Module : 4

Applications of Multivariable Calculus

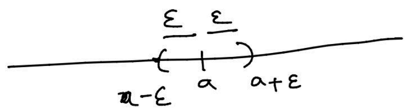
Taylor series

23 November 2020 12:59

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$ is Taylor series Expansion of $f(x) = e^x$ in powers of 'x' or about $x = 0$

Taylor Series Expansion of a function $f(x)$ about $x = a$ or in powers of $x - a$:

$x = a$



$(a-\varepsilon, a+\varepsilon)$

for $\varepsilon > 0$, $(a-\varepsilon, a+\varepsilon)$ is called as neighbourhood of the point $x = a$

$f(a)$ ✓

→ Taylor series Expansion of $f(x)$ about $x = a$ is used to find the functional values in the neighbourhood of $x = a$

→ If $f(x)$ is continuous and has successive derivatives at $x = a$, then the Taylor series Expansion of $f(x)$ about the point $x = a$ (or) in powers of '$x - a$' is

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots\cdots$$

Ex: Taylor series Expansion of $f(x) = e^x$ about $x = 0$

$f(x) = e^x$, $a = 0$

$f(a) = 1$, $f'(x) = e^x$, $f'(a) = 1$

$f''(x) = e^x$, $f''(a) = 1$

$f'''(x) = e^x$, $f'''(a) = 1$

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots\cdots$$

$$e^x = 1 + (x-0)(1) + \frac{(x-0)^2}{2!}(1) + \cdots\cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$$

→ find Taylor series Expansion of $f(x) = \sin x$, $\cos x$, $\sinh x$, $\cosh x$

→ find Taylor series Expansion of $f(x) =$ sin x, cos x, ...

about $x=0$

→ obtain the Taylor series expansion of $f(x) = \log_e(1+x)$ about $x=0$

Sol: $f(x) = \log(1+x)$, $a = 0$

$f(0) = 0$

$f'(x) = \frac{1}{1+x}$, $f''(x) = -\frac{1}{(1+x)^2}$, $f'''(x) = \frac{2}{(1+x)^3}$, ...

$f'(0) = 1$, $f''(0) = -1$, $f'''(0) = 2$, ...

Taylor Series Expansion of $f(x) = \log(1+x)$ about $x=0$ is

$$f(x) = f(0) + (x-0)f'(0) + \frac{(x-0)^2}{2!}f''(0) + \frac{(x-0)^3}{3!}f'''(0) + \cdots\cdots$$

$$\log(1+x) = 0 + x + \frac{x^2}{2}(-1) + \frac{x^3}{6}(2) + \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots$$

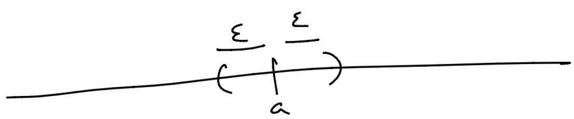
→ obtain the Taylor series Expansion of $\sin 2x$ about $x = \frac{\pi}{4}$

Sol: $\sin 2x = 1 - \left(x - \frac{\pi}{4}\right)^2 \cdot \frac{1}{2!}(4) + 0 + \frac{16\left(x-\frac{\pi}{4}\right)^4}{4!} + \cdots\cdots$

→ $f(x) = e^{x \sin x}$ in powers of 'x'.

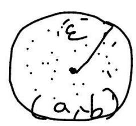
Sol: $f(x) = 1 + x\sin x + \frac{(x \sin x)^2}{2!} + \cdots\cdots$

$$f(x) = 1 + x\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] + \frac{x^2}{2}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]^2 + \cdots$$



$(a-\varepsilon, a+\varepsilon) \to x=a$

$(x-a)^2 + (y-b)^2 \leq \varepsilon^2$ is the



neighbourhood of the point (a,b)

Taylor Series Expansion of $f(x,y)$ about $x=a$ & $y=b$ (or) is

Taylor Series Expansion of $f(x,y)$ about $x=a$ & $y=b$ in powers of $(x-a)$ & $(y-b)$

If $f(x,y)$ possess continuous partial derivatives of n^{th} order in any neighbourhood of the point (a,b), then, the Taylor series Expansion of $f(x,y)$ about $x=a$ & $y=b$ is

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots\cdots$$

→ Expand $f(x,y) = e^x \sin y$ in powers of 'x' & 'y'

Sol: $a=0,\ b=0, \quad f(x,y) = e^x \sin y$

$f(a,b) = f(0,0) = 0$

$f_x = e^x \sin y, \quad f_x(a,b) = f_x(0,0) = 0$

$f_{xx} = e^x \sin y, \quad f_{xx}(a,b) = f_{xx}(0,0) = 0$

$f_y = e^x \cos y, \quad f_y(a,b) = f_y(0,0) = 1$

$f_{yy} = -e^x \sin y, \quad f_{yy}(a,b) = f_{yy}(0,0) = 0$

$f_x = \frac{\partial f}{\partial x}$

$f_y = \frac{\partial f}{\partial y}$

$f_{xx} = \frac{\partial^2 f}{\partial x^2}$

$f_{yy} = \frac{\partial^2 f}{\partial y^2}$

$$f_{xy} = \frac{\partial f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$
$$= (f_y)_x$$
$$= (e^x \cos y)_x$$
$$f_{xy} = e^x \cos y$$

$f_{xy}(a,b) = f_{xy}(0,0) = 1$

Taylor series of $f(x,y)$ in powers of $x-a$, & $y-b$ is

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)\right.$$

$$+2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\Big] + \cdots$$

$$e^x \sin y = 0 + x(0) + y(1) + \frac{1}{2!}\Big[x^2(0) + 2(x)(y)(1) + y^2(0)\Big] + \cdots$$

$$e^x \sin y = y + xy + \cdots$$

$\rightarrow$ $$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \frac{1}{2!}\Big[(x-a)^2 f_{xx}(a,b)$$
$$+ 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\Big] + \frac{1}{3!}\Big[(x-a)^3 f_{xxx}(a,b) +$$
$$3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\Big] + \cdots$$

$\rightarrow$ Expand $f(x,y) = e^x \log(1+y)$ in terms of x & y upto 3rd degree using Taylor series

Sol $a = 0$, $b = 0$

$f(x,y) = e^x \log(1+y)$, $f(0,0) = 0$

$f_x = e^x \log(1+y)$, $f_x(0,0) = 0$

$f_{xx} = e^x \log(1+y)$, $f_{xx}(0,0) = 0$

$f_{xxx} = e^x \log(1+y)$, $f_{xxx}(0,0) = 0$

$f_y = \dfrac{e^x}{1+y}$, $f_y(0,0) = 1$

$f_{yy} = \dfrac{-e^x}{(1+y)^2}$, $f_{yy}(0,0) = -1$

$f_{yyy} = \dfrac{2e^x}{(1+y)^3}$, $f_{yyy}(0,0) = 2$

$f_{xy} = (f_y)_x = \left(\dfrac{e^x}{1+y}\right)_x = \dfrac{e^x}{1+y}$, $f_{xy}(0,0) = 1$

$$f_{xxy} = \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = (f_{xy})_x = \left(\frac{e^x}{1+y}\right)_x = \frac{e^x}{1+y}$$

$$f_{xxy}(0,0) = 1$$

$$f_{xyy} = \frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) = (f_{yy})_x = \left[\frac{-e^x}{(1+y)^2}\right]_x$$

$$f_{xyy} = -\frac{e^x}{(1+y)^2}, \quad f_{xyy}(0,0) = -1$$

$$f(x,y) = y + \frac{1}{2!}(2xy - y^2) + \frac{1}{3!}(3x^2y - 3xy^2 + 2y^3)$$

$$= y + xy - \frac{y^2}{2} + \frac{x^2y}{2} - \frac{xy^2}{2} + \frac{y^3}{3}$$

→ $f(x,y) = e^y \log(1+x)$ in powers of 'x' & 'y'

→ $f(x,y) = e^{xy}$ in the neighbourhood of $(1,1)$

→ Maxima & Minima of a function of two variables :

temperature — u at 'p' (x,y,z) $\quad u(x,y,z)$

u may depend on its position or sometimes on another factor i.e time also.

$u(x,y,z,t)$

→ Def: let $f(x,y)$ be a function of two variables

→ $f(x,y)$ is said to have maximum or minimum at $x=a$, $y=b$ if $f(a,b) > f(a+h, b+k)$ (or) $f(a,b) < f(a+h, b+k)$ respectively where h & k are small values.

→ Extreme value: $f(a,b)$ is said to be an extreme value, if it is a maximum value or minimum value.

→ [illegible] conditions for $f(x,y)$ to have a maximum or it is a maximum value or minimum.

→ The necessary conditions for $f(x,y)$ to have a maximum or minimum at (a,b) are $f_x(a,b)=0$, $f_y(a,b)=0$

→ Sufficient conditions:

Let $f_x(a,b)=0$, $f_y(a,b)=0$, $\frac{\partial^2 f}{\partial x^2}(a,b) = f_{xx}(a,b) = l$

$\frac{\partial^2 f}{\partial x \partial y}(a,b) = f_{xy}(a,b) = m$, $\frac{\partial^2 f}{\partial y^2}(a,b) = f_{yy}(a,b) = n$

Then

(i) $f(a,b)$ is a maximum value if $ln-m^2>0$ & $l<0$

(ii) $f(a,b)$ is a minimum value if $ln-m^2>0$ & $l>0$

(iii) $f(a,b)$ is not an extreme value if $ln-m^2<0$

(iv) If $ln-m^2=0$, then $f(x,y)$ fails to have max or min value or needs further investigation.

→ Working rule:

Let $f(x,y)$ be given function

1) Find f_x and f_y and equate to zero

$f_x=0$, $f_y=0$

Solve these eqns for x & y

Let (a_1,b_1), (a_2,b_2), ... (a_n,b_n) are the pair of values

2) Find $l = f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $m = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$, $n = \frac{\partial^2 f}{\partial y^2} = f_{yy}$

3) i) If $ln-m^2>0$ & $l<0$ at (a_1,b_1) then (a_1,b_1) is point of max. & $f(a_1,b_1)$ is max value

ii) If $ln-m^2>0$ & $l>0$ at (a_1,b_1) then (a_1,b_1) is point of min & $f(a_1,b_1)$ is min value.

iii) If $ln-m^2<0$ at (a_1,b_1) then (a_1,b_1) is saddle point, i.e there is no max or min at (a_1,b_1)

iv) If $ln-m^2=0$ at (a_1,b_1), then no conclusion can be drawn there is no max or min at ....

iv) If $ln-m^2=0$ at (a_1,b_1), then no conclusion can be drawn

Similarly, we check the above 4 steps to the remaining pair of points (a_2,b_2), ... (a_n,b_n)

→ Find the relative maximum and minimum of the following functions.

1) $f(x,y) = x^3+y^3-3axy$

Sol: $f_x = 3x^2-3ay$, $f_y = 3y^2-3ax$

Now $f_x=0$, $f_y=0$

$3x^2-3ay=0$, $3y^2-3ax=0$

$x^2=ay$, $y^2=ax$

$y=\frac{x^2}{a}$

$y^2=ax \Rightarrow \left(\frac{x^2}{a}\right)^2=ax \Rightarrow x^4-a^3x=0$

$\Rightarrow x(x^3-a^3)=0$

$\Rightarrow x=0, x=a$

$x=0 \Rightarrow y=\frac{x^2}{a} \Rightarrow y=0 \Rightarrow (0,0)$

$x=a \Rightarrow y=\frac{a^2}{a} \Rightarrow y=a \Rightarrow (a,a)$

$l = f_{xx} = 6x$, $m=\frac{\partial^2 f}{\partial x \partial y} = -3a$, $n = f_{yy} = 6y$

$ln-m^2 = (6x)(6y)-(-3a)^2 = 36xy-9a^2$

At $(0,0)$, $ln-m^2 = -9a^2<0$, $l=6x=0$

$\therefore$ $(0,0)$ is a saddle point since $ln-m^2<0$ at $(0,0)$

At (a,a), $ln-m^2 = 36(a)(a) - 9a^2 = 27a^2 > 0$

$l = 6x = 6a > 0$ if $a > 0$

$6a < 0$ if $a < 0$

$\Rightarrow l > 0$ if $a > 0$ & $l < 0$ if $a < 0$

$\therefore$ If $a > 0$, f has minimum at (a,a) & min value is $f(a,a) = a^3 + a^3 - 3a^3 = -a^3$

If $a < 0$, f has maximum at (a,a) & max value is $f(a,a) = a^3 + a^3 - 3a^3 = -a^3$

$\rightarrow f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

Sol $f_x = 3x^2 + 3y^2 - 6x$

$f_y = 6xy - 6y$

$f_x = 0$, $f_y = 0$

$\Rightarrow 3x^2 + 3y^2 - 6x = 0 \Rightarrow x^2 + y^2 - 2x = 0$

$\Rightarrow 6xy - 6y = 0 \Rightarrow y(x-1) = 0$

$\Rightarrow y = 0, x = 1$

$y = 0 \Rightarrow x^2 + y^2 - 2x = 0 \Rightarrow x^2 - 2x = 0 \Rightarrow x = 0, 2$

$(0,0)$ $(2,0)$

$x = 1 \Rightarrow x^2 + y^2 - 2x = 0 \Rightarrow 1 + y^2 - 2 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

$(1,1)$, $(1,-1)$

$l = \frac{\partial^2 f}{\partial x^2} = 6x - 6$, $m = 6y$, $n = 6x - 6$

At $(0,0)$, $ln - m^2 = (6x-6)^2 - 3y^2 = 36 > 0$

$l = 6x - 6 = -6 < 0$

At $(0,0)$, $ln - m^2 > 0$, $l < 0$, Hence $(0,0)$ is point of maximum

At $(0,0)$, $ln-m^2>0$, $l<0$, Hence $(0,0)$ is point of maximum

& max value is $f(0,0) = 4$

At $(2,0)$, $ln-m^2 = (6(2)-6)^2-3(0)^2 = 36>0$

$l = 6x-6 = 12-6>0$

At $(2,0)$, $ln-m^2>0$ & $l>0$, Hence $(2,0)$ is point of minimum

& min value is $f(2,0) = 0$

At $(1,1)$, $ln-m^2 = -36<0$

$\therefore$ $(1,1)$ is a saddle point.

At $(1,-1)$, $ln-m^2 = -36<0$

$(1,-1)$ is a saddle point.

→ 3) $f(x,y) = x^2y+xy^2-axy$.

4) $f(x,y) = x^3+y^3-63(x+y)+12xy$

5) $f(x,y) = x^2y^2-5x^2-8xy-5y^2$.

→ 6) find three positive numbers whose sum is 100 & whose product is maximum.

sol: let x, y, z be three positive numbers

given $x+y+z=100$

& xyz is maximum

$f(x,y,z) = xyz$

$f(x,y) = xy(100-x-y) = 100xy-x^2y-xy^2$

$f_x = 100y-2xy-y^2$

$f_y = 100x-x^2-2xy$.

$f_x=0$, $f_y=0$

$$y^2 + 2xy - 100y = 0 \quad — \text{(a)}$$

$$x^2 + 2xy - 100x = 0 \quad — \text{(b)}$$

$$\text{(a)} - \text{(b)} \Rightarrow y^2 - x^2 - 100y + 100x = 0$$

$$\Rightarrow (y-x)(y+x) - 100(y-x) = 0$$

$$\Rightarrow (y-x)[x+y-100] = 0$$

$$\Rightarrow y = x, \quad x+y-100 = 0$$

$\underline{\underline{x=y}}$ $\quad$ (a) $\Rightarrow x^2 + 2x^2 - 100x = 0 \Rightarrow 3x^2 - 100x = 0$

$$x(3x-100) = 0$$

$$x = 0, \quad x = \frac{100}{3}$$

since $x = y$ $\quad (0,0) \quad \left(\frac{100}{3}, \frac{100}{3}\right)$

$\underline{\underline{x+y=100}}$

$$x = 100 - y$$

$$\text{(a)} \Rightarrow y^2 + 2(100-y)y - 100y = 0$$

$$\Rightarrow y^2 + 200y - 2y^2 - 100y = 0$$

$$\Rightarrow -y^2 + 100y = 0$$

$$y(y-100) = 0$$

$$y = 0, \quad y = 100$$

$y = 0, \quad x = 100 \Rightarrow (100, 0)$

$y = 100, \quad x = 0 \Rightarrow (0, 100)$

$(0,0) \quad \left(\frac{100}{3}, \frac{100}{3}\right), \quad (100, 0), \quad (0, 100)$

$l = -2y, \quad m = 100 - 2x - 2y, \quad n = -2x$

At $(0,0)$, $ln - m^2 < 0 \Rightarrow (0,0)$ is saddle point

[illegible] is saddle point

At $(0,0)$, $ln-m < 0$ $\Rightarrow$

At $(100,0)$, $ln-m^2 < 0$, $\Rightarrow$ $(100,0)$ is Saddle point

At $(0,100)$, $ln-m^2 < 0$, $\Rightarrow$ $(0,100)$ is Saddle point.

At $\left(\frac{100}{3}, \frac{100}{3}\right)$, $ln-m^2 = 4\left(\frac{100}{3}\right)\left(\frac{100}{3}\right) - \left(100 - \frac{200}{3} - \frac{200}{3}\right)^2 > 0$

$\therefore$ Product is max is $x = \frac{100}{3}$, $y = \frac{100}{3}$

$$z = 100 - x - y = 100 - \frac{100}{3} - \frac{100}{3} = \frac{100}{3}$$

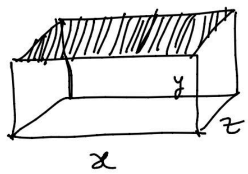
$\therefore$ $x = \frac{100}{3}$, $y = \frac{100}{3}$, $z = \frac{100}{3}$

→ A rectangular box open at the top is to have volume of 32 cubic units. Find the dimensions of the box requiring least material for its construction.

Sol: Let x, y, z be the dimensions of the box

Let 's' be the surface of the box

'v' be volume of the box



Surface $S = xy + 2yz + 2zx$

$V = xyz = 32$

$\Rightarrow z = \frac{32}{xy}$

Surface area $S = xy + 2y \cdot \frac{32}{xy} + 2x \cdot \frac{32}{xy}$

$$S = xy + \frac{64}{x} + \frac{64}{y}$$

$S_x = y - \frac{64}{x^2}$, $S_y = x - \frac{64}{y^2}$

$S_x = 0$, $S_y = 0$ $\Rightarrow$ $y - \frac{64}{x^2} = 0$, $x - \frac{64}{y^2} = 0$

$\Rightarrow y = \frac{64}{\;}$, $x = \frac{64}{\;}$

$\Rightarrow y = \frac{64}{x^2}$, $x = \frac{64}{y^2}$

$\Rightarrow x = \frac{64}{\frac{(64)^2}{x^4}}$ $\Rightarrow x = \frac{x^4}{64}$

$\Rightarrow x^4 - 64x = 0$

$x(x^3 - 64) = 0 \Rightarrow x = 0, x = 4$

$x = 0$ ✗ since $y = \infty$

$x = 4 \Rightarrow y = \frac{64}{16} \Rightarrow y = 4$

$(4,4)$

$l = S_{xx} = \frac{128}{x^3}$, $n = S_{yy} = \frac{128}{y^3}$, $S_{xy} = 1 = m$

At $(4,4)$, $ln - m^2 = \frac{128}{x^3} \cdot \frac{128}{y^3} - 1 = \frac{(128)^2}{4^3 4^3} - 1 > 0$

$l = \frac{128}{x^3} = \frac{128}{4^3} > 0$

$ln - m^2 > 0$, $l > 0$ at $(4,4)$

$\therefore$ Surface area is minimum at $x = 4$ & $y = 4$

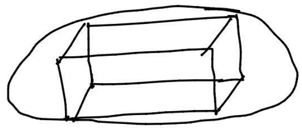
$xyz = 32$

$z = \frac{32}{xy} = 2$

$x = 4, y = 4$ & $z = 2$ are the dimensions with which we can construct a rectangular box with least material.

$\rightarrow$ Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Sol: $2x, 2y, 2z$



$$V = (2x)(2y)(2z) \to \max$$

$$\Rightarrow z^2 = c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

$$V^2 = 64x^2y^2z^2$$

$$f = V^2 = 64x^2y^2c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

f_x, f_y, f_{xx} f_{yy} f_{xy}

$V = ?$ $\quad V = \frac{8abc}{\sqrt{27}}$

→ Absolute Extrema:

Find the absolute maxima and absolute minima of $f(x,y) = x^2+4y^2-2x^2y+4$ on the rectangle $-1 \le x \le 1$, $-1 \le y \le 1$

Sol: $f(x,y) = x^2+4y^2-2x^2y+4$

$f_x = 0$, $f_y = 0$

$\Rightarrow f_x = 2x - 4xy$, $\quad f_y = 8y - 2x^2$

$\Rightarrow 2x - 4xy = 0$, $\quad 8y - 2x^2 = 0$

$\Rightarrow x(1-2y) = 0$, $\quad x^2 - 4y = 0$

$x = 0$, $y = \frac{1}{2}$

$x = 0 \Rightarrow y = 0 \Rightarrow (0,0)$

$y = \frac{1}{2} \Rightarrow x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2} \Rightarrow \left(\sqrt{2}, \frac{1}{2}\right), \left(-\sqrt{2}, \frac{1}{2}\right)$

$l = f_{xx} = 2-4y$, $\quad m = -4x$, $\quad n = 8$

At $(0,0)$, $ln - m^2 = (2-4y)(8) - (-4x)^2 = 16 > 0$

$l = 2 > 0$

At $(0,0)$, $ln - m^2 > 0$, $l > 0$, thus f has relative minimum at $(0,0)$ & relative minimum value is $f(0,0) = 4$

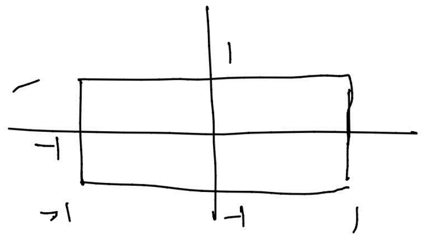
At $(0,0)$, $ln - m^2 > 0$, $l > 0$, then f has relative minimum at $(0,0)$
& relative minimum value is $f(0,0) = 4$

At $(\sqrt{2}, \frac{1}{2})$, $ln - m^2 = (2-2)(8) - (-4\sqrt{2})^2 < 0$

$(\sqrt{2}, \frac{1}{2})$ is a saddle point

At $(-\sqrt{2}, \frac{1}{2})$, $ln - m^2 = (2-2)(8) - (+4\sqrt{2})^2 < 0$

$(-\sqrt{2}, \frac{1}{2})$ is also a saddle point.



$-1 \leq x \leq 1$, $-1 \leq y \leq 1$

The right boundary $x = 1$, $-1 \leq y \leq 1$

$$f(x,y) = x^2 + 4y^2 - 2x^2y + 4$$

$$f(1,y) = 1 + 4y^2 - 2y + 4 = 4y^2 - 2y + 5$$

let $g(y) = f(1,y) = 4y^2 - 2y + 5$

$$g'(y) = 8y - 2$$

$$g'(y) = 0 \Rightarrow 8y - 2 = 0 \Rightarrow y = \frac{1}{4}$$

$g(1) = 7$

$g(-1) = 11$

$$g\left(\frac{1}{4}\right) = 4\left(\frac{1}{4}\right)^2 - 2\left(\frac{1}{4}\right) + 5 = \frac{19}{4} = 4.75$$

The left boundary: $x = -1$, $-1 \leq y \leq 1$

$$f(-1,y) = 1 + 4y^2 - 2y + 4$$

let $h(y) = f(-1,y) = 4y^2 - 2y + 5$

$$h'(y) = 0 \Rightarrow y = \frac{1}{4}$$

$$h\left(\frac{1}{4}\right) = 4.75$$

$h(1) = 7$

$h(-1) = 11$

upper boundary $y = 1$, $-1 \leq x \leq 1$

$$f(x,y) = x^2 + 4y^2 - 2x^2y + 4$$

$$f(x,1) = x^2 + 4 - 2x^2 + 4 = -x^2 + 8$$

$p(1) = 7$

$p(-1) = 7$

$f(x,1) = x^2+4-2x^2+4 = -x^2+8$

let $p(x) = f(x,1) = -x^2+8$

$p'(x) = 0 \Rightarrow x = 0$

$p(0) = 8$

$p(-1) = 7$

<u>lower boundary</u> $y=-1, \quad -1 \le x \le 1$

$f(x,-1) = 3x^2+8$

$q(x) = 3x^2+8$

$q'(x) = 6x = 0 \Rightarrow x = 0$

$q(0) = 8$

$q(1) = 11$

$q(-1) = 11$

The functional values at the boundary

$g(-1) = f(1,-1) = 11$

$g\left(\frac{1}{4}\right) = f\left(1,\frac{1}{4}\right) = 4.75$

$g(1) = f(1,1) = 7$

$h(-1) = f(-1,-1) = 11$

$h\left(\frac{1}{4}\right) = f\left(-1,\frac{1}{4}\right) = 4.75$

$h(1) = f(-1,1) = 7$

$p(-1) = f(-1,1) = 7$

$p(0) = f(0,1) = 8$

$p(1) = f(1,1) = 7$

$q(-1) = f(-1,-1) = 11$

$q(0) = f(0,-1) = 8$

$q(1) = f(1,-1) = 11$

$f(0,0) = 4$

Among all the above functional values '4' is minimum i.e at $(0,0)$

$\therefore$ Absolute minimum occurs at $(0,0)$ & absolute minimum value is 4

& Absolute max occurs at $(1,-1)$ & $(-1,-1)$ & Absolute max value is 11

& Absolute max value is 11

→ find absolute max & min of $f(x,y) = 2x^2 - y^2 + 6y$ on the disk $x^2+y^2 \le 16$

→ Constrained maxima or minima:

finding maxima or minima for a given function $f(x,y)$ under some condition $\phi(x,y)$

→ Lagrange's Multiplier method for finding max or min:

Suppose it is required to find Extremum of the function $f(x,y,z)$ subject to the condition $\phi(x,y,z)=0$ — (1)

Working rule:

1) Form Lagrangean function $F(x,y,z)$ as

$$F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z)$$

where 'λ' is Lagrangean multiplier

2) find $\frac{\partial F}{\partial x}=0$, $\frac{\partial F}{\partial y}=0$, $\frac{\partial F}{\partial z}=0$

$$\frac{\partial F}{\partial x}=0 \Rightarrow \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \quad — (2)$$

$$\frac{\partial F}{\partial y}=0 \Rightarrow \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \quad — (3)$$

$$\frac{\partial F}{\partial z}=0 \Rightarrow \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \quad — (4)$$

3) Solving (1), (2), (3) & (4) we get values of x, y, z which will be stationary values.

→ Note: To find maxima and minima of $f(x,y,z)$ subject to the conditions $\phi(x,y,z)=0$ & $\psi(x,y,z)=0$, then form Lagrangian function as the conditions $\phi(x,y,z)=0$ & $\psi(x,y,z)=0$

Lagrangian function as

$$F(x,y,z) = f(x,y,z) + \lambda_1\phi(x,y,z) + \lambda_2\psi(x,y,z)$$

λ_1, λ_2 are Lagrangian multipliers.

→ find the minimum value of $x^2+y^2+z^2$, given that $xyz=a^3$.

Sol: $f(x,y,z) = x^2+y^2+z^2$

$\phi(x,y,z) = xyz - a^3 = 0$ — ①

Form Lagrangian function as

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$$

$$F(x,y,z) = x^2+y^2+z^2+\lambda(xyz-a^3)$$

$$\frac{\partial F}{\partial x} = 2x+\lambda(yz)$$

$$\frac{\partial F}{\partial y} = 2y+\lambda(xz)$$

$$\frac{\partial F}{\partial z} = 2z+\lambda(xy)$$

$\frac{\partial F}{\partial x}=0 \Rightarrow 2x+\lambda yz=0$ — ② $\Rightarrow 2x=-\lambda yz \Rightarrow \frac{x}{yz}=\frac{-\lambda}{2}$

$\frac{\partial F}{\partial y}=0 \Rightarrow 2y+\lambda xz=0$ — ③ $\Rightarrow \frac{y}{xz}=\frac{-\lambda}{2}$

$\frac{\partial F}{\partial z}=0 \Rightarrow 2z+\lambda xy=0$ — ④ $\Rightarrow \frac{z}{xy}=\frac{-\lambda}{2}$

From ②, ③, & ④

$$\frac{x}{yz}=\frac{y}{xz}=\frac{z}{xy}=-\frac{\lambda}{2}$$

$$\left.\begin{array}{l}\frac{x}{y\cancel{z}}=\frac{y}{x\cancel{z}} \Rightarrow x^2=y^2 \\ \frac{y}{\cancel{x}z}=\frac{z}{\cancel{x}y} \Rightarrow y^2=z^2\end{array}\right\} \begin{array}{l}x^2=y^2=z^2 \\ \Rightarrow x=y=z\end{array}$$

from ① $xyz=a^3$

$xz \quad xy$

from (1) $\quad xyz = a^3$

$\Rightarrow x^3 = a^3 \Rightarrow x = a$

$\Rightarrow x = y = z = a$

$\therefore$ The stationary point is (a, a, a)

& the minimum value of f is $a^2 + a^2 + a^2 = 3a^2$.

→ find maximum & minimum values of $x+y+z$ subject to

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$$

Sol: $\quad f(x,y,z) = x+y+z$

$\phi(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0$

The Lagrangian function is $F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

$$F(x,y,z) = (x+y+z) + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right)$$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 1 - \frac{\lambda}{x^2} = 0 \quad — (2) \quad \Rightarrow x = \pm\sqrt{\lambda}$

$\frac{\partial F}{\partial y} = 0 \Rightarrow 1 - \frac{\lambda}{y^2} = 0 \quad — (3) \quad \Rightarrow y = \pm\sqrt{\lambda}$

$\frac{\partial F}{\partial z} = 0 \Rightarrow 1 - \frac{\lambda}{z^2} = 0 \quad — (4) \quad \Rightarrow z = \pm\sqrt{\lambda}$

$\Rightarrow x = y = z$

(1) $\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

$\frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} = 1 \quad \Rightarrow \quad \sqrt{\lambda} = 3 \quad \Rightarrow \quad \lambda = 9$

$x = \pm\sqrt{\lambda} \Rightarrow x = \pm\sqrt{9} \Rightarrow x = \pm 3$

$y = \pm\sqrt{x} \Rightarrow y = \pm 3$

$z = \pm\sqrt{x} \Rightarrow z = \pm 3$

$(3, 3, 3)$ & $(-3, -3, -3)$

The max value is $3+3+3$ is 9

min value is $-3-3-3$ is -9.

→ Find the volume of largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $a, b, c > 0$

Sol V is volume of rectangular parallelopiped

$$\phi(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

Lagrangian function is $F = V + \lambda\phi$

$$F = V + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

Let $2x, 2y, 2z$ be the dimensions of the rectangular parallelopiped

$$V = 8xyz$$

$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 8yz + \frac{2x}{a^2}\lambda = 0$ — (2) $\Rightarrow \frac{a^2yz}{x} = -\frac{\lambda}{4}$

$\frac{\partial F}{\partial y} = 0 \Rightarrow 8xz + \frac{2y}{b^2}\lambda = 0$ — (3) $\Rightarrow \frac{b^2xz}{y} = -\frac{\lambda}{4}$

$\frac{\partial F}{\partial z} = 0 \Rightarrow 8xy + \frac{2z}{c^2}\lambda = 0$ — (4) $\Rightarrow \frac{c^2xy}{z} = -\frac{\lambda}{4}$

$$\Rightarrow \frac{a^2yz}{x} = \frac{b^2xz}{y} = \frac{c^2xy}{z} = -\frac{\lambda}{4}$$

$$\frac{a^2 y \cancel{z}}{x} = \frac{b^2 x \cancel{z}}{y} \Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2}$$

$$\frac{b^2 xz}{y} = \frac{c^2 xy}{z} \Rightarrow \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$\Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}.$$

$$(1) \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{x^2}{a^2} + \frac{\cancel{y}x^2}{a^2} + \frac{x^2}{a^2} = 1$$

$$\frac{3x^2}{a^2} = 1 \Rightarrow x^2 = \frac{a^2}{3}$$

$$x = \pm \frac{a}{\sqrt{3}}$$

lly $y = \pm \frac{b}{\sqrt{3}}$ & $z = \pm \frac{c}{\sqrt{3}}$

Here $x = \frac{a}{\sqrt{3}}$, $y = \frac{b}{\sqrt{3}}$, $z = \frac{c}{\sqrt{3}}$ $\left(x \neq \frac{-a}{\sqrt{3}}, y \neq \frac{-b}{\sqrt{3}}, z \neq \frac{-c}{\sqrt{3}}\right)$

$$\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$$

$$V = \frac{8abc}{3\sqrt{3}}$$

→ Find the point on the plane $3x+2y+z-12=0$ which is nearest to the origin.

Sol : Let $P(x,y,z)$ be a point on the plane $3x+2y+z-12=0$

Let $O(0,0,0)$ be the origin.

distance b/w O & P is

distance b/w O & P is

$$OP = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2} = \sqrt{x^2+y^2+z^2}$$

$$(OP)^2 = x^2+y^2+z^2$$

$\phi = 3x+2y+z-12=0$

let $f = (OP)^2 = x^2+y^2+z^2$

This f should be minimum

The Lagrangian function is

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$$

$$F = x^2+y^2+z^2+\lambda(3x+2y+z-12)$$

Ans: $\left(\frac{18}{7}, \frac{12}{7}, \frac{6}{7}\right)$

→ find max value of $u = x^2y^3z^4$ if $2x+3y+4z = a$