Formulas

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\sin(x)\sin(y) = \tfrac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\cos(x)\cos(y) = \tfrac{1}{2}[\cos(x-y) + \cos(x+y)]$$

$$\sin(x)\cos(y) = \tfrac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$$\cos(x)\sin(y) = \tfrac{1}{2}[\sin(x+y) - \sin(x-y)]$$

1. Find (a) $L\{sin2tsin3t\}$ (b) $L\{cos^2 2t\}$ (c) $L\{e^{2t}cos^2 t\}$ (d) $L\{sin2tcost\}$.

a) $L\{\sin 2t \sin 3t\}$

Consider $\sin 2t \sin 3t = \frac{1}{2}\left[\cos(-t) - \cos 5t\right]$

$$= \frac{1}{2}\left[\cos t - \cos 5t\right]$$

$$\therefore L\{\sin 2t \sin 3t\} = \frac{1}{2} L\left[\cos t - \cos 5t\right]$$

$$= \frac{1}{2}\left[L(\cos t) - L(\cos 5t)\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right]$$

b) $L(\cos^2 2t)$

consider $\cos^2 2t = \dfrac{1+\cos 4t}{2}$

$$\therefore L(\cos^2 2t) = L\left[\frac{1+\cos 4t}{2}\right]$$

$$= \frac{1}{2}\left[L(1) + L(\cos 4t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right]$$

c) $L\{e^{2t}\cos^2 t\}$

$$L(\cos^2 t) = L\left[\frac{1+\cos 2t}{2}\right]$$

$$= \frac{1}{2}\left[L(1) + L(\cos 2t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$$

$$\therefore L\{e^{2t}\cos^2 t\} = \frac{1}{2}\left[\frac{1}{s-2} + \frac{s-2}{(s-2)^2+4}\right]$$

d) $L\{\sin 2t \cos t\}$

consider $\sin 2t \cos t = \frac{1}{2}[\sin 3t + \sin t]$

$$L\{\sin 2t \cos t\} = \frac{1}{2}[L(\sin 3t) + L(\sin t)]$$

$$= \frac{1}{2}\left[\frac{3}{s^2+9} + \frac{1}{s^2+1}\right] //$$

Multiplication by Powers of t:-

$$\text{If } L[f(t)] = F(s), \text{ then } L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n F^n(s) \text{ where } n = 1,2,3,....$$

Proof :–

$$\text{We have } F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Then by Leibnitz's rule for differentiating under the integral sign,

$$\frac{dF}{ds} = F'(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\,dt$$

$$= \int_0^\infty -te^{-st} f(t)\,dt = -\int_0^\infty e^{-st}[tf(t)]dt$$

$$= -L[tf(t)]$$

$$\text{Thus } L[tf(t)] = -\frac{dF}{ds} = -F'(s) \qquad (1)$$

which proves the theorem for $n = 1$.

Assume the theorem true for $n = k$, *i.e. assume*

$$\int_0^\infty e^{-st}\left[t^k f(t)\right] dt = (-1)^k f^{(k)}(s) \qquad (2)$$

Then

$$\frac{d}{ds}\int_0^\infty e^{-st}\left[t^k f(t)\right] dt = (-1)^k f^{(k+1)}(s)$$

or by Leibnitz's rule,

$$-\int_0^\infty e^{-st}\left[t^{k+1} f(t)\right] dt = (-1)^k f^{(k+1)}(s)$$

$$\int_0^\infty e^{-st}\left[t^{k+1} f(t)\right] dt = (-1)^{k+1} f^{(k+1)}(s) \qquad (3)$$

It follows that if (2) *is true, i.e. if the theorem holds for* $n = k$, *then* (3) *is true, i.e. the theorem holds for* $n = k + 1$. *But by* (1) *the theorem is true for* $n = 1$. *Hence it is true for* $n = 1 + 1 = 2$ *and* $n = 2 + 1 = 3$, *etc., and thus for all* $+Ve$ int*eger values of* n.

$$Find \quad (a)\ L[t\sin at]\ (b)\ L[t^2\cos at].$$

Solution :−

(a)

$$W.K.T \quad L[\sin at] = \frac{a}{s^2+a^2}$$

$$\therefore\ L[t\sin at] = (-1)^1 \frac{d}{ds} L[\sin at]$$

$$= -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}.$$

(b)

$$W.K.T \quad L[\cos at] = \frac{s}{s^2+a^2}$$

$$\therefore\ L[t^2\cos at] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right) = \frac{2s^3-6a^2s}{(s^2+a^2)^3}.$$

Division by t:-

If $L[f(t)] = F(s)$, then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$ provided the integral exists.

Proof :-

$$\text{We have } F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Integrating both sides with respect to s from s to ∞.

$$\int_s^\infty F(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt\right] ds$$

$$= \int_0^\infty \int_s^\infty f(t)\, e^{-st}\,ds\,dt \quad [\text{Changing the order of integration}]$$

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\,ds\right] dt \quad [\because t \text{ is independent of } s]$$

$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt = \int_0^\infty e^{-st}\frac{f(t)}{t}\,dt = L\left[\frac{1}{t} f(t)\right]$$

$$Find\ \ the\ \ Laplace\ \ transform\ \ of\ \ (i)\ \frac{(1-e^t)}{t}$$

$$(ii)\ \frac{\cos at - \cos bt}{t}.$$

$Sol:-$

$$Since\ \ L(1-e^t) = L(1) - L(e^t) = \frac{1}{s} - \frac{1}{s-1} = F(s)$$

$$\therefore\ L\left[\frac{1-e^t}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds = \left[\log s - \log(s-1)\right]_s^\infty$$

$$= \left[\log\left(\frac{\cancel{s}}{s-1}\right)\right]_s^\infty = -\log\left[\frac{1}{1-\frac{1}{s}}\right] = \log\left(\frac{s-1}{s}\right)$$

$\cancel{s}(1-1/s)$

$$= \left[\log\left(\frac{1}{1-1/s}\right)\right]_s^\infty = 0 - \log\left(\frac{1}{1-1/s}\right) = -\log\left(\frac{s}{s-1}\right)$$

$$= \log\left(\frac{s-1}{s}\right)$$

(iii) $L\left\{\frac{\cos at - \cos bt}{t}\right\}$

Sol:-

$$L(\cos at - \cos bt) = L(\cos at) - L(\cos bt)$$

$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{\cancel{s^2}\left(1+\frac{a^2}{s^2}\right)}{\cancel{s^2}\left(1+b^2/s^2\right)}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right) \quad //$$