CAT 2 Portions

Module - 3

↳ maxima/minima of two variables | G, T

10 ↳ Constrained maxima/minima / G, T (vectorform)
(Lagrange multiplier)

Module - 4 (Multiple Integrals)

↳ Area by double integration | G, T

10 ↳ Volume by " " | G, T

↳ " " Triple integration | G, T

↳ changing the order of integration

↳ changing the variable

Cartesian ⇄ polar

~~" ⇄ spherical/cylindrical~~

G/T

Module - 5 (Special functions)

10 ↳ Beta/Gamma function } G

↳ Error function

↳ Dirichlet integral

Module-5

?

Gamma function $\Gamma(n) \to$ Gamma of n

$3! = 3\times2\times1 = 6$

$5! = 5\times4\times3\times2\times1 = 120$

5

$2.5 \quad 3.5$

$0.5! =$

$7.5!$

$n!$

$\frac{dy}{dx} + cy = 0$ | $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin(x)$

$\frac{d^\alpha y}{dx^\alpha}$

$y = \sin(x)$

$\frac{dy}{dx} = \cos(x)$

$\frac{d^{1/2}y}{dx^{1/2}}$ $\quad$ $\frac{d^\alpha y}{dx^\alpha}$

$\frac{d^{\sqrt{2}}y}{dx^{\sqrt{2}}}$

$\Gamma(n) = (n-1)!$ ✓ $\quad 0.5! = \Gamma(1.5)$

$\Gamma(5) = 4!$

$n! = n \times (n-1) \times (n-2) \times \cdots 1$

$0.5! = \Gamma(1.5)$

$$\Gamma(n) = \int_0^\infty \underbrace{e^{-x} x^{n-1} \, dx}_{\hookrightarrow \text{Euler definition}}, \quad n > 0$$

$$\Gamma(3) = \int_0^\infty e^{-x} \frac{x^2}{u} \frac{dx}{dv}$$

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)}, \quad z \neq 0, -1, -2, \cdots = (z-1)!$$

$$\Gamma(3) = \lim_{n\to\infty} \frac{n!\, n^3}{3\times 4 \cdots (3+n)} = 6$$

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \right]$$

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-z} e^t dt, \; c > 0, \Re(z) > 0, \; i = \sqrt{-1}$$

$\Gamma(3) = 2! = 2$

$\Gamma(1) = 1$

$\Gamma(n+1) = n\Gamma(n)$

$\Gamma(n+1) = n!$ if n is intiger

$\Gamma(1/2) = \sqrt{\pi}$

$\Gamma(1/2) = \int_0^\infty e^{-x} x^{1/2-1} dx = \int_0^\infty e^{-x} x^{-1/2} dx$

$\Gamma(1/2) = \int_0^\infty e^{-y} y^{-1/2} dy$

$\Gamma(1/2)\,\Gamma(1/2) = \int_0^\infty e^{-x} x^{-1/2} dx \int_0^\infty e^{-y} y^{-1/2} dy$

$\int_0^4 x\,dx = \int_0^4 t\,dt$

$x = t^2$ $\quad = 2\int_0^\infty e^{-t^2} dt$

x	0	∞
t	0	∞

$dx = 2t\,dt$

$dx = 2\sqrt{x}\,dt$

$x^{-1/2} dx = 2dt$

$= \left[2\int_0^\infty e^{-x^2} dx\right]\left[2\int_0^\infty e^{-y^2} dy\right]$

$$\frac{d}{dx}\sin(x)\Big|_{x=\pi/2} = \frac{d}{dt}\sin(t)\Big|_{t=\pi/2}$$

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 2\int_0^\infty e^{-x^2}dx \times 2\int_0^\infty e^{-y^2}dy$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

$$= 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2}\,\underbrace{r}_{J}\,d\theta\,dr$$

$$= 4\int_0^\infty e^{-r^2}\,r\,\frac{\pi}{2}\,dr$$

$$= 2\pi\int_0^\infty e^{-r^2}\,r\,dr$$

$r^2 = s$
$2r\,dr = ds$

r	0	∞
s	0	∞

$$= \pi\int_0^\infty e^{-s}\,ds$$

$$= \pi\,\frac{e^{-s}}{-1}\Big|_0^\infty = \pi\left(\frac{0-1}{-1}\right) = \pi$$

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$$\left[\Gamma(1/2)\right]^2 = \pi$$

$$\Rightarrow \Gamma(1/2) = \sqrt{\pi}$$

Evaluate $\Gamma\left(\frac{5}{2}\right)$

$\Gamma(n) = (n-1)\Gamma(n-1)$ ✓ $\quad \Gamma(n+1) = n\Gamma(n)$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)$$

$$= \frac{3}{2} \times \frac{1}{2} \times \Gamma(1/2)$$

$$= \frac{3}{4}\sqrt{\pi}$$

Evaluate $\Gamma\left(-\frac{27}{2}\right)$

$$\Gamma\left(-\frac{27}{2}\right) = \Gamma\left(-\frac{29}{2}+1\right) = -\frac{29}{2}\Gamma\left(-\frac{29}{2}\right)$$

$$= -\frac{29}{2} \cdot -\frac{31}{2} \cdot -\frac{33}{2} \cdot$$

$$\Gamma\left(\frac{1}{2}\right) = \Gamma\left(\frac{-1}{2}+1\right)$$

$$\sqrt{\pi} = \frac{-1}{2}\Gamma(-1/2) = \frac{-1}{2}\times\frac{-3}{2}\times\frac{-5}{2}\times\frac{-7}{2}\times\frac{-9}{2}\times\frac{-11}{2}$$
$$\times\frac{-13}{2}\times\frac{-15}{2}\times\frac{-17}{2}\times\frac{-19}{2}\times\frac{-21}{2}$$
$$\times\frac{-23}{2}\times\frac{-25}{2}\times\frac{-27}{2}\Gamma\left(\frac{-27}{2}\right)$$

$$\Gamma\left(-\frac{27}{2}\right) = \frac{\sqrt{\pi}\quad 2^{14}}{1\times3\times5\cdots\times27}$$

$\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})$; $\Gamma(-\frac{7}{2})$; $\int_0^\infty x^{\frac{1}{2}} e^{-3x^5} dx$;

$$3x^5 = y$$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$x^{-7/2} = \frac{y^{1/5 \cdot -7/2}}{3^{1/5 \cdot -7/2}}$$
$$= \frac{y^{-\frac{33}{10}}}{3^{-33/10}}$$

$$3x^5 = y \Rightarrow x^5 = y/3$$
$$x = \frac{y^{1/5}}{3^{1/5}}$$
$$15x^4 dx = dy$$
$$dx = \frac{dy}{15x^4}$$
$$dx = \frac{1}{15}x^{-4}dy$$
$$x^{1/2}dx = \frac{1}{15}x^{-7/2}dy$$

$\Gamma(n) = (n-1)!$ $\quad n$ is +ve integer

$\Gamma(n) = (n-1)\Gamma(n-1)$

$\Gamma(1/2) = \sqrt{\pi}$

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx \quad n>0$

Beta function

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad , \quad m,n>0$$

$x = \sin^2\theta$

$$\beta(m,n) = \beta(n,m)$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \,d\theta$$

Evaluate $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx$ $\quad 1-x^4 = t$

$$= \int_0^1 (1-x^4)^{-1/2}\,dx$$

take $x^2 = \sin\theta$ $\quad \Rightarrow (1-x^4)^{-1/2} = (\cos^2\theta)^{-1/2}$

$\quad = \dfrac{1}{\cos\theta}$

$2x\,dx = \cos\theta\,d\theta$

$dx = \dfrac{\cos\theta}{2\sqrt{\sin\theta}}\,d\theta$

x	0	1
θ	0	$\pi/2$

$x^2 = \sin\theta$

$$= \int_0^{\pi/2} \frac{1}{\cos\theta} \frac{\cos\theta}{2\sqrt{\sin\theta}}\, d\theta$$

$$= \frac{1}{4} \cdot 2\int_0^{\pi/2} \cos^0\theta \, \sin^{-1/2}\theta \, d\theta$$

$2n-1=0$

$2m-1=-1/2$

$$= \frac{1}{4}\beta(1/2, 1/4)$$

$n = 1/2$

$m = 1/4$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{1}{4} \frac{\Gamma(1/2)\Gamma(1/4)}{\Gamma(3/4)} = \frac{\sqrt{\pi}}{4} \frac{\Gamma(1/4)}{\Gamma(3/4)}$$

$$= \frac{\sqrt{\pi}}{4} \frac{(\Gamma(1/4))^2}{\pi\sqrt{2}}$$

$\Gamma(1/4)\Gamma(3/4) = \pi\sqrt{2}$

$\Gamma(3/4) = \frac{\pi\sqrt{2}}{\Gamma(1/4)}$

$$= \frac{1}{8\sqrt{2\pi}}(\Gamma(1/4))^2$$

$$\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx$$

$1-x^4 = t$

$-4x^3\,dx = dt$

$\frac{1}{\sqrt{1-x^4}} = \frac{1}{\sqrt{t}}$

$dx = \frac{-dt}{4x^3}$

$= \frac{-dt}{4(1-t)^{3/4}}$

x	0	1
t	1	0

$x^4 = 1-t$

$x = (1-t)^{1/4}$

$= -\frac{1}{4}(1-t)^{-3/4}$

$x^3 = (1-t)^{3/4}$

$$= \frac{1}{4}\int_0^1 t^{-1/2}(1-t)^{-3/4}\,dt$$

$m-1 = -1/2 \Rightarrow m = \frac{1}{2}$

$n-1 = -3/4 \Rightarrow n = 1/4$

$$= \frac{1}{4}\beta(1/2, 1/4)$$

Evaluate $\int_0^{\pi/2} \sqrt{\tan(\theta)}\,d\theta$

$$= \frac{1}{2}\left[2\int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\,d\theta\right] = \frac{1}{2}\beta\left(\frac{1}{4}, \frac{3}{4}\right)$$

$2m-1 = 1/2 \Rightarrow m = 3/4$

$2n-1 = -1/2 \Rightarrow n = 1/4$

$= \frac{1}{2}\frac{\Gamma(1/4)\Gamma(3/4)}{\Gamma(1)}$

$= \frac{1}{2}\Gamma(1/4)\Gamma(3/4)$

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$$\int_0^1 x\,dx = \int_0^{\pi/2} \sin\theta\cos\theta\,d\theta$$

$\frac{1}{2}$ $\qquad$ $\frac{1}{2}$

Prove that $\int_0^1 \frac{x^2\,dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{\pi}{4\sqrt{2}}$

$$\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx$$

$$\frac{\cancel{\sin\theta}}{\cancel{\cos\theta}}$$

$x^2 = \sin\theta$

$$= \frac{1}{4} 2\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta \quad = \frac{1}{4}\beta\left(\tfrac{1}{2}, \tfrac{3}{4}\right)$$

$$\int_0^1 \frac{dx}{\sqrt{1+x^4}} \qquad x^2 = \tan\theta$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{1}{\sqrt{\sin(2\theta)}}\,d\theta \qquad 2\theta = \phi$$

$$= \frac{1}{\sqrt{2}} \frac{1}{4}\left[2\int_0^{\pi/2} \sin^{-1/2}\phi \, d\phi\right]$$

$$= \frac{1}{4\sqrt{2}} \beta\left(\tfrac{1}{2}, \tfrac{1}{4}\right)$$

$$= \frac{1}{4}\beta\left(\tfrac{1}{2}, \tfrac{3}{4}\right) \times \frac{1}{4\sqrt{2}} \beta\left(\tfrac{1}{2}, \tfrac{1}{4}\right)$$

$$= \frac{1}{16\sqrt{2}} \frac{\Gamma(1/2)\cancel{\Gamma(3/4)}}{\Gamma(5/4)} \frac{\Gamma(1/2)\Gamma(1/4)}{\cancel{\Gamma(3/4)}}$$

$$= \frac{1}{16\sqrt{2}} \pi \frac{\Gamma(1/4)}{\Gamma(5/4)} = \frac{\pi}{16\sqrt{2}} \frac{\cancel{\Gamma(1/4)}}{\frac{1}{4}\cancel{\Gamma(1/4)}}$$

$$\Gamma\left(1+\tfrac{1}{4}\right)$$

$$= \frac{\pi}{4\sqrt{2}}$$

$$\Gamma(n+1) = n\Gamma(n)$$

$\Gamma(1/4)\,\Gamma(3/4)$

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin(\pi n)}$$ Euler's Reflection formula

$$\Gamma\left(\frac{3}{4}\right)\Gamma(1-3/4) = \frac{\pi}{\sin\left(\frac{3\pi}{4}\right)}$$

$$\Gamma(3/4)\Gamma(1/4) = \frac{\pi}{1/\sqrt{2}} = \pi\sqrt{2}$$

$$\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}$$

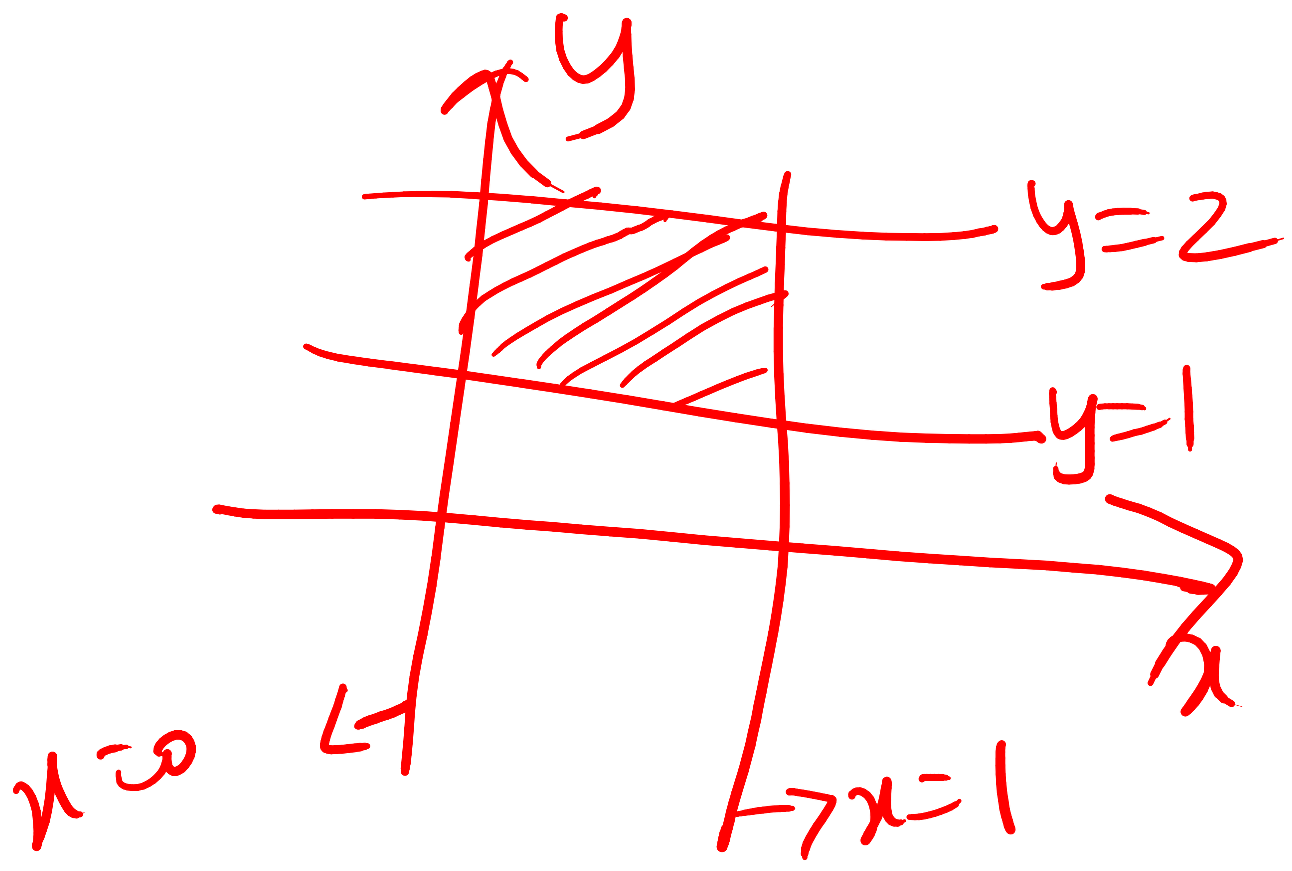
$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin(\pi n)} \quad B(n, 1-n)$$

$$\int_0^\infty \frac{dy}{1+y^4}.$$

$$\int_0^1\int_1^2 x+y \, dy \, dx$$

$y \to 1$ to 2

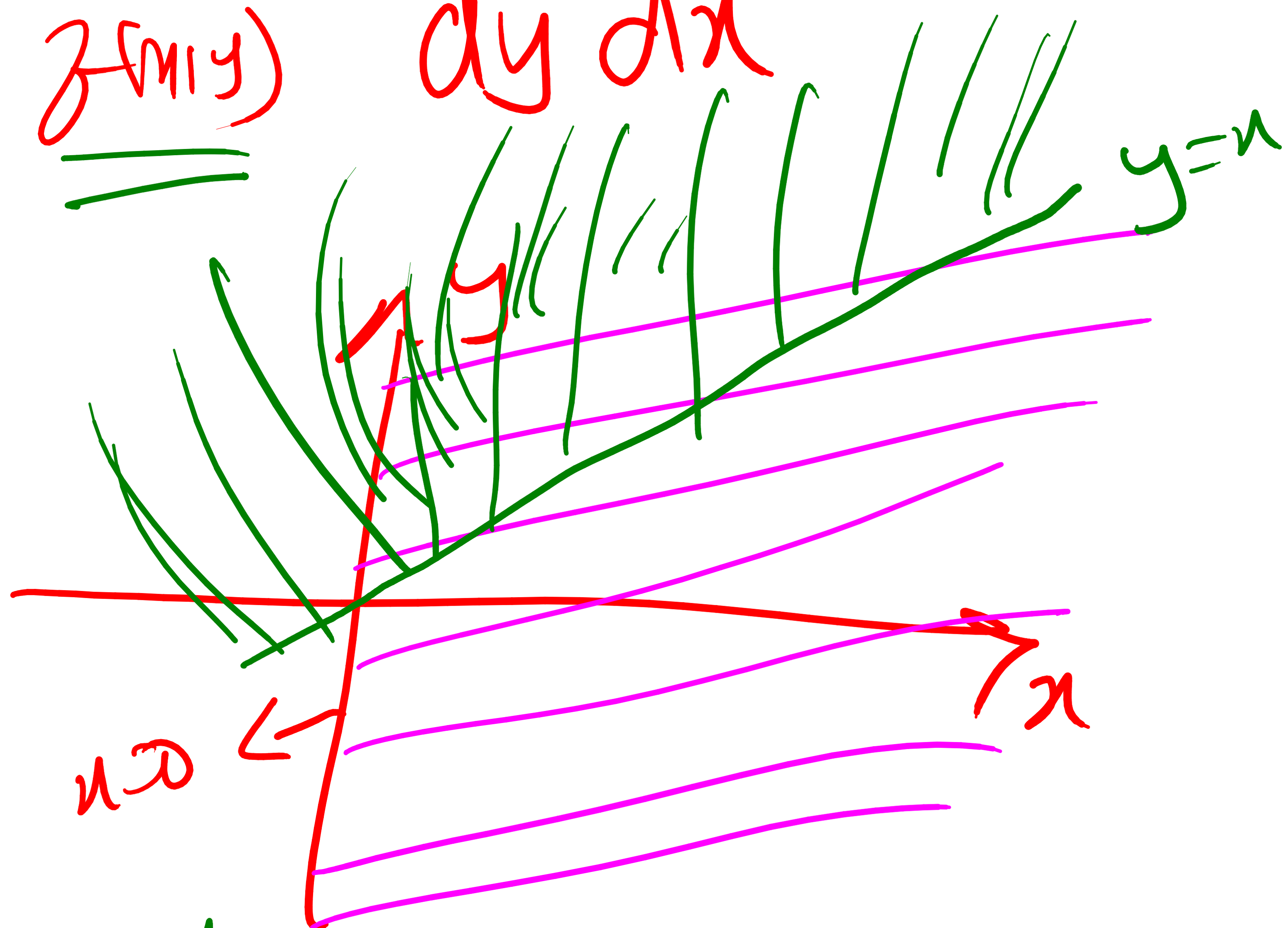
$x \to 0$ to 1

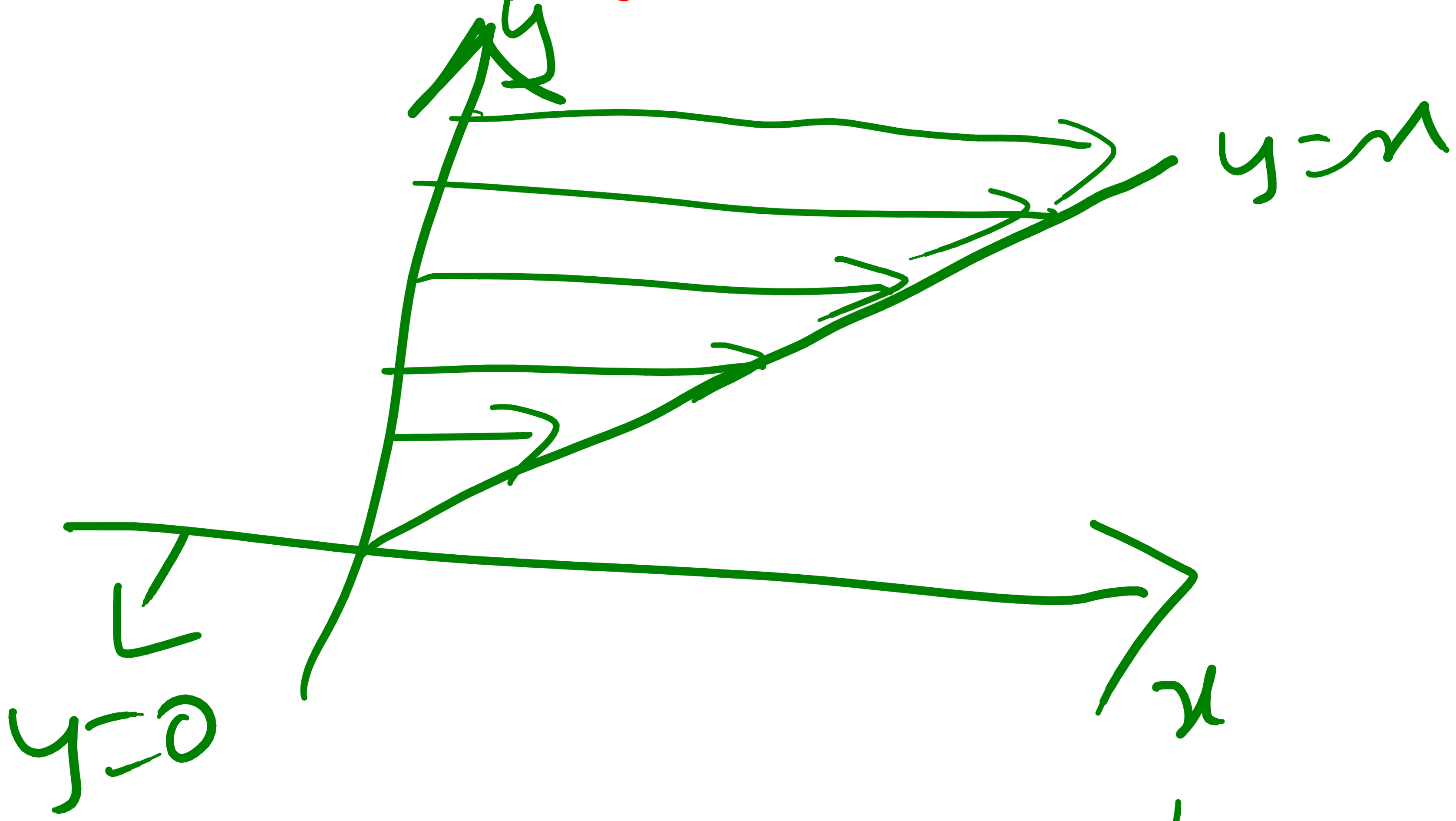


$$\int_0^\infty\int_x^\infty f(x,y) \, dy \, dx$$

$y \to x$ to ∞

$x \to 0$ to ∞





$y \to 0$ to ∞

$x \to 0$ to y

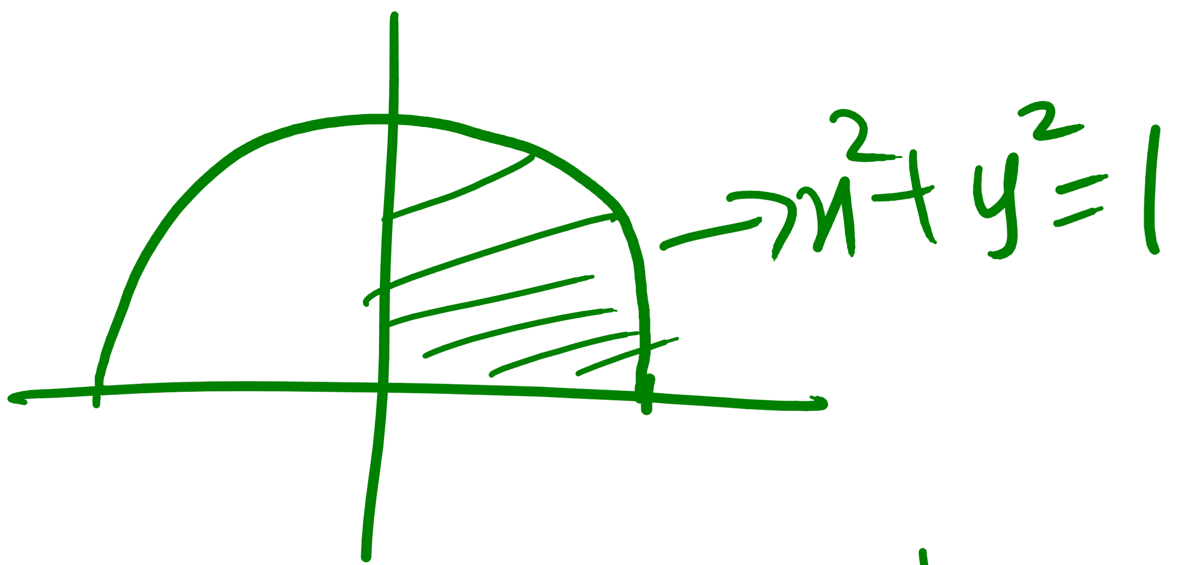
$$\int_0^\infty\int_x^\infty \cdots \, dy \, dx = \int_0^\infty\int_0^y \cdots \, dx \, dy$$

$x = r\cos\theta \qquad y = r\sin\theta$

$\iint e^{+(x^2+y^2)}\,dy\,dx$

$x = 1 \qquad r = \sec\theta$

$x^2+y^2 = 4 \qquad r = 2$



$\int_0^1 \int_0^{\pi/2} e^{r^2}\, r\,d\theta\,dr \qquad \int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\,dx\,dy$ (Correct)

$\int x^2\,dx \qquad x = \sin\theta$

$dx = \cos\theta\,d\theta$

$\iint dy\,dx$

$x = r\cos\theta$

$y = r\sin\theta$

$dy\,dx = |J|\,dr\,d\theta$

$$J = \begin{vmatrix} \frac{\partial y}{\partial \theta} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial x}{\partial r} \end{vmatrix}$$

xy

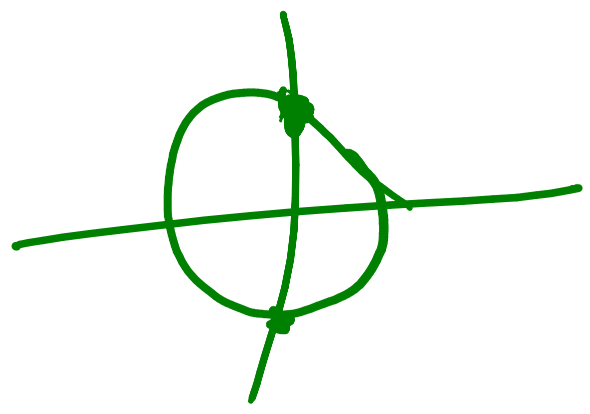
$f(x,y)$

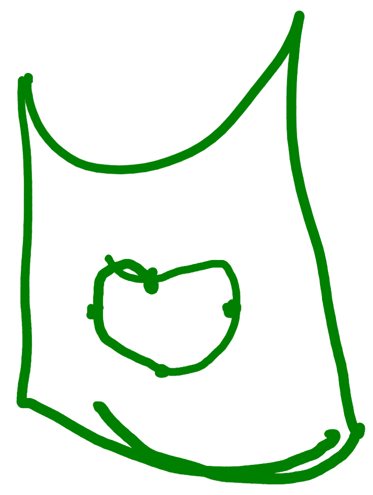
$x^2+y^2=4$

$g(x,y)=C$

$h(x,y) = f + \lambda g$

$= xy + \lambda(x^2+y^2-4)$





$$\left.\begin{aligned} h_x &= 0 \\ h_y &= 0 \\ h_\lambda &= 0 \end{aligned}\right\}$$

(x, y)

(x_1, y_1) $f(x_1, y_1)$

(x_2, y_2) $f(x_2, y_2)$

(x_3, y_3) $f(x_3, y_3)$

$\beta(m, 1/2) = 2^{2m-1}\beta(m, m)$

LHS

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta$$

$$\beta(m,\tfrac{1}{2}) = \frac{\Gamma(m)\Gamma(\tfrac{1}{2})}{\Gamma(m+\tfrac{1}{2})}$$

$$\beta(m,\tfrac{1}{2}) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta$$

RHS

$$\beta(m,m) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2m-1}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1} \, d\theta$$

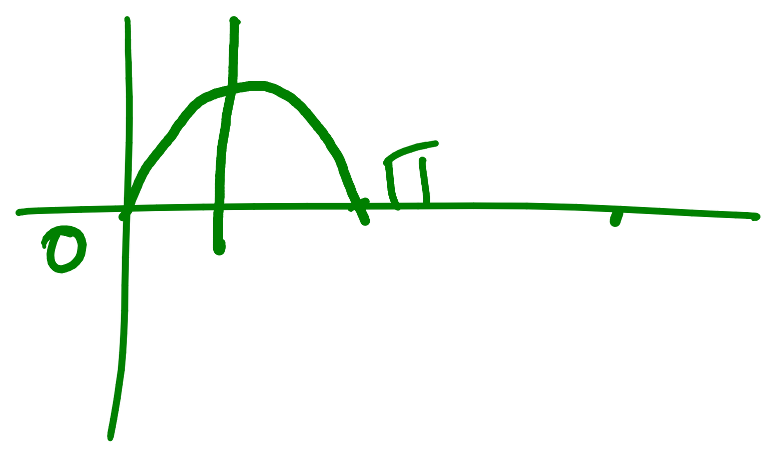
$$= 2\int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^{2m-1} d\theta$$

$$= \frac{2}{2^{2m-1}} \int_0^{\pi/2} \sin^{2m-1} 2\theta \, d\theta$$

$$2\theta = \phi$$
$$2\,d\theta = d\phi$$

θ	0	$\pi/2$
ϕ	0	π

$$= \frac{1}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1}\phi \, d\phi$$



$$= \frac{2}{2^{2m-1}} \int_0^{\pi/2} \sin^{2m-1}\phi \, d\phi$$

$$\int_{-a}^{a}$$

$$\beta(m,m) = \frac{1}{2^{2m-1}} \beta(m, 1/2)$$

$$\beta(m, 1/2) = 2^{2m-1} \beta(m,m)$$

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin(\pi n)}$$

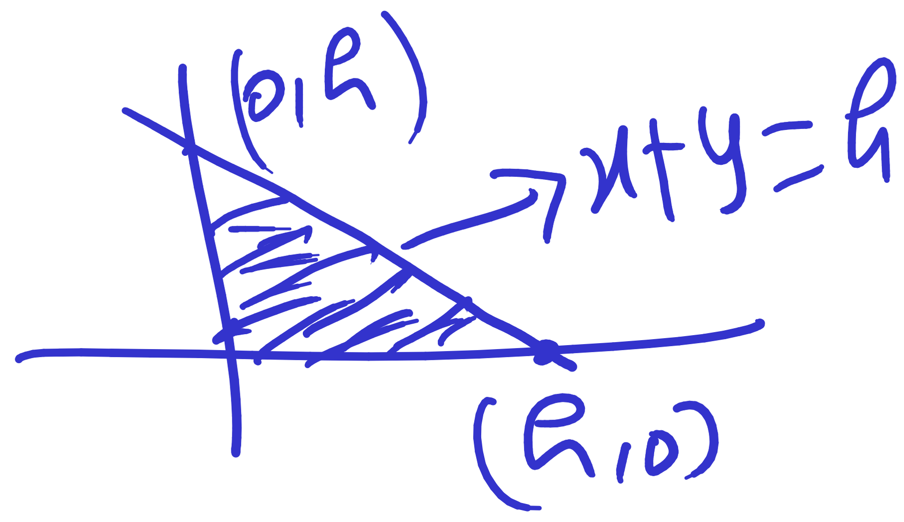
Try to prove

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma(2m)$$

Dirichlet's Integral

$$\iint_R x^{m-1} y^{n-1}\, dx\, dy = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} h^{m+n}$$

$x>0,\ y>0,\ x+y \le h$



$$\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)} h^{l+m+n}$$

$x>0,\ y>0,\ z>0,\ x+y+z \le h$

Example 7.48. *Evaluate the integral* $\iiint x^{l-1} y^{m-1} z^{n-1}\, dxdydz$ *where x, y, z are all positive with condition,* $(x/a)^p + (y/b)^q + (z/c)^r \le 1$. (U.P.T.U., 2005 S)

$\left(\frac{x}{a}\right)^p = u \Rightarrow x = a u^{1/p} \Rightarrow dx = a \frac{1}{p} u^{1/p - 1} du$

$\left(\frac{y}{b}\right)^q = v \Rightarrow y = b v^{1/q} \Rightarrow dy = \frac{b}{q} v^{1/q - 1} dv$

$\left(\frac{z}{c}\right)^r = w \Rightarrow z = c w^{1/r} \Rightarrow dz = \frac{c}{r} w^{1/r - 1} dw$

$$\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$$

$$= \iiint (a u^{1/p})^{l-1} (b v^{1/q})^{m-1} (c w^{1/r})^{n-1} \; \frac{a}{p} u^{1/p-1} du \; \frac{b}{q} v^{1/q-1} dv \; \frac{c}{r} w^{1/r-1} dw$$

$x = a u^{1/p}$

$y = b v^{1/q}$

$z = c w^{1/r}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \frac{a^l\, b^m\, c^n}{pqr} \iiint u^{l/p-1}\, v^{m/q-1}\, w^{n/r-1}\, du\, dv\, dw \qquad u+v+w \le 1$$

$$= \frac{a^l b^m c^n}{pqr} \; \frac{\Gamma(l/p)\,\Gamma(m/q)\,\Gamma(n/r)}{\Gamma\left(\frac{l}{p}+\frac{m}{q}+\frac{n}{r}+1\right)}$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$$

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$$

$$\operatorname{erf}(\infty) = 1$$

x	0	∞
t	0	∞

$$x^2 = t$$

$$2x\,dx = dt$$

$$dx = \frac{dt}{2x}$$

$$dx = \frac{1}{2} t^{-1/2} dt$$

$$\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{2} \int_0^\infty e^{-t}\, t^{-1/2} dt$$

$$= \frac{1}{\sqrt{\pi}} \Gamma(1/2) = 1$$

PT $$\frac{d}{dx}\left[\operatorname{erf}(ax)\right] = \frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$$

$$\frac{d}{dx}\left[\operatorname{erf}(ax)\right] = \frac{d}{dx}\left[\frac{2}{\sqrt{\pi}} \int_0^{ax} e^{-a^2x^2} dx\right]$$

differentiation under integral