Module - 2

Functions of two variables

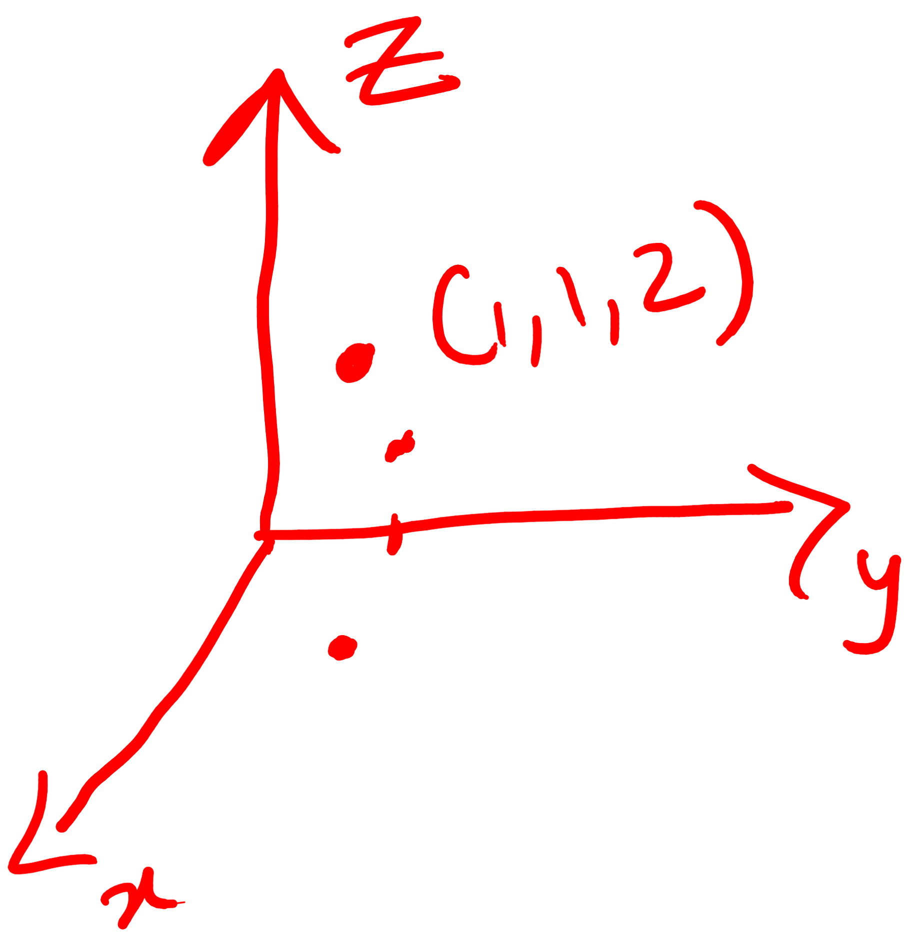
$z = f(x,y)$

$z = x + y$

$\mathbb{R}^2 \to \mathbb{R}$

$(0,0,0)$

$(0,1,1)$ $\quad$ $(1,1,2)$



$y = f(x)$

$\mathbb{R} \to \mathbb{R}$

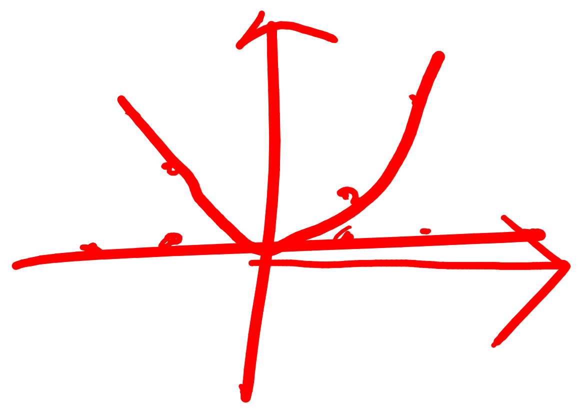
$y = x^2$

$(0,0)$

$(1,1)$

$(-1,1)$

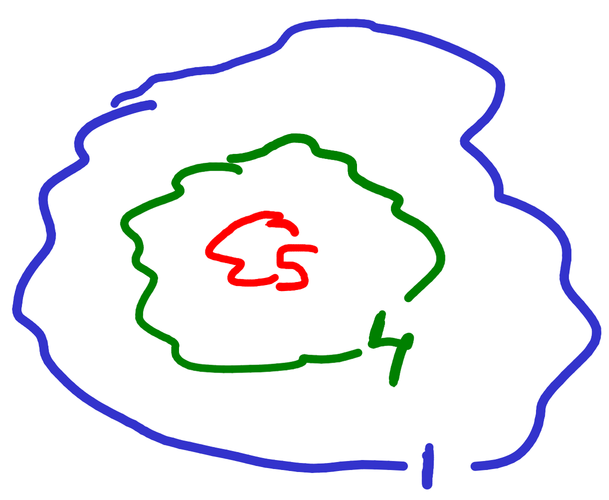
$(2,4)$ $\quad$ $(-2,4)$

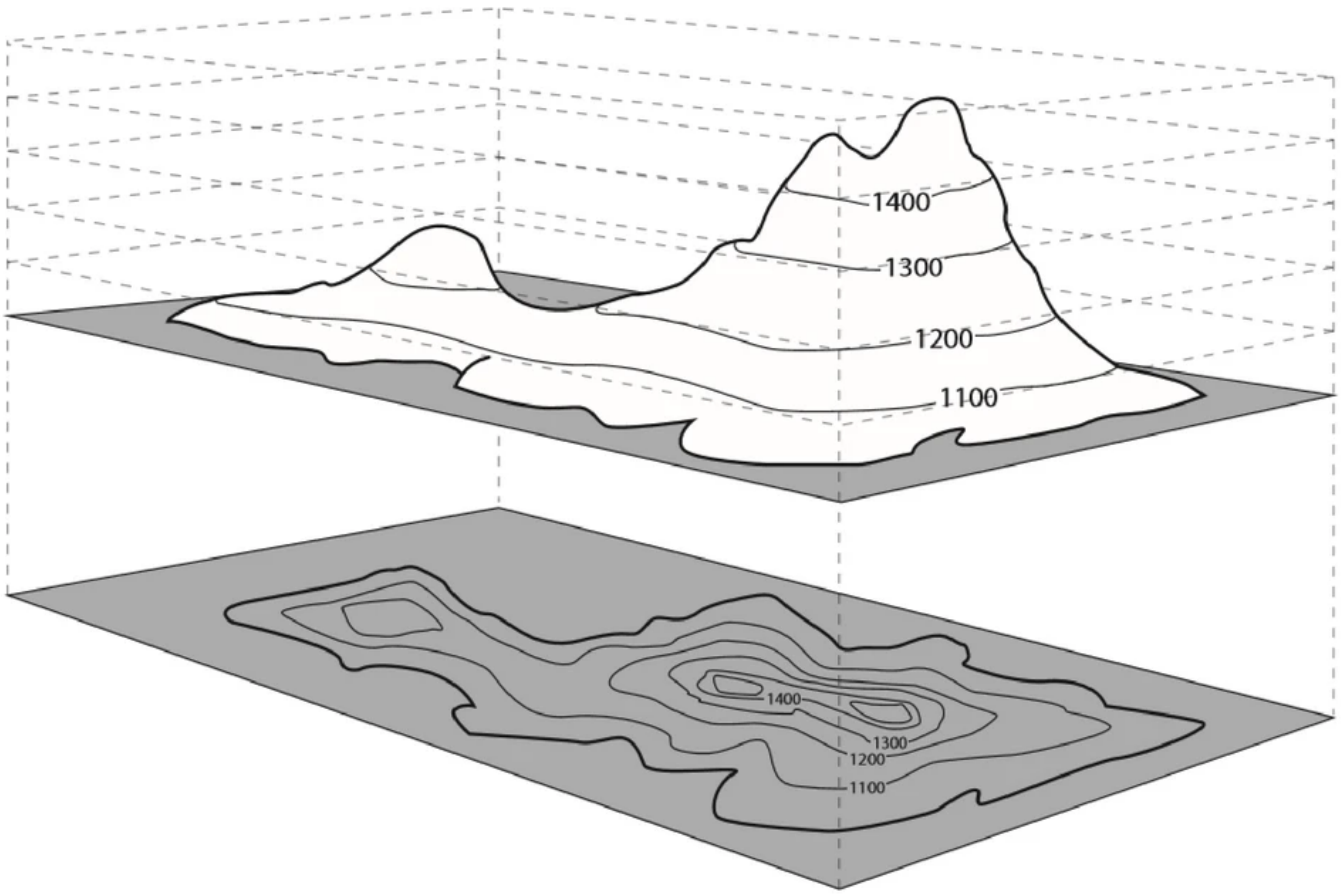


$\mathbb{R}^n \to \mathbb{R}$

$u = x + y + z$ $\qquad$ $(1,1,1,3)$

$\mathbb{R}^3 \to \mathbb{R}$




1400
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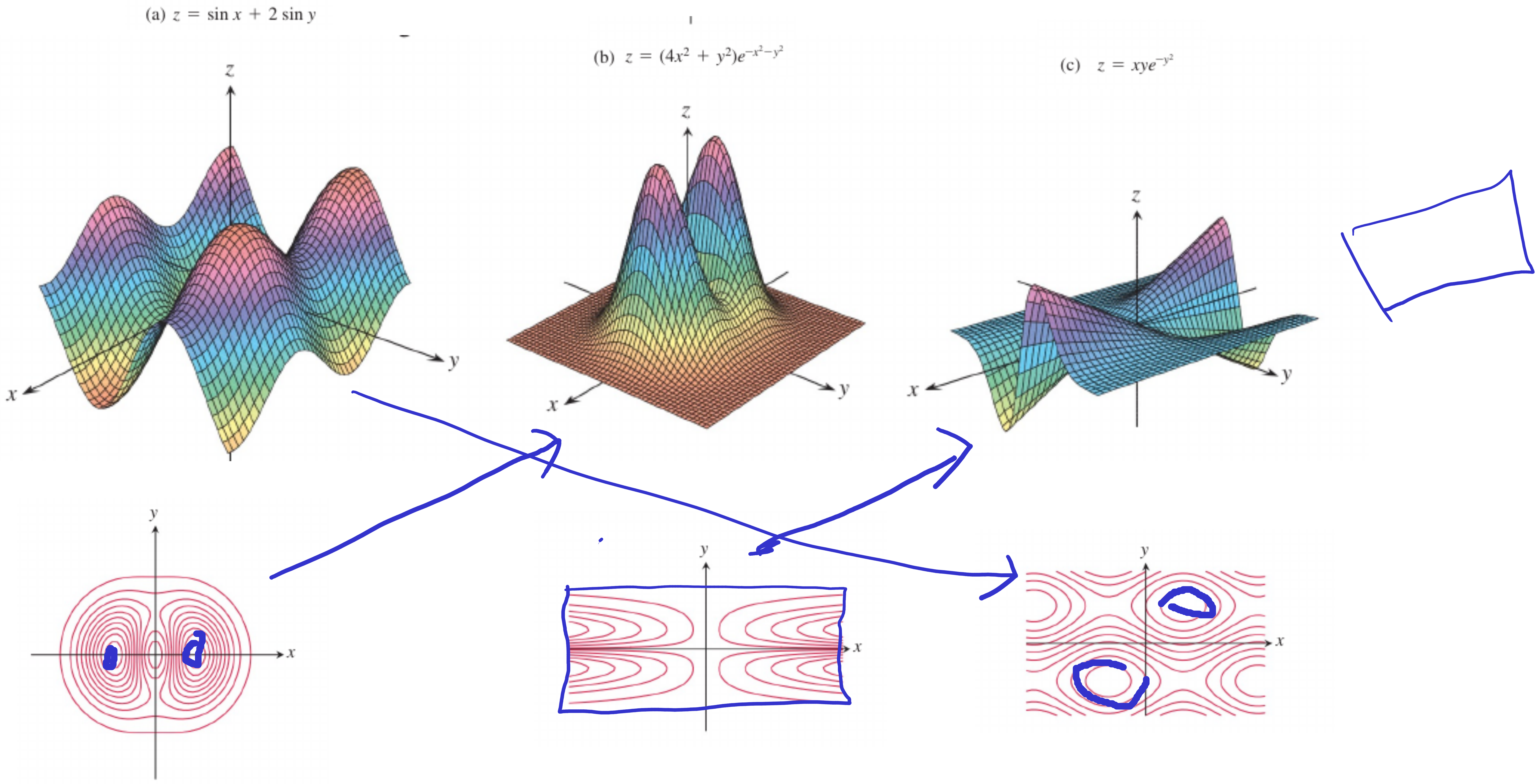
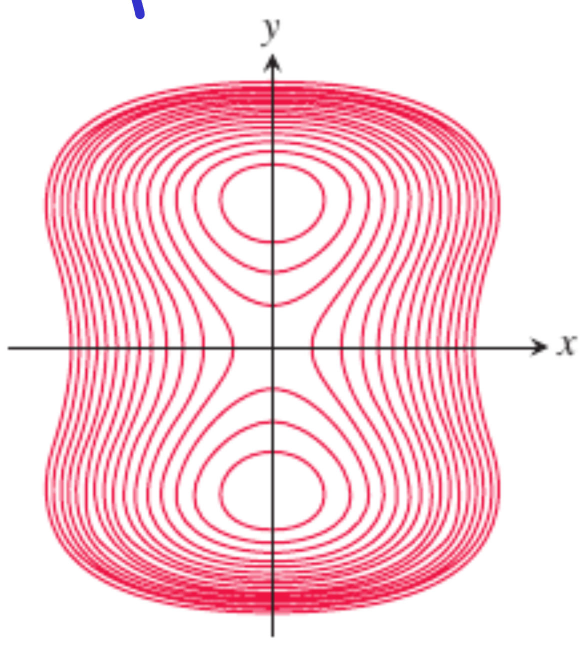
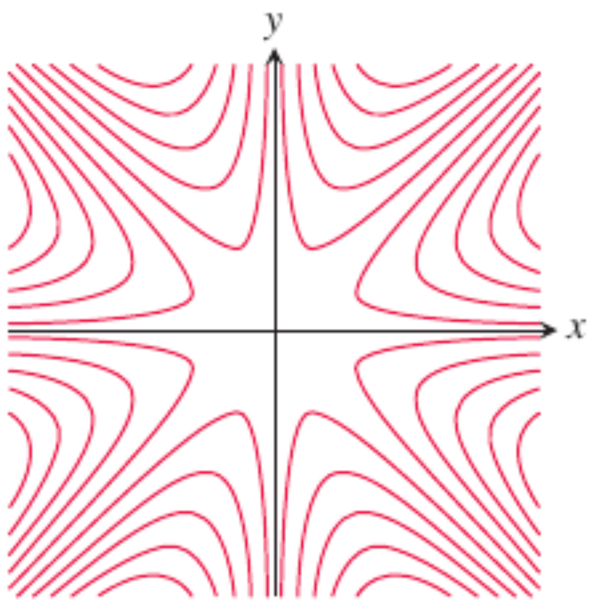

(a) $z = \sin x + 2 \sin y$
(b) $z = (4x^2 + y^2)e^{-x^2-y^2}$
(c) $z = xye^{-y^2}$
z
x
y

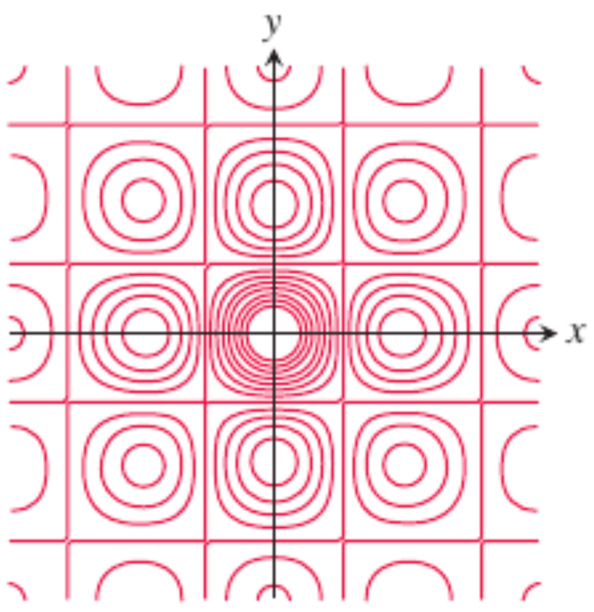
31. F



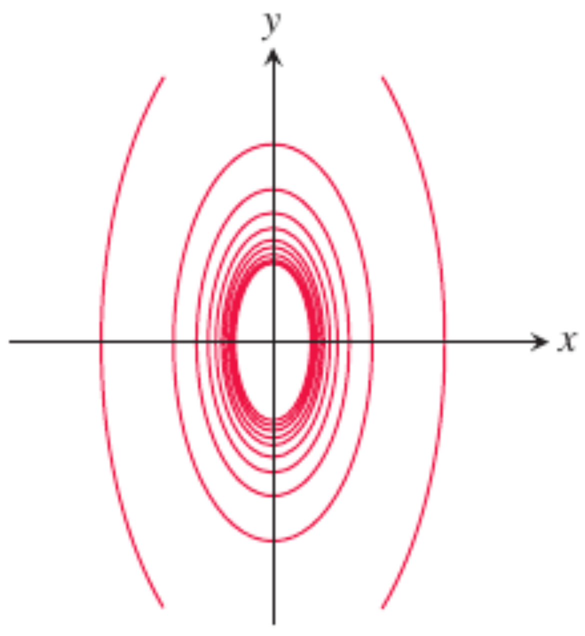
32. e



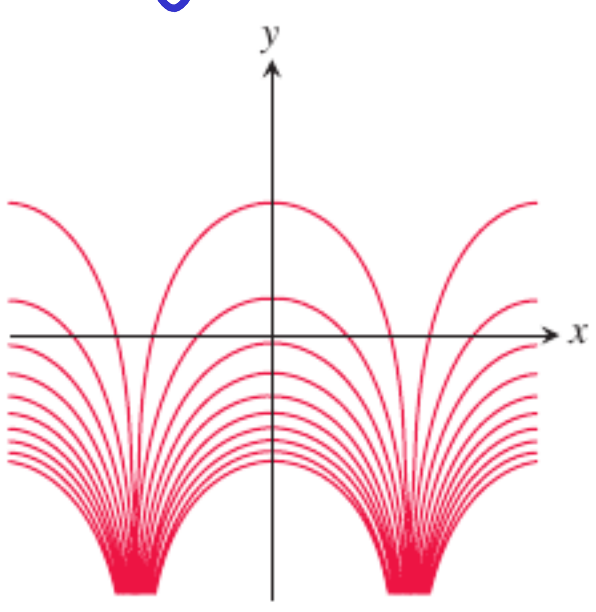
33. a



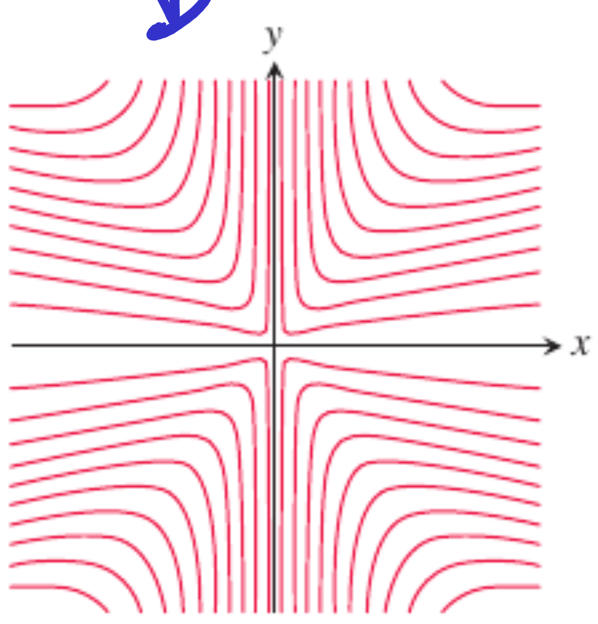
34. c



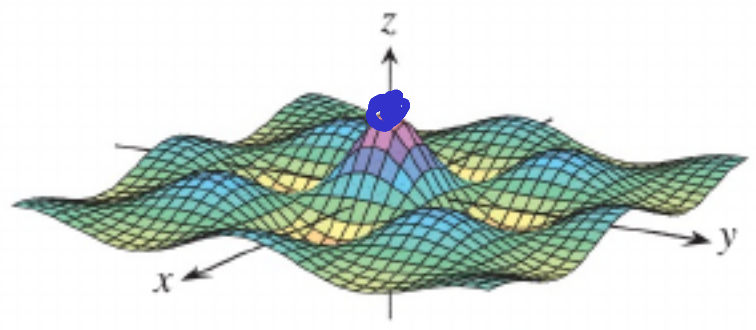
35. d



36. b

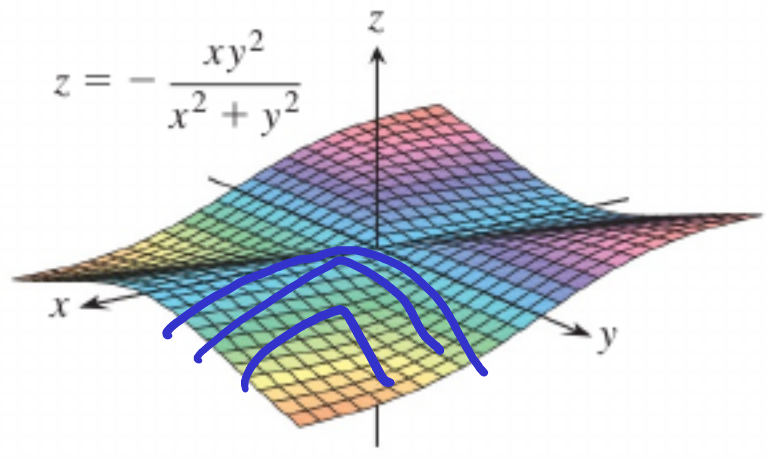


a.



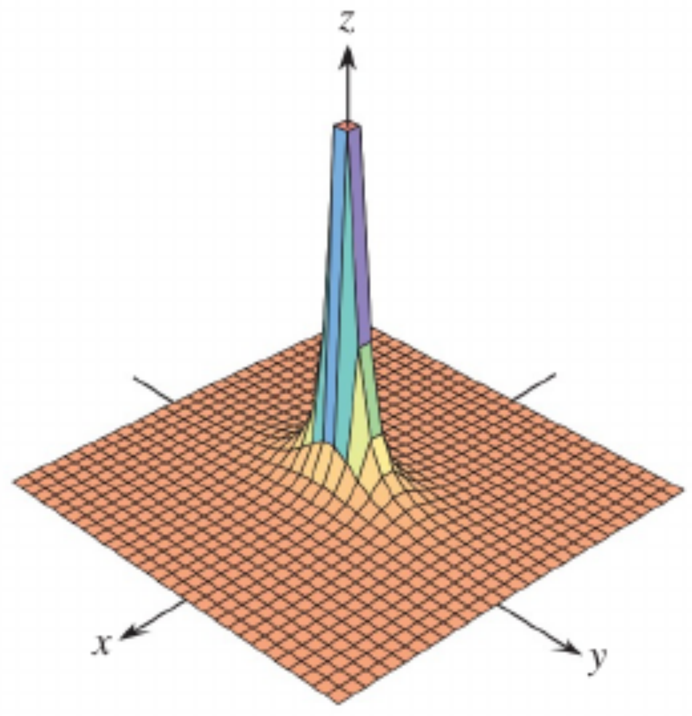
$z = (\cos x)(\cos y)\, e^{-\sqrt{x^2+y^2}/4}$

b.



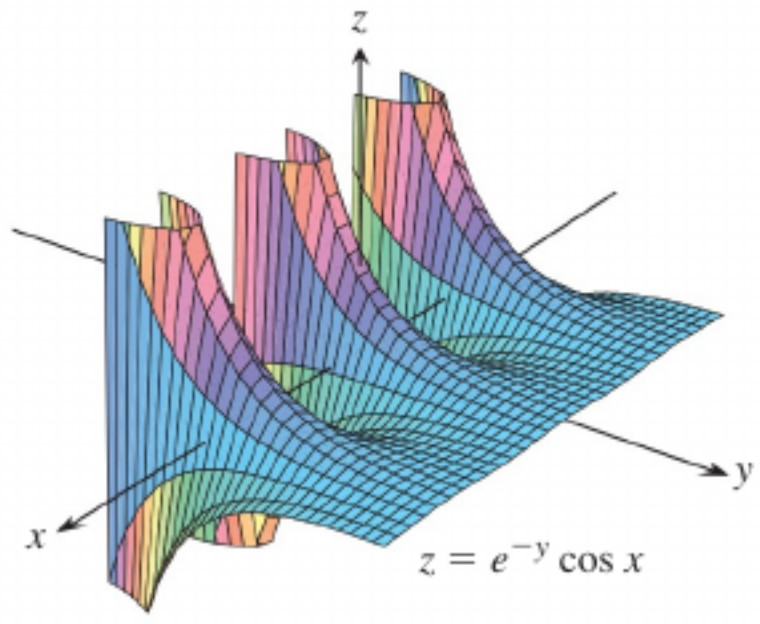
$z = -\dfrac{xy^2}{x^2+y^2}$

c.



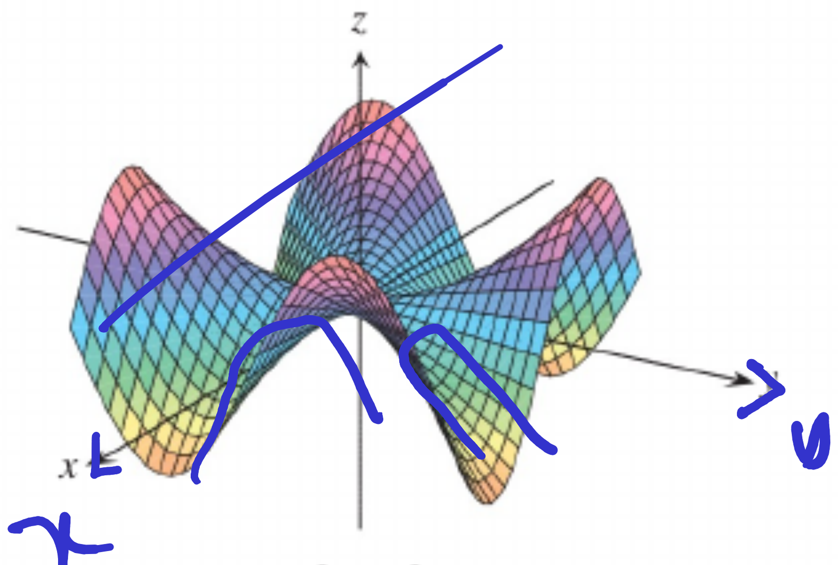
$z = \dfrac{1}{4x^2+y^2}$

d.



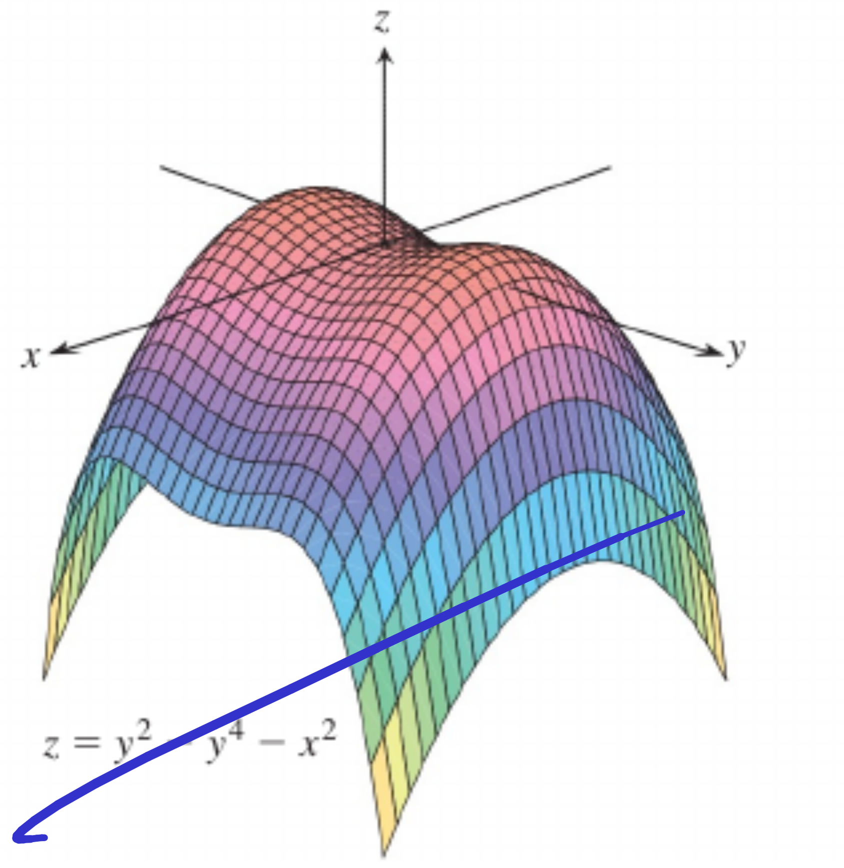
$z = e^{-y}\cos x$

e.

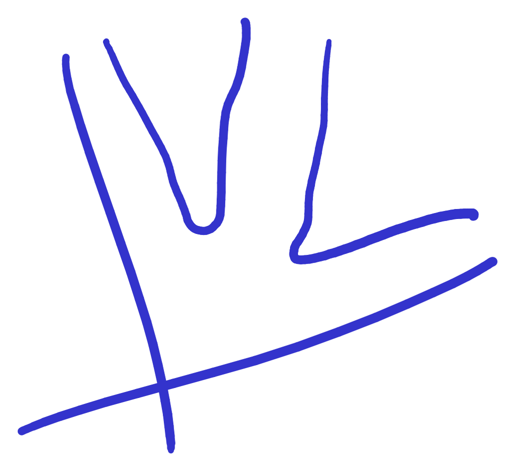


$z = \dfrac{xy(x^2-y^2)}{x^2+y^2}$

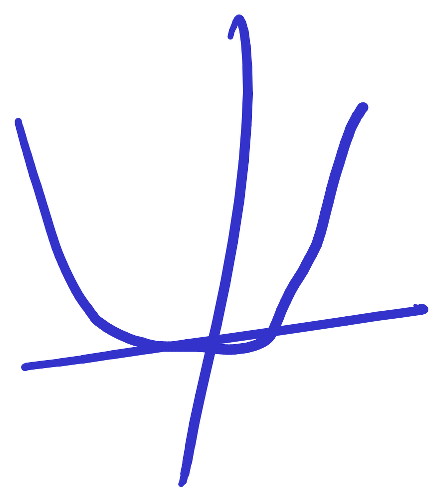
f.



$z = y^2 - y^4 - x^2$



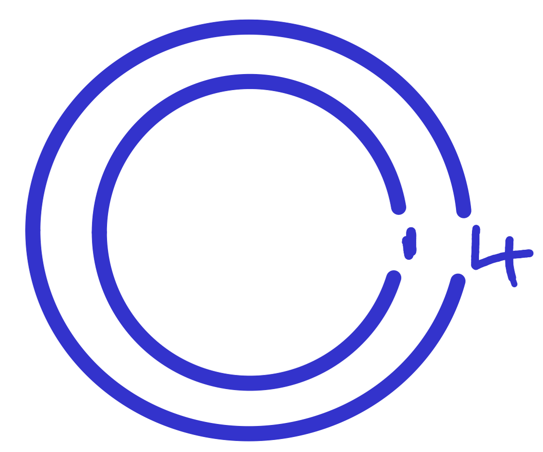
$y = x^2$



$z = x^2 + y^2$

$z = 1 \quad x^2 + y^2 = 1$

$z = 4$



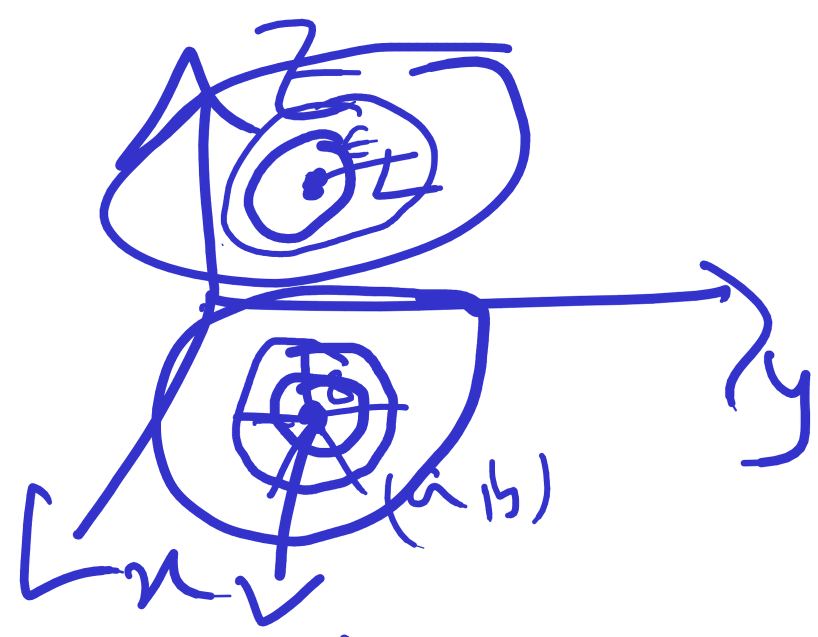
$$\left.\begin{array}{l}\text{existence of limit}\\ \text{Continuity}\\ \text{derivative}\end{array}\right\}$$

$\lim\limits_{x \to a} f(x) = L \qquad \lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x)$ ✓

$|f(x) - L| < \underline{\epsilon}, \quad |x - a| < \underline{\delta}$

5

$\lim\limits_{\substack{x \to a \\ y \to b}} \underline{f(x,y)} = L$



(a,b)

$|f(x,y) - L| < \underline{\epsilon}$

$\underline{\sqrt{(x-a)^2 + (y-b)^2}} < \underline{\delta}$

$$\lim_{x\to 5} x^2 = 25$$

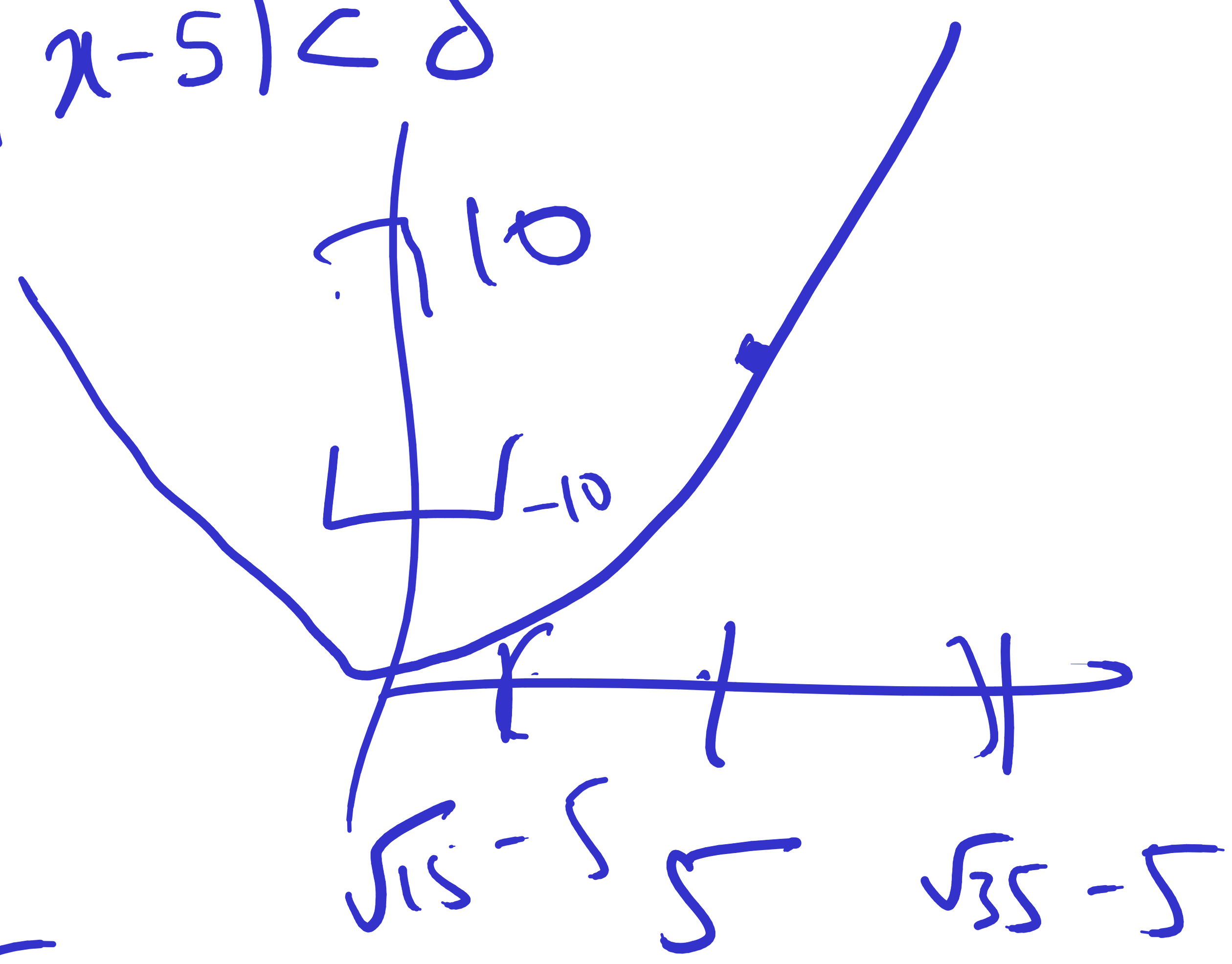
$|x-5| < \delta$

$|x^2-25| < \epsilon$

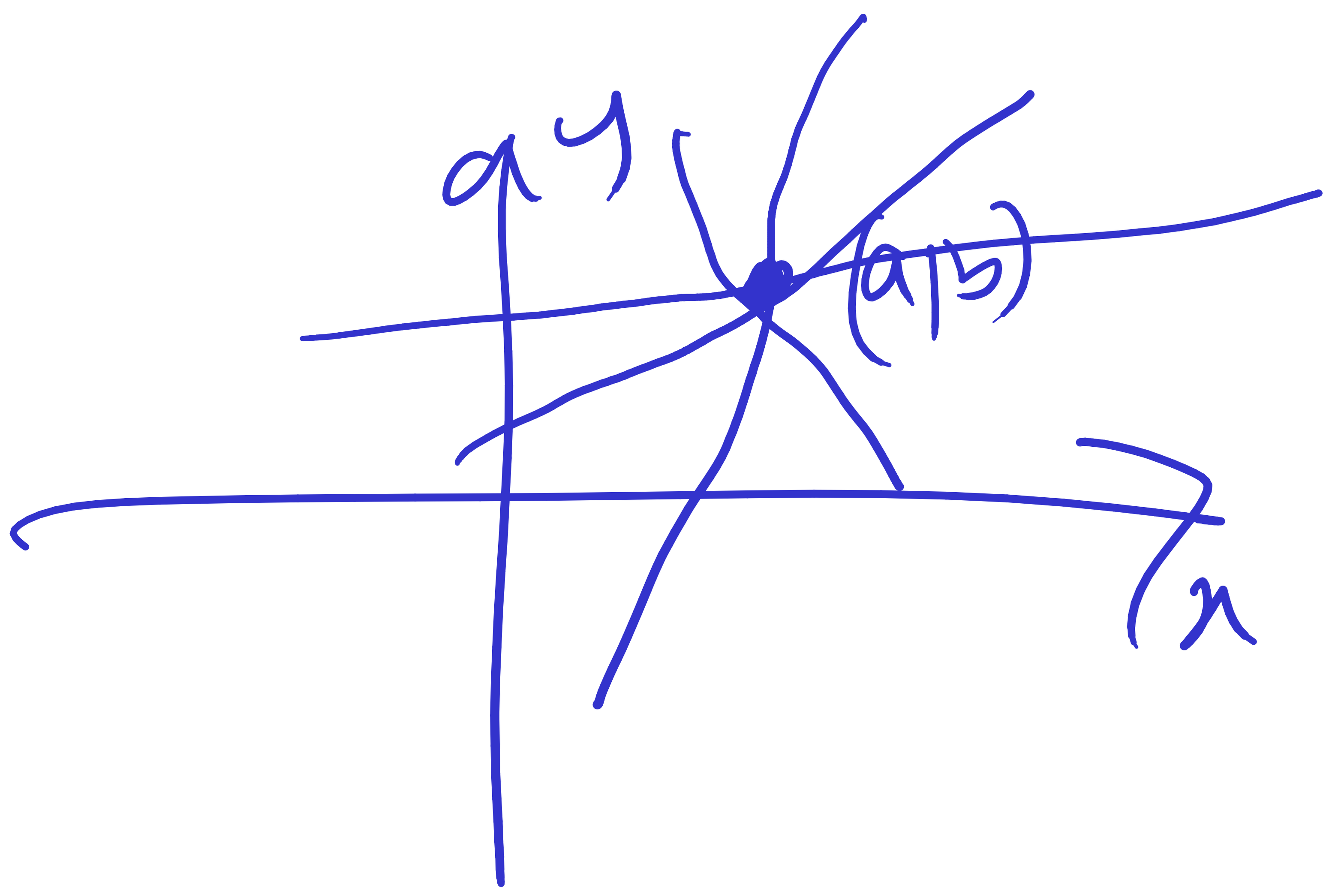
$-10 < x^2 - 25 < 10$

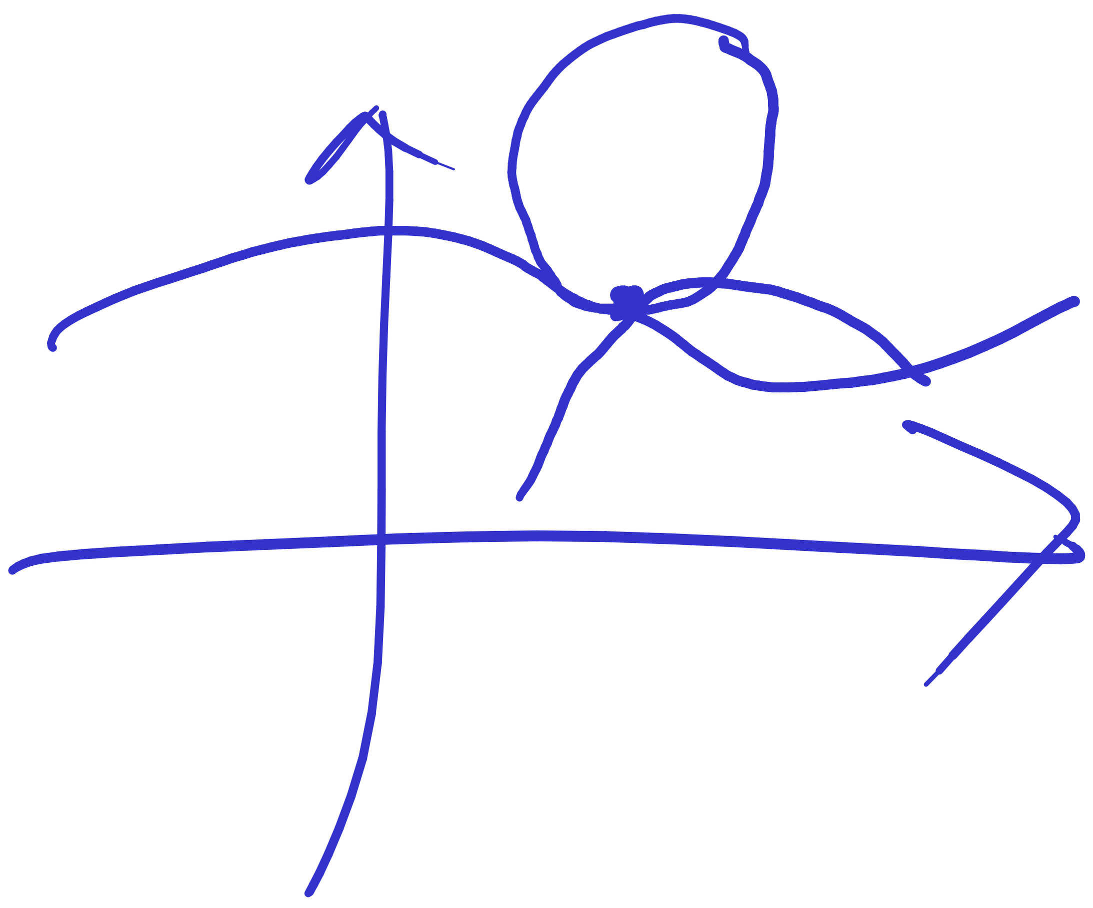
$15 < x^2 < 35$

$\sqrt{15} < x < \sqrt{35}$



$\sqrt{15}-5 < \underline{x-5} < \sqrt{35}-5$





$y = x$

$y = 2x$

$y = 3x$

$\underline{y = mx}$

(0,0)

$\lim_{\substack{x\to 0 \\ y\to 0}} f(x,y)$

(1,1)

$1, \sqrt{3}+2$

$y = x$

$y-1 = m(x-1)$ $y = x^2$

$\lim_{x\to 0} f(x)$

$y = 2-x$

$y = 2x-1$

$y = m(x-1)+1$ $y = 1, x = 1$

$$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = -3 \qquad \lim_{x\to 5} x^2 = 25$$

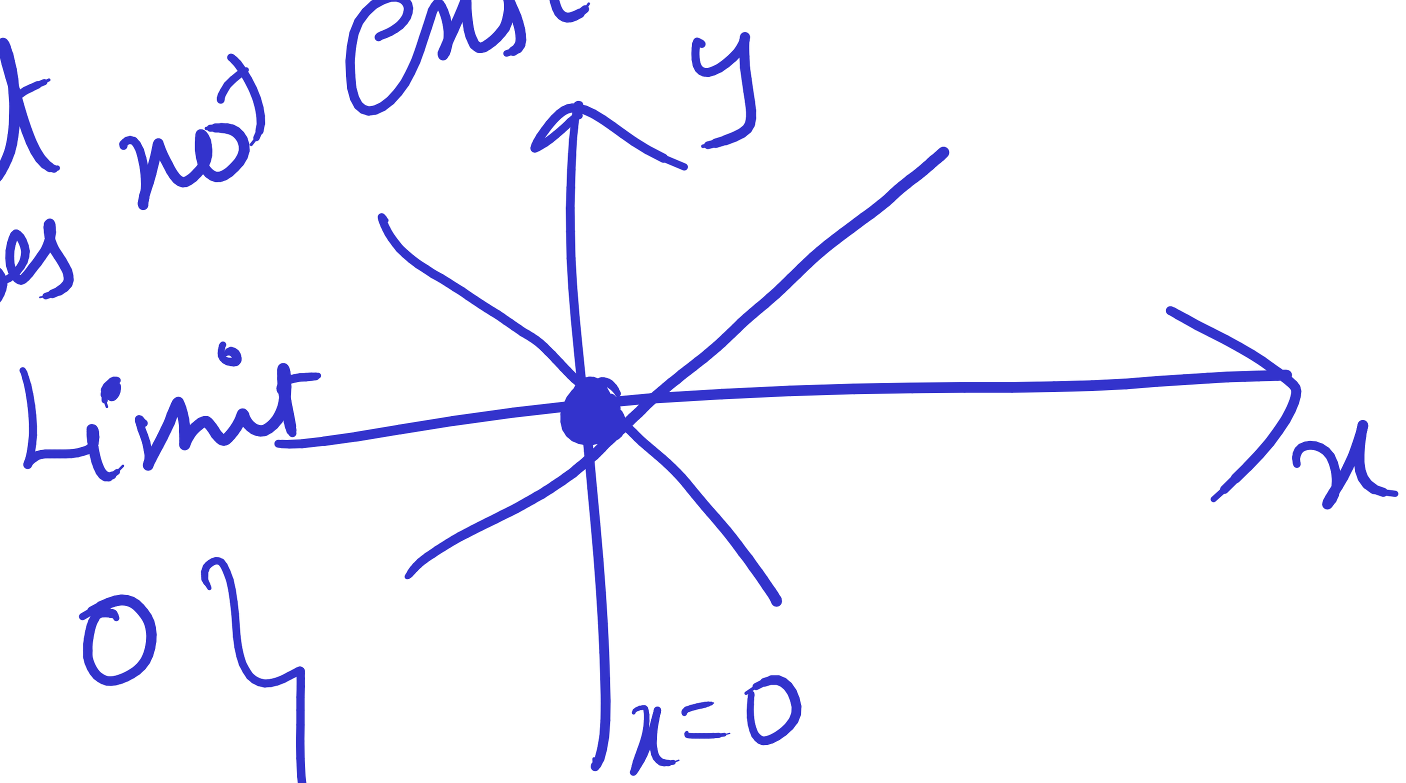
$$\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = 5$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}. = \frac{x(x-y)}{\sqrt{x}-\sqrt{y}} = \frac{x(\sqrt{x}+\sqrt{y})\cancel{(\sqrt{x}-\sqrt{y})}}{\cancel{(\sqrt{x}-\sqrt{y})}}$$

$$= 0$$

$$\lim_{(x,y)\to(0,0)} f(x, y) = \frac{y}{x},$$

limit does not exist



$$\left.\begin{array}{ll} \text{Along } x \text{ axis} & 0 \\ \text{Along } y \text{ axis} & \infty \end{array}\right\}$$

Along $y=x$ line $\quad 1$

Along $y=-x$ line $\quad -1$

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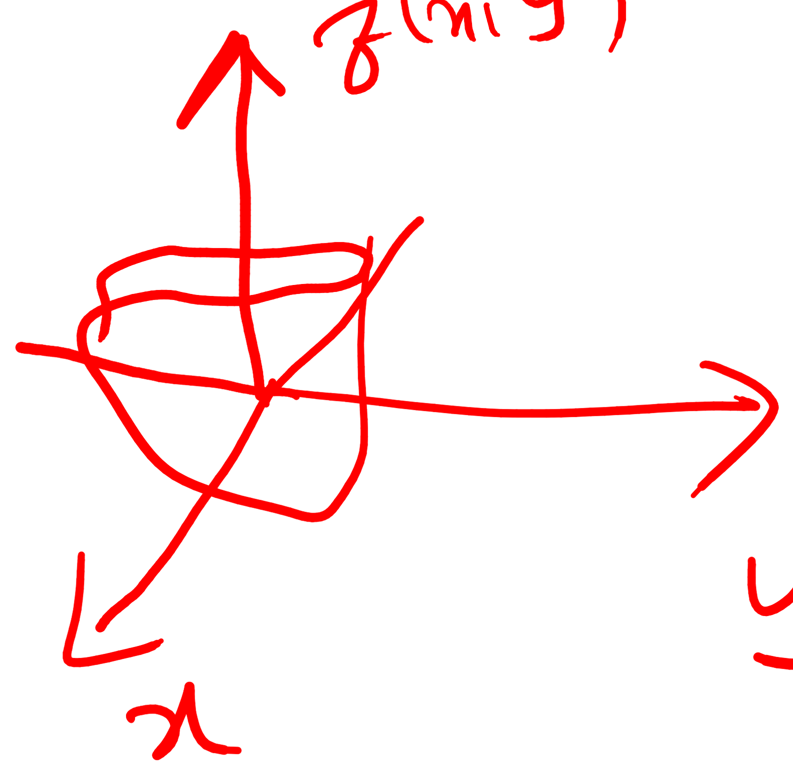
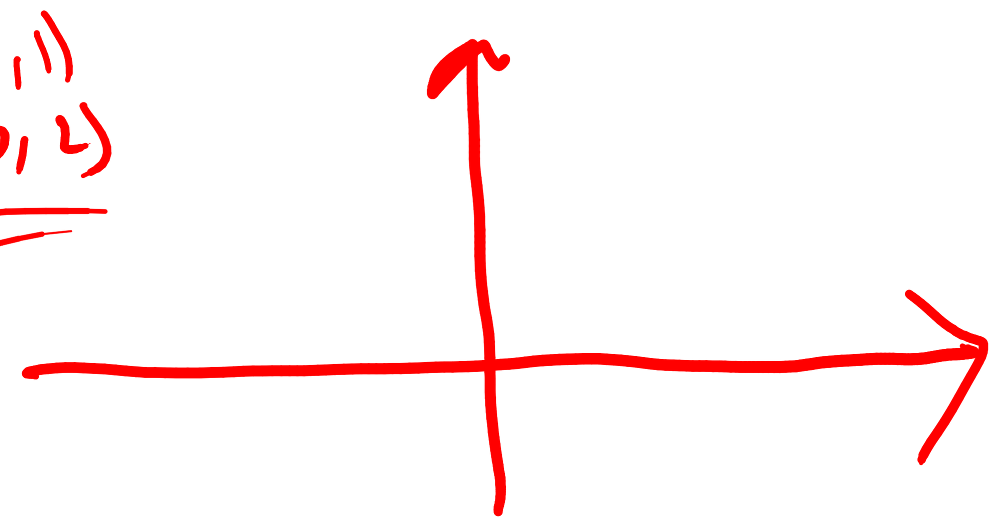
$f(x,y) : \mathbb{R}^2 \to \mathbb{R}$



(1, 1, 2)
(2, 4, 6)

$f(x) : \mathbb{R} \to \mathbb{R}$

(1,1)
(0,2)

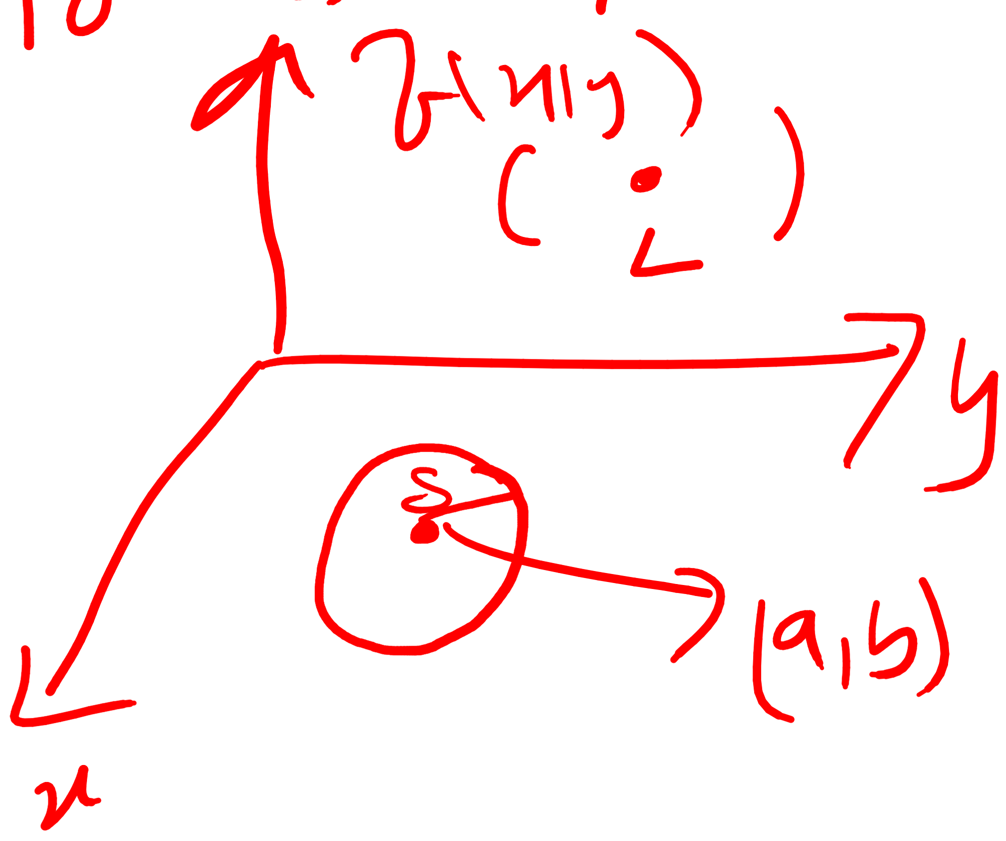


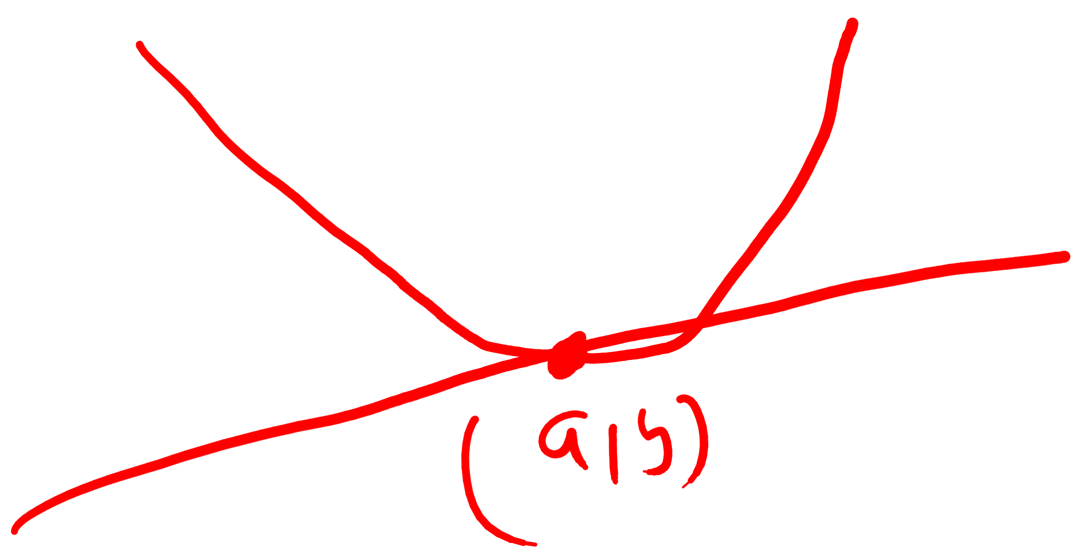
$\mathbb{R}^n \to \mathbb{R}$

Level curves

$$\lim_{\substack{x \to a \\ y \to b}} f(x,y) = L$$

$|f(x,y) - L| < \epsilon$ whenever $\sqrt{(x-a)^2 + (y-b)^2} < \delta$

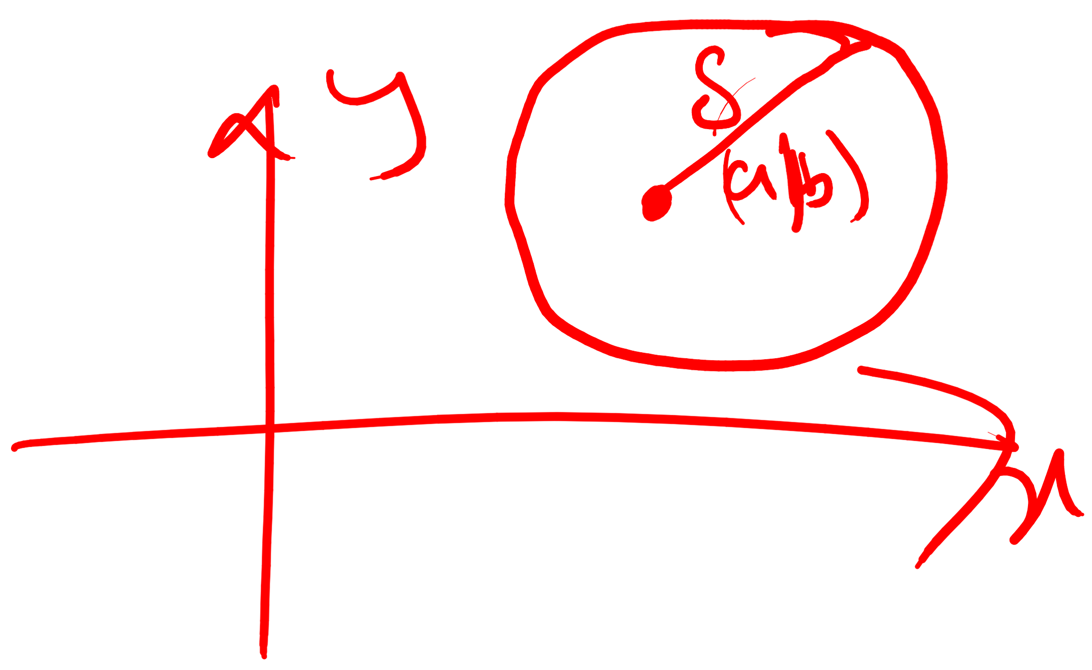


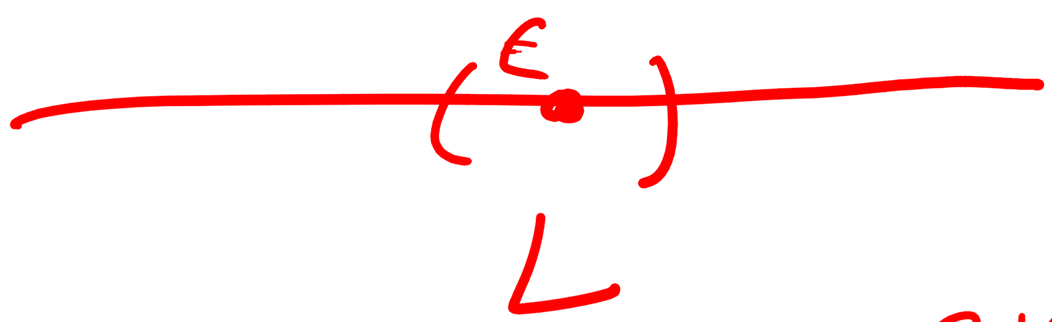


(0,0) $f(x,y)$

$y = mx \Rightarrow$

$y = mx^2 \to$ $y = mx^3$





$f(x,y)$

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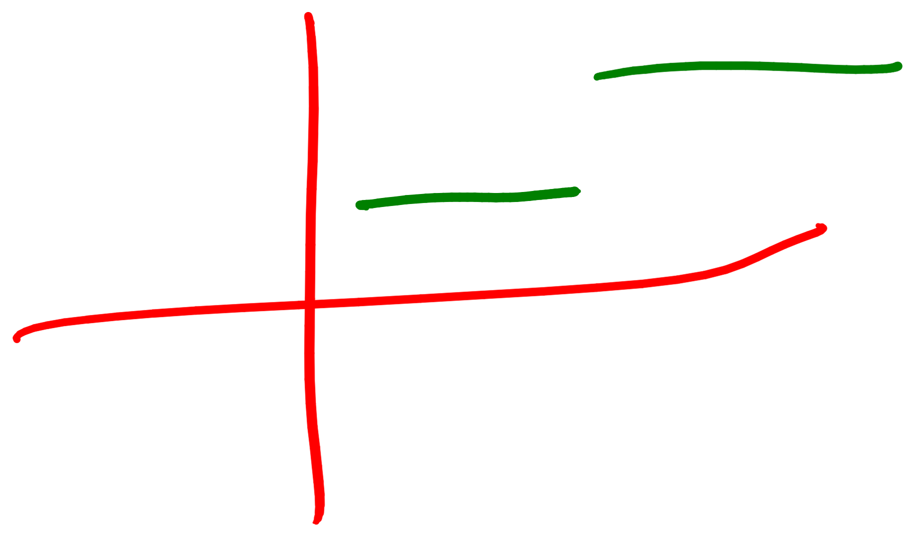
Continuity

$\lim_{x\to a} f(x) = f(a)$



$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

DEFINITION A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

DEFINITION The **partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0)** is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

x^2y $\frac{\partial z}{\partial x}$ $\frac{dz}{dx}$

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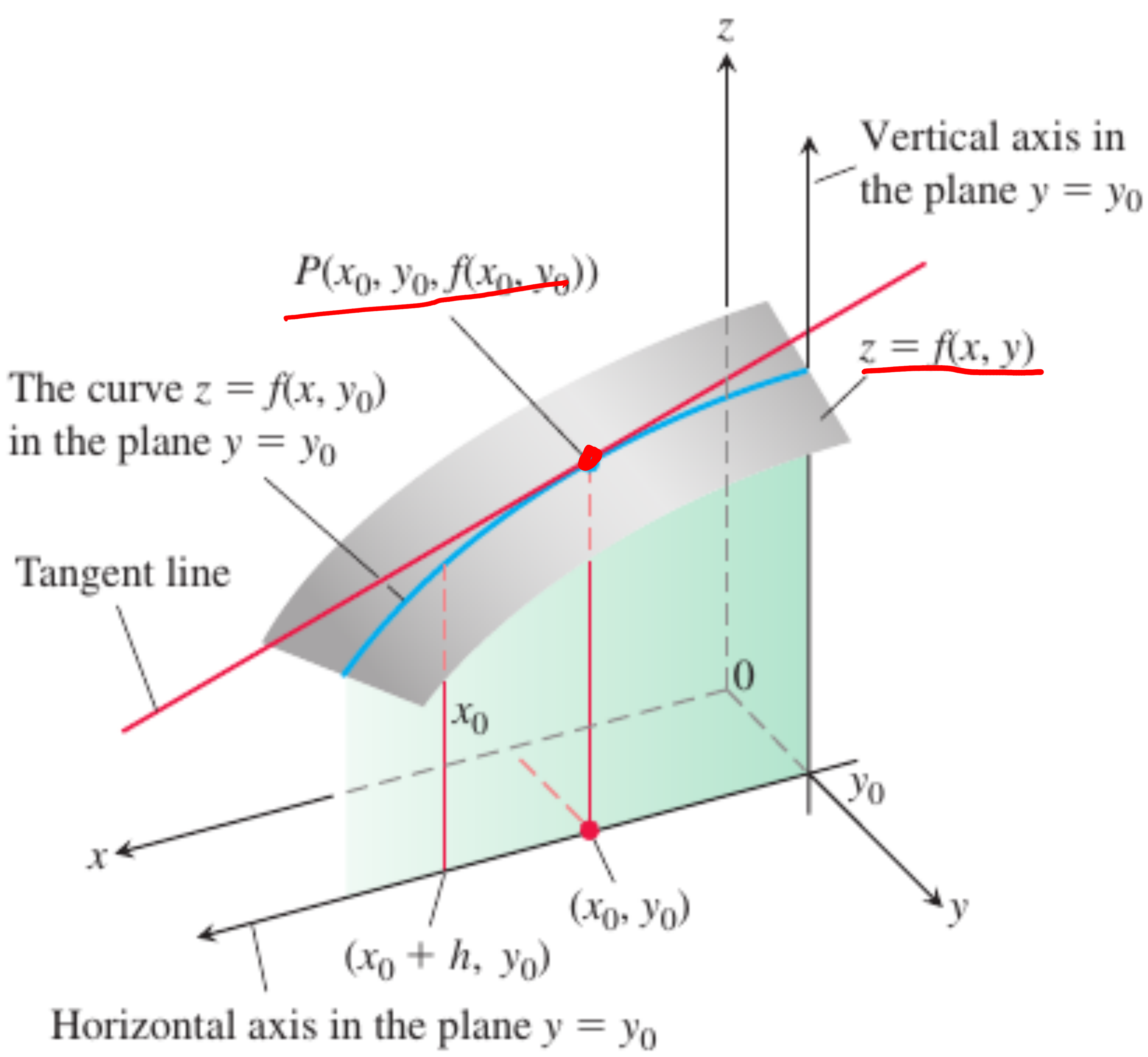
DEFINITION **The partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0) is**

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.

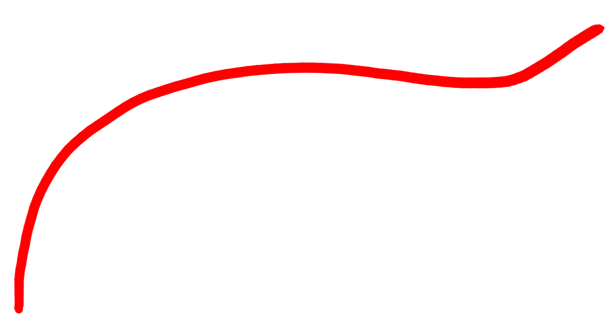


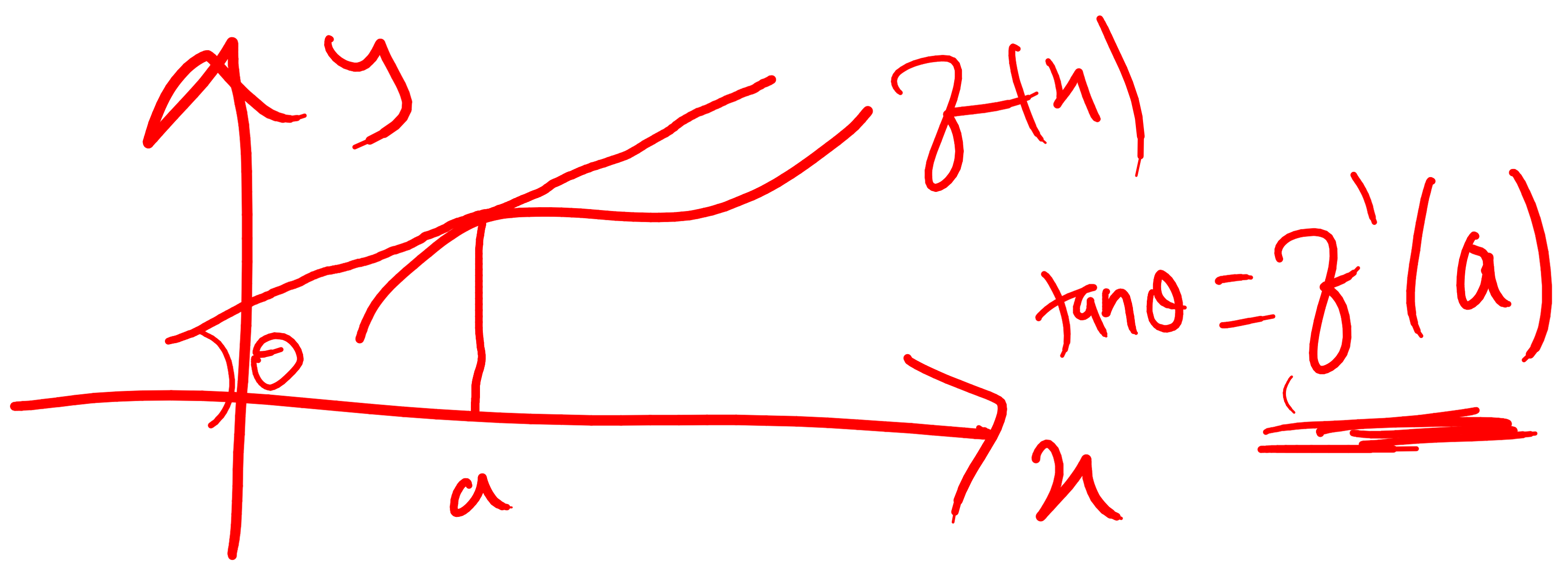
$$\left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)}$$

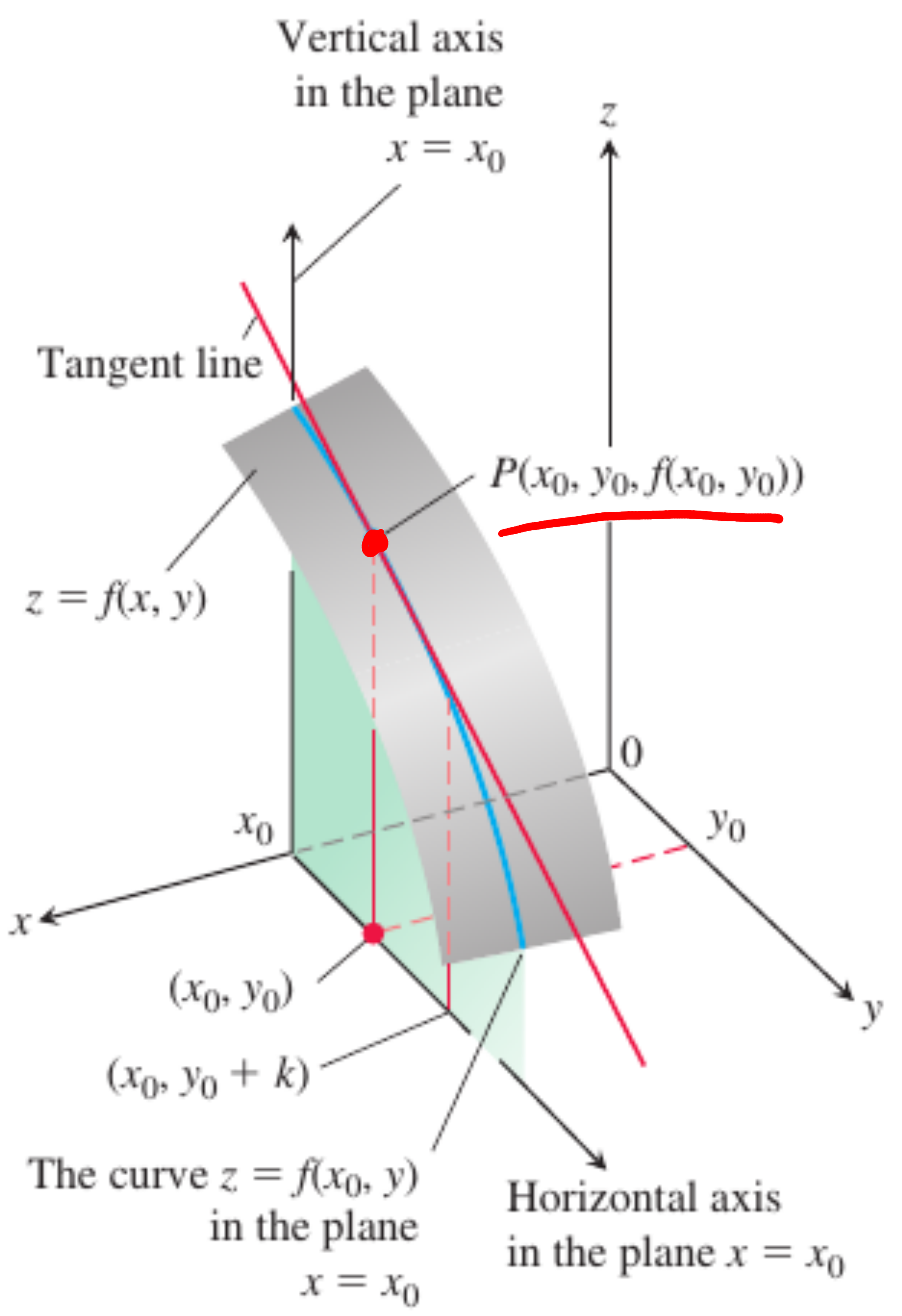
$x^2 y$

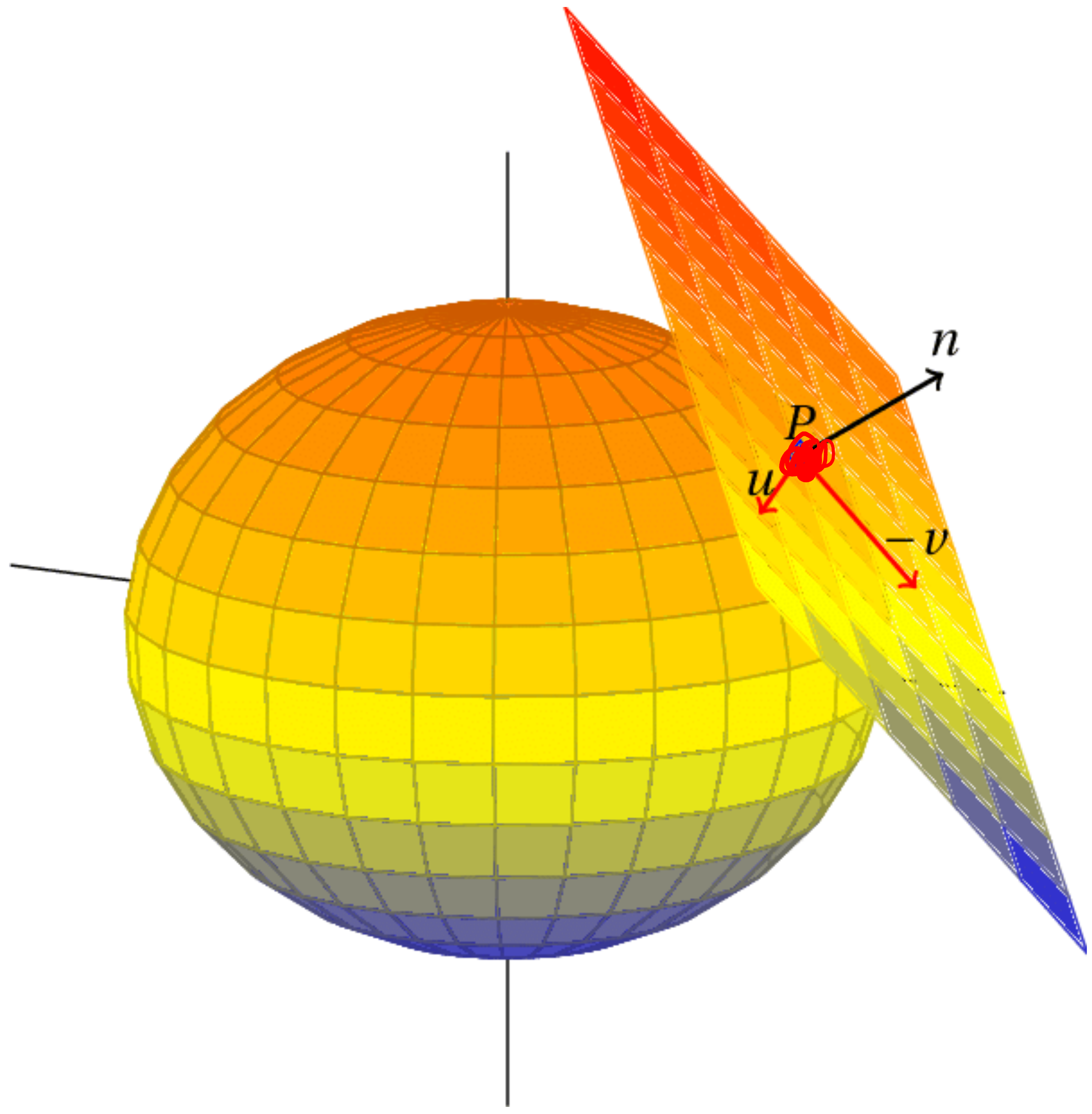
$\frac{dz}{dx}$ $\frac{\partial z}{\partial x}$

$\frac{\partial z}{\partial y}$



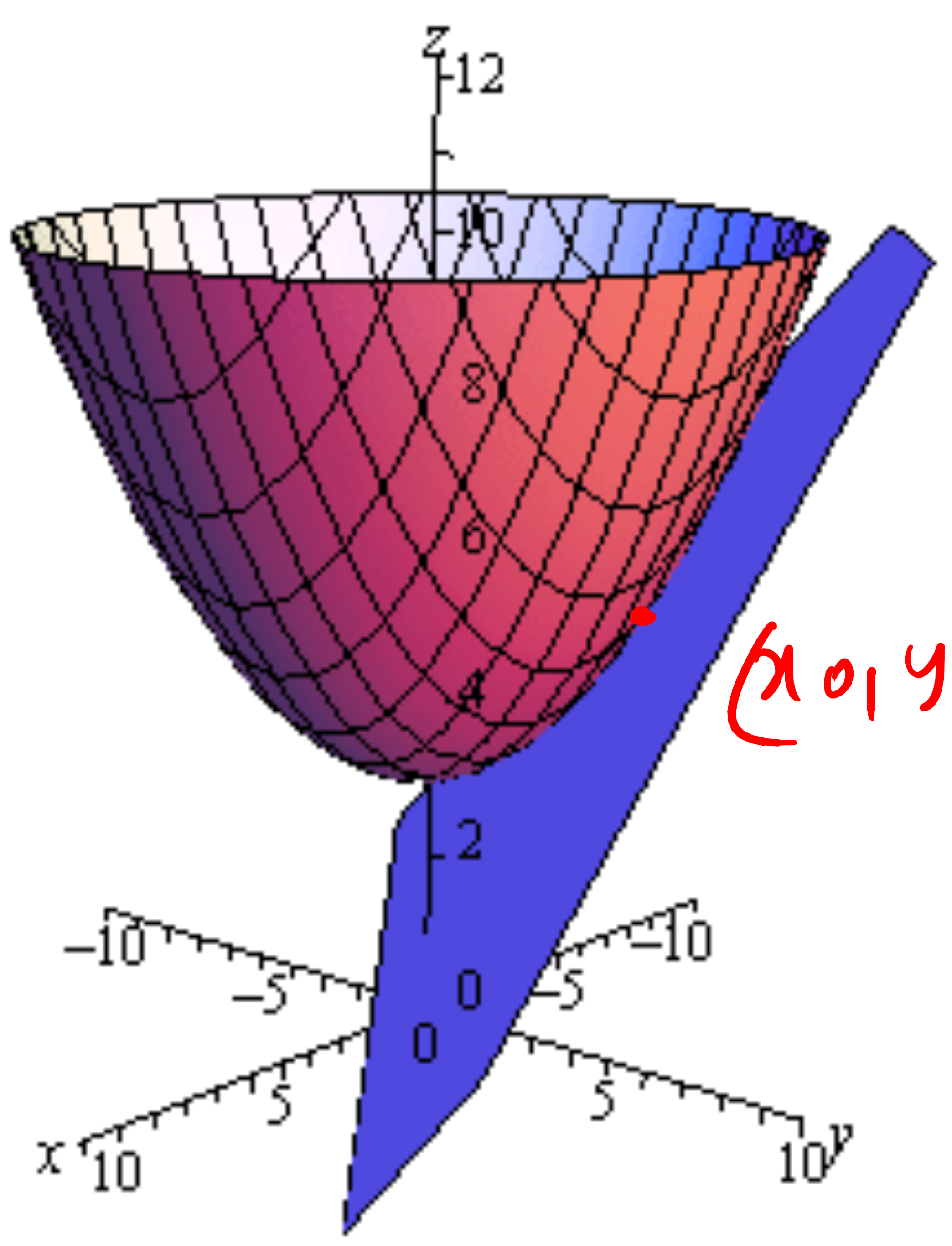






$$\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial y}$$



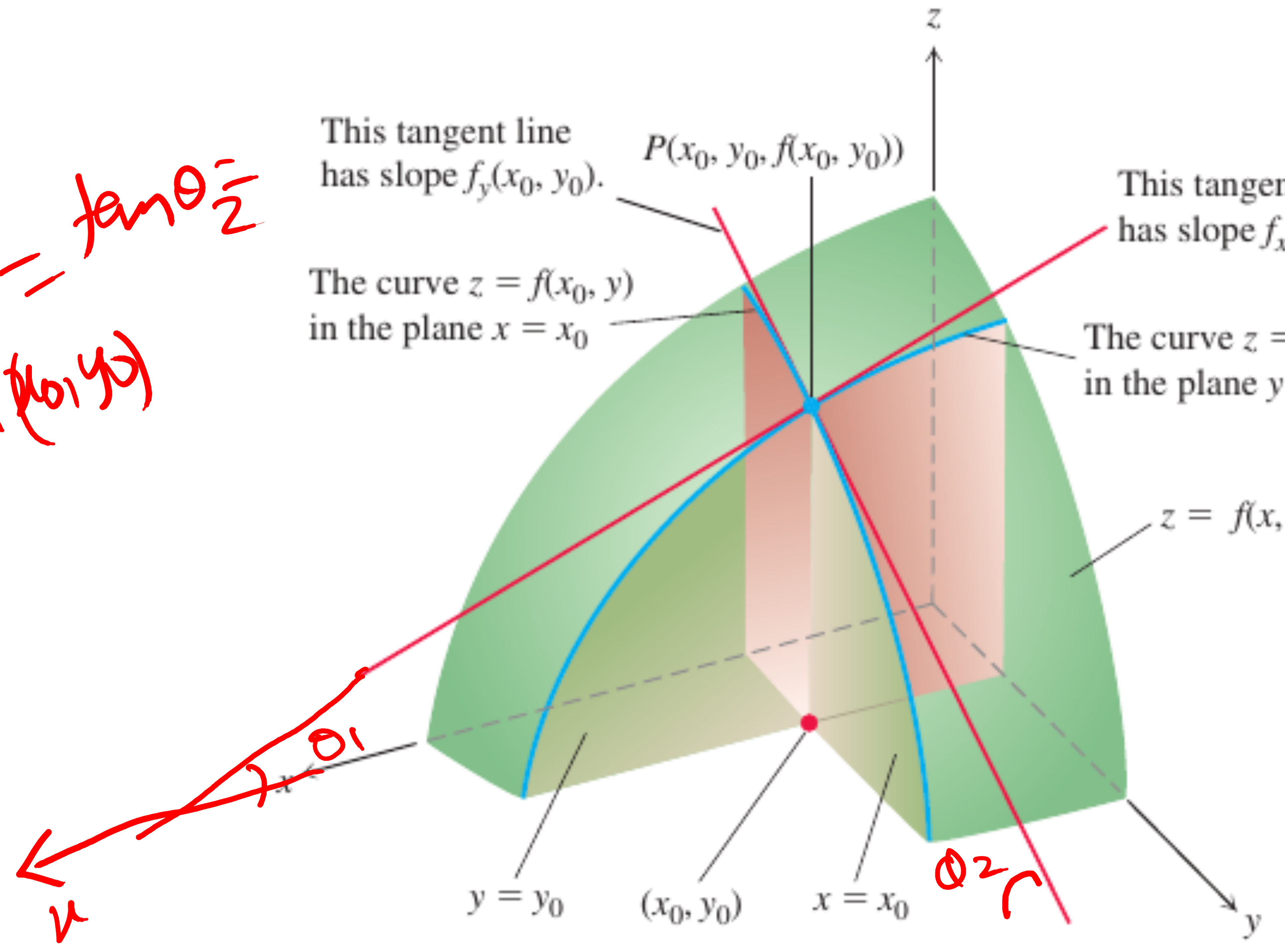
θ_1 (x axis)

$$\frac{\partial z}{\partial x} = \tan \theta_1$$

θ_2 (y axis)

$$\frac{\partial z}{\partial y} = \tan \theta_2$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \tan\theta_2$$



$$= \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$$

$\tan\theta_1$

EXAMPLE 1 Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

$$f(x,y) = x^2 + 3xy + y - 1$$

$$\frac{\partial f}{\partial x} = 2x + 3y \qquad \frac{\partial f}{\partial y} = 3x + 1$$

$$\left.\frac{\partial f}{\partial x}\right|_{(4,-5)} = 8 - 15 = -7 \qquad \frac{\partial f}{\partial y} = 13$$

EXAMPLE 2 Find $\partial f/\partial y$ as a function if $f(x, y) = y \sin xy$.

$\sin(2x)$

$\cos(2x)\,2$

$$\frac{\partial z}{\partial y} = y\cos(xy)\,x + \sin(xy)\,(1)$$

Find f_x and f_y as functions if

$$f(x, y) = \frac{2y}{y + \cos x}. \quad = 2y\left[\frac{1}{y+\cos x}\right]$$

$\left(\frac{1}{x}\right)'$

$-\frac{1}{x^2}$

$$f_x = \frac{\partial z}{\partial x} = 2y\left[\frac{-1}{[y+\cos x]^2}(-\sin x)\right]$$

$$= \frac{2y\sin(x)}{[y+\cos(x)]^2}$$

$$\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$$

$$\frac{\partial z}{\partial y} = \frac{(y+\cos x)2 - 2y(1)}{(y+\cos x)^2}$$

$$= \frac{2\cos(x)}{(y+\cos x)^2}$$

Find $\partial z/\partial x$ if the equation

$$yz - \ln z = x + y$$

$z(x,y)$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{1}{y - 1/z}$$

$y = x^2$

$f(x,y) = x^2 + y^2 = 4$

$$\frac{dy}{dx} = \frac{\partial y}{\partial x}$$

$$= \frac{\partial f/\partial y}{\partial f/\partial x}$$

$x^2 + y^2 = 4$

$2x + 2y\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u \log u$ where $\log u = (x^3 + y^3)/(3x + 4y)$.

$$\log u = \frac{x^3+y^3}{3x+4y}$$

Diff p.w.r.t x

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{(3x+4y)3x^2 - (x^3+y^3)3}{(3x+4y)^2}$$

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{9x^3+12x^2y-3x^3-3y^3}{(3x+4y)^2} = \frac{6x^3+12x^2y-3y^3}{(3x+4y)^2}$$

Diff p.w.r.t y

$$\frac{1}{u}\frac{\partial u}{\partial y} = \frac{(3x+4y)3y^2-(x^3+y^3)4}{(3x+4y)^2}$$

$$= \frac{9xy^2+12y^3-4x^3-4y^3}{(3x+4y)^2}$$

$$= \frac{9xy^2+8y^3-4x^3}{(3x+4y)^2}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = xu\left[\left(\frac{6x^3+12x^2y-3y^3}{(3x+4y)^2}\right)\right] + yu\left[\frac{9xy^2+8y^3-4x^3}{(3x+4y)^2}\right]$$

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = u\, 2\log u$

$\underline{\underline{u_x, u_y}}$

$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

$\frac{\partial^2 u}{\partial x^2}$ $\qquad$ $\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x}$ $\qquad$ $\frac{\partial^2 u}{\partial y^2}$

$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$ $\qquad$ $= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$ $\qquad$ $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$

Example 5.15. *If* $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, *prove that* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$.

(Rajasthan, 2006 ; Calicut, 2005)

and $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u\cos 2u}{4\cos^3 u}$. *(P.T.U., 2006)*

Example 5.10. *If* $x = e^{r\cos\theta}\cos(r\sin\theta)$ *and* $y = e^{r\cos\theta}\sin(r\sin\theta)$, *prove that* $\frac{\partial x}{\partial\theta} = -r\frac{\partial y}{\partial r}, \frac{\partial y}{\partial\theta} = r\frac{\partial x}{\partial r}$.

Hence show that $\frac{\partial^2 x}{\partial\theta^2}+r\frac{\partial x}{\partial r}+r^2\frac{\partial^2 x}{\partial r^2} = 0$.

$z(x,y)$ $\qquad$ $\underline{\underline{z_x}}$, z_y, z_{xy}, z_{xx}, z_{yy} $(z_y)_y$

$z_{xy} = z_{yx}$

$\frac{\partial^3 z}{\partial x^2\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\right)$ z_{yx}, $\underline{\underline{z_{xxy}}}$ $\qquad$ z_{xy} $\qquad$ $z_{xx} = (z_x)_x$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^n(a)$$

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

$f(x) = e^x$ $\quad x = 1$

		$x=1$
$f(x)$	e^x	e
$f'(x)$	e^x	e
$f''(x)$	e^x	e

$$e^x = e + \frac{(x-1)}{1!} e + \frac{(x-1)^2}{2!} e + \cdots$$

		$x=0$
$f(x)$	e^x	1
$f'(x)$	e^x	1
$f''(x)$	e^x	1

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$f(x,y) \quad x=a \quad , y=b$

$$f(x, y) = f(a, b) + [(x - a) f_x (a, b) + (y - b) f_y (a, b)] + \frac{1}{2!} [(x - a)^2 f_{xx} (a, b) + 2(x - a)(y - b) f_{xy} (a, b) + (y - b)^2 f_{yy}(a, b)] + \ldots$$

$$+ \frac{1}{3!} [(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)] + \ldots$$

Expand $e^x \log(1+y)$ in powers of x and y upto terms of third degree.

$f(x, y) = e^x \log(1 + y)$	$f(0, 0) = 0$
$f_x(x, y) = e^x \log(1 + y)$	$f_x(0, 0) = 0$
$f_y(x, y) = e^x \dfrac{1}{1+y}$	$f_y(0, 0) = 1$
$f_{xx}(x, y) = e^x \log(1 + y)$	$f_{xx}(0, 0) = 0$
$f_{xy}(x, y) = e^x \dfrac{1}{1+y}$	$f_{xy}(0, 0) = 1$
$f_{yy}(x, y) = -e^x (1 + y)^{-2}$	$f_{yy}(0, 0) = -1$
$f_{xxx}(x, y) = e^x \log(1 + y)$	$f_{xxx}(0, 0) = 0$
$f_{xxy}(x, y) = e^x \dfrac{1}{1+y}$	$f_{xxy}(0, 0) = 1$
$f_{xyy}(x, y) = -e^x (1 + y)^{-2}$	$f_{xyy}(0, 0) = -1$
$f_{yyy}(x, y) = 2e^x (1 + y)^{-3}$	$f_{yyy}(0, 0) = 2$

$$f(x,y) = f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2!}\{x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\} + \frac{1}{3!}\{x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\} + \ldots$$

$$e^x \log(1+y) = 0 + x(0) + y(1) + \frac{1}{2!}\{x^2(0) + 2xy(1) + y^2(-1)\} + \frac{1}{3!}\{x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)\} + \ldots$$

$$= y + xy - \frac{1}{2}y^2 + \frac{1}{2}(x^2y - xy^2) + \frac{1}{3}y^3 + \ldots$$

$(x-a)$ $(y-b)$

Expand $x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$ using Taylor's theorem.

$(1,-2)$ At $(1,-2)$

	At $(1,-2)$	
$f(x,y) = x^2y + 3y - 2$	-10	
$f_x = 2xy$	-4	
$f_y = x^2+3$	4	f_{xxxx}
$f_{xx} = 2y$	-4	f_{xxxy}
$f_{yy} = 0$	0	f_{xxyy}
$f_{xxx} = 0$	0	f_{xyyy}
$f_{xxy} = 2$	2	f_{yyyy}
$f_{xy} = 2x$	2	

$$f(x,y) = f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b)$$

$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$$+ \frac{1}{3!}\left[3(x-a)^2(y-b) f_{xxy}(a,b)\right]$$

$$= -10 + (x-1)(-4) + (y+2)4 + \frac{1}{2}\left[(x-1)^2(-4) + 2(x-1)(y+2)2 + (y+2)^2(0)\right]$$

$$+ \frac{1}{6}\, 3(x-1)^2(y+2)2$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2).$$

(1, 1)

Example 5.34. *Expand $f(x, y) = \tan^{-1}(y/x)$ in powers of $(x-1)$ and $(y-1)$ upto third-degree terms. Hence compute $f(1.1, 0.9)$ approximately.* (V.T.U., 2010 ; J.N.T.U., 2006 ; U.P.T.U., 2006)

$a=1,\ b=1$

$\rightarrow (1,1)$

$x,\ y$

$(0,0) \rightarrow$

$(0.1, -0.1)$

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} - \frac{1}{2}\{(x-1)-(y-1)\} + \frac{1}{4}\{(x-1)^2-(y-1)^2\} - \frac{1}{12}\{(x-1)^3 + 3(x-1)^2(y-1) - 3(x-1)(y-1)^2 - (y-1)^3\} + \ldots$$

$$\tan^{-1}\left(\frac{0.9}{1.1}\right) = 0.6857$$

Jacobian

$$\int \sqrt{1-x^2}\, dx$$

$x = \cos(t)$

$dx = -\sin(t)\, dt$

$1-x^2 = t^2$

$$= \int \sin(t)\,(-\sin(t))\, dt$$

$$\int_0^1 \int_1^2 (x^2+y^2)\, dx\, dy = \int_0^1 \left.\frac{x^3}{3} + y^2 x\right|_1^2 dy$$

$$= \int_0^1 \left(\frac{8}{3} + 2y^2\right) - \left(\frac{1}{3} + y^2\right) dy$$

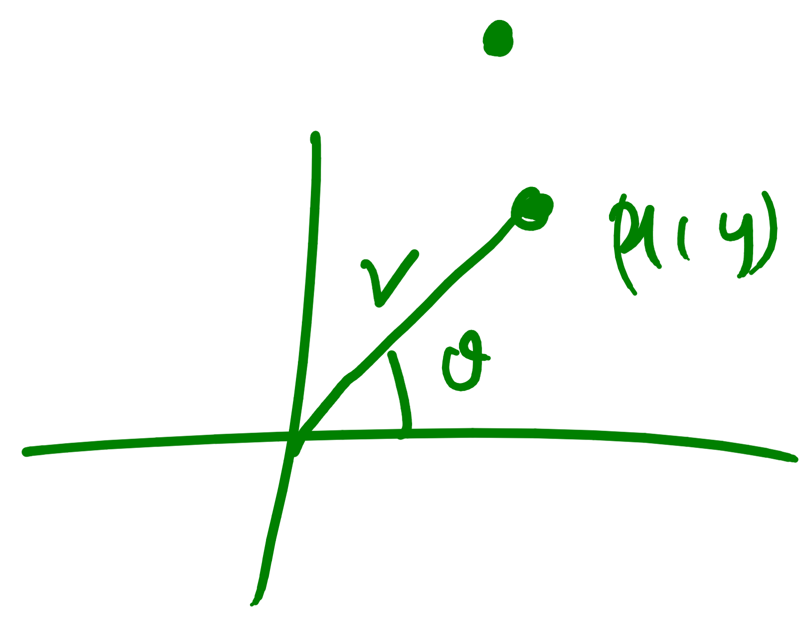
$$= \int_0^1 \frac{7}{3} + y^2 \; dy$$

$$= \frac{7}{3}y + \frac{y^3}{3}\Bigg|_0^1 = \left(\frac{7}{3} + \frac{1}{3}\right) - 0$$

$$= \frac{8}{3}$$

$x = r\sin(\theta)$ $\quad$ $y = r\cos(\theta)$

$dxdy = ?$ $\quad$ $(x, y) \to (r, \theta)$



$dxdy = |J| \, dr d\theta$ $\quad$ $(x, y) \to (r, \theta)$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$(u, v, w) \to (x, y, z)$

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

$x = r\cos\theta \qquad y = r\sin\theta$

$\dfrac{\partial x}{\partial r} = \cos\theta \qquad \dfrac{\partial y}{\partial r} = \sin\theta$

$\dfrac{\partial x}{\partial \theta} = -r\sin\theta \qquad \dfrac{\partial y}{\partial \theta} = r\cos\theta$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r\cos^2\theta + r\sin^2\theta$

$= r$

$dx\,dy = r\,dr\,d\theta$

$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z,$ J

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$\frac{\partial(x,z,y)}{\partial(\phi,\rho,z)}$$

$$= \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho$$

4 Questions

module 1 – 10
– 5

module 2 – 10
– 5

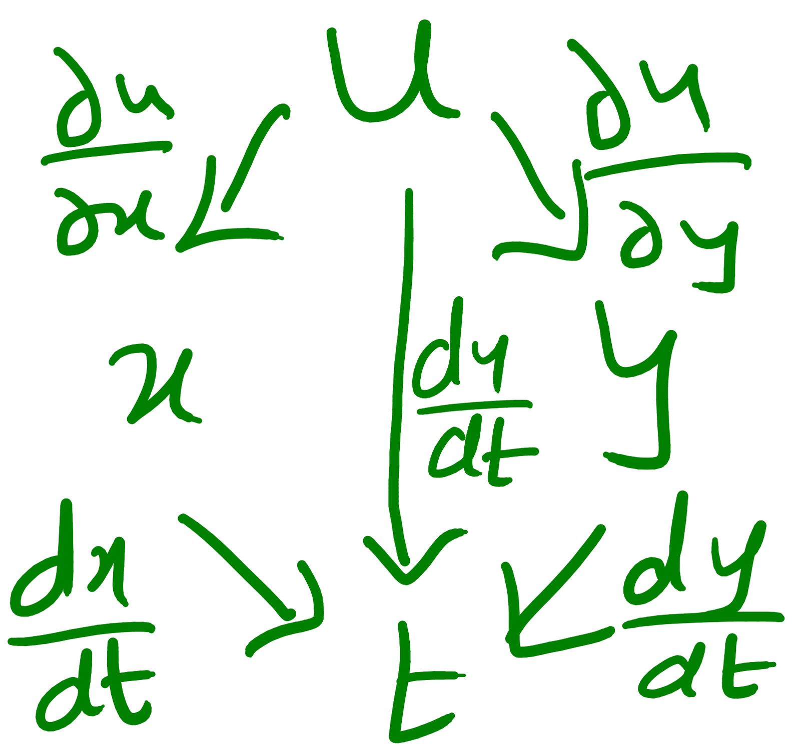
$u = \sin\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$

$u(t)$ $\quad \frac{du}{dt}$

$$u = \sin\left(\frac{e^t}{t^2}\right) \qquad \frac{dy}{dt} = \cos\left(\frac{e^t}{t^2}\right) \cdot \frac{t^2 e^t - e^t\, 2t}{(t^2)^2}$$

$$= \frac{t-2}{t^3}\, e^t \cos\left(\frac{e^t}{t^2}\right)$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$



$$= \cos\left(\frac{x}{y}\right)\frac{1}{y}\, e^t + \cos\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right) 2t$$

$z = u^2 + v^2$, $u = at^2$, $v = 2at$

find $\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}$

$u = x^2 + y^2 + z^2$

$x = e^{2t}$

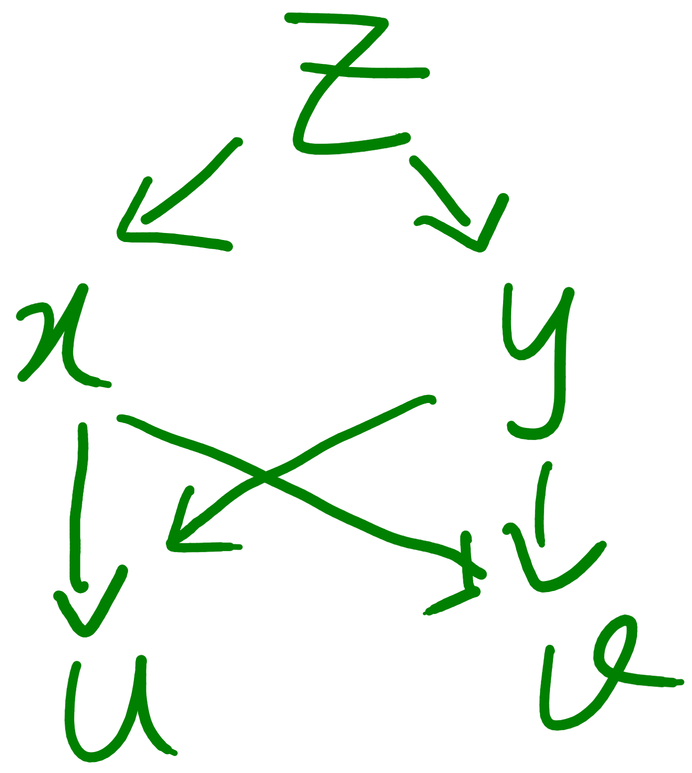
$y = e^{2t}\cos(3t)$

$z = e^{2t}\sin(3t)$

Find $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial u}{\partial z}\dfrac{dz}{dt}$

$z = f(x,y)$, $x = e^u\cos(v)$

$y = e^u\sin(v)$



PT $x\dfrac{\partial z}{\partial v} + y\dfrac{\partial z}{\partial u} = e^{2u}\left(\dfrac{\partial z}{\partial y}\right)$

$z(u, v)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= z_x x_u + z_y y_u$$

$$= \frac{\partial z}{\partial x} e^u\cos(v) + \frac{\partial z}{\partial y} e^u \sin(v)$$

$\frac{5}{6}$ $\frac{8}{4}$ $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(z)$

z'

z_x $\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial z} = 1$

$\cancel{\dfrac{\partial z}{\partial x}}$ $\dfrac{1}{\cancel{\partial z/\partial x}}$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial z}{\partial x} e^{u}(-\sin\theta) + \frac{\partial z}{\partial y}(e^{u}\cos\theta)$$

$$= -\frac{\partial z}{\partial x} e^{u}\sin(\theta) + \frac{\partial z}{\partial y} e^{u}\cos(\theta)$$

$$x\frac{\partial z}{\partial \theta} + y\frac{\partial z}{\partial u} = e^{u}\cos(\theta)\left[\cancel{-\frac{\partial z}{\partial x} e^{u}\sin(\theta)} + \frac{\partial z}{\partial y} e^{u}\cos(\theta)\right]$$

$$+ e^{u}\sin(\theta)\left[\cancel{e^{u}\cos(\theta)\frac{\partial z}{\partial x}} + e^{u}\sin(\theta)\frac{\partial z}{\partial y}\right]$$

$$= e^{2u}\cos^2(\theta)\frac{\partial z}{\partial y} + e^{2u}\sin^2(\theta)\frac{\partial z}{\partial y}$$

$$= e^{2u}\frac{\partial z}{\partial y}[1]$$

Try $x+y = 2e^{\theta}\cos\phi$, $x-y = 2ie^{\theta}\sin\phi$

PT $$\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = 4xy\frac{\partial^2 u}{\partial x\,\partial y}$$ $u(x,y)$

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$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)$$

$$x = e^{\theta} e^{i\phi} \qquad y = e^{\theta} e^{-i\phi}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\underline{e^{\theta} e^{i\phi}} + \frac{\partial u}{\partial y} e^{\theta} e^{-i\phi}$$

$$\frac{\partial u}{\partial \theta} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

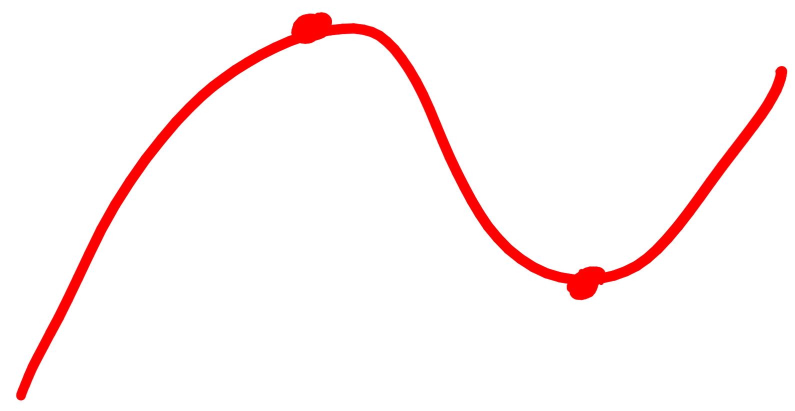
$$\frac{\partial}{\partial \theta} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

maxima/minima of two variables

$y = f(x)$

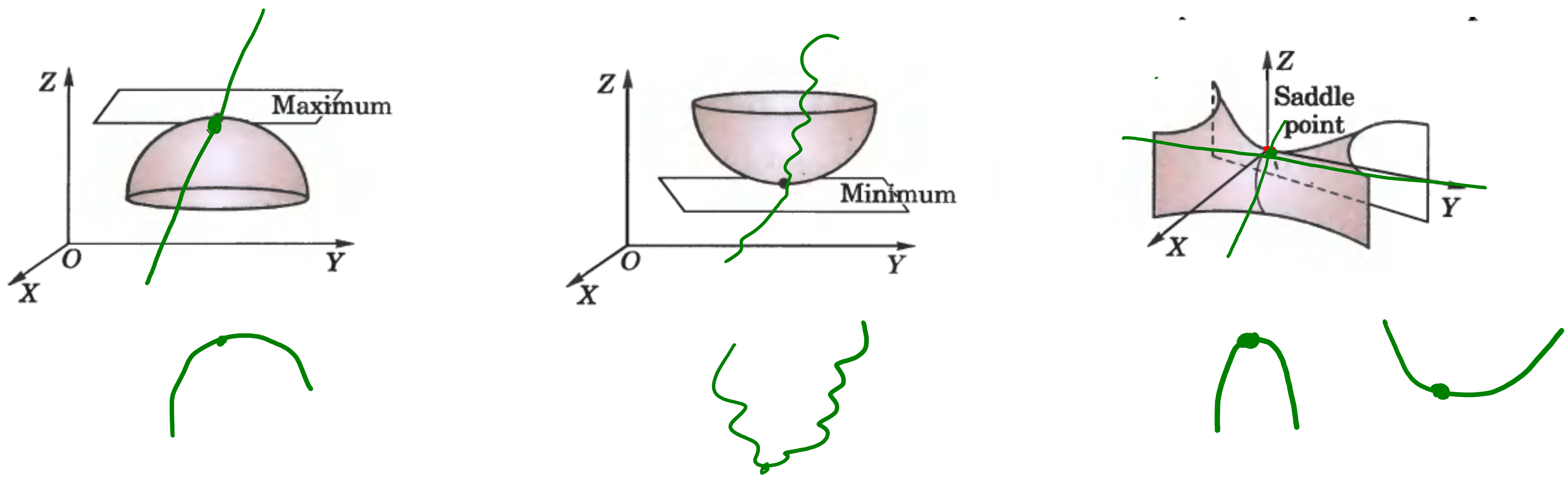
$f'(x) = 0$ critical points



$f''(x) > 0$

< 0

$z = f(x,y)$



$z = f(x,y)$

$\dfrac{\partial f}{\partial x} = 0, \quad \dfrac{\partial f}{\partial y} = 0$ $\quad (x,y) \to$ critical points

$r = \dfrac{\partial^2 f}{\partial x^2} \qquad s = \dfrac{\partial^2 f}{\partial x \partial y} \qquad t = \dfrac{\partial^2 f}{\partial y^2}$

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$$\begin{cases} rt-s^2<0 \rightarrow \text{saddle point} \\ rt-s^2=0 \rightarrow \text{This method is not suitable} \\ rt-s^2>0,\ r>0 \rightarrow \text{minimum point} \\ \qquad\qquad\ \ r<0 \rightarrow \text{maximum point} \end{cases}$$

Examine the following function for extreme values :

$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2.$

$f_x = 4x^3 - 4x + 4y = 0$ — ①

$f_y = 4y^3 + 4x - 4y = 0$ — ②

$4(x^3+y^3)=0$

$\Rightarrow \underline{x=-y}$

① $\Rightarrow 4x^3 - 4x - 4x = 0$

$4x^3 - 8x = 0 \Rightarrow 4x(x^2-2)=0$

$x=0,\ x=\sqrt{2},\ x=-\sqrt{2}$

$\Rightarrow y=0,\ y=-\sqrt{2},\ y=\sqrt{2}$

The critical points are

$(0,0)\quad (\sqrt{2},-\sqrt{2})\quad (-\sqrt{2},\sqrt{2})$

$z_{xx} = r = 12x^2 - 4$

$z_{xy} = s = 4$

$z_{yy} = t = 12y^2 - 4$

$rt - s^2 = (12x^2 - 4)(12y^2 - 4) - 16$

At $(0,0)$

$rt - s^2 = 0$ (This method is not suitable)

At $(\sqrt{2}, -\sqrt{2})$

$rt - s^2 = 386 > 0$, $r = 84 > 0$ minimum point

At $(-\sqrt{2}, \sqrt{2})$

$rt - s^2 = 386 > 0$, $r > 0$ minimum point

$z(x, y)$

$z_x = 0$ Critical points
$z_y = 0$

$r = z_{xx}$ $s = z_{xy}$ $t = z_{yy}$

$rt - s^2 = 0$ need further investigation

$rt - s^2 < 0$ saddle

$rt - s^2 > 0$, $r > 0$ minimum

$r < 0$ Maximum

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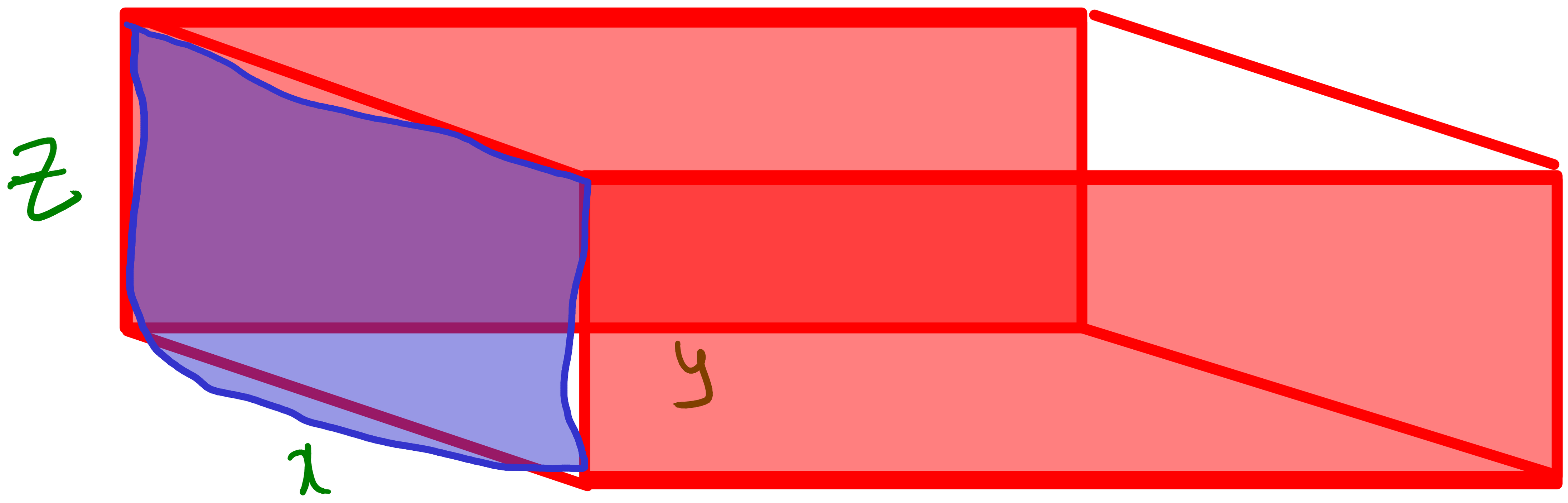
Example 5.45. *A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions of the box requiring least material for its construction.* (Kurukshetra, 2006 ; P.T.U., 2006 ; U.P.T.U., 2005)



$V = xyz = 32$

$S = yz + yz + xz + xz + xy$

$\Rightarrow z = \frac{32}{xy}$

$$S = xy + 2yz + 2xz$$

$$S = xy + \frac{2(32)}{x} + \frac{2(32)}{y}$$

$$S = xy + 64\left(\frac{1}{x} + \frac{1}{y}\right)$$

$$S_x = y - \frac{64}{x^2} = 0 \quad \Rightarrow y = \frac{64}{x^2}$$

$$S_y = x - \frac{64}{y^2} = 0$$

$$x - \frac{64}{(64/x^2)^2} = 0 \quad \Rightarrow x - \frac{1}{64}x^4 = 0$$

$$x\left(1 - x^3/64\right) = 0$$

$x = 0$ or $x = 4$
not possible
$\Rightarrow y = 4$
$z = 2$

Lagrange multiplier method

$\underline{\underline{f}}$, $\underline{\underline{g}}$

$f = xy + 2yz + 2xz$

$g = xyz - 32$

$h = f + \lambda g$ → Lagrange multiplier

$h = \underbrace{xy + 2yz + 2xz}_{f} + \lambda \underbrace{(xyz - 32)}_{g}$

$h_x = 0 \Rightarrow y + 2z + \lambda yz = 0$ — (1)

$h_y = 0 \Rightarrow x + 2z + \lambda xz = 0$ — (2)

$h_z = 0 \Rightarrow 2y + 2x + \lambda xy = 0$ — (3)

$h_\lambda = 0 \Rightarrow xyz - 32 = 0$ — (4)

(1) $\times x \Rightarrow xy + 2xz + \lambda xyz = 0$

(2) $\times y \Rightarrow xy + 2yz + \lambda xyz = 0$

(3) $\times z \Rightarrow 2yz + 2xz + \lambda xyz = 0$

$xyz = 32$

$x(x)\left(\frac{x}{2}\right) = 32$

$x^3 = 64$

$x = 4$

$\Rightarrow y = 4$

$\Rightarrow z = 2$

$xy + 2xz = -32\lambda$

$xy + 2yz = -32\lambda$

$2yz + 2xz = -32\lambda$

$\Rightarrow x = y$

$z = y/2 = \frac{x}{2}$

Given $x+y+z=a$, find the maximum valve of $x^m y^n z^p$.

$f = x^m y^n z^p \qquad g = x+y+z-a$

$h = x^m y^n z^p + \lambda(x+y+z-a)$

$$\left.\begin{aligned} h_x &= m x^{m-1} y^n z^p + \lambda = 0 \\ h_y &= x^m n y^{n-1} z^p + \lambda = 0 \\ h_z &= x^m y^n p z^{p-1} + \lambda = 0 \end{aligned}\right\}$$

$h_\lambda = x+y+z-a = 0$

$$\underline{m x^{m-1} y^n z^p = x^m n y^{n-1} z^p} = x^m y^n p z^{p-1}$$

$m x^{m-1} y^n = x^m n y^{n-1}$

$my = nx$

$\Rightarrow y = \dfrac{nx}{m}$

$n y^{n-1} z^p = y^n p z^{p-1}$

$nz = py$

$\Rightarrow z = \dfrac{py}{n} = \dfrac{p}{n}\dfrac{nx}{m}$

$= \dfrac{px}{m}$

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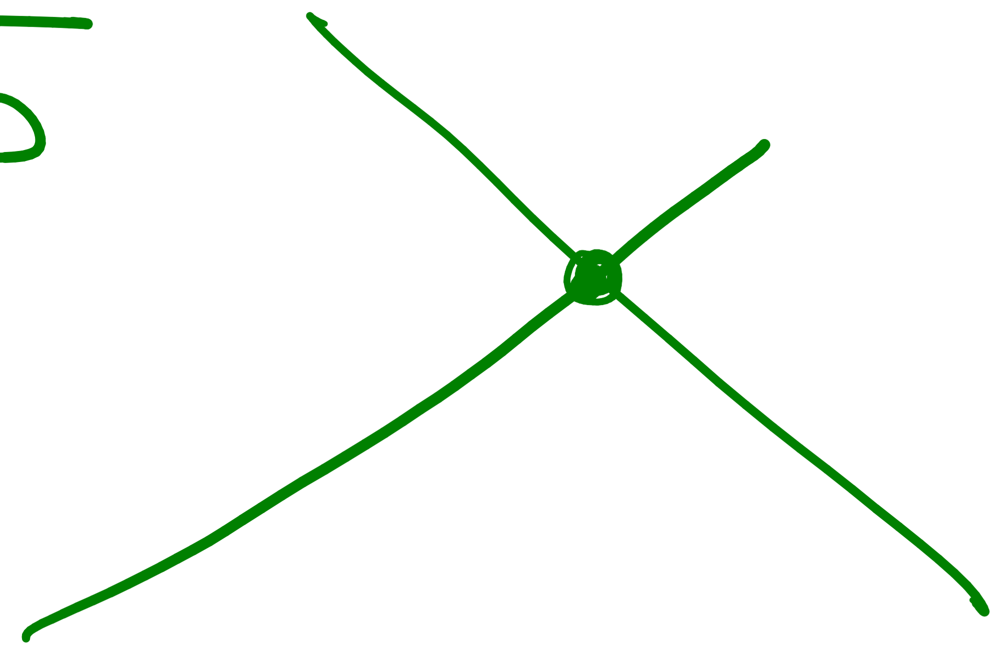
$$x+y+z=a$$

$$x+\frac{nx}{m}+\frac{px}{m}=a$$

$$x=\frac{am}{(m+n+p)} \Rightarrow y=\frac{an}{m+n+p}$$

$$\Rightarrow z=\frac{ap}{m+n+p}$$

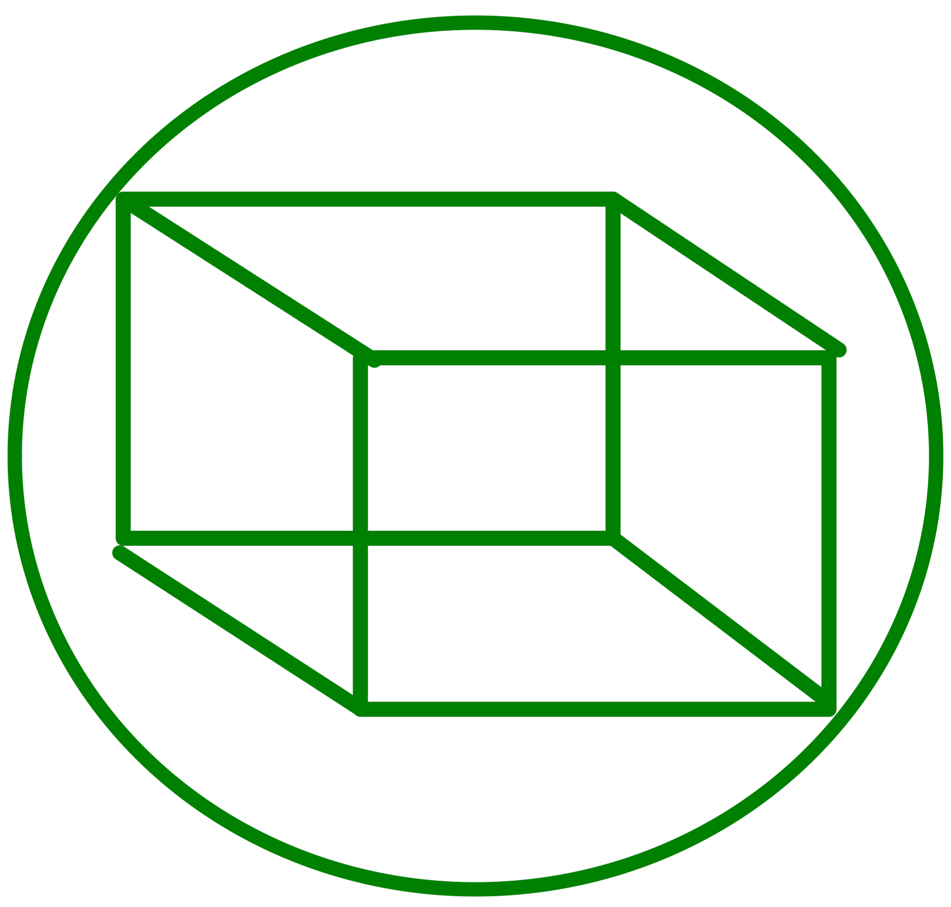


$$\text{max.val}=x^m y^n z^p$$

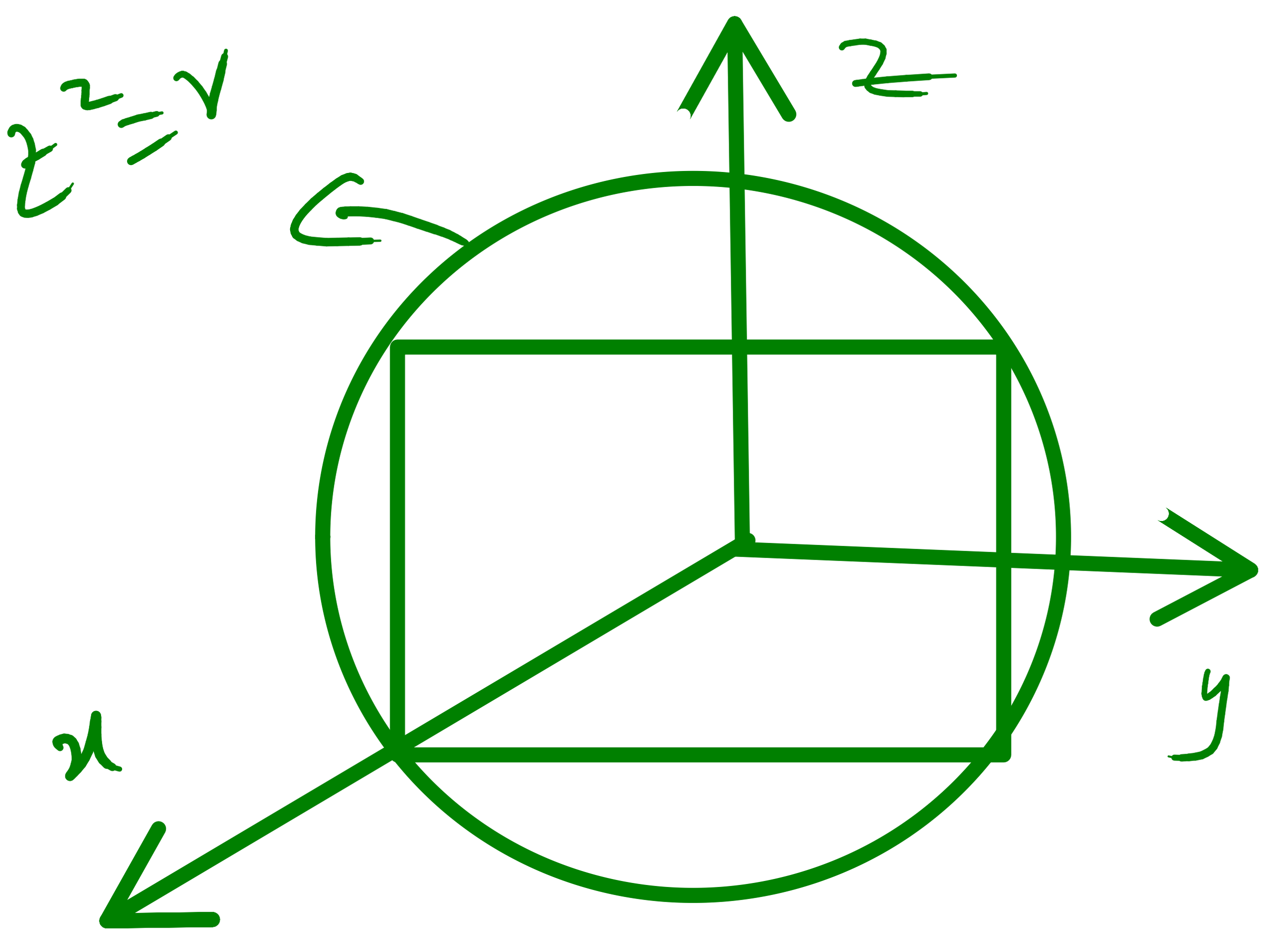
$$=\left(\frac{am}{m+n+p}\right)^m \left(\frac{an}{m+n+p}\right)^n \left(\frac{ap}{m+n+p}\right)^p$$

$$=\frac{a^{m+n+p}\, m^m\, n^n\, p^p}{(m+n+p)^{m+n+p}}$$

Example 5.48. *Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.* (Kurukshetra, 2006 ; U.P.T.U., 2004)



$x^2+y^2+z^2=r^2$



$2x, 2y, 2z$

$V = \frac{8xyz}{8}$

$x^2 + y^2 + z^2 = r^2$

$g = x^2 + y^2 + z^2 - r^2$