Find $L\{t\}$

Sol:- WKT $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$L\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

$u = t \qquad \int dv = \int e^{-st}\, dt$

$du = dt \qquad v = \frac{e^{-st}}{-s}$

$$\therefore L\{t\} = \int_0^\infty e^{-st}\, t\, dt = t\left(-\frac{e^{-st}}{s}\right)\Bigg|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, dt$$

$$= 0 + \frac{1}{s}\int_0^\infty e^{-st}\, dt = \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_0^\infty = -\frac{1}{s^2}(0-1)$$

$$= 1/s^2, \quad s > 0$$

2. Find $L\{t^n\}$, n > 0.

Sol:- WKT $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$L\{t^n\} = \int_0^\infty e^{-st}\, t^n\, dt$$

$u = t^n$, $\int dv = \int e^{-st}\,dt$

$du = n t^{n-1}\,dt$, $v = \dfrac{e^{-st}}{-s}$

$$\therefore L\{t^n\} = \int_0^\infty e^{-st}\, t^n\, dt$$

$$= t^n\left(-\frac{e^{-st}}{s}\right)\Bigg|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, n t^{n-1}\,dt$$

$$= 0 + \frac{n}{s}\int_0^\infty e^{-st}\, t^{n-1}\,dt$$

ie $\boxed{L\{t^n\} = \frac{n}{s} L\{t^{n-1}\}}$

When $n=1$ $L\{t^1\} = \frac{1}{s} L\{1\} = \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2}$, $s>0$

$n=2$, $L\{t^2\} = \frac{2}{s} L(t) = \frac{2}{s} \cdot \frac{1}{s^2} = \frac{2!}{s^3}$, $s>0$

$n=3$ $L\{t^3\} = \frac{3}{s} L\{t^2\} = \frac{3}{s} \cdot \frac{2}{s} \cdot \frac{1}{s^2} = \frac{3!}{s^4}$

$\vdots$

In general

$$L\{t^n\} = \frac{n!}{s^{n+1}}, \quad n>0$$

3. Find $L\{t^n\} = \frac{\cancel{n!}\ \Gamma(n+1)}{s^{n+1}}$, $s>0$, n > $\cancel{0}$ -1, Hence find $L\{t^{-\frac{1}{2}}\}$

sol:-

WKT $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$L\{t^n\} = \int_0^\infty e^{-st} t^n\,dt$$

Let $u = st \Rightarrow t = \frac{u}{s}$

$du = s\,dt$

when $t=0$, $u=0$

when $t\to\infty$, $u\to\infty$

$$\int_0^\infty x^{n-1} e^{-x}\,dx = \Gamma(n)$$

$$\therefore L\{t^n\} = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$= \frac{1}{s^{n+1}} \underbrace{\int_0^\infty u^n e^{-u}\,du}_{} = \frac{\Gamma(n+1)}{s^{n+1}}, \quad n>-1 \ \& \ s>0$$

$$L\{t^{-1/2}\} = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-1/2+1}}$$

$$= \frac{\Gamma(1/2)}{s^{1/2}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}, \quad s>0$$

$$L\left(t^{1/2}\right) \quad \left(\underbrace{\Gamma(n+1) = n\Gamma(n)}\right)$$

$$= \frac{\Gamma(1/2+1)}{s^{1/2+1}} = \frac{\frac{1}{2}\Gamma(1/2)}{s^{3/2}}$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}$$

Properties of Laplace Transforms

➢ <u>*Linearity Property :-*</u>

$L\{c_1 f(t)\} + L\{c_2 g(t)\}$

$c_1 L\{f(t)\} + c_2 L\{g(t)\}$

$c_1 F(s) + c_2 G(s)$

$$L\{c_1 f(t) + c_2 g(t)\} = c_1 F(s) + c_2 G(s)$$

$$If\ L[f(t)] = F(s),\ L[g(t)] = G(s)\ where\ c_1\ and\ c_2\ are\ constants.$$

Example:

$$Find\ L\{4e^{5t} + 6t^3 - 3\sin 4t + 2\cos 2t\}$$

Sol:-

$= L\{4e^{5t}\} + L\{6t^3\} - L\{3\sin 4t\} + L\{2\cos 2t\}$

$= 4L\{e^{5t}\} + 6L\{t^3\} - 3L\{\sin 4t\} + 2L\{\cos 2t\}$

$= 4 \cdot \frac{1}{s-5} + 6 \cdot \frac{3!}{s^4} - 3 \cdot \frac{4}{s^2+16} + 2 \cdot \frac{s}{s^2+4}$

$$= \frac{4}{s-5} + \frac{36}{s^4} - \frac{12}{s^2+16} + \frac{2s}{s^2+4}$$

- ***Translation Properties:-***

The first translation or Shifting property:

$$If\ L\left[f\left(t\right)\right] = F(s),\ then\ L\left[e^{at} f(t)\right] = F(s-a).$$

$$\text{Pr}\, oof :-$$

$$We\ have\ L\left[f\left(t\right)\right] = \int_0^\infty e^{-st} f\left(t\right) dt = F(s)$$

$$Then\ L\left[e^{at} f(t)\right] = \int_0^\infty e^{-st}\left[e^{at} f\left(t\right)\right] dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt = F(s-a).$$

$$Find\ \ (a)\ L\left(t^2 e^{3t}\right),\ (b)\ L\left(e^{-2t} \sin 4t\right).$$

a) $L\{t^2 e^{3t}\}$

$$L\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$$

$$L\{t^2 e^{3t}\} = \frac{2}{(s-3)^3}$$

b) $L\{e^{-2t}\sin 4t\}$

$$L\{\sin 4t\} = \frac{4}{s^2+16}$$

$$L\{e^{-2t}\sin 4t\} = \frac{4}{(s+2)^2+16} = \frac{4}{s^2+4+4s+16}$$

$$= \frac{4}{s^2+4s+20} \quad //$$

Find

1) $L\{e^{4t}\cos 5t\}$

2) $L\{3\cos 6t - 5\sin 6t\}$

Second translation or Shifting property:

$$If \quad L\left[f(t)\right] = F(s) \quad and \quad g(t) = \begin{cases} f(t-a) & t \rangle a \\ 0 & t \langle a \end{cases},$$

$$then \quad L\left[g(t)\right] = e^{-as}F(s).$$

Proof :−

$$L\left[g(t)\right] = \int_0^\infty e^{-st} g(t)\,dt = \int_0^a e^{-st} g(t)\,dt + \int_a^\infty e^{-st} g(t)\,dt$$

$$= \int_0^a e^{-st}\left(0\right)dt + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f\left(t-a\right)dt$$

$$Let \ u = t - a \Rightarrow t = u + a$$

$$Now \ du = dt$$

$$when \ t = a,\ u = 0 \ \ \& \ \ when \ t \to \infty,\ u \to \infty$$

$$\therefore \ L\left[g\left(t\right)\right] = \int_0^\infty e^{-s(u+a)} f(u)\,du = e^{-as}\int_0^\infty e^{-su} f(u)\,du = e^{-as}F(s)$$

$$Find\ L\left[f(t)\right]\ if\ f(t)=\begin{cases}\cos\left(t-\dfrac{2\pi}{3}\right) & t\rangle\dfrac{2\pi}{3} \\ 0 & t\langle\dfrac{2\pi}{3}\end{cases},$$

$Solution:-$

$$Since\ L(\cos t)=\frac{s}{s^2+1},\ it\ follows\ with\ a=\frac{2\pi}{3},\ that$$

$$L\left[f(t)\right]=\frac{se^{-\frac{2\pi s}{3}}}{s^2+1}$$

- ***Change of scale property:-***

$$If\ \ L\left[f(t)\right]=F(s),\ then\ \ L\left[f(at)\right]=\frac{1}{a}F\left(\frac{s}{a}\right).$$

$$\text{Proof}:-$$

$$L\left[f(at)\right]=\int_0^{\infty}e^{-st}f(at)\,dt$$

$$Let\ u=at\Rightarrow t=\frac{u}{a}$$

$$i.e.\ du=a\,dt\Rightarrow dt=\frac{du}{a}$$

$$when\ t=0,\ u=0\ \&\ when\ t\to\infty,\ u\to\infty$$

$$\therefore\ L\left[f(at)\right]=\int_0^{\infty}e^{-s\left(\frac{u}{a}\right)}f(u)\frac{du}{a}=\frac{1}{a}\int_0^{\infty}e^{-\frac{su}{a}}f(u)\,du=\frac{1}{a}F\left(\frac{s}{a}\right).$$

$$Find\, L\left(\frac{\sin at}{at}\right), given\ that\, L\left(\frac{\sin t}{t}\right) = \tan^{-1}\left(\frac{1}{s}\right).$$

$$Solution: -$$

$$L\left(\frac{\sin at}{at}\right) = \frac{1}{a}\tan^{-1}\left(\frac{1}{\left(\frac{s}{a}\right)}\right) = \frac{1}{a}\tan^{-1}\left(\frac{a}{s}\right)$$

Laplace Transform of Derivatives:-

$$If\ \ L\left[f(t)\right] = F(s),\ then\ \ S.T\ \ L\left[f'(t)\right] = sF(s) - f(0).$$

$$\Pr oof :-$$

$$L\left[f'(t)\right] = \int_0^\infty e^{-st} f'(t)\,dt$$

$$U\sin g\ \ \text{int}\, egration\ \ by\ \ parts,\ we\ \ have$$

$$L\left[f'(t)\right] = \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= \left[0 - f(0)\right] + sF(s)$$

$$= sF(s) - f(0).$$

$$If\ L\left[f(t)\right]=F(s),\ then\ L\left[f''(t)\right]=s^2F(s)-sf(0)-f'(0).$$

$$\Pr oof:-$$

$$Let\ L\left[g'(t)\right]=sL\left[g(t)\right]-g(0)=sG(s)-g(0)$$

$$Let\ g(t)=f''(t),\ then$$

$$\begin{aligned}
L\left[f''(t)\right]&=sL\left[f'(t)\right]-f'(0)\\
&=s\left[sL\left[f(t)\right]-f(0)\right]-f'(0)\\
&=s^2L\left[f(t)\right]-sf(0)-f'(0)\\
&=s^2F(s)-sf(0)-f'(0)
\end{aligned}$$

In general, we have

$$\mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0).$$