PARTIAL FRACTIONS

Find $\mathcal{L}^{-1}\left\{\dfrac{3s+7}{s^2-2s-3}\right\}$.

$$\frac{3s+7}{s^2-2s-3} = \frac{3s+7}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$

Multiplying by $(s-3)(s+1)$, we obtain

$$3s + 7 = A(s+1) + B(s-3) = (A+B)s + A - 3B$$

Equating coefficients, $A+B = 3$ and $A - 3B = 7$; then $A = 4$, $B = -1$,

$$\frac{3s+7}{(s-3)(s+1)} = \frac{4}{s-3} - \frac{1}{s+1}$$

and
$$\mathcal{L}^{-1}\left\{\frac{3s+7}{(s-3)(s+1)}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$
$$= 4e^{3t} - e^{-t}$$

Find $\mathcal{L}^{-1}\left\{\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$.

$$\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{s-3}$$

$\frac{7}{2} L^{-1}\left(\frac{1}{s-3}\right)$

$\frac{7}{2} e^{3t}$

$$\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1/6}{s+1}+\frac{-4/3}{s-2}+\frac{7/2}{s-3}\right\}$$

$$= -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$$

$-\frac{4}{3} e^{2t}$

$-\frac{1}{6} L^{-1}\left(\frac{1}{s+1}\right) = -\frac{1}{6} e^{-t}$

$-\frac{4}{3} L^{-1}\left(\frac{1}{s-2}\right) =$

$$= \frac{A(s-2)^3 + B(s+1) + C(s+1)(s-2) + D(s+1)(s-2)^2}{(s+1)(s-2)^3}$$

Find $\mathcal{L}^{-1}\left\{\dfrac{5s^2-15s-11}{(s+1)(s-2)^3}\right\}$

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{(s-2)^3} + \frac{C}{(s-2)^2} + \frac{D}{s-2}$$

$$\mathcal{L}^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{-1/3}{s+1} + \frac{-7}{(s-2)^3} + \frac{4}{(s-2)^2} + \frac{1/3}{s-2}\right\}$$

$$= -\frac{1}{3}e^{-t} - \frac{7}{2}t^2 e^{2t} + 4t\,e^{2t} + \frac{1}{3}e^{2t}$$

$$-\frac{1}{3}L^{-1}\left(\frac{1}{s+1}\right) = -\frac{1}{3}e^{-t}$$

$$-7L^{-1}\left(\frac{1}{(s-2)^3}\right) = -7e^{2t}L^{-1}\left(\frac{1}{s^3}\right)$$

$$= -\frac{7}{2}e^{2t}L^{-1}\left(\frac{2!}{s^3}\right) = -\frac{7}{2}t^2e^{2t}$$

Consider

$$\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$= \frac{A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2)}{(s+1)(s-2)(s-3)}$$

i.e., $2s^2-4 = A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2)$

$s=-1$

$-2 = 12A \Rightarrow A = -\frac{2}{12} = -1/6$

$s=2$, $4 = -3B \Rightarrow B = -4/3$

$s=3$, $14 = 4C \Rightarrow C = 7/2$

Consider

$$5s^2-15s-11 = A(s-2)^3 + B(s+1) + C(s+1)(s-2)^2 + D(s+1)(s-2)$$

Let $s=2$

$$-21 = 3B \Rightarrow B = -7$$

Let $s=-1$

$$9 = -27A \Rightarrow A = -1/3$$

Let $s=0$ $\quad -11 = -8A + B - 2C + 4D$

$$-11 = -8\left(-\frac{1}{3}\right) - 7 - 2C + 4D \Rightarrow -2C + 4D = -11 - 8/3 + 7$$
$$= -20/3 \quad \text{—}(1)$$

Let $s=1$, $\quad -21 = -A + 2B - 2C + 2D$

$$= -\left(-\frac{1}{3}\right) + 2(-7) - 2C + 2D$$

$$-2C + 2D = -21 - 1/3 + 14 = -22 \quad \text{—}(2)$$

Solving (1) & (2)

$C = 4 \qquad D = 1/3$

$$3s+1 = \frac{A(s^2+1)+(Bs+C)(s-1)}{(s-1)(s^2+1)}$$

Find $\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}$

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s-1} + \frac{-2s+1}{s^2+1}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= 2e^t - 2\cos t + \sin t$$

Consider

$$3s+1 = A(s^2+1) + (Bs+C)(s-1)$$

$s=1$, $4 = 2A \Rightarrow \boxed{A=2}$

$s=0$, $1 = A - C \Rightarrow 1 = 2 - C \Rightarrow \boxed{C=1}$

$s=-1$, $-2 = 2A + (-B+C)(-2)$

$= 2A - 2(-B+C) = 2A + 2B - 2C$

$2(2) + 2B - 2(1) = -2$

$2B = -2 - 2 = -4$

$\boxed{\Rightarrow B = -2}$

$$L^{-1}\left[\frac{2}{s-1}+\frac{-2s+1}{s^2+1}\right]$$

$$= L^{-1}\left[\frac{2}{s-1}+\frac{-2s}{s^2+1}+\frac{1}{s^2+1}\right]$$

$$= 2L^{-1}\left[\frac{1}{s-1}\right]-2L^{-1}\left[\frac{s}{s^2+1}\right]+L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= 2e^{t}-2\cos t+\sin t$$

Find $\mathcal{L}^{-1}\left\{\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

Multiplying by $(s^2+2s+2)(s^2+2s+5)$,

$$\begin{aligned} s^2+2s+3 &= (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2) \\ &= (A+C)s^3 + (2A+B+2C+D)s^2 + (5A+2B+2C+2D)s + 5B + 2D \end{aligned}$$

Then $A+C=0$, $2A+B+2C+D=1$, $5A+2B+2C+2D=2$, $5B+2D=3$. Solving, $A=0$, $B=\frac{1}{3}$, $C=0$, $D=\frac{2}{3}$. Thus

$$\mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/3}{s^2+2s+2}+\frac{2/3}{s^2+2s+5}\right\}$$

$$= \frac{1}{3}e^{-t}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$$

$$+\frac{\cancel{2}}{3}\,\frac{e^{-t}}{\cancel{2}}L^{-1}\left(\frac{2}{s^2+4}\right)$$

$$= \frac{1}{3}e^{-t}\sin t + \frac{2}{3}\cdot\frac{1}{2}e^{-t}\sin 2t$$

$$= \frac{1}{3}e^{-t}\sin t + \frac{1}{3}e^{-t}\sin 2t = \frac{1}{3}e^{-t}(\sin t + \sin 2t)$$