Module : 3

Multivariable Calculus

Functions of two variables

02 November 2020 09:48

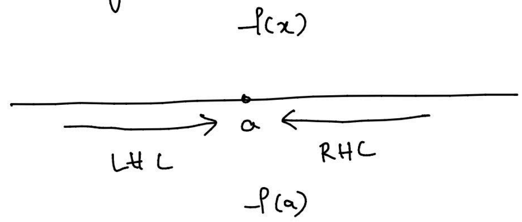
$y = f(x)$

$z = f(x,y) \rightarrow$ function of two variables

z ↓ dependent variable

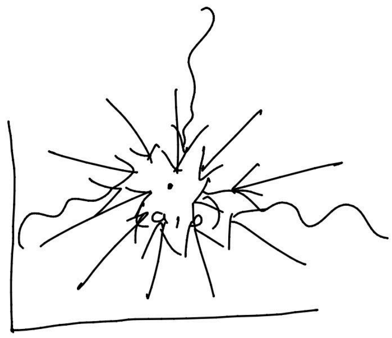
x, y ↓ independent variables

$z = f(x,y) \rightarrow$ represents a surface.



→ Limit :-

$f(x,y)$

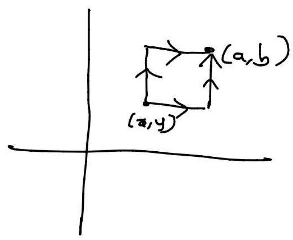


→ The function $f(x,y)$ is said to have a limit 'l' at the point (a,b) as $x \to a$ & $y \to b$ in any direction, $f(x,y)$ should approach a unique value 'l'.

$$\lim_{(x,y)\to(a,b)} f(x,y) = l$$

→ finding limit of $f(x,y)$ at $(x,y) = (a,b)$, $(a,b) \neq (0,0)$

1) $$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{\substack{x\to a \\ y \to b}} f(x,y)$$



If $$\lim_{x\to a}\left[\lim_{y\to b} f(x,y)\right] = \lim_{y\to b}\left[\lim_{x\to a} f(x,y)\right]$$

then the limit of $f(x,y)$ at (a,b) Exists

2) finding the limit of $f(x,y)$ at $(x,y) = (0,0)$

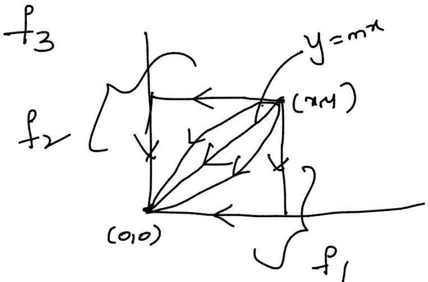
1) If $$\underbrace{\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right]}_{f_1} = \underbrace{\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right]}_{f_2}$$

If $f_1 = f_2$

f_1 $\qquad\qquad$ f_2

2) put $y = mx$, As $x \to 0$, $y \to 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x, mx) = f_3$$



If $f_1 = f_2 \neq f_3$

the limit do not exits

If $f_1 = f_2 = f_3$. then go to next step.

4) put $y = mx^2$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x, mx^2) = f_4$$

If $f_1 = f_2 = f_3 \neq f_4$ then limit do not exists

If $f = f_2 = f_3 = f_4$ then we check for $y = mx^n$

If $f_1 = f_2 = f_3 = \cdots$ then limit exists.

① If $f(x,y) = \dfrac{x+y}{2x-y}$, find $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$

sol: $f_1 = \displaystyle\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = \lim_{x\to 0}\left[\lim_{y\to 0} \frac{x+y}{2x-y}\right] = \lim_{x\to 0}\left[\frac{\cancel{x}}{2\cancel{x}}\right]$

$\Rightarrow \quad f_1 = \dfrac{1}{2}$

$f_2 = \displaystyle\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = \lim_{y\to 0}\left[\lim_{x\to 0} \frac{x+y}{2x-y}\right] = \lim_{y\to 0}\left[\frac{y}{-y}\right] = -1$

$f_2 = -1$

$f_1 \neq f_2$

$\therefore$ limit do not exists.

2) find $\displaystyle\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 y}{x^4+y^2}$

sol $f_1 = \displaystyle\lim_{x\to 0}\left[\lim_{y\to 0} \frac{x^2 y}{x^4+y^2}\right] = 0$

$$f_2 = \lim_{y\to 0}\left[\lim_{x\to 0} \frac{x^2 y}{x^4+y^2}\right] = 0$$

$$f_1 = f_2$$

$$f_3 = \lim_{x\to 0} f(x, mx) = \lim_{x\to 0} \frac{x^2(mx)}{x^4+m^2x^2} = \lim_{x\to 0} \frac{mx}{x^2+m^2} = 0$$

$$f_1 = f_2 = f_3$$

$$f_4 = \lim_{x\to 0} f(x, mx^2) = \lim_{x\to 0} \frac{x^2(mx^2)}{x^4+m^2x^4} = \lim_{x\to 0} \frac{x^4 m}{x^4+m^2x^4} = \frac{m}{1+m^2}$$

which depends on 'm'

for different values of 'm' we get different limit values

Hence $f_1 = f_2 = f_3 \neq f_4$

$\therefore$ The limit donot Exists.

$\rightarrow \lim_{\substack{x\to 0\\ y\to 0}} x^3+y^3$

$\rightarrow \lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2-x^2}{y^2+x^2}$

$\rightarrow \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3-y^3}{x^2+y^2}$

$\rightarrow \lim_{\substack{x\to 1\\ y\to 2}} \frac{x^2+2y}{x+y}$

$$f_1 = \lim_{x\to 1}\left[\lim_{y\to 2} \frac{x^2+2y}{x+y}\right] = \lim_{x\to 1}\left[\frac{x^2+4}{x+2}\right] = \frac{5}{3}$$

$$f_2 = \lim_{y\to 2}\left[\lim_{x\to 1} \frac{x^2+2y}{x+y}\right] = \lim_{y\to 2}\left[\frac{1+2y}{1+y}\right] = \frac{5}{3}$$

$$f_1 = f_2$$

$\therefore$ The limit Exists at $(x,y) = (1,2)$.

$l_1 = l_2$

$\therefore$ The limit exists at $(x,y) = (1,2)$.

$\rightarrow$ $\lim\limits_{\substack{x \to \infty \\ y \to 3}} \dfrac{2x-3}{x^3+4y^3}$

$f(x)$ $x=a$

If $\lim\limits_{x \to a} f(x)$ exists

$f(a)$ exists

$\lim\limits_{x \to a} f(x) = f(a)$

$\rightarrow$ Continuity:

The function $f(x,y)$ is said to be continuous at point (a,b) if $\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y)$ exists & $f(a,b)$ exists & $\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y) = f(a,b)$.

$\rightarrow$ Test the function $f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2}, & x \neq 0, y \neq 0 \\ 0, & x=0, y=0 \end{cases}$ is continuous at $(0,0)$ or not.

Sol:

$(a,b) = (0,0)$

$f(a,b) = f(0,0) = 0$

$f(x,y) = \dfrac{x^3-y^3}{x^2+y^2}$

$f_1 = \lim\limits_{x \to 0} \left[\lim\limits_{y \to 0} \dfrac{x^3-y^3}{x^2+y^2} \right] = \lim\limits_{x \to 0} [x] = 0$

$f_2 = \lim\limits_{y \to 0} \left[\lim\limits_{x \to 0} \dfrac{x^3-y^3}{x^2+y^2} \right] = \lim\limits_{y \to 0} [-y] = 0$

$f_1 = f_2$

$f_3 = \lim\limits_{x \to 0} f(x, mx) = \lim\limits_{x \to 0} \left[\dfrac{x^3 - m^3x^3}{x^2+m^2x^2} \right] = \lim\limits_{x \to 0} \left[\dfrac{x(1-m^3)}{1+m^2} \right] = 0$

$f_1 = f_2 = f_3$

$f_4 = \lim\limits_{x \to 0} f(x, mx^2) = \lim\limits_{x \to 0} \left[\dfrac{x^3 - m^3x^6}{x^2+m^2x^4} \right] = \lim\limits_{x \to 0} \left[\dfrac{x - m^3x^4}{1+m^2x^2} \right] = 0$

$f_1 = f_2 = f_3 = f_4 = 0$

$|||^{ly}$ $f_1 = f_2 = f_3 = f_4 = f_5 = \cdots = 0$

$\|^{ly}$ $f_1 = f_2 = f_3 = f_4 = f_5 = \cdots = 0$

$\therefore \lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) = 0$

$f(0,0) = 0$

$\therefore \lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) = f(0,0) = 0$

$\therefore$ $f(x,y)$ is continuous at $(0,0)$.

2) Discuss the continuity of $f(x,y) = \begin{cases} \dfrac{x}{\sqrt{x^2+y^2}}, & x \neq 0, y \neq 0 \\ 2, & x = 0, y = 0 \end{cases}$

at the origin.

Sol: $f(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$

$f(0,0) = 2$

$f_1 = \lim_{x \to 0} \left[\lim_{y \to 0} \dfrac{x}{\sqrt{x^2+y^2}} \right] = \lim_{x \to 0} 1 = 1$

$f_2 = \lim_{y \to 0} \left[\lim_{x \to 0} \dfrac{x}{\sqrt{x^2+y^2}} \right] = 0$

$f_1 \neq f_2$

$\therefore$ The limit do not exists.

$\therefore$ The function is not continuous at $(0,0)$

$\rightarrow$ $f(x,y) = \begin{cases} \dfrac{x^2+xy+x+y}{x+y}, & (x,y) \neq (2,-2) \\ 4, & (x,y) = (2,-2) \end{cases}$ check continuity at $(2,-2)$

Sol: $f(2,-2) = 4$

$f_1 = \lim_{x \to 2} \left[\lim_{y \to -2} \dfrac{x^2+xy+x+y}{x+y} \right] = \lim_{x \to 2} \left[\lim_{y \to -2} \dfrac{x(x+y)+1(x+y)}{x+y} \right]$

$$\lim_{x\to 2}\left[\lim_{y\to -2} \frac{\ldots}{x+y}\right]$$

$$= \lim_{x\to 2}\left[\lim_{y\to -2} (x+1)\right] = 3$$

$$f_2 = \lim_{y\to -2}\left[\lim_{x\to 2} (x+1)\right] = 3$$

$$f_1 = f_2 = 3$$

$\therefore \lim_{\substack{x\to 2\\ y\to -2}} f(x,y)$ Exists

$$\lim_{\substack{x\to 2\\ y\to -2}} f(x,y) \neq f(2,-2)$$

$\therefore$ The function is not continuous.

→ $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & x\neq 0, y\neq 0 \\ 0, & x=0, y=0 \end{cases}$ at $(0,0)$

→ $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & x\neq 0, y\neq 0 \\ 0, & x=0, y=0 \end{cases}$ at $(0,0)$

→ $f(x,y) = \begin{cases} \dfrac{x^2+xy}{x+y^2}, & x\neq 0, y\neq 0 \\ 0 & x=0, y=0 \end{cases}$ at $(0,0)$

→ $f(x,y) = \begin{cases} 2x^2+y, & (x,y)\neq (1,2) \\ 0, & (x,y) = (1,2) \end{cases}$ at $(1,2)$.

→ **Partial differentiation:**

Let $z = f(x,y)$ be a function of two variables x & y.

If $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$ Exists then this limit is called

If $\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$ Exists then this limit is called as partial derivative of $f(x,y)$ w.r.to x & it is denoted by

$\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

IIly the partial derivate of $f(x,y)$ w.r.to 'y' is defined as

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

In general, if $z = f(x_1, x_2, \ldots x_n)$

Then the partial derivative of z w.r.to x_i

$$\frac{\partial z}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \ldots x_i + \Delta x_i, \ldots x_n) - f(x_1, x_2, \ldots x_n)}{\Delta x_i}$$

$i = 1, 2, \ldots n$

If $z = f(x,y)$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = p$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = q$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ only if $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are continuous.

→ If $U = \frac{1}{}$, $x^2+y^2+z^2 \neq 0$, then prove that

→ If $U = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$, $x^2+y^2+z^2 \neq 0$, then prove that

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

$\sqrt{x}$

Sol: $U = (x^2+y^2+z^2)^{-1/2}$

$$\frac{\partial U}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x+0+0) = -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial U}{\partial y} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(0+2y+0) = -y(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial U}{\partial z} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(0+0+2z) = -z(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial}{\partial x}\left[-x(x^2+y^2+z^2)^{-3/2}\right]$$

$$= -\left[-\frac{3}{\cancel{2}}x(x^2+y^2+z^2)^{-5/2}(\cancel{2}x) + (x^2+y^2+z^2)^{-3/2}\cdot 1\right]$$

$$- (x^2+y^2+z^2)^{-5/2}\left[-3x^2 + x^2+y^2+z^2\right]$$

$$\frac{\partial^2 U}{\partial x^2} = -(x^2+y^2+z^2)^{-5/2}\left[-2x^2+y^2+z^2\right]$$

Illy $$\frac{\partial^2 U}{\partial y^2} = -(x^2+y^2+z^2)^{-5/2}\left[-2y^2+x^2+z^2\right]$$

$$\frac{\partial^2 U}{\partial z^2} = -(x^2+y^2+z^2)^{-5/2}\left[-2z^2+x^2+y^2\right]$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = -(x^2+y^2+z^2)^{-5/2}\left[-2x^2+y^2+z^2-2y^2+x^2+z^2 -2z^2+x^2+y^2\right]$$

$$= 0.$$

→ If $U = \log(x^3+y^3+z^3-3xyz)$ then P.T

→ If $U = \log(x^3+y^3+z^3-3xyz)$ then P.T

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 U = \frac{-9}{(x+y+z)^2}.$$

Sol: $U = \log(x^3+y^3+z^3-3xyz)$

$$\frac{\partial U}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$$

$$\frac{\partial U}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz}(3y^2-3xz)$$

$$\frac{\partial U}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}(3z^2-3xy)$$

$$\frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z} = \frac{3x^2-3yz+3y^2-3xz+3z^2-3xy}{x^3+y^3+z^3-3xyz} \quad — (1)$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 U = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)U$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left[\frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}\right]$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left[\frac{3\cancel{(x^2+y^2+z^2-xy-yz-zx)}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-zx)}}\right]$$

$$= \left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right]\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= 3\cdot\left(-\frac{1}{(x+y+z)^2}\right)+3\left(\frac{-1}{(x+y+z)^2}\right)+3\left(-\frac{1}{(x+y+z)^2}\right)$$

$$= \frac{-9}{(x+y+z)^2}$$

→ If $x^x y^y z^z = e$. Then show that at $x=y=z$ $\frac{\partial^2 z}{\partial x \partial y} = -(x\log xe)^{-1}$

Sol

$x^x y^y z^z = e$

$\log(x^x y^y z^z) = \log e$

$\log x^x + \log y^y + \log z^z = 1$

$x\log x + y\log y + z\log z = 1$ —— (1)

Diff (1) partially w.r.to 'x'.

$x \cdot \frac{1}{x} + \log x \cdot 1 + 0 + z \cdot \frac{1}{z}\frac{\partial z}{\partial x} + \log z \frac{\partial z}{\partial x} = 0$

$(1+\log x) = -\frac{\partial z}{\partial x}(1+\log z)$

$\frac{\partial z}{\partial x} = -\frac{(1+\log x)}{1+\log z}$ —— (a)

Diff (1) partially w.r.to 'y'.

$0 + y\cdot\frac{1}{y} + \log y \cdot 1 + z\cdot\frac{1}{z}\frac{\partial z}{\partial y} + \log z \frac{\partial z}{\partial y} = 0$

$\frac{\partial z}{\partial y} = -\frac{(1+\log y)}{1+\log z}$ —— (b)

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] = \frac{\partial}{\partial x}\left[-\frac{(1+\log y)}{1+\log z}\right]$$

$$= -(1+\log y)\frac{\partial}{\partial x}\left[\frac{1}{1+\log z}\right]$$

$$= -(1+\log y)\left[\frac{-1}{(1+\log z)^2}\right]\frac{1}{z}\frac{\partial z}{\partial x}$$

$$= \frac{(1+\log y)}{(1+\log z)^2}\cdot\frac{1}{z}\cdot\left[-\frac{(1+\log x)}{(1+\log z)}\right]$$

At $x=y=z$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\cancel{(1+\log x)}}{\cancel{(1+\log x)^2}}\cdot\frac{1}{x}\frac{(1+\cancel{\log x})}{(1+\log x)}$$

-2, -1 = -1

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-1}{x(1+\log x)} = -\frac{1}{x(\log e + \log x)}$$

$$= -\frac{1}{x \log xe} = -(x \log xe)^{-1}$$

→ If $u = \log(x^2+y^2+z^2)$ ~~then~~ P.T $(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 2$.

→ Chain rule of partial differentiation

If $y = f(x)$, $x = g(t)$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

→ If $z = f(x,y)$ where $x = \phi(u,v)$, $y = \psi(u,v)$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

This is called as chain rule of partial differentiation.

→ If $z = f(x,y)$, $x = \phi(t)$, $y = \psi(t)$

Then $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \quad \text{--- (1)}$$

This $\frac{dz}{dt}$ is called as total differential coefficient.

(1) ⇒ $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$ → Total differential of 'z'.

→ If $u = f(y-z, z-x, x-y)$ Then find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$.

sol let $y - z = r$ ⇒ $r(y,z)$

Sol let $y-z=r \Rightarrow r(y,z)$

$z-x=s \Rightarrow s(x,z)$

$x-y=t \Rightarrow t(x,y)$

$u=f(r,s,t)$, $r=r(y,z)$, $s=s(x,z)$, $t=t(x,y)$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial x} \quad \text{(By chain rule)}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$r=y-z$, $s=z-x$, $t=x-y$

$\frac{\partial r}{\partial x}=0$, $\frac{\partial s}{\partial x}=-1$, $\frac{\partial t}{\partial x}=1$

$\frac{\partial r}{\partial y}=1$, $\frac{\partial s}{\partial y}=0$, $\frac{\partial t}{\partial y}=-1$

$\frac{\partial r}{\partial z}=-1$, $\frac{\partial s}{\partial z}=1$, $\frac{\partial t}{\partial z}=0$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}(0)+\frac{\partial u}{\partial s}(-1)+\frac{\partial u}{\partial t}(1)=-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}(1)+\frac{\partial u}{\partial s}(0)+\frac{\partial u}{\partial t}(-1)=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}(-1)+\frac{\partial u}{\partial s}(1)+\frac{\partial u}{\partial t}(0)=-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

→ If $u=f\left(e^{y-z}, e^{z-x}, e^{x-y}\right)$, then find $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$.

Sol: let $r=e^{y-z} \Rightarrow r=r(y,z)$

$s=e^{z-x} \Rightarrow s=s(z,x)$

$$t = e^{x-y} \Rightarrow t = t(x,y)$$

$$u = f(r,s,t)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$e^{y} e^{-z}$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$r = e^{y-z}$ $\quad$ $s = e^{z-x}$ $\quad$ $t = e^{x-y}$

$\frac{\partial r}{\partial x} = 0$ $\quad$ $\frac{\partial s}{\partial x} = -e^{z-x}$ $\quad$ $\frac{\partial t}{\partial x} = e^{x-y}$

$\frac{\partial r}{\partial y} = e^{y-z}$ $\quad$ $\frac{\partial s}{\partial y} = 0$ $\quad$ $\frac{\partial t}{\partial y} = -e^{x-y}$

$\frac{\partial r}{\partial z} = -e^{y-z}$ $\quad$ $\frac{\partial s}{\partial z} = e^{z-x}$ $\quad$ $\frac{\partial t}{\partial z} = 0$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(-e^{z-x}\right) + \frac{\partial u}{\partial t}\left(e^{x-y}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left(e^{y-z}\right) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}\left(-e^{x-y}\right)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\left(-e^{y-z}\right) + \frac{\partial u}{\partial s}\left(e^{z-x}\right) + \frac{\partial u}{\partial t}(0)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

1) If $u = f(x^2+2yz,\ y^2+2zx)$ then find $(y^2-zx)\frac{\partial u}{\partial x} + (x^2-yz)\frac{\partial u}{\partial y} + (z^2-xy)\frac{\partial u}{\partial z}$

Sol:

$$u = f(r,s)$$

$r = x^2+2yz,$ $\quad$ $s = y^2+2zx$

$\frac{\partial r}{\partial x} = 2x$ $\quad$ $\frac{\partial s}{\partial x} = 2z$

$\frac{\partial r}{\partial y} = 2z$ $\quad$ $\frac{\partial s}{\partial y} = 2y$

$\frac{\partial r}{\partial z} = 2y$ $\quad$ $\frac{\partial s}{\partial z} = 2x$

$$\frac{\partial r}{\partial z} = 2y \qquad \frac{\partial s}{\partial z} = 2x$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = 2x\frac{\partial u}{\partial r} + 2z\frac{\partial u}{\partial s}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = 2z\frac{\partial u}{\partial r} + 2y\frac{\partial u}{\partial s}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} = 2y\frac{\partial u}{\partial r} + 2x\frac{\partial u}{\partial s}$$

$$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - zy)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z}$$

$$(y^2 - zx)\left[2x\frac{\partial u}{\partial r} + 2z\frac{\partial u}{\partial s}\right] + (x^2 - zy)\left[2z\frac{\partial u}{\partial r} + 2y\frac{\partial u}{\partial s}\right] + (z^2 - xy)\left[2y\frac{\partial u}{\partial r} + 2x\frac{\partial u}{\partial s}\right]$$

$$\frac{\partial u}{\partial r}\left[\cancel{2xy^2} - \cancel{2x^2 z} + \cancel{2x^2 z} - \cancel{2z^2 y} + \cancel{2yz^2} - \cancel{2xy^2}\right] +$$
$$\frac{\partial u}{\partial s}\left[\cancel{2y^2 z} - \cancel{2z^2 x} + \cancel{2x^2 y} - \cancel{2y^2 z} + \cancel{2z^2 x} - \cancel{2x^2 y}\right] = 0$$

→ If $\phi(cx - az, cy - bz) = 0$, show that $ap + bq = c$ where
$p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

→ If $u = r^m$ and $r^2 = x^2 + y^2 + z^2$ then find $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

Sol:

$$r^2 = x^2 + y^2 + z^2 \quad - (1)$$

Diff (1) w.r.to 'x' partially

$$2r\frac{\partial r}{\partial x} = 2x + 0 + 0$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \quad - (a)$$

lly diff (1) w.r.to y & z partially we get

$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad - (b) \qquad , \qquad \frac{\partial r}{\partial z} = \frac{z}{r} \quad - (c)$$

$$u = r^m$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = m r^{m-1}\cdot\frac{x}{r} = m r^{m-2} x \quad \text{— (i)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} = m r^{m-1}\frac{y}{r} = m r^{m-2} y$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} = m r^{m-1}\frac{z}{r} = m r^{m-2} z.$$

$$\frac{\partial^2 u}{\partial x^2} = m\left[(m-2) r^{m-3}\frac{\partial r}{\partial x}\cdot x + r^{m-2}\cdot 1\right]$$

$$= m\left[(m-2) r^{m-3}\cdot\frac{x}{r} x + r^{m-2}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = m\left[(m-2) r^{m-4} x^2 + r^{m-2}\right]$$

lly $$\frac{\partial^2 u}{\partial y^2} = m\left[(m-2) r^{m-4} y^2 + r^{m-2}\right]$$

$$\frac{\partial^2 u}{\partial z^2} = m\left[(m-2) r^{m-4} z^2 + r^{m-2}\right]$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = m\left[(m-2) r^{m-4}(x^2+y^2+z^2) + 3 r^{m-2}\right]$$

$$= m\left[(m-2) r^{m-4} r^2 + 3 r^{m-2}\right]$$

$$= m\left[(m-2) r^{m-2} + 3 r^{m-2}\right]$$

$$= m r^{m-2}\left[m-2+3\right]$$

$$= m(m+1) r^{m-2}.$$

→ If $u = f(r,s,t)$, $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$

find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

→ If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$ then

P.T $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$

P.T $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$

→ If $x = r\cos\theta$, $y = r\sin\theta$, then show that $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$

& $\frac{1}{r}\frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x}$.

→ If $u = x \log xy$, where $x^3+y^3+3xy = 1$ find $\frac{du}{dx}$

Sol

$u = x\log xy$

$x^3+y^3+3xy = 1$

Diff w.r.to 'x'

$3x^2 + 3y^2\frac{dy}{dx} + 3\left[x\frac{dy}{dx} + y\right] = 0$

$3x^2+3y = (-3y^2-3x)\frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{(x^2+y)}{x+y^2}$

$u = f(x,y)$

$y = y(x)$

$u = f(x,y)$

$x = x(t)$

$y = y(t)$

$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$

$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$

$u = \underset{u}{x}\,\underset{v}{\log xy}$

$\frac{\partial u}{\partial x} = x \cdot \frac{1}{xy} y + \log xy \cdot 1 = 1 + \log xy$

$\frac{\partial u}{\partial y} = x \cdot \frac{1}{xy} x = \frac{x}{y}$

$u = f(x)$

$\frac{du}{dx} = 1 + \log xy + \left(\frac{x}{y}\right)\left[-\frac{(x^2+y)}{x+y^2}\right]$

→ If $z = u^2+v^2$ $u = at^2$, $v = 2at$, find $\frac{dz}{dt}$

$\rightarrow$ If $z = f(x,y)$ & $z = 0$ then $dz = 0$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$\frac{\partial z}{\partial x} dx = -\frac{\partial z}{\partial y} dy$$

$$\boxed{\frac{dy}{dx} = \frac{-\partial z/\partial x}{\partial z/\partial y}}$$

$\rightarrow$ If $x^y = y^x$ then find $\frac{dy}{dx}$.

Sol:

$$x^y = y^x$$

$$x^y - y^x = 0$$

$$f = x^y - y^x = 0 \Rightarrow f = 0 \text{ then } df = 0$$

f is function of x, & y

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$0 = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \rightarrow \frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$$

$$f = x^y - y^x$$

$$\frac{\partial f}{\partial x} = y x^{y-1} - y^x \log y \qquad x^y = y^x$$

$$\frac{\partial f}{\partial y} = x^y \log x - x y^{x-1}$$

$$\frac{dy}{dx} = \frac{-y x^{y-1} + y^x \log y}{x^y \log x - x y^{x-1}}$$

$$\frac{dy}{dx} = \frac{-y x^{y-1} + x^y \log y}{x^y \log x - x y^y} \quad (\because x^y = y^x)$$

$$\frac{dy}{dx} = \frac{-yx^{y-1} + x^y \log y}{x^y \log x - \frac{x\, x^y}{y}} \qquad (\; - x - y \;)$$

$$= \frac{\cancel{x^y}\left[-\frac{y}{x} + \log y\right]}{\cancel{x^y}\left[\log x - \frac{x}{y}\right]} = \frac{y}{x}\left(\frac{-y + x\log y}{y\log x - x}\right)$$

→ If $(\sin y)^x = (\cos x)^y$ then find $\frac{dy}{dx}$

→ Jacobian:

Let $u = u(x,y)$ and $v = v(x,y)$ are the relation b/w the old variables x, y and new variable u, v

Then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called as Jacobian of u, v with respect to x, y

& it is denoted by $J\left[\frac{u,v}{x,y}\right]$ or $\frac{\partial(u,v)}{\partial(x,y)}$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

If $u = u(x,y,z)$, $v = v(x,y,z)$, $w = w(x,y,z)$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

is called as Jacobian of u, v, w w.r.to x, y, z.

→ Properties:

1) $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$

1) $\dfrac{\partial(u,v)}{\partial(x,y)} \times \dfrac{\partial(x,y)}{\partial(u,v)} = 1$

If $J = \dfrac{\partial(u,v)}{\partial(x,y)}$, $J' = \dfrac{\partial(x,y)}{\partial(u,v)}$ thus $JJ' = 1$

2) If u, v are functions of r, s & r, s are functions of x, y.

thus $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{\partial(u,v)}{\partial(r,s)} \times \dfrac{\partial(r,s)}{\partial(x,y)}$

→ If $x = r\cos\theta$, $y = r\sin\theta$, find $\dfrac{\partial(x,y)}{\partial(r,\theta)}$ & $\dfrac{\partial(r,\theta)}{\partial(x,y)}$

Also show that $\dfrac{\partial(x,y)}{\partial(r,\theta)} \times \dfrac{\partial(r,\theta)}{\partial(x,y)} = 1$

Sol: $x = r\cos\theta$, $y = r\sin\theta$

$\dfrac{\partial(x,y)}{\partial(r,\theta)} = ?$ $\qquad \dfrac{\partial(r,\theta)}{\partial(x,y)} = ?$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r\cos^2\theta + r\sin^2\theta = r$$

$\dfrac{\partial(r,\theta)}{\partial(x,y)} = ?$

r & θ should be expressed interms of x & y.

$x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2+y^2}$ —— (a)

$\dfrac{y}{x} = \dfrac{r\sin\theta}{r\cos\theta} \Rightarrow \tan\theta = \dfrac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\dfrac{y}{x}\right)$ —— (b)

$\dfrac{\partial r}{\partial x} = \dfrac{1 \cdot 2x}{2\sqrt{x^2+y^2}} = \dfrac{x}{\sqrt{x^2+y^2}}$, $\dfrac{\partial r}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}, \quad \frac{\partial\theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\,\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}}$$

$$= \frac{\cancel{x^2+y^2}}{\cancel{(x^2+y^2)}\sqrt{x^2+y^2}} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = r\cdot\frac{1}{r} = 1.$$

→ If $u = \dfrac{x+y}{1-xy}$, $\theta = \tan^{-1}x + \tan^{-1}y$ find $\dfrac{\partial(u,\theta)}{\partial(x,y)} = 0$

→ If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

then find $\dfrac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$

Since $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} \cdot \dfrac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = 1$ ($\because$ By 1st prop)

$$\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = \frac{1}{\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}}$$

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

$\partial x = \sin\theta\cos\phi$, $\dfrac{\partial x}{\partial\theta} = r\cos\theta\cos\phi$, $\dfrac{\partial x}{\partial\phi} = -r\sin\theta\sin\phi$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \ , \ \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi \ , \ \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi \ , \ \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi \ , \ \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$$

$$\frac{\partial z}{\partial r} = \cos\theta \ , \ \frac{\partial z}{\partial \theta} = -r\sin\theta \ , \ \frac{\partial z}{\partial \phi} = 0$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\left[r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi\right] + r\sin\theta\left[r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right]$$

$$= \cos\theta\left[r^2\sin\theta\cos\theta\right] + r^2\sin\theta\left[\sin^2\theta\right]$$

$$= r^2\sin\theta\cos^2\theta + r^2\sin\theta\sin^2\theta$$

$$= r^2\sin\theta\,(1)$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

Now $\dfrac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = \dfrac{1}{\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}} = \dfrac{1}{r^2\sin\theta}$.

→ If $x+y+z=u$, $y+z=uv$ and $z=uvw$

then find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol: $x+y+z=u$, $y+z=uv$ $\Big\}$ $x+uv=u \Rightarrow x=u-uv$

$y+z=uv$

$\dots$ $u - uv - uvw$

Sol: $x+y+z=u$
$y+z=uv$
$z=uvw$

$y+z=uv$
$y+uvw=uv \Rightarrow y=uv-uvw$
$z=uvw$

$\frac{\partial x}{\partial u}=1-v$, $\frac{\partial x}{\partial v}=-u$, $\frac{\partial x}{\partial w}=0$

$\frac{\partial y}{\partial u}=v-vw$, $\frac{\partial y}{\partial v}=u-uw$, $\frac{\partial y}{\partial w}=-uv$

$\frac{\partial z}{\partial u}=vw$, $\frac{\partial z}{\partial v}=uw$, $\frac{\partial z}{\partial w}=uv$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}=\begin{vmatrix}1-v & -u & 0\\ v-vw & u-uw & -uv\\ vw & uw & uv\end{vmatrix}$$

$$=(1-v)\left[uv(u-uw)+u^2vw\right]+u\left[uv(v-vw)+uv^2w\right]+0$$

$$=(1-v)\left[u^2v-\cancel{u^2wv}+\cancel{u^2wv}\right]+u\left[uv^2-\cancel{uv^2w}+\cancel{uv^2w}\right]$$

$$u^2v-\cancel{u^2v^2}+\cancel{u^2v^2}=u^2v.$$

→ If $u=x^2-y^2$, $v=2xy$, $x=r\cos\theta$, $y=r\sin\theta$, then find $\frac{\partial(u,v)}{\partial(r,\theta)}$

Sol: $$\frac{\partial(u,v)}{\partial(r,\theta)}=\frac{\partial(u,v)}{\partial(x,y)}\times\frac{\partial(x,y)}{\partial(r,\theta)}$$

$\frac{\partial u}{\partial x}=2x$, $\frac{\partial u}{\partial y}=-2y$, $\frac{\partial v}{\partial x}=2y$, $\frac{\partial v}{\partial y}=2x$

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}2x & -2y\\ 2y & 2x\end{vmatrix}=4x^2+4y^2$$

$\frac{\partial x}{\partial r}=\cos\theta$, $\frac{\partial x}{\partial \theta}=-r\sin\theta$, $\frac{\partial y}{\partial r}=\sin\theta$, $\frac{\partial y}{\partial \theta}=r\cos\theta$

∂x | | $\cos\theta$ $-r\sin\theta$ |

$\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = \ldots$ $\frac{\partial y}{\partial r}$ ...

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$
$$= r$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = 4(x^2+y^2) \times r = 4r(x^2+y^2)$$
$$= 4r(r^2\cos^2\theta + r^2\sin^2\theta)$$
$$= 4r^3.$$

→ If $x = e^r\sec\theta$, $y = e^r\tan\theta$, S.T, $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

→ If $x = \frac{u^2}{v}$, $y = \frac{v^2}{u}$, find $\frac{\partial(u,v)}{\partial(x,y)}$

→ Functional dependence:

Let $u = f(x,y)$ & $v = g(x,y)$ be two differentiable functions. Suppose these functions are connected by a relation $F(u,v) = 0$ where F is differentiable, then we say u & v are functionally dependent on one another, if the partial derivatives u_x, u_y, v_x, v_y are all not zero at a time.

→ If the functions u & v (functions of x, y) are functionally dependent then

$$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

→ If $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$ then u & v are not functionally dependent.

→ If $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$, find $\frac{\partial(u,v)}{\partial(x,y)}$

Hence prove that u & v are functionally dependent & find the relation b/w them.

Hence ...
find the relation b/w them.

Sol: $\frac{\partial(u,v)}{\partial(x,y)} = 0$

$\Rightarrow$ u & v are functionally dependent

$v = \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}u$

$\Rightarrow \boxed{v = \tan^{-1}u}$

→ Determine whether the functions $u = e^x \sin y$, $v = e^x \cos y$ are functionally dependent or not. If so find the relation.

Sol: $u = e^x \sin y$, $v = e^x \cos y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{vmatrix}$$

$$= -e^{2x}\sin^2 y - e^{2x}\cos^2 y = -e^{2x} \neq 0$$

u & v are not functionally dependent.

2) $u = \frac{x}{y}$, $v = \frac{x+y}{x-y}$

3) $u = xy + yz + zx$, $v = x^2+y^2+z^2$ & $w = x+y+z$.

$u = f(r)$, $x = r\cos\theta$, $y = r\sin\theta$

$r^2 = x^2+y^2$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$2r\partial r = 2x \Rightarrow \frac{\partial r}{} = x$

$\partial u = \frac{\partial u}{}\frac{\partial r}{}$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial r}\frac{\partial r}{\partial x}$$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial y}{\partial x} = f'(r)\frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(f'(r)\frac{x}{r}\right)$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{x}{r}\left[f''(r)\frac{\partial r}{\partial x}\right] + f'(r)\left[\frac{r\cdot 1 - x\cdot\frac{\partial r}{\partial x}}{r^2}\right]$$