① JACOBIANS

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

1) If $x = r\cos\theta$, $y = r\sin\theta$,

✓ find (i) $\frac{\partial(x,y)}{\partial(r,\theta)}$ ✓, (ii) $\frac{\partial(r,\theta)}{\partial(x,y)}$ ✓

$J \cdot J^{1} = 1$

Properties

1) $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$

2) $\frac{\partial(u,v)}{\partial(r,s)}$ u+v fun of x+y, x+y are funs of r+s

$= \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)}$

$u = f(x,y)$

$v = f(x,y)$

$x = f(r,s)$

$y = f(r,s)$

Sol:-

Given $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\int_{y=a}^{y=b}\int_{x=f_1(y)}^{x=f_2(y)} f(x,y)\,dx\,dy$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$= r$

$$\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=a} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

(ii) $\frac{\partial(r,\theta)}{\partial(x,y)} = ?$

WKT $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

ie $r \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$$

$u = f(x,y)$, $v = f(x,y)$ $y = f(r,\theta)$

2) If $u = 2xy$, $v = x^2 - y^2$ and $x = r\cos\theta$, $x = f(r,\theta)$

$y = r\sin\theta$. Evaluate $\frac{\partial(u,v)}{\partial(r,\theta)}$ with out actual substitution

Sol:- Given $u = 2xy$, $v = x^2 - y^2$ & $x = r\cos\theta$, $y = r\sin\theta$

$\frac{\partial u}{\partial x} = 2y$ | $\frac{\partial v}{\partial x} = 2x$ | $\frac{\partial x}{\partial r} = \cos\theta$

$\frac{\partial u}{\partial y} = 2x$ | $\frac{\partial v}{\partial y} = -2y$ | $\frac{\partial x}{\partial \theta} = -r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial y}{\partial \theta} = r\cos\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (-4y^2 - 4x^2)(r\cos^2\theta + r\sin^2\theta)$$

$$= -4(x^2+y^2)\, r(1)$$

$$\therefore \frac{\partial(u,v)}{\partial(r,\theta)} = -4r^3$$

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1 If $u = \frac{x+y}{x-y}$, & $v = \tan^{-1}x + \tan^{-1}y$ find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ (2)

Sol:-

Given $u = \frac{x+y}{x-y}$ & $v = \tan^{-1}x + \tan^{-1}y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial x} = \frac{(x-y)(1) - (x+y)(1)}{(x-y)^2} = \frac{\cancel{x} - y - \cancel{x} - y}{(x-y)^2}$$

$$= \frac{-2y}{(x-y)^2} \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial y} = \frac{(x-y)(1) - (x+y)(-1)}{(x-y)^2} = \frac{x - \cancel{y} + x + \cancel{y}}{(x-y)^2}$$

$$= \frac{2x}{(x-y)^2}$$

$$v = \tan^{-1}x + \tan^{-1}y$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

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$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{-2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{-2y}{(1+y^2)(x-y)^2} - \frac{2x}{(1+x^2)(x-y)^2}$$

$$= \frac{-2}{(x-y)^2}\left[\frac{y}{(1+y^2)} + \frac{x}{(1+x^2)}\right]$$

$$= -\frac{2}{(x-y)^2}\left[\frac{y(1+x^2)+x(1+y^2)}{(1+x^2)(1+y^2)}\right]$$

$$= \frac{-2}{(x-y)^2}\left[\frac{y+x^2y+x+xy^2}{(1+x^2)(1+y^2)}\right]$$

$$= \frac{-2}{(x-y)^2}\left[\frac{(x+y)+xy(x+y)}{(1+x^2)(1+y^2)}\right]$$

$$= \frac{-2}{(x-y)^2}\left[\frac{(x+y)(1+xy)}{(1+x^2)(1+y^2)}\right]$$

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4. Find the Jacobian $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ of the transformation ③

$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

Sol:

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$x = r\sin\theta\cos\phi$

$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$

$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$

$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$

$y = r\sin\theta\sin\phi$

$\frac{\partial y}{\partial r} = \sin\theta\sin\phi$

$\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi.$

$\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$

$z = r\cos\theta$

$\frac{\partial z}{\partial r} = \cos\theta$

$\frac{\partial z}{\partial \theta} = -r\sin\theta$

$\frac{\partial z}{\partial \phi} = 0$

$$\therefore J = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\left[r^2\cos\theta\sin\theta\cos^2\phi + r^2\cos\theta\sin\theta\sin^2\phi\right]$$
$$+ r\sin\theta\left[r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right]$$
$$= r^2\cos^2\theta\sin\theta\cos^2\phi + r^2\cos^2\theta\sin\theta\sin^2\phi$$
$$+ r\sin\theta\left[r\sin^2\theta(\cos^2\phi + \sin^2\phi)\right]$$

$$= r^2\sin\theta\left[\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi\right]$$
$$+ r^2\sin\theta\left[\sin^2\theta(1)\right]$$

$$= r^2\sin\theta\left[\cos^2\theta(\cos^2\phi+\sin^2\phi)\right] + r^2\sin\theta\left[\sin^2\theta\right]$$

$$= r^2\sin\theta\left[\cos^2\theta+\sin^2\theta\right] = r^2\sin\theta$$

HW

1) If $u = \frac{y^2}{x}$, $v = \frac{x^2}{y}$, find $\frac{\partial(u,v)}{\partial(x,y)}$.

2) If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$ show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$.

CW

5) If $x+y+z = u$, $y+z = uv$, $z = uvw$, prove
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$$

Sol:- Given
$x+y+z = u$, $y+z = uv$ & $z = uvw$

From $x+y+z = u$
$$x + uv = u$$
$$\boxed{x = u - uv = u(1-v)}$$

From $y+z = uv$
$$y + uvw = uv$$
$$y = uv - uvw$$
$$\boxed{y = uv(1-w)}$$

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$$\boxed{z = uvw}$$

(4)

$$\frac{\partial x}{\partial u} = 1 - v \qquad \frac{\partial y}{\partial u} = v(1-w) \qquad \frac{\partial z}{\partial u} = vw$$

$$\frac{\partial x}{\partial v} = -u \qquad \frac{\partial y}{\partial v} = u(1-w) \qquad \frac{\partial z}{\partial v} = uw$$

$$\frac{\partial x}{\partial w} = 0 \qquad \frac{\partial y}{\partial w} = -uv \qquad \frac{\partial z}{\partial w} = uv$$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= u^2 v \begin{vmatrix} 1-v & -1 & 0 \\ v(1-w) & 1-w & -1 \\ vw & w & 1 \end{vmatrix}$$

$$R_2' \Rightarrow R_2 + R_3$$

$$= u^2 v \begin{vmatrix} 1-v & -1 & 0 \\ v & 1 & 0 \\ vw & w & 1 \end{vmatrix}$$

$$= u^2 v(1 - \cancel{v} + \cancel{v}) = u^2 v$$