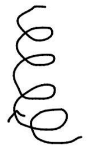
Module : 7

Vector Integration

Vector integration

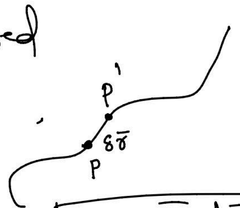
31 December 2020 07:57

→ **Line integral :**



Any integral which is to be evaluated along a curve is called as line integral.

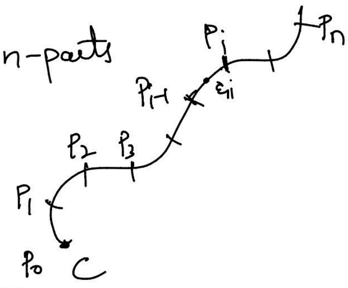
Let $\bar{f}$ be a continuous vector point function defined at every point of a curve 'c' in space.



$\boxed{W = \bar{F}\cdot d\bar{r}}$

$\bar{F}$ $\quad \bar{F}\cdot\delta\bar{r}$ $\quad \int \bar{F}\cdot d\bar{r}$

Divide the curve 'c' into n-parts at the points $P_0, P_1, P_2, \ldots P_n$



Let ξ_i be any point on the part of the curve $P_{i-1}P_i$

Now consider the sum $\quad \bar{f}(\xi_i)\cdot\delta r_i$

$$\sum_{i=1}^{n} \bar{f}(\xi_i)\cdot\delta r_i) \quad \text{where} \quad \delta r_i = r_i - r_{i-1}$$

The limit of this sum as $n\to\infty$ if exists is called as line integral of $\bar{f}$ along c & it is denoted by $\int_C \bar{F}\cdot d\bar{r}$

<u>Note:</u> 1) If $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, $d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$$\int_C \bar{f}\cdot d\bar{r} = \int_C f_1dx + f_2dy + f_3dz$$

2) When the path of the integration (path of the curve) is a closed curve, then the line integral of $\bar{f}$ over C is denoted by $\oint_C \bar{f}\cdot d\bar{r}$

3) If $\bar{f}$ represents the force acting on a moving particle along the curve AB. Then the work done during the displacement $d\bar{r}$ is $\bar{f}\cdot d\bar{r}$

Total work done from A to B $= \int_C \bar{f}\cdot d\bar{r}$

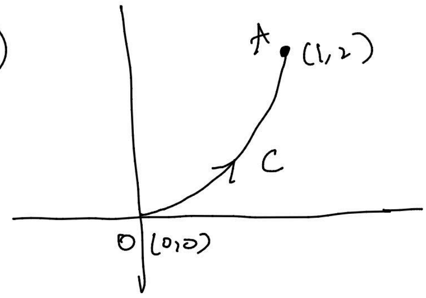
4) If the force is conservative then $\int_C \bar{f}\cdot d\bar{r}$ is independent of the path of the integration (i.e the integration depends only the end points of the curve).

5) $\bar{f}$ is conservative whenever $\bar{f}$ is an irrotational vector.

e.g. $\bar{F} = 2xy\bar{i} - y^2\bar{j}$. Evaluate $\int \bar{f}\cdot d\bar{r}$ where C is the

5) $\bar{f}$ is conservative...

→ If $\bar{f} = 3xy\bar{i} - y^2\bar{j}$. Evaluate $\int_C \bar{f}\cdot d\bar{r}$ where C is the curve $y = 2x^2$ in the xy-plane from $(0,0)$ to $(1,2)$



Sol: $\int_C \bar{f}\cdot d\bar{r} = \int_C (3xy\bar{i} - y^2\bar{j})\cdot(dx\bar{i} + dy\bar{j})$

$= \int_C (3xy\,dx - y^2\,dy)$

$y = 2x^2$

$dy = 4x\,dx$

$$\int_C \bar{f}\cdot d\bar{r} = \int_{x=0}^{1} 3x(2x^2)\,dx - (2x^2)^2\,4x\,dx.$$

$$= \int_0^1 (6x^3 - 16x^5)\,dx = \left(\frac{6x^4}{4} - \frac{16x^6}{6}\right)_0^1$$

$$= \left(\frac{6}{4} - \frac{16}{6}\right) = -\frac{7}{6}.$$

→ If $\bar{f} = (4xy - 3x^2z^2)\bar{i} + 2x^2\bar{j} - 2x^3z\bar{k}$. Then Prove that $\int_C \bar{f}\cdot d\bar{r}$ is independent of the path of the curve.

Sol: $\bar{f}$ is conservative (i.e independent of the path) when $\text{curl}\,\bar{f} = 0$.

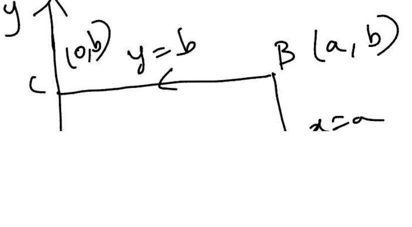
$$\text{curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix} = \bar{i}(0-0) - \bar{j}(-6x^2z + 6x^2z) + \bar{k}(4x - 4x)$$

$$= \bar{0}$$

$\Rightarrow \bar{f}$ is conservative

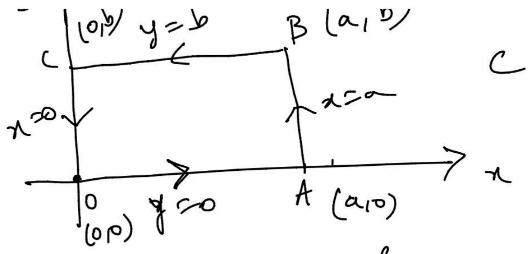
→ If $\bar{f} = (x^2+y^2)\bar{i} - 2xy\bar{j}$. Evaluate $\oint_C \bar{f}\cdot d\bar{r}$ where C is a rectangle in xy-plane bounded by the lines $x=0, x=a$ $y=0$ & $y=b$

Sol:



C : OABCO

Sol:



C : OABCO

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_{OABCO} \bar{F}\cdot d\bar{r} \quad \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CO} \bar{F}\cdot d\bar{r} \quad \text{--- (1)}$$

$$\bar{F}\cdot d\bar{r} = \left((x^2+y^2)\bar{i} - 2xy\bar{j}\right)\cdot(dx\bar{i}+dy\bar{j}) = (x^2+y^2)dx - 2xy\,dy.$$

Along OA

along OA $y=0$

$\Rightarrow dy = 0$ & $x \to 0$ to a.

$$\bar{F}\cdot d\bar{r} = (x^2+0)dx - 0 = x^2dx$$

$$\int_{OA} \bar{F}\cdot d\bar{r} = \int_{x=0}^{a} x^2 dx = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3} \quad \text{--- (a)}$$

Along AB

along AB $x=a$

$dx = 0$

& y varies from 0 to b

$$\bar{F}\cdot d\bar{r} = (a^2+y^2)(0) - 2ay\,dy$$

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_{y=0}^{b} -2ay\,dy = -(ay^2)_0^b = -ab^2.$$

Along BC

$y=b$

$\Rightarrow dy = 0$

& x varies from a to 0

$$\bar{F}\cdot d\bar{r} = (x^2+b^2)dx - (2xb)(0) = (x^2+b^2)dx$$

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{x=a}^{0} (x^2+b^2)dx = \left(\frac{x^3}{3}+b^2x\right)_a^0 = 0 - \frac{a^3}{3} - ab^2.$$

Along CO

$x = 0$

Along CO

$x = 0$

$\Rightarrow dx = 0$

& y varies from b to 0

$\bar{f} \cdot d\bar{r} = (0 + y^2)(0) + x(0) y \, dy = 0$

$$\int_{CO} \bar{f} \cdot d\bar{r} = 0$$

$$(1) \Rightarrow \oint_C \bar{f} \cdot d\bar{r} = \cancel{\frac{a^3}{3}} + (-ab^2) - \cancel{\frac{a^3}{3}} - ab^2 + 0 = -2ab^2$$

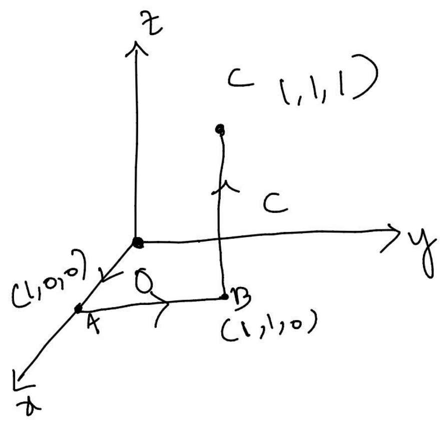
→ Find the workdone in moving a particle in the force field $\bar{f} = 3x^2 \bar{i} + \bar{j} + z\bar{k}$ along the straight line $(0,0,0)$ to $(2,1,3)$.

Ans $\dfrac{27}{2}$

→ If $\bar{f} = (x^2 - 27)\bar{i} - 6yz\bar{j} + 8xz^2\bar{k}$, evaluate $\int_C \bar{f} \cdot d\bar{r}$ from the point $(0,0,0)$ to $(1,1,1)$ i.e along the straight line from $_O(0,0,0)$ to $_A(1,0,0)$, then $_A(1,0,0)$ to $_B(1,1,0)$ & $_B(1,1,0)$ to $(1,1,1)_C$

H.W

Sol: 'C' is the curve joining the st. line from $O(0,0,0)$ to $A(1,0,0)$ & then $A(1,0,0)$ to $B(1,1,0)$ & $B(1,1,0)$ to $C(1,1,1)$



eq of OA

$$\frac{x-0}{1-0} = \frac{y-0}{0-0} = \frac{z-0}{0-0} = t$$

$x = t, \; y = 0, \; z = 0$

$\Rightarrow dx = t, \; dy = 0, \; dz = 0$

for $x = 0$, $y = 0$, $z = 0$ $\Rightarrow t = 0$

$$\int_C \bar{f} \cdot d\bar{r} = \int_{OABC} \bar{f} \cdot d\bar{r} = \int_{OA} \bar{f} \cdot d\bar{r} + \int \bar{f} \cdot d\bar{r} + \int \bar{f} \cdot d\bar{r} \quad \text{—— (1)}$$

$z = 0$

for $x = 1, y = 0, z = 0 \Rightarrow t = 1$

$\int_{AB} \bar{f} \cdot dr + \int_{BC} \bar{f} \cdot dr$ —— ①

& $\bar{f} \cdot d\bar{r} = (x^2 - 2z)\,dx - 6y\,dy + 8xz^2\,dz$

Along OA

$x = t, y = 0, z = 0$

& $dx = dt, dy = 0, dz = 0$

$t \to 0$ to 1

$$\int_{OA} \bar{f} \cdot d\bar{r} = \int_{t=0}^{1} (x^2 - 27)\,dx - 6y\,dy + 8xz^2\,dz$$

$$= \int_0^1 (t^2 - 27)\,dt - 0 + 0 = \left(\frac{t^3}{3} - 27t\right)_0^1 = \frac{1}{3} - 27 = -\frac{80}{3}$$

Along AB

$A(1,0,0)$ $B(1,1,0)$

$$\text{Eq of } AB = \frac{x-1}{1-1} = \frac{y-0}{1-0} = \frac{z-0}{0-0} = t$$

$\Rightarrow x = 1, y = t, z = 0$

$dx = 0, dy = dt, dz = 0$

$x = 1, y = 0, z = 0 \Rightarrow t = 0$, $x = 1, y = 1, z = 0 \Rightarrow t = 1$

$\Rightarrow t \to 0$ to 1

$$\int_{AB} \bar{f} \cdot d\bar{r} = \int_{t=0}^{1} (x^2 - 27)\,dx - 6y\,dy - 8xz^2\,dz$$

$$= \int_{t=0}^{1} (1 - 0)(0) - 6t\,dt - 8(1)(0)\,0 = -\int_0^1 6t\,dt$$

$$= -6\left(\frac{t^2}{2}\right)_0^1 = -3$$

Along BC

B (1,1,0) C(1,1,1)

Eq of BC is

$$\frac{x-1}{1-1} = \frac{y-1}{1-1} = \frac{z-0}{1-0} = t$$

$$\Rightarrow x = 1, \quad y = 1, \quad z = t$$

$$dx = 0, \quad dy = 0, \quad dz = dt$$

$$\begin{matrix} x = 1 \\ y = 1 \\ z = 0 \end{matrix} \Rightarrow t = 0 \quad , \quad \begin{matrix} x = 1 \\ y = 1 \\ z = 1 \end{matrix} \Rightarrow t = 1$$

$$\int_{BC} \bar{f} \cdot d\bar{r} = \int_{t=0}^{1} (x^2 - 27)\,dx - 6y\,dy + 8xz^2\,dz$$

$$= \int_{t=0}^{1} (1-27)(0) - 6(1)(0) + 8(1)t^2\,dt$$

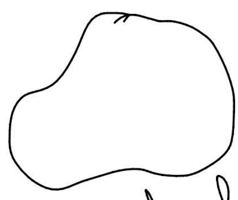
$$= \int_0^1 -8t^2\,dt = \left(+8\frac{t^3}{3}\right)_0^1 = +\frac{8}{3}$$

$$① \Rightarrow \int_C \bar{f} \cdot d\bar{r} = -\frac{80}{3} - 3 + \frac{8}{3} = -3 - \frac{72}{3} = -27$$

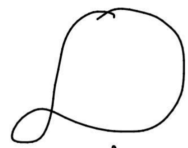
→ Green's theorem :

(Relation b/w line integral and a double integral)

If 'R' is a region bounded by a simple closed curve 'c' in the xy-plane & if $M(x,y)$ & $N(x,y)$ are continuous & differentiable in the region R, then



Simple closed curve



not a simple closed curve.

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

Here 'c' is traversed in anticlock wise direction.

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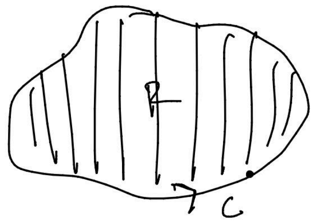
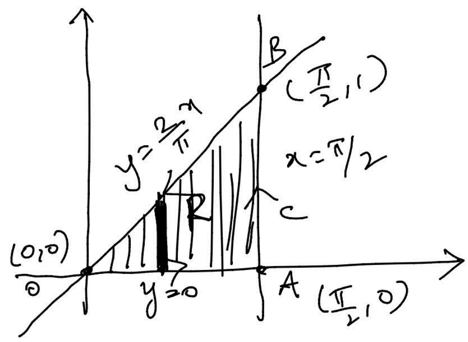


→ Evaluate by Green's theorem $\oint_C (y-\sin x)\,dx + \cos x\,dy$ where 'c' is the triangle enclosed by the lines $y=0$, $y=\frac{2}{\pi}x$ & $x=\frac{\pi}{2}$

Sol:



$$\oint_C \bar{F}\cdot d\bar{r} = \oint_C M\,dx + N\,dy$$

$$\oint_C (y-\sin x)\,dx + \cos x\,dy = \oint_C M\,dx + N\,dy$$

$$M = y-\sin x, \quad N = \cos x$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -\sin x$$

By Green's theorem $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$\oint_C (y-\sin x)\,dx + \cos x\,dy = \iint_R (-\sin x - 1)\,dx\,dy$$

$$= \int_{x=0}^{\pi/2} \int_{y=0}^{\frac{2}{\pi}x} (-\sin x - 1)\,dy\,dx.$$

$$= \int_{x=0}^{\pi/2} (-\sin x - 1)\left(y\right)_{y=0}^{\frac{2}{\pi}x} dx$$

$$= -\int_0^{\pi/2} (\sin x + 1)\left(\frac{2}{\pi}x\right) dx = -\frac{2}{\pi}\int_0^{\pi/2} (x\sin x + x)\,dx$$

$$= -\int_0 (\sin x + 1)\left(\frac{2}{\pi}x\right) dx = -\frac{2}{\pi}\int_0 (x\sin x + x)\, dx$$

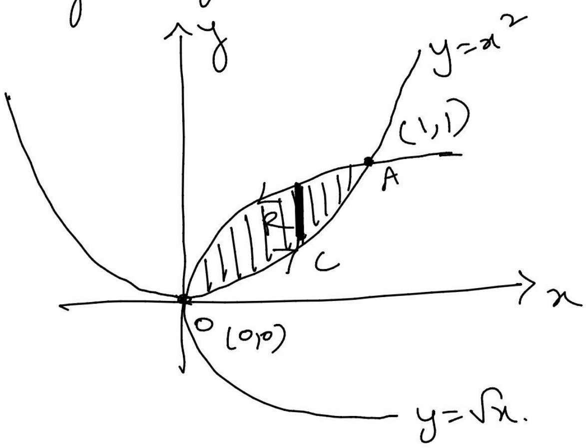
$$= -\frac{2}{\pi}\left[-x\cos x + \sin x + \frac{x^2}{2}\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[-\frac{\pi}{2}(0) + 1 + \frac{\pi^2}{8} - 0\right]$$

$$= -\frac{2}{\pi}\left\{1 + \frac{\pi^2}{8}\right\}.$$

Exercise : do verification part

→ Verify Green's theorem in plane for $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where 'C' is the boundary of the region $y = \sqrt{x}$ & $y = x^2$



Sol

$$\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \oint_C M\,dx + N\,dy$$

$M = 3x^2 - 8y^2$, $N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y$, $\frac{\partial N}{\partial x} = -6y$

By Green's theorem $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \iint_R (-6y + 16y)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} 10y\,dy\,dx.$$

$$= \int_{x=0}^{1} 10\left(\frac{y^2}{2}\right)_{y=x^2}^{\sqrt{x}} dx = 5\int_0^1 (x - x^4)\,dx$$

$$= 5\left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1 = 5\left(\frac{1}{2} - \frac{1}{5}\right)$$

$$\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \frac{3}{2}$$

Verification: $\quad \bar{f} = (3x^2 - 8y^2)\bar{i} + (4y - 6xy)\bar{j}$

$$\oint_C \bar{f}\cdot d\bar{r} = \oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$\oint_C \bar{f}\cdot d\bar{r} = \oint_{OAO} \bar{f}\cdot d\bar{r} = \int_{OA} \bar{f}\cdot d\bar{r} + \int_{AO} \bar{f}\cdot d\bar{r} \quad \text{—— (1)}$$

Along OA

Along OA, $\quad y = x^2$

$dy = 2x\,dx$

$x : 0 \to 1$

$$\int_{OA} \bar{f}\cdot d\bar{r} = \int_{x=0}^{1} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_{x=0}^{1} (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)\,2x\,dx$$

$$= \left(x^3 - \frac{8x^5}{5} + \frac{8x^4}{4} - \frac{12x^5}{5}\right)_0^1$$

$$= \left(1 - \frac{8}{5} + 2 - \frac{12}{5}\right) = -1$$

Along AO

along AO, $\quad y = \sqrt{x}$

$x = y^2$

$dx = 2y\,dy$

$y : 1 \to 0$

$$\int_{AO} \bar{f}\cdot d\bar{r} = \int_{y=1}^{0} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$\int_{1}^{0} (3y^4 - 8y^2)\,2y\,dy + (4y - 6y^3)\,dy$$

$$= \left(6\frac{y^6}{6} - 16\frac{y^4}{4} + \frac{4y^2}{2} - \frac{6y^4}{4}\right)_{1}^{0} \quad y=1$$

$$= 0 - \left(1 - 4 + 2 - \frac{3}{2}\right) = \frac{5}{2}$$

$$① \Rightarrow \oint_C \bar{F}\cdot d\bar{r} = \int_{OAO} \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AO} \bar{F}\cdot d\bar{r}$$

$$= -1 + \frac{5}{2} = \frac{3}{2}$$

L.H.S = R.H.S

Hence Green's Theorem is verified.

→ verify Green's theorem for $\oint_C (xy+y^2)\,dx + x^2\,dy$ where 'c' is curve bounded by $y=x$ & $y=x^2$.

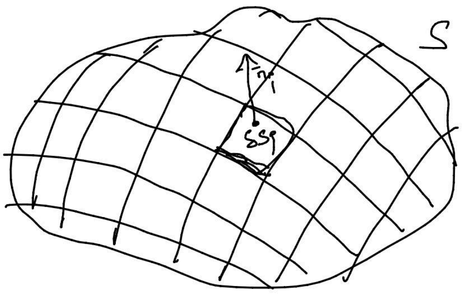
→ verify Green's theorem for $\oint_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy$ where 'c' is a square with vertices $(0,0)$, $(2,0)$, $(2,2)$ & $(0,2)$

→ Evaluate $\oint_C \bar{F}\cdot d\bar{r}$ where $C: x^2+y^2=a^2$ & $\bar{F} = \sin y\,\bar{i} + x(1+\cos y)\bar{j}$ Ans : πa^2

Surface integral

11 January 2021 09:20

Any integral which is evaluated over a surface is called as surface integral and it is denoted by $\iint_S \bar{F}\cdot\bar{n}\,ds$ or $\int_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{n}$ is the unit outward normal drawn to the surface 'S'.



Let 'S' be a smooth surface & $\bar{F}$ be a continuous vector point function defined over 'S'.

Now divide 'S' in to 'n' sub regions $\delta S_1, \delta S_2, \ldots \delta S_n$

Let P_i be any point on δS_i & $\bar{n}_i$ be the unit normal vector drawn to δS_i

Consider the sum $\sum_{i=1}^{n}\left(\bar{F}_i\cdot d\bar{A}_i\right)$ $\qquad d\bar{A}_i = \bar{n}_i \delta A_i$

where δA_i is the area of δS_i

If the above sum tends to a finite limit as $n \to \infty$ then it is called as surface integral of $\bar{F}$ over S & it is denoted by $\iint_S \bar{F}\cdot\bar{n}\,ds$

→ $\iint_S \bar{F}\cdot\bar{n}\,ds$ represents the rate of flow of fluid through the surface 'S'.

→ Cartesian form

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_S F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy.$$

If $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$$\bar{n}\,ds = \bar{i}\,dy\,dz + \bar{j}\,dx\,dz + \bar{k}\,dx\,dy$$

→ Let 'S' be a surface in space over which the surface integral

→ Let 's' be a surface in space over which the surface integral is to be evaluated

1) If R_1 be the projection of 's' on to xy-plane

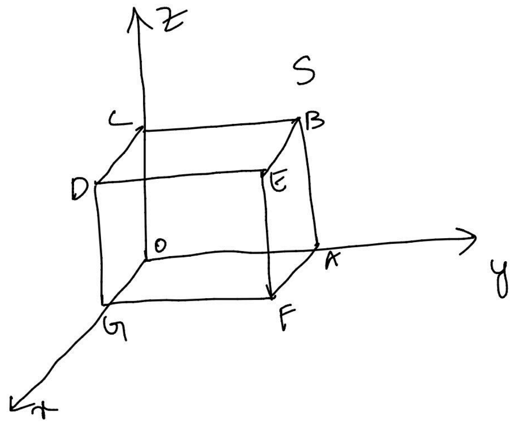
then $\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_1} \frac{\bar{F}\cdot\bar{n}\,dx\,dy}{|\bar{n}\cdot\bar{k}|}$

2) If R_2 be the projection of 's' on to yz-plane

then $\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_2} \frac{\bar{F}\cdot\bar{n}\,dy\,dz}{|\bar{n}\cdot\bar{i}|}$

3) If R_3 be the projection of 's' on to xz-plane

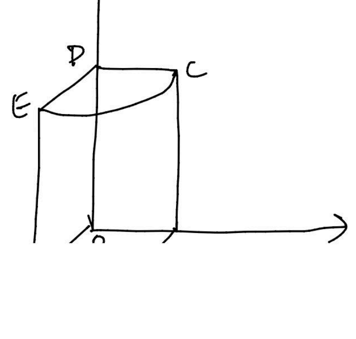
then $\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_3} \frac{\bar{F}\cdot\bar{n}\,dx\,dz}{|\bar{n}\cdot\bar{j}|}$



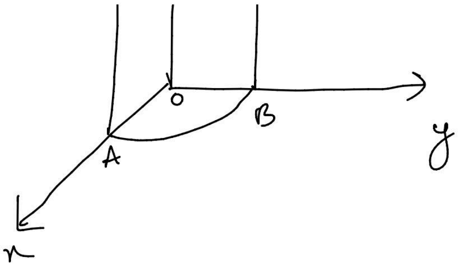
S_1 : OAFG – ~~xz~~ xy plane
S_2 : OABC – yz ''
S_3 : OCDG → xz ''
S_4 : BCDE
S_5 : DEFG
S_6 : ABEF

S_1, S_2, S_3 are on coordinate planes, so, we don't any projection for these, & S_4, S_5, & S_6 are in space. So we project S_4 on to xy plane, S_5 on to yz plane & S_6 on to xz-plane. but S_4 should not be projected on to either yz or zx plane since after projection we should get a closed curve.

2)



S_1 : OAB – xy plane
S_2 : OBCD – yz plane
S_3 : OAED – xz plane
S_4 : EDC



S_3 : OAED – xz plane.

S_4 : EDC

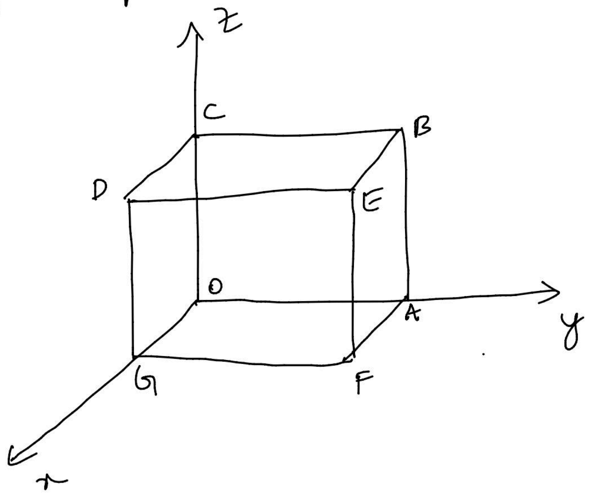
S_5 : ABCE

S_4 is to be projected onto xy-plane but not onto yz or zx-plane

S_5 is " " " either yz or zx-plane but not onto xy-plane.

→ Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$ & S is the surface of the cube bounded by $x=0, x=a, y=0, y=a, z=0$ & $z=a$

Sol



S_1 : OAFG → $z=0$

S_2 : OABC → $x=0$

S_3 : OCDG → $y=0$

S_4 : AFEB → $y=a$

S_5 : BEDC → $z=a$

S_6 : DEFG → $x=a$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{S_1} \bar{F}\cdot\bar{n}\,ds + \iint_{S_2} \bar{F}\cdot\bar{n}\,ds + \iint_{S_3} \bar{F}\cdot\bar{n}\,ds + \iint_{S_4} \bar{F}\cdot\bar{n}\,ds + \iint_{S_5} \bar{F}\cdot\bar{n}\,ds + \iint_{S_6} \bar{F}\cdot\bar{n}\,ds \quad \text{——— (1)}$$

S_1 : OAFG

$z = 0$

$\bar{n} = -\bar{k}$, $ds = dx\,dy$

$\bar{F}\cdot\bar{n} = (4xz\bar{i} - y^2\bar{j} + yz\bar{k})\cdot(-\bar{k}) = -yz = 0 \quad (\because z=0)$

$$\iint_{S_1} \bar{F}\cdot\bar{n}\,ds = \iint_{S_1} \bar{F}\cdot\bar{n}\,dx\,dy = 0$$

S_2 : OABC

$x = 0$

$\bar{n} = -\bar{i}$, $ds = dy\,dz$

$\bar{F}\cdot\bar{n} = -4xz = 0 \quad (\because x=0)$

$$\iint_{S_2} \bar{F}\cdot\bar{n}\,ds = 0$$

S_3 : OCDG

$y = 0$

$\bar{n} = -\bar{j}$

$\bar{F}\cdot\bar{n} = y^2 = 0 \quad (\because y = 0)$

$$\iint_{S_3} \bar{F}\cdot\bar{n}\,ds = 0$$

S_4 : AFEB

$y = a$

$\bar{n} = \bar{j}$, $\bar{F}\cdot\bar{n} = -y^2 = -a^2 \quad (\because y = a)$

let R_1 be projection of S_4 onto xz plane

$$\iint_{S_4} \bar{F}\cdot\bar{n}\,ds = \iint_{R_1} \frac{\bar{F}\cdot\bar{n}\,dx\,dz}{|\bar{n}\cdot\bar{j}|}$$

$$= \int_{x=0}^{a}\int_{z=0}^{a} \frac{-a^2\,dx\,dz}{|\bar{j}\cdot\bar{j}|} = -a^2(a)(a) = -a^4.$$

S_5 : BEDC

$z = a$

$\bar{n} = \bar{k}$

$\bar{F}\cdot\bar{n} = yz = ay \quad (\because z = a)$

let R_2 be the projection of S_5 on to xy-plane

$$\iint_{S_5} \bar{F}\cdot\bar{n}\,ds = \iint_{R_2} \frac{\bar{F}\cdot\bar{n}\,dx\,dy}{|\bar{n}\cdot\bar{k}|} = \int_{x=0}^{a}\int_{y=0}^{a} \frac{ay\,dx\,dy}{1}$$

$$= \int_{x=0}^{a} a\left(\frac{y^2}{2}\right)_{y=0}^{a} dx = \int_{x=0}^{a} \frac{a^3}{2}\,dx = \frac{a^4}{2}.$$

S_6 : DEFG

$x=0$

S_6 : DEFG

$x = a$

$\bar{n} = \bar{i}$

$\bar{F}\cdot\bar{n} = 4xz = 4az$

Let R_3 be projection of S_6 on to yz-plane

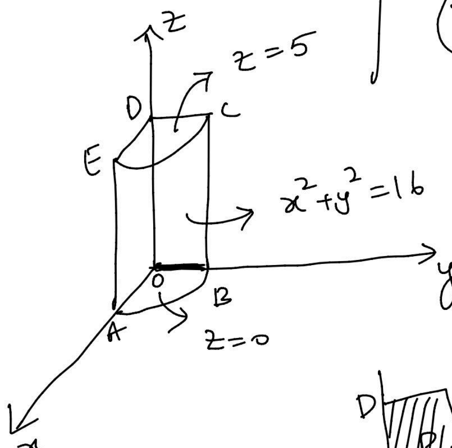
$$\iint_{S_6} \bar{F}\cdot\bar{n}\,ds = \iint_{R_3} \frac{\bar{F}\cdot\bar{n}\,dy\,dz}{|\bar{n}\cdot\bar{i}|} = \int_{y=0}^{a}\int_{z=0}^{a} \frac{4az\,dy\,dz}{1}$$

$$= \int_{y=0}^{a} 4a\left(\frac{z^2}{2}\right)_{z=0}^{a} dy = 2a^3\cdot a = 2a^4$$

$$(1) \Rightarrow \iint_S \bar{F}\cdot\bar{n}\,ds = 0+0+0-a^4+\frac{a^4}{2}+2a^4 = \frac{3a^4}{2}.$$

→ Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = z\bar{i}+x\bar{j}-3y^2z\bar{k}$ and 'S' is a surface of the cylinder $x^2+y^2=16$ included in the first octant b/w $z=0$ & $z=5$.

$x^2+y^2=16$
$x=0$
$y=0$
$z=0$
$z=5$



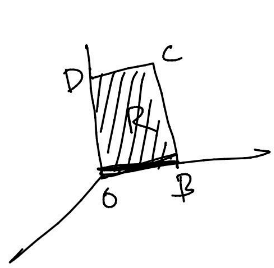
Sol:

$S : x^2+y^2-16$

The unit outward normal to 'S' is

$$\bar{n} = \frac{\nabla S}{|\nabla S|} = \frac{2x\bar{i}+2y\bar{j}}{\sqrt{4x^2+4y^2}}$$

$$\bar{n} = \frac{\cancel{2}(x\bar{i}+y\bar{j})}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x\bar{i}+y\bar{j}}{\sqrt{16}} \quad (\because x^2+y^2=16)$$



$$\boxed{\bar{n} = \frac{x\bar{i}+y\bar{j}}{4}}$$

$$\bar{F}\cdot\bar{n} = (z\bar{i}+x\bar{j}-3y^2z\bar{k})\cdot\left(\frac{x\bar{i}+y\bar{j}}{4}\right) = \frac{xz+xy}{4}$$

Let R be the projection of 'S' onto the yz-plane.

$$\iint \bar{F}\cdot\bar{n}\,ds = \iint_R \frac{\bar{F}\cdot\bar{n}\,dy\,dz}{|\bar{n}\cdot\bar{i}|}$$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \frac{F\cdot n\,dy\,dz}{|\bar{n}\cdot\bar{i}|}$$

$$= \int_{y=0}^{4}\int_{z=0}^{5} \frac{\frac{xz+xy}{\cancel{4}}}{\left|\left(\frac{x\bar{i}+y\bar{j}}{\cancel{4}}\right)\cdot\bar{i}\right|}\,dy\,dz = \int_{y=0}^{4}\int_{z=0}^{5} \frac{\cancel{x}(y+z)\,dz\,dy}{\cancel{x}}$$

$$= \int_{y=0}^{4}\left(yz+\frac{z^2}{2}\right)_{z=0}^{5}dy = \int_{y=0}^{4}\left(5y+\frac{25}{2}\right)dy$$

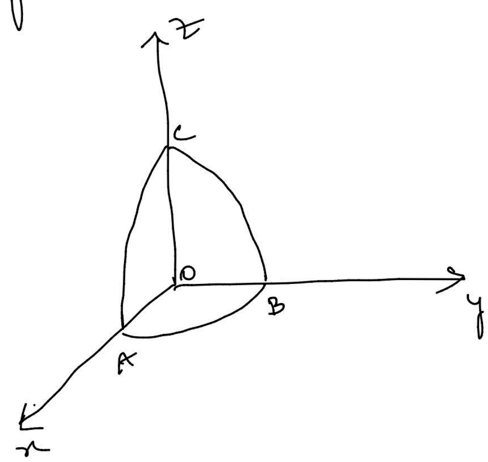
$$\left(\frac{5y^2}{2}+\frac{25}{2}y\right)_0^4 = \frac{16\times 5}{2}+\frac{25}{2}\times 4$$

$$= 90.$$

→ Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$, $\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$ and S is part of the surface of the sphere $x^2+y^2+z^2=1$ which lies in the first octant.

Sol: $S: x^2+y^2+z^2=1$

$\boxed{S: ABC}$



Ans : $\frac{3}{8}$.

→ Volume integral:

Let F be a vector point function & V be the volume enclosed by the surface 'S' then the volume integral of F over V is defined by $\iiint_V \bar{F}\,dv$ (or) $\iiint_V \bar{F}\,dx\,dy\,dz$.

→ Let $\bar{f} = 2z\bar{i} - x\bar{j} + y\bar{k}$, evaluate $\iiint_V \bar{F}\,dv$ where 'V' is the volume bounded by $x=0, y=0, x=2, y=4,$

is the volume bounded by $x=0, y=0, \ldots, y=\ldots$

$z=x^2$, $z=2$.

Sol: $\iiint \bar{F}\,dv = \int_{x=0}^{2}\int_{y=0}^{4}\int_{z=x^2}^{2} (2z\bar{i} - x\bar{j} + y\bar{k})\,dz\,dy\,dx$.

$$\int_{x=0}^{2}\int_{y=0}^{4} \left(z^2\bar{i} - xz\bar{j} + yz\bar{k}\right)_{z=x^2}^{2} dy\,dx$$

$$\int_{x=0}^{2}\int_{y=0}^{4} (4\bar{i} - 2x\bar{j} + 2y\bar{k}) - (x^4\bar{i} - x^3\bar{j} + x^2y\bar{k})\,dy\,dx$$

$$\int_{x=0}^{2} \left[(4y\bar{i} - 2xy\bar{j} + y^2\bar{k}) - \left(x^4y\bar{i} - x^3y\bar{j} + \frac{x^2y^2}{2}\bar{k}\right)\right]_{y=0}^{4} dx$$

$$\int_{x=0}^{2}$$

Ans: $\frac{32}{15}(3\bar{i} + 5\bar{k})$.

→ Evaluate $\iiint \nabla F\,dv$ where V is the volume bounded by $x=0$, $y=0$, $z=0$, $2x+2y+z=4$ & $F = x^2y + yz^2$

→ <u>Gauss - Divergence theorem</u>:

(Relation b/w surface integral & a volume integral)

Let 'S' be a closed surface enclosing the volume V. If $\bar{F}$ is a continuous & differentiable vector point function then

$$\boxed{\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \operatorname{div}\bar{F}\,dv}$$

where $\bar{n}$ is unit outward normal drawn to the surface 's'.

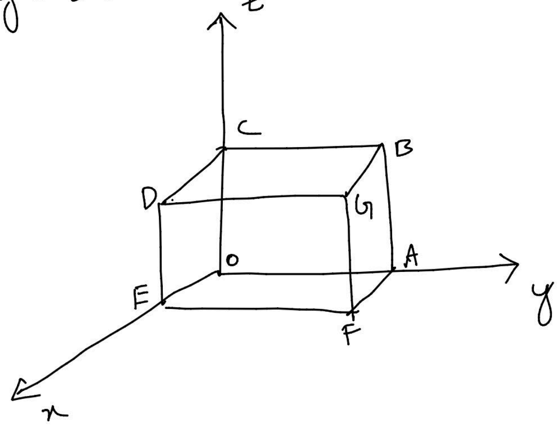
Cartesian form $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$$\iint_S F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$

→ Verify Gauss divergence theorem for $\bar{F} = (x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}$ taken over the surface of the cube bounded by the coordinate planes and $x = y = z = a$.

↓
xy-plane
yz- 〃
zx-plane

sol:



OABC – S_1 — $x = 0$
OCDE – S_2 — $y = 0$
OAFE – S_3 — $z = 0$
DEFG – S_4 — $x = a$
ABGF – S_5 — $y = a$
BCDG – S_6 — $z = a$

$\bar{F} = (x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}$

$$\operatorname{div}\bar{F} = \frac{\partial}{\partial x}(x^3 - yz) + \frac{\partial}{\partial y}(-2x^2y) + \frac{\partial}{\partial z}(z) = 3x^2 - 2x^2 + 1 = x^2 + 1$$

By Gauss-divergence theorem

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \operatorname{div}\bar{F}\,dv$$

$$= \iiint_V (x^2+1)\,dx\,dy\,dz = \int_{x=0}^{a}\int_{y=0}^{a}\int_{z=0}^{a} (x^2+1)\,dz\,dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{a} (x^2+1)\,a\,dy\,dx = \int_{x=0}^{a} (x^2+1)\,a^2\,dx$$

$$= \left(\frac{x^3}{3} + x\right)_0^a a^2 = \left(\frac{a^3}{3} + a\right)a^2$$

$$= \frac{a^5}{3} + a^3$$

Verification:

$$\iint \bar{F}\cdot\bar{n}\,ds = \iint_{S_1} \bar{F}\cdot\bar{n}\,ds + \iint_{S_2} \bar{F}\cdot\bar{n}\,ds + \iint_{S_3} \bar{F}\cdot\bar{n}\,ds + \iint_{S_4} \bar{F}\cdot\bar{n}\,ds + \iint_{S_5} \bar{F}\cdot\bar{n}\,ds + \iint_{S_6} \bar{F}\cdot\bar{n}\,ds$$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{S_1} \bar{F}\cdot\bar{n}\,ds + \iint_{S_2} \bar{F}\cdot\bar{n}\,ds + \iint_{S_3} \bar{F}\cdot\bar{n}\,ds + \iint_{S_4} \cdots \iint_{S_5} \cdots \iint_{S_6} \cdots$$

S_1: OABC

$$\bar{F} = (x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}$$

$x = 0$

$\bar{n} = -\bar{i}$

$\bar{F}\cdot\bar{n} = -x^3 + yz = yz \quad (\because x = 0)$

$ds = dy\,dz$

$$\iint_{S_1} \bar{F}\cdot\bar{n}\,ds = \int_{y=0}^{a}\int_{z=0}^{a} yz\,dy\,dz = \frac{a^4}{4}$$

S_2: OCDE

$y = 0$

$\bar{n} = -\bar{j}$

$\bar{F}\cdot\bar{n} = 2x^2y = 0 \quad (\because y = 0)$

$$\iint_{S_2} \bar{F}\cdot\bar{n}\,ds = 0$$

S_3: OAFE

$z = 0$

$\bar{n} = -\bar{k}$

$\bar{F}\cdot\bar{n} = -z = 0 \quad (\because z = 0)$

$$\iint_{S_3} \bar{F}\cdot\bar{n}\,ds = 0$$

S_4: DEFG

$x = a$, $\bar{F}\cdot\bar{n} = x^3 - yz = a^3 - yz$

$\bar{n} = \bar{i}$

Let R_1 be the projection of S_4 onto yz-plane

$$\iint_{S_4} \bar{F}\cdot\bar{n}\,ds = \iint_{R_1} \frac{\bar{F}\cdot\bar{n}\,dy\,dz}{|\bar{n}\cdot\bar{i}|} = \int_{y=0}^{a}\int_{z=0}^{a} \frac{(a^3 - yz)\,dy\,dz}{1}$$

$$\iint_{S_4} \bar{F}\cdot\bar{n}\,ds = a^5 - \frac{a^4}{4}$$

S_5: ABGF

$y = a$

$\bar{n} = \bar{j}$

$\bar{F}\cdot\bar{n} = -2x^2y = -2ax^2 \quad (\because y = a)$

Let R_2 be the projection of S_5 on to xz-plane

$$\int^{a}\int^{a} -2ax^2\,dx\,dz = -\frac{2}{3}a^5$$

Let R_2 be the projection of S_5 on to ...

$$\iint_{S_5} \bar{F}\cdot\bar{n}\,ds = \iint_{R_2} \frac{\bar{F}\cdot\bar{n}\,dx\,dz}{|\bar{n}\cdot\bar{j}|} = \int_{x=0}^{a}\int_{z=0}^{a} \frac{-2ax^2}{1}\,dx\,dz = -\frac{2}{3}a^5$$

S_6 : BCDG

$z = a$

$\bar{n} = \bar{k}$, $\bar{F}\cdot\bar{n} = z = a$

Let R_3 be the projection of S_6 on to xy-plane

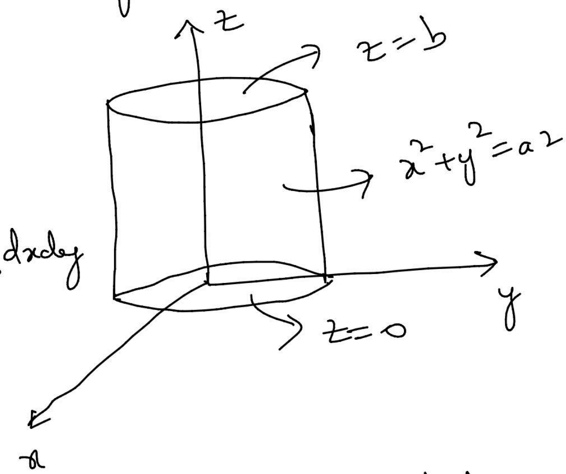
$$\iint_{S_6} \bar{F}\cdot\bar{n}\,ds = \iint_{R_3} \frac{\bar{F}\cdot\bar{n}\,dx\,dy}{|\bar{n}\cdot\bar{k}|} = \int_{x=0}^{a}\int_{y=0}^{a} \frac{a\,dx\,dy}{1} = a^3.$$

$$(1) \Rightarrow \iint_S \bar{F}\cdot\bar{n}\,ds = \cancel{\frac{a^4}{4}} + 0 + 0 + a^5 - \cancel{\frac{a^4}{4}} - \frac{2}{3}a^5 + a^3 = \frac{a^5}{3} + a^3$$

L.H.S = R.H.S

Hence Gauss - divergence theorem is verified.

→ Evaluate $\iint_S x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$, where 's' is the closed surface consisting of the cylinder $x^2+y^2=a^2$ & the the circular discs $z=0$, $z=b$.



Sol: $\iint x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$

$$= \iint F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy$$

$F_1 = x^3$, $F_2 = x^2y$, $F_3 = x^2z$

By Gauss - divergence theorem

$$\iint_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$

$$\iint_S x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy = \iiint_V (3x^2 + x^2 + x^2)\,dx\,dy\,dz$$

$$= \iiint_V 5x^2\,dx\,dy\,dz$$

$$= \iiint_V 5x^2 \ldots$$

$$= \int_{x=-a}^{a} \int_{z=0}^{b} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 5x^2 \, dy\, dz\, dx.$$

By changing to cylindrical coordinates

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

$r \to 0$ to a

$\theta \to 0$ to 2π

$z \to 0$ to b

$$\iint \bar{F}\cdot\bar{n}\, ds = \int_{r=0}^{a} \int_{\theta=0}^{2\pi} \int_{z=0}^{b} 5r^2\cos^2\theta \; r\, dz\, d\theta\, dr$$

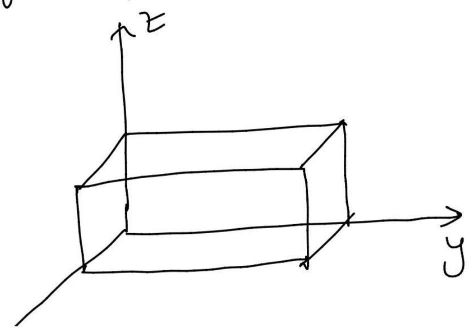
$$= \int_{r=0}^{a} \int_{\theta=0}^{2\pi} 5r^3\cos^2\theta \, b \, d\theta\, dr = \int_{r=0}^{a} \int_{\theta=0}^{2\pi} 5r^3 b \left(\frac{1+\cos 2\theta}{2}\right) d\theta\, dr$$

$$= \frac{5b}{2} \int_{r=0}^{a} r^3 \left(\theta + \frac{\sin 2\theta}{2}\right)_{\theta=0}^{2\pi} dr$$

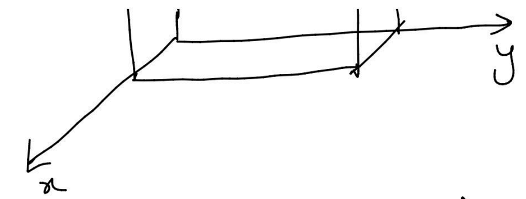
$$\frac{5b}{2} \int_{r=0}^{a} r^3 \left(2\pi + 0 - 0\right) dr$$

$$5b\pi \left(\frac{r^4}{4}\right)_0^a = \frac{5b\pi a^4}{4}$$

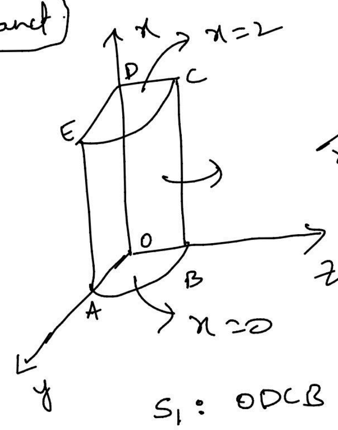
→ Verify Gauss divergence theorem for $\bar{F} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ taken over a rectangular parallelopiped $0 \le x \le a, \quad 0 \le y \le b, \quad 0 \le z \le c$.

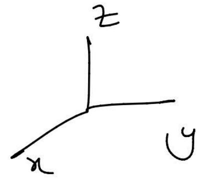


Ans $abc(a+b+c)$.



→ Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = 2x^2y\,\bar{i} - y^2\bar{j} + 4xz^2\bar{k}$ where 's' is the surface bounded by $y^2+z^2=9$, $x=2$, $x=0$, $y=0$, $z=0$ $\boxed{\text{in first octant}}$





Sol: $\bar{F} = 2x^2y - y^2\bar{j} + 4xz^2\bar{k}$

$\text{div}\bar{F} = 4xy - 2y + 8xz.$

By Gauss-divergence theorem

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint \text{div}\,\bar{F}\,dv$$

$$= \int_{x=0}^{2}\int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (4xy - 2y + 8xz)\,dx\,dy\,dz$$

S_1: ODCB — $y=0$
S_2: OAB — $x=0$
S_3: ODEA — $z=0$
S_4: EDC — $x=2$
S_5: ABCE — $y^2+z^2=9$

By changing to cylindrical coordinates.

$y = r\cos\theta$, $z = r\sin\theta$, $x = x$ | $r \to 0$ to 3
$dy\,dz\,dx = r\,dr\,d\theta\,dx$ | $\theta \to 0$ to $\frac{\pi}{2}$
| $x \to 0$ to 2

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \int_{r=0}^{3}\int_{\theta=0}^{\pi/2}\int_{x=0}^{2} (4xr\cos\theta - 2r\cos\theta - 8xr\sin\theta)\,r\,dr\,d\theta\,dx$$

Ans $= \underline{180}$

→ using G.D.T, evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$ & the 'S' is the surface of the sphere $x^2+y^2+z^2=a^2$.

Ans:

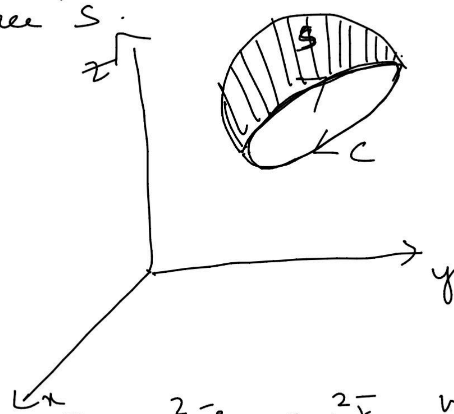
→ **Stoke's theorem**

(Relation b/w a line integral and a surface integral.)

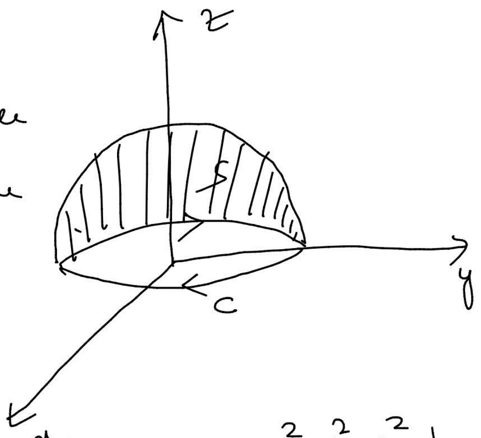
Let 's' be an open surface bounded by a simple closed curve 'c'. If $\bar{F}$ is any differentiable vector point function, then

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$

where 'c' is traversed in anti-clockwise direction & $\bar{n}$ is the unit outward normal drawn to the surface 's'.



→ Verify stoke's theorem for $\bar{f} = (2x-y)\bar{i} - z^2y\bar{j} - zy^2\bar{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ bounded by its projection on xy-plane.



sol: The boundary 'c' of the surface 's' is projection of 's' on to xy-plane

i,e $C: x^2+y^2=1,\ z=0$

$S: x^2+y^2+z^2-1$

By stoke's theorem

$$\oint_C \bar{f}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{f}\cdot\bar{n}\,ds \quad -(1)$$

$$\text{curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -z^2y & -zy^2 \end{vmatrix} = \bar{i}(-2yz+2yz) - \bar{j}(0-0) + \bar{k}(0+1)$$

$$= \bar{k}$$

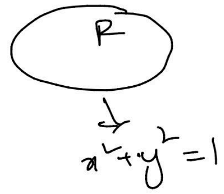
$$\bar{n} = \frac{\nabla S}{|\nabla S|} = \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\bar{i}+y\bar{j}+z\bar{k}}{1}$$

$$\text{curl}\,\bar{f}\cdot\bar{n} = z$$

∴ L R be the projection of 's' onto xy-plane

$\text{curl}\, f \cdot n = z$

let R be the projection of s onto xy-plane



$$\iint_S \text{curl}\, \bar{f}\cdot\bar{n}\, ds = \iint_R \frac{\text{curl}\, \bar{f}\cdot\bar{n}\, dxdy}{|\bar{n}\cdot\bar{k}|}$$

$$= \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{\cancel{z}\, dxdy}{\cancel{z}}$$

$$= \int_{x=-1}^{1} \left(2\sqrt{1-x^2}\right) dx$$

$$= 2\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_{-1}^{1}$$

$$= 2\left[0 + \frac{\pi}{4} - \left(0 - \frac{\pi}{4}\right)\right]$$

$$\oint_C \bar{f}\cdot d\bar{r} = \iint_S \text{curl}\, \bar{f}\cdot\bar{n}\, ds = \pi$$

<u>Verification:</u> $\oint_C \bar{F}\cdot d\bar{r}$ $\quad C: x^2+y^2=1$

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_C f_1 dx + f_2 dy + f_3 dz = \oint_C (2x-y)dx - z^2 y\, dy - zy^2 dz$$

on C, $z=0$ & $dz=0$

$$\Rightarrow \oint_C \bar{F}\cdot d\bar{r} = \oint_C (2x-y)dx - 0 - 0 = \oint_C (2x-y)dx$$

parametric eq of 'c' $- x^2+y^2=1$ is

$x = \cos t$, $y = \sin t$

$dx = -\sin t\, dt$, $dy = \cos t\, dt$

$t \to 0$ to 2π

$$\oint_C \bar{F}\cdot d\bar{r} = \int_{t=0}^{2\pi} (2\cos t - \sin t)(-\sin t)dt$$

$$= \int_{t=0}^{2\pi} (-2\sin t\cos t + \sin^2 t)dt$$

$$\int_{t=0}^{2\pi}\left(-\sin 2t + \frac{1-\cos 2t}{2}\right)dt$$

2π

$$\int_{t=0} \left(\sin 2t + \frac{1-\cos 2t}{2} \right) \dots$$

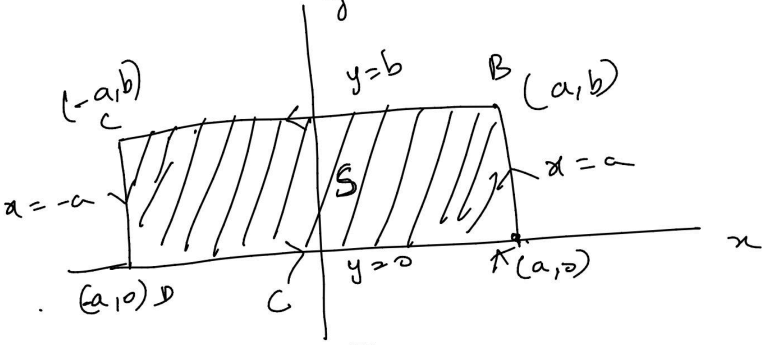
$$\oint_C \bar{f}\cdot d\bar{r} = \left(\frac{\cos 2t}{2} + \frac{t}{2} + \frac{\sin 2t}{4} \right)_{t=0}^{2\pi} = \pi$$

L.H.S = R.H.S

Hence Stoke's theorem is verified.

→ verify stoke's theorem for $\bar{f} = (x^2+y^2)\bar{i} - 2xy\bar{j}$ taken round the rectangle bounded by the lines $x = \pm a$, $y=0$, $y=b$

Sol:



$$\text{curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$\text{curl}\,\bar{f} = \bar{i}(0) - \bar{j}(0) + \bar{k}(-2y-2y) = -4y\bar{k}$$

$$\bar{n} = \bar{k}$$

$$\text{curl}\,\bar{f}\cdot\bar{n} = -4y$$

$$\iint_S \text{curl}\,\bar{f}\cdot\bar{n}\,ds = \int_{x=-a}^{a}\int_{y=0}^{b} -4y\,dx\,dy = \int_{x=-a}^{a} -2(y^2)_{y=0}^{b} dx = -2b^2\int_{-a}^{a} dx$$

$$\iint_S \text{curl}\,\bar{f}\cdot\bar{n}\,ds = -4ab^2$$

By stokes theorem $\oint_C \bar{f}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{f}\cdot\bar{n}\,ds$

$$= -4ab^2.$$

verification

$$\oint_C \bar{f}\cdot d\bar{r} = \int_{ABCDA} \bar{f}\cdot d\bar{r} = \int_{AB} \bar{f}\cdot d\bar{r} + \int_{BC} \bar{f}\cdot d\bar{r} + \int_{CD} \bar{f}\cdot d\bar{r} + \int_{DA} \bar{f}\cdot d\bar{r}$$

Along AB

$x = a$

$dx = 0$

$y \to 0$ to b

$$\int \bar{f}\cdot d\bar{r} = \int (x^2+y^2)\,dx - 2xy\,dy = \int_{y=0}^{b} 0 - 2ay\,dy = \left(-ay^2\right)_0^b$$

$$ab^2.$$

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_{AB} (x^2+y^2)\,dx - 2xy\,dy \quad = \int \quad y=0$$

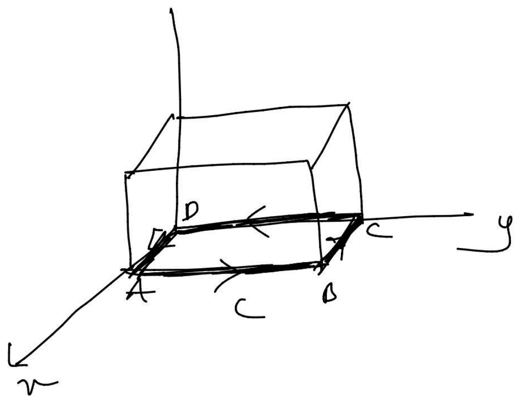
$$= -ab^2.$$

H.W

(1) verify stoke's theorem for $\bar{A} = y^2\bar{i} + xy\bar{j} - xz\bar{k}$ where S is the hemisphere $x^2+y^2+z^2 = a^2$, $z \geq 0$.

(2) verify stoke's theorem for $\bar{F} = (y-z+2)\bar{i} + (yz+4)\bar{j} - xz\bar{k}$ where S is the surface of the cube $x=0$, $y=0$, $x=2$, $y=2$, $z=2$ above xy-plane

$$\iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$



Differentiation

$(cu)' = cu' \qquad (c \text{ constant})$

$(u + v)' = u' + v'$

$(uv)' = u'v + v'u$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} \qquad \text{(Chain rule)}$$

$(x^n)' = nx^{n-1}$

$(e^x)' = e^x$

$(a^x)' = a^x \ln a$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$

$(\cot x)' = -\csc^2 x$

$(\sinh x)' = \cosh x$

$(\cosh x)' = \sinh x$

$$(\ln x)' = \frac{1}{x}$$

$$(\log_a x)' = \frac{\log_a e}{x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}$$

$$(\arctan x)' = \frac{1}{1 + x^2}$$

$$(\text{arccot } x)' = -\frac{1}{1 + x^2}$$

Integration

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \tan x\,dx = -\ln|\cos x| + c$$

$$\int \cot x\,dx = \ln|\sin x| + c$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + c$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + c$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \text{arcsinh}\,\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \text{arccosh}\,\frac{x}{a} + c$$

$$\int \sin^2 x\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + c$$

$$\int \cos^2 x\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c$$

$$\int \text{tau}^2 x\,dx = \tan x - x + c$$

$$\int \cot^2 x\,dx = -\cot x - x + c$$

$$\int \ln x\,dx = x\ln x - x + c$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + c$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + c$$

Some Constants

$$e = 2.71828\ 18284\ 59045\ 23536$$
$$\sqrt{e} = 1.64872\ 12707\ 00128\ 14685$$
$$e^2 = 7.38905\ 60989\ 30650\ 22723$$

$$\pi = 3.14159\ 26535\ 89793\ 23846$$
$$\pi^2 = 9.86960\ 44010\ 89358\ 61883$$
$$\sqrt{\pi} = 1.77245\ 38509\ 05516\ 02730$$

$$\log_{10} \pi = 0.49714\ 98726\ 94133\ 85435$$
$$\ln \pi = 1.14472\ 98858\ 49400\ 17414$$
$$\log_{10} e = 0.43429\ 44819\ 03251\ 82765$$
$$\ln 10 = 2.30258\ 50929\ 94045\ 68402$$

$$\sqrt{2} = 1.41421\ 35623\ 73095\ 04880$$
$$\sqrt[3]{2} = 1.25992\ 10498\ 94873\ 16477$$
$$\sqrt{3} = 1.73205\ 08075\ 68877\ 29353$$
$$\sqrt[3]{3} = 1.44224\ 95703\ 07408\ 38232$$
$$\ln 2 = 0.69314\ 71805\ 59945\ 30942$$
$$\ln 3 = 1.09861\ 22886\ 68109\ 69140$$

$$\gamma = 0.57721\ 56649\ 01532\ 86061$$
$$\ln \gamma = -0.54953\ 93129\ 81644\ 82234$$
(see Sec. 5.6)
$$1^\circ = 0.01745\ 32925\ 19943\ 29577 \text{ rad}$$
$$1 \text{ rad} = 57.29577\ 95130\ 82320\ 87680^\circ = 57^\circ 17' 44.806''$$

Polar Coordinates

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x}$$
$$dx\,dy = r\,dr\,d\theta$$

Series

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m \quad (|x| < 1)$$
$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$
$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$$
$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!}$$
$$\ln(1-x) = -\sum_{m=1}^{\infty} \frac{x^m}{m} \quad (|x| < 1)$$
$$\arctan x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2m+1} \quad (|x| < 1)$$

Greek Alphabet

α	Alpha	ν	Nu
β	Beta	ξ	Xi
γ, Γ	Gamma	o	Omicron
δ, Δ	Delta	π	Pi
ϵ, ε	Epsilon	ρ	Rho
ζ	Zeta	σ, Σ	Sigma
η	Eta	τ	Tau
$\theta, \vartheta, \Theta$	Theta	υ, Υ	Upsilon
ι	Iota	ϕ, φ, Φ	Phi
κ	Kappa	χ	Chi
λ, Λ	Lambda	ψ, Ψ	Psi
μ	Mu	ω, Ω	Omega

Vectors

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$
$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$
$$\operatorname{div} \mathbf{v} = \nabla\cdot\mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$
$$\operatorname{curl} \mathbf{v} = \nabla\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

The End🙂