Module : 6

Vector Differentiation

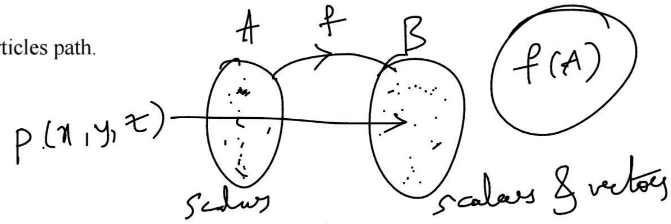
Vector differential Calculus

29 December 2020 09:55

Vector valued functions and motion in space:

When a particle moves through space during a time interval I, we think of the particle coordinates as functions defined on I, $x = f(t)$, $y = g(t)$ and $z = h(t)$, $t \epsilon I$

The points $(x, y, z) = f(t), g(t), h(t)), t\epsilon I$, make up the curve in space that we call the particles path.



A curve in space can also be represented in vector form, ~~the vector~~

$\overrightarrow{r(t)} = \overrightarrow{OP} = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ $\quad \boxed{\vec{r} = x\bar{i} + y\bar{j} + z\bar{k}}$

$\overrightarrow{r(t)} = \overrightarrow{OP}$ is the position vector.

Definition:

- Let $\overrightarrow{r(t)} = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ be a vector function with domain D, and l is a vector. We say that $\vec{r}$ has limit l at t approcaches t_0 and write $\lim_{t\to t_0} \overrightarrow{r(t)} = l$, if for every number $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $t \in D$ is $\left|\overrightarrow{r(t)} - l\right| < \varepsilon$, whenever $0 < |t - t_0| < \delta$.
- A vector function $\overrightarrow{r(t)} = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ has a derivative at t, if f, g and h are derivatives at t. The derivative is the vector

$$\overrightarrow{r'(t)} = \frac{d\vec{r}}{dt} = \lim_{\Delta t\to 0} \frac{\overrightarrow{r(t+\Delta t)} - \overrightarrow{r(t)}}{\Delta t} \longleftarrow \frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

$$= \frac{df}{dt}\vec{i} + \frac{dg}{dt}\vec{j} + \frac{dh}{dt}\vec{k} \Longrightarrow \text{tangent vector.}$$

$\boxed{\begin{array}{l}\text{As } t \to t_0 \\ f(t) \to \text{unique value}\end{array}}$

- If $\vec{r}$ is the position vector of a particle moving along a smooth curve in a space, then $v(t) = \frac{d\vec{r}}{dt}$ is the particle velocity, tangent to the curve. At any time 't', the direction of v is the particle speed, and the derivative of $\frac{dv}{dt}$ when exists, is the particles acceleration vector.

Note:

As $t \to t_0$ $\quad \boxed{\overrightarrow{r(t)} \to f}$

1. Velocity is the derivative of position vector: $\vec{v} = \frac{d\vec{r}}{dt}$ ⟵ tangent vector
2. Speed is the magnitude of the velocity: Speed $= |\vec{v}|$
3. Acceleration is the derivative of the velocity $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$
4. The unit vector $\frac{\vec{v}}{|\vec{v}|}$ is the direction of motion at time 't'.

$\longrightarrow \; t_0 \; \longleftarrow$

Problem:

Find the velocity, speed and acceleration of a particle whose motion in space is given by the position vector $\overrightarrow{r(t)} = (2cost)\vec{i} + (2sint)\vec{j} + (5cos^2t)\vec{k}$.

$\frac{d\bar{r}}{dt} \to$ tangent vector (velocity)

Properties:

1) Let ϕ be any scalar function & $\bar{a}$ be a vector function

then $\frac{d}{dt}(\phi\bar{a}) = \frac{d\phi}{dt}\bar{a} + \phi\frac{d\bar{a}}{dt}$.

2) If λ is a const, $\frac{d}{dt}(\lambda\bar{a}) = \lambda\frac{d\bar{a}}{dt}$, $\bar{a}$ is a vector function

3) $\frac{d}{dt}(\bar{a} \pm \bar{b}) = \frac{d\bar{a}}{dt} \pm \frac{d\bar{b}}{dt}$, $\bar{a}, \bar{b}$ are vector functions

4) $\frac{d}{dt}(\bar{a}\cdot\bar{b}) = \frac{d\bar{a}}{dt}\cdot\bar{b} + \bar{a}\cdot\frac{d\bar{b}}{dt}$, $\bar{a}, \bar{b}$ are vector functions

5) $\frac{d}{dt}(\bar{a}\times\bar{b}) = \frac{d\bar{a}}{dt}\times\bar{b} + \bar{a}\times\frac{d\bar{b}}{dt}$ ״

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→ Vector differential operator (∇) — def

The vector differential operator ∇ is defined as

$$\nabla \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}.$$

Gradient of a scalar point function:

Let ϕ be a scalar point function, then the gradient of a scalar point function ϕ is denoted by grad ϕ or $\nabla\phi$

$$\text{grad}\,\phi = \nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\phi$$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

properties:

If ϕ & ψ are two scalar point functions

1) $\text{grad}(\phi \pm \psi) = \text{grad}\,\phi \pm \text{grad}\,\psi$.

2) $\text{grad}(\phi\psi) = \psi\,\text{grad}\,\phi + \phi\,\text{grad}\,\psi$

3) $\text{grad}\left(\frac{\phi}{\psi}\right) = \frac{\psi\,\text{grad}\,\phi - \phi\,\text{grad}\,\psi}{\psi^2}$, $\psi \neq 0$.

4) $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, then $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

If ϕ is a scalar function, then $\phi(x,y,z)$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz.$$

$$= \left(\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}\right)\cdot\left(\bar{i}\,dx + \bar{j}\,dy + \bar{k}\,dz\right)$$

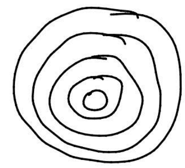
$$\boxed{d\phi = \nabla\phi\cdot d\bar{r}}$$

If $\phi(x,y,z) = c$ denotes level surfaces

→ let $\phi(x,y,z) = c$ denotes level surfaces

Ex: $\phi(x,y,z) = x^2+y^2+z^2$

$x^2+y^2+z^2 = c$ — level surface



then $d\phi = 0$

$\Rightarrow \boxed{\nabla\phi \cdot d\bar{r} = 0}$

$\Rightarrow \nabla\phi$ is $\perp^r$ to $d\bar{r}$

$d\bar{r}$ is the tangent vector.

$\Rightarrow$ $\boxed{\nabla\phi \text{ is a normal vector}}$

→ If $\bar{a}$ is a vector, then $\frac{\bar{a}}{|\bar{a}|}$ is the unit vector.

→ find a unit normal vector to the given surface $x^2y + 2xz = 4$ at the point $(2,-2,3)$.

Sol: $\phi = x^2y + 2xz - 4$

normal vector to ϕ is $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$$\nabla\phi = \bar{i}(2xy + 2z) + \bar{j}(x^2) + \bar{k}(2x)$$

$$\nabla\phi\Big|_{(2,-2,3)} = \bar{i}(-8+6) + \bar{j}(4) + \bar{k}(4)$$

$$\nabla\phi = -2\bar{i} + 4\bar{j} + 4\bar{k}$$

$$\text{unit normal vector} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-2\bar{i}+4\bar{j}+4\bar{k}}{\sqrt{4+16+16}}$$

→ Evaluate the angle b/w the normals to the surface $xy = z^2$ at the points $(4,1,2)$ & $(3,3,-3)$.

Sol:

$$\phi = xy - z^2$$

Let $\bar{n}_1$ be the normal to ϕ at $(4,1,2)$

$\bar{n}_2$ be the normal to ϕ at $(3,3,-3)$

$$\bar{n}_1 = \nabla\phi\Big|_{(4,1,2)} \quad , \quad \bar{n}_2 = \nabla\phi\Big|_{(3,3,-3)}$$

$$\bar{n}_1 = \left(\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}\right)\Big|_{(4,1,2)} \quad , \quad \bar{n}_2 = \left(\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}\right)\Big|_{(3,3,-3)}$$

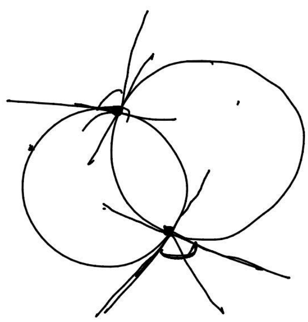
$$\bar{n}_1 = \bar{i} + 4\bar{j} - 4\bar{k} \quad , \quad \bar{n}_2 = 3\bar{i} + 3\bar{j} + 6\bar{k}$$

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|} \Rightarrow \cos\theta = \frac{3 + 12 - 24}{\sqrt{1+16+16}\sqrt{9+9+36}}$$

$$\cos\theta = \frac{-9}{\sqrt{33}\sqrt{54}}$$

$$\theta = \cos^{-1}\left(\frac{-9}{\sqrt{33}\sqrt{54}}\right).$$

→ Find the angle of intersection of the surfaces $x^2+y^2+z^2=29$ & $x^2+y^2+z^2+4x-6y-8z-47=0$ at the point $(4,-3,2)$



→ Divergence of a vector point function

Let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ be a vector point function then the divergence of a vector point function $\bar{f}$ is denoted by $\text{div}\,\bar{f}$ or $\nabla\cdot\bar{f}$

$$\nabla\cdot\bar{f} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left(f_1\bar{i} + f_2\bar{j} + f_3\bar{k}\right)$$

$$\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$$

$\nabla \cdot f = (\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z}$

$$\boxed{\operatorname{div} \bar{f} = \nabla \cdot \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.}$$

$\bar{f} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$

→ Solenoidal vector:
A vector $\bar{f}$ is said to be solenoidal vector if $\operatorname{div} \bar{f} = 0$.

→ $\operatorname{div}(\bar{f} \pm \bar{g}) = \operatorname{div} \bar{f} \pm \operatorname{div} \bar{g}$.

→ find the value of 'n' if $\bar{f} = r^n \bar{r}$ is solenoidal.

sol: $\operatorname{div} \bar{f} = 0 \Rightarrow$ then $\bar{f}$ is solenoidal.

$$\bar{f} = r^n \bar{r}.$$

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

Diff partially w.r.to $x, y, \& z$.

$$2r\frac{\partial r}{\partial x} = 2x, \quad 2r\frac{\partial r}{\partial y} = 2y, \quad 2r\frac{\partial r}{\partial z} = 2z$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r} \qquad f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$$

$$\bar{f} = \underline{\underline{r^n \bar{r}}} = r^n (x\bar{i} + y\bar{j} + z\bar{k}) = \underbrace{r^n x}_{f_1}\bar{i} + \underbrace{r^n y}_{f_2}\bar{j} + \underbrace{r^n z}_{f_3}\bar{k}$$

$$\operatorname{div} \bar{f} = \nabla \cdot \bar{f} = \frac{\partial}{\partial x}(\underbrace{r^n x}_{f_1}) + \frac{\partial}{\partial y}(\underbrace{r^n y}_{f_2}) + \frac{\partial}{\partial z}(\underbrace{r^n z}_{f_3})$$

$$= \sum \frac{\partial}{\partial x}(r^n x)$$

$$= \sum \left(r^n \cdot 1 + x\, n\, r^{n-1} \frac{\partial r}{\partial x}\right)$$

$$= \sum \left(r^n + x\, n\, r^{n-1} \cdot \frac{x}{r}\right)$$

$$= \sum \left(r^n + n r^{n-2} x^2\right)$$

$$= r^n + n r^{n-2} x^2 + r^n + n r^{n-2} y^2 + r^n + n r^{n-2} z^2$$

$$= r^n + n r^{n-2} x^2 + r^n + n r^{n-2} y^2 + r^n + n r^{n-2} z^2$$

$$= 3r^n + n r^{n-2}(x^2+y^2+z^2)$$

$$3r^n + n r^{n-2}(r^2)$$

$$3r^n + n r^n$$

$$\operatorname{div} \bar{f} = r^n(3+n)$$

$\bar{f}$ is solenoidal $\Rightarrow \operatorname{div} \bar{f} = 0$

$$\Rightarrow r^n(3+n) = 0$$

$$\rightarrow n+3 = 0$$

$$\boxed{n = -3}$$

→ find $\nabla \cdot \left(\frac{\bar{r}}{r^3}\right)$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

→ Directional derivative:

$\frac{\partial f}{\partial x}$ → rate of change of f in the direction of x

$\frac{\partial f}{\partial y}$ → " " " " " y

$\frac{\partial f}{\partial z}$ → " " " " " z

→ Let $\phi(x,y,z)$ be a scalar point function, then the derivative of ϕ in the direction of $\bar{a}$ is called as directional derivative & it is defined as $\nabla\phi \cdot \hat{e}$ where $\hat{e}$ is the unit vector in the direction of $\bar{a}$ i.e $\hat{e} = \frac{\bar{a}}{|\bar{a}|}$.

→ $\boxed{\text{Directional derivative} = \nabla\phi \cdot \hat{e}}$

→ Directional derivative of ϕ in the direction of x i

$$\nabla\phi \cdot \hat{e}$$

i is unit vector in the direction of x.

$$\nabla\phi \cdot \bar{i} = \left(\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}\right)\cdot \bar{i} = \frac{\partial \phi}{\partial x}$$

. . . . . . ative is $= |\nabla\phi|$

→ Maximum directional derivative is $= |\nabla\phi|$

Minimum " " " $= -|\nabla\phi|$

→ Find the directional derivative of $f = xy + yz + zx$ in the direction of $\bar{i} + 2\bar{j} + 2\bar{k}$ at the point $(1,2,0)$.

Sol: Directional derivative $= \nabla f \cdot \hat{e}$

$\hat{e}$ is the unit vector in the direction of $\bar{a} = \bar{i} + 2\bar{j} + 2\bar{k}$

$$\hat{e} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{1+4+4}} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{3}$$

$$\nabla f \cdot \hat{e} = \left(\bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}\right) \cdot \left(\frac{\bar{i} + 2\bar{j} + 2\bar{k}}{3}\right)$$

$$= \left(\bar{i}(y+z) + \bar{j}(x+z) + \bar{k}(x+y)\right) \cdot \left(\frac{\bar{i} + 2\bar{j} + 2\bar{k}}{3}\right)$$

$$\nabla f \cdot \hat{e}\Big|_{(1,2,0)} = \left[2\bar{i} + \bar{j} + 3\bar{k}\right] \cdot \left[\frac{\bar{i} + 2\bar{j} + 2\bar{k}}{3}\right]$$

$$= \frac{2+2+6}{3} = \frac{10}{3}$$

→ Find the directional derivative of $f = x^2 - y^2 + 2z^2$ at $P(1,2,3)$ in the direction of $\overline{PQ}$ where Q is $(5,0,4)$.

Sol: $\overline{PQ} = \overline{OQ} - \overline{OP}$

$\overline{OP} = \bar{i} + 2\bar{j} + 3\bar{k}$

$\overline{OQ} = 5\bar{i} + 4\bar{k}$

$\overline{PQ} = 4\bar{i} - 2\bar{j} + \bar{k}$

directional derivative $= \nabla f \cdot \hat{e}$, $\hat{e} = \dfrac{\overline{PQ}}{|\overline{PQ}|} = \dfrac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{16+4+1}}$

$$\nabla f\Big|_{(1,2,3)} = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}\Big|_{(1,2,3)}$$

$$= \bar{i}(2x) + \bar{j}(-2y) + \bar{k}(4z)\Big|_{(1,2,3)}$$

$$\nabla f\Big|_{(1,2,3)} = 2\bar{i} - 4\bar{j} + 12\bar{k}$$

$$\nabla f \cdot \hat{e}\Big|_{(1,2,3)} = (2\bar{i} - 4\bar{j} + 12\bar{k}) \cdot \left(\frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{21}}\right) =$$

→ Find the greatest value of the directional derivative of $f = x^2 y z^3$ at $(2,1,-1)$.

Sol Greatest directional derivative $= |\nabla f|$

→ If the temperature at any point in space is given by $t = xy + yz + zx$, find the direction in which the temperature changes most rapidly with distance from $(1,1,1)$ & determine the maximum rate of change.

Sol $t = xy + yz + zx$

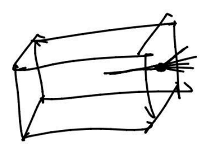
The greatest rate of increase of t at any point is given by the direction of ∇t.

Note: 1) The maximum rate of change of f occurs in the direction of ∇f & maximum rate is $|\nabla f|$

2) The minimum rate of change of f occurs in the direction of $-\nabla f$ & minimum rate is $-|\nabla f|$.

→ Divergence of a vector $= \nabla \cdot \bar{F}$

→ Let $\bar{V} = V_x\bar{i} + V_y\bar{j} + V_z\bar{k}$ be the velocity vector of any moving fluid, then $\nabla \cdot \bar{V}$ represents the rate of change of flow per unit volume at the point.



→ If $\bar{V}$ represents electric flux, then the amount of flux diverging per unit volume is

→ If $\bar{v}$ [illegible] amount of flux diverging per unit volume is $\nabla \cdot \bar{v}$.

→ Show that $\bar{u} = (2x^2+8xy^2z)\bar{i} + (3x^3y-3xy)\bar{j} - (4y^2z^2+2x^3z^3)\bar{k}$ is not solenoidal.

sol:

→ Curl of a vector

Let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ be a continuously differentiable vector function, then curl of of $\bar{f}$ is denoted by curl $\bar{f}$ (or) $\nabla \times \bar{f}$ & it is defined as

$$\text{curl } \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

→ Curl represents the angular rotation of the vector.

If $\bar{\omega}$ be an angular velocity about a fixed axis & $\bar{v}$ is the velocity at any point $P(x,y,z)$, then

$$\boxed{\bar{\omega} = \frac{1}{2} \text{curl } \bar{v}}$$

The angular velocity of rotation at any point is equal to half the curl of velocity vector.

→ Irrotational vector:

$\bar{f}$ is said to be irrotational vector if $\boxed{\text{curl } \bar{f} = 0}$

→ Note: If curl $\bar{f} = 0$ then $\exists$ a scalar function $\phi \ni$ $\bar{f} = \nabla\phi$.

$$\text{curl } \bar{f} = \nabla \times \bar{f} = \nabla \times \nabla\phi$$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \end{vmatrix}$$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = 0$$

→ If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, then p.T $r^n \bar{r}$ is irrotational.

Sol: $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$, $r^2 = x^2+y^2+z^2$.

$\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$r^n \bar{r} = r^n(x\bar{i} + y\bar{j} + z\bar{k}) = xr^n\bar{i} + yr^n\bar{j} + zr^n\bar{k}$

$$\text{curl } r^n\bar{r} = \nabla \times (r^n \bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xr^n & yr^n & zr^n \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(zr^n) - \frac{\partial}{\partial z}(yr^n)\right] - \bar{j}\left[\frac{\partial}{\partial x}(zr^n) - \frac{\partial}{\partial z}(xr^n)\right] + \bar{k}\left[\frac{\partial}{\partial x}(yr^n) - \frac{\partial}{\partial y}(xr^n)\right]$$

$$= \bar{i}\left[znr^{n-1}\frac{\partial r}{\partial y} - ynr^{n-1}\frac{\partial r}{\partial z}\right] + \bar{j}\left[znr^{n-1}\frac{\partial r}{\partial x} - xnr^{n-1}\frac{\partial r}{\partial z}\right] + \bar{k}\left[ynr^{n-1}\frac{\partial r}{\partial x} - xnr^{n-1}\frac{\partial r}{\partial y}\right]$$

$$= \bar{i}\left[znr^{n-1}\frac{y}{r} - ynr^{n-1}\frac{z}{r}\right] - \bar{j}\left[znr^{n-1}\frac{x}{r} - xnr^{n-1}\frac{z}{r}\right] + \bar{k}\left[ynr^{n-1}\frac{x}{r} - xnr^{n-1}\frac{y}{r}\right]$$

$\nabla \times (r^n \bar{r}) = \bar{0}$

$\Rightarrow r^n \bar{r}$ is an irrotational vector.

→ If $\bar{F} = (x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}$, then show that $\bar{F} \cdot \text{curl } \bar{F} = 0$

→ show that the vector $(x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$ is
... potential.

→ show that the vector $(x^2-yz)i + (y^2-zx)j + (z^2-xy)k$ is irrotational and find its scalar potential.

Sol: $\bar{F} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$$\text{curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix} = \bar{i}(-x+x) - \bar{j}(-y+y) + \bar{k}(-z+z)$$

$$= \bar{0}$$

curl $\bar{F} = 0 \Rightarrow \bar{F}$ is irrotational.

⇒ If curl $\bar{F} = 0$ then ∃ a scalar point function ϕ →

$\underline{\bar{F} = \nabla\phi}$

$$(x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = x^2-yz, \quad \frac{\partial\phi}{\partial y} = y^2-zx, \quad \frac{\partial\phi}{\partial z} = z^2-xy. \quad \text{—— (1)}$$

$$\frac{\partial\phi}{\partial x} = x^2-yz$$

Integrate w.r.to 'x'

$$\phi = \frac{x^3}{3} - xyz + f(y,z) \quad \text{—— (2)}$$ where $f(y,z)$ is an integration constant.

Diff partially w.r.to 'y'

$$\frac{\partial\phi}{\partial y} = 0 - xz + \frac{\partial}{\partial y}f(y,z)$$

from (1) ⇒ $y^2 - \cancel{zx} = -\cancel{xz} + \frac{\partial}{\partial y}f(y,z)$

$$\frac{\partial}{\partial y}f(y,z) = y^2$$

Integrate w.r.to 'y'

$f(y,z) = \frac{y^3}{3} + g(z)$, where $g(z)$ is integration constant

(2) ⇒ $$\phi = \frac{x^3}{3} - xyz + \frac{y^3}{3} + g(z) \quad \text{—— (3)}$$

Diff partially w.r.to 'z'

$$\frac{\partial\phi}{} = 0 - xy + 0 + \frac{\partial}{\partial z}g(z)$$

$$\frac{\partial\phi}{\partial z} = 0 - xy + 0 + \frac{\partial}{\partial z} g(z)$$

from ① $z^2 - \cancel{xy} = -\cancel{xy} + \frac{\partial}{\partial z} g(z)$

$$\frac{d}{dz} g(z) = z^2$$

Integrate w.r.to 'z'

$$g(z) = \frac{z^3}{3} + c$$

③ ⇒ $\boxed{\phi = \frac{x^3}{3} - xyz + \frac{y^3}{3} + \frac{z^3}{3} + c.}$

→ find the constants a, b, c so that the vector $\bar{F} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$ is irrotational. Also find the scalar point function.

Ans $a = 4,\ b = 2,\ c = -1$

$$\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4zx + c.$$

→ Operators:

1) $\nabla \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

2) $\bar{a}\cdot\nabla \equiv (\bar{a}\cdot\bar{i})\frac{\partial}{\partial x} + (\bar{a}\cdot\bar{j})\frac{\partial}{\partial y} + (\bar{a}\cdot\bar{k})\frac{\partial}{\partial z}$.

If ϕ is scalar function & $\bar{f}$ is a vector function.

$$(\bar{a}\cdot\nabla)\phi = (\bar{a}\cdot\bar{i})\frac{\partial\phi}{\partial x} + (\bar{a}\cdot\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}\cdot\bar{k})\frac{\partial\phi}{\partial z}$$

3) $\bar{a}\times\nabla \equiv (\bar{a}\times\bar{i})\frac{\partial}{\partial x} + (\bar{a}\times\bar{j})\frac{\partial}{\partial y} + (\bar{a}\times\bar{k})\frac{\partial}{\partial z}$.

$$(\bar{a}\times\nabla)\phi = (\bar{a}\times\bar{i})\frac{\partial\phi}{\partial x} + (\bar{a}\times\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}\times\bar{k})\frac{\partial\phi}{\partial z}$$

$$(\bar{a}\times\nabla)\cdot\bar{f} = (\bar{a}\times\bar{i})\cdot\frac{\partial\bar{f}}{\partial x} + (\bar{a}\times\bar{j})\cdot\frac{\partial\bar{f}}{\partial y} + (\bar{a}\times\bar{k})\cdot\frac{\partial\bar{f}}{\partial z}.$$

$$(\bar{a}\times\nabla)\times\bar{f} = (\bar{a}\times\bar{i})\times\frac{\partial\bar{f}}{\partial x} + (\bar{a}\times\bar{j})\times\frac{\partial\bar{f}}{\partial y} + (\bar{a}\times\bar{k})\times\frac{\partial\bar{f}}{\partial z}$$

→ $\nabla\cdot = \bar{i}\cdot\frac{\partial}{\partial x} + \bar{j}\cdot\frac{\partial}{\partial y} + \bar{k}\cdot\frac{\partial}{\partial z}$ (divergence)

4) $\nabla\cdot \equiv \bar{i}\cdot\frac{\partial}{\partial x} + \bar{j}\cdot\frac{\partial}{\partial y} + \bar{k}\cdot\frac{\partial}{\partial z}$ (divergence)

$\bar{f}$ is vector function

$$\nabla\cdot\bar{f} = \bar{i}\cdot\frac{\partial \bar{f}}{\partial x} + \bar{j}\cdot\frac{\partial \bar{f}}{\partial y} + \bar{k}\cdot\frac{\partial \bar{f}}{\partial z}.$$

5) $\nabla\times \equiv \bar{i}\times\frac{\partial}{\partial x} + \bar{j}\times\frac{\partial}{\partial y} + \bar{k}\times\frac{\partial}{\partial z}$ (curl)

$$\nabla\times\bar{f} = \bar{i}\times\frac{\partial \bar{f}}{\partial x} + \bar{j}\times\frac{\partial \bar{f}}{\partial y} + \bar{k}\times\frac{\partial \bar{f}}{\partial z}$$

→ Laplacian operator: (∇^2)

$$\nabla^2 = \nabla\cdot\nabla = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)$$

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.}$$

→ P.T $\operatorname{div}(\operatorname{grad} r^m) = m(m+1)r^{m-2}$, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

sol:
$$\begin{aligned}
\operatorname{div}(\operatorname{grad} r^m) &= \nabla\cdot(\nabla r^m)\\
&= \nabla^2 r^m\\
&= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^m
\end{aligned}$$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow r = |\bar{r}|$, $r^2 = x^2 + y^2 + z^2$

$$\frac{\partial r}{\partial x} = \frac{x}{r},\quad \frac{\partial r}{\partial y} = \frac{y}{r},\quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\begin{aligned}
\operatorname{div}(\operatorname{grad} r^m) &= \sum \frac{\partial^2}{\partial x^2} r^m\\
&= \sum \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} r^m\right) = \sum \frac{\partial}{\partial x}\left[m r^{m-1}\frac{\partial r}{\partial x}\right]\\
&= \sum \frac{\partial}{\partial x}\left[m r^{m-1}\cdot\frac{x}{r}\right]\\
&= \sum \frac{\partial}{\partial x}\left[m r^{m-2}\cdot x\right]\\
&= \sum m\left(r^{m-2}\cdot 1 + x\cdot(m-2)r^{m-3}\frac{\partial r}{\partial x}\right)\\
&= \sum m\left(r^{m-2} + x(m-2)r^{m-3}\frac{x}{r}\right)\\
&= \sum m\left(r^{m-2} + (m-2)r^{m-4}x^2\right)
\end{aligned}$$

$= \sum m (r^{m-2} + (m-2) r^{m-4} x^2)$

$= m r^{m-2} + m(m-2) r^{m-4} x^2 + m r^{m-2} + m(m-2) r^{m-4} y^2$

$+ m r^{m-2} + m(m-2) r^{m-4} z^2.$

$= 3m r^{m-2} + m(m-2) r^{m-4} [x^2 + y^2 + z^2]$

$3m r^{m-2} + m(m-2) r^{m-4} r^2$

$3m r^{m-2} + m(m-2) r^{m-2}$

$r^{m-2} m [3 + m - 2] = m(m+1) r^{m-2}.$

→ find $\nabla^2 (\log r)$

→ If $\bar{a}$ is a vector differentiable function & ϕ is a scalar function, then

i) P.T $\operatorname{div}(\phi \bar{a}) = (\operatorname{grad} \phi) \cdot \bar{a} + \phi \operatorname{div} \bar{a}$

(or). $\nabla \cdot (\phi \bar{a}) = (\nabla \phi) \cdot \bar{a} + \phi (\nabla \cdot \bar{a})$.

$\nabla \cdot \equiv \bar{i} \cdot \frac{\partial}{\partial x} + \bar{j} \cdot \frac{\partial}{\partial y} + \bar{k} \cdot \frac{\partial}{\partial z}$

$\equiv \sum \bar{i} \cdot \frac{\partial}{\partial x}$

sol: $\operatorname{div}(\phi(\bar{a})) = \nabla \cdot (\phi \bar{a}) = \sum \bar{i} \cdot \frac{\partial}{\partial x} (\phi \bar{a})$

$= \sum \bar{i} \cdot \left[\frac{\partial \phi}{\partial x} \bar{a} + \phi \frac{\partial \bar{a}}{\partial x} \right]$

$= \sum \left(\bar{i} \cdot \frac{\partial \phi}{\partial x} \bar{a} \right) + \sum \left(\bar{i} \cdot \phi \frac{\partial \bar{a}}{\partial x} \right)$

$= \sum \left(\bar{i} \frac{\partial \phi}{\partial x} \right) \cdot \bar{a} + \sum \left(\bar{i} \cdot \frac{\partial \bar{a}}{\partial x} \right) \phi$

$= \nabla \phi \cdot \bar{a} + (\nabla \cdot \bar{a}) \phi$

→ 2) $\operatorname{curl}(\phi \bar{a}) = \nabla \times (\phi \bar{a}) = (\operatorname{grad} \phi) \times \bar{a} + \phi \operatorname{curl} \bar{a}$

3) $\operatorname{grad}(\bar{a} \cdot \bar{b}) = (\bar{b} \cdot \nabla) \bar{a} + (\bar{a} \cdot \nabla) \bar{b} + \bar{b} \times \operatorname{curl} \bar{a} + \bar{a} \times \operatorname{curl} \bar{b}$.

4) $\nabla \cdot (\bar{a} \times \bar{b}) = \bar{b} \cdot (\nabla \times \bar{a}) - \bar{a} \cdot (\nabla \times \bar{b})$

5) $\text{curl}\,(\bar{a} \times \bar{b}) = \bar{a}\,\text{div}\,\bar{b} - \bar{b}\,\text{div}\,\bar{a}$

6) $\text{curl}[\text{grad}\,\phi] = 0$

7) $\text{div}[\text{curl}\,\bar{f}] = 0$ $\qquad \bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$