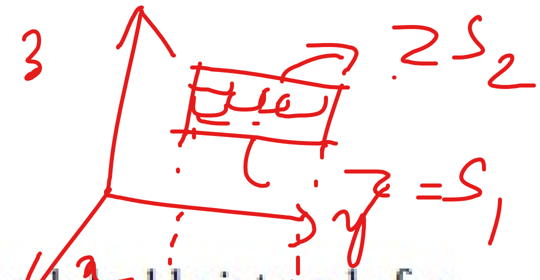


TRIPLE INTEGRALS

Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables. Let's first deal with the simplest case where f is defined on a rectangular box:

$$\boxed{1} \qquad B = \{(x, y, z) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d,\ r \leqslant z \leqslant s\}$$

The first step is to divide B into sub-boxes. We do this by dividing the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, dividing $[c, d]$ into m subintervals of width Δy, and dividing $[r, s]$ into n subintervals of width Δz. The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box B into lmn sub-boxes

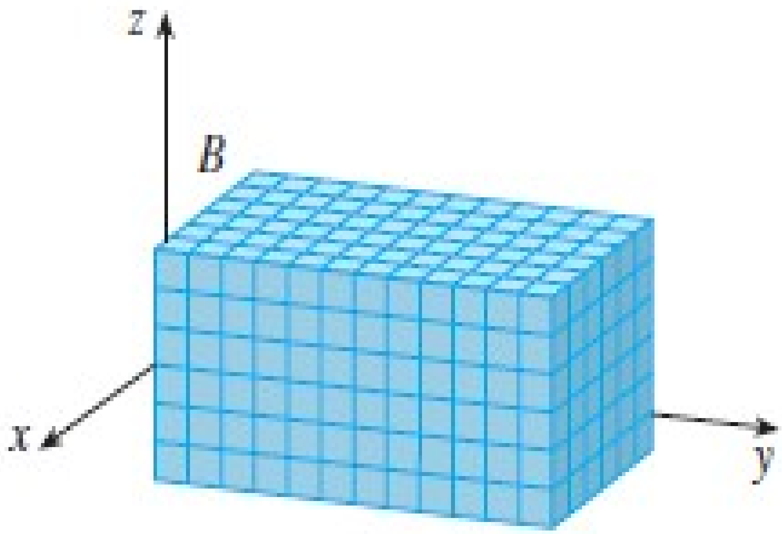
$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

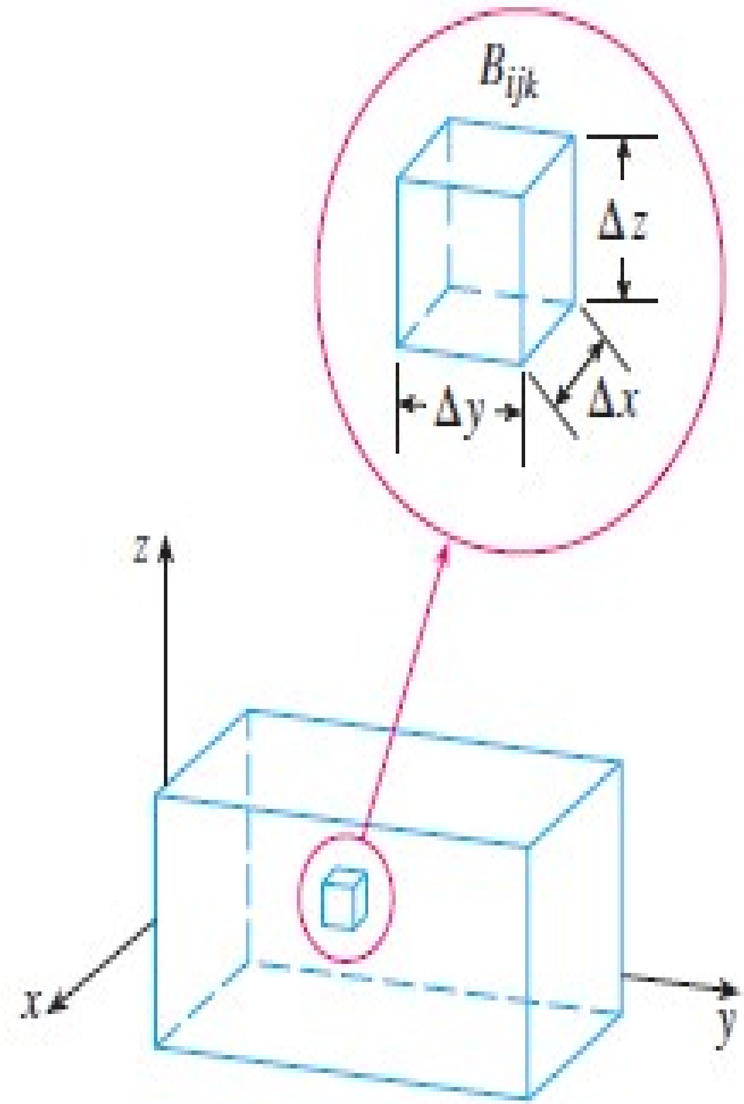
which are shown in Figure 1. Each sub-box has volume $\Delta V = \Delta x \, \Delta y \, \Delta z$.

Then we form the **triple Riemann sum**

$$\boxed{2} \qquad \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \, \Delta V$$

where the sample point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is in B_{ijk}. By analogy with the definition of a double integral. we define the triple integral as the limit of the triple Riemann sums in (2).


z
B
x
y


B_{ijk}
Δz
Δy
Δx
z
x
y

3 DEFINITION The triple integral of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

if this limit exists.

4 FUBINI'S THEOREM FOR TRIPLE INTEGRALS If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

Problem 1: Evaluate the triple integral $\iiint_B xyz^2\, dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

SOLUTION We could use any of the six possible orders of integration. If we choose to integrate with respect to x, then y, and then z, we obtain

$$\iiint_B xyz^2\, dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dx\, dy\, dz = \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2}\right]_{x=0}^{x=1} dy\, dz$$

$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\, dz = \int_0^3 \left[\frac{y^2z^2}{4}\right]_{y=-1}^{y=2} dz$$

$$= \int_0^3 \frac{3z^2}{4}\, dz = \frac{z^3}{4}\bigg]_0^3 = \frac{27}{4}$$

We restrict our attention to continuous functions f and to certain simple types of regions. A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

5 $$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is the projection of E onto the xy-plane as shown in Figure 2. Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$.

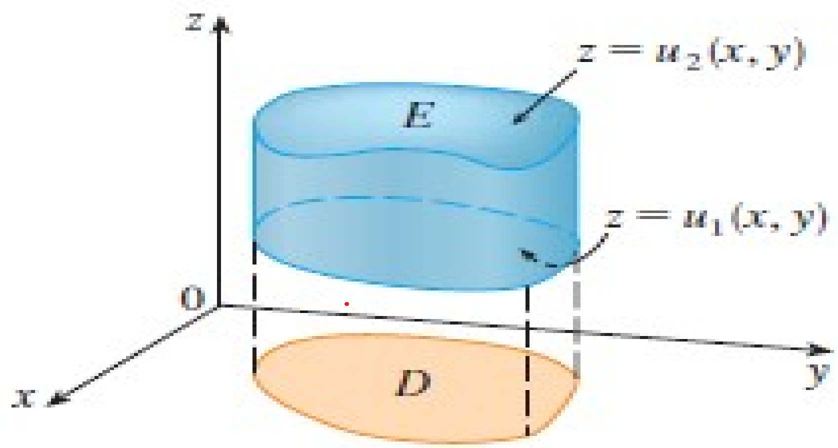


FIGURE
A type 1 solid region

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

In particular, if the projection D of E onto the xy-plane is a type I plane region (as in Figure), then

$$E = \{(x, y, z) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x),\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

and Equation 6 becomes

$y =$

7 $$\boxed{\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dy\, dx}$$

$y =$

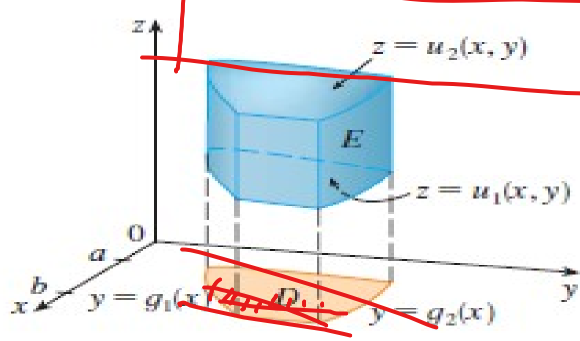


FIGURE
A type 1 solid region where the projection D is a type I plane region

If, on the other hand, D is a type II plane region (as in Figure), then

$$E = \{(x, y, z) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y),\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

and Equation 6 becomes

$y = d$ $x =$

8
$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dx\, dy$$

$y = c$ $x =$

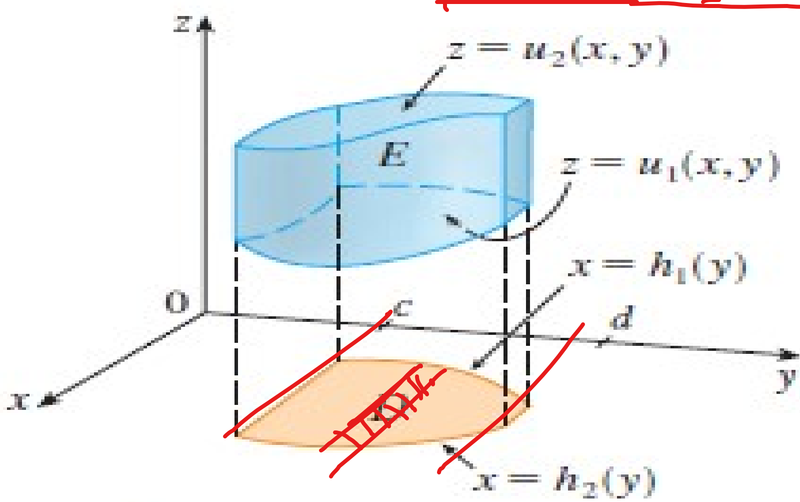


FIGURE
A type 1 solid region with a type II projection

Problem : Evaluate $\iiint_E z\, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

SOLUTION When we set up a triple integral it's wise to draw *two* diagrams: one of the solid region E (see Figure 5) and one of its projection D on the xy-plane (see Figure 6). The lower boundary of the tetrahedron is the plane $z = 0$ and the upper

$0 \le z \le 1 - x - y$

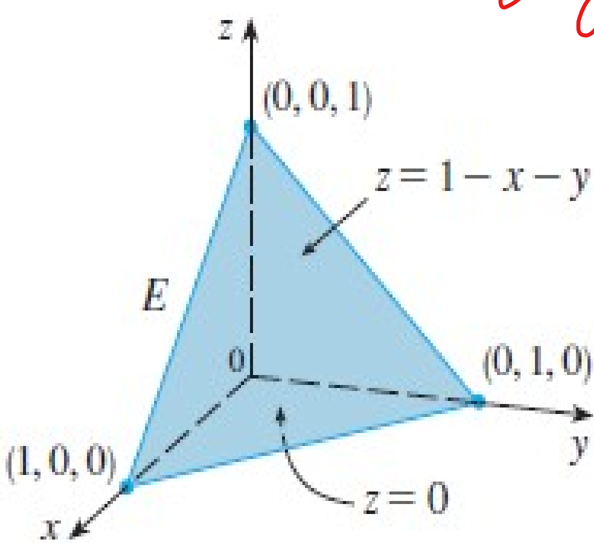


FIGURE 1

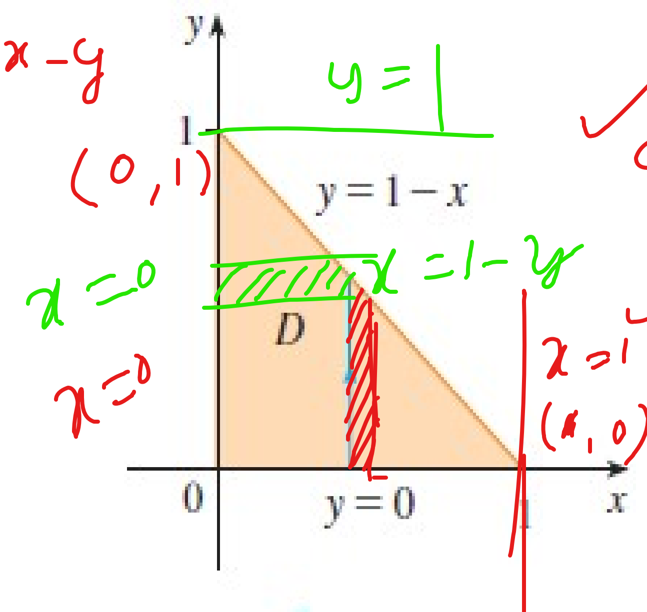


FIGURE 2

$dz\,dy\,dx$

$dz\,dx\,dy$

$0 \le y \le 1 - x, \quad 0 \le x \le 1$

boundary is the plane $x + y + z = 1$ (or $z = 1 - x - y$), so we use $u_1(x, y) = 0$ and $u_2(x, y) = 1 - x - y$ in Formula 7. Notice that the planes $x + y + z = 1$ and $z = 0$ intersect in the line $x + y = 1$ (or $y = 1 - x$) in the xy-plane. So the projection of E is the triangular region shown in Figure 6, and we have

$$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x,\ 0 \leqslant z \leqslant 1 - x - y\}$$

This description of E as a type 1 region enables us to evaluate the integral as follows:

$$\iiint_E z\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2} \right]_{z=0}^{z=1-x-y} dy\, dx$$

$$= \tfrac{1}{2} \int_0^1 \int_0^{1-x} (1 - x - y)^2\, dy\, dx = \tfrac{1}{2} \int_0^1 \left[-\frac{(1 - x - y)^3}{3} \right]_{y=0}^{y=1-x} dx$$

$$= \tfrac{1}{6} \int_0^1 (1 - x)^3\, dx = \frac{1}{6} \left[-\frac{(1 - x)^4}{4} \right]_0^1 = \frac{1}{24}$$

Problem : Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

SOLUTION The tetrahedron T and its projection D on the xy-plane are shown in Figures 1 and 2 . The lower boundary of T is the plane $z = 0$ and the upper boundary is the plane $x + 2y + z = 2$, that is, $z = 2 - x - 2y$.

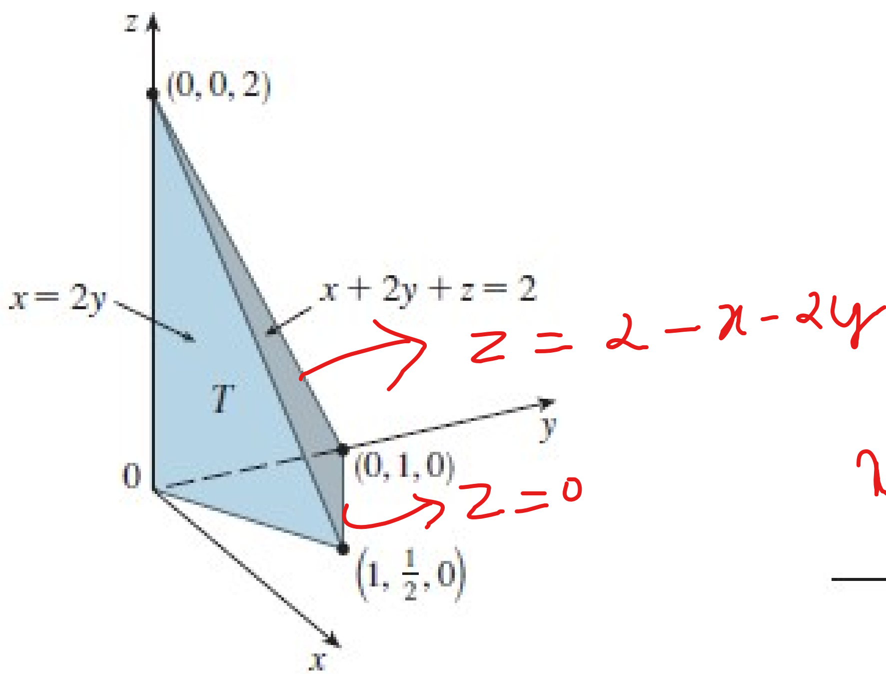


FIGURE 1

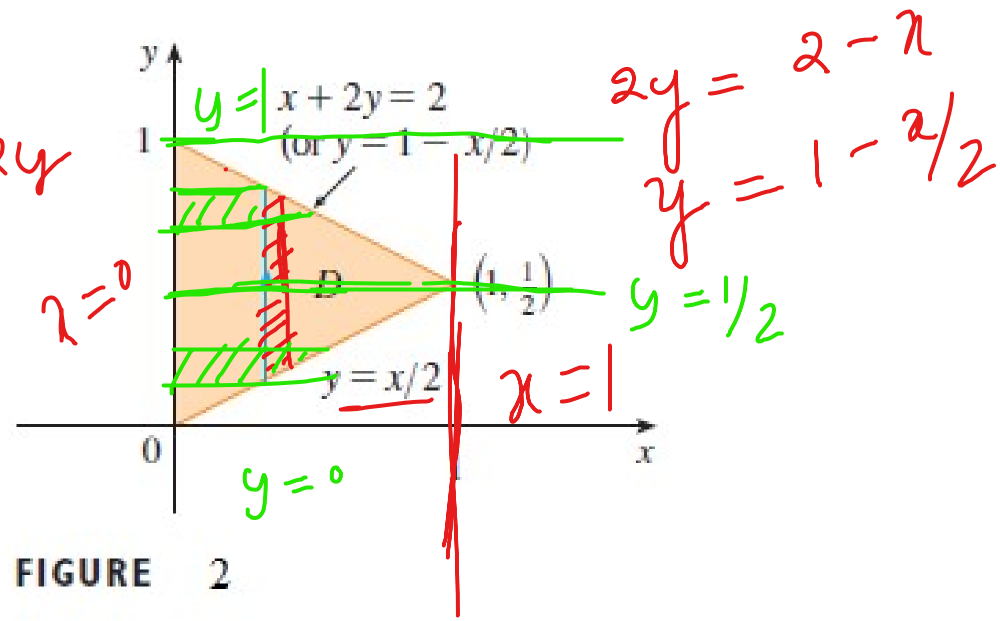


FIGURE 2

Therefore we have

$$V(T) = \iiint\limits_T dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\, dy\, dx$$

$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx = \frac{1}{3}$$

TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES

CYLINDRICAL COORDINATES

In the **cylindrical coordinate system**, a point P in three-dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P. (See Figure 1.)

To convert from cylindrical to rectangular coordinates, we use the equations

1

$$\boxed{x = r\cos\theta \qquad y = r\sin\theta \qquad z = z}$$

whereas to convert from rectangular to cylindrical coordinates, we use

$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x}$

$z = z$

2

$$\boxed{r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z}$$

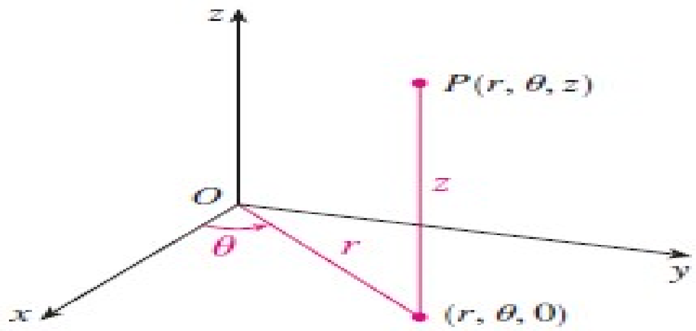
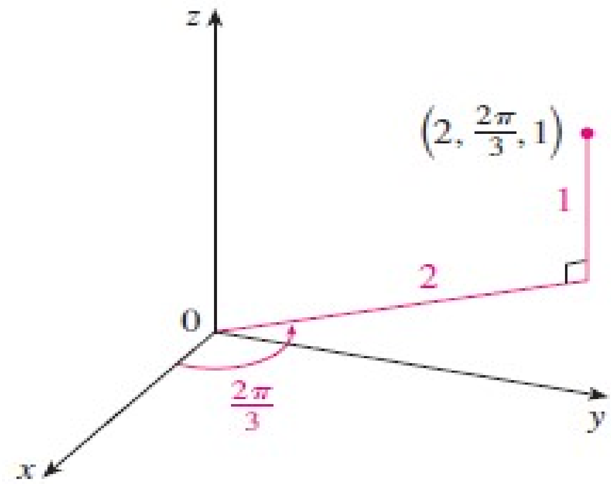


FIGURE 1
The cylindrical coordinates of a point



Problem :

(a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.
(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

SOLUTION
(a) The point with cylindrical coordinates $(2, 2\pi/3, 1)$ is plotted in Figure . From Equations 1, its rectangular coordinates are

$$x = 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1$$

$$y = 2\sin\frac{2\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$z = 1$$

Thus the point is $\left(-1, \sqrt{3}, 1\right)$ in rectangular coordinates.

(b) From Equations 2 we have

$$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$$

$$\tan\theta = \frac{-3}{3} = -1 \qquad \text{so} \qquad \theta = \frac{7\pi}{4} + 2n\pi$$

$$z = -7$$

Therefore one set of cylindrical coordinates is $\left(3\sqrt{2}, 7\pi/4, -7\right)$. Another is $\left(3\sqrt{2}, -\pi/4, -7\right)$. As with polar coordinates, there are infinitely many choices.

NOTE :-

Cylindrical coordinates are useful in problems that involve symmetry about an axis, and the z-axis is chosen to coincide with this axis of symmetry. For instance, the axis of the circular cylinder with Cartesian equation $x^2 + y^2 = c^2$ is the z-axis. In cylindrical coordinates this cylinder has the very simple equation $r = c$. (See Figure) This is the reason for the name "cylindrical" coordinates.

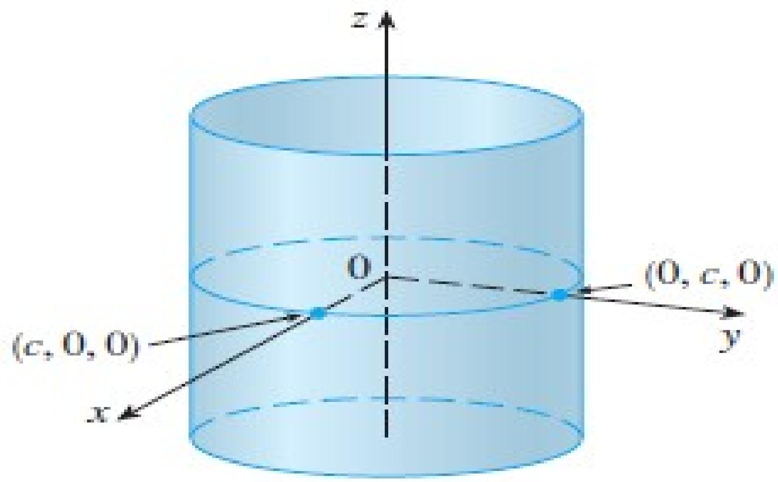


FIGURE
$r = c$, a cylinder

EVALUATING TRIPLE INTEGRALS WITH CYLINDRICAL COORDINATES

$$\underbrace{dx\,dy\,dz}_{dV} = r\,dz\,dr\,d\theta$$

$$\iiint_E f(x,y,z)\,dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=u_1(r\cos\theta,\, r\sin\theta)}^{z=u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

Problem :- Evaluate $\boxed{\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx.}$

SOLUTION This iterated integral is a triple integral over the solid region

$$E = \left\{(x, y, z) \mid -2 \leqslant x \leqslant 2,\ -\sqrt{4 - x^2} \leqslant y \leqslant \sqrt{4 - x^2},\ \sqrt{x^2 + y^2} \leqslant z \leqslant 2\right\}$$

and the projection of E onto the xy-plane is the disk $x^2 + y^2 \leqslant 4$. The lower surface of E is the cone $z = \sqrt{x^2 + y^2}$ and its upper surface is the plane $z = 2$. (See Figure)
This region has a much simpler description in cylindrical coordinates:

$$E = \left\{(r, \theta, z) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant 2,\ r \leqslant z \leqslant 2\right\}$$

Therefore, we have

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx = \iiint_E (x^2 + y^2)\, dV$$

$$= \int_0^{2\pi} \int_0^2 \int_r^2 r^2\, r\, dz\, dr\, d\theta$$

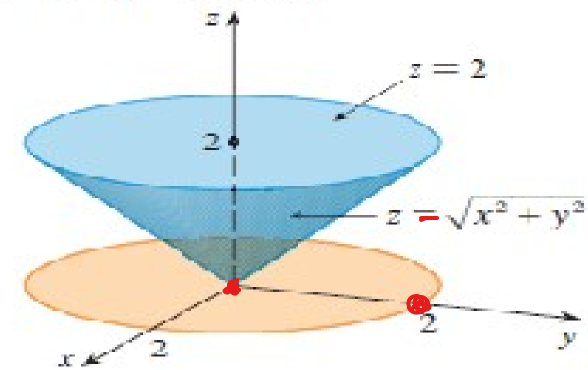
$$= \int_0^{2\pi} d\theta \int_0^2 r^3(2 - r)\, dr$$

$$= 2\pi\left[\tfrac{1}{2}r^4 - \tfrac{1}{5}r^5\right]_0^2 = \tfrac{16}{5}\pi$$

$(z)_r^2$

$r^3(2-r)$

$\left(\frac{2r^4}{4_2} - \frac{r^5}{5}\right)_0^2$



FIGURE

TRIPLE INTEGRALS IN SPHERICAL COORDINATES

Another useful coordinate system in three dimensions is the *spherical coordinate system*. It simplifies the evaluation of triple integrals over regions bounded by spheres or cones.

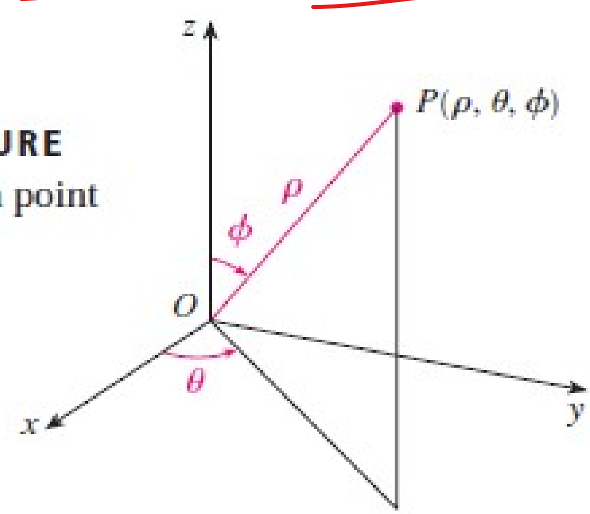
SPHERICAL COORDINATES

The **spherical coordinates** (ρ, θ, ϕ) of a point P in space are shown in Figure , where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP. Note that

$$\rho \geqslant 0 \qquad 0 \leqslant \phi \leqslant \pi$$

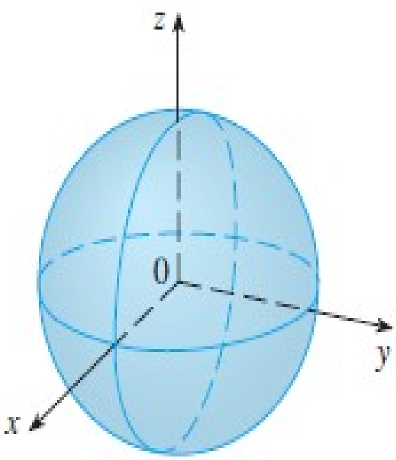
FIGURE

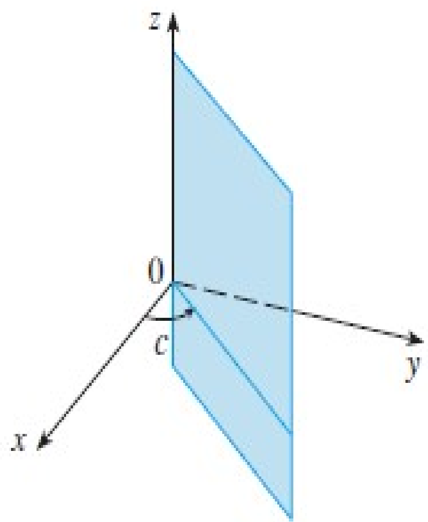
The spherical coordinates of a point

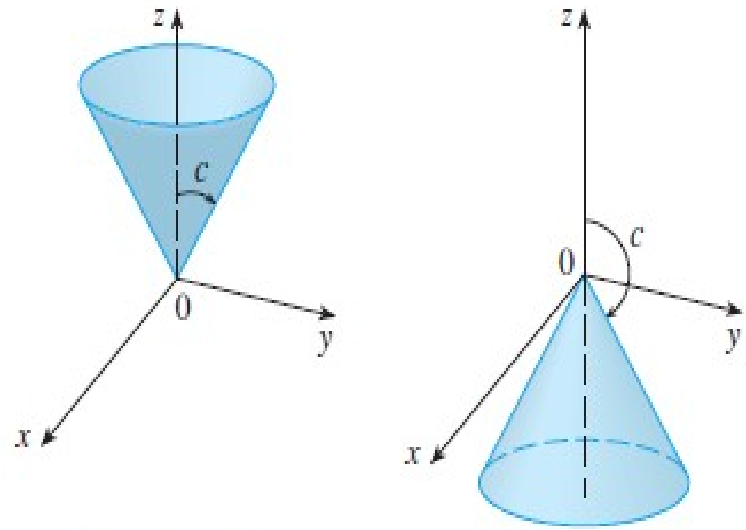


NOTE :-

The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point. For example, the sphere with center the origin and radius c has the simple equation $\rho = c$ (see Figure 1); this is the reason for the name "spherical" coordinates. The graph of the equation $\theta = c$ is a vertical half-plane (see Figure 2), and the equation $\phi = c$ represents a half-cone with the z-axis as its axis (see Figure 3).







$0 < c < \pi/2$ $\quad$ $\pi/2 < c < \pi$

FIGURE 1 $\rho = c$, a sphere **FIGURE 2** $\theta = c$, a half-plane **FIGURE 3** $\phi = c$, a half-cone

To convert from spherical to rectangular coordinates, we use the equations

$$\boxed{x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi}$$

Also, the distance formula shows that

$$\boxed{\rho^2 = x^2 + y^2 + z^2}$$

$$\iiint_E f(x, y, z)\, dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\}$$

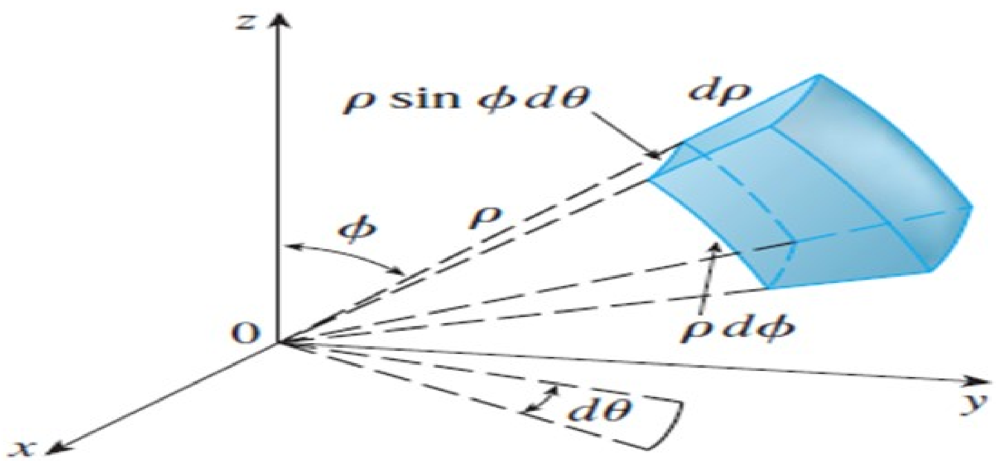


FIGURE
Volume element in spherical coordinates: $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

$dz\,dy\,dx = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

Problem:- Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where B is the unit ball:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\}$$

SOLUTION Since the boundary of B is a sphere, we use spherical coordinates:

$$B = \{(\rho, \theta, \phi) \mid 0 \leqslant \rho \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant \pi\}$$

In addition, spherical coordinates are appropriate because

$$x^2 + y^2 + z^2 = \rho^2$$

Thus (3) gives

$$\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV = \int_0^\pi \int_0^{2\pi} \int_0^1 e^{(\rho^2)^{3/2}} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \int_0^\pi \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_0^1 \rho^2 e^{\rho^3}\, d\rho$$

$$= \left[-\cos\phi\right]_0^\pi (2\pi) \left[\tfrac{1}{3}e^{\rho^3}\right]_0^1 = \tfrac{4}{3}\pi(e-1)$$

$-(\cos\phi)_0^\pi$

$-(\cos\pi - \cos 0)$

$-((-1) - 1)$

2

$\int_0^1 \rho^2 e^{\rho^3}\, d\rho$

$u = \rho^3$

$du = 3\rho^2\, d\rho$

when $\rho = 0$, $u = 0$,

$\rho = 1$, $u = 1$,

$= \frac{1}{3}\int_0^1 3\rho^2 e^{\rho^3}\, d\rho = \frac{1}{3}\int_0^1 e^u\, du$

$= \frac{1}{3}\left(e^{u}\right)_0^1$

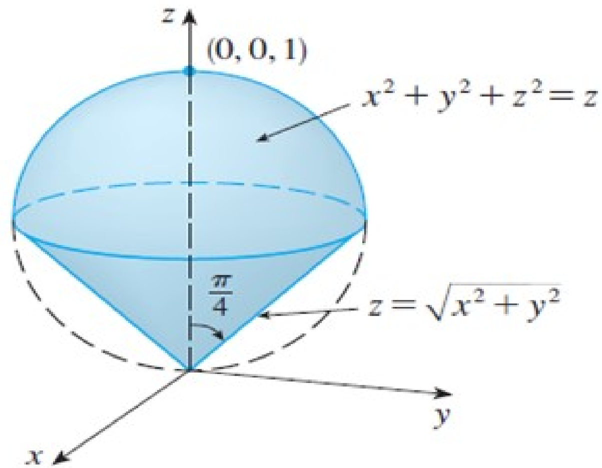
$= 1/3 (e - 1)$

Problem : Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. (See Figure)

$x^2+y^2+z^2-z=0$

$(-u,-v,-w)$

$(0,0,1/2)$



$x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$u=0, \quad v=0$

$2w=-1$

$w=-1/2$

SOLUTION Notice that the sphere passes through the origin and has center $(0, 0, \frac{1}{2})$. We write the equation of the sphere in spherical coordinates as

$$\rho^2 = \rho \cos \phi \qquad \text{or} \qquad \rho = \cos \phi$$

$\rho^2 = \rho \cos \phi$

$\rho = \cos \phi$

The equation of the cone can be written as

$$\rho \cos \phi = \sqrt{\rho^2 \sin^2\phi \, \cos^2\theta + \rho^2 \sin^2\phi \, \sin^2\theta} = \rho \sin \phi$$

This gives $\sin \phi = \cos \phi$, or $\phi = \pi/4$. Therefore the description of the solid E in spherical coordinates is

$\phi = \pi/4$

$$E = \{(\rho, \theta, \phi) \mid 0 \leqslant \theta \leqslant 2\pi, \; 0 \leqslant \phi \leqslant \pi/4, \; 0 \leqslant \rho \leqslant \cos \phi\}$$

$-\int_0^{\pi/4} -\sin\phi \cos^3\phi \, d\phi = -\int_1^{1/\sqrt{2}} u^3 \, du$

when $\phi = 0 \Rightarrow u = 1$

$\phi = \pi/4 \Rightarrow u = \frac{1}{\sqrt{2}}$

$u = \cos\phi$

$du = -\sin\phi \, d\phi$

$= -\left(\frac{u^4}{4}\right)_1^{1/\sqrt{2}}$

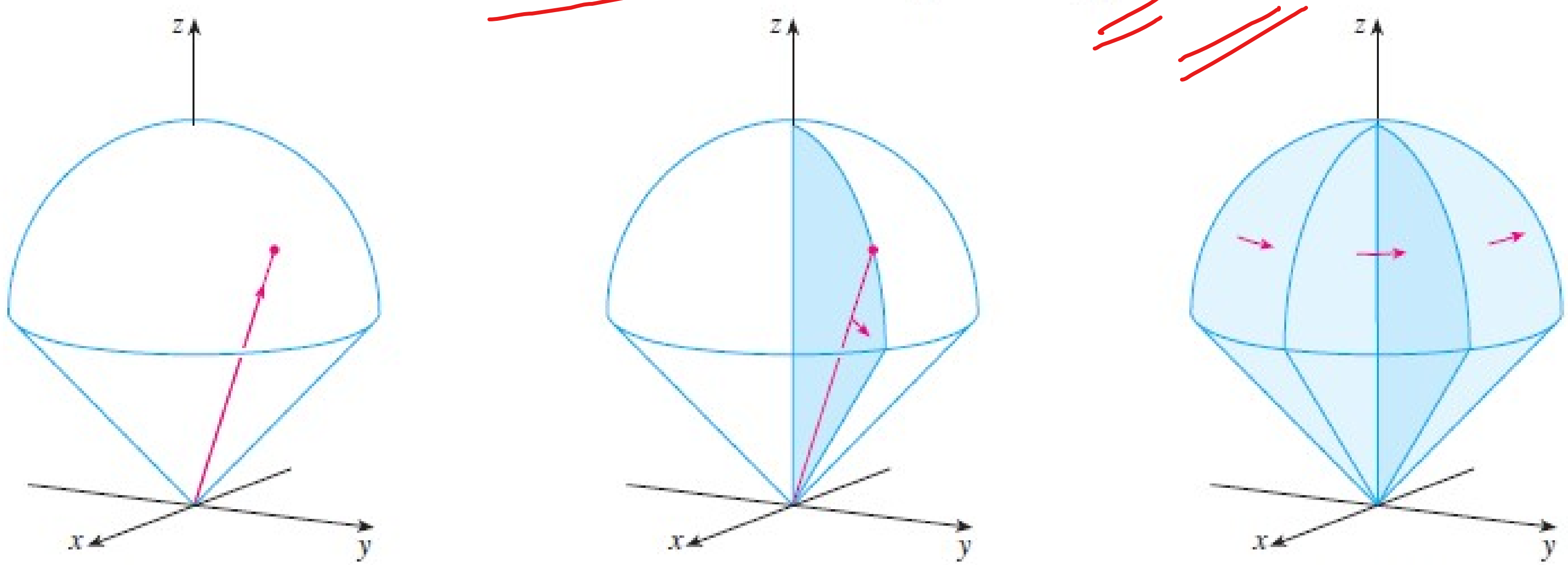
Figure shows how E is swept out if we integrate first with respect to ρ, then ϕ, and then θ. The volume of E is

$$V(E) = \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$(\theta)_0^{2\pi}$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=\cos\phi} d\phi$$

$$= \frac{2\pi}{3} \int_0^{\pi/4} \sin\phi \cos^3\phi \, d\phi = \frac{2\pi}{3}\left[-\frac{\cos^4\phi}{4}\right]_0^{\pi/4} = \frac{\pi}{8}$$



ρ varies from 0 to $\cos\phi$ while ϕ and θ are constant.

ϕ varies from 0 to $\pi/4$ while θ is constant.

θ varies from 0 to 2π.

Gamma and Beta functions

$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt$

Definition.
For x positive we define the **Gamma function** by

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\, dt.$$

t^{n-1}

This integral cannot be easily evaluated in general, therefore we first look at the Gamma function at two important points. We start with $n = 1$:

$$\Gamma(1) = \int_0^\infty t^0 e^{-t}\, dt = \int_0^\infty e^{-t}\, dt = \left[-e^{-t}\right]_0^\infty$$
$$= \lim_{t\to\infty}\left(\frac{-1}{e^t}\right) - (-1) = 0 + 1 = 1.$$

Now we look at the value at $n = 1/2$:

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t}\, dt = \left| \begin{array}{c} y = \sqrt{t} \\ 2dy = t^{-1/2}\, dt \\ t = y^2 \end{array} \right| = 2\int_0^\infty e^{-y^2}\, dy = \sqrt{\pi}.$$

Show that $\Gamma(\cancel{x}+1)=\cancel{x\Gamma(x)}$ $n\Gamma(n)$

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma(\cancel{x}+1) = \int_0^\infty t^{n} e^{-t}\,dt = \left| \begin{matrix} f = t^{n} & g' = e^{-t} \\ f' = n t^{n-1} & g = -e^{-t} \end{matrix} \right|$$

$$= \left[-t^{n} e^{-t} \right]_0^\infty - \int_0^\infty -n t^{n-1} e^{-t}\,dt = -\lim_{t\to\infty} \left(\frac{t^x}{e^t} \right) + n \int_0^\infty t^{n-1} e^{-t}\,dt$$

$$= 0 + n\Gamma(n) = n\Gamma(n).$$

Definition.

For m, n positive we define the **Beta function** by

$$\beta(\mathrm{m}, \mathrm{n}) = \int_0^1 t^{\mathrm{m}-1} (1-t)^{\mathrm{n}-1}\,dt.$$

Using the substitution $u = 1 - t$ it is easy to see that

$u = 1 - t$

$t = 1 - u$

$$B(\mathrm{m}, \mathrm{n}) = B(\mathrm{n}, \mathrm{m}).$$

$du = -dt$

$t = 0$, $u = 1$ | when $t = 1$, $u = 0$

$$= \int_0^1 u^{n-1} (1-u)^{m-1}\,du = \int_1^0 (1-u)^{m-1} u^{n-1} (-du)$$

$= \beta(n, m)$

Pr operties of beta function :

1) $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta.$

Sol :-

Put $t = \sin^2\theta$ *in the definition* , $dt = 2\sin\theta\cos\theta\, d\theta$

When $t = 0,\ \theta = 0$ *and when* $t = 1,\ \theta = \frac{\pi}{2}$

$$\beta(m,n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}.2\sin\theta\cos\theta\, d\theta$$

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta.$$

Relation between Beta and Gamma Functions :

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Sol :

$$\Gamma(m) = \int_0^\infty e^{-t} t^{m-1} dt$$

Put $t = x^2$, *to get* ,

$$\Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-1} dx$$

Similarly , $\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1} dy$

$$\therefore \quad \Gamma(m)\Gamma(n) = 4\int_0^\infty e^{-x^2} x^{2m-1} dx \,.\int_0^\infty e^{-y^2} y^{2n-1} dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dxdy \qquad (A)$$

Let $x = r\cos\theta$, $y = r\sin\theta$ *and* $dxdy = rdrd\theta$.

$t = x^2$

$dt = 2x\,dx$

when $t = 0$, $x =$

when $t \to$, $x \to \infty$

$$\Gamma(m) = \int_0^\infty e^{-x^2} (x^2)^{m-1} 2x\,dx$$

$$= 2\int_0^\infty e^{-x^2} x^{2m-1} dx$$

(r, θ)

$$Here\ \ r = 0\ \ to\ \ r = \infty\ \ and\ \ \theta = 0\ \ to\ \theta = \frac{\pi}{2}$$

$$Equation\ \ (A)\ becomes$$

$$\Gamma(m)\Gamma(n) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2} r^{2m-1} \cos^{2m-1}\theta\, r^{2n-1} \sin^{2n-1}\theta\, dr r d\theta$$

$$= 2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1}\, dr . 2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta$$

$$= \Gamma(m+n)\beta(m,n)$$

$$\therefore \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \because\ \Gamma(n) = 2\int_0^{\infty} e^{-t^2} t^{2n-1} dt.$$

$\Pr ove\ that$

$$(a)\ \iint_D x^{l-1}y^{m-1}dxdy = \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)}h^{l+m}\ \ where\ D\ is\ the\ domain\ x\ge 0, y\ge 0$$

$and\ \ x+y\le h.$

$$(b)\ \iiint_V x^{l-1}y^{m-1}z^{n-1}dxdydz = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}$$

$where\ V\ is\ the\ region\ x\ge 0, y\ge 0, z\ge 0\ and\ x+y+z\le 1.$

$Sol:-$

$(a)\ Putting\ \ x/h = X\ \ and\ \ y/h = Y, we\ see\ that\ the\ given\ \text{int}\ egral$

$$= \iint_{D'} (hX)^{l-1}(hY)^{m-1}h^2 dXdY\ \ where\ D'\ is\ the\ domain\ X\ge 0, Y\ge 0\ and\ X+Y\le 1.$$

$$= h^{l+m}\int_0^1\int_0^{1-X} X^{l-1}Y^{m-1}dYdX = h^{l+m}\int_0^1 X^{t-1}\left[\frac{Y^m}{m}\right]_0^{1-X} dX$$

$$= \frac{h^{l+m}}{m}\int_0^1 X^{l-1}(1-X)^m dX = \frac{h^{l+m}}{m}\beta(l, m+1)$$

$$= \frac{h^{l+m}}{m}\cdot\frac{\Gamma(l)\Gamma(m+1)}{\Gamma(l+m+1)} = h^{l+m}\frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)}\ \ [\therefore \Gamma(m+1)/m = \Gamma(m)]$$

(b) *Taking* $y+z \le 1-x(=h;\, say)$, *the triple* integral

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^{l-1} y^{m-1} z^{n-1}\, dz\, dy\, dx$$

$$= \int_0^1 x^{l-1} \left[\int_0^h \int_0^{h-y} y^{m-1} z^{n-1}\, dz\, dy \right] dx$$

$$= \int_0^1 x^{l-1} \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} h^{m+n}\, dx ....[By\, (a)]$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \int_0^1 x^{l-1} (1-x)^{m+n}\, dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \beta(l, m+n+1)$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \cdot \frac{\Gamma(l)\Gamma(m+n+1)}{\Gamma(l+m+n+1)} = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}.$$

$Evaluate\quad the\quad \text{int}\ egral\quad \iiint x^{l-1} y^{m-1} z^{n-1} dxdydz\quad where\quad x, y, z\ are\ all$

$positive\quad with\quad condition\quad , (x/a)^p + (y/b)^q + (z/c)^r \leq 1.$

$Sol\ :-$

$$Put\quad (x/a)^p = u\quad i.e.\ x = au^{\frac{1}{p}}\quad so\ that\quad dx = \frac{a}{p} u^{\frac{1}{p}-1} du$$

$$(y/a)^q = v\quad i.e.\ y = bv^{\frac{1}{q}}\quad so\ that\quad dy = \frac{b}{q} v^{\frac{1}{q}-1} dv$$

$$(z/c)^r = w\quad i.e.\ z = cw^{\frac{1}{r}}\quad so\ that\quad dz = \frac{c}{r} w^{\frac{1}{r}-1} dw$$

$$Then\quad \iiint x^{l-1} y^{m-1} z^{n-1} dxdydz$$

$$= \iiint (au^{\frac{1}{p}})^{t-1} (bv^{\frac{1}{q}})^{m-1} (cw^{\frac{1}{r}})^{n-1} \left(\frac{a}{p}\right) u^{\frac{1}{p}-1} \left(\frac{b}{q}\right) v^{\frac{1}{q}-1} \left(\frac{c}{r}\right) w^{\frac{1}{r}-1} dudvdw$$

$$= \frac{a^l b^m c^n}{pqr} \iiint u^{\frac{l}{p}-1} v^{\frac{m}{q}-1} w^{\frac{n}{r}-1} dudvdw\quad where\quad u + v + w \leq 1.$$

$$= \frac{a^l b^m c^n}{pqr} \frac{\Gamma(l/p)\Gamma(m/q)\Gamma(n/r)}{\Gamma(l/p + m/q + n/r + 1)}$$