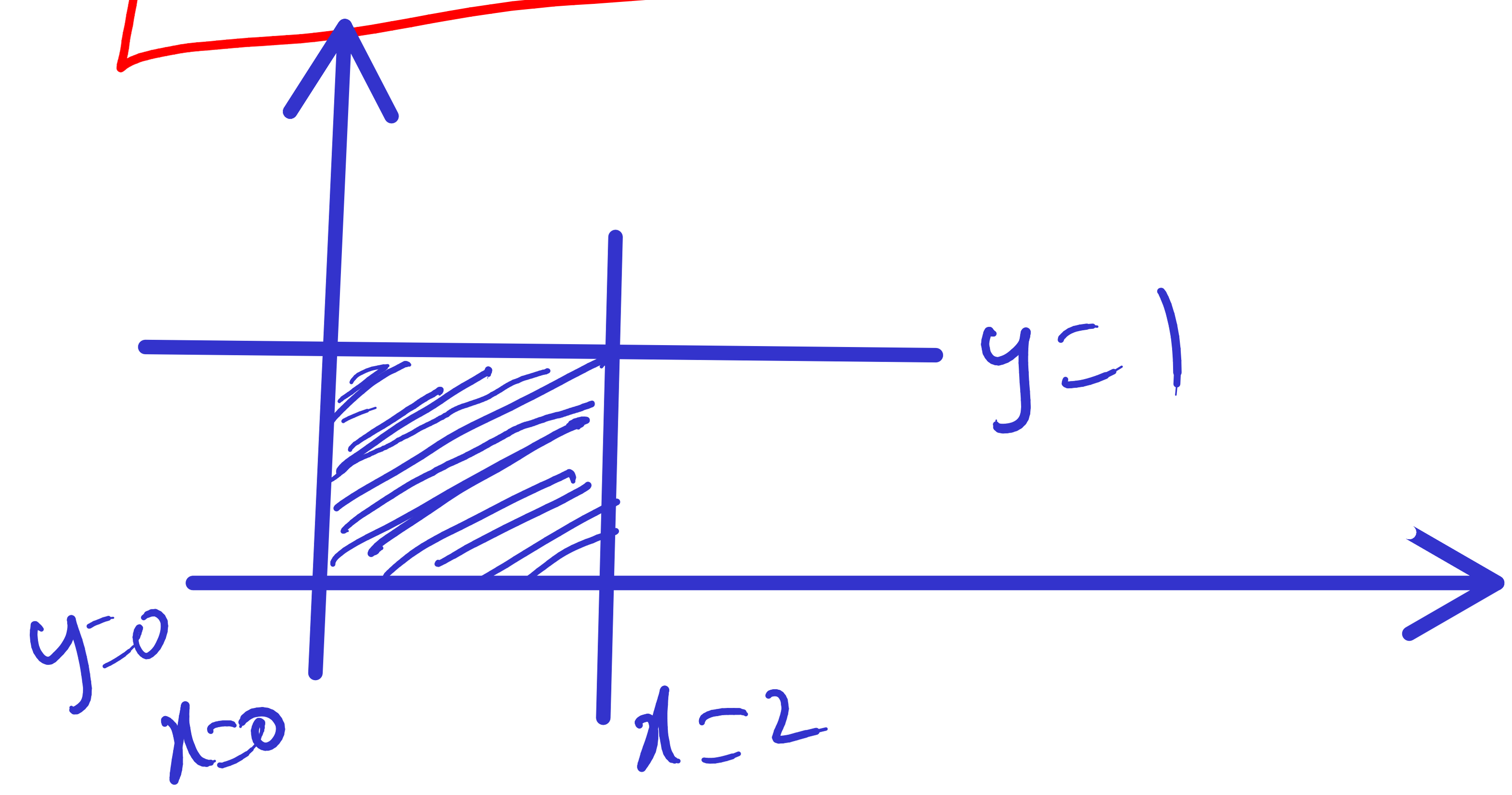
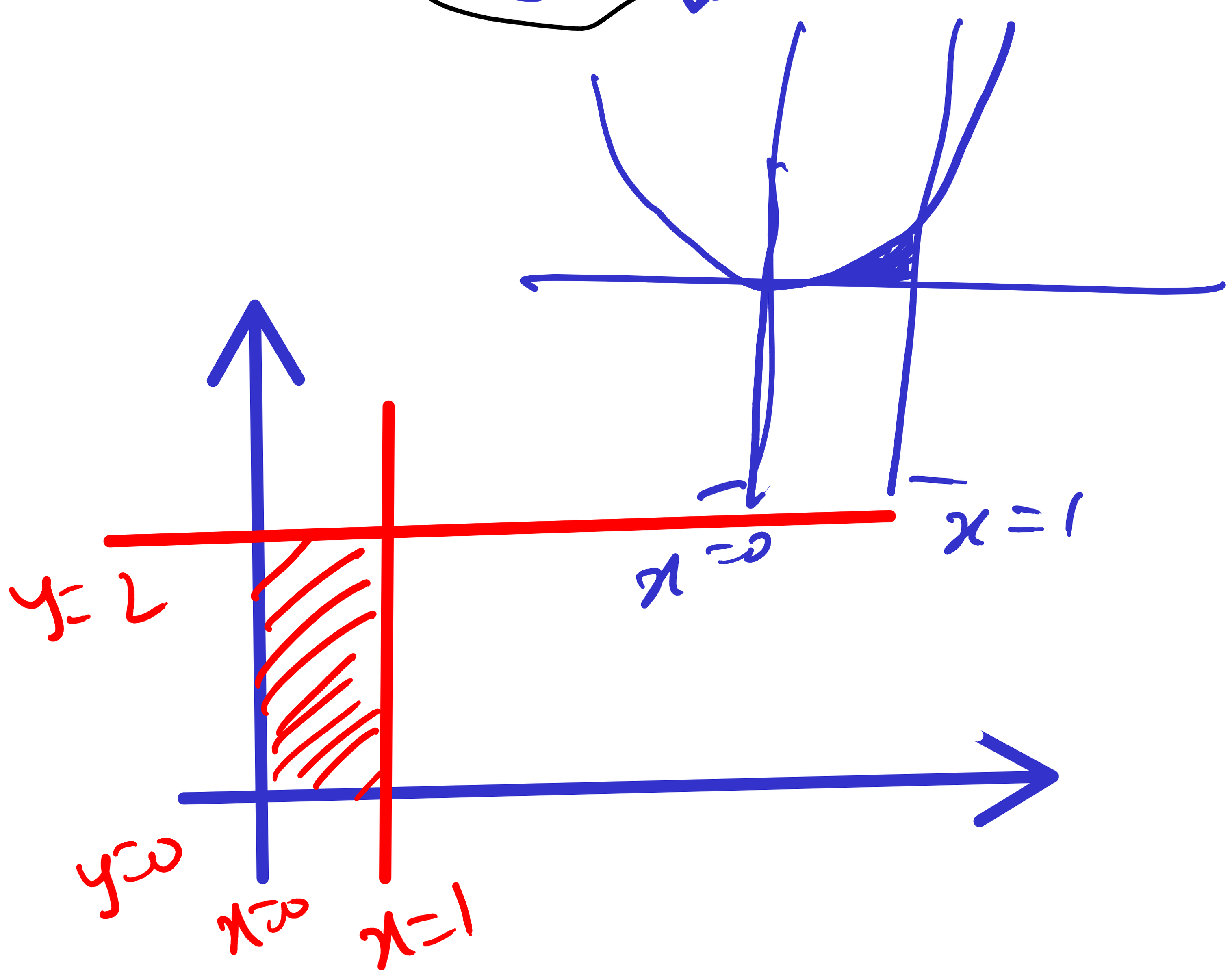
Multiple Integrals

$$\boxed{\int_0^1 \boxed{\int_0^2 x+y \; dy} \; dx}$$

$$\int_0^1 x^2 \, dx = \frac{1}{3}$$





$$\int_0^1 \left(\int_0^2 x+y \; dy \right) dx$$

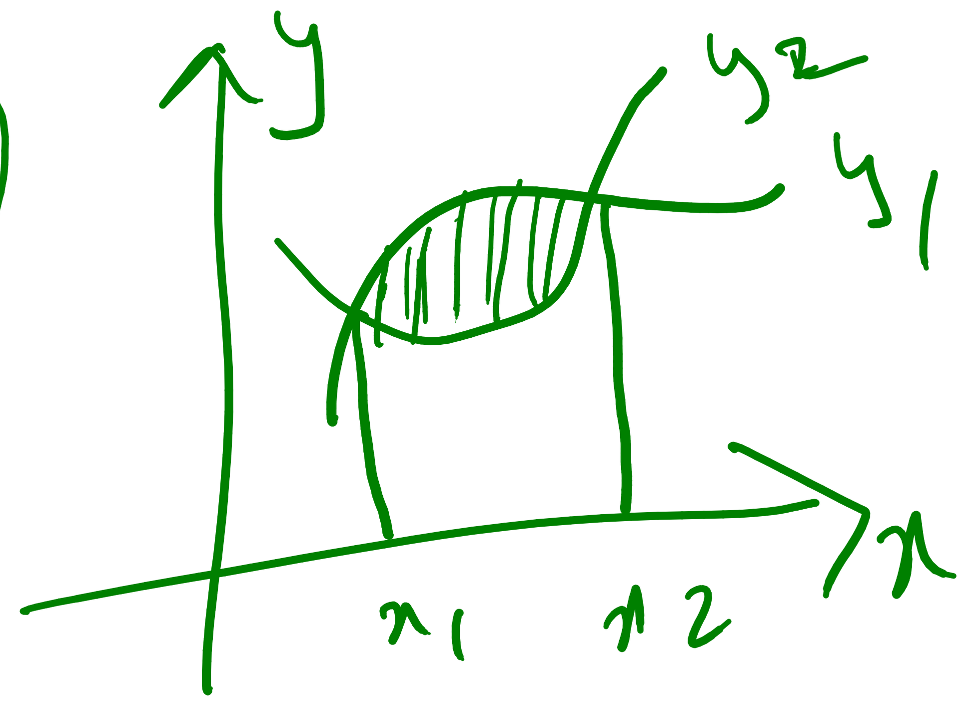
$$= \int_0^1 xy + \frac{y^2}{2} \Big|_0^2 \, dx$$

$$= \int_0^1 (2x+2) - (0+0) \; dx = x^2 + 2x \Big|_0^1$$

$$= 3 - 0$$

$$= 3$$

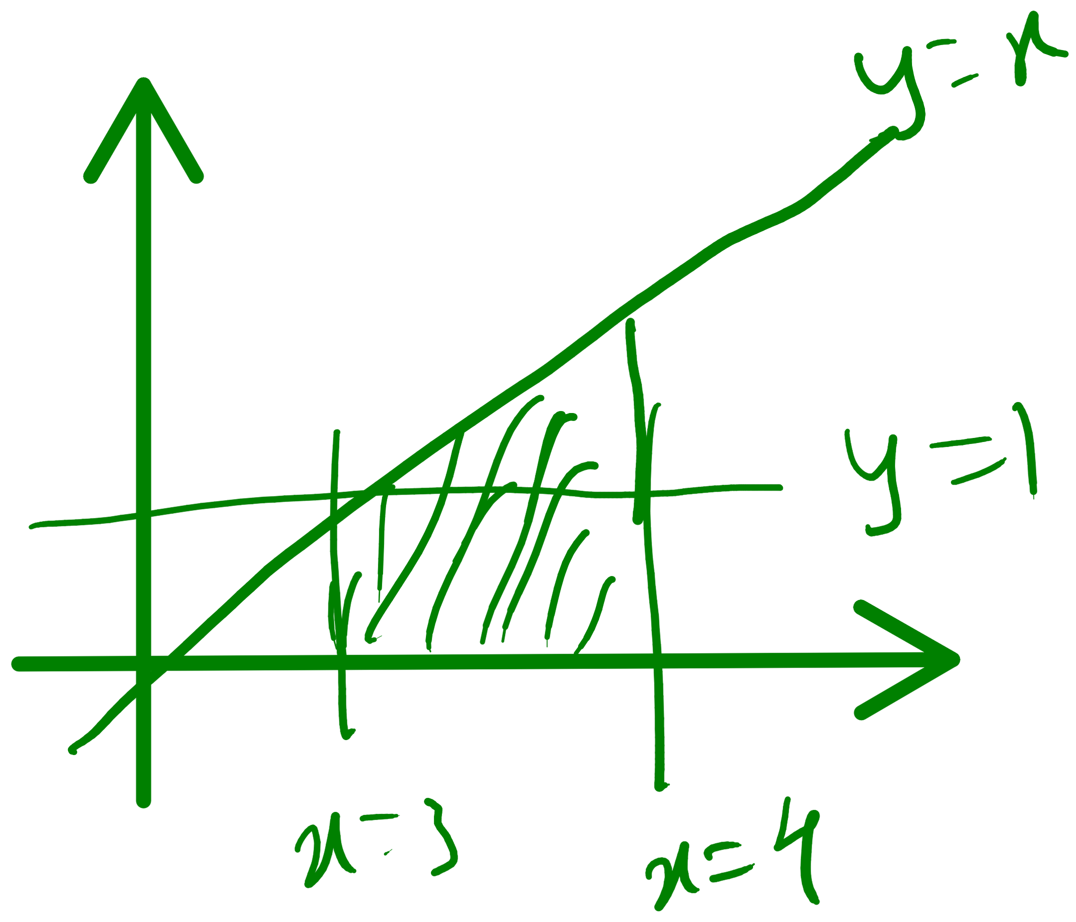
$$\int_{x_1}^{x_2} (y_1 - y_2)\, dx = \int_{x_1}^{x_2} \int_{y_2}^{y_1} 1 \cdot dy\, dx$$



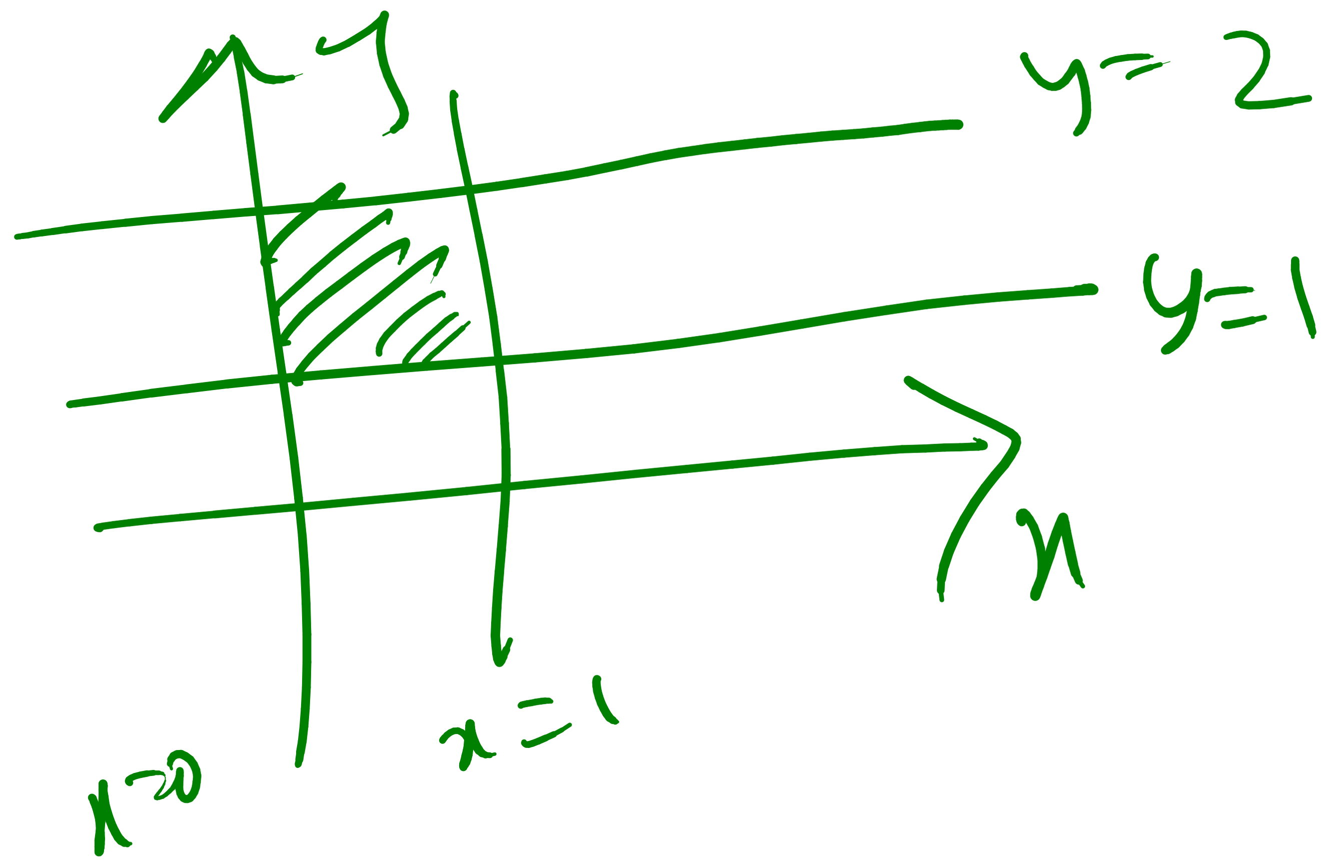
$$= \int_{x_1}^{x_2} y \Big|_{y_2}^{y_1} dx$$

$$= \int_{x_1}^{x_2} y_1 - y_2 \, dx$$

$$\int_3^4 x\, dx = \frac{x^2}{2}\Big|_3^4 = \frac{7}{3}$$



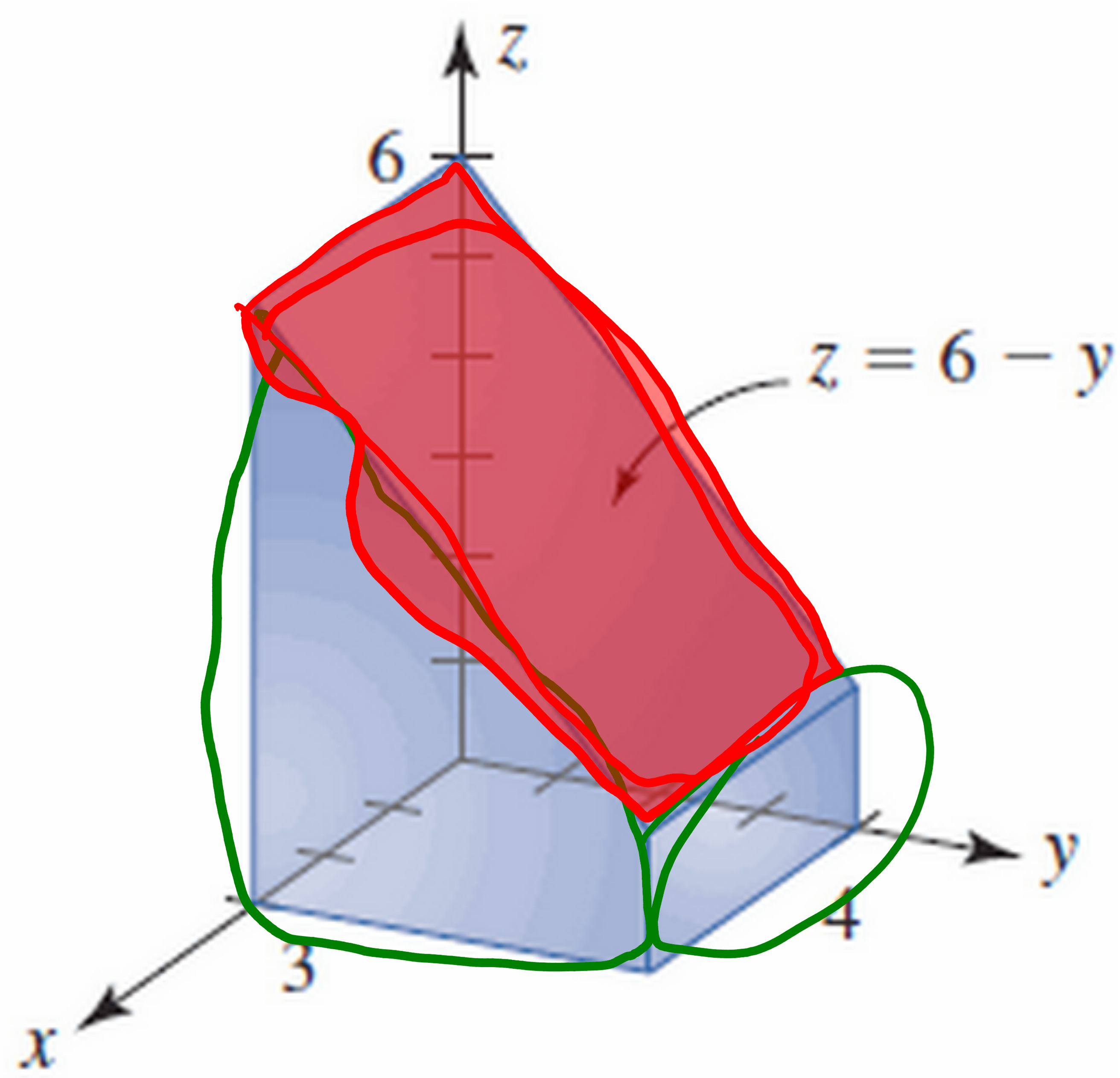
$$\int_0^1 \int_1^2 (x+y)\, dy\, dx$$



$$\int_3^4 dx$$

Use a double integral to find the volume of the solid shown in the figure.

[ ] cu units

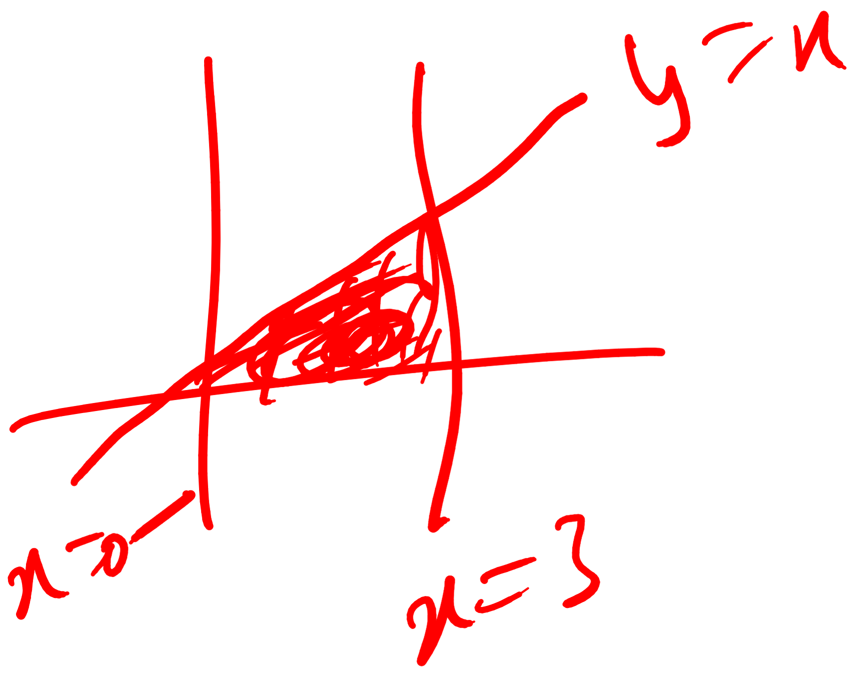


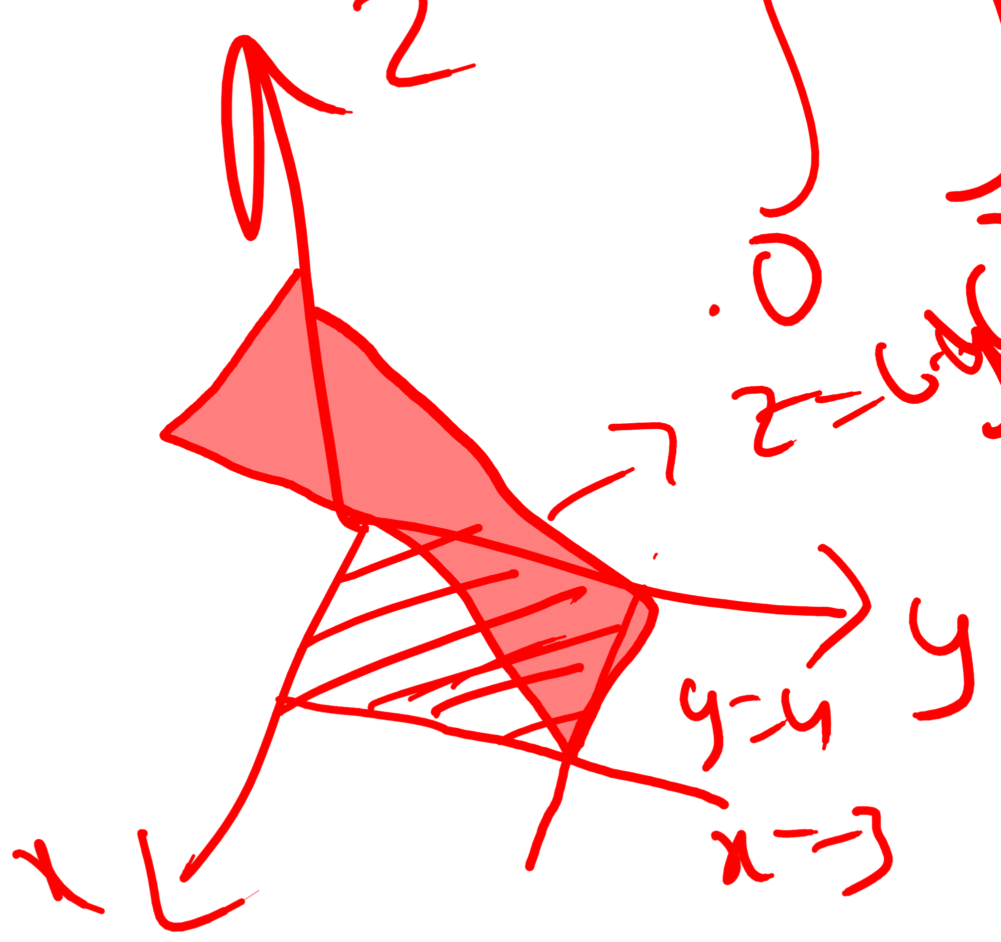
$$\int_0^3 \int_0^4 6-y \, dy \, dx$$

$x=0 \to yz$ plane

$x=3$

$$\int_0^3 x \, dx$$





$$\int_0^3 \int_0^4 6-y \, dy \, dx$$

$x=0 \quad x=3$

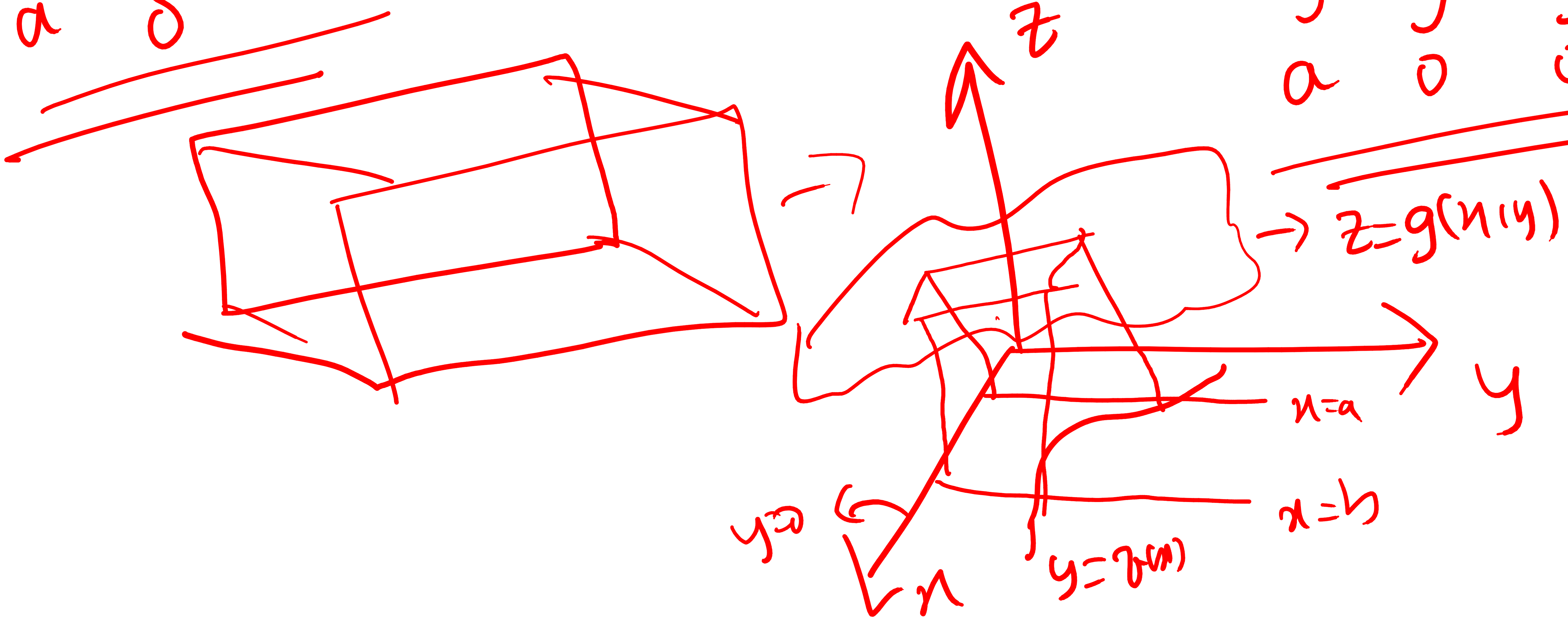
$y=0 \quad y=4$

$z=6-y$

$\int_a^b dx \to$ length

$\int_a^b f(x)\,dx \to$ area $= \int_a^b \int_0^{f(x)} dy \, dx$

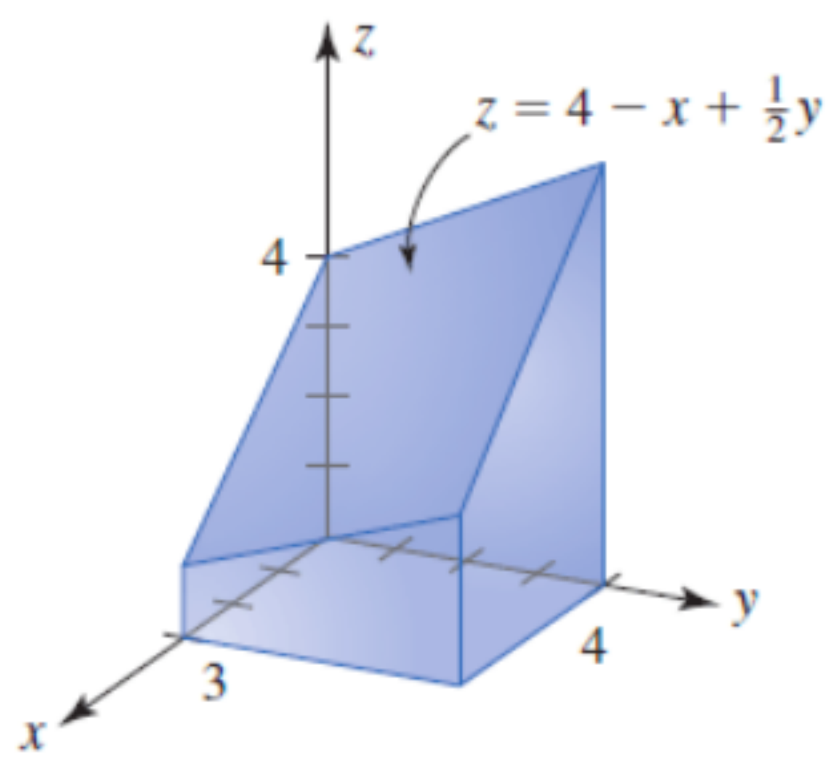
$$\int_a^b \int_0^{\gamma(x)} g(x,y)\,dy\,dx \rightarrow \text{volume} = \int_a^b \int_0^{\gamma(x)} \int_0^{g(x,y)} dz\,dy\,dx$$



0/1 points | Previous Answers TanApCalcBr10 8.6.027.

Use a double integral to find the volume of the solid shown in the figure.

cu units



$$\int_0^3 \int_0^4 \left(4 - x + \frac{y}{2}\right) dy\,dx$$

$$\int_0^3 \int_0^4 \int_0^{4-x+\frac{y}{2}} dz\,dy\,dx$$

$$\int_0^3 \int_0^4 \gamma(x,y)\,dy\,dx = \int_0^4 \int_0^3 \gamma(x,y)\,dx\,dy$$

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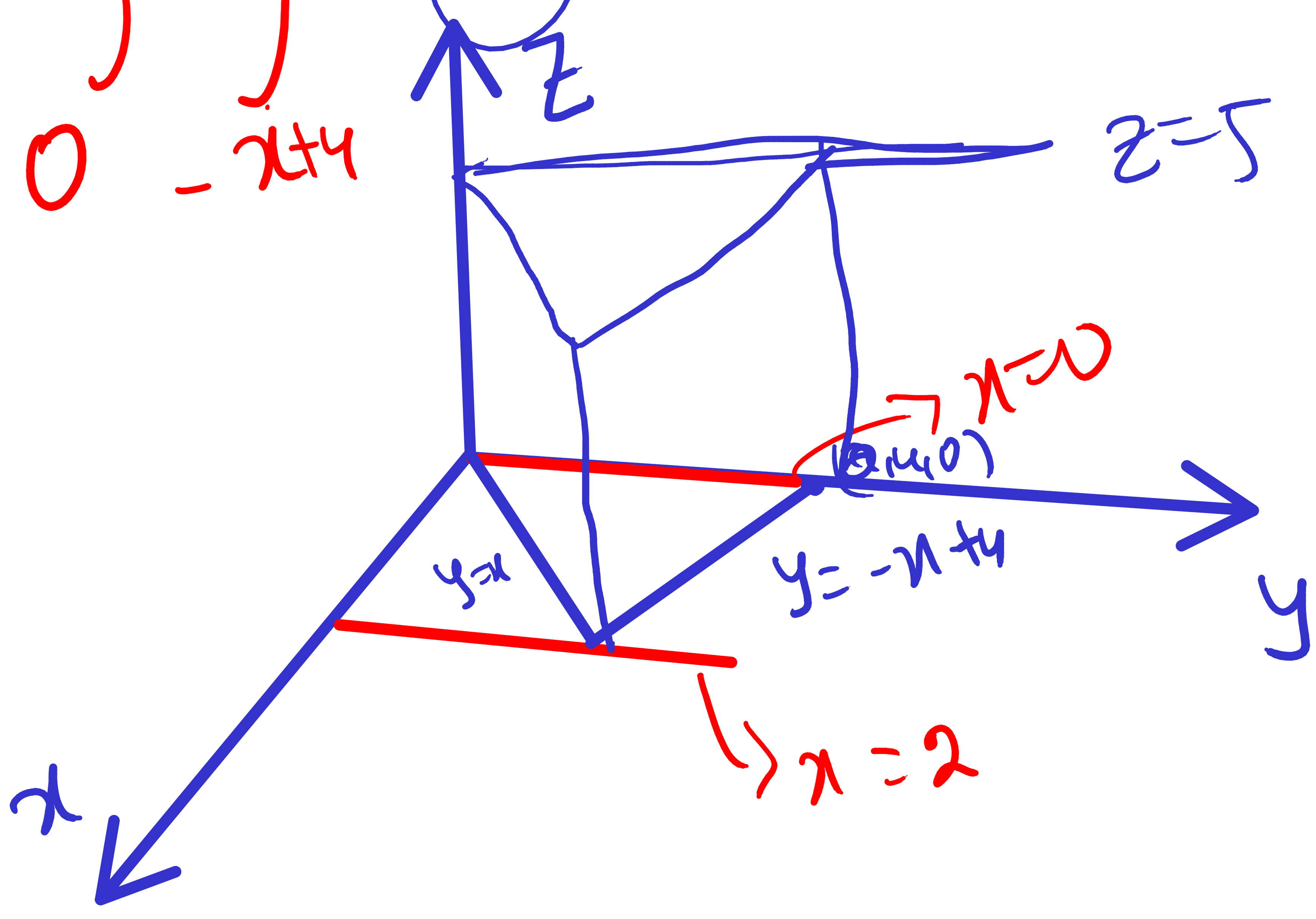
CALCULUS

BMAT101L

$$\int_0^2 \int_{-x+4}^{x} 5 \, dy \, dx$$



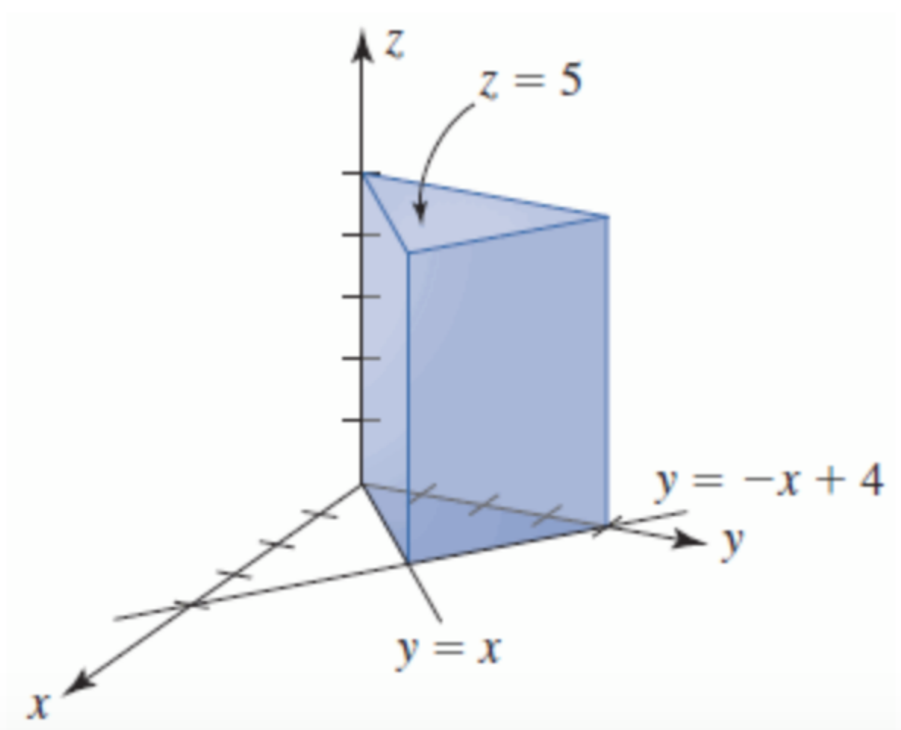
$x \to 0$ to 2

$y \to -x+4$ to x

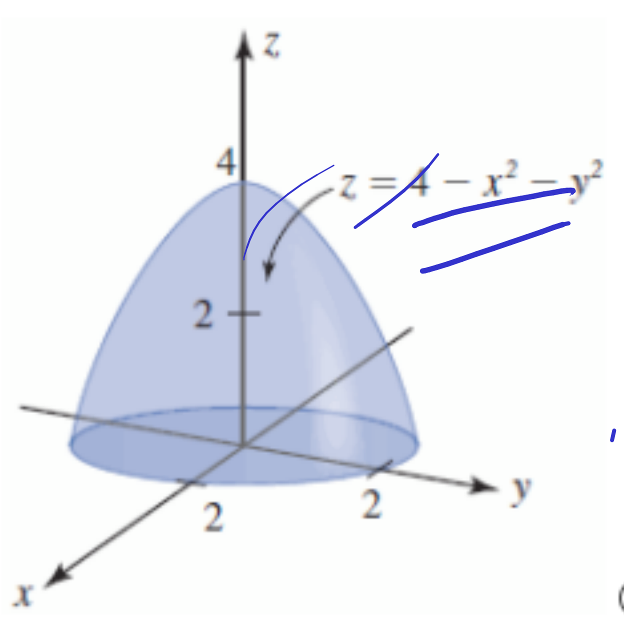
$z \to 5$

Use a double integral to find the volume of the solid shown in the figure.

cu units



Use a double integral to find the volume of the solid shown in the figure.

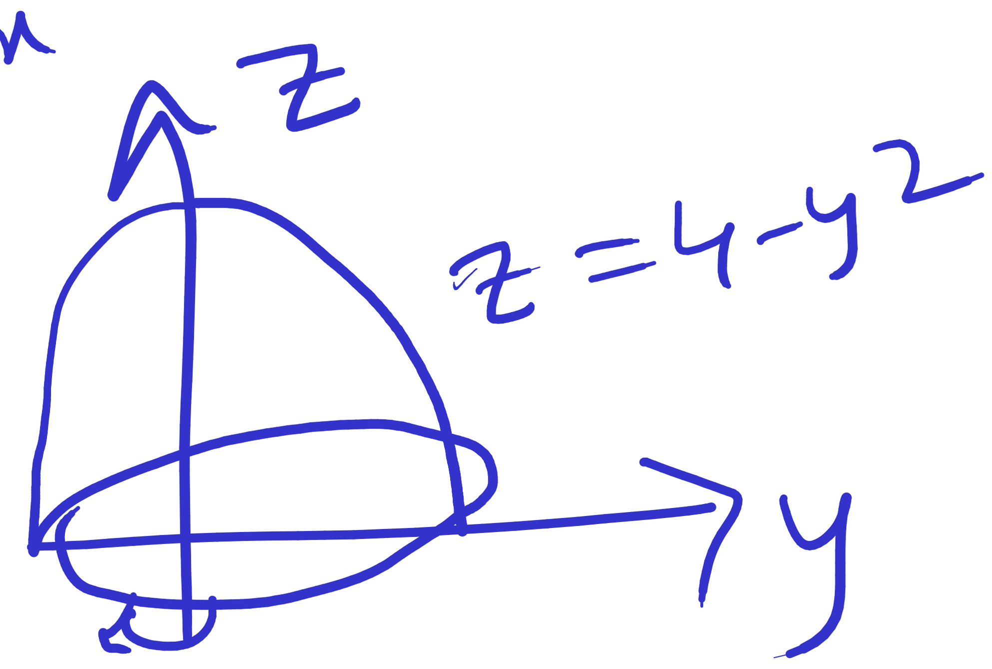


cu units

$$\iint 4-x^2-y^2 \, dy \, dx$$

$x = -2$ to 2

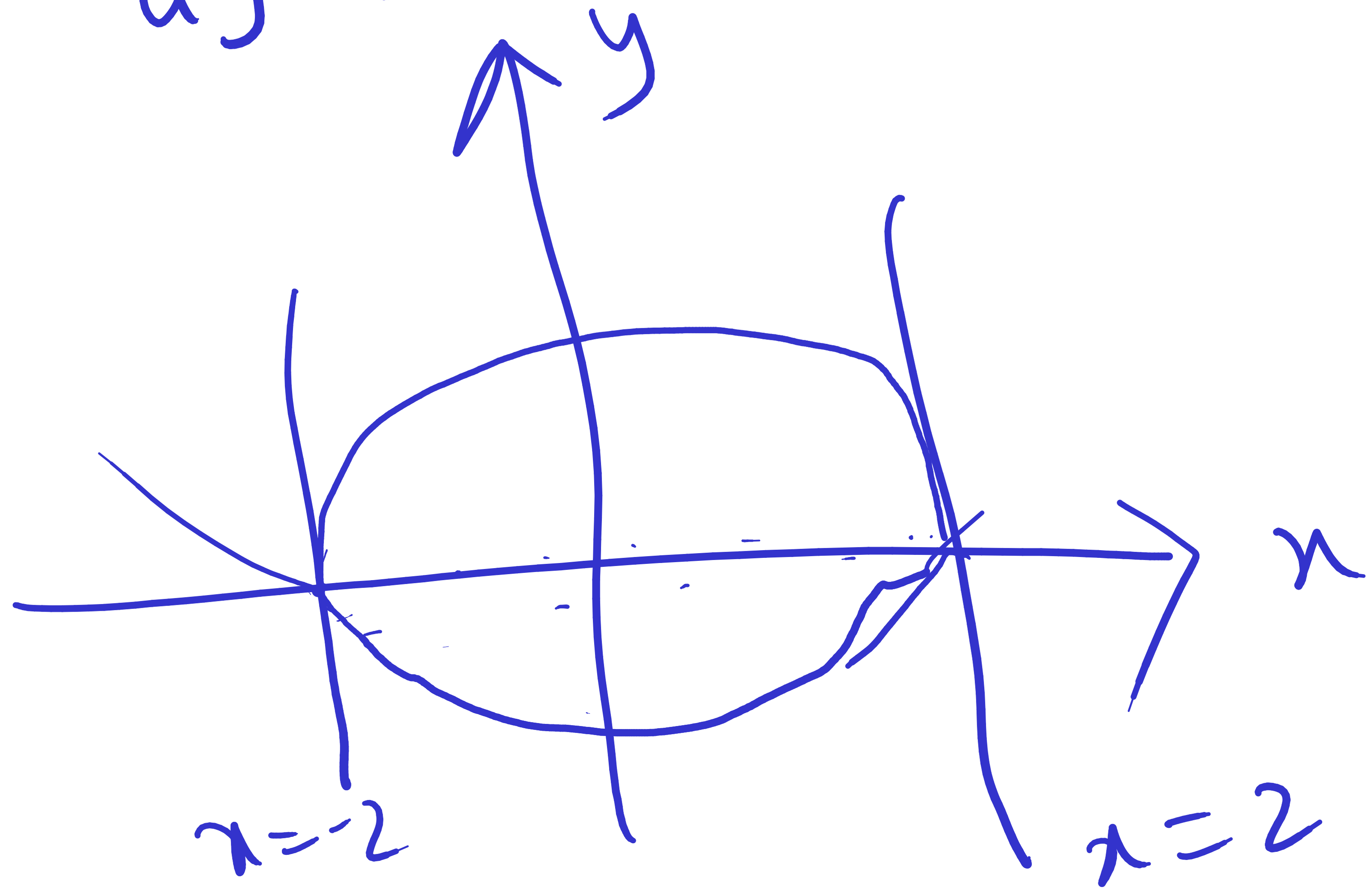
$y = -2$ to 2



$$\int_0^4 \pi (4-y^2)^2 \, dy$$

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 4-x^2-y^2 \, dy \, dx$$

$$\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$$



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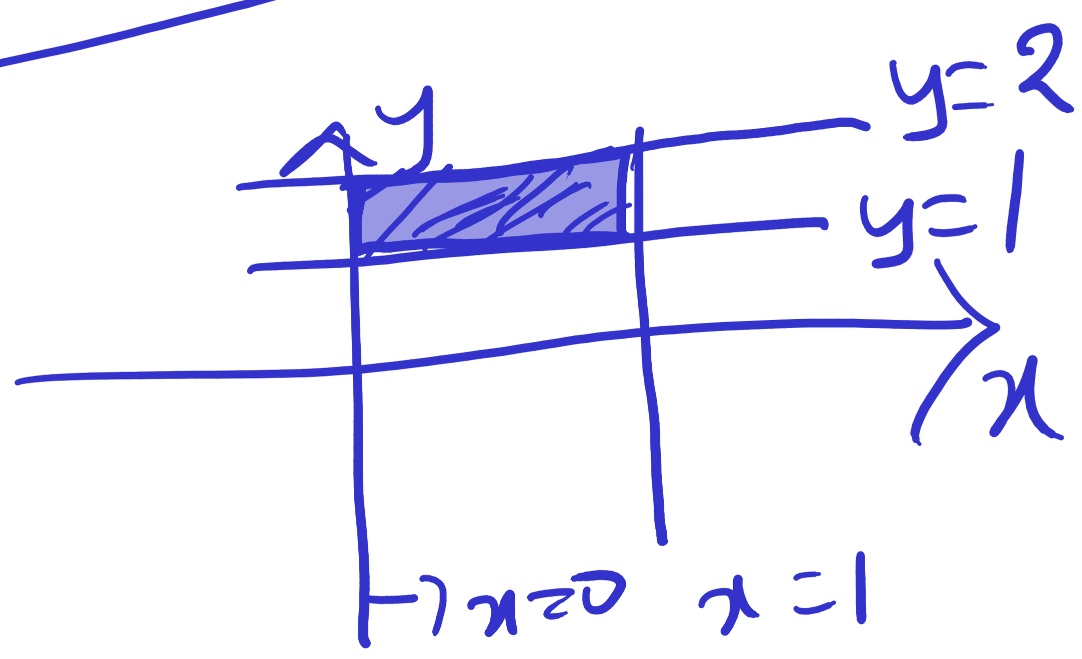
Change the order of integration

$$\int_1^2 \int_0^1 (x^2+y^2)\,dx\,dy$$

$x \to 0$ to 1
$y \to 1$ to 2



$$= \int_0^1 \int_1^2 (x^2+y^2)\,dy\,dx$$

$y \to 1$ to 2
$x \to 0$ to 1

$$\int_1^2 \int_0^1 (x^2+y^2)\,dx\,dy = \int_0^1 \int_1^2 (x^2+y^2)\,dy\,dx$$

$$\int_1^2 \left. \frac{x^3}{3} + xy^2 \right|_0^1 dy$$

$$\int_1^2 \left(\frac{1}{3}+y^2\right) dy$$

$$\left. \frac{1}{3}y + \frac{y^3}{3} \right|_1^2 = \left(\frac{2}{3}+\frac{8}{3}\right) - \left(\frac{1}{3}+\frac{1}{3}\right)$$

$$= \frac{8}{3} \;//$$

$$= \int_0^1 \left. x^2y + \frac{y^3}{3} \right|_1^2 dx$$

$$= \int_0^1 \left(2x^2+\frac{8}{3}\right) - \left(x^2+\frac{1}{3}\right) dx$$

$$= \int_0^1 x^2 + \frac{7}{3}\, dx$$

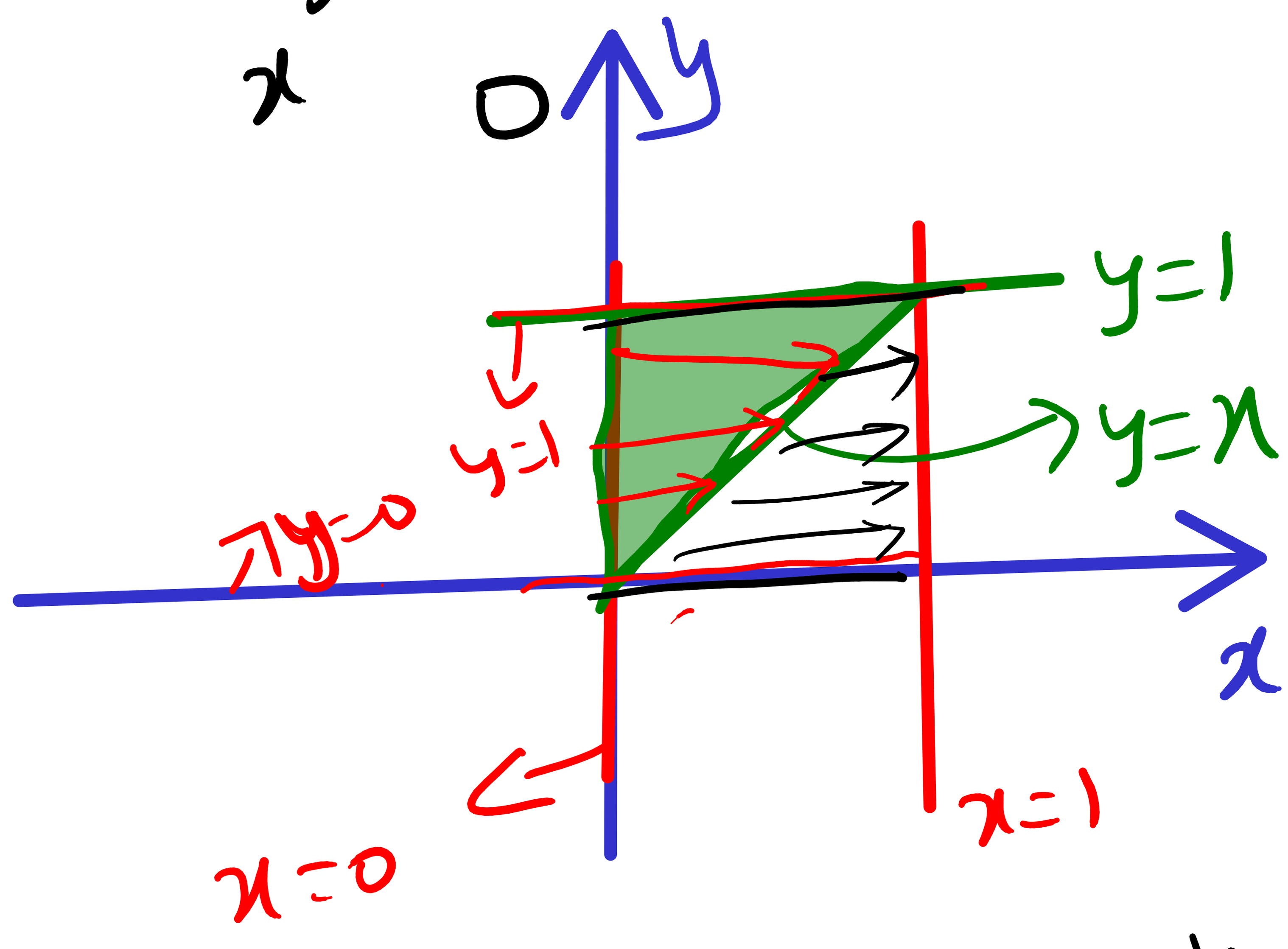
$$= \left. \frac{x^3}{3} + \frac{7}{3}x \right|_0^1$$

$$= \frac{1}{3} + \frac{7}{3} = 8/3 \;//$$

$$\int_0^1\int_x^1 xy\,dy\,dx = \int_0^1\int_0^y xy\,dx\,dy \neq \int_x^1\int_0^1 xy\,dx\,dy$$

$x \to 0$ to 1 Constant

$y \to x$ to 1 Varying



$y \to$ constant

$x \to$ varying

$y \to 0$ to 1

$x \to 0$ to y

y constant

$y \to 0$ to 1

$x \to y$ to 1

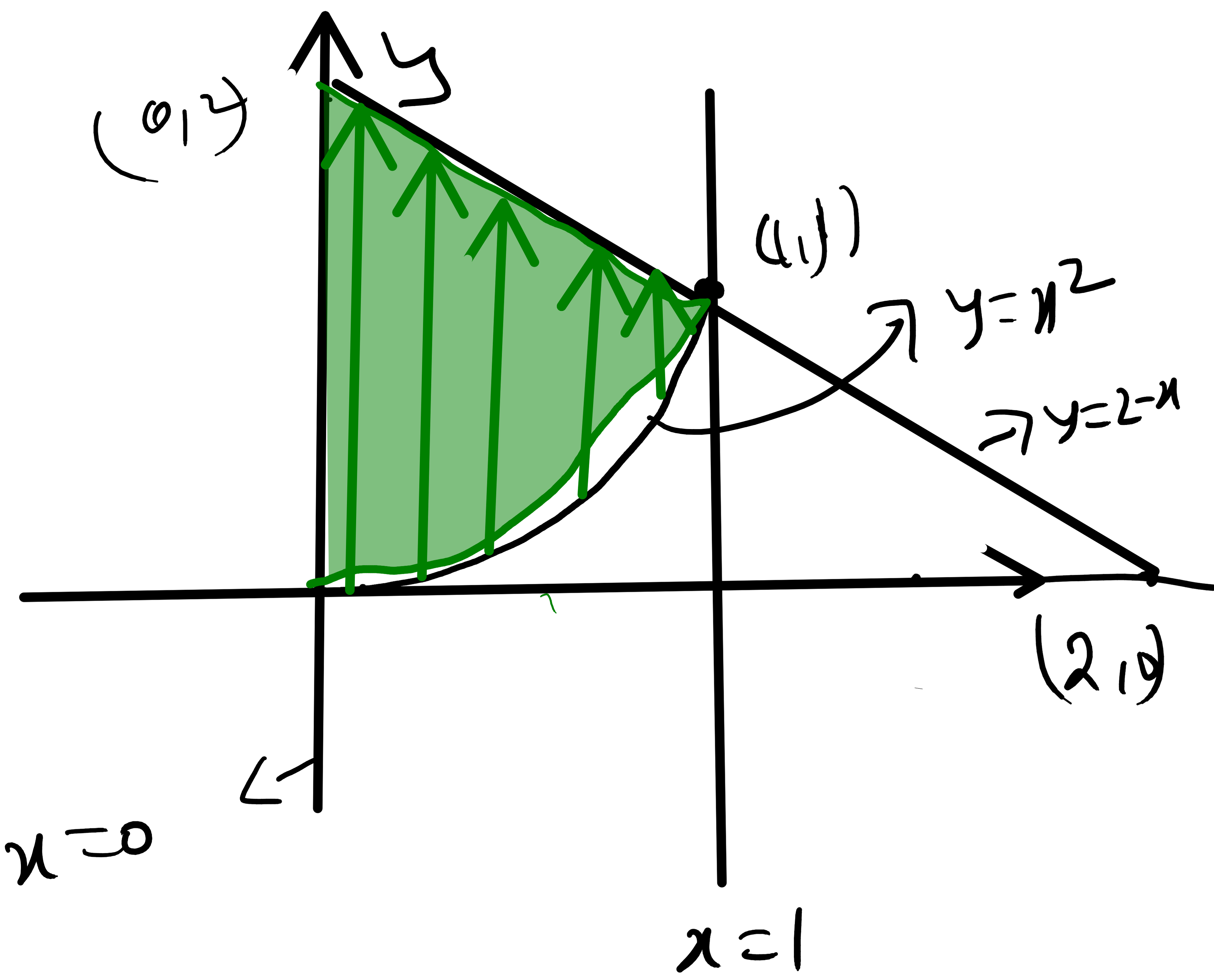
Change the order of Integration

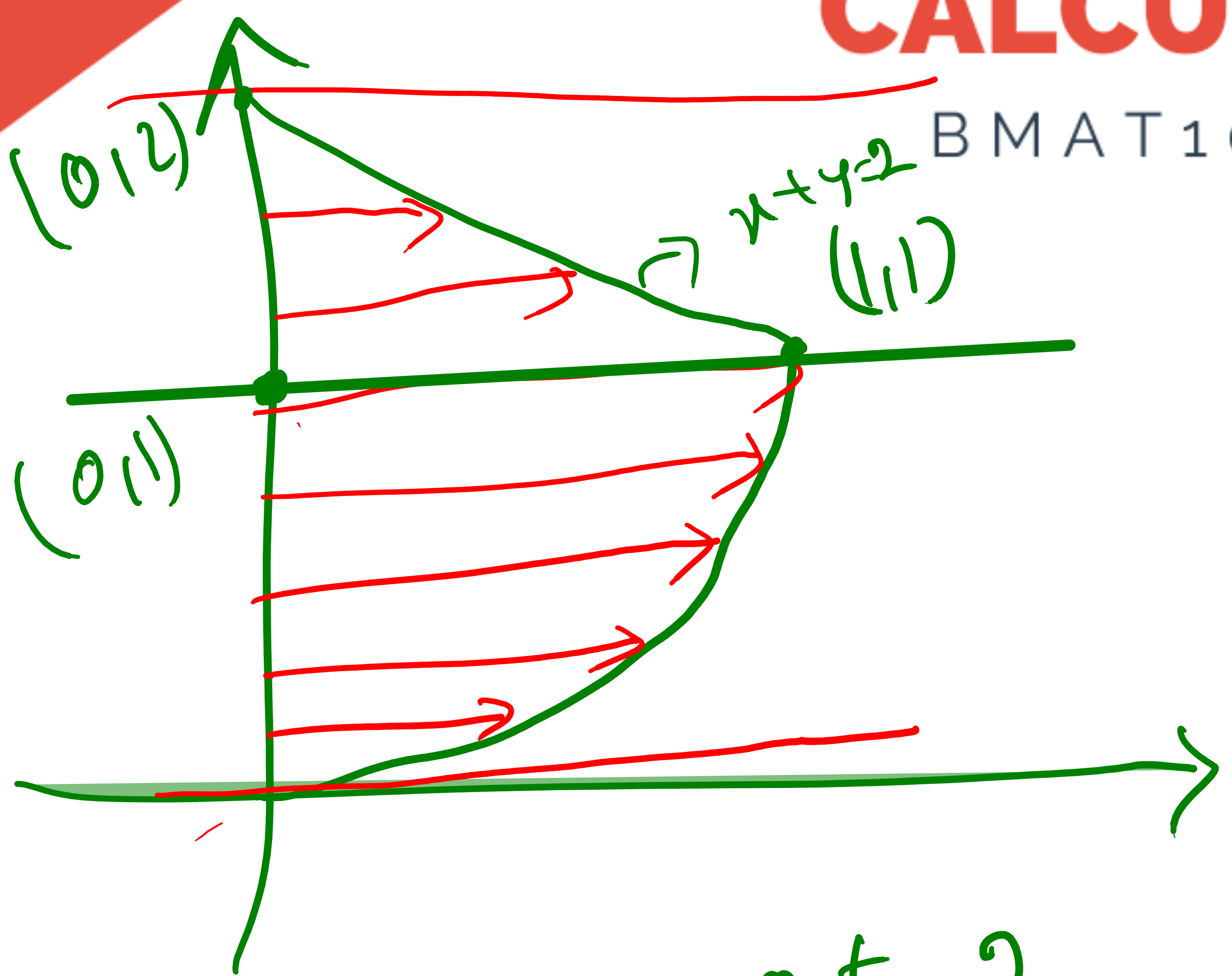
$$\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx$$

$x \to 0$ to 1

$y \to x^2$ to $2-x$

$y =$





y constant
x varying

$y \to 0$ to 2
$x \to 0$ to

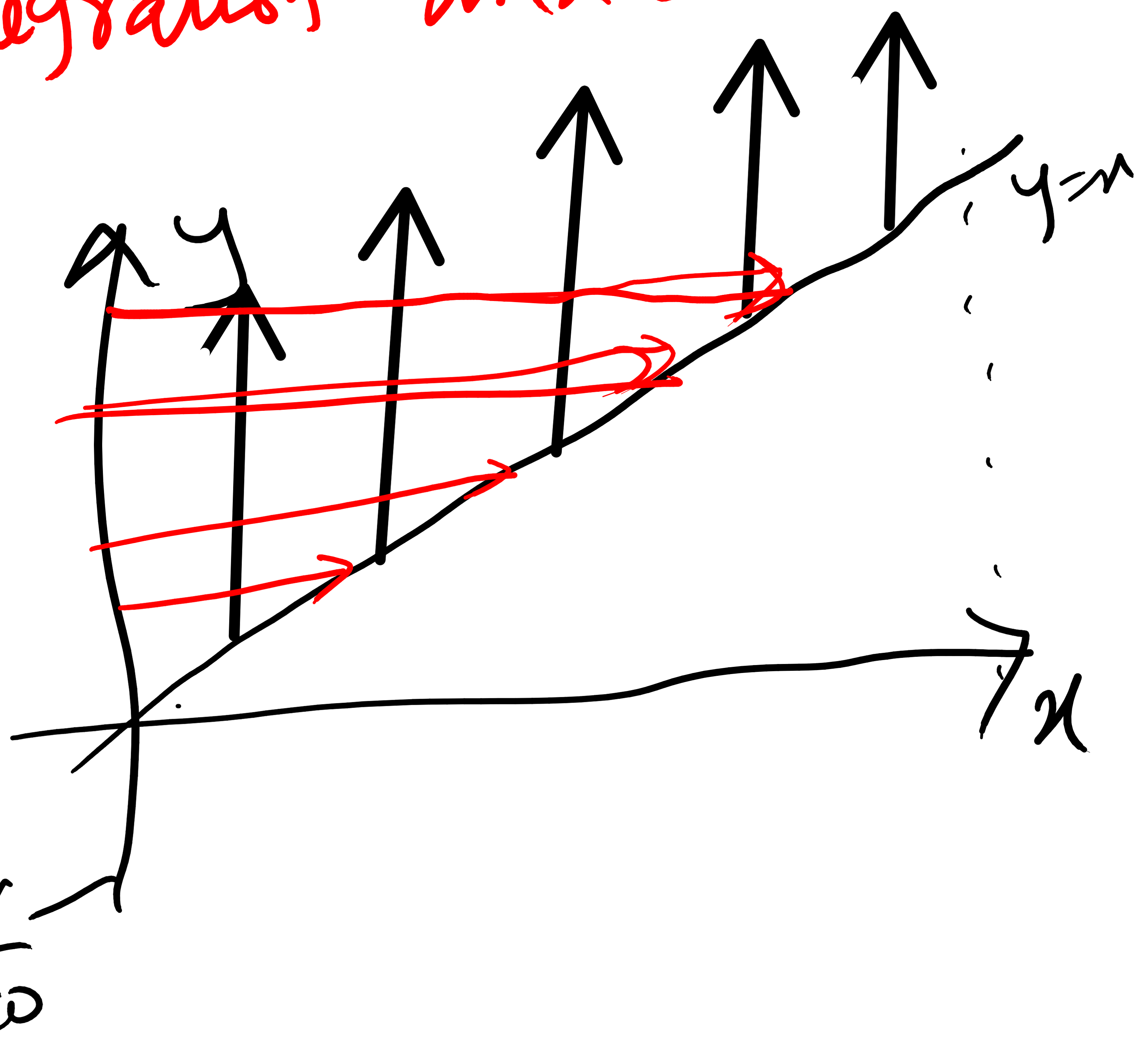
$y \to 0$ to 1
$x \to 0$ to $\sqrt{y}$

$y \to 1$ to 2
$x \to 0$ to $2-y$

$$\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx = \int_0^1 \int_0^{\sqrt{y}} xy\, dx\, dy + \int_1^2 \int_0^{2-y} xy\, dx\, dy$$

Change the order of integration and evaluate

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$$



$x \to 0$ to ∞
$y \to x$ to ∞

$y \to 0$ to ∞
$x \to 0$ to y

$$\int_0^{\infty}\int_x^{\infty}\frac{e^{-y}}{y}\,dy\,dx = \int_0^{\infty}\int_0^{y}\frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_0^{\infty}\frac{e^{-y}}{y}\,x\Big|_0^{y}\,dy$$

$$= \int_0^{\infty} e^{-y}\,dy = \frac{e^{-y}}{-1}\Big|_0^{\infty}$$

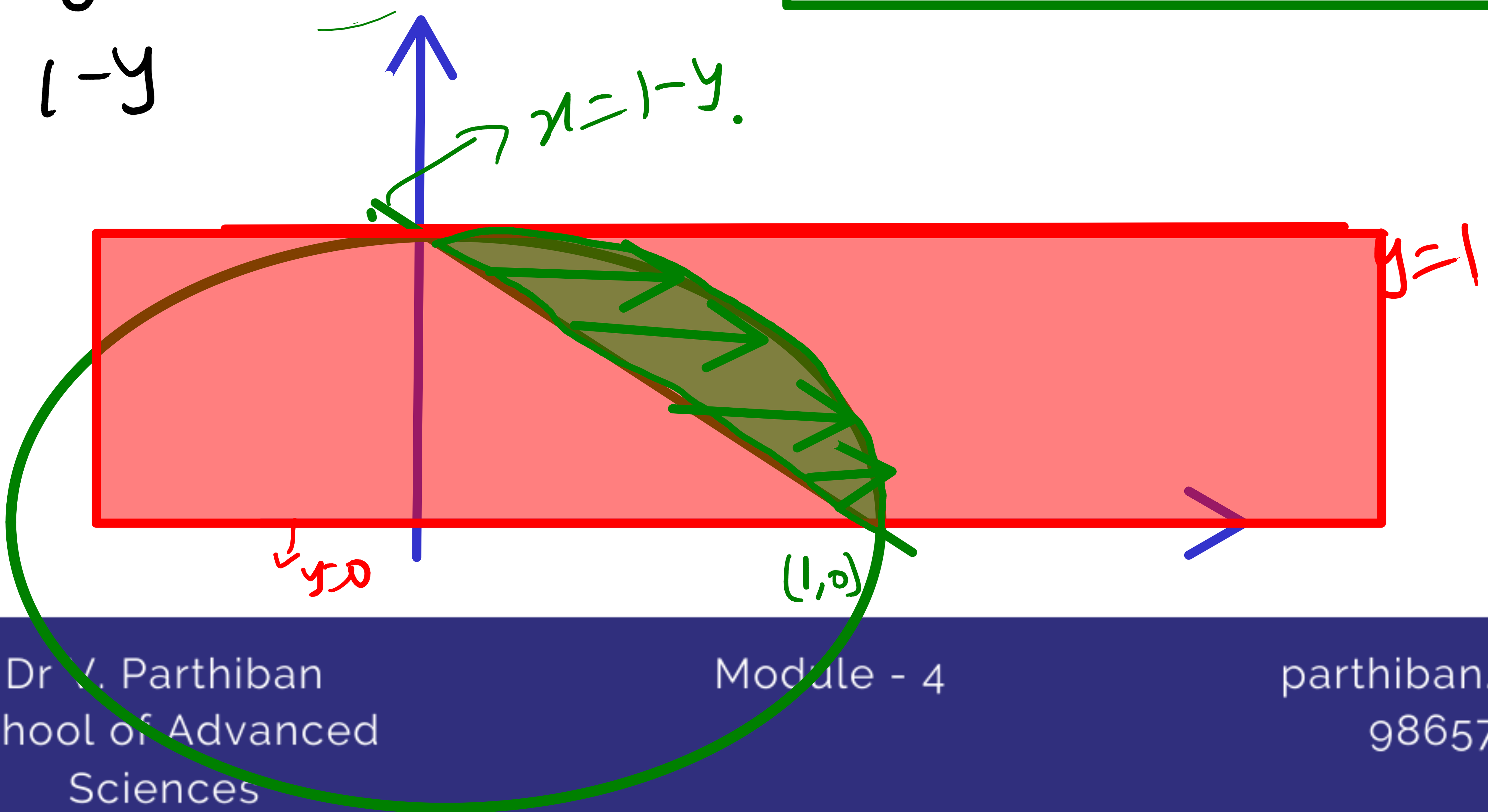
$$= \frac{0-1}{-1} = 1$$

1) $\int_1^3\int_0^{6/x} x^2\,dy\,dx$

2) $\int_0^1\int_{1-y}^{\sqrt{1-y^2}} y\,dx\,dy$

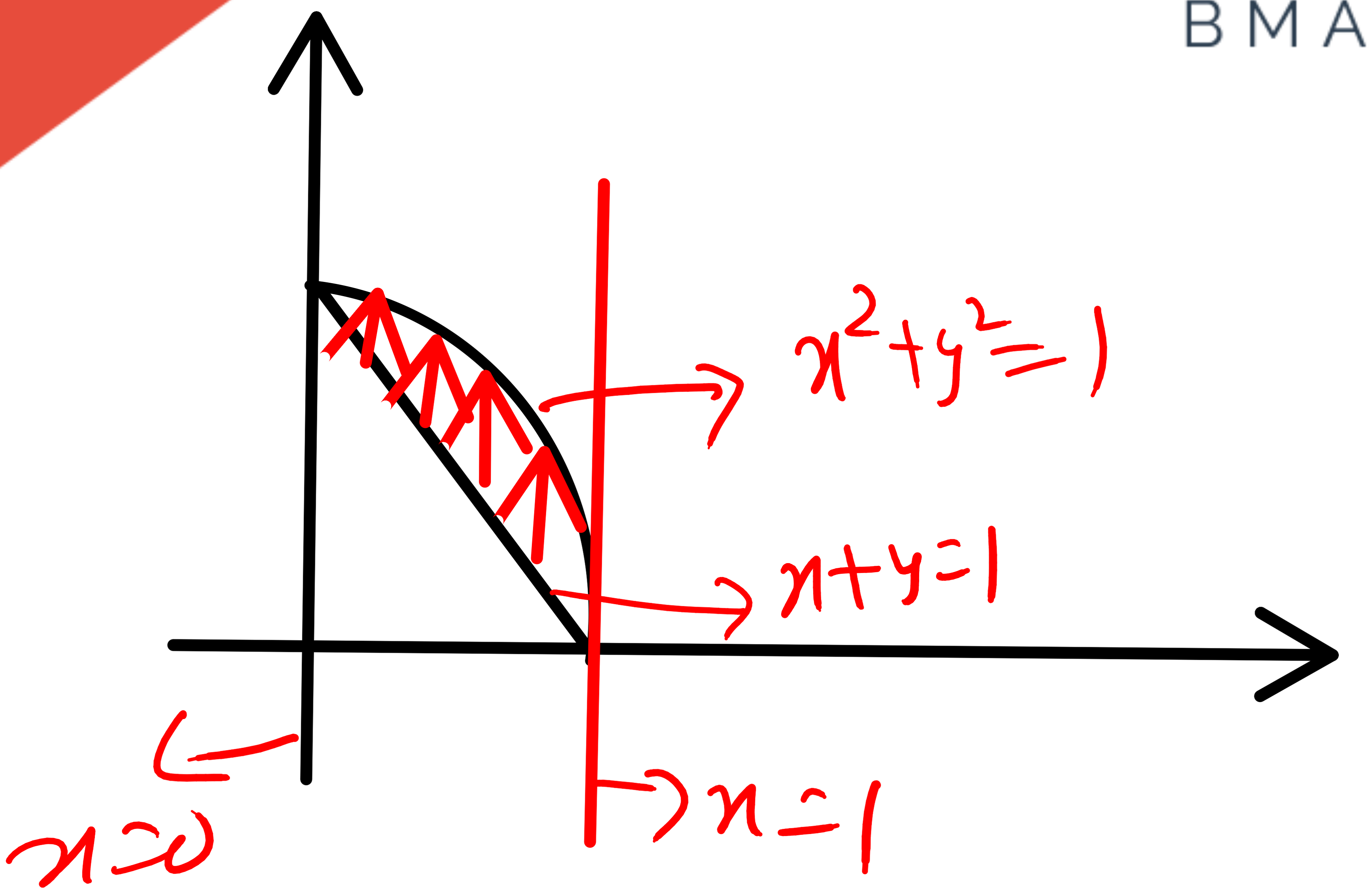
$y \to 0$ to 1

$x \to 1-y$ to $\sqrt{1-y^2}$



$x=\sqrt{1-y^2}$

$x^2+y^2=1$



$x \to 0$ to 1

$y \to 1-x$ to $\sqrt{1-x^2}$

$$\int_0^1 \int_{1-y}^{\sqrt{1-y^2}} y \, dx \, dy = \int_0^1 \int_{1-x}^{\sqrt{1-x^2}} y \, dy \, dx$$

$$\Rightarrow \int_0^1 xy \Big|_{1-y}^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 \frac{y^2}{2} \Big|_{1-x}^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 y\sqrt{1-y^2} - (1-y)y \, dy$$

$$= \int_0^1 \frac{1-x^2 - (1-x)^2}{2} dx$$

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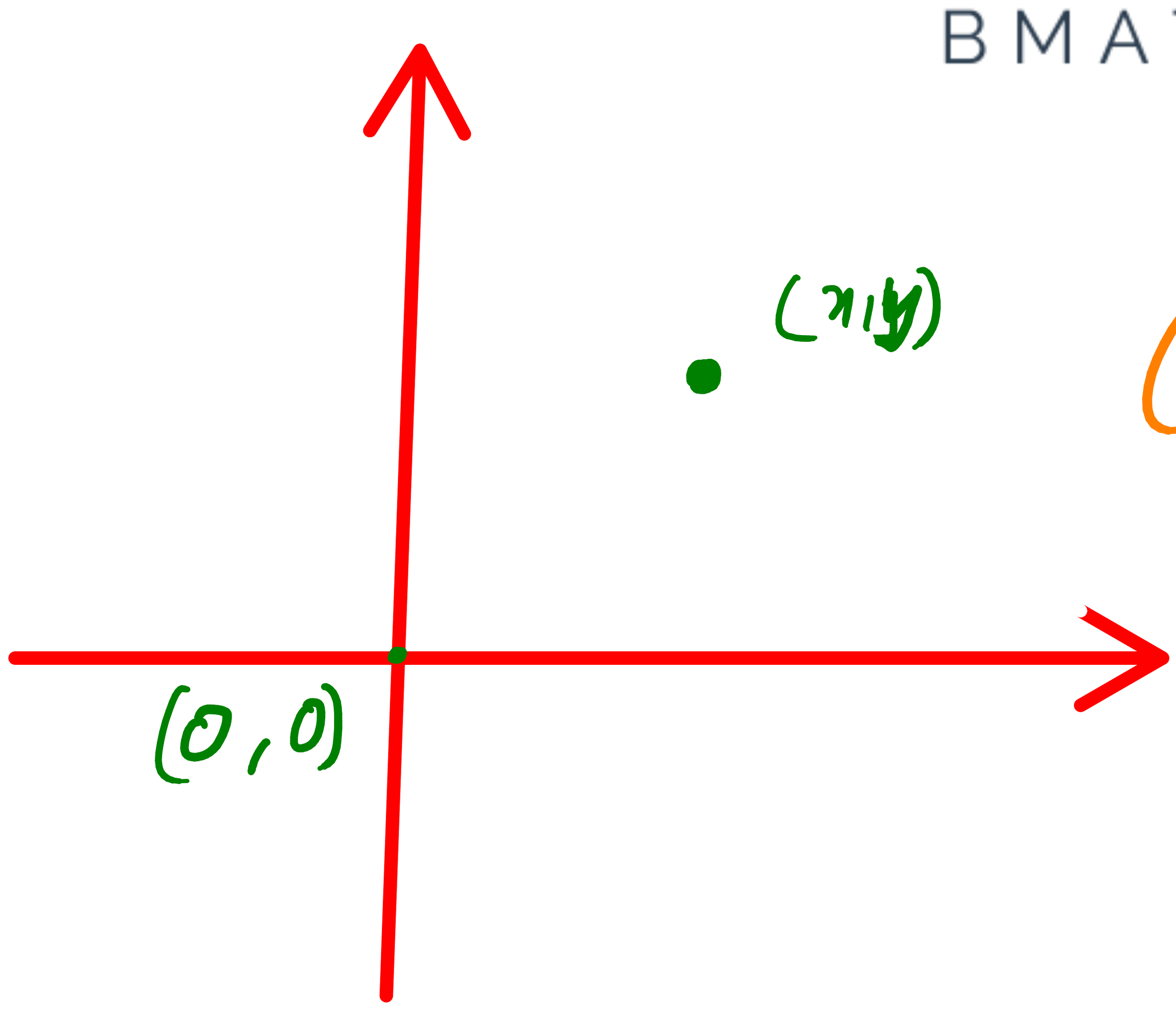
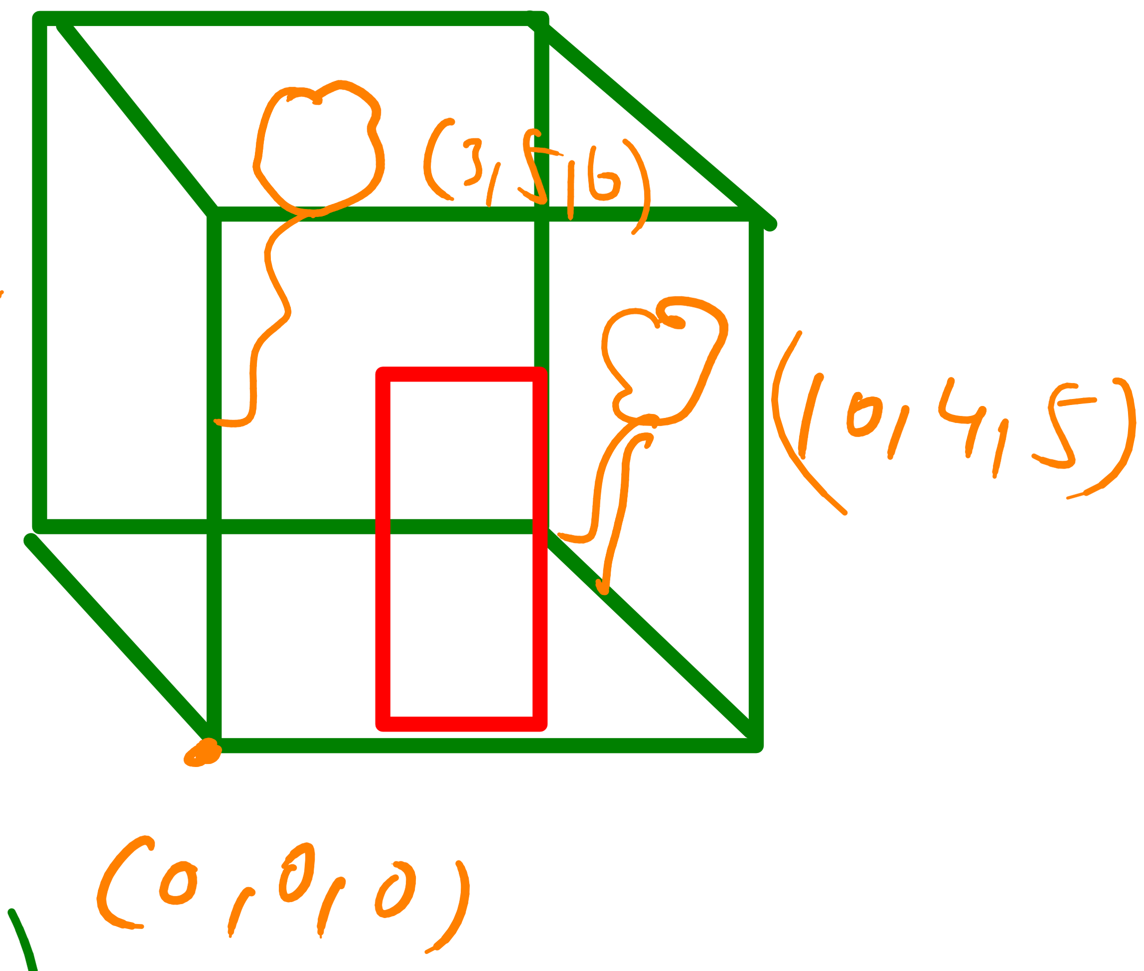
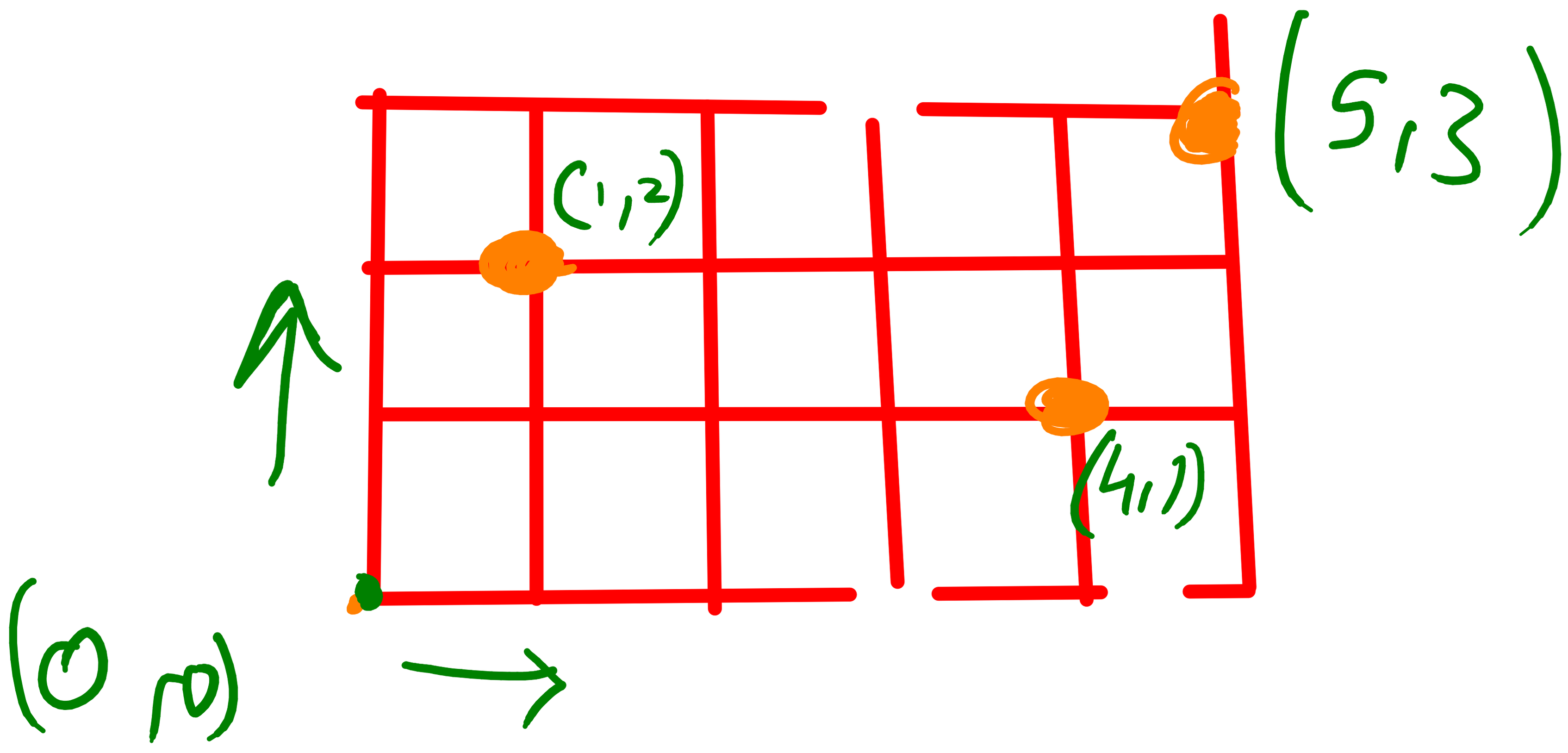
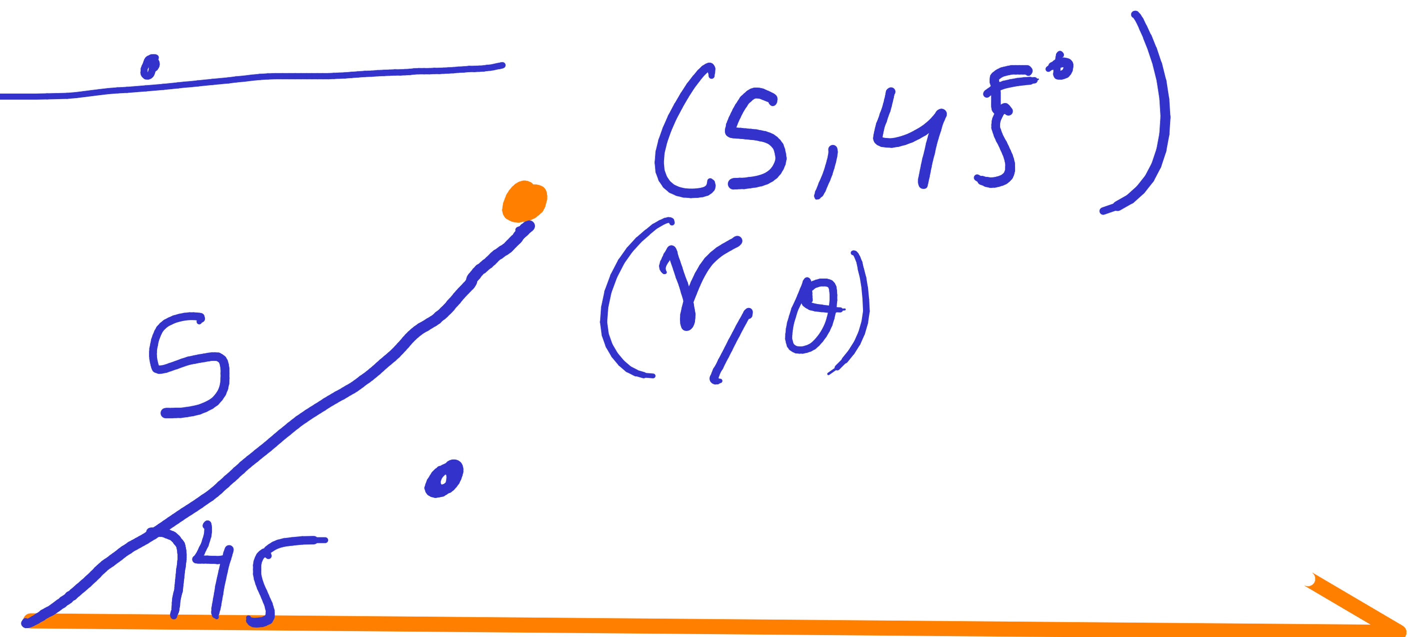


Cartesian Coordinate

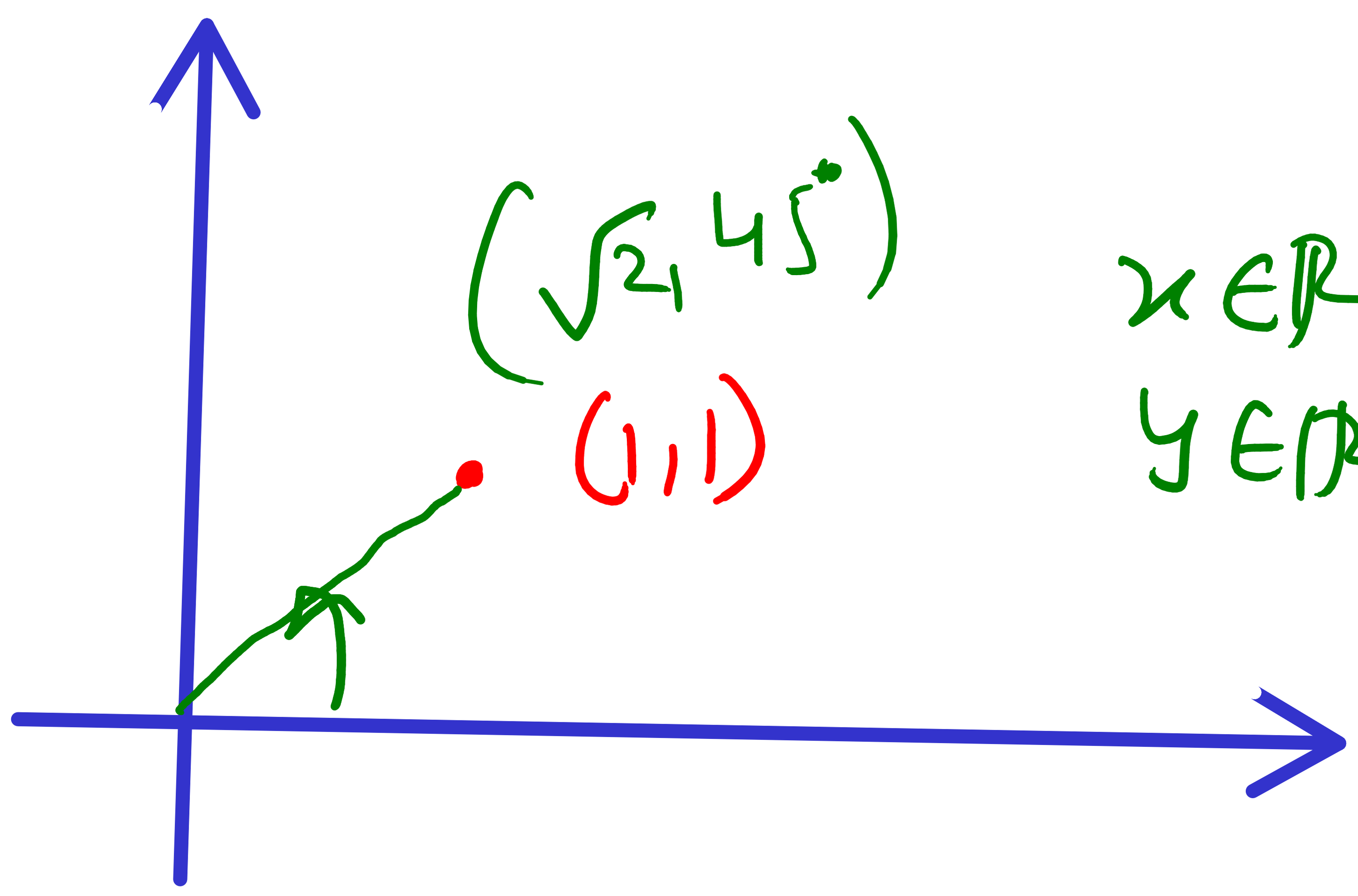




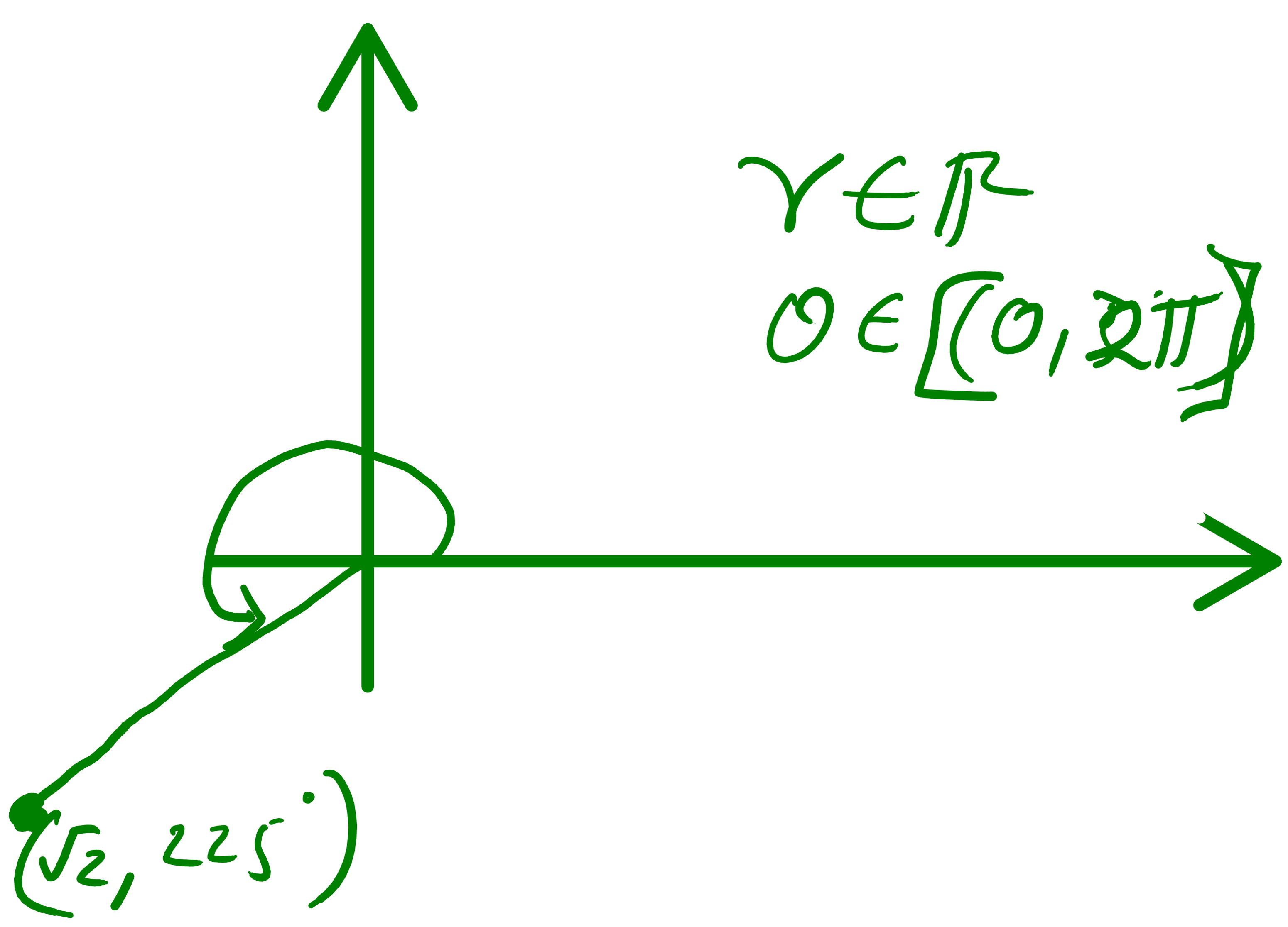
Polar coordinates



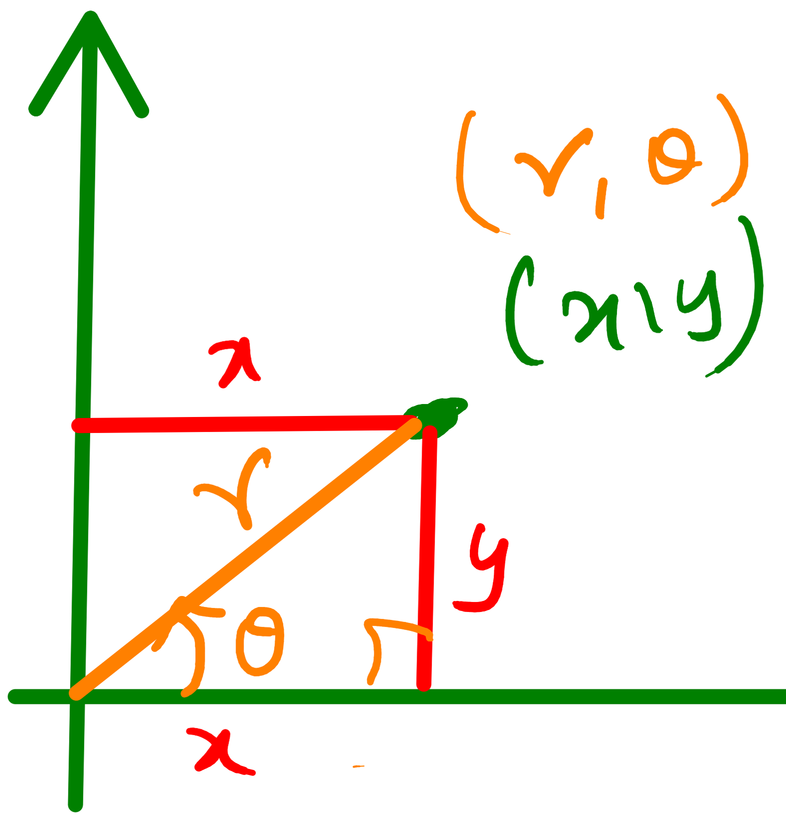
initial line



$x \in \mathbb{R}$
$y \in \mathbb{R}$



$r \in \mathbb{R}$
$\theta \in [0, 2\pi]$



$$r = \sqrt{x^2+y^2}$$

$$\tan\theta = y/x$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$$

$$r = \sqrt{x^2+y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$(5, 10)$

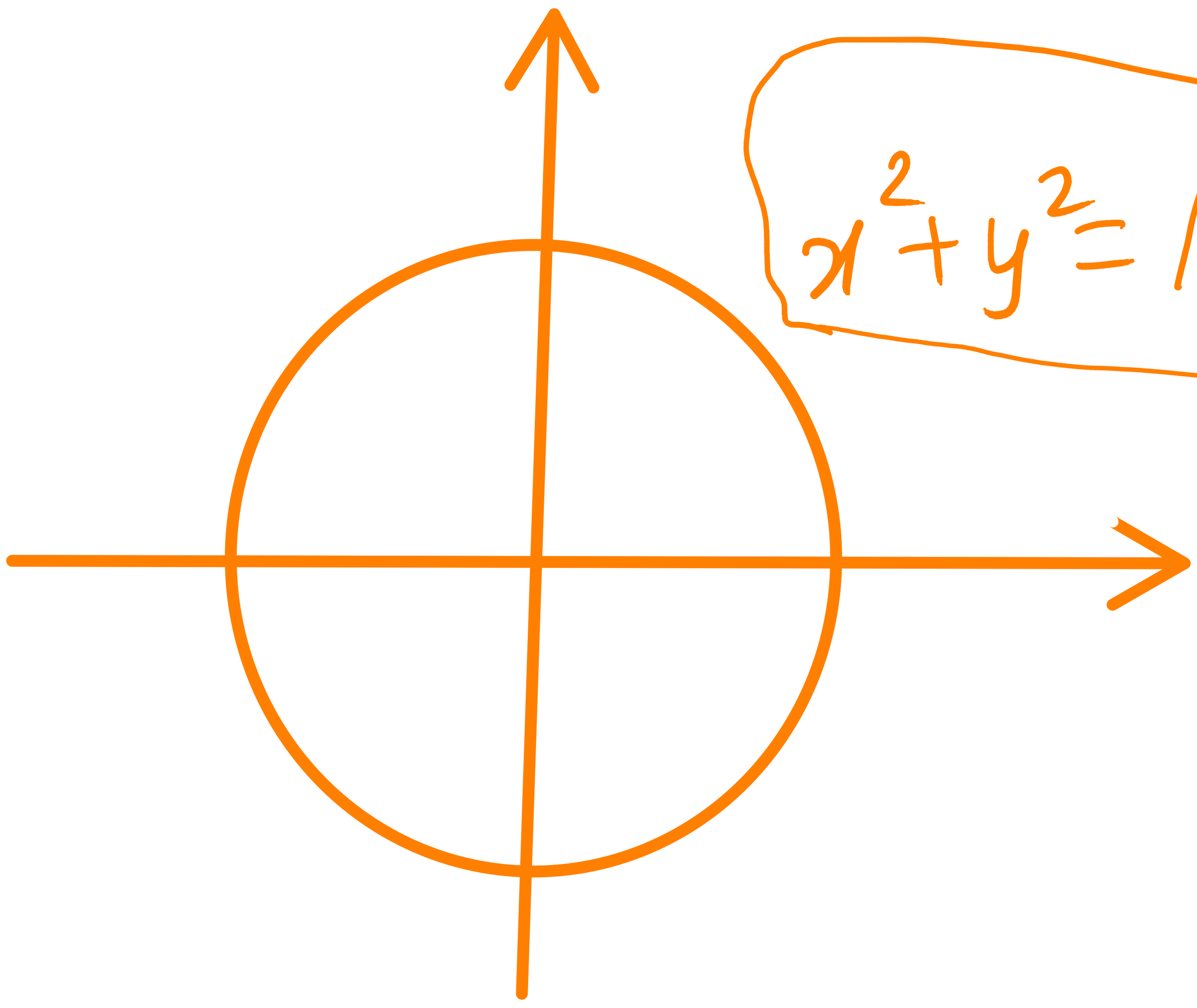
$\sqrt{125}$

$(\cancel{15}, \tan^{-1}(2))$

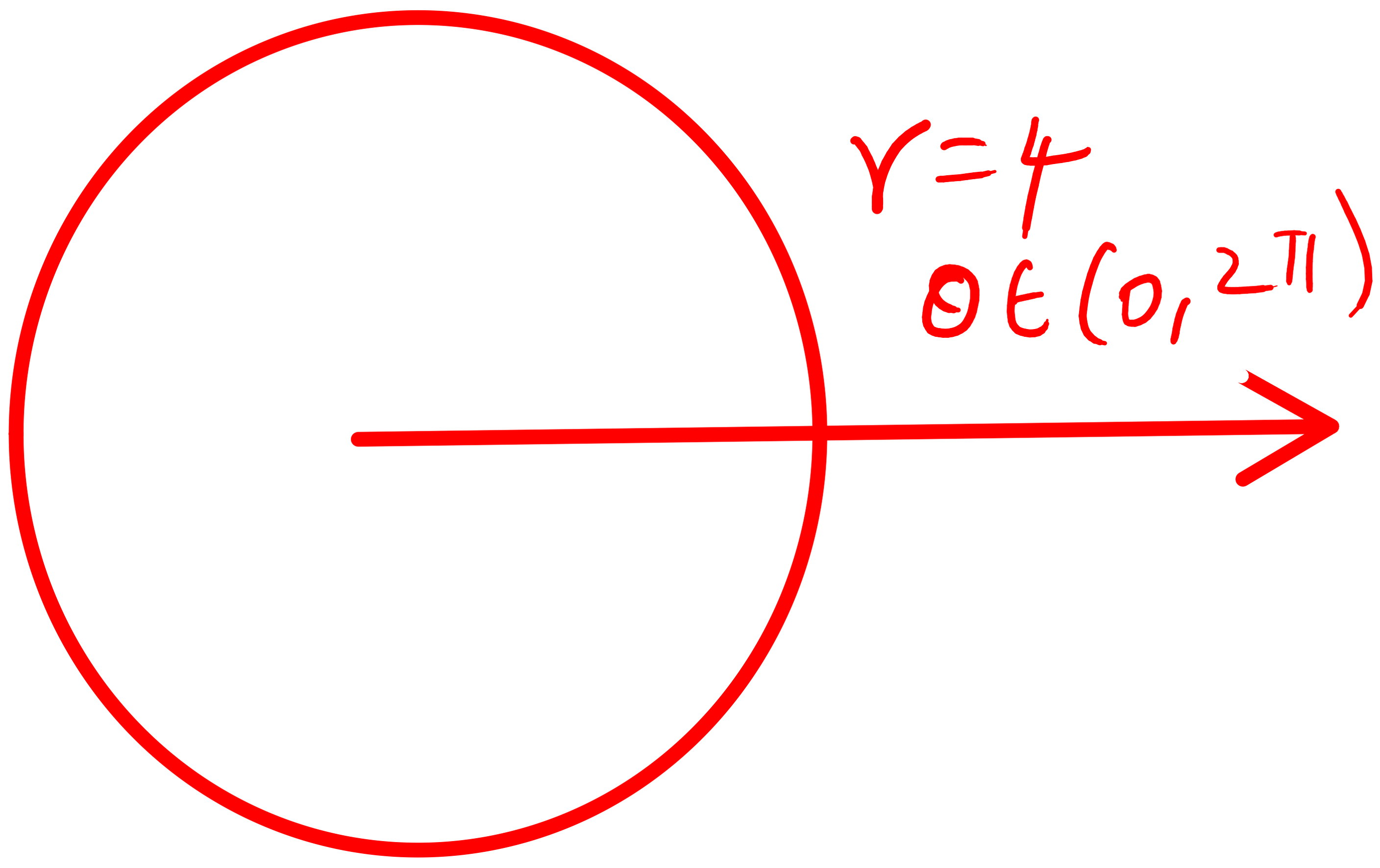
$(11.180, 63.43^\circ)$

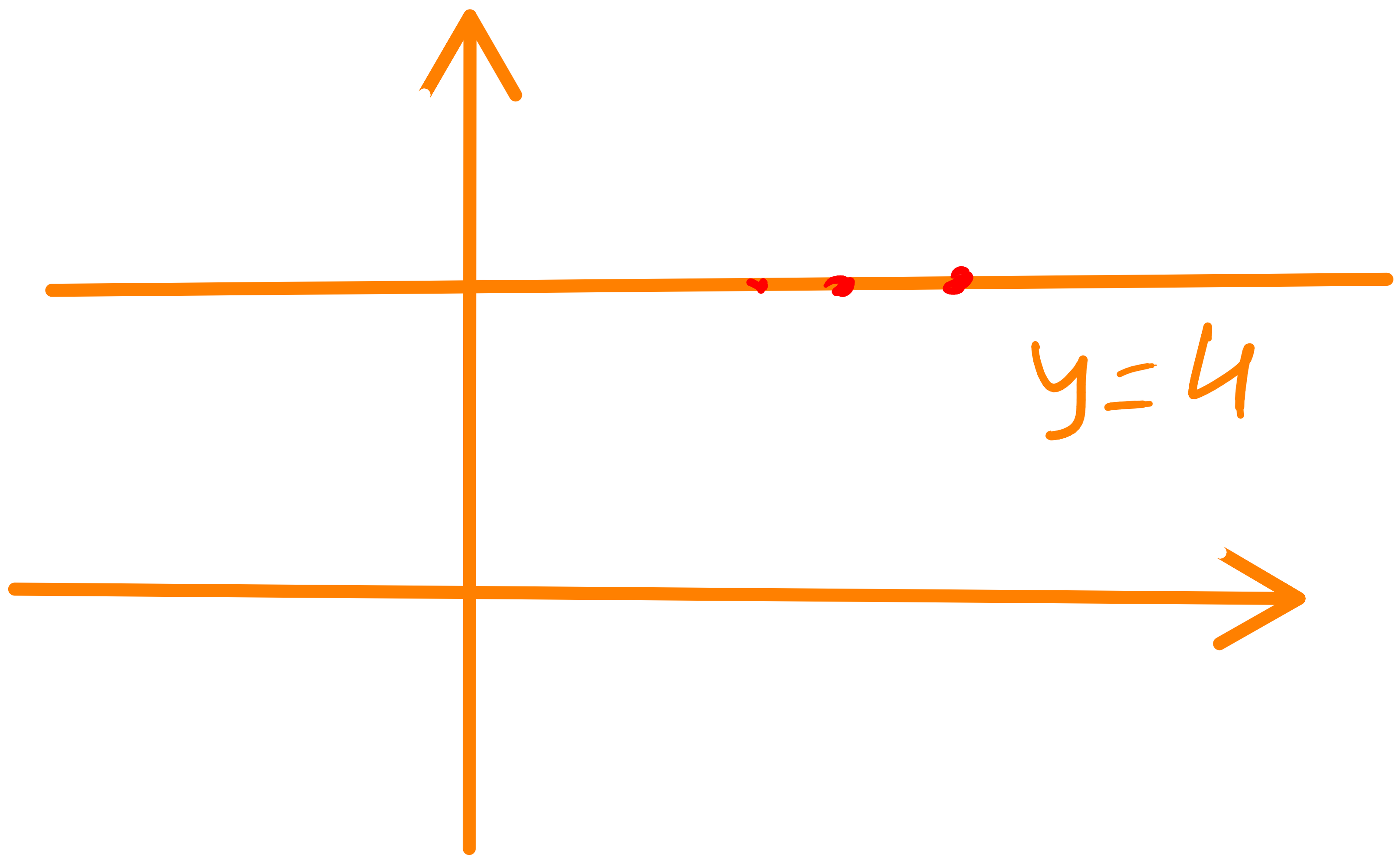
$x = 11.180 \times \cos(63.43^\circ)$
$= 5$

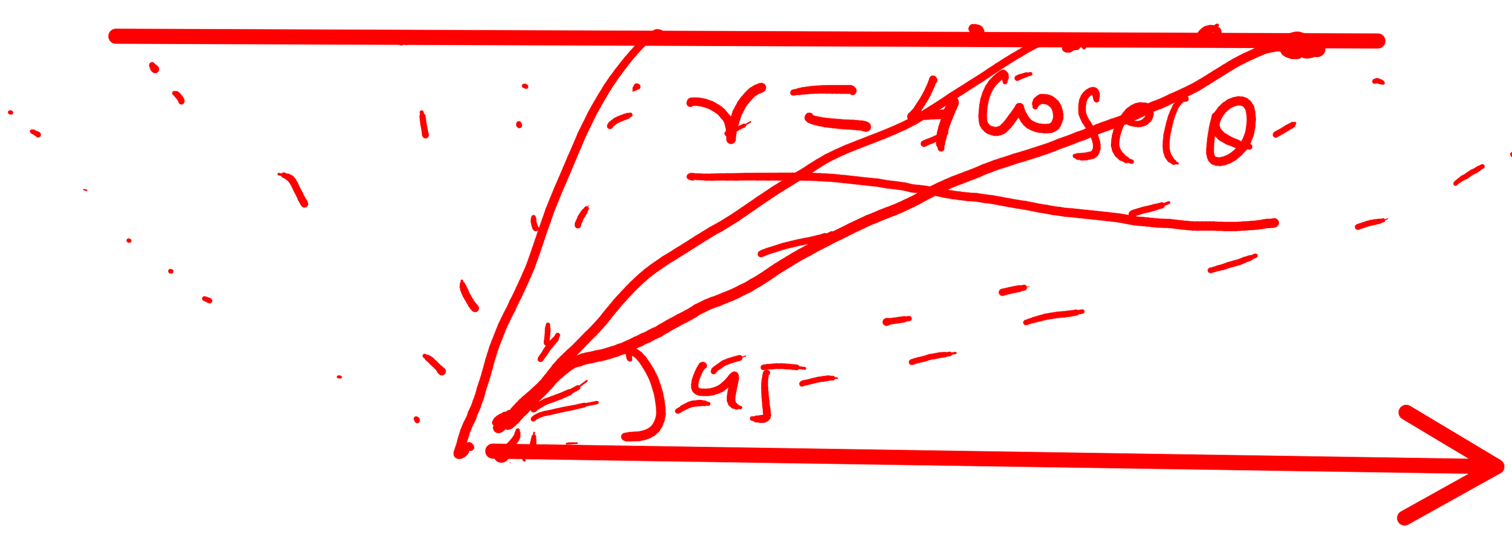
$y = 11.180 \times \sin(63.43^\circ)$
$= 10$



$$x^2+y^2=16$$







$$(x-a)^2+(y-b)^2=c^2$$

$$x=r\cos\theta \qquad y=r\sin\theta$$

$$(r\cos\theta-a)^2+(r\sin\theta-b)^2=c^2$$

$$\underline{r^2\cos^2\theta}+a^2-2ar\cos\theta+r^2\sin^2\theta+b^2-2br\sin\theta=c^2$$

$$r^2+a^2+b^2-2(ra\cos\theta+rb\sin\theta)=c^2$$

$$r^2+\underbrace{a^2+b^2-c^2}_{k}=2r(a\cos\theta+b\sin\theta)$$

$$r=\frac{-2(a\cos\theta+b\sin\theta)}{2}\pm\sqrt{}$$

$$r=a\cos\theta+b\sin\theta$$

$$(x,y)\to(r,\theta)$$

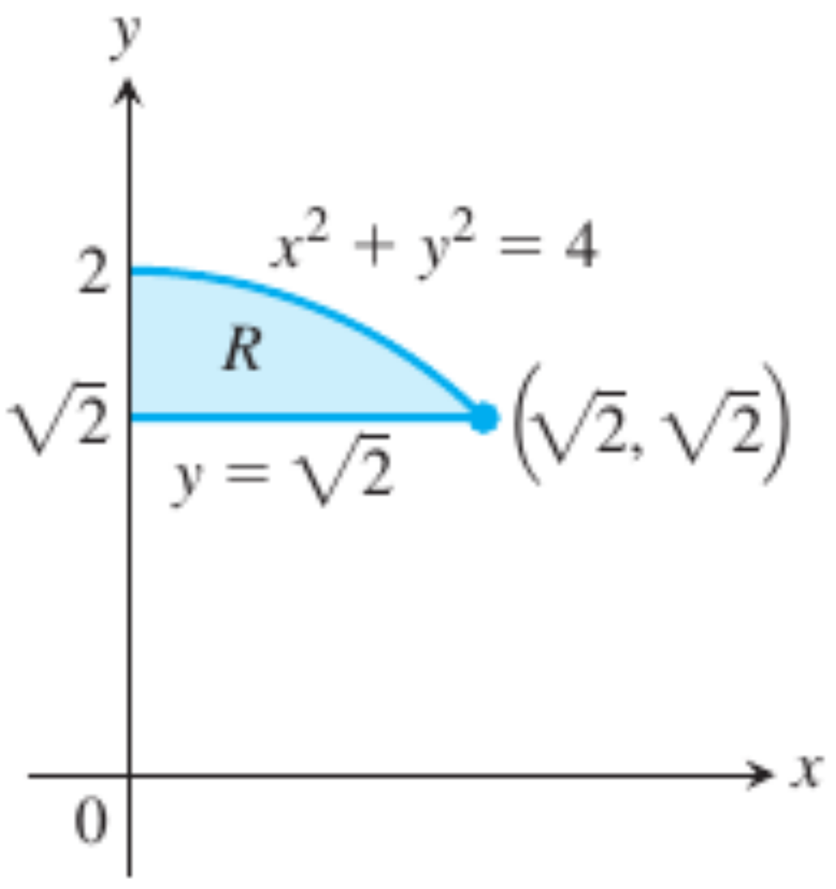
$$r=\sqrt{x^2+y^2}$$

$$\theta=\tan^{-1}(y/x)$$

$$(r,\theta)\to(x,y)$$

$$x=r\cos\theta$$

$$y=r\sin\theta$$

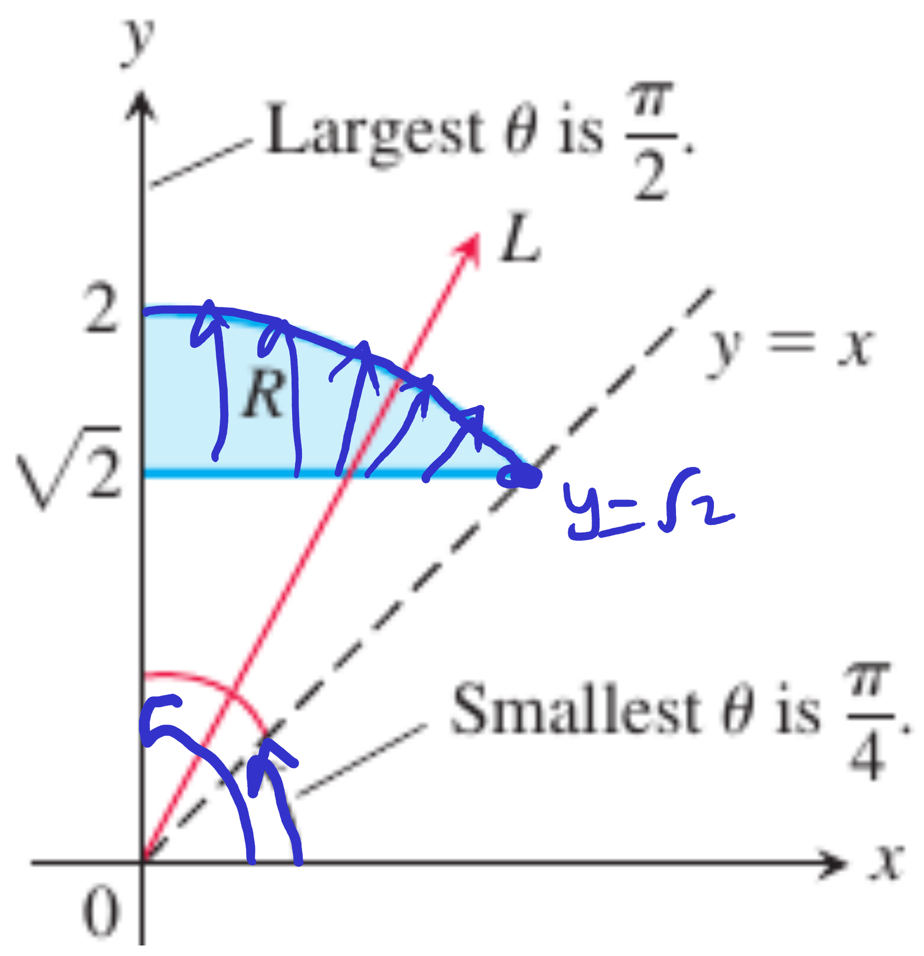


(a)

$$\int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-y^2}} dx\,dy = \int_{\sqrt{2}}^{2} \sqrt{4-y^2}\,dy$$

$y \to \sqrt{2}$ to 2

$x \to 0$ to $\sqrt{4-y^2}$



$x \to 0$ to $\sqrt{2}$

$y \to \sqrt{2}$ to $\sqrt{4-x^2}$

$\theta \to \pi/4$ to $\pi/2$

$r \to \sqrt{2}\,\mathrm{cosec}(\theta)$ to 2

$$\int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-y^2}} dx\,dy = \int_{\pi/4}^{\pi/2}\int_{\sqrt{2}\,\mathrm{cosec}(\theta)}^{2} |J|\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\int_{\sqrt{2}\,\mathrm{cosec}(\theta)}^{2} r\,dr\,d\theta$$

$\int x\,dx$

$x = r^2$

$dx = 2r\,dr$

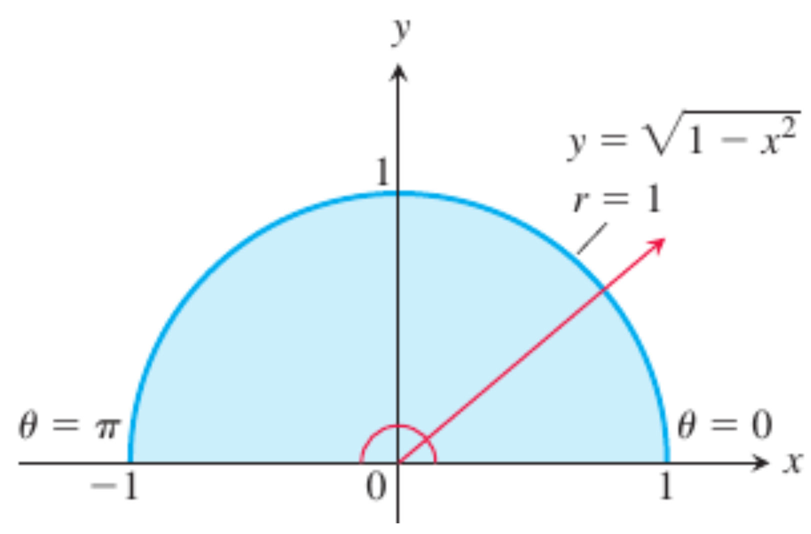
$(x,y) \to (r,\theta)$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

EXAMPLE 3 Evaluate

$$\iint_R e^{x^2+y^2}\, dy\, dx,$$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$ (Figure 15.27).



$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} e^{x^2+y^2}\, dy\, dx$$

$\theta \to 0$ to π

$r \to 0$ to 1

$$\| $$

$$\int_0^{\pi}\int_0^{1} e^{r^2}\, r\, dr\, d\theta$$

$\int f(nx)\, dx$

$\int f(x)\, dx = F(x)$

$\int f(nx)\, dx = \dfrac{F(nx)}{n}$

$\int e^{2x}\, dx$

$\dfrac{e^{2x}}{2}$

$\int f(x^2)\, dx \quad f(x)$

$x = r\cos\theta \qquad y = r\sin\theta$

$$\begin{vmatrix} \dfrac{\partial x}{\partial r} = \cos\theta & \dfrac{\partial y}{\partial r} = \sin\theta \\ \dfrac{\partial x}{\partial \theta} = -r\sin\theta & \dfrac{\partial y}{\partial \theta} = r\cos\theta \end{vmatrix} = r$$

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Evaluate the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx.$$

$$= \int_0^1 \left. x^2 y + \frac{y^3}{3} \right|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 x^2\sqrt{1-x^2} + \frac{(1-x^2)\sqrt{1-x^2}}{3}\, dx$$

$x = r\cos\theta$

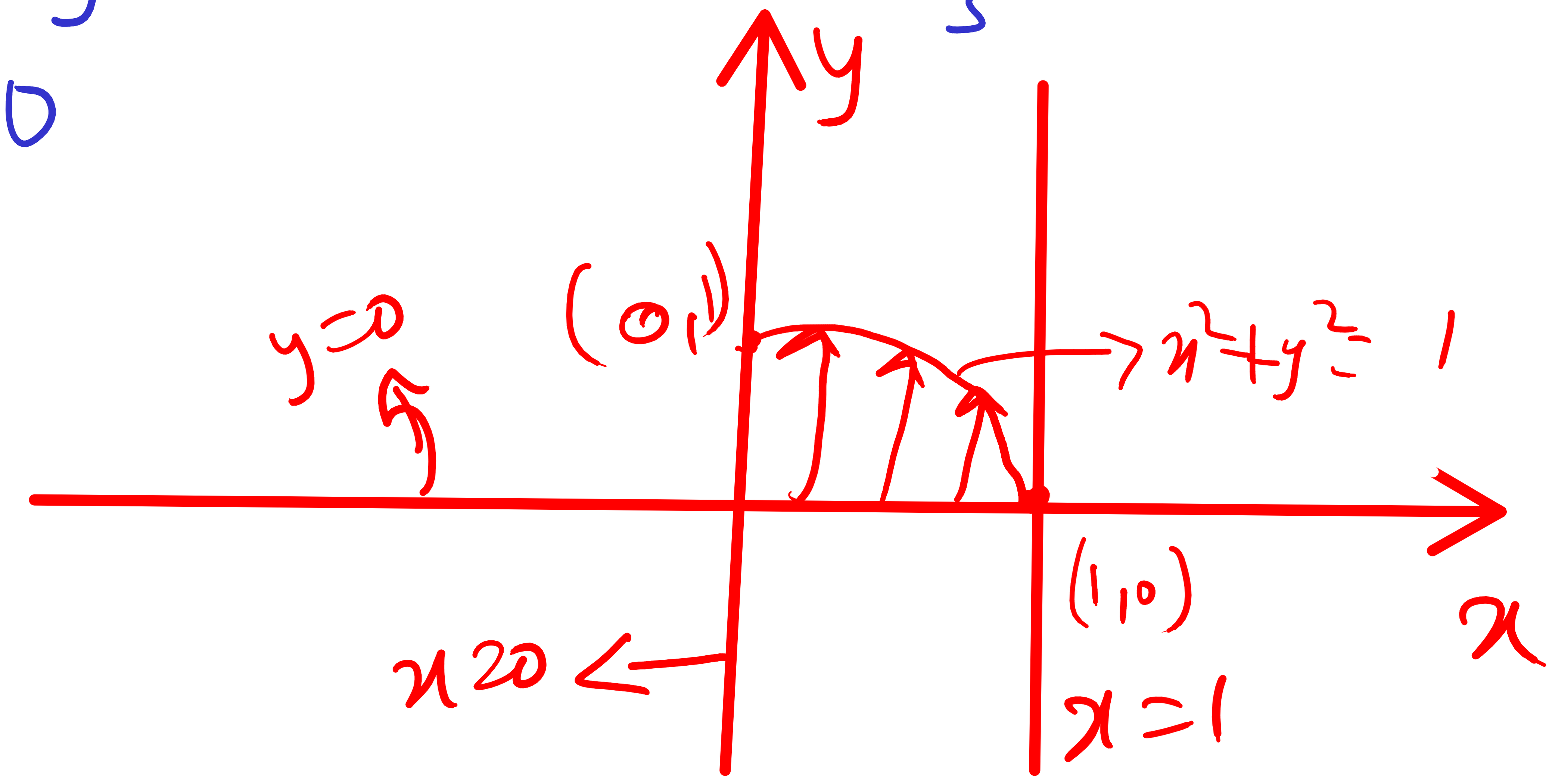
$y = r\sin\theta$

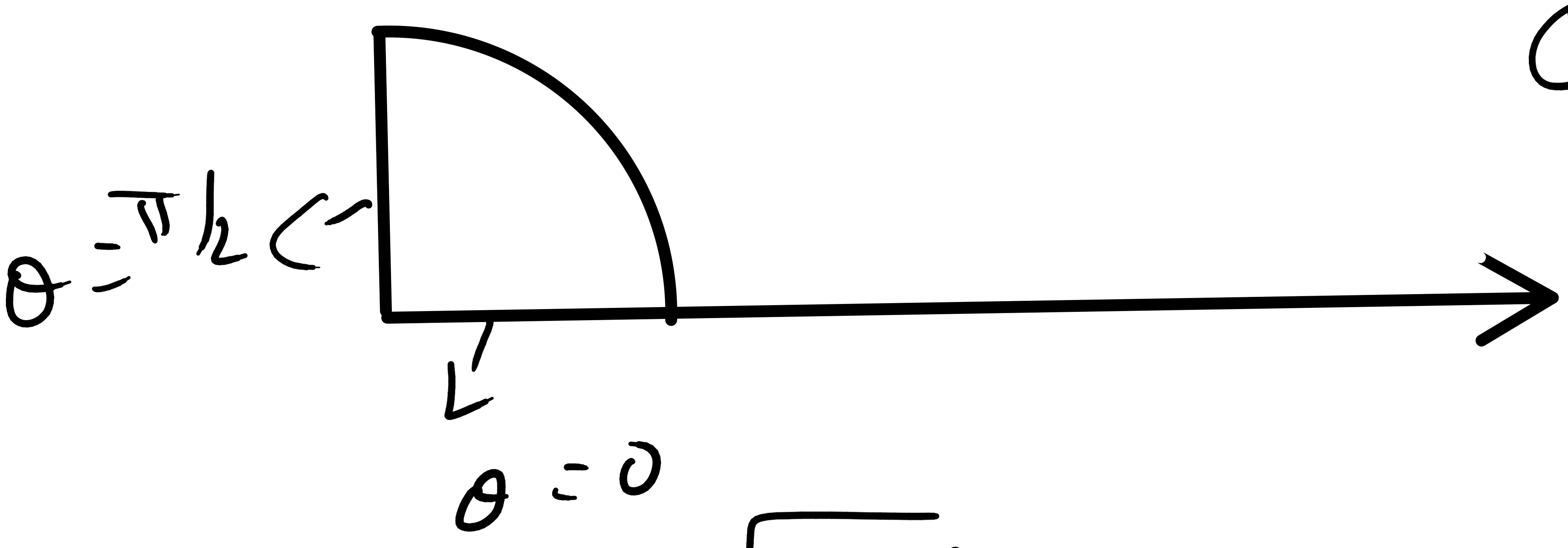
$dy\,dx = |J|\, dr\, d\theta$

$= r\, dr\, d\theta$

$x \to 0$ to 1

$y \to 0$ to $\sqrt{1-x^2}$





$\theta \to 0$ to $\pi/2$

$r \to 0$ to 1

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx = \int_0^{\pi/2} \int_0^1 r^2\, r\, dr\, d\theta$$

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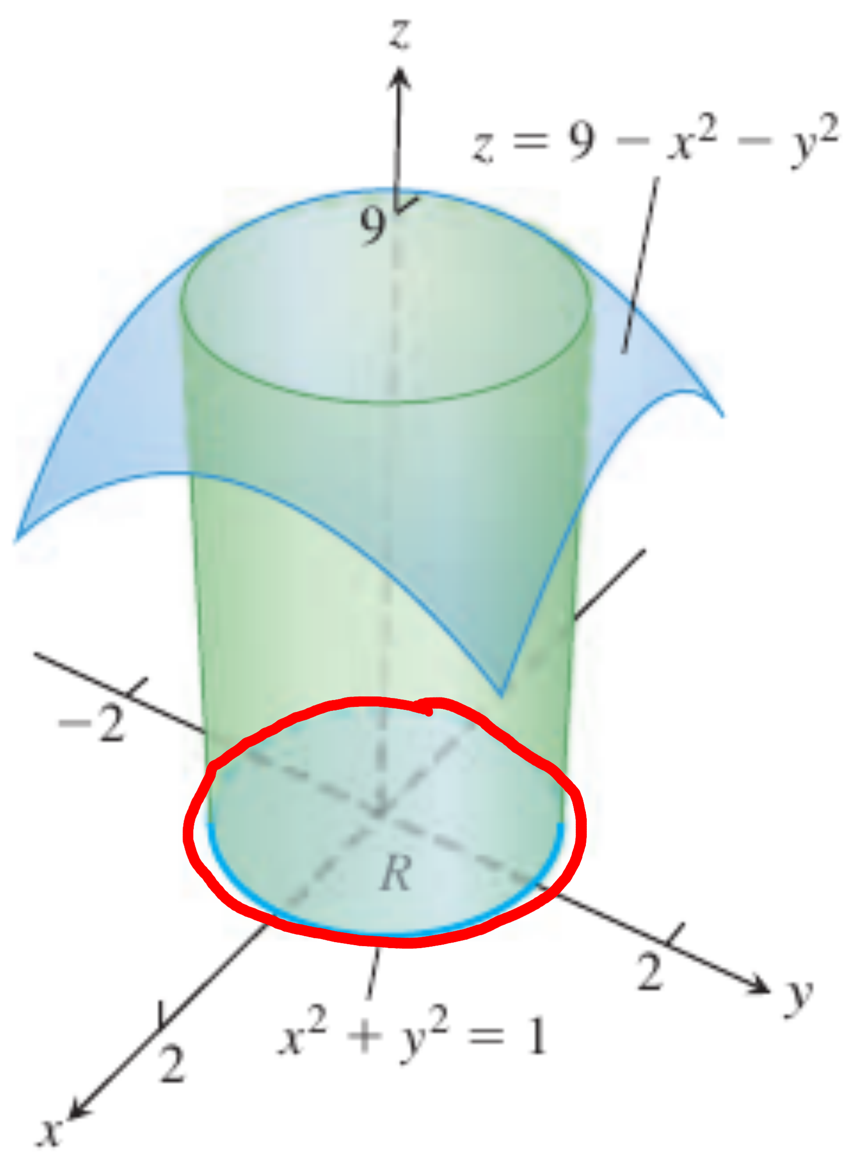
$$= \int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta = \int_0^{\pi/2} \frac{r^4}{4}\Big|_0^1 d\theta$$

$$= \frac{1}{4}\left[\theta\right]_0^{\pi/2} = \pi/8$$

EXAMPLE 5 Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.

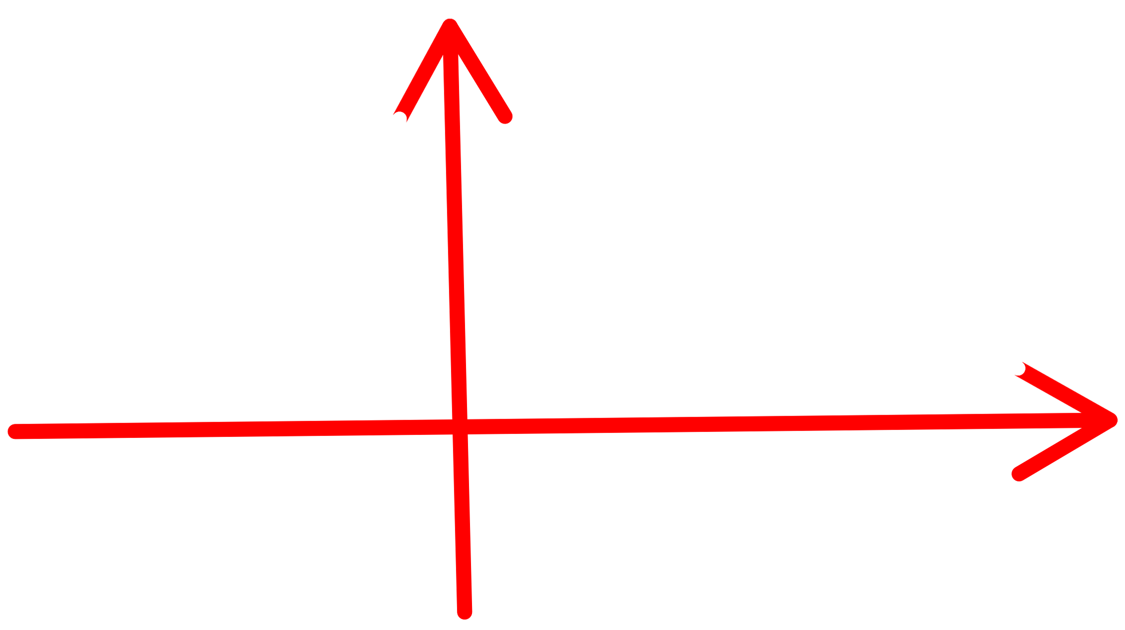


$$V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (9-x^2-y^2)\,dy\,dx$$

$\theta \to 0$ to 2π

$r \to 0$ to 1

$$= \int_0^{2\pi}\int_0^1 (9-r^2)\,r\,dr\,d\theta$$

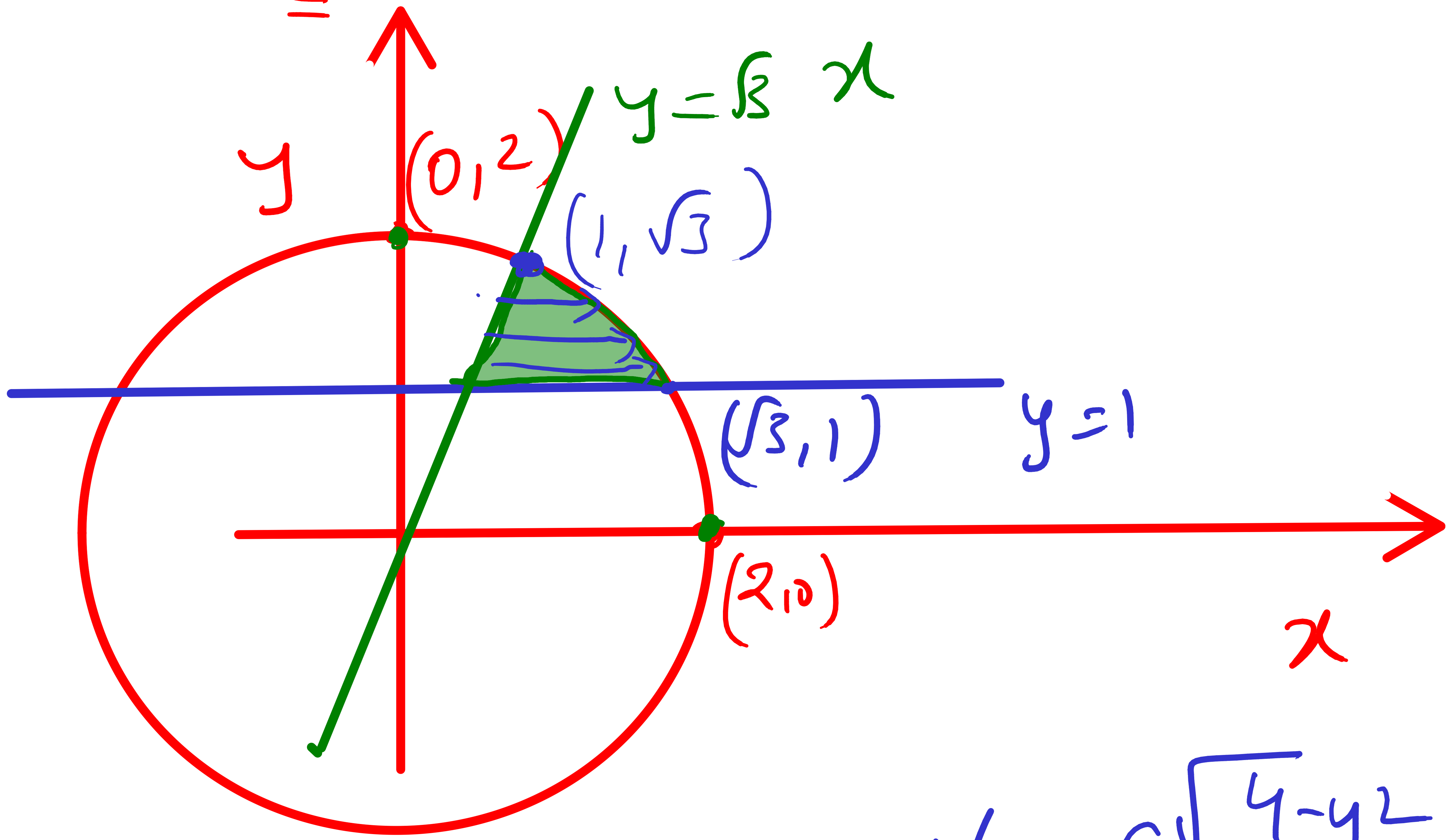


$$= \int_0^{2\pi} \frac{9r^2}{2} - \frac{r^4}{4}\Big|_0^1 d\theta$$

$$= \int_0^{2\pi}\left(\frac{9}{2} - \frac{1}{4}\right) d\theta = \frac{17}{4}\left[\theta\right]_0^{2\pi}$$

$$= 17\pi/2$$

EXAMPLE 6 Using polar integration, find the area of the region R in the xy-plane enclosed by the circle $x^2 + y^2 = 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.



$y \to 1$ to $\sqrt{3}$

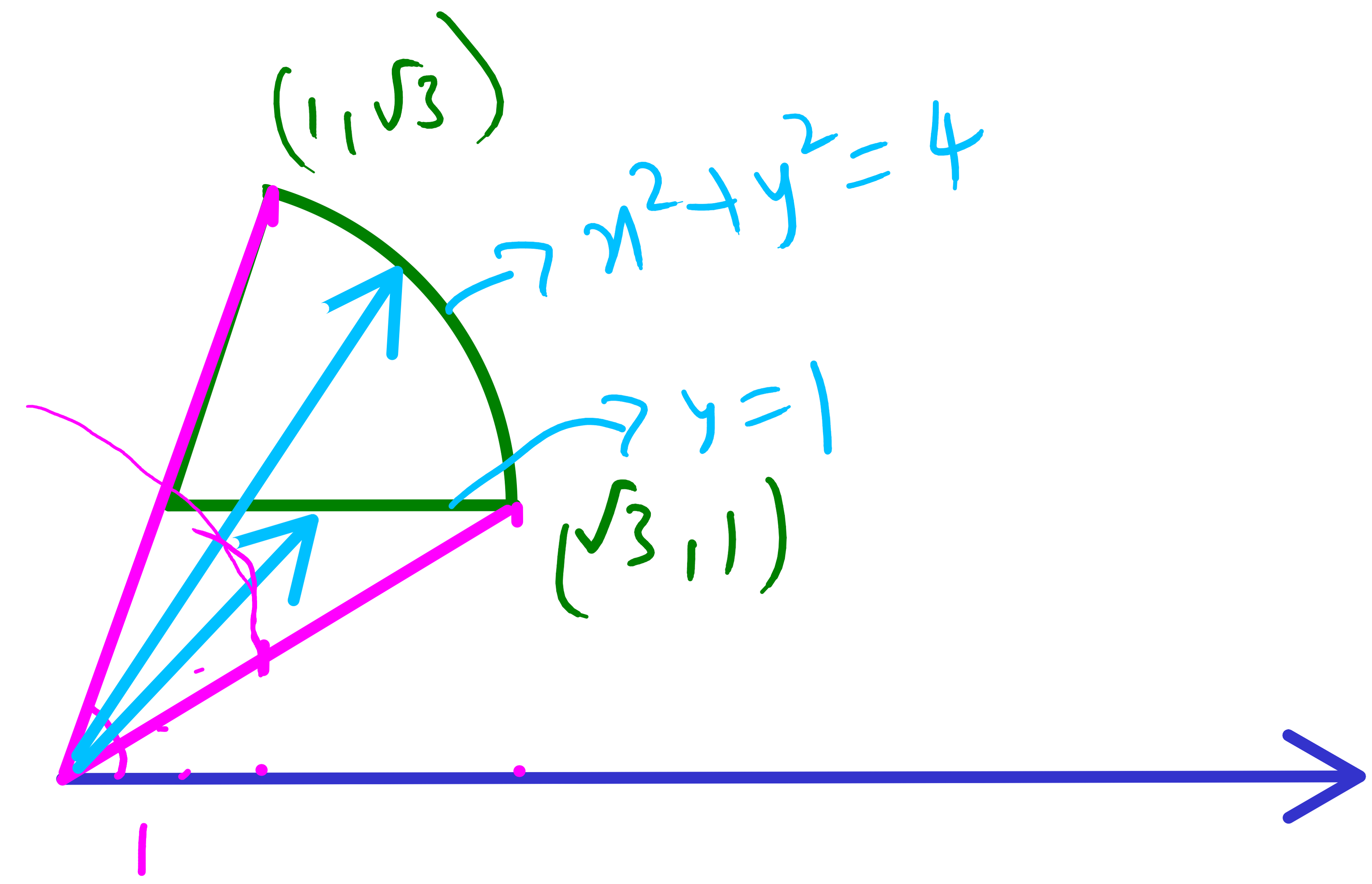
$x \to \dfrac{y}{\sqrt{3}}$ to $\sqrt{4-y^2}$

$$\text{Area} = \int_{1}^{\sqrt{3}} \int_{y/\sqrt{3}}^{\sqrt{4-y^2}} dx\, dy$$

$y = 1$

$r\sin\theta = 1$

$r = \operatorname{cosec}\theta$



$\tan\theta_1 = 1/\sqrt{3}$

$\theta_1 = \tan^{-1}(1/\sqrt{3})$

$= \pi/6$

$\tan\theta_2 = \sqrt{3}$

$\theta_2 = \tan^{-1}(\sqrt{3})$

$= \pi/3$

$\theta \to \pi/6$ to $\pi/3$

$r \to \operatorname{cosec}\theta$ to 2

$$\text{Area} = \int_{\pi/6}^{\pi/3}\int_{\operatorname{cosec}\theta}^{2} r\,dr\,d\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{r^2}{2}\Big|_{\operatorname{cosec}\theta}^{2} d\theta = \int_{\pi/6}^{\pi/3} 2 - \frac{\operatorname{cosec}^2\theta}{2}\,d\theta$$

$$= 2\theta + \frac{\cot\theta}{2}\Big|_{\pi/6}^{\pi/3}$$

$$= \left[\frac{2\pi}{3} + \frac{1/\sqrt{3}}{2}\right] - \left[\frac{2\pi}{6} + \frac{\sqrt{3}}{2}\right]$$

$$= \frac{\pi - \sqrt{3}}{3}$$

$$\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2}\,dy\,dx \qquad \int_0^1 \int_x^{\sqrt{2-x^2}} (x+2y)\,dy\,dx$$

$$\int_a^b f(x)\,dx \rightarrow \text{Area}$$

$$\int_a^b \int_{f_1(x)}^{f_2(x)} dy\,dx$$

$$\int_a^b \int_{f_1(x)}^{f_2(x)} \underline{\underline{g(x,y)}}\,dy\,dx$$

$$\| \quad \text{Volume}$$

$$\int_a^b \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} dz\,dy\,dx$$

$$I = \int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,\underset{\mathrm{I}}{dz}\,\underset{\mathrm{II}}{dy}\,\underset{\mathrm{III}}{dx}$$

$$\int_{-1}^{1} \int_0^z \int_{x-z}^{x+z} (x+y+z)\,\cancel{dx\,dy\,dz}\,\overset{dy\,dx\,dz}{}.$$

$$= \int_{-1}^{1} \int_0^z xy + \frac{y^2}{2} + yz \Big]_{x-z}^{x+z} dx\,dz$$

$$= \int_{-1}^{1} \int_0^z (x+z)\,2z + 2xz\;dx\;dz$$

$$= 2\int_{-1}^{1} \frac{x^2 z}{2} + xz^2 + \frac{x^2 z}{2}\Big|_0^z \, dz$$

$$= 2\int_{-1}^{1} \left(\frac{z^3}{2} + z^3 + \frac{z^3}{2}\right) dz$$

$$= 4\left[\frac{z^4}{4}\right]_{-1}^{1} = 0$$

EXAMPLE 1 Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

$$\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$z = x^2 + 3y^2$

$z = 8 - x^2 - y^2$

$x^2 + 3y^2 = 8 - x^2 - y^2$

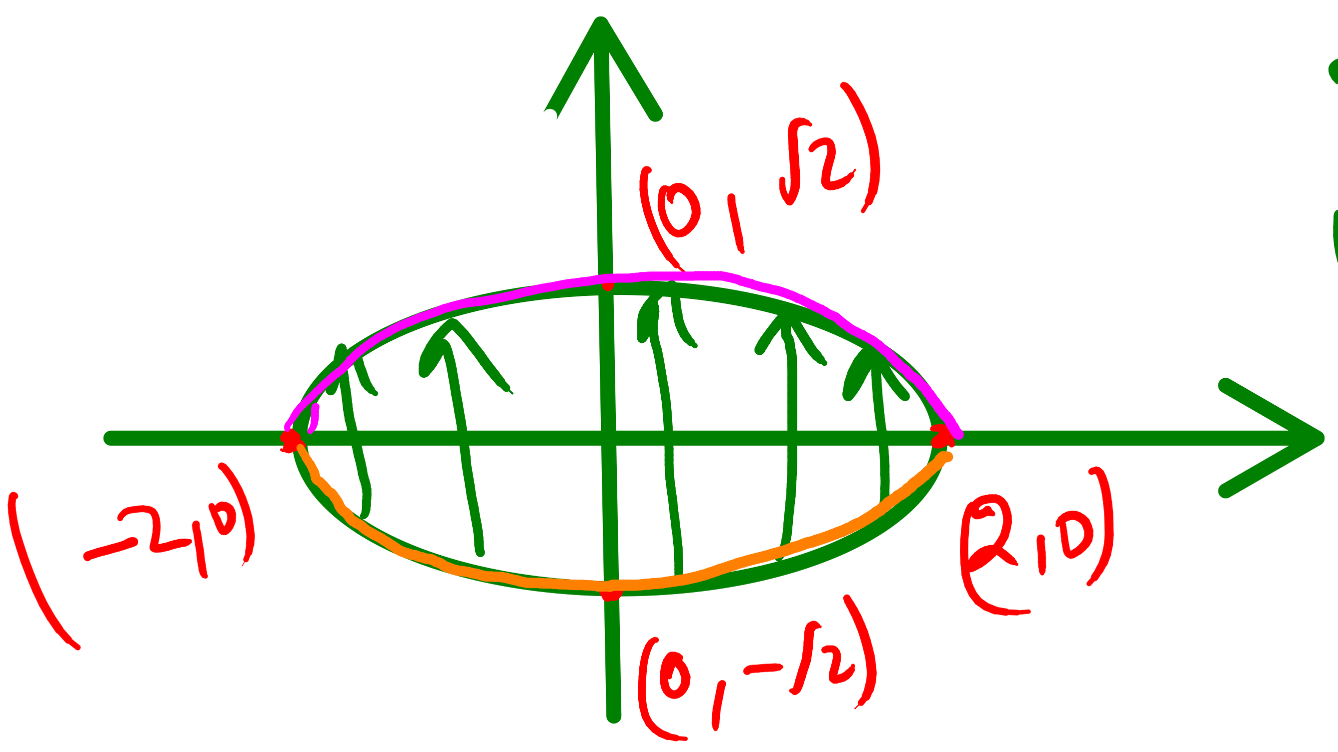
$2x^2 + 4y^2 = 8$

$x^2 + 2y^2 = 4$

$$\boxed{\frac{x^2}{4} + \frac{y^2}{2} = 1}$$

$\frac{y^2}{2} = 1 - \frac{x^2}{4}$

$y = \sqrt{\frac{4-x^2}{2}}$



$x \to -2$ to 2

$y \to -\sqrt{\frac{4-x^2}{2}}$ to $\sqrt{\frac{4-x^2}{2}}$

$$V = \iiint_D dz\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\, dy\, dx$$

$$= \int_{-2}^{2} \left[(8 - 2x^2)y - \frac{4}{3}y^3 \right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx$$

$$= \int_{-2}^{2} \left(2(8 - 2x^2)\sqrt{\frac{4 - x^2}{2}} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2} \right) dx$$

$$= \int_{-2}^{2} \left[8\left(\frac{4 - x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2} \right] dx = \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4 - x^2)^{3/2}\, dx$$

$$= 8\pi\sqrt{2}. \quad \text{After integration with the substitution } x = 2\sin u$$

2

$z = 8 - x^2 - y^2$

$z = x^2 + 3y^2$

$x = \sqrt{8 - y^2 - z}$

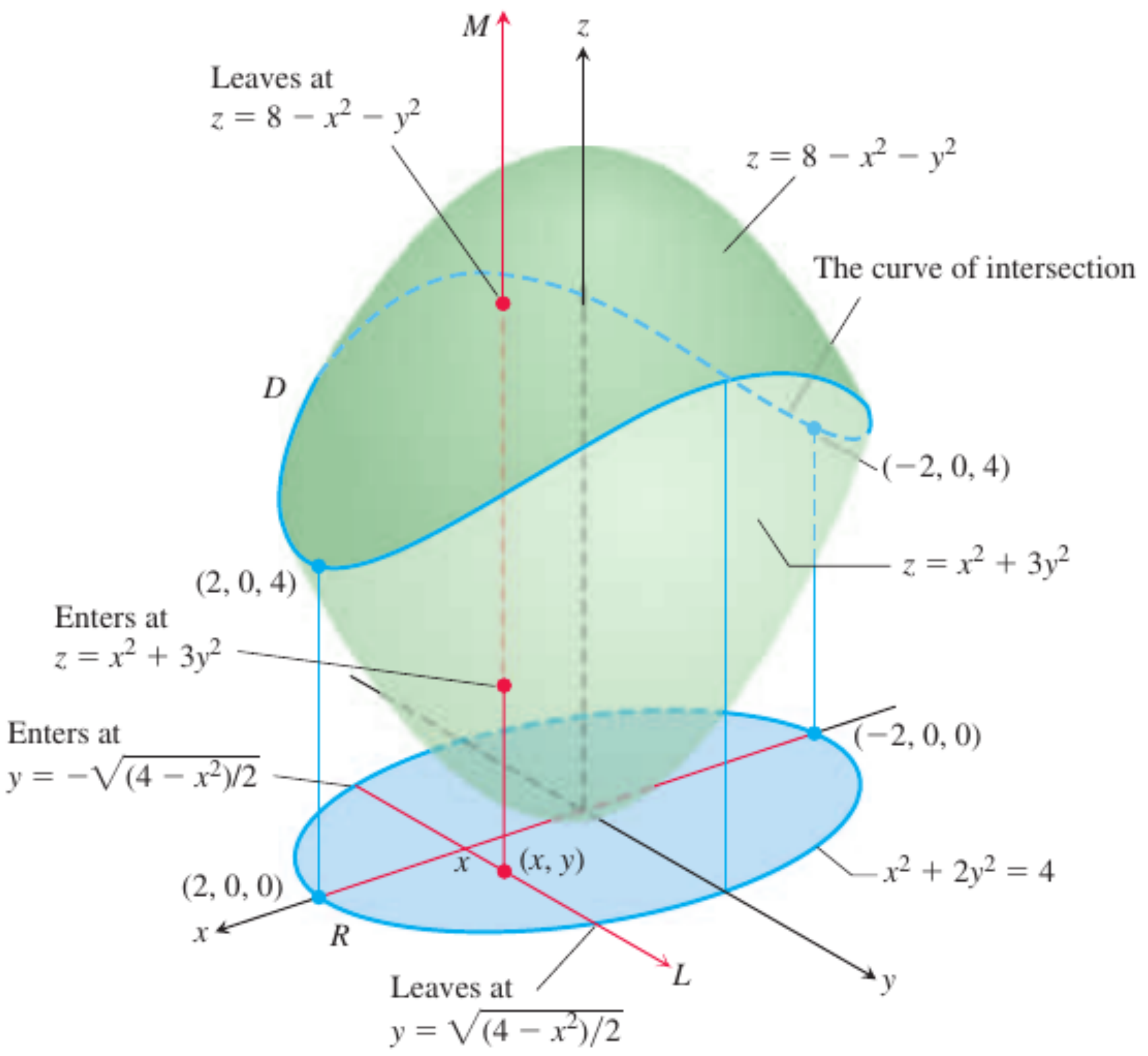
$x = \sqrt{z - 3y^2}$

$8 - y - z = z - 3y^2$

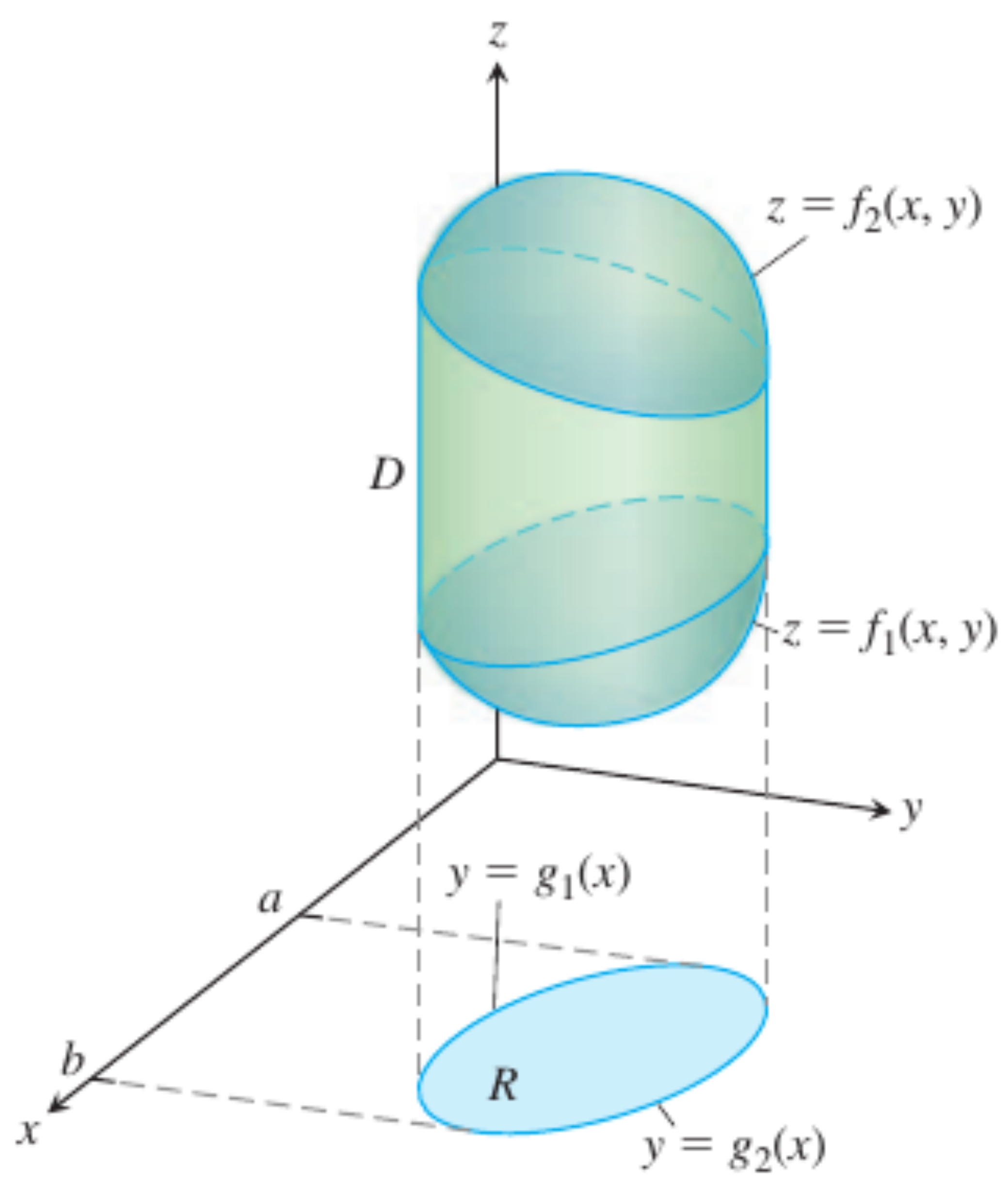
$2z - 2y^2 = 8$

$2y^2 = 2z - 8$

$y^2 = z - 4$

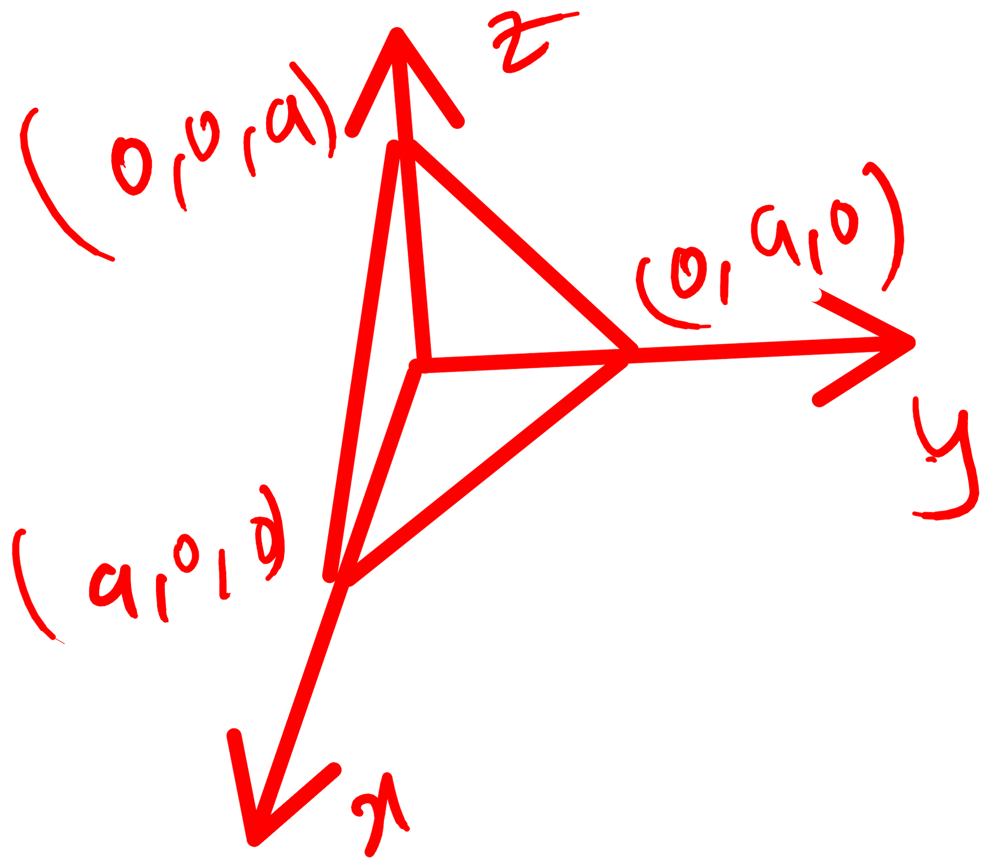


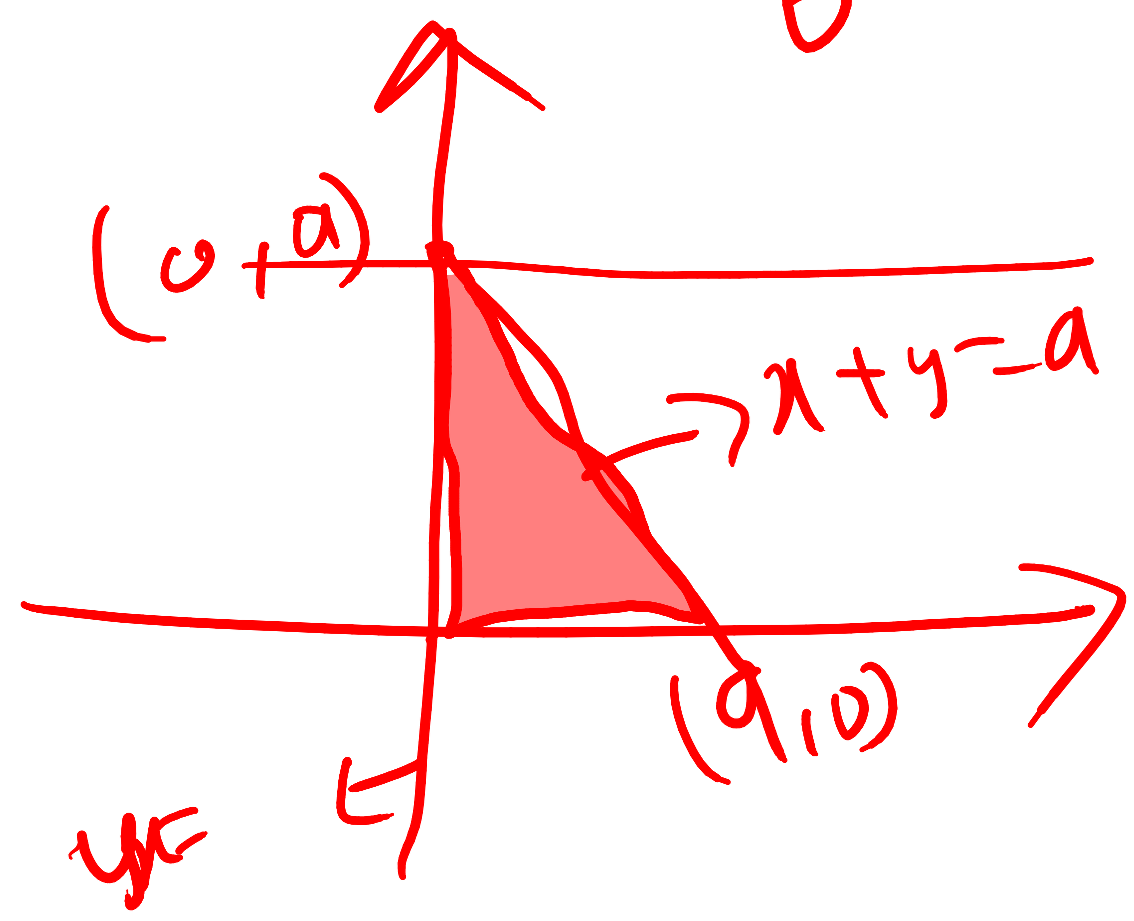
$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\, dy\, dx.$$



Calculate the volume of the solid bounded by the planes $x = 0$, $y = 0$, $x + y + z = a$ and $z = 0$.

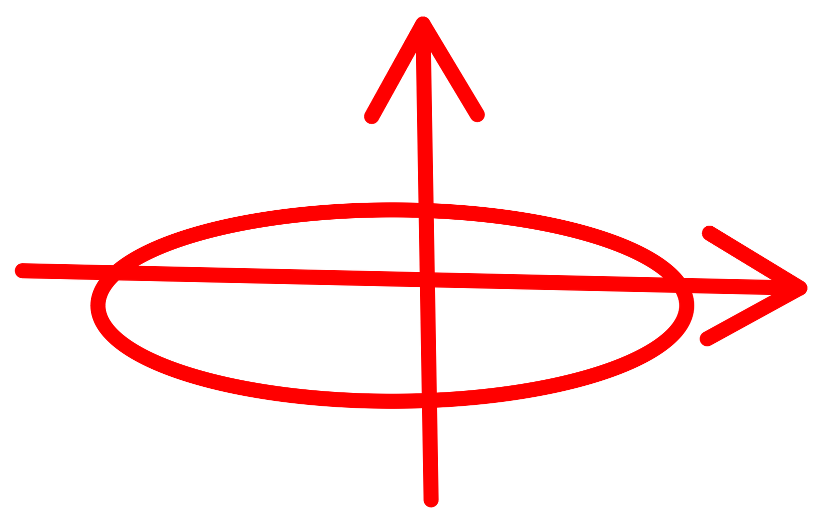
$$V = \int_0^a \int_0^{a-x} \int_0^{a-x-y} dz\, dy\, dx = \frac{a^3}{6}$$





Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

$$\int_{-a}^{a} \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} \int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b}}} dz\, dy\, dx$$



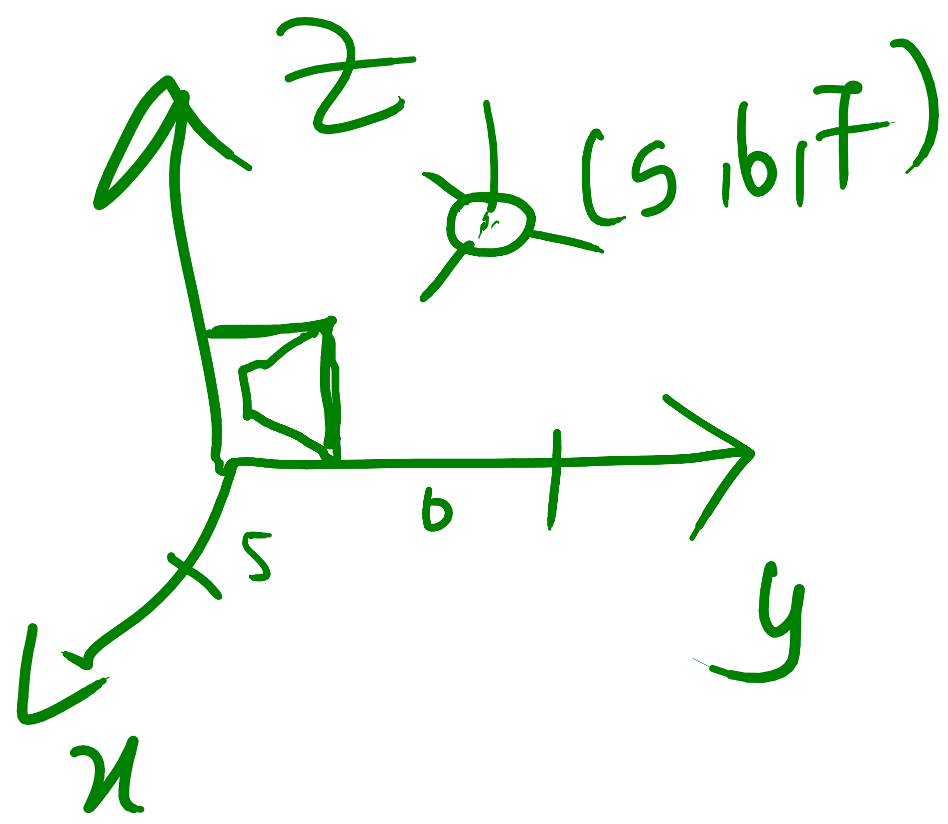
$x = r\cos\theta$

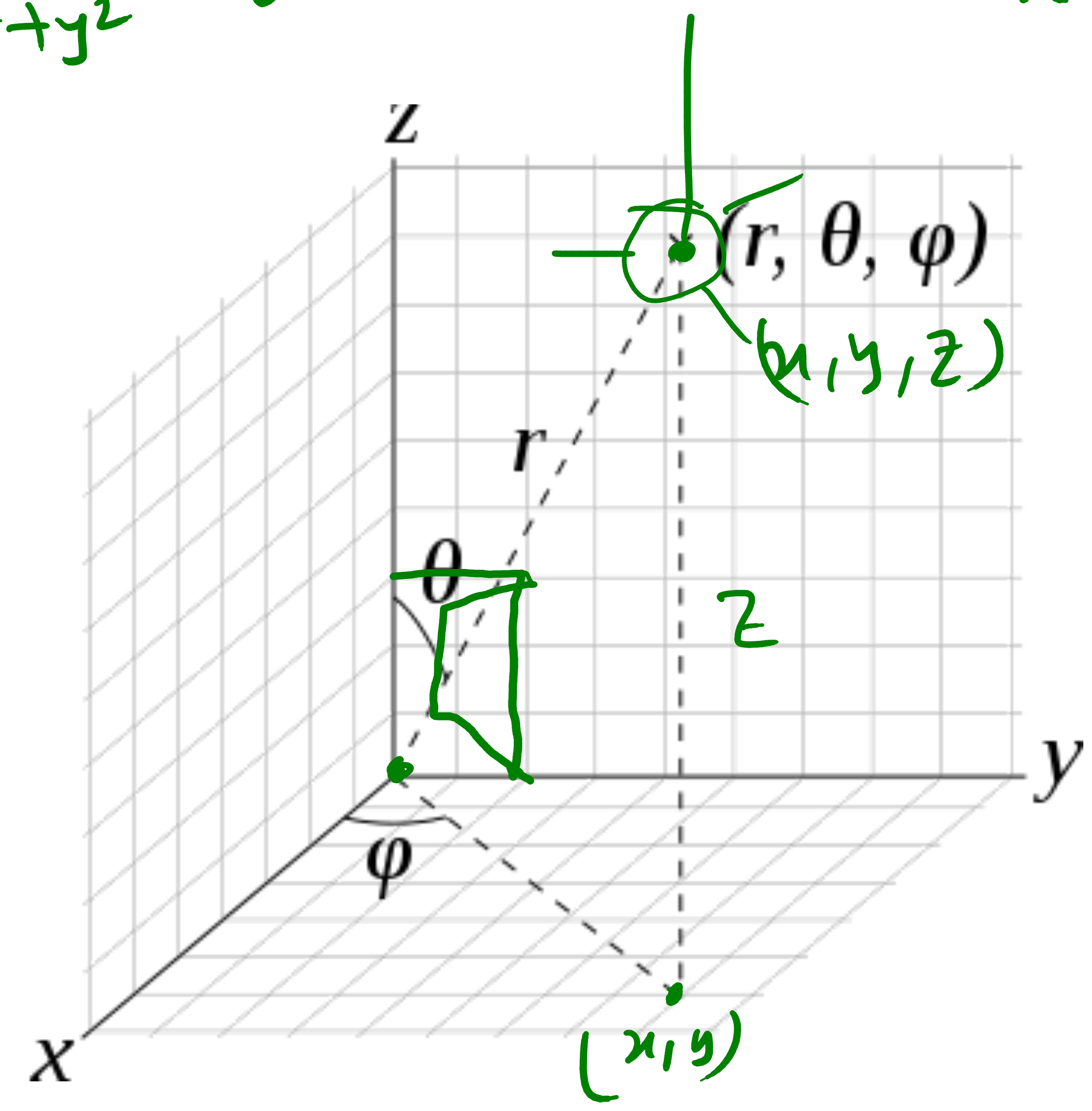
$y = r\sin\theta$

$(x, y) \quad (r, \theta)$

(x, y, z)

$r = \sqrt{x^2+y^2} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$





Spherical Coordinate system

$x = r\cos\phi\sin\theta$

$y = r\sin\phi\cos\theta$

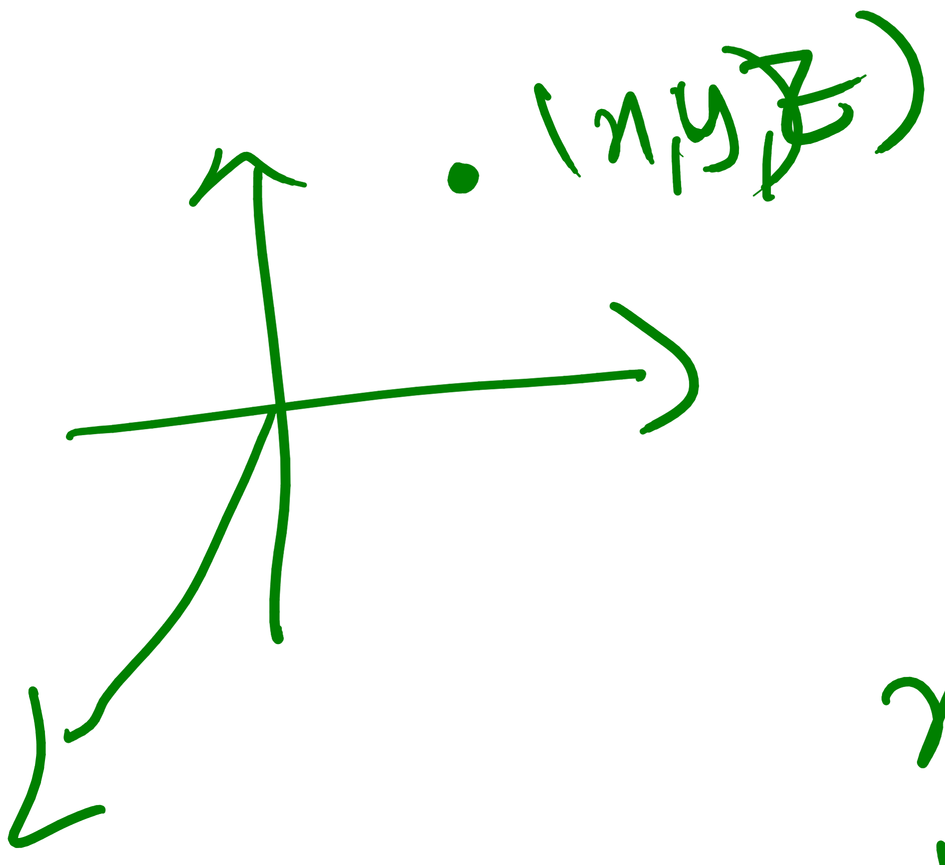
$z = r\cos\theta$

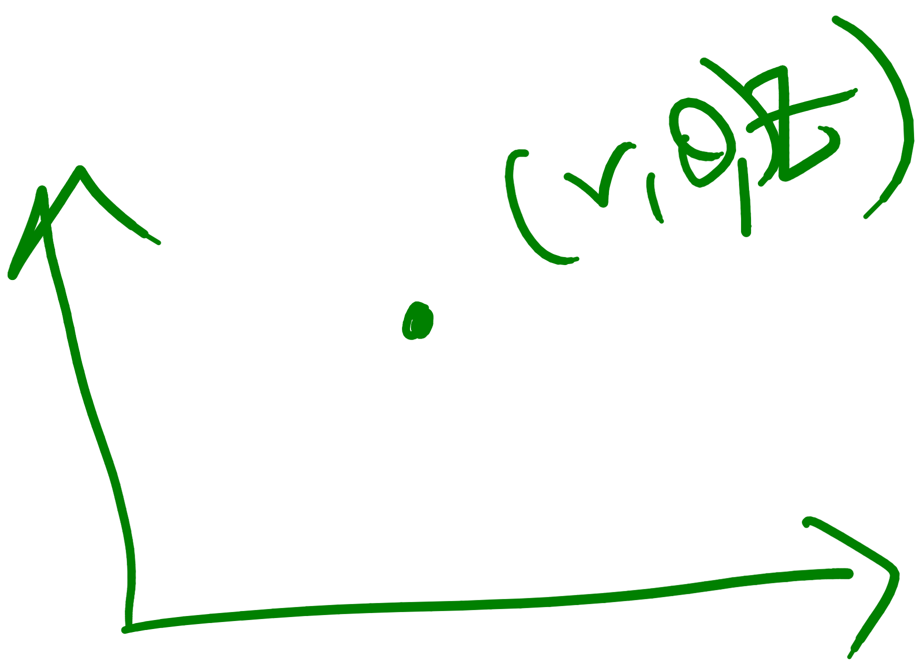
$$r = \sqrt{x^2+y^2+z^2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$\phi = \begin{cases} \tan^{-1}(y/x), & x>0 \\ \tan^{-1}(y/x)+\pi, & x<0 \\ \pi/2 & x=0 \end{cases}$$

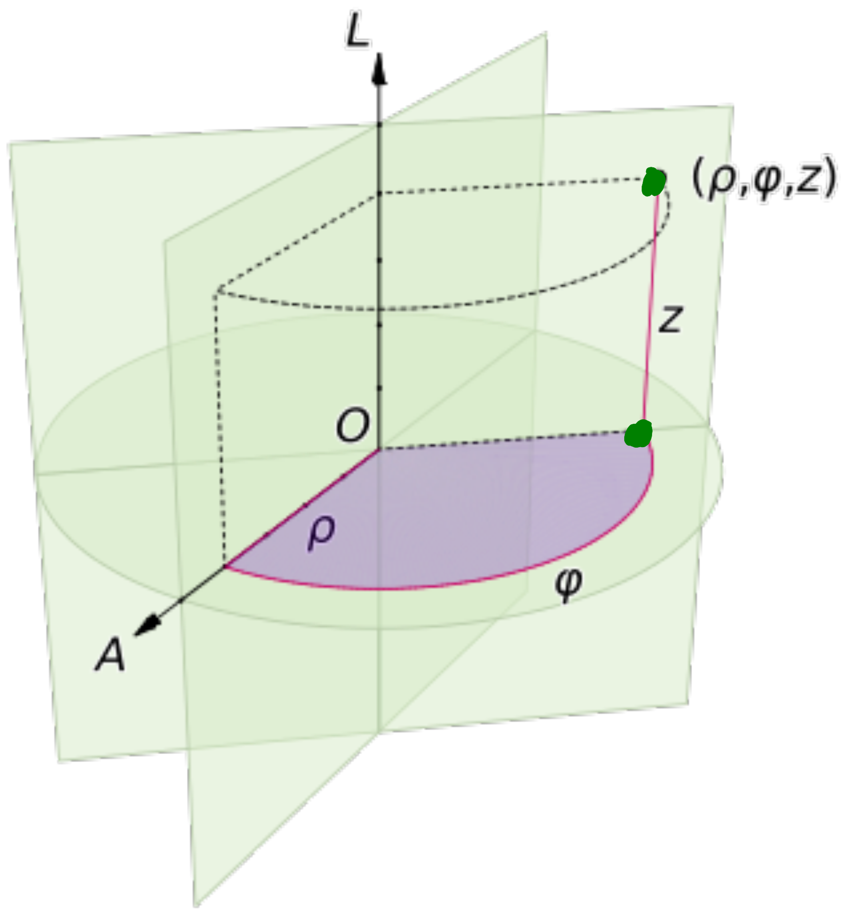
Cylindrical Coordinate





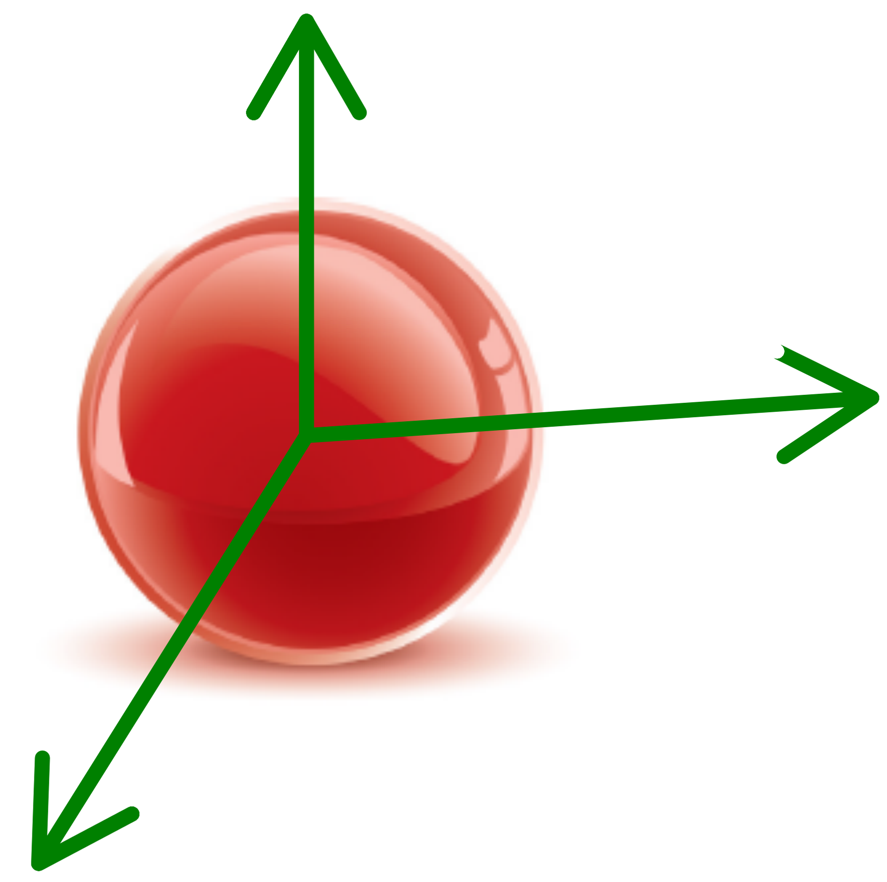
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

$$r = \sqrt{x^2+y^2}$$
$$\theta = \tan^{-1}(y/x)$$
$$z = z$$

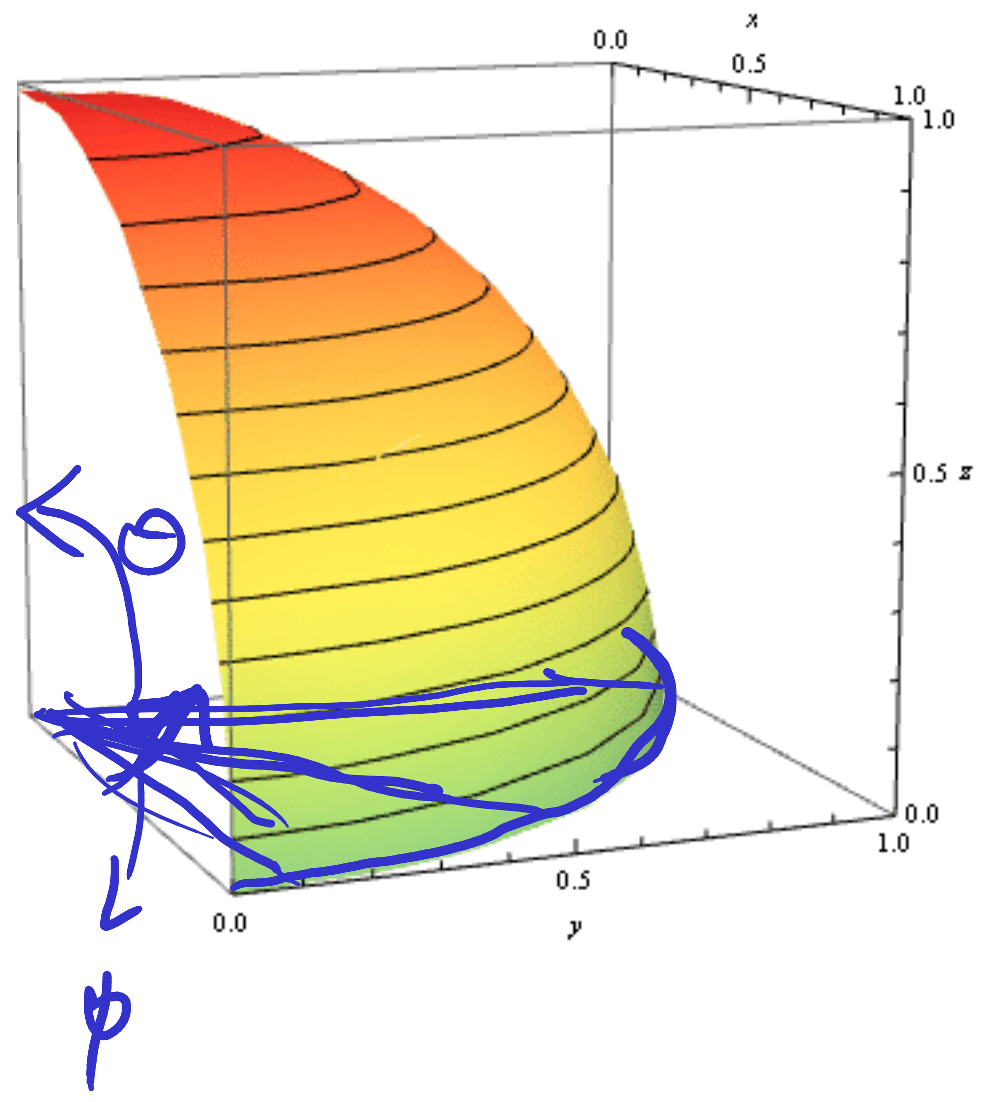


Find, by triple integration, the volume of the sphere $x^2 + y^2 + z^2 = a^2$.

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$



$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$



Spherical coordinates.

$$= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a |J|\, dr\, d\theta\, d\phi$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$x = r\cos\phi\sin\theta$

$y = r\sin\phi\cos\theta$

$z = r\cos\theta$

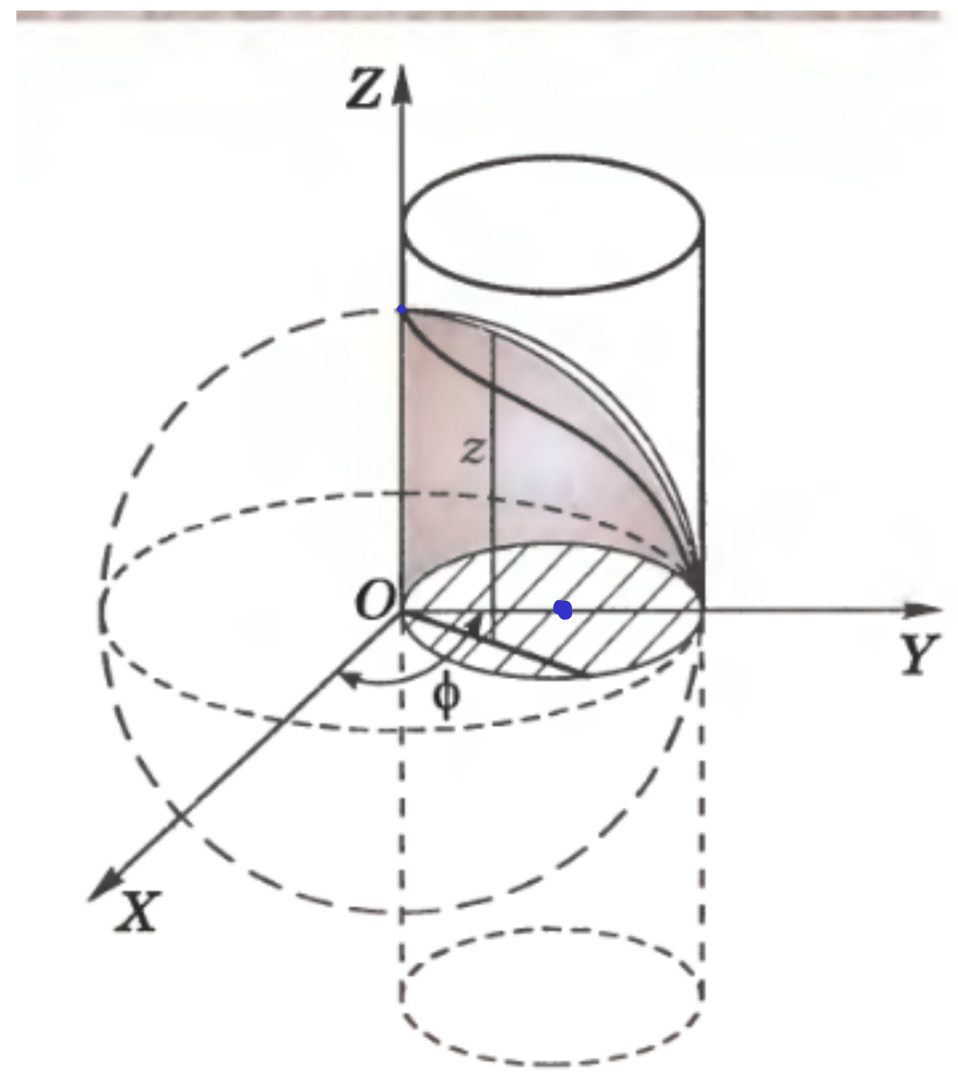
$$= \begin{vmatrix} \cos\phi\sin\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2 \sin\theta$$

$$\text{volume} = 8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{4}{3} \pi a^3$$

Example 7.28. *Find the volume of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ay$.* (Rohtak, 2003)



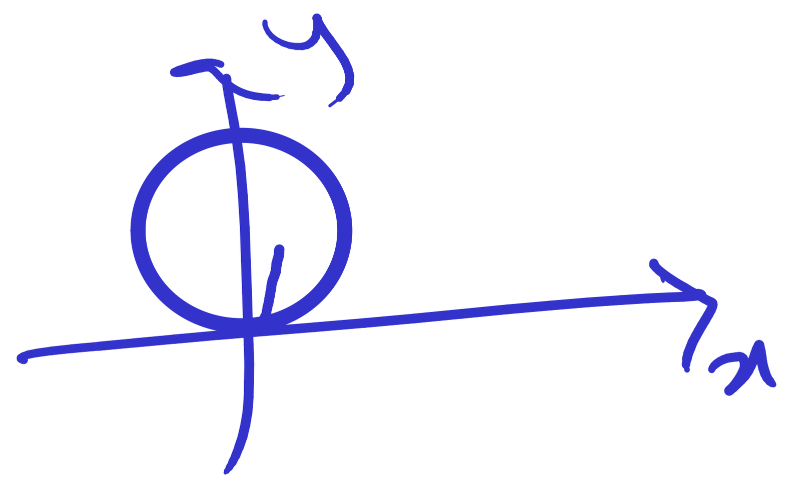
$$x^2 + y^2 = ay$$

$$x^2 + y^2 - ay = 0$$

$$x^2 + y^2 - ay + \frac{a^2}{4} - \frac{a^2}{4} = 0$$

$$x^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{4}$$

$$\left(0, \frac{a}{2}\right) \quad \frac{a}{2}$$



$$V = 2 \int \int \int$$

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