Module : 1

Application of Multivariable Calculus

Introduction

22 September 2020 10:54

Applications of Single variable Calculus:

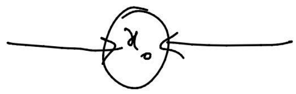
$y = f(x)$

x – independent

y – dependent

Derivative $f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \checkmark \text{ Exist.}$$



$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}, \quad v \neq 0$$

Differentiable on an interval (a,b) or $[a,b]$

A function $y = f(x)$ is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a,b]$ if it is differentiable on the interior (a,b) and if the limits

$$\lim_{h\to 0+} \frac{f(a+h)-f(a)}{h} \quad \text{(Right - hand derivative at a)}$$

$$\lim_{h\to 0-} \frac{f(b+h)-f(b)}{h} \quad \text{(Left - hand derivative at b)}$$

exist at the end points.

1) find the derivative of $y = (x^2+1)(x^2+3)$

2) find the derivative of $y = \frac{t^2-1}{t^3+1}$

Acceleration: Rate of change of velocity w.r to time is called acceleration

Rate of change – $\frac{d}{dt}$

Rate of change — $\frac{d}{dt}$

Acceleration is $a = \frac{dv}{dt}$, v is velocity.

Velocity: If 's' is displacement, then rate of change of displacement is velocity $v = \frac{ds}{dt}$

Chain rule of differentiation

$$y = f(u) \quad \& \quad u = g(t)$$

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt}$$

Maxima & minima of function in an interval :-

1) find $f'(x)$

2) $f'(x) = 0$

The roots of $f'(x) = 0$ are called as critical points

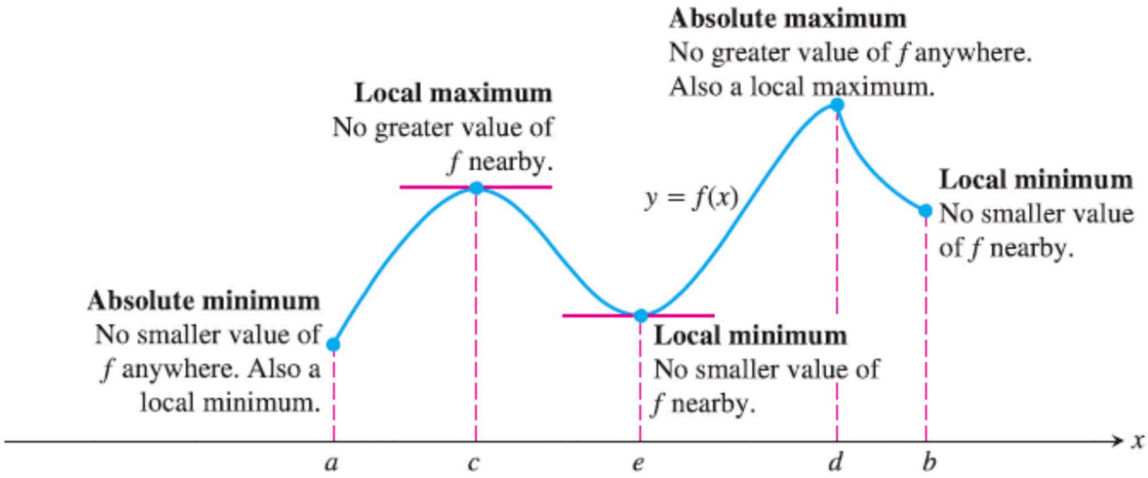
let the critical points $x = a_1, a_2, \ldots$

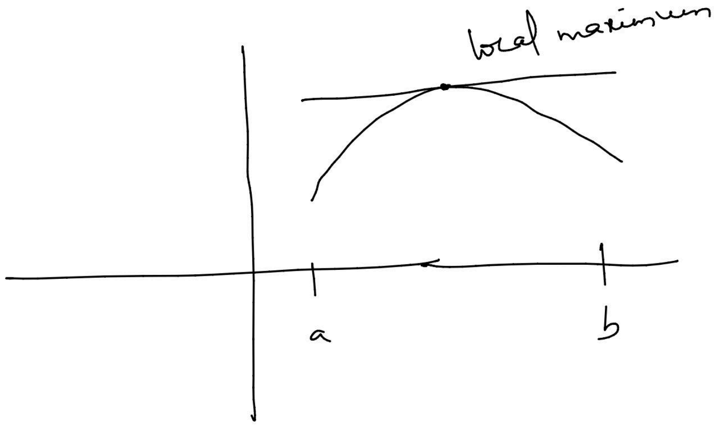
3) find $f''(x)$

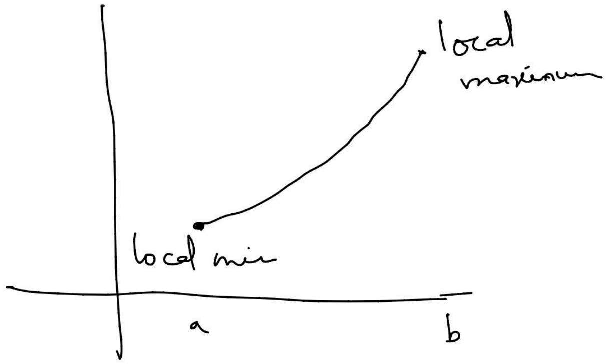
4) if $f''(a_1) > 0$ — f is minimum at a_1

if $f''(a_2) < 0$ — f is maximum at a_2

⋮





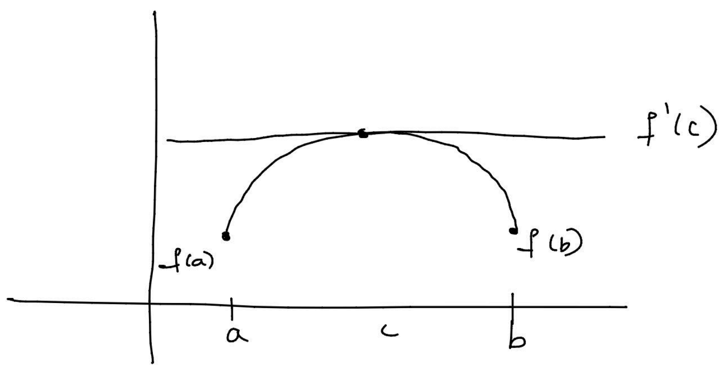


Rolle's theorem:

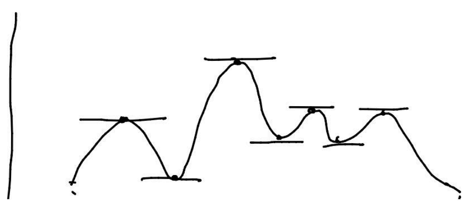
1) let $f(x)$ be continuous in $[a,b]$ & differentiable in (a,b)

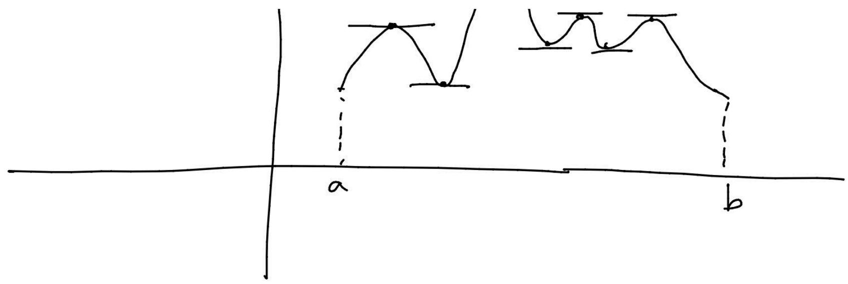
& if $f(a)=f(b)$ then $\exists$ at least one point c in (a,b) $\ni$

$f'(c)=0$.



$f'(c)=0$





Geometrical interpretation

1) $f(x)$ is continuous in $[a,b]$

2) At each & every point 'c' in (a,b) i,e $a < c < b$, there exists a unique tangent at $(c, f(c))$.

3) $f(a) = f(b)$ $\Rightarrow$ The end points of the curve lies at the same hight from x-axis.

Then by Rolle's theorem $\exists$ atleast one point $c \in (a,b) \ni$ $f'(c) = 0$.

i,e there exist atleast one point 'c' in (a,b) such that the slope of $f(x)$ at $(c, f(c))$ is parallel to x-axis.

① Verify Rolle's theorem for $f(x) = (x+2)^3(x-3)^4$ in $[-2, 3]$
a, b

Sol: $f(x) = (x+2)^3(x-3)^4$ is a polynomial function

Every polynomial function is continuous & differentiable everywhere in particular $[-2, 3]$

$$f(a) = f(-2) = 0$$

$$f(b) = f(3) = 0$$

$$f(a) = f(b)$$

Hence Rolle's theorem is applicable

$\therefore$ $\exists$ atleast one point 'c' in $(-2, 3)$ $\ni$ $f'(c) = 0$

Verification: $f'(c) = 0$, $f(x) = (x+2)^3(x-3)^4$

$$f'(x) = 3(x+2)^2(x-3)^4 + 4(x+2)^3(x-3)^3$$

$$f'(x) = 3(x+2)^2(x-3)^4 + 4(x+2)^3(x-3)^3$$
$$= (x+2)^2(x-3)^3\left[3(x-3) + 4(x+2)\right]$$
$$= (x+2)^2(x-3)^3(3x-9+4x+8)$$
$$f'(x) = (x+2)^2(x-3)^3(7x-1)$$

$c \in (a,b)$

$$f'(c) = (c+2)^2(c-3)^3(7c-1) = 0$$
$$\Rightarrow c = -2,\ 3,\ \frac{1}{7}$$
$$\Rightarrow c = \frac{1}{7} \in (-2,3)$$

Hence Rolle's ~~theorem~~ is verified

2) Verify Rolle's theorem $f(x) = 2x^3 + x^2 - 4x - 2$ in $[-\sqrt{2}, \sqrt{2}]$

3) $f(x) = 2x^3 + x^2 - 4x - 2$ in $[-\sqrt{3}, \sqrt{3}]$

4) Verify whether Rolle's theorem is applicable for the following functions or not

a) $f(x) = \tan x$ in $[0, \pi]$

b) $f(x) = \frac{1}{x^2}$ in $[-1, 1]$

c) $f(x) = \frac{1}{x^2}$ in $[1, 2]$

5) $f(x) = (x-a)^m (x-b)^n$ where m, n are +ve integers in (a,b).

Sol: $f(x)$ is continuous & differentiable everywhere

Hence f is continuous in $[a,b]$

f is differentiable in (a,b)

$f(a) = 0$, $f(b) = 0$

$\Rightarrow f(a) = f(b)$

$\therefore$ Rolle's theorem is applicable

... one point c in (a,b)

$\therefore$ Rolle's theorem is applicable

<u>verification</u> By Rolle's theorem $\exists$ atleast one point c in (a,b)

$\ni f'(c) = 0$ $\qquad f(x) = (x-a)^m (x-b)^n$

$$f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m (x-b)^{n-1}$$

$$f'(x) = (x-a)^{m-1}(x-b)^{n-1}\left[m(x-b) + n(x-a)\right]$$

$$(x-a)^{m-1}(x-b)^{n-1}\left[mx - mb + nx - na\right]$$

$$f'(x) = (x-a)^{m-1}(x-b)^{n-1}\left[(m+n)x - (mb+na)\right]$$

$$f'(c) = 0 \Rightarrow \underbrace{(c-a)^{m-1}(c-b)^{n-1}}\left[(m+n)c - (mb+na)\right] = 0$$

$$(m+n)c = mb+na$$

$$c = \frac{mb+na}{m+n}$$

$\Rightarrow$ c divides internally a & b in the ratio $m:n$

$\Rightarrow c \in (a,b)$

$\Rightarrow$ Rolle's theorem is verified.

$\rightarrow$ $f(x) = \dfrac{\sin x}{e^x}$, $[0, \pi]$

$\rightarrow$ $f(x) = \log\left[\dfrac{x^2+ab}{x(a+b)}\right]$ in (a,b), $a>0, b>0$.

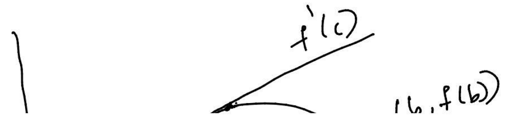
$\rightarrow$ $f(x) = |x|$ in $(-1,1)$

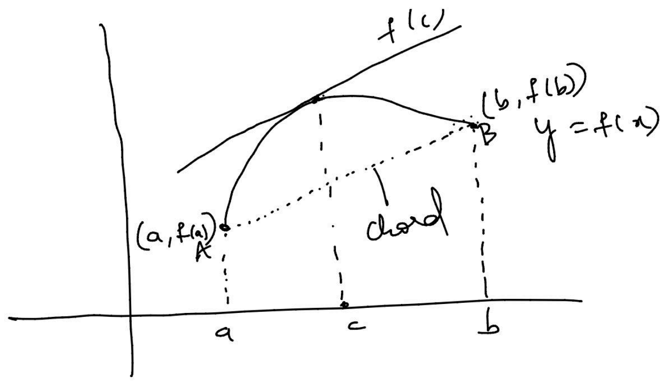
$\rightarrow$ <u>Lagrange's Mean Value theorem (or) Mean Value theorem</u> :-

1) Let $f(x)$ be continuous in $[a,b]$

2) $f(x)$ is differentiable in (a,b)

then $\exists$ atleast one point $c \in (a,b) \ni f'(c) = \dfrac{f(b)-f(a)}{b-a}$





slope of AB $= \dfrac{f(b) - f(a)}{b-a}$ $\qquad f'(c) = \dfrac{f(b) - f(a)}{b-a}$

$f'(c)$ is slope of tangent at $(c, f(c))$

Geometrical interpretation

1) If $f(x)$ is continuous in $[a,b]$

2) If there exist a unique tangent at $(c, f(c))$, $\forall\ c \in (a,b)$ then $\exists$ atleast one point $c \in (a,b)$ such that the slope of the tangent drawn at $(c, f(c))$ is parallel to slope of the chord joining the end points of the curve $(a, f(a))$ & $(b, f(b))$.

① Verify Lagrange's Mean Value theorem for $f(x) = x^3 - x^2 - 5x + 3$ in $[0,4]$

sol: $f(x)$ is continuous & differentiable everywhere

So continuous in $[0,4]$ & diff. in $(0,4)$

$\therefore$ Lagrange's mean value theorem can be applied

then $\exists$ atleast one point $c \in (0,4) \ni f'(c) = \dfrac{f(4) - f(0)}{4-0}$

$$\Rightarrow f'(c) = \frac{f(4) - f(0)}{4}$$

verification : $f(4) = 4^3 - 4^2 - 5(4) + 3 = 31$

$f(0) = 0 - 0 - 0 + 3 = 3$

$f'(x) = 3x^2 - 2x - 5$, $f'(c) = 3c^2 - 2c - 5$

$$f'(c) = \frac{f(4)-f(0)}{4} \Rightarrow 3c^2-2c-5 = \frac{31-3}{4}$$

$$3c^2-2c-5 = 7$$

$$3c^2-2c-12=0$$

$$c = \frac{2 \pm \sqrt{4+4\times3\times12}}{6} = c = \frac{2\pm\sqrt{148}}{6}$$

$$c = \frac{1\pm\sqrt{37}}{3}$$

$$c = \frac{1+\sqrt{37}}{3} \in (0,4) \quad , \quad c = \frac{1-\sqrt{37}}{3} \notin (0,4)$$

Hence Lagrange's Mean value theorem is verified

(2) $f(x) = \log_e x$ in $[1,e]$

(3) $f(x) = x^{1/3}$ in $[-1,1]$

(4) $f(x) = x(x-2)(x-3)$ in $(0,4)$

(5) $f(x) = \cos x$ in $[0,\frac{\pi}{2}]$

(6) If $a<b$, $a,b>0$ prove that $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ using Lagrange's mean value theorem & also deduce the following

a) $\frac{\pi}{4}+\frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4}+\frac{1}{6}$

b) $\frac{5\pi+4}{20} < \tan^{-1}2 < \frac{\pi+2}{4}$

Sol: Let $f(x) = \tan^{-1}x$ in (a,b)

$f(x)$ is continuous in $[a,b]$ & diff in (a,b)

Then by Lagrange's mean value theorem $\exists$ atleast one point $c\in(a,b)$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f'(x) = \frac{1}{1+x^2}, \quad f'(c) = \frac{1}{1+c^2}$$

$$f(b) = \tan^{-1} b \qquad f(a) = \tan^{-1} a$$

$$f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow \frac{1}{1+c^2} = \frac{\tan^{-1} b - \tan^{-1} a}{b-a} \quad \text{—— } (1)$$

$c \in (a,b)$

$$\Rightarrow \quad a < c < b$$

$$\Rightarrow \quad a^2 < c^2 < b^2$$

$$\Rightarrow \quad 1+a^2 < 1+c^2 < 1+b^2$$

$$\Rightarrow \quad \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

$$\Rightarrow \quad \frac{1}{1+a^2} > \frac{\tan^{-1} b - \tan^{-1} a}{b-a} > \frac{1}{1+b^2} \quad (\because \text{from } (1))$$

$$\Rightarrow \quad \frac{b-a}{1+a^2} > \tan^{-1} b - \tan^{-1} a > \frac{b-a}{1+b^2}$$

$$\Rightarrow \quad \frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2} \quad \text{—— } (2)$$

a) $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

$a = 1, \ b = \frac{4}{3}$ in (2)

$$\Rightarrow \quad \frac{\frac{4}{3}-1}{1+\frac{16}{9}} < \tan^{-1}\frac{4}{3} - \tan^{-1} 1 < \frac{\frac{4}{3}-1}{1+1^2}$$

$$\frac{1}{3} \times \frac{9}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{3} \times \frac{1}{2}$$

$$\frac{3}{25} + \frac{\pi}{4} < \tan^{-1}\frac{4}{3} < \frac{1}{6} + \frac{\pi}{4}$$

(2) for $x > 0$, S.T, $1+x < e^x < 1+xe^x$ by L.M.T.

$f(x) = e^x$ in $[0, x]$

$f(x) = e^x$ is continuous & differentiable everywhere

so continuous in $[0,x]$ & differentiable in $(0,x)$

Hence Lagrange's mean value theorem is applicable

By Lagrange's mean value theorem $\exists$ atleast one point $c \in (0,x)$

$$\Rightarrow \quad f'(c) = \frac{f(x) - f(0)}{x - 0}$$

$$f'(x) = e^x \quad , \quad f'(c) = e^c$$

$$e^c = \frac{e^x - 1}{x} \quad \text{— (1)}$$

$c \in (0, x)$

$0 < c < x$

$e^0 < e^c < e^x$

$$1 < \frac{e^x - 1}{x} < e^x \quad (\because \text{from } (1))$$

$x < e^x - 1 < xe^x$

$1 + x < e^x < 1 + xe^x$

Increasing and decreasing functions :-

→ If $f'(x) > 0 \ \forall \ x \in I$, then $f(x)$ is increasing in I.

→ If $f'(x) < 0 \ \forall \ x \in I$, then $f(x)$ is decreasing in I.

Critical point:

Critical point is a point where $f'(x) = 0$.

→ Find the critical points of $f(x) = x^3 - 12x - 5$ & identify the intervals on which f is increasing or decreasing.

Sol:
$$f(x) = x^3 - 12x - 5$$
$$f'(x) = 3x^2 - 12$$
$$f'(x) = 0 \Rightarrow 3x^2 - 12 = 0$$
$$\Rightarrow x^2 = 4$$
$$\therefore x = \pm 2 \text{ are the critical points}$$

$(-\infty, -2)$ $(-2, 2)$ $(2, \infty)$

for $x < -2$, $f'(x) > 0 \Rightarrow f$ is increasing in $(-\infty, -2)$

for $x > -2$ & $x < 2$, $f'(x) < 0 \Rightarrow f$ is decreasing in $(-2, 2)$

for $x > 2$, $f'(x) > 0 \Rightarrow f$ is increasing in $(2, \infty)$

→ Determine all the intervals in which the function $f(x) = -x^5 + \frac{5}{2}x^4 + \frac{40}{3}x^3 + 5$ is increasing or decreasing.

→ First derivative test for local extrema (maxima or minima)

Let $x = c$ is a critical point of $f(x)$

1) If $f'(x)$ changes sign from -ve to +ve at $x = c$, then $f(x)$ has local minimum at $x = c$ i,e

if $f'(x) < 0$ for $x < c$ & $f'(x) > 0$ for $x > c$, then f has local minimum or relative minimum at $x = c$

2) If $f'(x)$ changes sign from +ve to -ve at $x = c$, then

local minimum or ...

2) If $f'(x)$ changes sign from +ve to -ve at $x=c$, then $f(x)$ has local maximum at $x=c$ i,e

if $f'(x)>0$ for $x<c$ & $f'(x)<0$ for $x>c$, then f has local maximum or relative maximum at $x=c$.

→ Examine the function $f(x) = x^3-3x+3$ for maximum & minimum values.

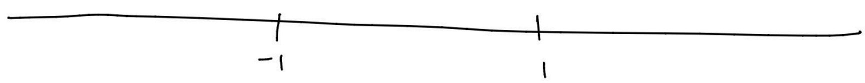
Sol:

$$f'(x) = 3x^2-3$$

$$f'(x)=0 \Rightarrow 3x^2-3=0$$

$$\Rightarrow x^2=1$$

$$\Rightarrow x = \pm 1 \text{ are critical points.}$$



for $x<-1$, $f'(x)>0$ ⇒ f is increasing in $(-\infty,-1)$

for $x>-1$ & $x<1$, $f'(x)<0$ ⇒ f is decreasing in $(-1,1)$

At $x=-1$, $f'(x)$ changes sign from +ve to -ve

then $f(x)$ has local maximum at $x=-1$

& the local maximum is $f(-1) = -1+3+3 = 5$

for $x>1$, $f'(x)>0$ ⇒ f is increasing. in $(1,\infty)$

At $x=1$, $f'(x)$ changes sign from -ve to +ve

then $f(x)$ has local minimum at $x=1$

& the local minimum is $f(1) = 1-3+3 = 1$

→ $f(x) = \sin^2 x$, $0<x<\pi$

$$f'(x) = \sin 2x$$

$$f'(x)=0 \Rightarrow \sin 2x = 0$$

$$2x = 0, \pm\pi, \pm 2\pi, -----$$

$$x = 0, \pm\frac{\pi}{2}, \pm\pi, -----$$

since interval is $(0, \pi)$, $x = \frac{\pi}{2}$ is the only critical point.

for $x < \frac{\pi}{2}$, $f'(x) > 0 \Rightarrow f$ is increasing in $(0, \frac{\pi}{2})$

for $x > \frac{\pi}{2}$, $f'(x) < 0 \Rightarrow f$ is decreasing in $(\frac{\pi}{2}, \pi)$

$f'(x)$ changes sign from +ve to −ve

$\therefore$ f has local maximum at $x = \frac{\pi}{2}$ &

local maximum value is $f(\frac{\pi}{2}) = 1$.

→ Concavity:

let $f(x)$ be a differentiable function, then

1) f is concave upward in I if $f'(x)$ is increasing in I

2) f is concave downward in I if $f'(x)$ is decreasing in I.

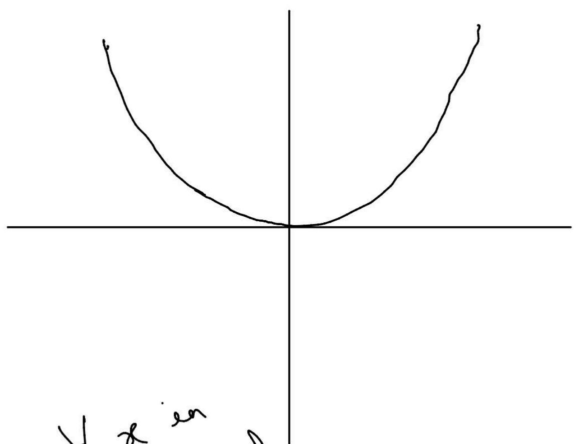
Second derivative test for concavity:

1) If $f'' > 0$ on I, then f is concave upward in I

2) If $f'' < 0$ on I, then f is concave downward in I.

I is any interval.

Ex: $y = x^2$



$f(x) = x^2$

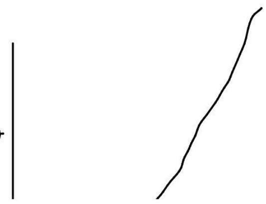
$f''(x) = 2 > 0$ $\forall$ x in any interval

$\Rightarrow f''(x) = 2 > 0$ $\forall$ $x \in (-\infty, \infty)$

Ex: $y = x^3$

$f(x) = x^3$

point of inflexion

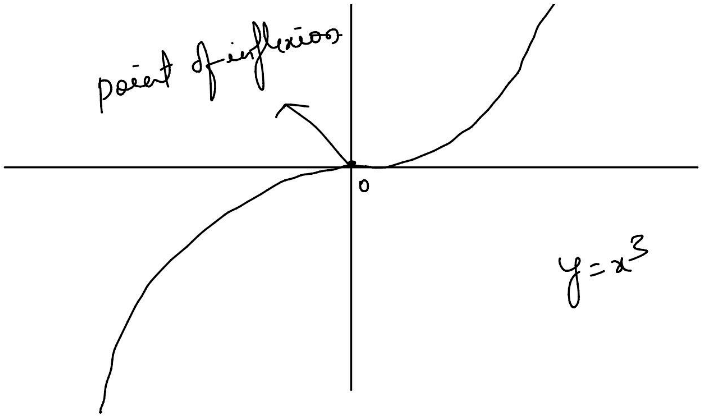


Ex: $y = x^3$

$f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = 6x$



in $(0, \infty)$, $f'' > 0 \Rightarrow f$ is concave up

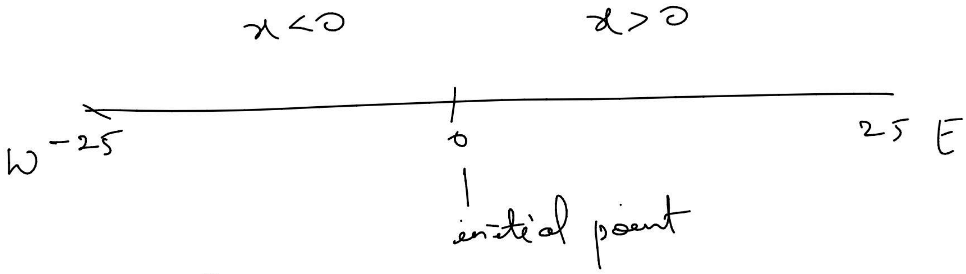
in $(-\infty, 0)$ $f'' < 0 \Rightarrow f$ is concave downward.

→ Note: when $f''(c) = 0$ then $x = c$ is said to be point of inflexion.

→ Suppose that the elevation above sea level of a road is given by $E(x) = 500 + \cos\frac{x}{4} + \sqrt{3}\sin\frac{x}{4}$ where x is in miles.

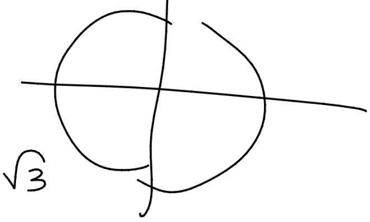
Assume that if x is +ve we are to the east of the initial point of measurement & if x is −ve " " " west " " ". If we start 25 miles to the west of initial point of measurement & drive until we are 25 miles to the east of initial point of measurement, how many miles of our drive were we driving up incline.

sol:



$(-25, 25)$

$E(x) = 500 + \cos\frac{x}{4} + \sqrt{3}\sin\frac{x}{4}$



$E'(x) = -\frac{1}{4}\sin\frac{x}{4} + \frac{\sqrt{3}}{4}\cos\frac{x}{4}$

$E'(x) = 0 \Rightarrow \frac{1}{4}\sin\frac{x}{4} = \frac{\sqrt{3}}{4}\cos\frac{x}{4} \Rightarrow \tan\frac{x}{4} = \sqrt{3}$

$\tan\frac{x}{4} = \sqrt{3}$

$\tan\frac{x}{4} = \tan\theta$

$\tan\frac{x}{4} = \tan\frac{\pi}{}$

$$\tan\frac{x}{4} = \ldots$$

$$\tan\frac{x}{4} = \tan\frac{\pi}{3} \qquad\qquad \tan\frac{x}{4} = \cancel{\tan}\,\theta$$

$$\tan\frac{x}{4} = \tan\frac{4\pi}{3} \qquad\qquad \frac{x}{4} = 2n\pi + \theta$$

$$\frac{x}{4} = 2n\pi + \frac{\pi}{3} \Rightarrow x = 8n\pi + \frac{4\pi}{3}, \quad n = 0, \pm1, \pm2, \ldots$$

$$\frac{x}{4} = 2n\pi + \frac{4\pi}{3} \Rightarrow x = 8n\pi + \frac{16\pi}{3}, \quad n = 0, \pm1, \pm2, \ldots$$

$n = 0 \Rightarrow x = \frac{4\pi}{3} = 4.186 \in (-25, 25)$

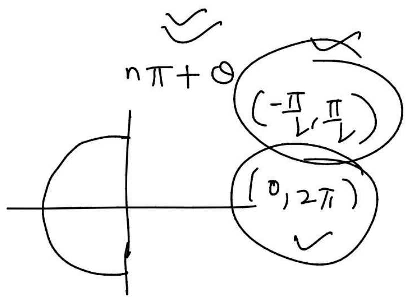
$x = \frac{16\pi}{3} = 16.746 \in (-25, 25)$

$n = 1 \Rightarrow x = \frac{28}{3}\pi = 29.306 \notin (-25, 25)$ $\qquad x = 8n\pi + \frac{4\pi}{3}$

$x = \frac{40}{3}\pi = 41.86 \notin (-25, 25)$ $\qquad x = 8n\pi + \frac{16\pi}{3}$

$n = -1 \Rightarrow x = \frac{-20\pi}{3} = -20.933 \in (-25, 25)$

$x = -\frac{8\pi}{3} = -8.373 \in (-25, 25)$



$n = -2 \quad x \notin (-25, 25)$

$x = -8.373, -20.933, 4.186, 16.746$ are the critical points

for $x \in (-25, -20.933)$, $E'(x) > 0$, $E(x)$ is increasing

$x \in (-20.933, -8.373)$, $E'(x) < 0$, " " decreasing

$x \in (-8.373, 4.186)$, $E'(x) > 0$, $E(x)$ is increasing

$x \in (4.186, 16.746)$, $E'(x) < 0$, $E(x)$ is decreasing

$x \in (16.746, 25)$, $E'(x) > 0$, $E(x)$ is increasing

$x \in (16.746, 25)$, $E'(x) > 0$, $E(x)$ is increasing =

In $(-25, -20.933)$, $(-8.373, 4.186)$, $(16.746, 25)$, $E(x)$ is increasing.

$(-20.933+25) + (4.186 + 8.373) + (25 - 16.746)$

$= 24.8$ miles (approx)

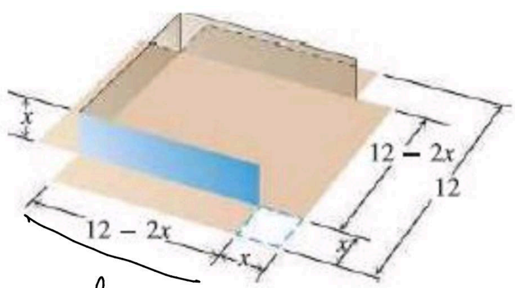
we drive up 24.8 miles on the incline.

→ Second derivative test for maxima (or) minima

Let $x = c$ be the critical point of the function $f(x)$

1) If $f''(c) > 0$, then $f(x)$ has local minima at $x = c$
2) If $f''(c) < 0$, then $f(x)$ has local maxima at $x = c$
3) If $f''(c) = 0$, then $f(x)$ has either max or local minima or neither.

1) An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?



Let 'x' be the dimension of the square's cut

$l = 12 - 2x$

$b = 12 - 2x$

$h = x$

$V = lbh = (12-2x)^2 x$ should be max

$V' = 2(12-2x)(-2)x + (12-2x)^2$

$$V' = 2(12-2x)(-2)x + (12-2x)^2$$

$$V' = 0 \Rightarrow -4x(12-2x) + (12-2x)^2 = 0$$

$$(12-2x)\left[12-2x-4x\right] = 0$$

$$x = 6, 2$$

$x = 2$, $V'' = -4x(-2) + (12-2x)(-4) + 2(12-2x)(-2)$

$V''(2) = -48 < 0$

V is max at $x = 2$

→ $f(x) = \sin x(1+\cos x)$ in $(0, \pi)$, find max & min values.

Sol

$f'(x) = \sin x(-\sin x) + \cos x(1+\cos x)$

$f'(x) = -\sin^2 x + \cos^2 x + \cos x$.

$f'(x) = 0 \Rightarrow \cos^2 x - \sin^2 x + \cos x = 0$

$\Rightarrow \cos 2x + \cos x = 0$

$x = \frac{\pi}{3}$ is critical point.

$f''(x) = -2\sin 2x - \sin x$.

$f''\left(\frac{\pi}{3}\right) < 0$

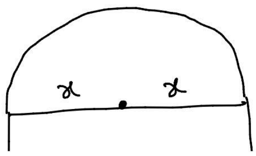
$\therefore$ f is max at $x = \frac{\pi}{3}$

& max value is $f\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{4}$

→ A window has the form of a rectangle surmounted by a semi circle. If the perimeter is 40 ft. Then find its dimensions so that the greatest amount of light is admitted.

Sol:

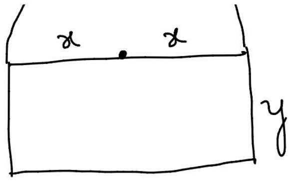
perimeter $= 2x + 2y + \pi x$



$2y = 40 - 2x - \pi x$

perimeter = $2x+2y+\pi x$

$40 = 2x+2y+\pi x$



$2y = 40 - 2x - \pi x$

$y = 20 - x - \frac{\pi x}{2}$

$A = \frac{1}{2}\pi x^2 + 2xy$

we have to find x & y such that A is maximum.

$A = \frac{1}{2}\pi x^2 + 2x\left(20 - x - \frac{\pi x}{2}\right)$

$\frac{dA}{dx} = \pi x + 40 - 4x - 2\pi x = -\pi x - 4x + 40$

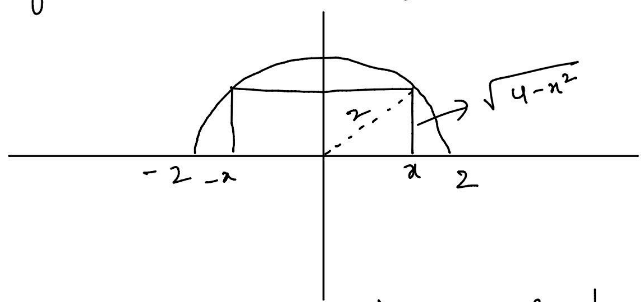
$\frac{dA}{dx} = 0 \Rightarrow x(\pi+4) = 40 \Rightarrow x = \frac{40}{\pi+4}$

$\frac{d^2A}{dx^2} = -\pi - 4 < 0$

$\therefore$ A is max at $x = \frac{40}{\pi+4}$

→ A rectangle is to be inscribed in a semi circle of radius '2'. what is the largest area the rectangle can have & what are the dimensions.

Sol:



$A = 2x\sqrt{4-x^2}$ since $l = 2x$ & $b = \sqrt{4-x^2}$

$\frac{dA}{dx} = \frac{\cancel{2}x \times -2x}{\cancel{2}\sqrt{4-x^2}} + 2\sqrt{4-x^2} = -\frac{2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2}$

$\frac{dA}{dx} = 0 \Rightarrow -\frac{2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2} = 0$

$\Rightarrow 2x^2 = 2(4-x^2)$

$\Rightarrow -2x^2 + 8 = 2x^2$

$4x^2 = 8$

$x^2 = 2$

$$x = \pm\sqrt{2}$$

$$\frac{d^2A}{dx^2} = -2\left[\frac{\sqrt{4-x^2}\,2x - x^2\frac{1}{\cancel{2}\sqrt{4-x^2}}\times -\cancel{2}x}{4-x^2}\right] + \cancel{2}\frac{1}{\cancel{2}\sqrt{4-x^2}}\times -2x$$

$$\frac{d^2A}{dx^2} = -\frac{4x\sqrt{4-x^2} + x^3/\sqrt{4-x^2}}{4-x^2} - \frac{2x}{\sqrt{4-x^2}}$$

At $x = \sqrt{2}$, $\frac{d^2A}{dx^2} = -\frac{4\sqrt{2}\sqrt{2} + 2\cancel{\sqrt{2}}/\cancel{\sqrt{2}}}{2} - \frac{2\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} = -\frac{8+2}{2} - 2$

$\frac{d^2A}{dx^2} < 0$ at $x = \sqrt{2}$

A is max at $x = \sqrt{2}$

$\therefore$ Max area is $A(\sqrt{2}) = 2\sqrt{2}\sqrt{4-2} = 4$ sq. units.

dimensions are $2x = 2\sqrt{2}$ & $\sqrt{4-x^2} = \sqrt{2}$

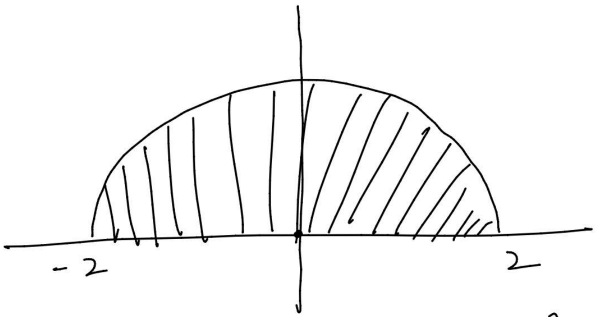
06 October 2020 11:30

Application of Single integral:

Def: If $f(x)$ is non-negative & integrable on $[a,b]$, then the area under the curve $y=f(x)$ in $[a,b]$ is called as definite integral $\int_a^b f(x)\,dx$.

→ Average value of f on $[a,b]$ is called as mean & is defined as $av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$.

→ find the average value of $f(x) = \sqrt{4-x^2}$ on $[-2,2]$



$y = \sqrt{4-x^2}$
$\Rightarrow x^2+y^2=4$

$$A = \int_{-2}^{2}\sqrt{4-x^2}\,dx = \frac{1}{2}\cdot\pi(2)^2 = 2\pi$$

$$Av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2+2}\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

$$= \frac{1}{4}\times 2\pi = \frac{\pi}{2}$$

→ Mean Value theorem for definite integrals:

If f is continuous on $[a,b]$ then at some point $c\in(a,b)$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

→ Fundamental theorem for integral calculus:

If f is cont on $[a,b]$ & $F(x) = \int_a^x f(t)\,dt$ & diff on (a,b)

then its derivative is $f(x)$

then its derivative is $f(x)$

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Use fundamental theorem of integral calculus find $\frac{dy}{dx}$.

1) $y = \int_a^x (t^3+1)\,dt$

$f(t) = t^3+1$, $f(x) = x^3+1$

$\frac{dy}{dx} = x^3+1$ by Fundamental theorem of calculus

2) $y = \int_x^5 3t\sin t\,dt$

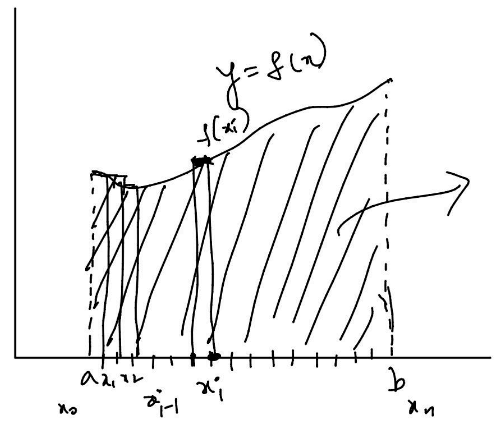
$= -\int_5^x 3t\sin t\,dt = \int_a^x f(t)\,dt\,dt$

$f(t) = -3t\sin t$, $f(x) = -3x\sin x$.

$\frac{dy}{dx} = f(x) = -3x\sin x$.

3) $y = \int_1^{x^2} \cos t\,dt$.

$\rightarrow$ $y = f(x)$ in $[a,b]$



$\int_a^b f(x)\,dx$

$\Delta x_i = x_i - x_{i-1}$

$[a,b]$ - n equal parts $x_0, x_1, \ldots\ldots$ x_n

$\Delta x_i = x_i - x_{i-1}$

$f(x_i)\,\Delta x_i$

Area $= f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_i)\Delta x_i + \cdots f(x_n)\Delta x_n$

$= \sum_{i=1}^{n} f(x_i)\Delta x_i$ $\quad$ n - no. of subdivisions of interval

As $n \to \infty$ such that $\Delta x_i \to 0$

$= \int_a^b f(x)\,dx$

Fundamental theorem of Calculus

If f is continuous at every point in $[a,b]$ & F is antiderivative of f on $[a,b]$ then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Net change theorem:

The net change in a function $F(x)$ over the interval $a \le x \le b$ is the integral of its rate of change.

$$F(b) - F(a) = \int_a^b F'(x)\,dx.$$

Ex: If an object with position $s(t)$, moves along a coordinate line, its velocity is $v(t) = s'(t)$

$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

The velocity of an object at time 't' during its motion was given by $v(t) = 160 - 32t$, find the displacement of the object during the time period $0 \le t \le 8$.

$\int_0^8 v(t)\,dt = 256 =$ displacement during $0 \le t \le 8$.

→ If f is continuous on a symmetric interval $[-a, a]$ &

a) f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

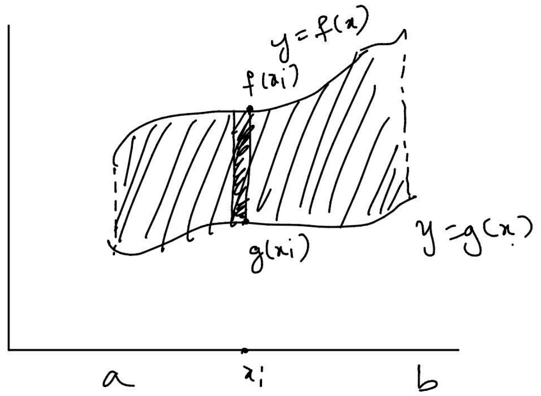
a) f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

b) f is odd then $\int_{-a}^{a} f(x)\,dx = 0$.

→ Area between the curves:

If f & g are two continuous functions with $f(x) \geq g(x)$ on $[a,b]$ then the area of the region between the curves $y = f(x)$ & $y = g(x)$

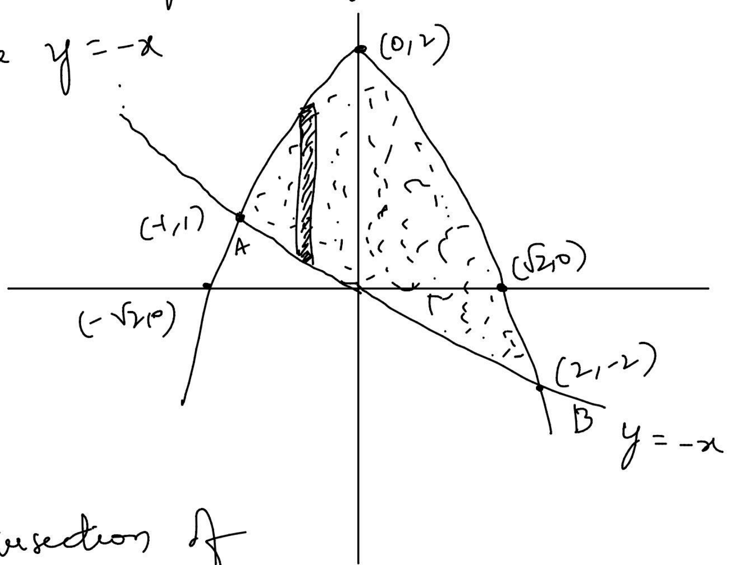
is $A = \int_{a}^{b} [f(x) - g(x)]\,dx$



$A = [f(x_i) - g(x_i)]\,\Delta x_i$

$$A = \int_{a}^{b} [f(x) - g(x)]\,dx$$

→ find the area of the region enclosed by the parabola $y = 2 - x^2$ & the line $y = -x$



$x = 0$ in $y = 2 - x^2$

$y = 2$

$(0,2)$

$y = 0$ in $y = 2 - x^2$

$x^2 = 2$

$x = \pm\sqrt{2}$

$(\sqrt{2},0)$ $(-\sqrt{2},0)$

points of intersection of

$y = 2 - x^2$ & $y = -x$

$-x = 2 - x^2 \Rightarrow x^2 - x - 2 = 0$

$\Rightarrow x = -1, 2$

$x = -1,\ y = 1,\ x = 2,\ y = -2$

$(-1,1) \quad (2,-2)$

$$A = \int_{x=-1}^{2} [f(x) - g(x)]\,dx = \int_{x=-1}^{2} [2 - x^2 - (-x)]\,dx$$

$$= \int_{-1}^{2} (2 - x^2 + x)\,dx = \left(2x - \frac{x^3}{3} + \frac{x^2}{2}\right)_{-1}^{2}$$

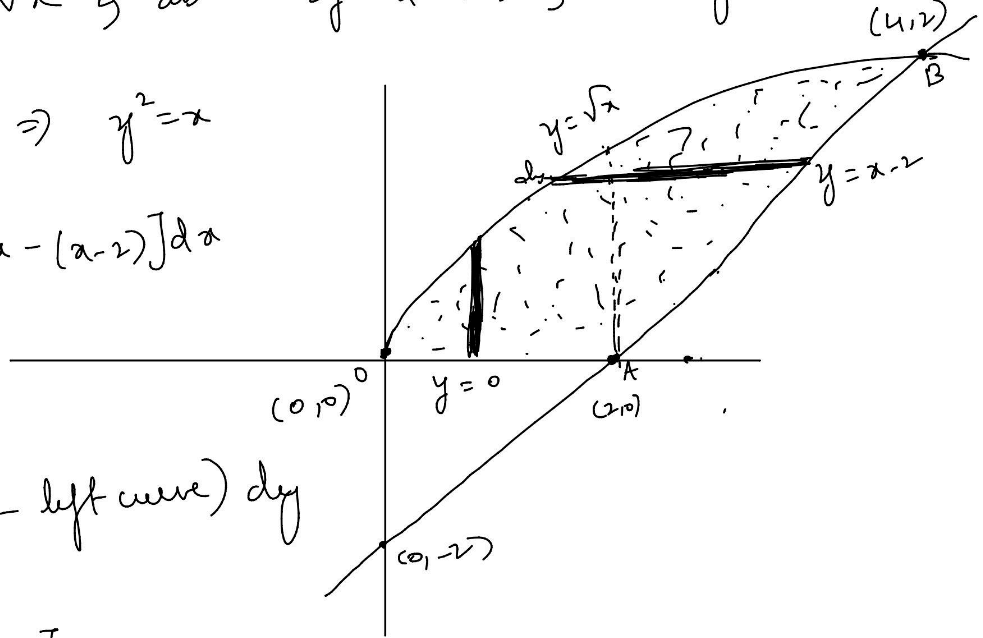
$A = 4.5$ sq units.

→ find the area of the region in the first quadrant bounded below by $y = \sqrt{x}$ & above by x-axis & line $y = x - 2$

Sol: $y = \sqrt{x} \Rightarrow y^2 = x$

$$A = \int_{x=0}^{2} (\sqrt{x} - 0)\,dx + \int_{x=2}^{4} [\sqrt{x} - (x-2)]\,dx$$

$$= \frac{10}{3}$$



$$A = \int_{y=0}^{2} [(\text{Right curve}) - \text{left curve})\,dy$$

$$A = \int_{y=0}^{2} [(y+2) - y^2]\,dy = \frac{10}{3}$$

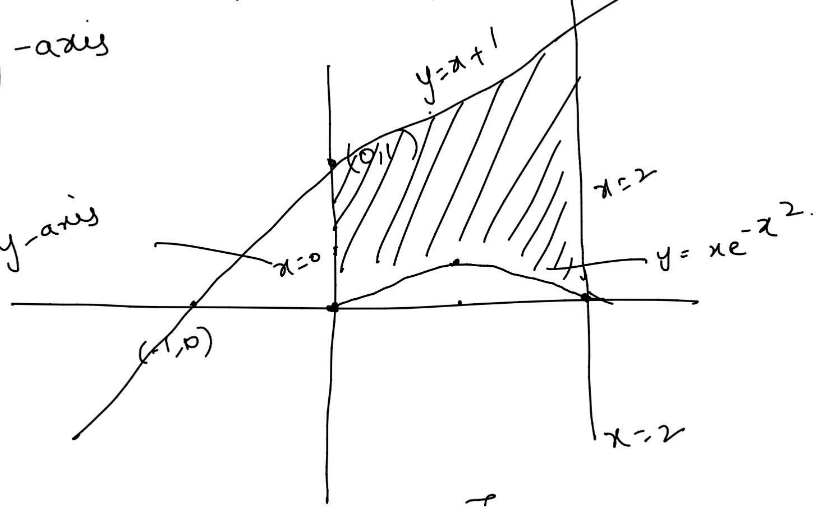
→ Determine the area of the region bounded by $y = xe^{-x^2}$, $y = x+1$, $x = 2$ & y-axis

$x = 0,\ y = 0$

$x = 1,\ y = \frac{1}{e}$

$x = 2,\ y = \frac{2}{e^4}$

Sol:



$x=2$

$$A = \int_{x=0}^{2} \left[(\text{upper curve}) - (\text{lower curve})\right] dx$$

$$A = \int_0^2 \left[(x+1) - x e^{-x^2}\right] dx$$

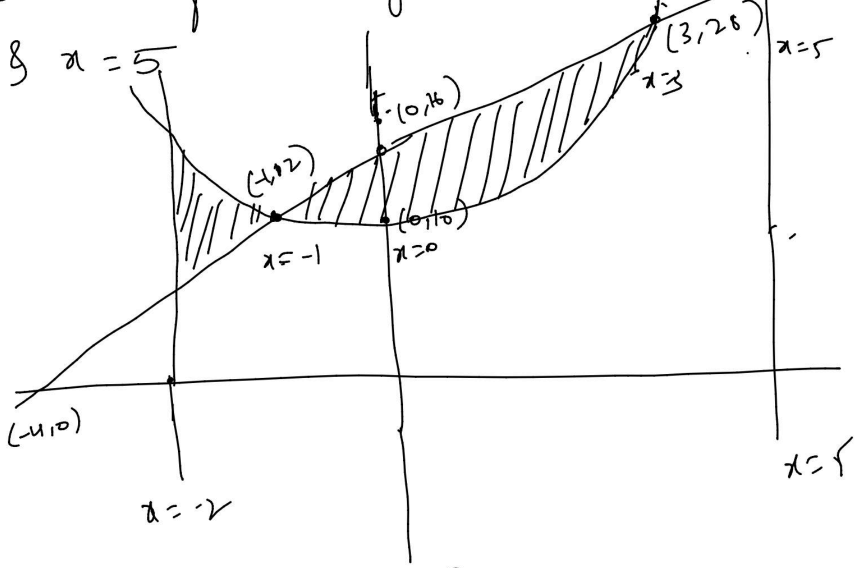
$$A = \left(\frac{x^2}{2} + x\right)_0^2 + \frac{1}{2}\int_0^2 (-2x)\, e^{-x^2}\, dx$$

$$= \left(\frac{4}{2} + 2\right) + \frac{1}{2}\left(e^{-x^2}\right)_0^2$$

$$= 4 + \frac{1}{2}\left(e^{-4} - 1\right) = \frac{7}{2} + \frac{1}{2}e^{-4} \text{ sq. units.}$$

→ Find the area of the region bounded by $y = 2x^2+10$, $y = 4x+16$, $x = -2$ & $x = 5$

Sol:



$$A = \int_{-2}^{-1} (\text{upper} - \text{lower})\, dx + \int_{-1}^{3} (\text{upper} - \text{lower})\, dx + \int_{3}^{5} (\text{upper} - \text{lower})\, dx$$

$$A = \int_{-2}^{-1} \left[(2x^2+10) - (4x+16)\right] dx + \int_{-1}^{3} \left[(4x+16) - (2x^2+10)\right] dx + \int_{3}^{5} \left[2x^2+10 - (4x+16)\right] dx$$

$$A = \frac{142}{3} \text{ sq. units}$$

→ Determine the area of region enclosed by $y = \sin x$, $y = \cos x$ & $x = \frac{\pi}{2}$, y-axis

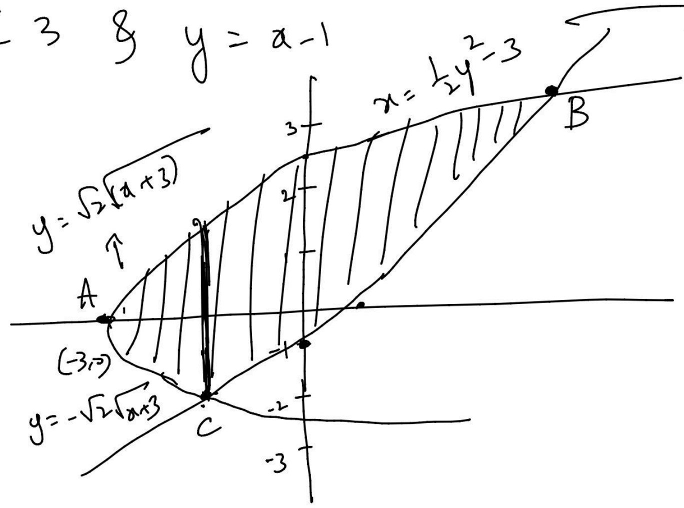
→ $x = \frac{1}{2}y^2 - 3$ & $y = x - 1$. 2, 3 → $y^2 = 2(x+3)$

$\rightarrow \quad x = \frac{1}{2}y^2 - 3 \quad \& \quad y = x - 1$

$y^2 = 2(x+3)$

$y = \pm\sqrt{2}\sqrt{x+3}$

Sol:



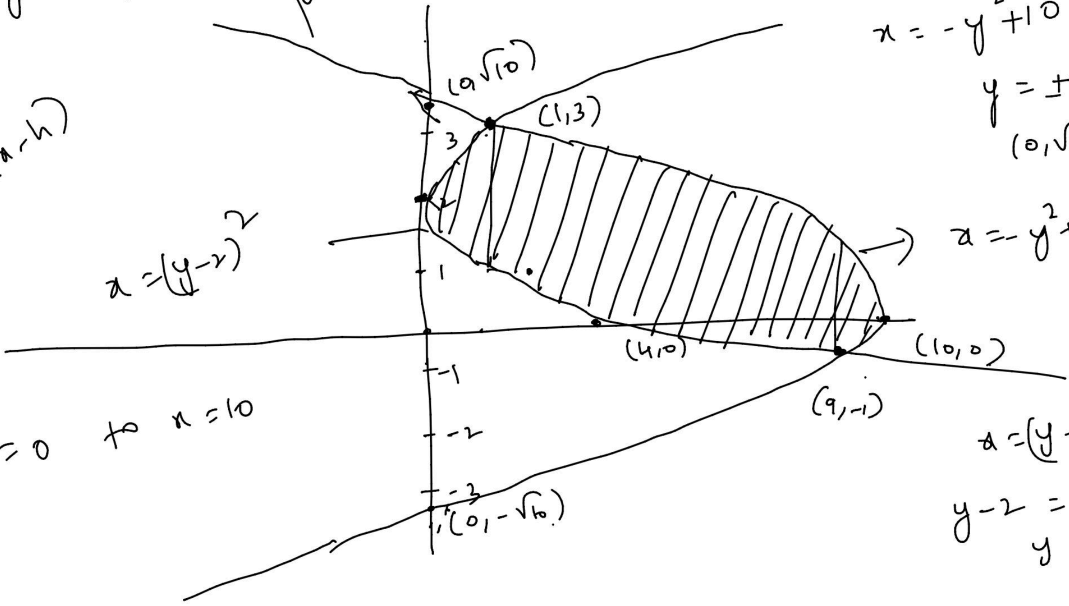
$A = 18$ sq units.

$\rightarrow \quad x = -y^2 + 10, \quad x = (y-2)^2$

$y^2 = 10 - x$

$(y-k)^2 = 4a(x-h)$

$y^2 = 4ax$



$x = -y^2 + 10$

$y = \pm\sqrt{10}$

$(0,\sqrt{10}) \quad (0,-\sqrt{10})$

$x = -y^2 + 10$

$x = 0$ to $x = 10$

$x = (y-2)^2$

$y - 2 = 0$

$y = 2$

points of intersection of parabolas are

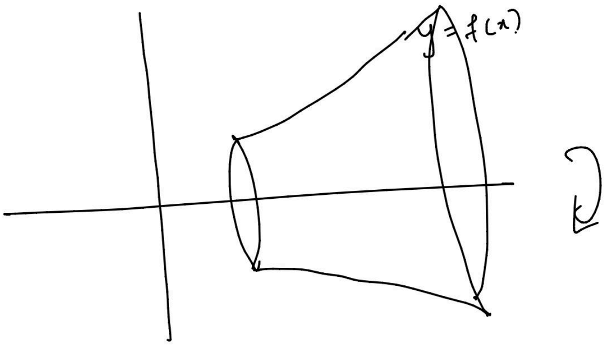
$(9,-1) \quad (1,3)$

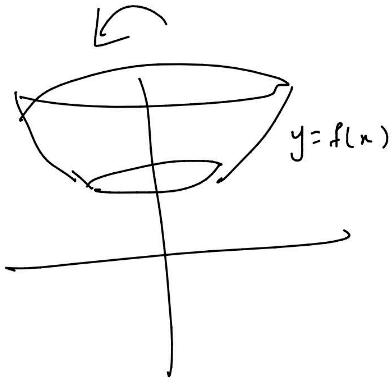
$$A = \int_{y=-1}^{3} (\text{right curve} - \text{left curve})\, dy$$

$$\int_{-1}^{3} \left[(-y^2+10) - (y-2)^2\right] dy = \frac{64}{3}$$

Volume of Solid of Revolution

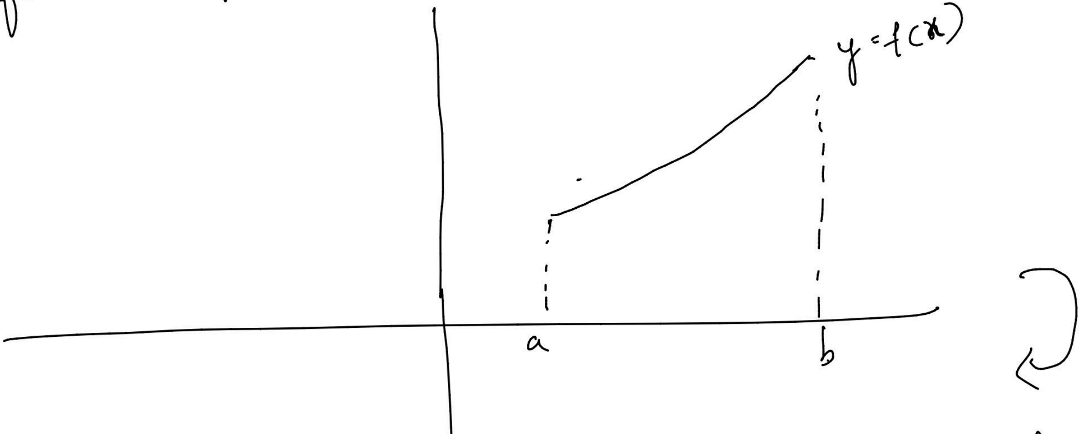
14 October 2020 07:18



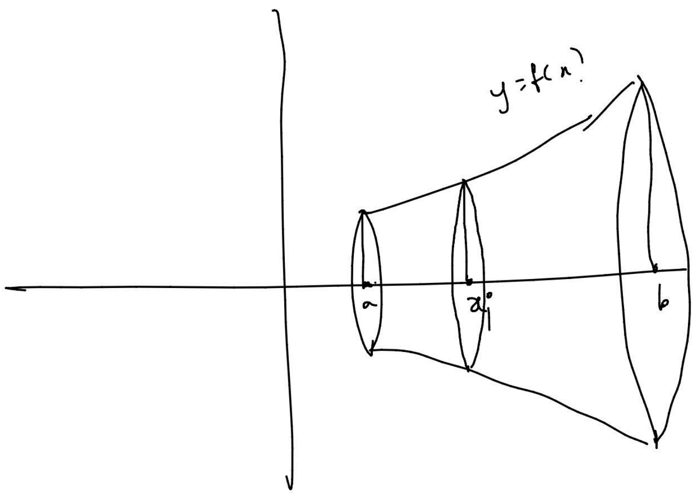


Solid of Revolution: The Solid formed by revolving a curve or a region about an axis is called as Solid of revolution. & The axis about which we revolve is called axis of revolution.
The volume of the solid formed after revolution of $y = f(x)$ about any axis is called volume of solid of revolution.

Volume of Solid of revolution: (Method of rings)



If the revolution about x-axis

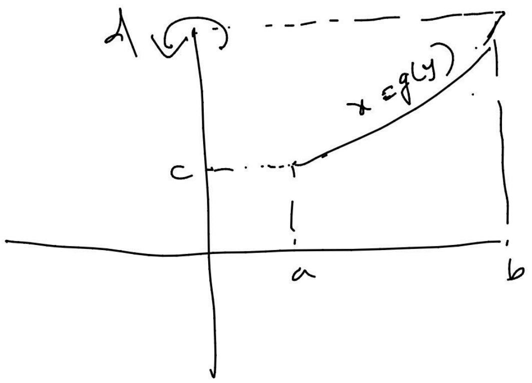


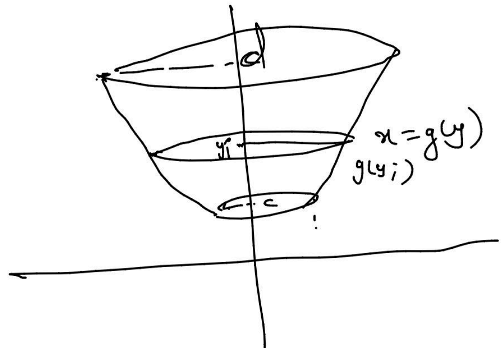
Area of ring $= \pi r^2$

$= \pi [f(x_i)]^2$

$$V = \int_a^b \pi [f(x_i)]^2 \, dx$$

If the revolution about y-axis



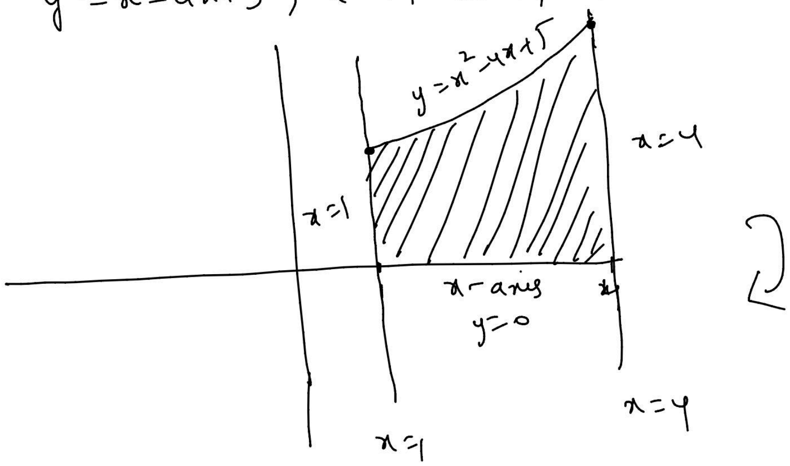


Area of ring $= \pi [g(y_i)]^2$

Volume $= \int_c^d \pi [g(y_i)]^2 \, dy$

→ find the volume of the solid obtained by rotating the region bounded by $y = x^2 - 4x + 5$, $x = 1$, $x = 4$, x-axis about x-axis

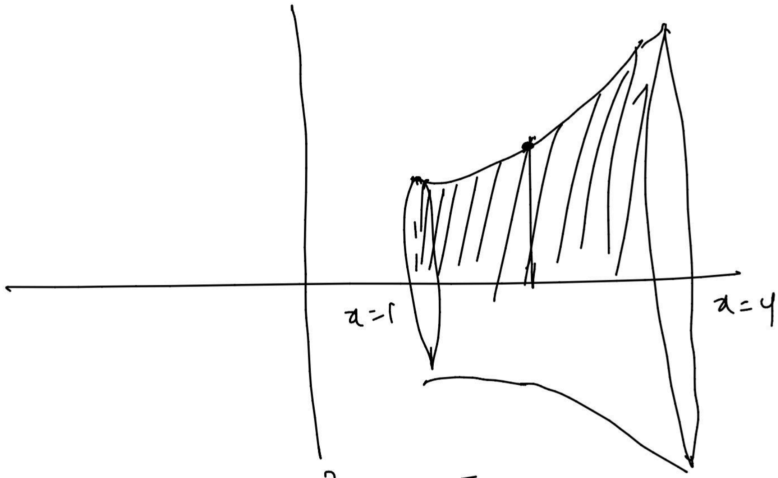
sol:



$y = x^2 - 4x + 5$

point of intersection of $y = x^2 - 4x + 5$, $x = 1$ ⇒ $y = 2 \rightarrow A(1, 2)$

" " " " , $x = 4$ ⇒ $y = 5 \rightarrow B(4, 5)$



Radius of ring $= x^2-4x+5$

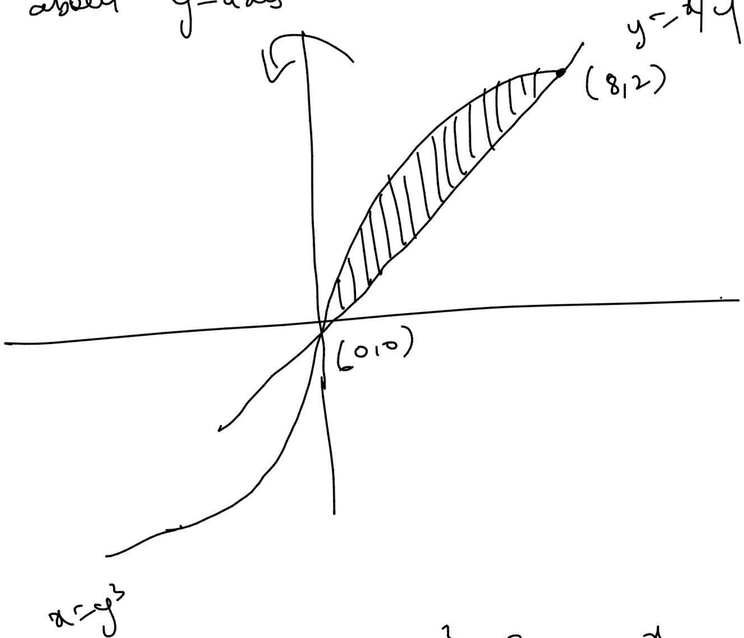
$$V = \int_{x=1}^{4} \pi\left[x^2-4x+5\right]^2 dx$$

$$V = \pi\int_{x=1}^{4}\left(x^4+16x^2+25-8x^3-40x+10x^2\right)dx$$

$$V = \frac{78\pi}{5} \text{ cubic units.}$$

→ Determine the volume of the solid obtained by rotating the region bounded by $y=\sqrt[3]{x}$ & $y=\frac{x}{4}$ that lies in the first quadrant about y-axis.

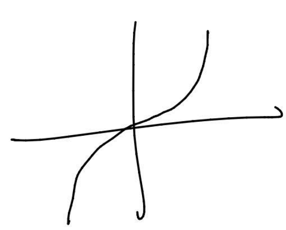
Sol:

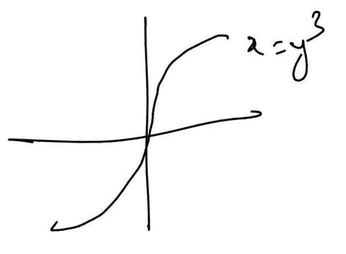


$y = \sqrt[3]{x}$

$y^3 = x$

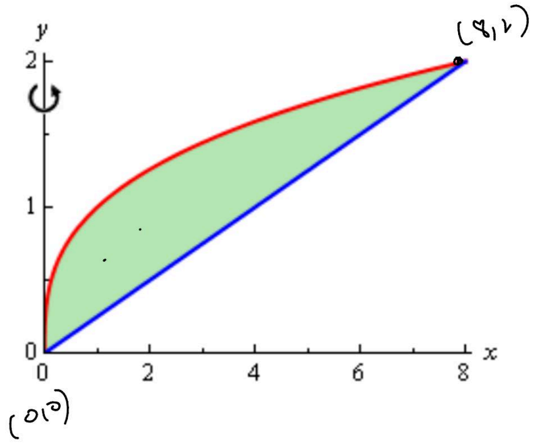
$y = x^3$

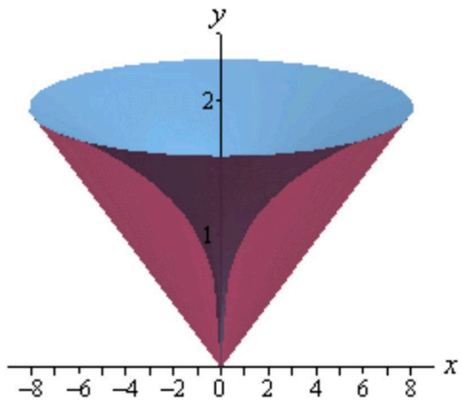


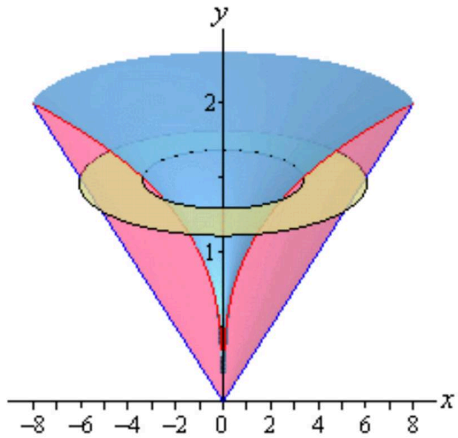


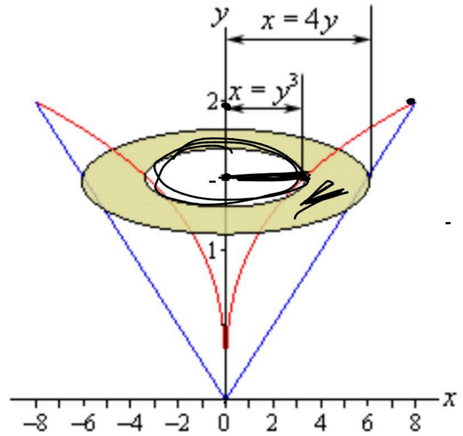
point of intersection of $x=y^3$ & $y=\frac{x}{4}$

O(0,0) (8,2)









inner radius $= y^3$

outer radius $= 4y$

$$A = \pi[\text{outer radius}]^2 - \pi[\text{inner radius}]^2$$

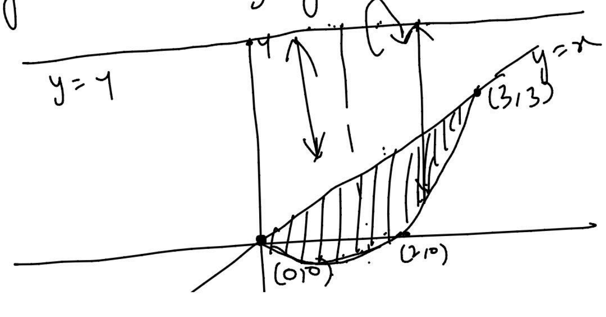
$$V = \int_{y=0}^{2} A\,dy = \int_0^2 \left\{\pi\left[(4y)^2\right] - \pi\left[(y^3)^2\right]\right\} dy$$

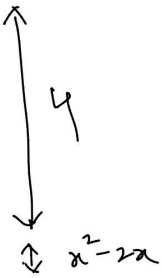
$$V = \pi \int_0^2 (16y^2 - y^6)\,dy$$

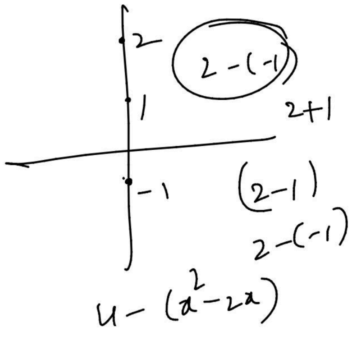
$$V = \frac{512}{21}\pi \text{ cubic units.}$$

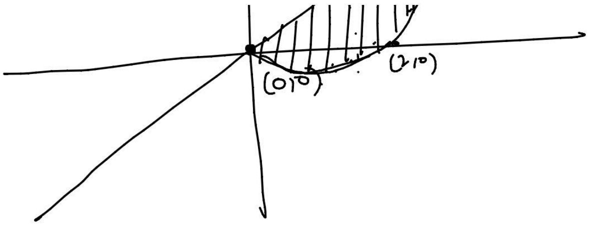
→ 3) $y = x^2 - 2x$ & $y = x$ about $y = 4$

sol:









x^2-2x

$2-(-1)$

$4-(x^2-2x)$

point of intersection of $y=x$ & $y=x^2-2x$ are $(0,0)$ & $(3,3)$

In $y=x^2-2x$, $x=0 \Rightarrow y=0$ $(0,0)$

, $y=0 \Rightarrow x=0,2$ $(2,0)$

Inner radius $= 4-x$

outer radius $= 4-(x^2-2x) = 4-x^2+2x$

$A = \pi\left[(\text{outer rad})^2 - (\text{Inner rad})^2\right]$

$V = \int_{x=0}^{3} \pi\left[(4-x^2+2x)^2 - (4-x)^2\right]dx. = \frac{153\pi}{5}$ cubic units

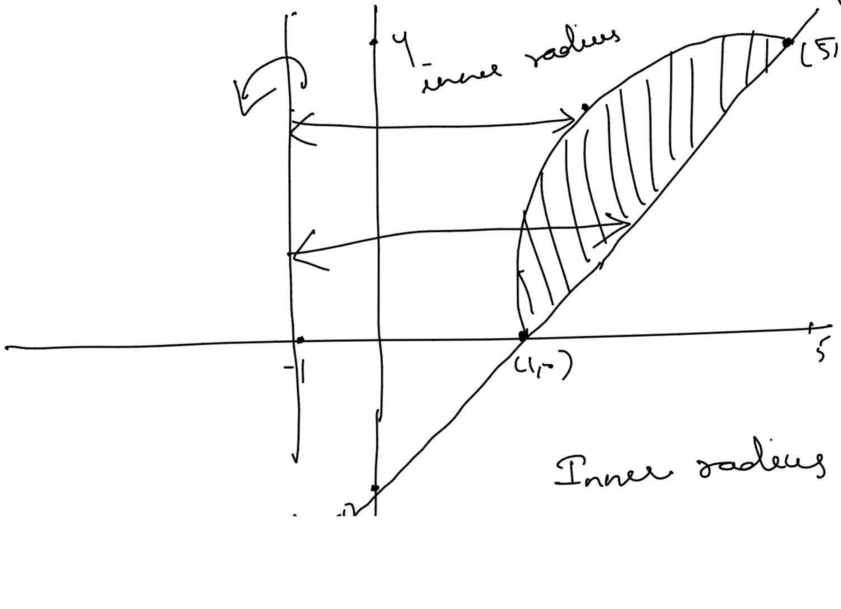
→ $y=2\sqrt{x-1}$, $y=x-1$ about $x=-1$

Sol: point of intersection of $y=x-1$ & $y=2\sqrt{x-1}$

$x-1=2\sqrt{x-1} \Rightarrow (x-1)^2=4(x-1)$

$(1,0)$ $(5,4)$

$\Rightarrow x=y+1$



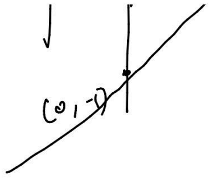
$y=2\sqrt{x-1}$

$\frac{y}{2}=\sqrt{x-1}$

$\left(\frac{y}{2}\right)^2=x-1$

$x=\frac{y^2}{4}+1$

Inner radius $= -1-\left(\frac{y^2}{4}+1\right)$



Inner radius $= -1 - \left(\frac{y^2}{4} + 1\right)$

Outer radius $= -1 - (y+1)$