$$\text{Re}\,sult\ :\int_0^\infty \frac{f(t)}{t}\,dt=\int_0^\infty F(s)\,ds. \qquad (1)$$

$F(s) = L\{f(t)\}$

$$Show\ \ that\ \int_0^\infty \frac{\sin t}{t}\,dt=\frac{\pi}{2}.$$

$Sol:-$

$$WKT\quad L[\sin t]=\frac{1}{s^2+1}=F(s)$$

$$\therefore \int_0^\infty \frac{\sin t}{t}\,dt=\int_0^\infty\left(\frac{1}{s^2+1}\right)ds=\left[\tan^{-1}s\right]_0^\infty=\frac{\pi}{2}.$$

Laplace Transform of Integrals : −

If $L\left[f(t)\right] = F(s)$, then $L\left[\int_0^t f(u)\,du\right] = \dfrac{F(s)}{s}$.

Proof : −

Let $g(t) = \int_0^t f(u)\,du$. Then $g'(t) = f(t)$ and $g(0) = 0$.

Taking the Laplace transform of both sides, we have

$$L\left[g'(t)\right] = L\left[f(t)\right]$$

$$sL\left[g(t)\right] - g(0) = F(s)$$

$$sL\left[g(t)\right] = F(s)$$

$$L\left[g(t)\right] = \frac{F(s)}{s}$$

$$\text{i.e. } L\left[\int_0^t f(u)\,du\right] = \frac{F(s)}{s}$$

$$Find \quad L\left[\int_0^t \frac{\sin u}{u}\,du\right]$$

Sol :-

$$W.K.T \quad L[\sin t] = \frac{1}{s^2+1}$$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \left(\frac{1}{s^2+1}\right) ds = \tan^{-1}\frac{1}{s}$$

$$= \left(\tan^{-1} s\right)_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} s$$

$$= \pi/2 - \tan^{-1} s$$

$$= \tan^{-1}\left(\frac{1}{s}\right)$$

$$\therefore L\left[\int_0^t \frac{\sin u}{u}\,du\right] = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

Evaluation of Integrals:-

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

Evaluate $(a)\ \int_0^\infty te^{-2t}\cos t\,dt$, $(b)\ \int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt$.

Solution :–

$$\int_0^\infty e^{-st}\, t\cos t\,dt$$

$$s = 2$$

(a)

$$L[t\cos t] = \int_0^\infty te^{-st}\cos t\,dt$$

$$= -\frac{d}{ds}L[\cos t] = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{s^2-1}{\left(s^2+1\right)^2}$$

$$= \frac{(s^2+1)(1)-s(2s)}{(s^2+1)^2}$$

Then letting $s = 2$, *we find* $\int_0^\infty te^{-2t}\cos t\,dt = \frac{3}{25}$.

$$= \frac{s^2+1-2s^2}{(s^2+1)^2} = \frac{-s^2+1}{(s^2+1)^2} = \frac{1-s^2}{(s^2+1)^2}$$

(b)

$$L\left[\frac{e^{-t}-e^{-3t}}{t}\right]=\int_s^\infty \left[L(e^{-t})-L(e^{-3t})\right]ds$$

$$\left\{\log\left[\frac{\cancel{s}(1+1/s)}{\cancel{s}(1+3/s)}\right]\right\}_s^\infty$$

$$=\int_s^\infty\left[\frac{1}{s+1}-\frac{1}{s+3}\right]ds$$

$$=\left[\log(s+1)-\log(s+3)\right]_s^\infty$$

$$=\left[\log\left(\frac{s+1}{s+3}\right)\right]_s^\infty=\left[0-\log\left(\frac{s+1}{s+3}\right)\right]$$

$$=\log\left(\frac{s+3}{s+1}\right)$$

$$or\quad \int_0^\infty e^{-st}\left(\frac{e^{-t}-e^{-3t}}{t}\right)dt=\log\left(\frac{s+3}{s+1}\right)$$

Taking the lim *it as* $s\to 0$, *we find* $\displaystyle\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt=\log 3.$

Laplace transform of periodic functions

Definition: A function $f(t)$ is said to be periodic with period T if $f(t+T) = f(t)$ for all t.

Note that if $f(t)$ is periodic with periodic T then $f(t+nt) = f(t)$ for all integer n and for all $t > 0$. Geometrically a function $f(t)$ is periodic with period T then the shape of its graph is the same in every interval $(nT, (n+1)T)$ for $n = 0, 1, 2, 3, \ldots$.

Example 1: It is well known that the circular functions sin at and cos at are periodic functions with period 2π.

Example 2: Geometrically, we can construct a function of given periodicity by T simply drawing a graph in the interval $(0, T)$ and repeat the same shape in every interval

$$(T, 2T), (2T, 3T), \ldots (nT, (n+1)T), \ldots$$

One of the main use of Laplace transformation is to study the transients of a signal is behaviour at infinity. Mostly all signals are periodic in nature. Hence it is important to study the behavior of a periodic function under the Laplace transform. When the function $f(t)$ is periodic its Laplace transform takes the following form.

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$If \; f(t) \; has \; period \;\; T \rangle 0 \; then$

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)dt}{1-e^{-sT}}.$$

$\Pr oof :-$

$$We \; have \; L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + ...$$

In the first integral let $t = u$, *in the second integral let* $t = u + T$, *in the third integral let* $t = u + 2T$, *etc.*

$$Then\ \ L[f(t)] = \int_0^T e^{-su} f(u)\,du + \int_0^T e^{-s(u+T)} f(u+T)\,du + \int_0^T e^{-s(u+2T)} f(u+2T)\,du + \ldots.$$

$$= \int_0^T e^{-su} f(u)\,du + e^{-sT}\int_0^T e^{-su} f(u)\,du + e^{-2sT}\int_0^T e^{-su} f(u)\,du + \ldots\ldots.$$

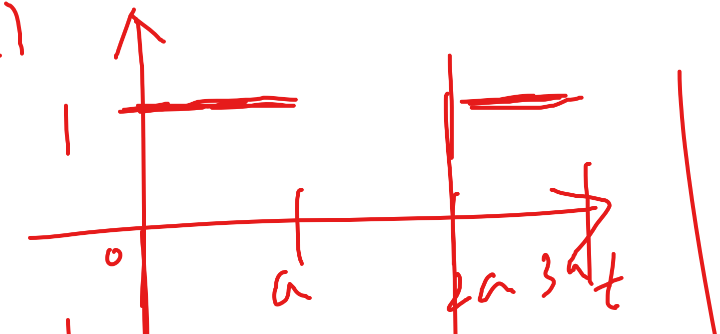
$$\because f(u) = f(u+T) = f(u+2T) = \ldots\ldots\ldots\ldots$$

$$= (1 + e^{-sT} + e^{-2sT} + \ldots\ldots.)\int_0^T e^{-su} f(u)\,du$$

$$= \frac{\int_0^T e^{-su} f(u)\,du}{1 - e^{-sT}} = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}\left[\because 1 + r + r^2 + r^3 + \ldots.. = \frac{1}{1-r},\ |r|\langle 1\right]$$

$$= \frac{1}{s}\,\frac{(1 - e^{-as})}{(1 + e^{-as})} = \frac{1}{s}\,\frac{(1 - e^{-as/2}\cdot e^{-as/2})}{(1 + e^{-as/2}\cdot e^{-as/2})} = \frac{1}{s}\,\frac{(e^{as/2} - e^{-as/2})}{(e^{as/2} + e^{-as/2})}$$

$$= \frac{1}{s}\,\frac{\cancel{e^{-as/2}}\left(\frac{1}{e^{-as/2}} - e^{-as/2}\right)}{\cancel{e^{-as/2}}\left(\frac{1}{e^{-as/2}} + e^{-as/2}\right)}$$



find the Laplace transform of a square wave (or) Meander.

The function representing the square wave is $f(t) = \begin{cases} 1 & \text{if } 0 < t < a \\ -1 & \text{if } a < t < 2a \end{cases}$

and extended periodically. $T = 2a$

$$\mathcal{L}(f(t)) = \frac{1}{(1-e^{-2as})}\int_0^{2a} e^{-st} f(t)\,dt. = \frac{l}{(l-e^{-sT})}\left\{\int_0^a e^{-st}.dt - \int_a^{2a} e^{-st}dt\right\}$$

$-2as$

$$= \frac{1}{(1-e^{-2as})}\left\{\left(\frac{e^{-st}}{-s}\right)\Big]_{t=0}^{t=a} \cancel{+}\, - \left(\frac{e^{-st}}{-s}\right)\Big]_{t=a}^{t=2a}\right\}$$

$$= \frac{(1-e^{-as})^2}{s(1-e^{-2as})} = \frac{1}{s}\tan h\left(\frac{as}{2}\right).$$

$$= \frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}}$$

$a^m \cdot a^n = a^{m+n}$

$$= \frac{1}{(1-e^{-2as})}\left[\left\{\frac{-e^{-as}}{s} - \left(-\frac{1}{s}\right)\right\} - \left\{\frac{-e^{-2as}}{s} - \left(-\frac{e^{-as}}{s}\right)\right\}\right]$$

$$= \frac{1}{''}\left[-\frac{e^{-as}}{s}+\frac{1}{s}+\frac{e^{-2as}}{s}-\frac{e^{-as}}{s}\right] = \frac{1}{''}\left(\frac{1+e^{-2as}-2e^{-as}}{s}\right)$$

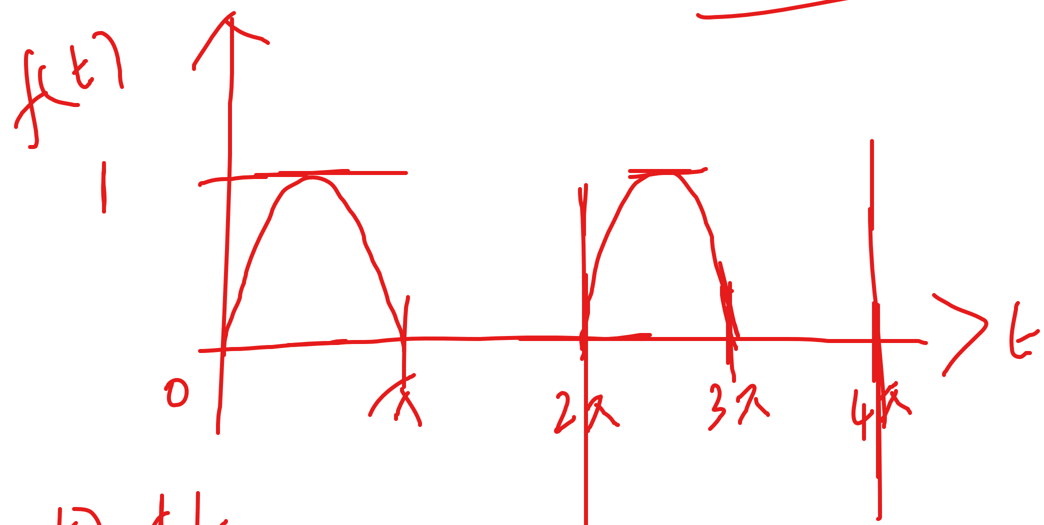
1) *Find the Laplace transform and draw the graph of*

$$f(t)=\begin{cases}\sin t & 0 \langle\, t \,\langle\, \pi \\ 0 & \pi \,\langle\, t \,\langle\, 2\pi\end{cases}$$

extended periodicaly with period 2π. (*or*)

Draw the graph and find the laplace transform of a half wave rectified sine *curve*

$T = 2\pi$



$$L[f(t)] = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \int_0^{\pi} e^{-st} \sin t \, dt + \int_{\pi}^{2\pi} e^{-st} (0) \, dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \int_0^{\pi} e^{-st} \sin t \, dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-st}}{s^2+1} \left(-s \sin t - \cos t \right) \right]_0^{\pi}$$

$$= \frac{1}{(1+s^2)(1-e^{-2\pi s})}\left[\left\{e^{-\pi s}(-s\sin\pi - \cos\pi)\right\} - \left\{e^{-s(0)}(-s\sin(0) - \cos(0))\right\}\right]$$

$$= \frac{1}{(1-e^{-2\pi s})(1+s^2)}\left[e^{-\pi s}+1\right]$$

$$= \frac{\cancel{(1+e^{-\pi s})}}{\cancel{(1+e^{-\pi s})}(1-e^{-\pi s})(1+s^2)} = \frac{1}{(1-e^{-\pi s})(s^2+1)}$$

2) $Find\ L[f(t)]\ where\ f(t)=\begin{cases} t & 0 \langle\ t \ \langle\ 1 \\ 0 & 1 \langle\ t \ \langle\ 2 \end{cases}\ and$

$f(t+\overset{T}{2})=f(t)\ for\ t\ \rangle\ 0.$

Sol:- Here $T=2$

WKT $L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$

For the given $f(t)$

$$L[f(t)] = \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}\, t\, dt + \int_1^2 e^{-st} \underset{0}{(0)}\, dt\right]$$

$$= \frac{1}{1-e^{-2s}}\int_0^1 t e^{-st}\, dt$$

Let $u = t$ $\qquad \int dv = \int e^{-st}\, dt$

$du = dt$ $\qquad v = \frac{e^{-st}}{-s}$

$$= \frac{1}{1-e^{-2s}}\left[t\left(-\frac{e^{-st}}{s}\right)\Bigg|_0^1 - \int_0^1 \frac{e^{-st}}{-s}\,dt \right]$$

$$= \frac{1}{1-e^{-2s}}\left[\left(-\frac{e^{-s}}{s}\right) - 0 + \frac{1}{s}\int_0^1 e^{-st}\,dt \right]$$

$$= \frac{1}{1-e^{-2s}}\left[-\frac{e^{-s}}{s} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)\Bigg|_0^1 \right]$$

$$= \frac{1}{1-e^{-2s}}\left[-\frac{e^{-s}}{s} - \frac{1}{s^2}\left(e^{-s} - 1\right) \right] = \frac{1 - e^{-s}(s+1)}{s^2\left(1-e^{-2s}\right)}$$

4. If $F(t) = t^2, 0 < t < 2$ and $F(t+2) = F(t)$, find $L\{F(t)\}$.

Sol:- Here $T = 2$

WKT

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$\therefore L[f(t)] = \frac{1}{1-e^{-2s}} \int_0^2 t^2 e^{-st}\, dt$$

$$= \frac{1}{1-e^{-2s}} \left[t^2 \left(\frac{e^{-st}}{-s} \right) - 2t \left(\frac{e^{-st}}{s^2} \right) + 2 \left(\frac{e^{-st}}{-s^3} \right) \right]_0^2$$

$$= \frac{1}{1-e^{-2s}} \left[\left(\frac{-4e^{-2s}}{s} - \frac{4e^{-2s}}{s^2} - \frac{2e^{-2s}}{s^3} \right) - \left(0 - 0 - \frac{2}{s^3} \right) \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{2 - 2e^{-2s} - 4se^{-2s} - 4s^2e^{-2s}}{s^3} \right]$$