

$z=0$ to $z=1-x-y$.

$x+y=1$



$\int_0^1 \int_0^{1-x} \int_0^{1-x-y}$

$z = 0$ to $z = 1 - x - y$.

$$\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} \frac{dz\,dy\,dx}{(x+y+z+1)^3}$$

$$\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} (x+y+z+1)^{-3}\,dz\,dy\,dx = \int_{x=0}^{1}\int_{y=0}^{1-x}\left[\frac{(x+y+z+1)^{-2}}{-2}\right]_{z=0}^{1-x-y} dy\,dx$$

$$= \frac{-1}{2}\int_{x=0}^{1}\int_{y=0}^{1-x}\left[(x+y+1-x-y+1)^{-2} - (x+y+0+1)^{-2}\right] dy\,dx.$$

$$= -\frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{1-x}\left[\frac{1}{4} - (x+y+1)^{-2}\right] dy\,dx$$

$$= \frac{-1}{2}\int_{x=0}^{1}\left[\frac{1}{4}y - \frac{(x+y+1)^{-1}}{-1}\right]_{y=0}^{1-x} dx = \frac{-1}{2}\int_{x=0}^{1}\left[\frac{1}{4}(1-x) + (x+1-x+1)^{-1} - (x+1)^{-1}\right] dx$$

$$= \frac{-1}{2}\left[\frac{1}{4}\left(x - \frac{x^2}{2}\right) - \frac{x}{2} - \log(x+1)\right]_{x=0}^{1}$$

$$= -\frac{1}{2}\left[\frac{1}{4}\times\frac{1}{2} - \frac{1}{2} - \log 2 - 0 - 0 - 0\right]$$

$$= -\frac{1}{2}\left[\frac{1}{8} - \frac{1}{2} - \log 2\right] = \frac{3}{16} + 4\log 2$$

→ $\iiint z^2\,dx\,dy\,dz$ taken over the volume bounded by surfaces $\underbrace{x^2+y^2=a^2}_{\text{cylinder}}$, $\underbrace{x^2+y^2=z}_{\text{paraboloid}}$ & $\underbrace{z=0}_{xy\text{-plane.}}$

surface

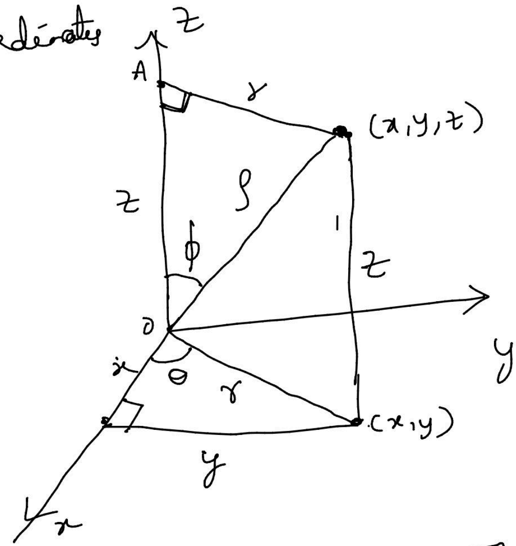
cylinder paraboloid xy-plane.

Sol

$$\left.\begin{aligned} &z=0 \qquad z=x^2+y^2 \\ &y=\pm\sqrt{a^2-x^2} \\ &x=-a \text{ to } a. \end{aligned}\right\}$$

→ **Change of variables to spherical coordinates:**

Let (ρ,θ,ϕ) be spherical coordinates



$\sin\phi=\dfrac{r}{\rho} \Rightarrow r=\rho\sin\phi$

$\cos\phi=\dfrac{z}{\rho} \Rightarrow z=\rho\cos\phi$

$\sin\theta=\dfrac{y}{r} \Rightarrow y=r\sin\theta$

$\cos\theta=\dfrac{x}{r} \Rightarrow x=r\cos\theta$

$x=r\cos\theta=\rho\cos\theta\sin\phi$

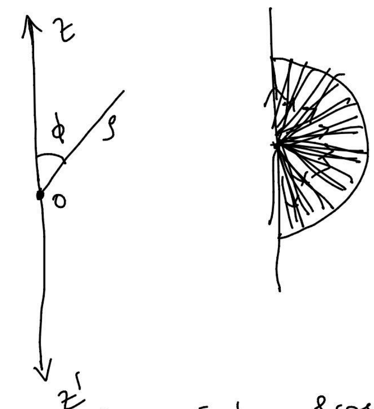
$y=r\sin\theta=\rho\sin\theta\sin\phi$

$z=\rho\cos\phi$

$$\Rightarrow \boxed{\begin{aligned} x&=\rho\cos\theta\sin\phi \\ y&=\rho\sin\theta\sin\phi \\ z&=\rho\cos\phi \end{aligned}}$$

ρ,θ,ϕ

$\rho\ge 0, \quad 0\le\phi\le\pi \qquad 0\le\theta\le 2\pi$



$(x,y,z)\to(\rho,\theta,\phi)$

$$J=\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=\begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{vmatrix}=\begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ & & -\rho\sin\phi \end{vmatrix}$$

$$J = \frac{\partial(\ldots)}{\partial(\rho,\theta,\phi)} \quad \begin{vmatrix} \ldots \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \ldots \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \ldots \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$J = \cos\phi\left[-\rho^2\sin^2\theta\sin\phi\cos\phi - \rho^2\cos^2\theta\sin\phi\cos\phi\right] - \rho\sin\phi\left[\rho\cos^2\theta\sin^2\phi + \rho\sin^2\theta\sin^2\phi\right]$$

$$= -\rho^2\cos^2\phi\sin\phi - \rho^2\sin^3\phi$$

$$= -\rho^2\sin\phi\left[\cos^2\phi + \sin^2\phi\right] = -\rho^2\sin\phi$$

$$J = -\rho^2\sin\phi$$

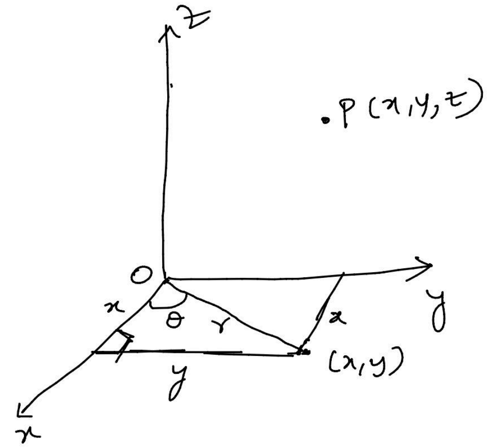
$$\iiint f(x,y,z)\,dx\,dy\,dz = \iiint F(\rho,\theta,\phi)\,|J|\,d\rho\,d\theta\,d\phi$$

$$= \iiint F(\rho,\theta,\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

$$\boxed{\begin{aligned} x &= \rho\cos\theta\sin\phi \\ y &= \rho\sin\theta\sin\phi \\ z &= \rho\cos\phi \end{aligned} \quad , \quad dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi.}$$

→ <u>Change of variables to cylindrical coordinates:</u>

cylindrical coordinates → (r,θ,z)



$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

$z = z$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}}$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \partial z & \partial z & \partial z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\frac{\partial(r,\theta,z)}{} \quad \begin{vmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & 1 \end{vmatrix}$$

$J = r$

(r, θ, z)

$r \to$ radius of cylinder

$\theta \to 0 \le \theta \le 2\pi$

$z \to$ height of the cylinder.

$\to \displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$ by changing to spherical coordinates.

Sol

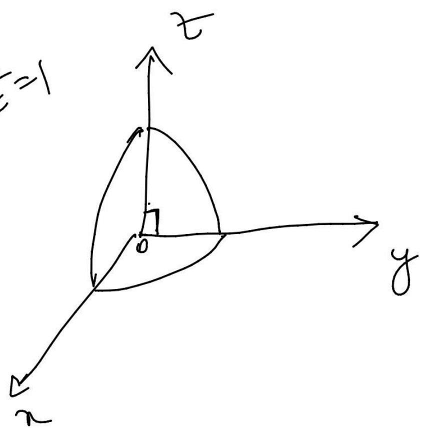
$x = 0$ to 1

$y = 0$ to $\sqrt{1-x^2}$

$z = 0$ to $\sqrt{1-x^2-y^2}$

$z = \sqrt{1-x^2-y^2}$

$x^2+y^2+z^2 = 1$



$x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$

$dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

$x^2+y^2+z^2 = \rho^2$

$\rho \to 0$ to 1

$\phi \to 0$ to $\frac{\pi}{2}$

$\theta \to 0$ to $\frac{\pi}{2}$

$$I = \int_{\rho=0}^{1} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \frac{\rho^2\sin\phi\,d\rho\,d\theta\,d\phi}{\sqrt{1-(x^2+y^2+z^2)}}$$

$$\int_{\rho=0}^{1} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \frac{\rho^2\sin\phi\,d\phi\,d\theta\,d\rho}{\sqrt{1-\rho^2}}$$

$$\int_0^1 \int^{\pi/2} \frac{-\rho^2}{\sqrt{1-\rho^2}} \left(\cos\phi\right)_{\phi=0}^{\pi/2} d\theta\,d\rho$$

$$\int_{\rho=0} \int_{\theta=0} \frac{-\rho\,(\cos\,)}{\sqrt{1-\rho^2}} \Big)_{\phi=0}$$

$$\int_{\rho=0}^{1} \int_{\theta=0}^{\pi/2} \frac{\rho^2}{\sqrt{1-\rho^2}}\, d\theta\, d\rho$$

$$\int_{\rho=0}^{1} \frac{\rho^2}{\sqrt{1-\rho^2}} \left(\theta\right)_{\theta=0}^{\pi/2} d\rho = \frac{\pi}{2} \int_{\rho=0}^{1} \frac{\rho^2+1-1}{\sqrt{1-\rho^2}}\, d\rho.$$

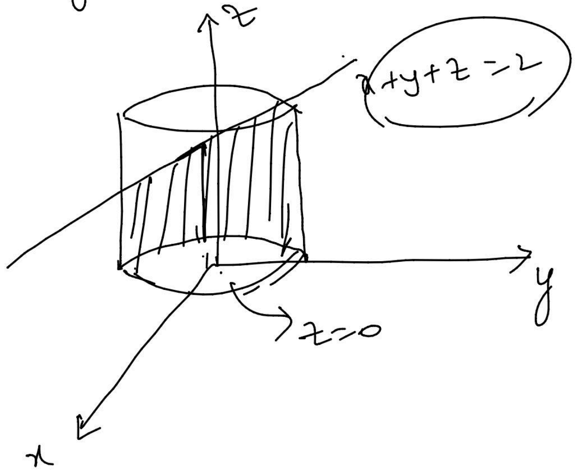
$$= \frac{\pi}{2}\left[\int_{\rho=0}^{1} \frac{1}{\sqrt{1-\rho^2}}\, d\rho - \int_{\rho=0}^{1} \sqrt{1-\rho^2}\, d\rho \right]$$

$$= \quad \underline{\text{Ans}} = ?$$

→ Evaluate $\iiint z\, dv$ over the region b/w the two planes $x+y+z=2$, & $z=0$ & the cylinder $x^2+y^2=1$

Sol:

$z=0 \quad , \quad z = 2-x-y$



$x = r\cos\theta$

$y = r\sin\theta$

$z = z \qquad , J = r$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

$r \to 0$ to 1

$\theta \to 0$ to 2π

$z = 0$ to $2-x-y = 2 - r\cos\theta - r\sin\theta.$

$$\int_{r=0}^{1} \int_{\theta=0}^{2\pi} \int_{z=0}^{2-r\cos\theta-r\sin\theta} z\, r\, dz\, d\theta\, dr = \int_{r=0}^{1} \int_{\theta=0}^{2\pi} r \left(z\right)_{z=0}^{2-r\cos\theta-r\sin\theta} d\theta\, dr$$

$$\int^{1} \int^{2\pi}$$

$z=0 \qquad \theta=0$

$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} r x\left[2 - r\cos\theta - r\sin\theta\right] d\theta\, dr$$

$$\int_{r=0}^{1}\int_{\theta=0}^{2\pi}\left(2r^2\cos\theta - r^3\cos^2\theta - r^3\sin\theta\cos\theta\right] d\theta\, dr$$

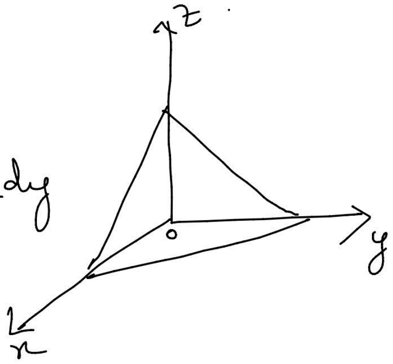
$\frac{1+\cos 2\theta}{2}$

$$\int_{r=0}^{1}\left[2r^2(\sin\theta) - r^3\left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right) + \frac{r^3}{4}\cos 2\theta\right]_{\theta=0}^{2\pi} dr$$

→ using spherical coordinates, evaluate $\iiint xyz\, dx\,dy\,dz$ taken over the volume bounded by the sphere in the first octant.

→ using cylindrical coordinates, evaluate $\iiint (x^2+y^2+z^2)\, dx\,dy\,dz$ over the region bounded by $0 \le z \le x^2+y^2 \le 1$.

→ find the volume of the tetrahedron bounded by the planes $x=0, y=0, z=0, \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$



sol: $V = \iiint dx\,dy\,dz$ (or) $V = \iint f(x,y)\,dx\,dy$

$z = 0$ to $c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

$y = 0$ to $b\left(1-\frac{x}{a}\right)$

$x = 0$ to a

$$V = \int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})}\int_{z=0}^{c(1-\frac{x}{a}-\frac{y}{b})} dz\,dy\,dx. \Rightarrow V = \int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})}\left(z\right)_{z=0}^{c(1-\frac{x}{a}-\frac{y}{b})} dy\,dx$$

$x=0 \quad y=0$

$$V = c\int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})} \left(1-\frac{x}{a}-\frac{y}{b}\right) dy\, dx$$

$$V = c\int_{x=0}^{a} \left(y - \frac{xy}{a} - \frac{y^2}{2b}\right)_{y=0}^{b(1-\frac{x}{a})} dx$$

$$V = c\int_{x=0}^{a} \left[b\left(1-\frac{x}{a}\right) - \frac{x}{a}b\left(1-\frac{x}{a}\right) - \frac{1}{2\cancel{b}}b^{\cancel{2}}\left(1-\frac{x}{a}\right)^2\right] dx$$

$$V = c\left[bx - \frac{bx^2}{2a} - \frac{b}{a}\frac{x^2}{2} + \frac{b}{a^2}\frac{x^3}{3} - \frac{b}{2}\left(x - \frac{x^2}{a} + \frac{x^3}{3a^2}\right)\right]_0^a$$

$$V = \frac{abc}{6} \text{ cubic units.}$$

(or)

$$V = \iint f(x,y)\, dx\, dy \qquad z = c\left(1-\frac{x}{a}-\frac{y}{b}\right)$$

$$V = \iint c\left(1-\frac{x}{a}-\frac{y}{b}\right) dx\, dy$$

→ find the volume bounded by xy-plane, the cylinder $x^2+y^2=1$ & the plane $x+y+z=3$

$z=0$

Sol: $V = \iiint dx\, dy\, dz$

$z = 0$ to $3-x-y$

$y = -\sqrt{1-x^2}$ to $\sqrt{1-x^2}$

$x = -1$ to 1

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

$r \to 0$ to 1

$\theta \to 0$ to 2π

$z \to 0$ to $3-x-y$

$dx\,dy\,dz = \ldots$

$\theta \to 0$ to 2π

$z \to 0$ to $3 - x - y$

0 to $3 - r\cos\theta - r\sin\theta$

$$V = \iiint r\,dr\,d\theta\,dz$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{z=0}^{3-r\cos\theta-r\sin\theta} r\,dz\,d\theta\,dr = \int_{r=0}^{1}\int_{\theta=0}^{2\pi} r\left(z\right)_{z=0}^{3-r\cos\theta-r\sin\theta} d\theta\,dr$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} r(3 - r\cos\theta - r\sin\theta)\,d\theta\,dr$$

$$= \int_{r=0}^{1}\left(3r\theta - r^2\sin\theta + r^2\cos\theta\right)_{\theta=0}^{2\pi} dr$$

$$= \int_{0}^{1}\left[\left(3r(2\pi) - 0 + \cancel{r^2}\right) - \left(0 - 0 + \cancel{r^2}\right)\right] dr$$

$$= \int_{0}^{1} 6r\pi\,dr = 6\pi\left(\frac{r^2}{2}\right)_{0}^{1} = 3\pi \text{ cubic units.}$$

Beta and Gamma Functions

22 December 2020 10:58

$\int_a^b f(x)\,dx$ — improper integral if either $a=\infty$ or $b=\infty$ or a & $b=\infty$ (or) a, b are finite but $f(x)$ becomes undefined in b/w a & b.

Improper integral of 1st kind

$\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^b f(x)\,dx$ (or) $\int_{-\infty}^\infty f(x)\,dx$

Improper integral of second kind

$\int_a^b f(x)\,dx$

a & b are finite but $f(x)$ is undefined at some $c \in (a,b)$

→ Beta function:

An improper integral of the form $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ $m>0, n>0$ is called as Beta function & it denoted by $\beta(m,n)$ or $B(m,n)$.

$$\boxed{B(m,n) = \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx} \quad m>0, n>0$$

→ Beta function converges for $0<m<1$ & $0<n<1$

Properties:

1) Symmetry property: $\beta(m,n) = \beta(n,m)$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

put $1-x = y \Rightarrow -dx = dy$

$x \to 0 \Rightarrow y \to 1$ & $x \to 1 \Rightarrow y \to 0$

$$\beta(m,n) = \int_1^0 (1-y)^{m-1} y^{n-1}(-dy) = \int_0^1 y^{n-1}(1-y)^{m-1}\,dy$$

$$= \beta(n,m)$$

$$\beta(m,n) = \beta(n,m)$$

$$\beta(m,n) = \beta(n,m)$$

2) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$.

Sol: $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$

$x \to 0 \Rightarrow \theta \to 0$

$x \to 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\,d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$$

$\rightarrow$ $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

Sol $\beta(m+1,n) + \beta(m,n+1) = \int_0^1 x^{m+1-1}(1-x)^{n-1}\,dx + \int_0^1 x^{m-1}(1-x)^{n+1-1}\,dx$

$$= \int_0^1 x^m (1-x)^{n-1}\,dx + \int_0^1 x^{m-1}(1-x)^n\,dx.$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\left[x+1-x\right]dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(m,n).$$

$\rightarrow$ $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

Sol: $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$

put $x = \frac{1}{1+y} \Rightarrow dx = \frac{-dy}{(1+y)^2}$

$x \to 0 \Rightarrow y \to \infty$

$x \to 1 \Rightarrow y \to 0$

$$\beta(m,n) = \int_{\infty}^{0} \left(\frac{1}{1+y}\right)^{m-1} \left(1 - \frac{1}{1+y}\right)^{n-1} \left(\frac{-dy}{(1+y)^2}\right)$$

$$= \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}}\, dy$$

$$= \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{n+m}}\, dy \qquad (\because \beta(m,n) = \beta(n,m))$$

→ show that $\displaystyle\int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right).$

Sol: $\displaystyle\int_0^{\pi/2} \sin^{(m-1)}\theta \cos^{(n-1)}\theta\, (\sin\theta\cos\theta)\, d\theta.$

$$\int_0^{\pi/2} (\sin^2\theta)^{\frac{m-1}{2}} (\cos^2\theta)^{\frac{n-1}{2}} \sin\theta\cos\theta\, d\theta.$$

let $\sin^2\theta = x$ $\qquad\qquad m-1 \quad n-1$

$\sin\theta\cos\theta\, d\theta = \frac{dx}{2}$

$\theta \to 0 \Rightarrow x \to 0$

$\theta \to \frac{\pi}{2} \Rightarrow x \to 1$

$$\int_0^1 x^{\frac{m-1}{2}} (1-x)^{\frac{n-1}{2}} \frac{dx}{2} = \frac{1}{2}\int_0^1 x^{\frac{m-1}{2}} (1-x)^{\frac{n-1}{2}}\, dx.$$

$$= \frac{1}{2}\int_0^1 x^{\frac{m+1}{2}-1} (1-x)^{\frac{n+1}{2}-1}\, dx$$

$$= \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right).$$

→ Express $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^2}}\, dx$ in terms of β-function.

sol:
$$\int_0^1 x(1-x^2)^{-1/2}dx \qquad x \quad (1-x)$$

let $x^2 = y$

$$dx = \frac{dy}{2x} = \frac{dy}{2\sqrt{y}} = \frac{dy}{2} y^{-1/2}$$

$x \to 0 \Rightarrow y \to 0$ $\qquad 0$

$x \to 1 \Rightarrow y \to 1$

$$\int_0^1 y^{1/2}(1-y)^{-1/2}\frac{dy}{2}y^{-1/2} = \frac{1}{2}\int_0^1 y^0 (1-y)^{-1/2}dy$$

$$= \frac{1}{2}\int_0^1 y^{1-1}(1-y)^{\frac{1}{2}-1}dy$$

$$\frac{1}{2}\beta\left(1,\frac{1}{2}\right).$$

$\to$ $$\int_0^3 \frac{dx}{\sqrt{9-x^2}}$$

sol:
$$\int_0^3 (9-x^2)^{-1/2}dx.$$

$$\frac{1}{3}\int_0^3 \left(1-\frac{x^2}{9}\right)^{-1/2}dx$$

$$\frac{x^2}{9} = y \qquad \Rightarrow x^2 = 9y$$

$$x = 3y^{1/2}$$

$$dx = \frac{3}{2}y^{-1/2}dy$$

$x \to 0 \Rightarrow y \to 0$

$x \to 3 \Rightarrow y \to 1$

$$\frac{1}{3}\int_0^1 (1-y)^{-1/2}\frac{3}{2}y^{-1/2}dy$$

$$\frac{1}{3}\times\frac{3}{2}\int_0^1 y^{-1/2}(1-y)^{-1/2}dy$$

$$\frac{1}{2}\int_0^1 y^{\frac{1}{2}-1}(1-y)^{\frac{1}{2}-1}\,dy = \frac{1}{2}\beta\left(\frac{1}{2},\frac{1}{2}\right).$$

$\rightarrow$ $$\int_0^a (a-x)^{m-1}x^{n-1}\,dx = a^{m+n-1}\beta(m,n).$$

$\rightarrow$ $$\int_0^1 x^m(1-x^n)^p\,dx = \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right).$$

$\rightarrow$ $$\frac{\beta(p,q+1)}{q} = \frac{\beta(p+1,q)}{p} = \frac{\beta(p,q)}{p+q}, \quad p>0,\ q>0.$$

Sol: $$\frac{\beta(p,q+1)}{q} = \frac{1}{q}\int_0^1 x^{p-1}(1-x)^{q+1-1}\,dx$$

$$= \frac{1}{q}\int_0^1 \underset{v}{x^{p-1}}\ \underset{u}{(1-x)^q}\,dx.$$

$$= \frac{1}{q}\left[\left((1-x)^q\frac{x^p}{p}\right)_0^1 - \int_0^1 q(1-x)^{q-1}(-1)\frac{x^p}{p}\,dx\right]$$

$$= \frac{1}{q}\left[0 + \frac{q}{p}\int_0^1 x^p(1-x)^{q-1}\,dx\right]$$

$$\frac{1}{p}\cdot\int_0^1 x^{p+1-1}(1-x)^{q-1}\,dx$$

$$\frac{1}{p}\beta(p+1,q)$$

$\rightarrow$ <u>Gamma function</u>:

Gamma function is defined as

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx, \quad n>0.$$

$\rightarrow$ $\Gamma(1) = 1$

$$\Gamma(1) = \int_0^\infty e^{-x}x^{1-1}\,dx = \int_0^\infty e^{-x}\,dx = 1$$

$\rightarrow \Gamma(1) = 1$

$$\Gamma(1) = \int_0^\infty e^{-x} x^{1-1} dx = \int_0^\infty e^{-x} dx = 1$$

$\rightarrow \Gamma(n+1) = n\Gamma(n)$ (or) $\longrightarrow n > -1$

$\Gamma(n) = (n-1)\Gamma(n-1)$, $\longrightarrow n > 0$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\left[x^{n-1}\left(-e^{-x}\right)\right]_0^\infty - \int_0^\infty (n-1)\, x^{n-2} \left(-e^{-x}\right) dx$$

$$0 + (n-1) \int_0^\infty x^{n-2} e^{-x} dx$$

$$(n-1) \int_0^\infty e^{-x} x^{(n-1)-1} dx = (n-1)\,\Gamma(n-1).$$

$\rightarrow \Gamma(n+1) = n\Gamma(n)$ //

$\rightarrow \Gamma(n) = (n-1)(n-2)(n-3) \cdots\cdots (n-r)\,\Gamma(n-r)$, $n-r > 0$

$\rightarrow \Gamma(n+1) = n!$, when 'n' is a positive integer.

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n)\\ &= n\Gamma(n-1+1) = n(n-1)\Gamma(n-1)\\ &= n(n-1)(n-2)\Gamma(n-2)\\ &\;\vdots\end{aligned}$$

$(n-r) > 0$

$$n(n-1)(n-2)(n-3)(n-4)\cdots\cdots 3\cdot 2\cdot 1\,\Gamma(1)$$

$$n(n-1)(n-2)(n-3)\cdots\cdots 3\cdot 2\cdot 1 = n!$$

$\rightarrow$ Relation between Beta and Gamma functions:

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad m > 0,\ n > 0$$

proof.

$$\Gamma(m) = \int_0^\infty e^{-x} x^{m-1} dx$$

Let $x = ut$

proof:

let $x = yt$

$dx = y\,dt$

$$\Gamma(m) = \int_0^\infty e^{-yt}(yt)^{m-1}\,y\,dt = \int_0^\infty y^m e^{-yt} t^{m-1}\,dt$$

$$\Gamma(m) = \int_0^\infty y^m e^{-yx} x^{m-1}\,dx$$

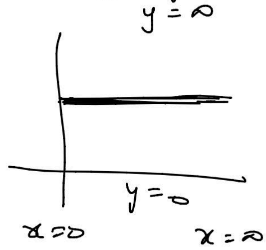
$$\frac{\Gamma(m)}{y^m} = \int_0^\infty e^{-yx} x^{m-1}\,dx \quad \text{——— (1)}$$

Multiply both sides with $e^{-y}.\,y^{m+n-1}$ & integrate w.r.to 'y' from 0 to ∞

$$\int_0^\infty \frac{\Gamma(m)}{y^m} e^{-y} y^{m+n-1}\,dy = \int_0^\infty \left\{ \int_0^\infty e^{-yx} x^{m-1} e^{-y} y^{m+n-1}\,dx \right\} dy$$

$$\Gamma(m)\int_0^\infty e^{-y} y^{n-1}\,dy = \int_0^\infty \left\{ \int_0^\infty e^{-y(1+x)} y^{m+n-1} x^{m-1}\,dx \right\} dy$$

$$\Gamma(m)\Gamma(n) = \int_0^\infty \left\{ \int_0^\infty e^{-y(1+x)} y^{m+n-1}\,dy \right\} x^{m-1}\,dx$$



(changing the order of integration).

$$\Gamma(m)\Gamma(n) = \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}} x^{m-1}\,dx \quad (\because \text{by (1)})$$

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \beta(m,n)$$

$$\Rightarrow \boxed{\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

$$\frac{\Gamma(m)}{y^m} = \int_0^\infty e^{-yx} x^{m-1}\,dx \quad \text{— (1)}$$

y	x	$m-1$	dx
$1+x$	y	$m+n-1$	dy

$$\frac{\Gamma(m+n)}{(1+x)^{m+n}}$$

$\rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\Gamma(m)\Gamma(n)$

$\rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Sol:

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$m = \frac{1}{2}, \quad n = \frac{1}{2}$$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{\Gamma(1)} = \left[\Gamma\left(\frac{1}{2}\right)\right]^2 \quad \text{--- (1)}$$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}\,dx = \int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx$$

let $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^{\pi/2} (\sin^2\theta)^{-1/2}(1-\sin^2\theta)^{-1/2}\, 2\sin\theta\cos\theta\,d\theta$$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\pi/2} d\theta = 2\left(\frac{\pi}{2}\right) = \pi$$

$$(1) \Rightarrow \beta\left(\frac{1}{2},\frac{1}{2}\right) = \left[\Gamma\left(\frac{1}{2}\right)\right]^2$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi \Rightarrow \boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

$$\rightarrow \boxed{\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}}$$

$$\rightarrow \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Sol

$$\left.\begin{aligned} \beta(m,n) &= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \\ \beta(m,n) &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \end{aligned}\right\}$$

$$\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

let $m+n=1$ then $m=1-n$

$$\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\Gamma(1-n)\,\Gamma(n)}{\Gamma(1)}$$

$$\Rightarrow \Gamma(n)\,\Gamma(1-n) = \int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}$$

$$\boxed{\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}}$$

→ $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

sol $\Gamma(n) = \int_0^\infty e^{-x}\,x^{n-1}\,dx$

$n=\frac{1}{2}$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-x}\,x^{\frac{1}{2}-1}\,dx \Rightarrow \sqrt{\pi} = \int_0^\infty e^{-x}\,x^{-1/2}\,dx.$$

let $x=y^2$ $x\to 0 \Rightarrow y\to 0$

$dx = 2y\,dy$ $x\to\infty \Rightarrow y\to\infty$

$$\sqrt{\pi} = \int_0^\infty e^{-y^2}\left(y^2\right)^{-1/2}2y\,dy$$

$$\boxed{\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-y^2}\,dy.}$$

→ $$\boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}$$

→ Compute $\Gamma\left(\frac{11}{2}\right)$ & $\Gamma\left(-\frac{1}{2}\right)$, $\Gamma\left(-\frac{7}{2}\right)$.

$\Gamma(n) = n(n-1)(n-2)\cdots(n-r)$
$(n-r)>0$

sol $\Gamma(n+1) = n\Gamma(n)$

Sol $\Gamma(n+1) = n\Gamma(n)$

$$\Gamma\left(\frac{11}{2}\right) = \Gamma\left(\frac{9}{2}+1\right) = \frac{9}{2}\Gamma\left(\frac{9}{2}\right)$$

$$= \frac{9}{2}\Gamma\left(\frac{7}{2}+1\right) = \frac{9}{2}\cdot\frac{7}{2}\Gamma\left(\frac{7}{2}\right) = \frac{9}{2}\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{9\cdot 7\cdot 5\cdot 3\cdot\sqrt{\pi}}{2^5}$$

→ $\Gamma\left(-\frac{1}{2}\right) = ?$

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} = -2\sqrt{\pi}$$

→ $n = 0$

$\Gamma(0) = \frac{\Gamma(0+1)}{0}$ → undefined

$\Gamma(-1) = \frac{\Gamma(-1+1)}{-1}$ → undefined

Ily $\Gamma(-2), \Gamma(-3), \Gamma(-4), \ldots$ are undefined.

→ $B(m+1, n) = \frac{m}{m+n}\beta(m,n)$, $m>0, n>0$

→ $\beta(m,n) = \beta(m+1,n) + \beta(m, n+1)$, $m>0, n>0$.

→ $\Gamma(n) = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx$, $n>0$.

Sol $\int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx$. $\qquad \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

$\log\frac{1}{x} = y$

$\Rightarrow e^y = \frac{1}{x} \Rightarrow -e^{-y} dy = -\frac{1}{x^2} dx$.

$\Rightarrow x = e^{-y}$

$$dx = -e^{-y}\,dy$$

$x \to 0 \Rightarrow y \Rightarrow \infty$

$x \to 1 \Rightarrow y \to 0$

$$\int_{\infty}^{0} (y)^{n-1}(-e^{-y}\,dy) = \int_{0}^{\infty} e^{-y} y^{n-1}\,dy. = \Gamma(n).$$

→ Evaluate

1) $\displaystyle\int_0^\infty x^{n-1} e^{-kx}\,dx.$ Ans : $\dfrac{\Gamma(n)}{k^n}$ $(n>0, k>0)$

2) $\displaystyle\int_0^\infty e^{-x^{\frac{1}{n}}}\,dx, \; n>0,$ Ans $n\,\Gamma(n).$

3) $\displaystyle\int_0^1 x^5 (1-x)^3\,dx$ Ans $\dfrac{1}{504}$

4) $\displaystyle\int_0^1 x^{5/2}(1-x^2)^{3/2}\,dx.$ Ans $\dfrac{8}{65}\cdot\dfrac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)}.$

→ $\displaystyle\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta\,d\theta$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$$

→ $\displaystyle\int_0^{\pi/2} \sin^7\theta\,d\theta$

$7 = 2m-1$

$2\times 4-1$

Sol: $\displaystyle\int_0^{\pi/2} \sin^{2(4)-1}\theta \cos^0\theta\,d\theta$

$$\int_0^{\pi/2} \sin^{2(4)-1}\theta \cos^{2\left(\frac{1}{2}\right)-1}\,d\theta$$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta = \frac{\beta(m,n)}{2}$$

$$= \frac{1}{2}\beta\left(4,\frac{1}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma(4)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(4+\frac{1}{2}\right)} = \frac{1}{2}\frac{3!\sqrt{\pi}}{\Gamma\left(\frac{9}{2}\right)}$$

$2!\,\sqrt{\cancel{\pi}} \times 2^4$

$$= \frac{1}{2}\,\frac{3!\,\cancel{\sqrt{\pi}} \times 2^4}{7\cdot5\cdot3\cdot1\,\cancel{\sqrt{\pi}}}$$

$\rightarrow$ $$\int_0^1 x^4\left(\log\frac{1}{x}\right)^3 dx.$$

$\rightarrow$ $$\int_0^\infty 3^{-4x^2}\,dx.$$

$$\int_0^\infty e^{\log_e 3^{-4x^2}}\,dx$$

$$= \int_0^\infty e^{-4x^2\log 3}\,dx.$$

put $2x\sqrt{\log 3} = y \Rightarrow 4x^2\log 3 = y^2$

$$dx = \frac{dy}{2\sqrt{\log 3}}$$

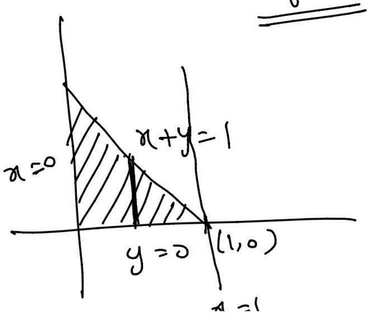
$$\int_0^\infty e^{-y^2}\frac{dy}{2\sqrt{\log 3}} = \frac{1}{2\sqrt{\log 3}}\int_0^\infty e^{-y^2}dy.$$

$$= \frac{1}{2}\,\frac{1}{\sqrt{\log 3}} \times \frac{\sqrt{\pi}}{2}.$$

$\rightarrow$ Prove that $$\iint_D x^p y^q\,dx\,dy = \frac{\Gamma(p+1)\,\Gamma(q+1)}{\Gamma(p+q+3)}.$$

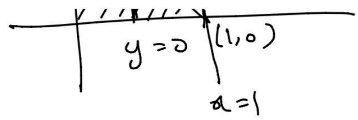
where $p>0, q>0$ & D is the domain $x \geq 0,\ y \geq 0$ & $\underline{x+y \leq 1}$.

Sol: $$\int_{x=0}^{1}\int_{y=0}^{1-x} x^p y^q\,dy\,dx$$



$$\int_0^1 x^p\left(\frac{y^{q+1}}{q+1}\right)_0^{1-x} dx$$

$$\int_{x=0}^{1} x^p \left(\frac{y^{q+1}}{q+1}\right)_{y=0}^{1-x} dx$$



$$\frac{1}{q+1}\int_{x=0}^{1} x^p (1-x)^{q+1}\, dx$$

$$\frac{1}{q+1}\int_{0}^{1} x^{(p+1)-1} (1-x)^{(q+2)-1}\, dx$$

$$\frac{1}{q+1}\beta(p+1, q+2) = \frac{1}{q+1}\,\frac{\Gamma(p+1)\Gamma(q+2)}{\Gamma(p+1+q+2)}$$

$$= \frac{1}{\cancel{q+1}}\,\frac{\Gamma(p+1)\cancel{(q+1)}\Gamma(q+1)}{\Gamma(p+q+3)}$$

$$= \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+3)}.$$

→ P.T $\displaystyle\iint_D x^{l-1} y^{m-1}\, dx\,dy = \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)}\, h^{l+m}$

where D : $x \geq 0$, $y \geq 0$ & $x+y \leq h$. $\boxed{X+Y \leq 1}$ ✓

Sol: let $x = Xh$ & $y = Yh$

then the domain D changes to D' : $X \geq 0$, $Y \geq 0$, $Xh + Yh \leq h$
$\Rightarrow X + Y \leq 1$

$$J = \frac{\partial(x,y)}{\partial(X,Y)} = \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} \end{vmatrix} = \begin{vmatrix} h & 0 \\ 0 & h \end{vmatrix} = h^2$$

$$dx\,dy = |J|\, dX\,dY$$
$$\Rightarrow dx\,dy = h^2\, dX\,dY$$

$$\iint x^{l-1} y^{m-1}\, dx\,dy = \int^{1}\!\!\int^{1-X} (Xh)^{l-1} (Yh)^{m-1} h^2\, dX\,dY$$

$$\iint x^{l-1} y^{m-1}\, dx\, dy = \int_{X=0}\int_{Y=0} (Xh)^{l-1} (Yh)^{m-1} h\, dX\, h\, dY$$

$$= \int_{X=0}^{1}\int_{Y=0}^{1-X} h^{l+m} X^{l-1} Y^{m-1}\, dY\, dX$$

$$= \int_{X=0}^{1} h^{l+m} X^{l-1} \left(\frac{Y^m}{m}\right)_{Y=0}^{1-X} dX$$

$$= \int_{X=0}^{1} \frac{h^{l+m}}{m} X^{l-1} (1-X)^m\, dX$$

$$= \frac{h^{l+m}}{m} \int_0^1 X^{l-1} (1-X)^m\, dX$$

$$= \frac{h^{l+m}}{m} \int_0^1 X^{l-1} (1-X)^{m+1-1}\, dX$$

$$= \frac{h^{l+m}}{m} \beta(l, m+1) = \frac{h^{l+m}}{m} \frac{\Gamma(l)\Gamma(m+1)}{\Gamma(l+m+1)}$$

$$= \frac{h^{l+m}}{\cancel{m}} \frac{\Gamma(l)\cancel{m}\Gamma(m)}{\Gamma(l+m+1)} = h^{l+m} \cdot \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)}$$

Dirichlet integral

→ show that
$$\boxed{\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}.}$$

where V is the region $x \geq 0,\ y \geq 0,\ z \geq 0$ & $x+y+z \leq 1$

Sol:

$x+y+z \leq 1$

$y+z \leq 1-x = h$ (say)

$y+z \leq h$

$z \leq h-y.$

$$\int_0^1 \int_0^{1-x} \int_{z=0}^{1-x-y} x^{l-1} y^{m-1} z^{n-1}\, dz\, dy\, dx.$$

$$\int_{x=0}^{1}\left\{\int_{y=0}^{h}\int_{z=0}^{h-y} x^{l-1} y^{m-1} z^{n-1}\, dz\, dy\right\} dx.$$

let $y = Yh$ & $z = Zh$.

$J = h^2$

$dy\,dz = h^2\, dY\, dZ$

$\boxed{Y = 0}$ for $y = 0$

$\boxed{Y = 1}$ for $y = h$

$\boxed{Z = 0}$ for $z = 0$

$\boxed{Z = 1-Y}$ for $z = h-y$.

$$\int_{x=0}^{1}\left[\int_{Y=0}^{1}\int_{Z=0}^{1-Y} x^{l-1}\, Y^{m-1} h^{m-1} Z^{n-1} h^{n-1} h^2\, dZ\, dY\right] dx$$

$$\int_{x=0}^{1}\int_{Y=0}^{1} x^{l-1} Y^{m-1} h^{m+n} \left(\frac{Z^n}{n}\right)_{Z=0}^{1-Y} dY\, dx$$

$$\frac{1}{n}\int_{x=0}^{1}\int_{Y=0}^{1} x^{l-1} Y^{m-1} h^{m+n} (1-Y)^n\, dY\, dx$$

$$\frac{1}{n}\int_{x=0}^{1} x^{l-1} h^{m+n}\left[\int_{Y=0}^{1} Y^{m-1}(1-Y)^{n+1-1}\, dY\right] dx.$$

$h = 1-x$

$$= \frac{1}{n}\int_{x=0}^{1} x^{l-1} h^{m+n}\, \beta(m, n+1)\, dx.$$

$$= \frac{1}{n}\int_{x=0}^{1} x^{l-1} (1-x)^{m+n}\, \beta(m, n+1)\, dx$$

$$\frac{\beta(m, n+1)}{n}\int_{x=0}^{1} x^{l-1}(1-x)^{m+n+1-1}\, dx.$$

$$\frac{\beta(m, n+1)}{n}\, \beta(l, m+n+1).$$

$\Gamma(m)\Gamma(n)\Gamma(l)$

$$\frac{\Gamma(m)\,\Gamma(n+1)}{n\,\Gamma(\cancel{m+n+1})} \cdot \frac{\Gamma(l)\,\Gamma(\cancel{m+n+1})}{\Gamma(l+m+n+1)} = \frac{\Gamma(m)\,\Gamma(n)\,\Gamma(l)}{\Gamma(l+m+n+1)}.$$

→ Evaluate $\iiint_V xyz\,dx\,dy\,dz$ taken over the volume of the tetrahedron given by $x \geq 0$, $y \geq 0$, $z \geq 0$ & $x+y+z \leq 1$.

Ans $\frac{1}{720}$.