Laplace Transform of the Heaviside Step Function

The Heaviside step function is a piecewise continuous function defined by

$$h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Figure 42.1 displays the graph of $h(t)$.

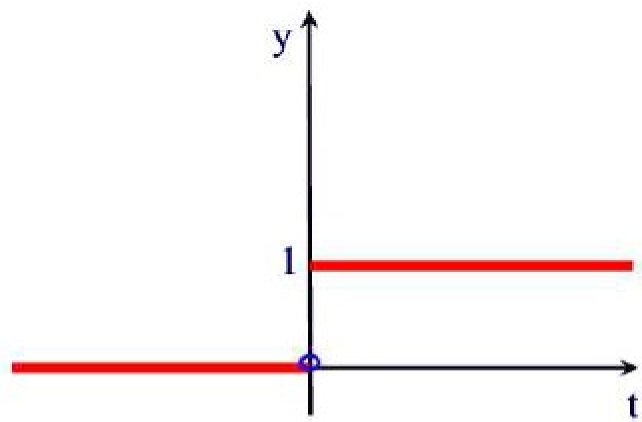


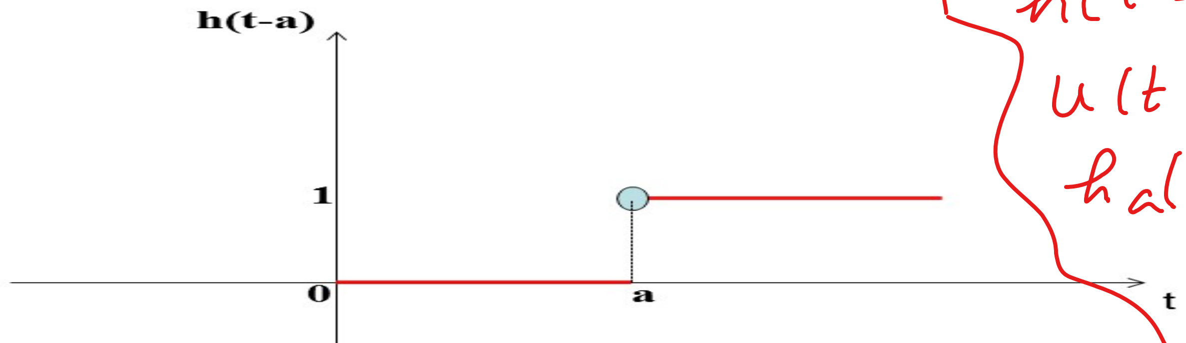
Figure 42.1

Taking the Laplace transform of $h(t)$ we find

$$\mathcal{L}[h(t)] = \int_0^\infty h(t)e^{-st}dt = \int_0^\infty e^{-st}dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s},\ s > 0.$$

A Heaviside function at $a > 0$ *is the shifted function* $h(t-a)$ (*a units to the right*). *It is defined as*

$$h(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}.$$



$h(t-a)$
$u(t-a)$
$h_a(t)$
$U_a(t)$

For this function, the Laplace transform is

$$L[h(t-a)] = \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty = \left[-\frac{1}{s}\left(e^{-st}\right)_a^\infty\right] = -\frac{1}{s}\left[0 - e^{-sa}\right] = \frac{e^{-as}}{s}.$$

$If\ \ L[f(t)] = F(s),\ \ then\ P.T\ L[\underbrace{f(t-a).u(t-a)}] = e^{-as}F(s).$

$\Pr oof:-$

$$L[f(t-a).u(t-a)] = \int_0^\infty e^{-st} f(t-a)u(t-a)\ dt$$

$$= \int_0^a e^{-st} f(t-a)(0)\ dt + \int_a^\infty e^{-st} f(t-a)\ dt$$

$$= \int_a^\infty e^{-st} f(t-a)\ dt$$

$Put\ u = t-a \ \Rightarrow \ t = u + a$

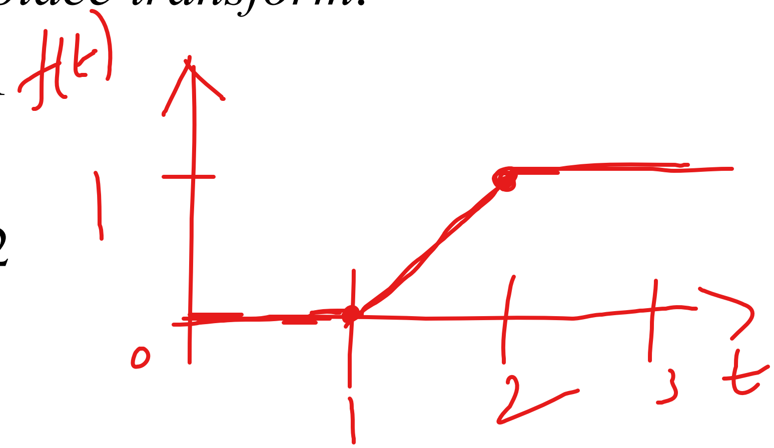
$Now\ du = dt$

$when\ t = a,\ u = 0\ \ \&\ \ when\ t \to \infty, u \to \infty$

$$= \int_0^\infty e^{-s(u+a)} f(u)\ du = e^{-sa}\int_0^\infty e^{-su} f(u)\ du = e^{-sa}F(s).$$

1) *For the given function draw its graph and express in terms of unit step function. Find also its Laplace transform.*

$$f(t)=\begin{cases}0, & 0<t<1\\ t-1, & 1<t<2\\ 1, & t>2\end{cases}$$



Solution :–

(*a*) *Its graph is* ?

$$\begin{aligned}f(t)&=(0)\left[u(t-0)-u(t-1)\right]+(t-1)\left[u(t-1)-u(t-2)\right]+(1)u(t-2)\\&=(t-1)\left[u(t-1)-u(t-2)\right]+u(t-2)\\&=(t-1)u(t-1)-(t-1)u(t-2)+u(t-2)\\&=(t-1)u(t-1)-u(t-2)[(t-1)-1]\\&=(t-1)u(t-1)-(t-2)u(t-2)\end{aligned}$$

Cont.......

$$W.K.T \;\; L[f(t-a)u(t-a)] = e^{-as}L[f(t)] = e^{-as}F(s).$$

$$Also L(t) = \frac{1}{s^2}. \Rightarrow F(s)$$

$$\therefore L[(t-1)u(t-1)] = e^{-s}.\frac{1}{s^2} \;\&\; L[(t-2)u(t-2)] = e^{-2s}.\frac{1}{s^2}.$$

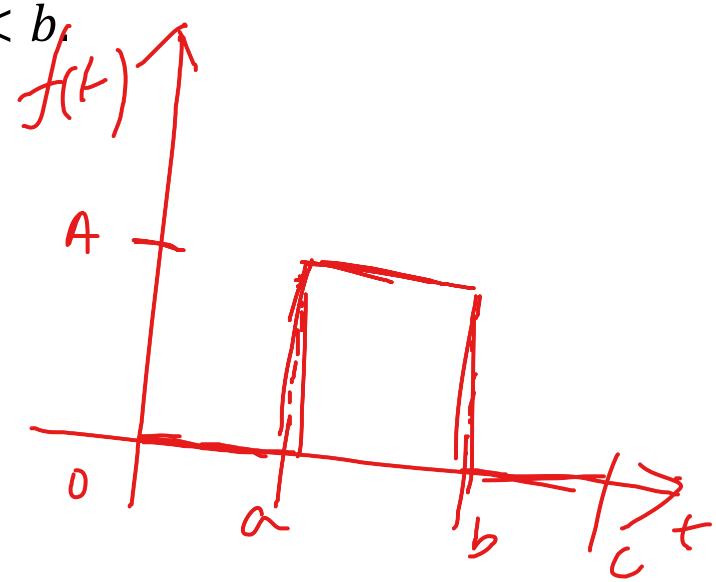
$$L[f(t)] = L[(t-1)u(t-1)-(t-2)u(t-2)] = \frac{e^{-s}-e^{-2s}}{s^2}. \;//$$

2) Express the given function in terms of unit step function and finds its Laplace Transform.

$$f(t) = \begin{cases} 0, & t < a \\ A, & a < t < b \\ 0, & t > b \end{cases}$$

Assume the constants $a, b,$ and A are positive with $a < b$.

Sol:-



$$f(t) = 0\,[\underbrace{u(t-0)}_{=0} - u(t-a)] + A\,[u(t-a) - u(t-b)] + \underbrace{0\,[u(t-b)]}_{=0} = A\,[u(t-a) - u(t-b)]$$

$$\mathcal{L}[f(t)] = A\,\mathcal{L}[u(t-a) - u(t-b)]$$

$$= A\left[\mathcal{L}[u(t-a)] - \mathcal{L}[u(t-b)]\right]$$

$$= A\left[\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right]$$

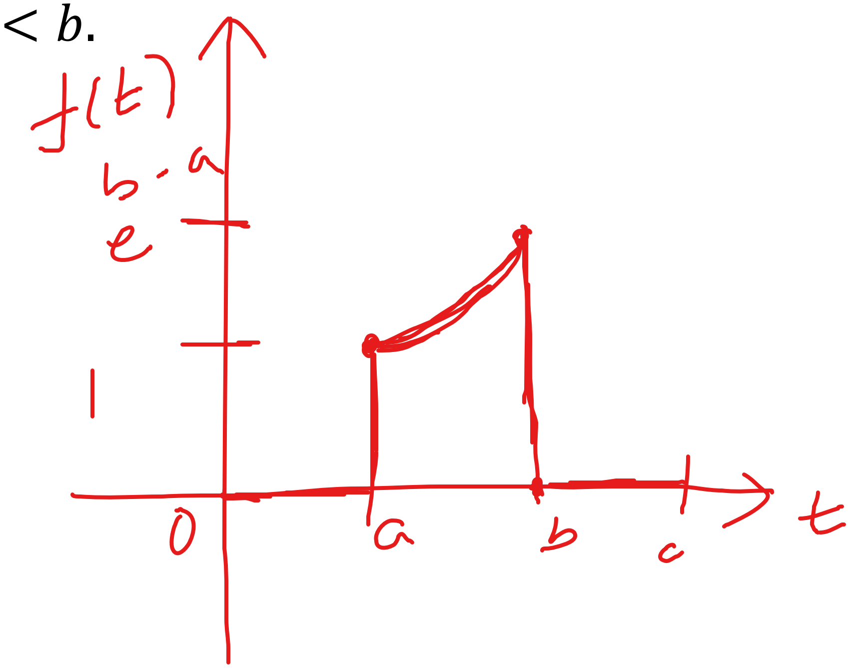
$$= A\left[\frac{e^{-as} - e^{-bs}}{s}\right]$$

3) Express the given function in terms of unit step function and finds its Laplace Transform.

$$f(t) = \begin{cases} 0\,, t < a \\ e^{t-a}, a < t < b \\ 0, t > b \end{cases}$$

Assume the constants a, b are positive with $a < b$.

Sol:-



Now

$$f(t) = 0\,[\underbrace{u(t-0) - u(t-a)}] + e^{t-a}[u(t-a) - u(t-b)] + 0\,[\underbrace{u(t-b)}]$$

$$= e^{t-a}[u(t-a) - u(t-b)]$$

Now

$$t - a = (b-a) + (t-b) \quad \text{— I}$$

Using I

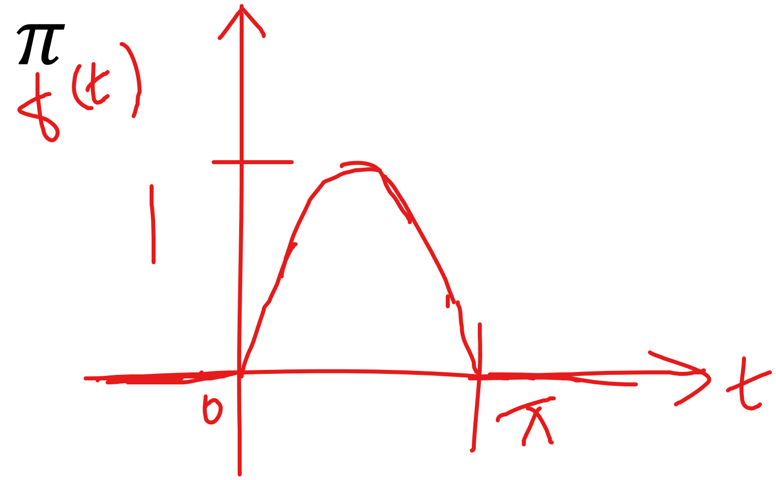
$$f(t) = e^{t-a}u(t-a) - e^{b-a}\cdot e^{t-b}u(t-b)$$

$$L[f(t)] = L\{e^{t-a}u(t-a)\} - e^{b-a}L\{e^{t-b}u(t-b)\}$$

$$= e^{-as}\cdot\frac{1}{s-1} - e^{b-a}e^{-bs}\frac{1}{s-1} = \frac{e^{-as} - e^{b-a-bs}}{s-1}$$

4) Express the given function in terms of unit step function and finds its Laplace Transform.

$$f(t) = \begin{cases} 0, t < 0 \\ sint, 0 < t < \pi \\ 0, t > \pi \end{cases}$$



Sol:-

$$f(t) = \underset{\overset{\Vert}{0}}{0}\left[u(t-0)\right] + sint\left[u(t-0) - u(t-\pi)\right] + 0\left[\underset{\overset{\Vert}{0}}{u(t-\pi)}\right]$$

$$= sint\left[u(t) - u(t-\pi)\right]$$

$$= sint\, u(t) - sint\, u(t-\pi)$$

$$= sint\, u(t) + sin(t-\pi)\, u(t-\pi)$$

$$sin(t-\pi) = -sint$$

$$sint = -sin(t-\pi)$$

$$L\{f(t)\} = L\left[\sin t \, u(t) + \sin(t-\pi)\, u(t-\pi)\right]$$

$$= L\left[\sin t \, u(t)\right] + L\left[\sin(t-\pi)\, u(t-\pi)\right]$$

$$= \frac{1}{s^2+1} + e^{-\pi s} \cdot \frac{1}{s^2+1}$$

$$= \frac{1 + e^{-\pi s}}{s^2+1} \quad //$$