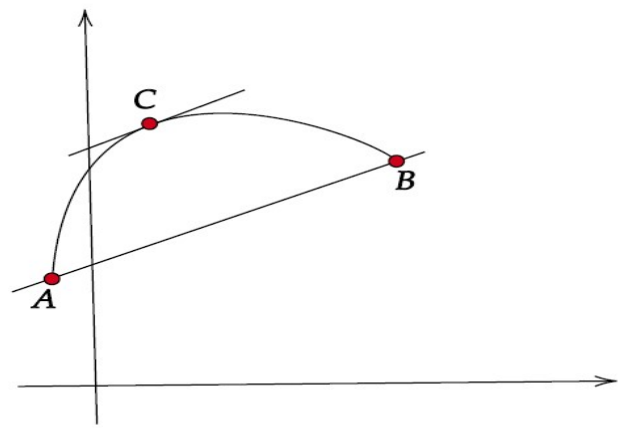
The Mean Value Theorem:-

Geometrically speaking, if the graph of a continuous function has a non vertical tangent line at every point between *A* and *B*, then there is at least one point *C* on the graph between *A* and *B* at which the tangent line is parallel to the secant line *AB*.



Mean Value Theorem for Derivatives

If f is continuous on a closed interval $[a, b]$ and differentiable on its interior (a, b), then there is at least one number c in (a, b) where

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

or, equivalently, where

$$f(b) - f(a) = f'(c)(b - a)$$

1. Find the number c guaranteed by the Mean Value Theorem for $f(x) = 2\sqrt{x}$ on $[1, 4]$

Sol:-

Given $f(x) = 2\sqrt{x}$ on $[1, 4]$

$$f'(x) = x^{-1/2} = \frac{1}{\sqrt{x}}$$

$$f'(c) = \frac{1}{\sqrt{c}} \quad \text{——} \ (1)$$

$$\frac{f(4) - f(1)}{4 - 1} = \frac{4 - 2}{3} = 2/3 \quad \text{——} \ (2)$$

From (1) & (2)

$$\frac{1}{\sqrt{c}} = 2/3 \Rightarrow c = 9/4$$

2. Let $f(x) = x^3 - x^2 - x + 1$ on $[-1, 2]$. Find all numbers c satisfying the conclusion to the Mean Value Theorem.

Sol:-

Given $f(x) = x^3 - x^2 - x + 1$ on $[-1, 2]$

$$f'(x) = 3x^2 - 2x - 1$$

$$f'(c) = 3c^2 - 2c - 1 \quad \text{— (1)}$$

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{3 - 0}{3} = 1 \quad \text{— (2)}$$

From (1) & (2), we get $3c^2 - 2c - 1 = 1$

$$3c^2 - 2c - 2 = 0 \Rightarrow c = \left(2 \pm \sqrt{28}\right)/6$$

$c_1 \simeq -0.55$ & $c_2 \simeq 1.22$, both numbers are in the interval $(-1, 2)$.

3. Let $f(x) = x^{\frac{2}{3}}$ on $[-8, 27]$. Show that the conclusion to the Mean Value Theorem fails and figure out why.

Sol:- Given $f(x) = x^{2/3}$ on $[-8, 27]$

$$f'(x) = \frac{2}{3x^{1/3}}$$

$$f'(c) = \frac{2}{3c^{1/3}} \quad \text{--- } (1)$$

$$\frac{f(27) - f(-8)}{27-(-8)} = \frac{9-4}{35} = 1/7 \quad \text{--- } (2)$$

From (1) & (2)

$$\frac{2}{3c^{1/3}} = \frac{1}{7} \Rightarrow c^{1/3} = 14/3$$

$$c = (14/3)^3 \simeq 102$$

But $c = 102$ is not in the $(-8, 27)$.

Further $f'(0)$ fails to exist.

$\therefore$ MVT fails.

Rolle's Theorem:-

If f is continuous on $[a, b]$ and differentiable on (a, b) and if $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

All 3 conditions of Rolle's theorem are necessary for the theorem to be true:

1. $f(x)$ is continuous on the closed interval $[a, b]$;
2. $f(x)$ is differentiable on the open interval (a, b);
3. $f(a) = f(b)$.

1. Check the validity of Rolle's theorem for the function $f(x) = \sqrt{1-x^2}$ on the segment $[-1, 1]$.

Sol:– Given $f(x) = \sqrt{1-x^2}$ on $[-1, 1]$.

The function value at the end pts

$$f(-1) = f(1) = 0$$

Hence, the derivative must be equal to zero at some pt on $(-1, 1)$

$$f'(x) = \frac{1}{\cancel{2}}(1-x^2)^{-1/2}(0 - \cancel{2}x)$$

$$= \frac{-x}{\sqrt{1-x^2}}$$

This shows that the derivative is zero at $x = 0 \in (-1, 1)$

$\therefore$ Rolle's theorem is valid for the given function on its given interval.

2. Check the validity of the Rolle's Theorem for the function $f(x) = \frac{x^2-4x+3}{x-2}$ on the segment $[1, 3]$.

Sol:- Given $f(x) = \dfrac{x^2-4x+3}{x-2}$ on $[1, 3]$.

The function value at the end pts are

$$f(1) = \frac{1^2-4(1)+3}{1-2} = 0$$

$$f(3) = \frac{3^2-4(3)+3}{3-2} = 0$$

We see that, when $x=2$, the function has a discontinuity, ie one of the three conditions of Rolle's theorem, namely the function f sh be continuous in (a,b) fails.

$\therefore$ Rolle's Theorem fails for the above case.