Module : 2

Laplace

Transforms

Laplace Transform

16 October 2020 09:44

Integral transform: An improper integral of the form $\int_{-\infty}^{\infty} K(s,t) f(t)\,dt$ is called integral transform, where $K(s,t)$ is called as kernel of transform.

If $K(s,t) = \begin{cases} e^{-st}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

$$\text{Integral transform} = \int_{-\infty}^{0} K(s,t) f(t)\,dt + \int_{0}^{\infty} K(s,t) f(t)\,dt$$

$$= 0 + \int_{0}^{\infty} e^{-st} f(t)\,dt$$

$$= \int_{0}^{\infty} e^{-st} f(t)\,dt = \text{Laplace transform.}$$

Def: If $f(t)$ is function defined for all $t > 0$ then the Laplace transform of $f(t)$ is denoted by $L\{f(t)\}$ (or) $F(s)$ & defined as

$$\boxed{L\{f(t)\} = F(s) = \int_{0}^{\infty} e^{-st} f(t)\,dt}$$

L — Laplace transform operator.

Inverse Laplace transform:

If $F(s)$ is Laplace transform of $f(t)$ then inverse Laplace transform of $F(s)$ is defined as

$$L^{-1}\{F(s)\} = f(t)$$

$L^{-1} \rightarrow$ inverse Laplace transform operator.

→ **Sufficient conditions for the Existence of Laplace transform:**

1) $f(t)$ must be piecewise continuous or continuous in a given interval

2) $f(t)$ must be of Exponential order.

→ **Piecewise continuous:** If $f(t)$ is defined in $[a,b]$ & if it is
(or) defined in that interval such that the
... continuous) | interval can be divided into a finite no. of ... $f(t)$ is continuous

→ Piecewise ... (or) (sectionally continuous) | ... defined in that ... interval can be divided in to a finite no. of subintervals in each of which $f(t)$ is continuous & has both left hand & right hand limits at the end points of the subinterval.

Ex : 1) $f(t) = \begin{cases} t^2, & 0 < t \le 5 \\ 2t+3, & t > 5 \end{cases}$ is piecewise continuous.

2) $f(t) = \frac{1}{t}$ in $(-1, 1)$ is not piecewise cont in any interval containing zero.

→ Exponential Order:

$f(t)$ is said to be a function of exponential order if

$$\lim_{t \to \infty} e^{-at} f(t) = \text{finite value.}$$

Ex: t^2, $\sin at$, e^{at} etc are functions of exponential order.

$$\lim_{t \to \infty} t^2 e^{-at} = \lim_{t \to \infty} \frac{t^2}{e^{at}} = \frac{\infty}{\infty} \text{ indeterminate form}$$

$$= \lim_{t \to \infty} \frac{2t}{ae^{at}} = \frac{\infty}{\infty} \quad \text{" "}$$

$$= \lim_{t \to \infty} \frac{2}{a^2 e^{at}} = 0 \text{ a finite value.}$$

2) $f(t) = e^{t^2}$

$$\lim_{t \to \infty} e^{-at} e^{t^2} = \lim_{t \to \infty} \frac{e^{t^2}}{e^{at}} = \frac{\infty}{\infty}$$

$$= \lim_{t \to \infty} \frac{2t e^{t^2}}{ae^{at}} = \frac{\infty}{\infty}$$

$$\lim_{t \to \infty} \frac{4t^2 e^{t^2}}{a^2 e^{at}} = \frac{\infty}{\infty}$$

$\Rightarrow f(t) = e^{t^2}$ is not a function of exponential order.

$\therefore L\{\frac{1}{t}\}$ & $L\{e^{t^2}\}$ do not exist

$\therefore$ $L\{\frac{1}{t}\}$ & $L\{e^{t^2}\}$ do not exist

→ Linearity property:

If $L\{f(t)\} = F(s)$ & $L\{g(t)\} = G(s)$ then

$L\{a f(t) \pm b g(t)\} = a F(s) \pm b G(s)$, a & b are constants.

$= a L\{f(t)\} \pm b L\{g(t)\}$

Proof:

$$L\{a f(t) \pm b g(t)\} = \int_0^\infty e^{-st} (a f(t) \pm b g(t))\, dt$$

$$= \int_0^\infty a e^{-st} f(t)\, dt \pm \int_0^\infty b e^{-st} g(t)\, dt$$

$$= a \int_0^\infty e^{-st} f(t) dt \pm b \int_0^\infty e^{-st} g(t) dt$$

$$= a F(s) \pm b G(s)$$

→ Laplace transform of elementary functions :-

1) $L\{K\} = \frac{K}{s}$, $s > 0$, K is any constant

Sol: $f(t) = K$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt \Rightarrow L\{K\} = \int_0^\infty e^{-st} K\, dt$$

$$L\{K\} = K \int_0^\infty e^{-st} dt = K \left(\frac{e^{-st}}{-s}\right)_0^\infty = -\frac{K}{s}(0-1) = \frac{K}{s}$$

2) $\boxed{L\{t\} = \frac{1}{s^2}}$, $s > 0$

Sol: $L\{t\} = \int_0^\infty \underset{v}{e^{-st}}\, \underset{u}{t}\, dt$

$$\int uv\, dx = u\int v\, dx - \int \left(u' \int v\, dx\right) dx$$

$$L\{t\} = t\int_0^\infty e^{-st} dt - \int \left(1 \cdot \int e^{-st} dt\right) dt$$

$$= \left[t\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} dt$$

[If $\lim_{t\to\infty} f(t) = 0$, $\lim_{t\to\infty} g(t) = \infty$, then $\lim_{t\to\infty} f(t) g(t) = 0$] X.

[If $\lim_{t\to\infty} f(t) = 0$, $\lim_{t\to\infty} g(t) = \infty$, then $\lim_{t\to\infty} f(t)g(t)$] ×

$$L\{t\} = (0-0) + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_0^\infty = -\frac{1}{s^2}(0-1) = \frac{1}{s^2}$$

$$\lim_{t\to\infty} \frac{t}{e^{st}} = \frac{\infty}{\infty} \qquad \lim_{t\to\infty} \frac{1}{se^{st}} = 0 \quad ✓$$

→ $L\{t^n\} = \dfrac{n!}{s^{n+1}}$, n is +ve integer.

sol:
$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\{t^n\} = \int_0^\infty e^{-st} t^n\,dt = \left[t^n\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty - \int_0^\infty n t^{n-1} \frac{e^{-st}}{-s}\,dt$$

$$= 0 + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1}\,dt$$

$$L\{t^n\} = \frac{n}{s} L\{t^{n-1}\} \quad \text{— (1)}$$

$$L\{t^{n-1}\} = \frac{n-1}{s} L\{t^{n-2}\}$$

$$L\{t^{n-2}\} = \frac{n-2}{s} L\{t^{n-3}\}$$

$$\vdots$$

$$L\{t^{n-(n-2)}\} = \frac{n-(n-2)}{s} L\{t^{n-(n-1)}\}$$

$$L\{t^{n-(n-1)}\} = \frac{n-(n-1)}{s} L\{t^{n-n}\}$$

$$(1) \Rightarrow L\{t^n\} = \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s} \cdots\cdots \frac{n-(n-2)}{s}\,\frac{(n-(n-1))}{s}\cdot L\{1\}$$

$$= \frac{n(n-1)(n-2)\cdots 2\cdot 1}{s^n}\cdot\frac{1}{s}$$

$$\boxed{L\{t^n\} = \frac{n!}{s^{n+1}}} \quad n \text{ is +ve integer.}$$

Gamma function:- → improper integral in the form $\int_0^\infty e^{-x} x^{n-1}\,dx$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx \qquad \Gamma - \text{Gamma.}$$

Properties:
1) $\Gamma(n+1) = n\Gamma(n)$, $n > 0$
2) $\Gamma(n+1) = n!$, n is +ve integer.
3) $\Gamma(1) = 1$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$
4) $\Gamma(0)$, $\Gamma(-1)$, $\Gamma(-2)$, $\Gamma(-3)$, ---- are undefined.

$\rightarrow$ $\boxed{L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}}$, $n \neq -1, -2, -3, -4, \ldots$

$L\{\frac{1}{t}\}$ $L\{t^{-1}\}$ $L\{t^{-2}\}$ $L\{t^{-3}\}$ ---- are not Exist

$\rightarrow$ $\boxed{L\{e^{at}\} = \frac{1}{s-a}}$, $s > a$

$\rightarrow$ $\boxed{L\{e^{-at}\} = \frac{1}{s+a}}$

$\rightarrow$ $L\{e^{iat}\} = \frac{1}{s-ia}$

$L\{\cos at + i\sin at\} = \frac{1}{s-ia} \times \frac{s+ia}{s+ia}$

$\Rightarrow L\{\cos at\} + L\{i\sin at\} = \frac{s+ia}{s^2 - i^2a^2}$

$\Rightarrow L\{\cos at\} + iL\{\sin at\} = \frac{s}{s^2+a^2} + \frac{ia}{s^2+a^2}$

Compare real & imag parts

$\boxed{L\{\cos at\} = \frac{s}{s^2+a^2}}$ $\boxed{L\{\sin at\} = \frac{a}{s^2+a^2}}$

$\boxed{L\{\cosh at\} = \frac{s}{s^2-a^2}}$ $\boxed{L\{\sinh at\} = \frac{a}{s^2-a^2}}$

$L\{\cosh at\} = \frac{s}{s^2-a^2}$ | $\quad \frac{\quad}{s-a^2}$

(1) find $L\{ e^{2t} - 3e^{-4t} + t^2 + 2t^{5/2} + \cos 3t - 4\sin 2t + \sinh 2t - \cosh 3t + 9\}$

$= L\{e^{2t}\} - 3L\{e^{-4t}\} + L\{t^2\} + 2L\{t^{5/2}\} + L\{\cos 3t\} - 4L\{\sin 2t\} + L\{\sinh 2t\} - L\{\cosh 3t\} + L\{9\}.$

$= \frac{1}{s-2} - 3\cdot\frac{1}{s+4} + \frac{2!}{s^{2+1}} + \frac{2\cdot\Gamma(\frac{5}{2}+1)}{s^{\frac{5}{2}+1}} + \frac{2s}{s^2+3^2} - \frac{4\cdot 2}{s^2+2^2} + \frac{2}{s^2-2^2} - \frac{s}{s^2-3^2} + \frac{9}{s}$ —— (1)

$\boxed{\sinh at = \frac{e^{at}-e^{-at}}{2}}$

$\cosh at = \frac{e^{at}+e^{-at}}{2}$

$\Gamma(\frac{5}{2}+1) = ?$

$\Gamma(n+1) = n\Gamma(n), \quad n>0$

$\Gamma(\frac{5}{2}+1) = \frac{5}{2}\Gamma(\frac{5}{2}) = \frac{5}{2}\Gamma(\frac{3}{2}+1) = \frac{5}{2}\cdot\frac{3}{2}\Gamma(\frac{3}{2})$

$= \frac{5}{2}\cdot\frac{3}{2}\Gamma(\frac{1}{2}+1) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})$

$\Gamma(\frac{5}{2}+1) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$

2) $L\{(t^2+1)^2\} = ?$

$= L\{t^4+1+2t^2\}$

$= L\{t^4\} + L\{1\} + L\{2t^2\}$

$= \frac{4!}{s^{4+1}} + \frac{1}{s} + \frac{2\cdot 2!}{s^{2+1}}$

3) $L\{(\sin t + \cos t)^2\} = ?$ $\qquad A = \frac{1}{s} + \frac{2}{s^2+4}$

4) $L\{\cos^3 2t\} = ?$

$\cos 3A = 4\cos^3 A - 3\cos A \Rightarrow \cos^3 A = \dfrac{\cos 3A + 3\cos A}{4}$

$A = 2t$

$\cos^3 2t = \dfrac{\cos 6t + 3\cos 2t}{4}$

$L\{\cos^3 2t\} = \frac{1}{4} L\{\cos 6t + 3\cos 2t\}$

$= \frac{1}{4}\left[\dfrac{s}{s^2+6^2} + 3\cdot\dfrac{s}{s^2+2^2}\right]$

$\boxed{\cosh 2A = 2\cosh^2 A - 1}$

5) $L\{\cosh^2 2t\} \longrightarrow \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2-16}\right)$

$\sinh^3 2t$

6) $L\{\sinh^3 2t\} \longrightarrow \frac{1}{8}\left(\frac{12}{s^2-36} - \frac{12}{s^2-4}\right)$

$(\sinh 2t)^3$

$\left(\dfrac{e^{2t} - e^{-2t}}{2}\right)^3$

7) $L\left\{\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3\right\}$

8) $L\{\sin 2t \cos t\}$ — multiply & divide by 2

9) $L\{3\cos 3t \cos 4t\}$

10) $L\{\cos t \cos 2t \cos 3t\}$

7) $L\left\{\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3\right\}$

$L\left\{t^{3/2} + \frac{1}{t^{3/2}} + 3t\cdot\frac{1}{\sqrt{t}} + 3\sqrt{t}\cdot\frac{1}{t}\right\}$

$L\{t^{3/2}\} + L\{t^{-3/2}\} + 3\cdot L\{t^{1/2}\} + 3L\{t^{-1/2}\}$

$\dfrac{\Gamma(\frac{3}{2}+1)}{s^{\frac{3}{2}+1}} + \dfrac{\Gamma(-\frac{3}{2}+1)}{s^{-\frac{3}{2}+1}} + \dfrac{3\Gamma(\frac{1}{2}+1)}{s^{\frac{1}{2}+1}} + \dfrac{3\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}}$

$\Gamma(n+1) = n\Gamma(n), \quad n > 0$

$\dfrac{\frac{3}{2}\Gamma(\frac{3}{2})}{s^{5/2}} + \dfrac{\Gamma(-\frac{1}{2})}{s^{-1/2}} + \dfrac{3\cdot\frac{1}{2}\Gamma(\frac{1}{2})}{s^{3/2}} + \dfrac{3\Gamma(\frac{1}{2})}{s^{1/2}}$

$s^{-1/2}$ $\quad s$

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{5/2}} + \frac{-2\sqrt{\pi}}{s^{-1/2}} + \frac{3\cdot\frac{1}{2}\cdot\sqrt{\pi}}{s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}}.$$

$\rightarrow$ $L\{\text{sen}\,2t\cos t\}$

Sol $\quad \frac{1}{2}\,2\,L\{\text{sen}\,2t\cos t\} = \frac{1}{2}L\{2\,\text{sen}\,2t\cos t\}$

$2\,\text{sen}A\cos B = \text{sen}(A+B) + \text{sen}(A-B)$

$$= \frac{1}{2}L\{\text{sen}(2t+t) + \text{sen}(2t-t)\}$$

$$= \frac{1}{2}L\{\text{sen}\,3t + \text{sen}\,t\}$$

$$= \frac{1}{2}\left[L\{\text{sen}\,3t\} + L\{\text{sen}\,t\}\right]$$

$$= \frac{1}{2}\left[\frac{3}{s^2+3^2} + \frac{1}{s^2+1^2}\right]$$

$\rightarrow$ $L\{\cos t\cos 2t\cos 3t\} = ?$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$$L\{\cos t\,\tfrac{1}{2}(2\cos 2t\cos 3t)\}$$

$$\frac{1}{2}L\{\cos t\,(\cos(2t+3t) + \cos(2t-3t))\}$$

$$\frac{1}{2}L\{\cos t\,(\cos 5t + \cos t)\}$$

$\cos^2 t = \frac{1+\cos 2t}{2}$

$$= \frac{1}{2}L\{\cos t\cos 5t + \cos^2 t\}$$

$$= \frac{1}{2}L\{\cos t\cos 5t\} + \frac{1}{2}L\{\cos^2 t\}$$

$$= \frac{1}{4}L\{2\cos t\cos 5t\} + \frac{1}{2}L\left\{\frac{1+\cos 2t}{2}\right\}$$

$$= \frac{1}{4}L\{\cos 6t + \cos 4t\} + \frac{1}{4}L\{1+\cos 2t\}$$

$$= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{s}{s^2+16}\right] + \frac{1}{4}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$$

$$= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{s}{s^2+16}\right] + \frac{1}{4}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$$

→ find $L\{f(t)\}$, $f(t) = |t-1| + |t+1|$, $t \geq 0$.

sol:
$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$t \geq 0 \rightarrow \quad 0 < t < 1, \quad t > 1$

<u>$0 < t < 1$</u>

$t < 1 \Rightarrow t-1 < 0, \quad t > 0$

$t + 1 > 0$

$|t-1| = -(t-1), \quad |t+1| = t+1$

$f(t) = |t-1| + |t+1| = -(t-1) + t + 1 = 2$

<u>$t > 1$</u>

$t > 1 \Rightarrow t - 1 > 0$

$|t-1| = t-1, \quad |t+1| = t+1$

$f(t) = |t-1| + |t+1| = t-1+t+1 = 2t$

$$f(t) = \begin{cases} 2, & 0 < t < 1 \\ 2t, & t > 1 \end{cases}$$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st} f(t)\,dt + \int_1^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}\, 2\,dt + \int_1^\infty e^{-st}\, 2t\,dt$$

$$= 2\left(\frac{e^{-st}}{-s}\right)_0^1 + 2\left[\left(t\,\frac{e^{-st}}{-s}\right)_1^\infty - \int_1^\infty 1 \cdot \frac{e^{-st}}{-s}\,dt\right].$$

$$= \frac{-2}{s}\left(e^{-s} - 1\right) - \frac{2}{s}\left(0 - e^{-s}\right) + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_1^\infty\Bigg]$$

$$\cancel{\frac{-2}{s}e^{-s}} + \frac{2}{s} + \cancel{\frac{2}{s}e^{-s}} - \frac{1}{s^2}\left(0 - e^{-s}\right)$$

$$= \frac{2}{s} + \frac{e^{-s}}{s^2}.$$

$\rightarrow$ $L\{\sin\sqrt{t}\} = ?$ $\qquad \left\{ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \int_0^\infty e^{-st}\sin\sqrt{t}\,dt \right\}$

$\rightarrow$ If $f(t) = \begin{cases} 1, & 0 < t < 2 \\ 2, & 2 < t < 4 \\ 3, & 4 < t < 6 \\ 0, & t > 6 \end{cases}$

Sol: $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 e^{-st} f(t)\,dt + \int_2^4 e^{-st} f(t)\,dt + \int_4^6 e^{-st} f(t)\,dt + \int_6^\infty e^{-st} f(t)\,dt$

$$L\{f(t)\} = \int_0^2 e^{-st}\cdot 1\,dt + \int_2^4 e^{-st}\cdot 2\,dt + \int_4^6 e^{-st}\cdot 3\,dt + \int_6^\infty e^{-st}\cdot 0\,dt$$

$$= ?$$

$\rightarrow$ First shifting property:– (First translation theorem)

If $L\{f(t)\} = F(s)$, then $L\{e^{at}f(t)\} = F(s-a)$

$$\Rightarrow L\{e^{at}f(t)\} = [F(s)]_{s\to s-a}$$

$$L\{e^{at}f(t)\} = [L\{f(t)\}]_{s\to s-a}$$

IIly $L\{e^{-at}f(t)\} = F(s+a)$

$$= [L\{f(t)\}]_{s\to s+a}.$$

1) $L\{e^{at}t^n\} = [L\{t^n\}]_{s\to s-a}$, n is +ve integer.

$$= \left(\frac{n!}{\quad}\right) \qquad = \frac{n!}{\;\;^{n+1}}$$

1) $L\{e^{at} t^n\}$ ...

$$= \left(\frac{n!}{s^{n+1}}\right)_{s \to s-a} = \frac{n!}{(s-a)^{n+1}}$$

2) $L\{e^{-at} t^n\} = \dfrac{n!}{(s+a)^{n+1}}$

3) $L\{e^{at} \sin bt\} = \dfrac{b}{(s-a)^2+b^2}$, $L\{e^{at} \cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}$

→ find $L\{e^{-3t}(2\cos 5t - 3\sin 5t)\}$

$$= L\{2e^{-3t}\cos 5t - 3e^{-3t}\sin 5t\}$$

$$= 2L\{e^{-3t}\cos 5t\} - 3L\{e^{-3t}\sin 5t\}$$

$$= 2\left[\left(L\{\cos 5t\}\right)_{s \to s+3}\right] - 3\left[\left(L\{\sin 5t\}\right)_{s \to s+3}\right]$$

$$= 2\left[\left(\frac{s}{s^2+25}\right)_{s \to s+3}\right] - 3\left[\left(\frac{5}{s^2+25}\right)_{s \to s+3}\right]$$

$$= \frac{2(s+3)}{(s+3)^2+25} - \frac{15}{(s+3)^2+25}$$

→ find $L\{e^{-3t}(\cos 4t + 3\sin 4t)\}$ → $\dfrac{s+15}{s^2+6s+25}$

→ find $L\{e^{4t}\sin 2t \cos t\}$

sol $= \left[L\{\sin 2t \cos t\}\right]_{s \to s-4}$ (By F.S. Property)

$$= \frac{1}{2}\left[L\{2\sin 2t\cos t\}\right]_{s \to s-4}$$

$$= \frac{1}{2}\left[L\{\sin 3t + \sin t\}\right]_{s \to s-4}$$

$$= \frac{1}{2}\left[\frac{3}{s^2+9} + \frac{1}{s^2+1}\right]_{s \to s-4}$$

$$= \frac{1}{2}\left[\frac{3}{(s-4)^2+9} + \frac{1}{(s-4)^2+1}\right].$$

$\rightarrow$ $L\{\cosh at \sin bt\}$ $\qquad\qquad$ $\cosh at = \frac{e^{at}+e^{-at}}{2}$

$L\left\{\left(\frac{e^{at}+e^{-at}}{2}\right)\sin bt\right\}$

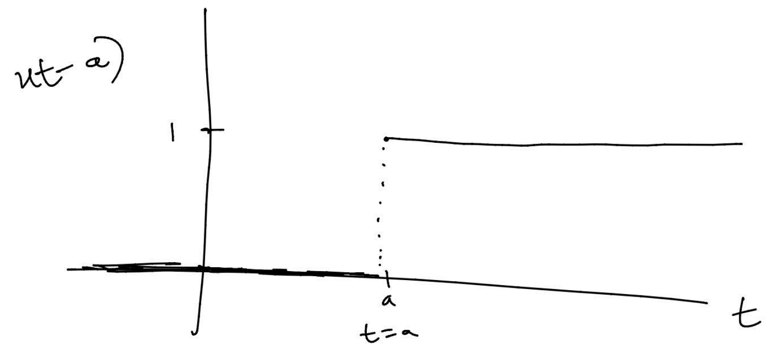
$\frac{1}{2} L\{e^{at}\sin bt + e^{-at}\sin bt\}$

$\rightarrow$ Unit step function : (or) Heaviside function:

The unit step function is denoted by $u(t-a)$ or $H(t-a)$ or $u_a(t)$ & it is defined as

$$u(t-a) = \begin{cases} 0, & \text{if } t<a \rightarrow 0<t<a \\ 1, & \text{if } t>a. \rightarrow a<t<\infty \end{cases}$$

This is unit step function at $t=a$



for $a=0 \rightarrow u(t) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}$

$$L\{u(t-a)\} = \int_0^{\infty} e^{-st} u(t-a)\,dt.$$

$$= \int_0^{a} e^{-st} u(t-a)\,dt + \int_a^{\infty} e^{-st} u(t-a)\,dt$$

$$\int^{\infty} e^{-st} \cdot 1 \, dt$$

$$= \int_0^a \ldots$$

$$= 0 + \int_a^\infty e^{-st} \cdot 1 \, dt$$

$$= \left(\frac{e^{-st}}{-s}\right)_a^\infty = -\frac{1}{s}\left(0 - e^{-as}\right) = \frac{e^{-as}}{s}$$

$$\boxed{L\{u(t-a)\} = \frac{e^{-as}}{s}}$$

→ Second shifting property:

If $L\{f(t)\} = F(s)$ & let $g(t) = \begin{cases} f(t-a) & , t > a \\ 0 & , t < a \end{cases}$, then

$$L\{g(t)\} = e^{-as} F(s).$$

Another form of second shifting property in terms of unit step function

If $L\{f(t)\} = F(s)$, then $L\{f(t-a)u(t-a)\} = e^{-as} F(s)$

→ If $g(t) = \begin{cases} \cos t(t - \frac{\pi}{3}) & , t > \frac{\pi}{3} \\ 0 & , t < \frac{\pi}{3} \end{cases}$ then find $L\{g(t)\}$

sol: $g(t) = \begin{cases} f(t-a) & , t > a \\ 0 & , t < a \end{cases}$

$f(t-a) = \cos(t - \frac{\pi}{3})$ $\quad a = \frac{\pi}{3}$

$f(t) = \cos t$

$F(s) = L\{f(t)\} = L\{\cos t\} = \frac{s}{s^2+1}$

By second shifting prop $L\{g(t)\} = e^{-as} F(s)$

$$L\{g(t)\} = e^{-\frac{\pi}{3}s} \cdot \frac{s}{s^2+1}$$

→ find $L\{t\, u(t-3)\}$

sol: $L\{f(t-a)u(t-a)\}$ $\quad a = 3$ $\quad f(t-a) = \cos(t-a)$

sol:

$L\{f(t-a)u(t-a)\}$ $\quad a=3$ $\quad f(t-a) = \cos(t-a)$

$L\{t\,u(t-3)\} = L\{(t-3+3)\,u(t-3)\}$ $\quad f(t) = \cos t$

$f(t-a) = f(t-3) = (t-3)+3$

$f(t) = t+3$

$F(s) = L\{f(t)\} = L\{t+3\} = \frac{1}{s^2} + \frac{3}{s}$

By second shif prop $L\{f(t-a)u(t-a)\} = e^{-as}F(s)$

$L\{(t-3+3)u(t-3)\} = e^{-3s}\left(\frac{1}{s^2}+\frac{3}{s}\right).$

$\rightarrow L\{e^{-3t}u(t-2)\}$

$= L\{e^{-3(t-2+2)}u(t-2)\}$

$= L\{e^{-3(t-2)}e^{-6}u(t-2)\}$

$= e^{-6}L\{e^{-3(t-2)}u(t-2)\}$ — (1)

$L\{e^{-3(t-2)}u(t-2)\} = L\{f(t-a)u(t-a)\}$

$\Rightarrow f(t-a) = f(t-2) = e^{-3(t-2)}$, $a=2$

$f(t) = e^{-3t}$

$F(s) = L\{f(t)\} = \frac{1}{s+3}$

By S.S.P $L\{f(t-a)u(t-a)\} = L\{e^{-3(t-2)}u(t-2)\} = e^{-2s}\cdot\frac{1}{s+3}$ — (2)

(1) $\Rightarrow L\{e^{-3t}u(t-2)\} = e^{-6}\cdot e^{-2s}\cdot\frac{1}{s+3}.$

$\rightarrow$ change of scale property:

If $L\{f(t)\} = F(s)$, then $L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$ &

$L\{f(\frac{t}{a})\} = aF(as).$

$\rightarrow$ If $L\{f(t)\} = \frac{1}{s}e^{-1/s}$, find $L\{e^{-t}f(3t)\}$

$\frac{1}{}\ e^{-1/s}$ (1)

$\rightarrow$ If $L\{f(t)\} = \frac{1}{s}$ ...

Sol: $F(s) = \frac{1}{s} e^{-1/s}$ —— (1)

$L\{e^{-t} f(3t)\} = \left[L\{f(3t)\}\right]_{s \to s+1}$ (∵ By First shifting prop)

$= \left[\frac{1}{3} F\left(\frac{s}{3}\right)\right]_{s \to s+1}$ (∵ By change of scale)

$= \left[\frac{1}{3} \frac{1}{s/3} e^{-1/s/3}\right]_{s \to s+1}$

$= \left[\frac{\cancel{3}}{\cancel{3}s} e^{-3/s}\right]_{s \to s+1}$

$L\{e^{-t} f(3t)\} = \left[\frac{1}{s+1} e^{-3/s+1}\right]$

$\rightarrow$ Laplace transform of derivatives:

If $L\{f(t)\} = F(s)$, then

$L\{f'(t)\} = sF(s) - f(0)$

$L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$

$L\{f'''(t)\} = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$

$\vdots$

$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots\cdots - f^{n-1}(0)$.

$\rightarrow$ Laplace transform of integrals:

If $L\{f(t)\} = F(s)$, then

$L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s} F(s)$

$L\left\{\int_0^t \int_0^t f(t)\,dt\,dt\right\} = \frac{1}{s^2} F(s)$

$\vdots$

$L\{\underbrace{\int_0^t \int_0^t \cdots \int_0^t}_{n\text{-times}} f(t)\,\underbrace{dt\,dt \cdots dt}_{n\text{times}}\} = \frac{1}{s^n} F(s)$.

→ using L.T of derivatives find

1) $L\{e^{at}\}$

Sol: $f(t) = e^{at}$, $f'(t) = ae^{at}$.

$f(0) = 1$

using L.T of derivatives $L\{f'(t)\} = sF(s) - f(0)$

$L\{f'(t)\} = sL\{f(t)\} - f(0)$

$\Rightarrow L\{ae^{at}\} = sL\{e^{at}\} - 1$

$\Rightarrow a\,L\{e^{at}\} = sL\{e^{at}\} - 1$

$L\{e^{at}\}(s-a) = 1$

$\Rightarrow L\{e^{at}\} = \frac{1}{s-a}$

→ $L\{\sin at\}$, $L\{t\sin at\}$ $= ?$

→ If $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$, then find $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$

Sol: let $f(t) = \sin\sqrt{t}$, $f(0) = 0$, $F(s) = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$

$f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$

L.T of derivatives $\Rightarrow$ $L\{f'(t)\} = sL\{f(t)\} - f(0)$

$= sF(s) - f(0)$

$L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s\cdot\frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}} - 0$

$\frac{1}{\cancel{2}} L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{\cancel{2}s^{1/2}} e^{-1/4s}$

$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{s^{1/2}} e^{-1/4s}$

→ If $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s^{3/2}}$, find $L\left\{\frac{1}{\sqrt{t\pi}}\right\}$

→ P.T $L\{\int_0^t e^{-t}\cos t\,dt\}$

$\rightarrow$ find $L\left\{\int_0^t e^{-t}\cos t\,dt\right\}$

$L\left\{\int_0^t e^{-t}\cos t\,dt\right\} = L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}F(s)$ — (1)

($\because$ L.T of integrals)

$f(t) = e^{-t}\cos t$

$F(s) = L\{f(t)\} = L\{e^{-t}\cos t\}$ ($\because$ F.S.P)

$= \left[L\{\cos t\}\right]_{s\to s+1}$

$= \left[\frac{s}{s^2+1}\right]_{s\to s+1}$

$F(s) = \frac{s+1}{(s+1)^2+1} = \frac{s+1}{s^2+2s+2}$

(1) $\Rightarrow$ $L\left\{\int_0^t e^{-t}\cos t\,dt\right\} = \frac{1}{s}F(s) = \frac{1}{s}\cdot\frac{s+1}{s^2+2s+2}$.

$\rightarrow$ Multiplication by t^n:

If $L\{f(t)\} = F(s)$, then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$, $n = 1, 2, 3, \ldots$

$\rightarrow$ Division by t

If $L\{f(t)\} = F(s)$, then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$.

$\rightarrow$ $L\{t e^{-t}\sin t\}$

Sol: $L\{e^{-t}(t\sin t)\} = L\{e^{-t}f(t)\} = \left[L\{f(t)\}\right]_{s\to s+1}$ — (1)

$f(t) = t\sin t$.

$L\{f(t)\} = L\{t\sin t\} = L\{t\,g(t)\}$

$= (-1)^1 \frac{d}{ds}G(s)$ — (2)

$g(t) = \sin t \Rightarrow G(s) = L\{g(t)\} = L\{\sin t\} = \frac{1}{s^2+1}$

$G(s) = \frac{1}{s^2+1}$

(2) $\Rightarrow$ $L\{f(t)\} = -\frac{d}{ds}G(s) = -\frac{d}{ds}\cdot\frac{1}{s^2+1}$

(2) $\Rightarrow$ $L\{f(t)\} = -\frac{d}{ds} G(s) = -\frac{d}{ds} \cdot \frac{1}{s^2+1}$

$L\{f(t)\} = -\left(-\frac{1}{(s^2+1)^2} \; 2s\right) = \frac{2s}{(s^2+1)^2}$

(1) $\Rightarrow$ $L\{e^{-t}\, t \sin t\} = \left[L\{f(t)\}\right]_{s\to s+1} = \left[\frac{2s}{(s^2+1)^2}\right]_{s\to s+1}$

$= \frac{2(s+1)}{((s+1)^2+1)^2}$

$\rightarrow$ $L\left\{\int_0^t t e^{-t} \sin 4t \, dt\right\}$

Sol: $L\left\{\int_0^t t e^{-t} \sin 4t \, dt\right\} = L\left\{\int_0^t f(t)\, dt\right\} = \frac{1}{s} F(s)$ — (1)

($\because$ By L.T of integrals)

where $f(t) = t e^{-t} \sin 4t$

$F(s) = L\{f(t)\} = L\{e^{-t}(t \sin 4t)\}$

$= L\{e^{-t} g(t)\}$ where $g(t) = t \sin 4t$

$F(s) = \left[L\{g(t)\}\right]_{s\to s+1}$ ($\because$ By F.S.P) — (2)

$L\{g(t)\} = L\{t \sin 4t\} = L\{t\, h(t)\}$ where $h(t) = \sin 4t$.

$L\{g(t)\} = (-1)^1 \frac{d}{ds} H(s)$ — (3)

$H(s) = L\{h(t)\} = L\{\sin 4t\} = \frac{4}{s^2+16}$

(3) $\Rightarrow$ $L\{g(t)\} = -\frac{d}{ds}\left(\frac{4}{s^2+16}\right)$

$= -4 \cdot \left(\frac{-1}{(s^2+16)^2} \; 2s\right) = \frac{8s}{(s^2+16)^2}$

$L\{g(t)\} = \frac{8s}{(s^2+16)^2}$

(2) $\Rightarrow$ $F(s) = \left[L\{g(t)\}\right]_{s\to s+1} = \left[\frac{8s}{(s^2+16)^2}\right]_{s\to s+1}$

$F(s) = 8(s+1) = \frac{8(s+1)}{\quad}$

$$F(s) = \frac{8(s+1)}{((s+1)^2+16)^2} = \frac{8(s+1)}{(s^2+2s+17)^2}$$

(1) $\Rightarrow L\{\int_0^t e^{-t}\, t\sin 4t\} = \frac{1}{s} F(s) = \frac{1}{s} \cdot \frac{8(s+1)}{(s^2+2s+17)^2}$.

→ find 1) $L\{t \sin at\}$

2) $L\{t \sin 3t \cos 2t\}$

3) $L\{t e^{2t} \sin 3t\}$

4) $L\{(1+te^{-t})^2\} = L\{1 + t^2 e^{-2t} + 2te^{-t}\}$

→ If $L\{t^{1/2}\} = \frac{\sqrt{\pi}}{2s^{3/2}}$, find $L\{t^{-1/2}\}$

sol: Let $f(t) = t^{-1/2}$

$$t f(t) = t\, t^{-1/2} = t^{1/2}$$

$$L\{t f(t)\} = (-1)^1 \frac{d}{ds} L\{f(t)\}$$

$$L\{t^{1/2}\} = -\frac{d}{ds} L\{t^{-1/2}\}$$

$$\frac{\sqrt{\pi}}{2s^{3/2}} = -\frac{d}{ds} L\{t^{-1/2}\}$$

$$\int \frac{\sqrt{\pi}}{2} s^{-3/2} ds = \int \left(-\frac{d}{ds} L\{t^{-1/2}\}\right) ds$$

$$\cancel{-}\cancel{2} \frac{\sqrt{\pi}}{\cancel{2}} s^{-1/2} = \cancel{-} L\{t^{-1/2}\}$$

$$L\{t^{-1/2}\} = \sqrt{\frac{\pi}{s}}$$

→ find $L\left\{\frac{\sin t}{t}\right\}$

sol $L\left\{\frac{\sin t}{t}\right\} = L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds$ — (1) (∵ Division by t)

where $f(t) = \sin t$

where $f(t) = \sin t$

$$F(s) = L\{f(t)\} = L\{\sin t\} = \frac{1}{s^2+1}$$

$$(1) \Rightarrow L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds = \left(\tan^{-1}s\right)_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}s$$

$$= \frac{\pi}{2} - \tan^{-1}s$$

$$L\left\{\frac{\sin t}{t}\right\} = \cot^{-1}s.$$

(2) $L\left\{\dfrac{e^{-at}-e^{-bt}}{t}\right\}$

Sol $L\left\{\dfrac{e^{-at}-e^{-bt}}{t}\right\} = L\left\{\dfrac{f(t)}{t}\right\} = \displaystyle\int_s^\infty F(s)\,ds$ —— (1)

where $f(t) = e^{-at} - e^{-bt}$

$$F(s) = L\{e^{-at} - e^{-bt}\} = \frac{1}{s+a} - \frac{1}{s+b}$$

$$(1) \Rightarrow L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)\right]_s^\infty$$

$$= \log 1 - \log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)$$

$$= -\log\left(\frac{s+a}{s+b}\right)$$

$$= \log\left(\frac{s+b}{s+a}\right).$$

$\rightarrow L\left\{\dfrac{\cos 2t-\cos 3t}{t}\right\} = \dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2+4}\right)$

$\rightarrow L\left\{\dfrac{1-e^t}{t}\right\} = \log\left(\dfrac{s-1}{s}\right)$

$\rightarrow L\left\{\int_0^t \dfrac{e^t \sin t}{t}\,dt\right\}$

sol: $\Rightarrow L\left\{\int_0^t \dfrac{e^t \sin t}{t}\,dt\right\} = L\left\{\int_0^t f(t)\,dt\right\} = \dfrac{1}{s}F(s)$ — ①

where $f(t) = \dfrac{e^t \sin t}{t}$

$F(s) = L\{f(t)\} = L\left\{\dfrac{e^t \sin t}{t}\right\} = L\left\{\dfrac{g(t)}{t}\right\} = \int_s^\infty G(s)\,ds$ — ②

where $g(t) = e^t \sin t$

$G(s) = L\{g(t)\} = L\{e^t \sin t\}$

$= [L\{\sin t\}]_{s\to s-1} = \left(\dfrac{1}{s^2+1}\right)_{s\to s-1}$

$G(s) = \dfrac{1}{(s-1)^2+1}$

② $\Rightarrow F(s) = \int_s^\infty G(s)\,ds = \int_s^\infty \dfrac{1}{(s-1)^2+1}\,ds = \left[\tan^{-1}(s-1)\right]_s^\infty$

$= \tan^{-1}\infty - \tan^{-1}(s-1)$

$= \dfrac{\pi}{2} - \tan^{-1}(s-1)$

$F(s) = \cot^{-1}(s-1)$

① $\Rightarrow L\left\{\int_0^t \dfrac{e^t \sin t}{t}\,dt\right\} = \dfrac{1}{s}F(s) = \dfrac{1}{s}\cot^{-1}(s-1).$

$\rightarrow L\left\{\dfrac{1-\cos at}{t}\right\} = ?$

$\rightarrow L\left\{e^{-3t}\int_0^t \dfrac{\sin t}{t}\,dt\right\} = \dfrac{1}{s+3}\cot^{-1}(s+3)$

Evaluation of integrals by using Laplace transform:

→ Evaluation of integrals by using Laplace transform:

① Evaluate $\int_0^\infty t e^{-3t} dt$.

Sol: $F(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-3t}\, t\, dt$

$\downarrow$ $L\{f(t)\}$ $\quad s = 3, \quad f(t) = t$.

$= L\{t\} = \frac{1}{s^2}$

$= \frac{1}{9}$

② $\int_0^\infty e^{-4t} \sin 3t\, dt$

$\int_0^\infty e^{-4t} \sin 3t\, dt = \int_0^\infty e^{-st} f(t) dt$

$s = 4, \quad f(t) = \sin 3t$.

$\int_0^\infty e^{-4t} \sin 3t\, dt = L\{f(t)\} = L\{\sin 3t\} = \frac{3}{s^2+9}$

$= \frac{3}{4^2+9} = \frac{3}{25}$

③ $\int_0^\infty \frac{\sin 2t}{t} dt$

$\int_0^\infty e^{-0t} \frac{\sin 2t}{t} dt = \int_0^\infty e^{-st} f(t)\, dt$

$s = 0, \quad f(t) = \frac{\sin 2t}{t}$

$\int_0^\infty \frac{\sin 2t}{t} dt = L\{f(t)\} = L\left\{\frac{\sin 2t}{t}\right\} = L\left\{\frac{g(t)}{t}\right\}$

$= \int_s^\infty G(s)\, ds$ ——— ①

where $g(t) = \sin 2t$

$G(s) = L\{g(t)\} = L\{\sin 2t\} = \frac{2}{s^2+4}$

① ⇒ $\int_0^\infty \frac{\sin 2t}{t} dt = \int \frac{2}{s^2+4} ds$

① $\Rightarrow$ $\int_0^{\infty} \frac{\sin 2t}{t}\, dt = \int_s^{\infty} \frac{2}{s^2+4}\, ds$

$= 2 \cdot \frac{1}{2} \left[\tan^{-1}\left(\frac{s}{2}\right) \right]_s^{\infty} = \frac{\pi}{2} - \tan^{-1}\frac{s}{2}$

$= \frac{\pi}{2} - \tan^{-1}(0) \quad (\because s = 0)$

$= \frac{\pi}{2}$.

$\rightarrow$ $\int_0^{\infty} t^2 e^{-4t} \sin 2t\, dt = \frac{11}{500}$

$\rightarrow$ $\int_0^{\infty} \left(\frac{e^{-at} - e^{-bt}}{t} \right) dt = \log \frac{b}{a}$

$\rightarrow$ $\int_0^{\infty} \left(\frac{\cos at - \cos bt}{t} \right) dt = \log \frac{1}{5}$.

Properties

16 October 2020 10:50

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n, \quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}}, \quad n=1,2,3,\ldots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at}, \quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t), \quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^{\infty}F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

Expressing a piecewise continuous function in terms of a unit step function or Heaviside function :- $u(t-a)$ (or) $H(t-a)$ or $u_a(t)$:-

that

or Heaviside function :- $u(t-a)$ (or) $H(t-a)$ or $u_a(t)$.

Let $f(t)$ be a piecewise continuous function defined such that

$$f(t) = \begin{cases} f_1(t), & a < t \le a_1 \\ f_2(t), & a_1 < t \le a_2 \\ f_3(t), & a_2 < t \le a_3 \\ f_4(t), & t > a_3 \end{cases}$$

then $f(t)$ can be expressed in terms of unit step function as

$$\boxed{f(t) = f_1(t)\left[u(t-a) - u(t-a_1)\right] + f_2(t)\left[u(t-a_1) - u(t-a_2)\right] + f_3(t)\left[u(t-a_2) - u(t-a_3)\right] + f_4(t)\,u(t-a_3)}$$

→ find the Laplace transform of $f(t) = \begin{cases} 0, & 0 < t < 1 \\ t-1, & 1 < t < 2 \\ t, & t > 2 \end{cases}$

by expressing it in terms of unit step function.

sol: $f(t) = \begin{cases} 0 & , 0 < t < 1 \\ t-1 & , 1 < t < 2 \\ t & , t > 2 \end{cases}$

$$f(t) = 0\left[u(t-0) - u(t-1)\right] + (t-1)\left[u(t-1) - u(t-2)\right] + t\,u(t-2)$$

$$= (t-1)u(t-1) - (t-1)u(t-2) + t\,u(t-2)$$

$$= (t-1)u(t-1) - u(t-2)\left[\cancel{t} - 1 - \cancel{t}\right]$$

$$f(t) = (t-1)u(t-1) + u(t-2)$$

$$L\{f(t)\} = L\{(t-1)u(t-1)\} - L\{u(t-2)\}$$
$$+$$

$f(t) = t, a = 1$

$F(s) = \frac{1}{s^2}$

$$L\{f(t)\} = e^{-s}\cdot\frac{1}{s^2} - e^{-2s}\cdot\frac{1}{s}.$$

$L[f(t-1)u(t-1)] = e^{-as}F(s)$

$f(t-a) = t-1$

$u(t-a) = u(t-1)$

→ $f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ \ldots & \pi < t < 2\pi \end{cases}$

$$\rightarrow \quad f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ \sin 2t, & \pi \le t < 2\pi \\ \sin 3t, & t > 2\pi \end{cases}$$

$\underline{Sol}$ $$f(t) = \sin t\,[u(t-0) - u(t-\pi)] + \sin 2t\,[u(t-\pi) - u(t-2\pi)] + \sin 3t\,[u(t-2\pi)]$$

$$f(t) = \sin t\, u(t) + u(t-\pi)[\sin 2t - \sin t] + u(t-2\pi)[\sin 3t - \sin 2t]$$

$$L\{f(t)\} = \underset{(1)}{L\{\sin(t-0)u(t-0)\}} + \underset{(2)}{L\{[\sin 2t - \sin t]\,u(t-\pi)\}} + \underset{(3)}{L\{[\sin 3t - \sin 2t]\,u(t-2\pi)\}} \longleftarrow (A)$$

(1) $L\{\sin(t-0)\,u(t-0)\}$

$f(t) = \sin t \quad , a = 0 \quad , F(s) = \dfrac{1}{s^2+1}$

$= e^{-0s} F(s) = \dfrac{1}{s^2+1}$

(2) $L\{\sin 2t - \sin t\}\,u(t-\pi)$

$L\{-\sin(2\pi - 2t) - \sin(\pi - t)\}\,u(t-\pi)$

$L\{\sin 2(t-\pi) + \sin(t-\pi)\}\,u(t-\pi) = L\{f(t-a)\,u(t-a)\}$

$a = \pi$

$f(t-\pi) = \sin 2(t-\pi) + \sin(t-\pi)$

$f(t) = \sin 2t + \sin t$

$F(s) = \dfrac{2}{s^2+4} + \dfrac{1}{s^2+1}$

$= e^{-as} F(s) = e^{-\pi s}\left(\dfrac{2}{s^2+4} + \dfrac{1}{s^2+1}\right)$

(3) $L\{(\sin 3t - \sin 2t)\,u(t-2\pi)\}$

$L\{-\sin(6\pi - 3t) + \sin(4\pi - 2t)\}\,u(t-2\pi)$

$L\{\sin 3(t-2\pi) - \sin 2(t-2\pi)]\,u(t-2\pi)$

$\sin(6\pi - 3t)$
$\sin(2\pi + 4\pi - 3t)$
$\sin(4\pi - 3t)$
$\sin(2\pi + 2\pi - 3t)$
$\sin(2\pi - 3t)$

$L\{\sin 3(t-2\pi) - \sin 2(t-2\pi)\}u(t-2\pi)$

$\sin(2\pi+2\pi-3t)$

$\sin(2\pi-3t)$

$-\sin 3t$

$L\{f(t-a)\,u(t-a)\}$

$f(t-2\pi) = \sin 3(t-2\pi) - \sin 2(t-2\pi)$

$a = 2\pi$

$f(t) = \sin 3t - \sin 2t$

$F(s) = L\{\sin 3t - \sin 2t\} = \dfrac{3}{s^2+9} - \dfrac{2}{s^2+4}$

$= e^{-as}F(s) = e^{-2\pi s}\cdot\left(\dfrac{3}{s^2+9} - \dfrac{2}{s^2+4}\right)$

→ Periodic functions:

A function $f(t)$ is said to be periodic function of period 'T'

if $f(t+T) = f(t)$

If $f(t)$ is periodic of period T, then $f(t+T) = f(t+2T) = f(t+3T) = \cdots = f(t)$

L.T of periodic function

If $f(t)$ is a periodic function of period 'T', then

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt.$$

sol: $f(t)$ is periodic of period T

$f(t) = f(t+T) = f(t+2T) = f(t+3T) = \cdots\cdots$

$$L\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$$

$$= \int_0^T e^{-st}f(t)\,dt + \int_T^{2T} e^{-st}f(t)\,dt + \int_{2T}^{3T} e^{-st}f(t)\,dt + \cdots\cdots$$

In the second integral, $t = u+T \Rightarrow dt = du$

$t = T \Rightarrow u = 0$

$t = 2T \Rightarrow u = T$

In the third integral, $t = u+2T$, $dt = du$

In the third integral, $t = u+2T$, $dt = du$

for $t = 2T$, $u = 0$

$t = 3T$, $u = T$

& So on.

$$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(u+T)} f(u+T)\,du + \int_0^T e^{-s(u+2T)} f(u+2T)\,du + \text{-------}$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-su} f(u+T)\,du + e^{-2sT}\int_0^T e^{-su} f(u+2T)\,du + \text{------}$$

$$f(u) = f(u+T) = f(u+2T) = f(u+3T) = \text{------}$$

$$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-su} f(u)\,du + e^{-2sT}\int_0^T e^{-su} f(u)\,du + \text{-----}$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-st} f(t)\,dt + e^{-2sT}\int_0^T e^{-st} f(t)\,dt + \text{----}$$

$$= \int_0^T e^{-st} f(t)\,dt \left[1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \text{----}\right]$$

$$= \int_0^T e^{-st} f(t)\,dt \left[\frac{1}{1-e^{-sT}}\right]$$

$$\boxed{L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt.}$$

→ find $L\{f(t)\}$, where $f(t)$ is a periodic function of period 2π given by $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi. \end{cases}$

Sol: $T = 2\pi$

$$\frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt = \frac{1}{1-e^{-s2\pi}}\int_0^{2\pi} e^{-st} f(t)\,dt$$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt = \frac{1}{1-e^{-s2\pi}} \int_0^{2\pi} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s\pi}} \left[\int_0^{\pi} e^{-st} \sin t\,dt + \int_{\pi}^{2\pi} e^{-st}\, 0\,dt \right]$$

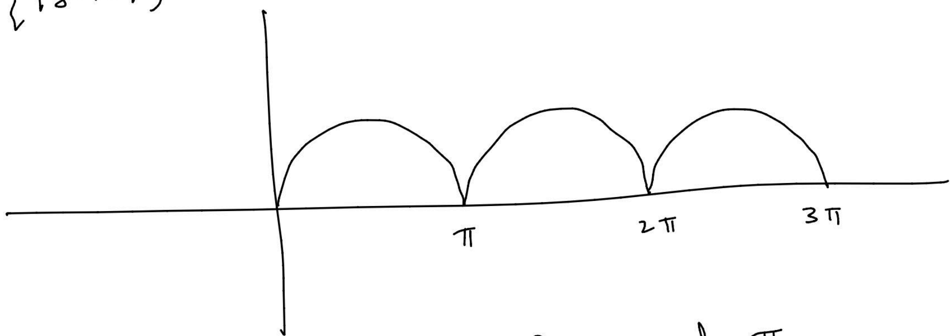
$$\boxed{\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx)}$$

$$= \frac{1}{1-e^{-2s\pi}} \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t) \right]_0^{\pi} + 0$$

$$= \frac{1}{(s^2+1)(1-e^{-2s\pi})} \left[\left(e^{-s\pi}(0-(-1)\right) - \left(1(0-1)\right) \right]$$

$$= \frac{1}{(s^2+1)(1-e^{-2s\pi})} \left[e^{-s\pi} + 1 \right] = \frac{e^{-s\pi}+1}{(s^2+1)(1-e^{-2s\pi})}.$$

→ find $L\{|\sin t|\}$.



$|\sin t|$ is periodic function of period π

$T = \pi$

$$L\{f(t)\} = \frac{1}{1-e^{-s\pi}} \int_0^{\pi} e^{-st} |\sin t|\,dt = \frac{1}{1-e^{-s\pi}} \int_0^{\pi} e^{-st} \sin t\,dt$$

?

→ find $L\{f(t)\}$, $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & 1 \le t < 2 \end{cases}$, $f(t) = f(t+2)$

→ Impulse function (or) Dirac delta function:

it is defined as

→ Impulse function (or) Dirac delta function:

Impulse function is denoted by $\delta(t-a)$ & it is defined as

$$\delta(t-a) = \lim_{\varepsilon \to 0} f_\varepsilon(t-a)$$

where $f_\varepsilon(t-a) = \begin{cases} 0, & \text{for } t < a \\ \frac{1}{\varepsilon}, & \text{for } a \le t \le a+\varepsilon \\ 0 & \text{for } t > a+\varepsilon. \end{cases}$ $\qquad \varepsilon > 0$

$$L\{\delta(t-a)\} = L\{\lim_{\varepsilon \to 0} f_\varepsilon(t-a)\} = \lim_{\varepsilon \to 0} L\{f_\varepsilon(t-a)\}$$

$$= \lim_{\varepsilon \to 0}\left[\int_0^\infty e^{-st} f_\varepsilon(t-a)\,dt\right]$$

$$= \lim_{\varepsilon \to 0}\left[\int_0^a e^{-st}\, 0\, dt + \int_a^{a+\varepsilon} e^{-st}\cdot\frac{1}{\varepsilon}\,dt + \int_{a+\varepsilon}^\infty e^{-st}\cdot 0\,dt\right]$$

$$= \lim_{\varepsilon \to 0}\int_a^{a+\varepsilon} e^{-st}\cdot\frac{1}{\varepsilon}\,dt = \lim_{\varepsilon \to 0}\frac{1}{\varepsilon}\left(\frac{e^{-st}}{-s}\right)_a^{a+\varepsilon}$$

$$= \lim_{\varepsilon \to 0}\frac{1}{-\varepsilon s}\left(e^{-s(a+\varepsilon)} - e^{-as}\right) = \frac{0}{0}$$

$$= \lim_{\varepsilon \to 0}\frac{e^{-as}\cdot(-\cancel{s}e^{-\varepsilon s}) - 0}{-\cancel{s}}$$

$$\lim_{\varepsilon \to 0} e^{-as}\cdot e^{-\varepsilon s} = e^{-as}$$

$$\boxed{L\{\delta(t-a)\} = e^{-as}}$$

for $a = 0 \Rightarrow L\{\delta(t)\} = 1$

Inverse Laplace transform

If $L\{f(t)\} = F(s)$ then the inverse Laplace transform of $F(s)$ is $f(t)$

$F(s)$ — Laplace transform

$f(t)$ — inverse Laplace transform

$\ldots\{F(s)\}$

$f(t)$ — inverse Laplace ~~transform~~

$$\boxed{f(t) = L^{-1}\{F(s)\}}$$

L^{-1} — inverse Laplace transform operator.

$\rightarrow$ $L\{k\} = \frac{K}{s}$, $L^{-1}\{\frac{K}{s}\} = K \Rightarrow L^{-1}\{\frac{1}{s}\} = 1$

$\rightarrow$ $L\{t\} = \frac{1}{s^2}$, $L^{-1}\{\frac{1}{s^2}\} = t$

$\rightarrow$ $L\{t^n\} = \frac{n!}{s^{n+1}}$, $L^{-1}\{\frac{1}{s^{n+1}}\} = \frac{t^n}{n!}$

$\rightarrow$ $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$, $L^{-1}\{\frac{1}{s^{n+1}}\} = \frac{t^n}{\Gamma(n+1)}$

$\rightarrow$ $L\{e^{at}\} = \frac{1}{s-a}$, $L^{-1}\{\frac{1}{s-a}\} = e^{at}$

$\rightarrow$ $L\{e^{-at}\} = \frac{1}{s+a}$, $L^{-1}\{\frac{1}{s+a}\} = e^{-at}$

$\rightarrow$ $L\{\sin at\} = \frac{a}{s^2+a^2}$, $L^{-1}\{\frac{1}{s^2+a^2}\} = \frac{1}{a}\sin at$

$\rightarrow$ $L\{\cos at\} = \frac{s}{s^2+a^2}$, $L^{-1}\{\frac{s}{s^2+a^2}\} = \cos at$

$\rightarrow$ $L\{\sinh at\} = \frac{a}{s^2-a^2}$, $L^{-1}\{\frac{1}{s^2-a^2}\} = \frac{1}{a}\sinh at$

$\rightarrow$ $L\{\cosh at\} = \frac{s}{s^2-a^2}$, $L^{-1}\{\frac{s}{s^2-a^2}\} = \cosh at$

$\rightarrow$ $L\{af(t) \pm bg(t)\} = aF(s) \pm bG(s)$, $L^{-1}\{aF(s) \pm bG(s)\} = af(t) \pm bg(t)$

$\rightarrow$ <u>F.S.P</u>

If $L\{f(t)\} = F(s)$, then

$L\{e^{at}f(t)\} = F(s-a)$

$L\{e^{-at}f(t)\} = F(s+a)$.

If $L^{-1}\{F(s)\} = f(t)$ then

$L^{-1}\{F(s-a)\} = e^{at}f(t)$
$= e^{at}L^{-1}\{F(s)\}$

$L^{-1}\{F(s+a)\} = e^{-at}f(t)$
$= e^{-at}L^{-1}\{F(s)\}$

$\rightarrow$ <u>S.S.P</u>

$L\{f(t)\} = F(s)$ &

<u>S.S.P</u>

If $L^{-1}\{F(s)\} = f(t)$ then

$f(t-a),\ t>a$

→ S.S.P

If $L\{f(t)\} = F(s)$ &

$$g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

$$L\{g(t)\} = e^{-as}F(s)$$

$$L\{f(t-a)\,u(t-a)\} = e^{-as}F(s)$$

L.T of derivatives

$$L\{f'(t)\} = sF(s) - f(0)$$

⋮

$$L\{f^n(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots f^{n-1}(0).$$

→ L.T of integrals

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}F(s)$$

→ Multiplication by t

$$L\{t f(t)\} = (-1)^1 \frac{d}{ds}F(s) = -F'(s)$$

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s) = (-1)^n F^n(s)$$

→ Division by t

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

If $\overline{L^{-1}}\{F(s)\} = f(t)$ then

$$L^{-1}\{e^{-as}F(s)\} = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

(or)

$$L^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a).$$

Multiplication by s

If $L^{-1}\{F(s)\} = f(t)$ &

$f(0) = f'(0) = f''(0) \cdots f^{n-1}(0) = 0$

then $L^{-1}\{sF(s)\} = f'(t)$

$$L^{-1}\{s^2F(s)\} = f''(t)$$

⋮

$$L^{-1}\{s^nF(s)\} = f^n(t).$$

Division by s

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\,dt.$$

Inverse L.T of derivatives

$$L^{-1}\{F'(s)\} = -t f(t)$$

$$L^{-1}\{F^n(s)\} = (-1)^n t^n f(t).$$

Inverse L.T of integrals

$$L^{-1}\left\{\int_s^\infty F(s)\,ds\right\} = \frac{f(t)}{t}.$$

→ Find the inverse L.T of the following.

1) $L^{-1}\left\{\frac{2s-5}{s^2-4}\right\}$

Sol: $L^{-1}\left\{\frac{2s}{s^2-4} - \frac{5}{s^2-4}\right\}$

$$2L^{-1}\left\{\frac{s}{s^2-4}\right\} - 5L^{-1}\left\{\frac{1}{s^2-4}\right\}$$

. .

$2L^{-1}\left\{\frac{s}{s^2-4}\right\} - 5L^{-1}\left\{\frac{1}{s^2-4}\right\}$

$2\cosh 2t - 5\cdot\frac{1}{2}\sinh 2t$.

2) $L^{-1}\left\{\frac{s^2+9s-9}{s^3-9s}\right\}$

$= L^{-1}\left\{\frac{s^2-9+9s}{s(s^2-9)}\right\} = L^{-1}\left\{\frac{s^2-9}{s(s^2-9)} + \frac{9s}{s(s^2-9)}\right\}$

$= L^{-1}\left\{\frac{1}{s} + \frac{9}{s^2-9}\right\}$

$L^{-1}\left\{\frac{1}{s}\right\} + 9L^{-1}\left\{\frac{1}{s^2-9}\right\}$

$= 1 + 9\cdot\frac{1}{3}\sinh 3t \quad = 1 + 3\sinh 3t$.

→ Finding inverse Laplace transform by Partial fractions

① find $L^{-1}\left\{\frac{4}{(s+1)(s+2)}\right\}$

$F(s) = \frac{4}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$

$\Rightarrow \quad 4 = A(s+2) + B(s+1)$

put $s=-1 \Rightarrow 4 = A \Rightarrow A = 4$

$s = -2 \Rightarrow 4 = -B \Rightarrow B = -4$

$F(s) = \frac{4}{s+1} - \frac{4}{s+2}$

$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{4}{s+1} - \frac{4}{s+2}\right\} = 4L^{-1}\left\{\frac{1}{s+1}\right\} - 4L^{-1}\left\{\frac{1}{s+2}\right\}$

$= 4e^{-t} - 4e^{-2t}$.

→ find $L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+25)}\right\}$

$F(s) = \frac{s^2}{\quad} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+25}$

$$F(s) = \frac{s^2}{(s^2+4)(s^2+25)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+25}$$

$$\Rightarrow s^2 = (As+B)(s^2+25) + (Cs+D)(s^2+4)$$

Compare coeff of $s^3 \Rightarrow 0 = A + C$ — (a)

" " " $s^2 \Rightarrow 1 = B + D$ — (b)

" " " $s \Rightarrow 0 = 25A + 4C$ — (c)

" " constants $\Rightarrow 0 = 25B + 4D$ — (d)

Solve (a) & (c) $\left.\begin{array}{l} A+C=0 \\ 25A+4C=0 \end{array}\right\}$ $A=0,\ C=0$

Solve (b) & (d) $B+D=1 \quad \times 4 \Rightarrow$

$25B+4D=0$

$$\begin{array}{r} 4B+\cancel{4D}=4 \\ -\ 25B+\cancel{4D}=0 \\ \hline -21B \quad = 4 \end{array}$$

$$B = -\frac{4}{21}$$

$$D = 1 - B = 1 + \frac{4}{21} = \frac{25}{21}$$

$$F(s) = \frac{0s - 4/21}{s^2+4} + \frac{0s + \frac{25}{21}}{s^2+25} = \frac{-4}{21(s^2+4)} + \frac{25}{21(s^2+25)}$$

$$L^{-1}\{F(s)\} = -\frac{4}{21} L^{-1}\left\{\frac{1}{s^2+4}\right\} + \frac{25}{21} L^{-1}\left\{\frac{1}{s^2+25}\right\}$$

$$= -\frac{4}{21}\cdot\frac{1}{2}\sin 2t + \frac{25}{21}\cdot\frac{1}{5}\sin 5t$$

$$= -\frac{2}{21}\sin 2t + \frac{5}{21}\sin 5t$$

$$\rightarrow L^{-1}\left\{\frac{1}{s(s^2+1)(s^2+4)}\right\}$$

$$\rightarrow L^{-1}\left\{\frac{5s-2}{s^2(s+2)(s-1)}\right\} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s-1}$$

First shifting property:

$e^{at} L^{-1}\{F(s)\}$

First shifting property:

If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\{F(s-a)\} = e^{at} L^{-1}\{F(s)\}$

$L^{-1}\{F(s+a)\} = e^{-at} L^{-1}\{F(s)\}$.

→ find $L^{-1}\left\{\frac{1}{(s+2)^2+16}\right\}$ $\quad F(s+2) = \frac{1}{(s+2)^2+16}$, $F(s) = \frac{1}{s^2+16}$

$= L^{-1}\{F(s+2)\}$

By F.S.P $L^{-1}\{F(s+a)\} = e^{-at} L^{-1}\{F(s)\}$, $a = 2$

$= e^{-2t} L^{-1}\left\{\frac{1}{s^2+16}\right\}$

$= e^{-2t} \frac{1}{4} \sin 4t$.

→ $L^{-1}\left\{\frac{3s-2}{s^2-4s+20}\right\}$

$s^2 - s + 1$

$s^2 - 2 \cdot s \cdot \frac{1}{2} + \frac{1}{4} - \frac{1}{4}$

$= L^{-1}\left\{\frac{3s-2}{s^2-4s+4+16}\right\} = L^{-1}\left\{\frac{3s-2}{(s-2)^2+16}\right\}$

$= L^{-1}\left\{\frac{3(s-2+2)-2}{(s-2)^2+16}\right\} = L^{-1}\left\{\frac{3(s-2)+4}{(s-2)^2+16}\right\}$

$= L^{-1}\{F(s-2)\}$ where $F(s-2) = \frac{3(s-2)+4}{(s-2)^2+16}$

$= e^{2t} L^{-1}\{F(s)\}$ ($\because$ By F.S.P) $\quad F(s) = \frac{3s+4}{s^2+16}$

$= e^{2t} L^{-1}\left\{\frac{3s+4}{s^2+16}\right\}$

$= e^{2t}\left[L^{-1}\left\{\frac{3s}{s^2+16}\right\} + L^{-1}\left\{\frac{4}{s^2+16}\right\}\right]$

$= e^{2t}\left[3\cos 4t + 4 \cdot \frac{1}{4} \sin 4t\right]$

→ $L^{-1}\left\{\frac{3s-14}{s^2-4s+8}\right\}$

$L^{-1}\left\{\frac{3s-14}{}\right\} = L^{-1}\left\{\frac{3s-14}{(s-2)^2+4}\right\} = L^{-1}\left\{\frac{3(s-2+2)-14}{(s-2)^2+4}\right\}$

sol: $L^{-1}\left\{\frac{3s-14}{s^2-4s+4+4}\right\} = L^{-1}\left\{\frac{3s-14}{(s-2)^2+4}\right\} = L^{-1}\left\{\frac{3(s-2+2)-14}{(s-2)^2+4}\right\}$

$= L^{-1}\left\{\frac{3(s-2)-8}{(s-2)^2+4}\right\} = L^{-1}\{F(s-2)\}$

$F(s-2) = \frac{3(s-2)-8}{(s-2)^2+4}$, $F(s) = \frac{3s-8}{s^2+4}$

$= e^{2t} L^{-1}\{F(s)\}$

$e^{2t} L^{-1}\left\{\frac{3s-8}{s^2+4}\right\} = e^{2t}\left[L^{-1}\left\{\frac{3s}{s^2+4}\right\} - L^{-1}\left\{\frac{8}{s^2+4}\right\}\right]$

$= e^{2t}\left[3\cos 2t - 8\cdot\frac{1}{2}\sin 2t\right]$

$e^{2t}(3\cos 2t - 4\sin 2t)$.

→ $L^{-1}\left\{\frac{s+3}{s^2-10s+29}\right\}$

→ $L^{-1}\left\{\frac{s}{s^4+4a^4}\right\}$ $\quad s^4+4a^4 = (s^2+2a^2)^2-(2as)^2$

$(s^2-2as+2a^2)(s^2+2as+2a^2)$

$L^{-1}\left\{\frac{s}{(s^2-2as+2a^2)(s^2+2as+2a^2)}\right\} = L^{-1}\left\{\frac{As+B}{s^2-2as+2a^2} + \frac{Cs+D}{s^2+2as+2a^2}\right\}$

→ Second shifting property:

If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$.

→ find $L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\}$

$= L^{-1}\left\{\frac{1}{s^2+1}\right\} + L^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$

$= \sin t + L^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$ — (1)

$L^{-1}\left\{e^{-\pi s}\cdot\frac{1}{s^2+1}\right\} = L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$ — (2)

$a = \pi$, $F(s) = \frac{1}{s^2+1}$

$$- \} \quad s^2+1 \}$$

$$a = \pi, \quad F(s) = \frac{1}{s^2+1}$$

$$f(t) = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$$f(t-a) = f(t-\pi) = \sin(t-\pi) = -\sin(\pi - t)$$

$$f(t-\pi) = -\sin t$$

$$(2) \Rightarrow L^{-1}\left\{e^{-\pi s} \cdot \frac{1}{s^2+1}\right\} = f(t-\pi)u(t-\pi) = -\sin t\, u(t-\pi).$$

$$(1) \Rightarrow L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\} = \sin t - \sin t\, u(t-\pi).$$

$\rightarrow$ find $L^{-1}\left\{\dfrac{3+5s}{s^2 e^{2s}}\right\}$

$$= L^{-1}\left\{e^{-2s} \cdot \frac{3+5s}{s^2}\right\} = L^{-1}\{e^{-as}F(s)\} \quad \text{--- } (1)$$

$$a = 2, \quad F(s) = \frac{3+5s}{s^2}$$

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{3+5s}{s^2}\right\} = L^{-1}\left[\frac{3}{s^2}\right] + L^{-1}\left[\frac{5}{s}\right]$$

$$f(t) = 3t + 5$$

$$f(t-2) = 3(t-2)+5 = 3t-1$$

By s.s.p $L^{-1}\left\{e^{-2s} \cdot \dfrac{3+5s}{s^2}\right\} = f(t-2)u(t-2)$

$$(3t-1)\,u(t-2).$$

$\rightarrow$ find $L^{-1}\left\{\dfrac{(s+1)\,e^{-\pi s}}{s^2+s+1}\right\} = L^{-1}\{e^{-as}F(s)\}$ --- (1)

$$a = \pi, \quad F(s) = \frac{s+1}{s^2+s+1}$$

$$f(t) = L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} = L^{-1}\left\{\frac{s+1}{\underbrace{s^2 + 2\cdot s\cdot \frac{1}{2} + \frac{1}{4}} - \frac{1}{4} + 1}\right\}$$

$$= L^{-1}\left\{\frac{s+1}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}}\right\}$$

$$\left\{ \left(s+\frac{1}{2}\right)^2 + \frac{3}{4} \right\}$$

$$f(t) = L^{-1}\left\{ \frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \right\} = L^{-1}\left\{ G\left(s+\frac{1}{2}\right) \right\}$$

$$\Rightarrow f(t) = e^{-\frac{1}{2}t} L^{-1}\{G(s)\} \quad \text{--- (2)}$$

$(\because$ By F.S.P$)$

where $G\left(s+\frac{1}{2}\right) = \dfrac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$$G(s) = \frac{s+\frac{1}{2}}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

(2) $\Rightarrow$ $$f(t) = e^{-\frac{1}{2}t} L^{-1}\left\{ \frac{s+\frac{1}{2}}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2} \right\}$$

$$= e^{-\frac{1}{2}t}\left[L^{-1}\left\{ \frac{s}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2} \right\} + \frac{1}{2} L^{-1}\left\{ \frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2} \right\} \right]$$

$$f(t) = e^{-\frac{1}{2}t}\left[\cos\frac{\sqrt{3}}{2}t + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}} \sin\frac{\sqrt{3}}{2}t \right]$$

$$f(t) = e^{-\frac{1}{2}t}\left[\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t \right]$$

(1) $\Rightarrow$ $$L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a) = f(t-\pi)u(t-\pi)$$

$$= e^{-\frac{1}{2}(t-\pi)}\left[\cos\frac{\sqrt{3}}{2}(t-\pi) + \frac{1}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}(t-\pi) \right] u(t-\pi).$$

$\rightarrow$ find $L^{-1}\left\{ \dfrac{e^{-2s}}{s^2+4s+5} \right\}$

$\rightarrow$ <u>Inverse L.T of derivatives</u>

If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\{F^{(n)}(s)\} = (-1)^n t^n f(t)$

$$\boxed{L^{-1}\{F'(s)\} = -t f(t)}$$

$\rightarrow$ find $L^{-1}\left\{ \log\left(1+\frac{1}{s^2}\right) \right\}$

$$F(s) = \log\left(1+\frac{1}{s^2}\right) = \log\left(\frac{s^2+1}{s^2}\right)$$

$\rightarrow$ find $L^{-1}\{\log(1+\frac{1}{s^2})\}$

$$F(s) = \log\left(1+\frac{1}{s^2}\right) = \log\left(\frac{s^2+1}{s^2}\right)$$

$$= \log(s^2+1) - \log s^2 = \log(s^2+1) - 2\log s$$

$$F'(s) = \frac{2s}{s^2+1} - \frac{2}{s}$$

By inverse L.T of derivatives $L^{-1}\{F'(s)\} = -t f(t)$

$$\Rightarrow L^{-1}\{F'(s)\} = -t\, L^{-1}\{F(s)\}$$

$$\Rightarrow L^{-1}\left\{\frac{2s}{s^2+1} - \frac{2}{s}\right\} = -t\, L^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\}$$

$$\Rightarrow L^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\} = -\frac{2}{t}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\} - L^{-1}\left\{\frac{1}{s}\right\}\right]$$

$$\boxed{L^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\} = -\frac{2}{t}\left[\cos t - 1\right]}$$

$\rightarrow$ find $L^{-1}\left\{\log\left(\frac{s+3}{s+4}\right)\right\}$

$$F(s) = \log\left(\frac{s+3}{s+4}\right) = \log(s+3) - \log(s+4)$$

$$F'(s) = \frac{1}{s+3} - \frac{1}{s+4}$$

By inverse L-T of derivatives $\boxed{L^{-1}\{F'(s)\} = -t\, L^{-1}\{F(s)\}}$

$$L^{-1}\left\{\frac{1}{s+3} - \frac{1}{s+4}\right\} = -t\, L^{-1}\left\{\log\left(\frac{s+3}{s+4}\right)\right\}$$

$$\Rightarrow \frac{e^{-3t} - e^{-4t}}{-t} = L^{-1}\left\{\log\left(\frac{s+3}{s+4}\right)\right\}$$

$\rightarrow$ find $L^{-1}\left\{\log\frac{s^2+1}{(s-1)^2}\right\}$

$\rightarrow$ find $L^{-1}\left\{\tan^{-1}\left(\frac{a}{s}\right) + \cot^{-1}\left(\frac{s}{b}\right)\right\}$

→ find $L^{-1}\{\ldots(s)\ldots\}$

sol: $F(s) = \tan^{-1}\left(\frac{a}{s}\right) + \cot^{-1}\left(\frac{s}{b}\right)$

$$F'(s) = \frac{1}{1+\frac{a^2}{s^2}} \times \frac{-a}{s^2} - \frac{1}{1+\frac{s^2}{b^2}} \times \frac{1}{b}$$

$$F'(s) = \frac{-a}{a^2+s^2} - \frac{b}{b^2+s^2}.$$

$$L^{-1}\{F'(s)\} = -t\,L^{-1}\{F(s)\}$$

$$L^{-1}\left\{\frac{-a}{s^2+a^2} - \frac{b}{s^2+b^2}\right\} = -t\,L^{-1}\left\{\tan^{-1}\left(\frac{a}{s}\right) + \cot^{-1}\left(\frac{s}{b}\right)\right\}$$

$$\Rightarrow \frac{-\sin at - \sin bt}{-t} = L^{-1}\left\{\tan^{-1}\left(\frac{a}{s}\right) + \cot^{-1}\left(\frac{s}{b}\right)\right\}$$

→ find $L^{-1}\left\{\cot^{-1}\left(\frac{s+a}{b}\right)\right\}$

→ $L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$ ✓

→ Inverse L.T of integrals:

If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{\int_s^\infty F(s)\,ds\right\} = \frac{f(t)}{t}$

$$t\,L^{-1}\left\{\int_s^\infty F(s)\,ds\right\} = L^{-1}\{F(s)\}$$

$$\boxed{L^{-1}\{F(s)\} = t\,L^{-1}\left\{\int_s^\infty F(s)\,ds\right\}}$$

✓ $\left\{\frac{F'(s)}{(F(s))^n}\right\}$

1) find $L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$

sol: $F(s) = \frac{1}{(s+1)^2}$

$$L^{-1}\{F(s)\} = t\,L^{-1}\left\{\int_s^\infty F(s)\,ds\right\}$$

$$L^{-1}\{F(s)\} = t\, L^{-1}\left\{\int_s F(s)\,ds\right\}$$

$$L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = t\, L^{-1}\left\{\int_s^{\infty} \frac{1}{(s+1)^2}\,ds\right\}$$

$$= t\, L^{-1}\left\{\left(-\frac{1}{s+1}\right)_s^{\infty}\right\} = t\, L^{-1}\left\{0 + \frac{1}{s+1}\right\}$$

$$= t\, L^{-1}\left\{\frac{1}{s+1}\right\} = t e^{-t}.$$

2) $L^{-1}\left\{\dfrac{2s}{(s^2-4)^2}\right\}$

$$\int \frac{f'(x)}{[f(x)]^n} = \frac{[f(x)]^{-n+1}}{-n+1}$$

sol: $F(s) = \dfrac{2s}{(s^2-4)^2}$

$$L^{-1}\{F(s)\} = t\, L^{-1}\left\{\int_s^{\infty} F(s)\,ds\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{2s}{(s^2-4)^2}\right\} = t\, L^{-1}\left\{\int_s^{\infty} \frac{2s}{(s^2-4)^2}\,ds\right\}$$

$$= t\, L^{-1}\left\{\left(-\frac{1}{s^2-4}\right)_s^{\infty}\right\} = t\, L^{-1}\left\{\frac{1}{s^2-4}\right\}$$

$$= t\cdot\frac{1}{2}\sinh 2t.$$

$\Rightarrow L^{-1}\left\{\dfrac{s+2}{(s^2+4s+5)^2}\right\}$

sol $F(s) = \dfrac{s+2}{(s^2+4s+5)^2}$

$$L^{-1}\{F(s)\} = t\, L^{-1}\left\{\int_s^{\infty} F(s)\,ds\right\}$$

$$L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} = \frac{t}{2} L^{-1}\left\{\int_s^{\infty} \frac{2(s+2)}{(s^2+4s+5)^2}\,ds\right\}$$

$$= \frac{t}{2} L^{-1}\left\{\left(-\frac{1}{s^2+4s+5}\right)_s^{\infty}\right\} = \frac{t}{2} L^{-1}\left\{\frac{1}{s^2+4s+5}\right\}$$

$$= \frac{t}{2} L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$

$$= \frac{t}{2} e^{-2t} L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{t}{2} e^{-2t} \sin t .$$

→ $L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\}$

→ $L^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\}$, $L^{-1}\left\{\int_s^\infty \log\left(\frac{u-1}{u+1}\right) du\right\}$

→ Multiplication by 's':

If $L^{-1}\{F(s)\} = f(t)$, $f(0) = 0$ then

$L^{-1}\{sF(s)\} = f'(t)$.

$\frac{F'(s)}{[F(s)]^n}$ ✓ $n>1$

→ find $L^{-1}\left\{\frac{s}{2s^2-1}\right\}$

sol: $L^{-1}\{sF(s)\} = L^{-1}\left\{s \cdot \frac{1}{2s^2-1}\right\}$

$F(s) = \frac{1}{2s^2-1}$

$f(t) = L^{-1}\left\{\frac{1}{2s^2-1}\right\} = \frac{1}{2} L^{-1}\left\{\frac{1}{s^2-\frac{1}{2}}\right\} = \frac{1}{2} L^{-1}\left\{\frac{1}{s^2-\left(\frac{1}{\sqrt{2}}\right)^2}\right\}$

$f(t) = \frac{1}{2}\left(\frac{1}{1/\sqrt{2}} \sinh \frac{1}{\sqrt{2}} t\right) = \frac{1}{\sqrt{2}} \sinh \frac{t}{\sqrt{2}}$

$f(0) = 0$

$L^{-1}\{sF(s)\} = f'(t) \Rightarrow L^{-1}\left\{\frac{s}{2s^2-1}\right\} = \frac{d}{dt}\left(\frac{1}{\sqrt{2}} \sinh \frac{t}{\sqrt{2}}\right)$

$= \frac{1}{\sqrt{2}} \cosh \frac{t}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2} \cosh \frac{t}{\sqrt{2}}$

→ find $L^{-1}\left\{\frac{s-3}{s^2+4s+13}\right\}$

$\rightarrow$ find $L^{-1}\left\{\dfrac{s-3}{s^2+4s+13}\right\}$

sol: $L^{-1}\left\{\dfrac{s}{s^2+4s+13}\right\} - 3L^{-1}\left\{\dfrac{1}{s^2+4s+13}\right\}$ — (1)

$\downarrow$

$L^{-1}\left\{\dfrac{s}{s^2+4s+13}\right\} = L^{-1}\{sF(s)\}$ where $F(s) = \dfrac{1}{s^2+4s+13}$

$f(t) = L^{-1}\left\{\dfrac{1}{s^2+4s+13}\right\}$

$= L^{-1}\left\{\dfrac{1}{(s+2)^2+9}\right\} = e^{-2t}L^{-1}\left\{\dfrac{1}{s^2+9}\right\}$

$f(t) = e^{-2t}\dfrac{1}{3}\sin 3t$, $f(0) = 0$

$L^{-1}\{sF(s)\} = f'(t)$

$L^{-1}\left\{\dfrac{s}{s^2+4s+13}\right\} = \dfrac{d}{dt}\left(\dfrac{e^{-2t}\sin 3t}{3}\right) = \dfrac{1}{3}\left[3e^{-2t}\cos 3t - 2e^{-2t}\sin 3t\right]$

$L^{-1}\left\{\dfrac{s-3}{s^2+4s+13}\right\} = \dfrac{1}{3}\left(3e^{-2t}\cos 3t - 2e^{-2t}\sin 3t\right) - 3\left(e^{-2t}\cdot\dfrac{1}{3}\sin 3t\right)$

$\rightarrow$ find $L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)^2}\right\}$

sol: $L^{-1}\left\{s\cdot\dfrac{s}{(s^2+a^2)^2}\right\} = L^{-1}\{s.F(s)\}$ — (1)

$F(s) = \dfrac{s}{(s^2+a^2)^2}$

$L^{-1}\{F(s)\} = t\,L^{-1}\left\{\int_s^\infty F(s)\,ds\right\}$

$f(t) = L^{-1}\{F(s)\} = \dfrac{t}{2}L^{-1}\left\{\int_s^\infty \dfrac{2s}{(s^2+a^2)^2}\,ds\right\} = \dfrac{t}{2}L^{-1}\left\{\left(\dfrac{-1}{(s^2+a^2)}\right)_s^\infty\right\}$

$= \dfrac{t}{2}L^{-1}\left\{\dfrac{1}{s^2+a^2}\right\}$

$$f(t) = \frac{t}{2} \cdot \frac{1}{a} \sin at \quad , \quad f(0) = 0$$

$$L^{-1}\{s \cdot F(s)\} = L^{-1}\left\{s \cdot \frac{s}{(s^2+a^2)^2}\right\} = f'(t) = \frac{1}{2a}\frac{d}{dt}(t \sin at)$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = \frac{1}{2a}\left[at \cos at + \sin at\right]$$

→ find $L^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\}$

→ Division by s:

If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\,dt$.

1) find $L^{-1}\left\{\frac{1}{s(s+2)}\right\}$

$$L^{-1}\left\{\frac{\frac{1}{s+2}}{s}\right\} = L^{-1}\left\{\frac{F(s)}{s}\right\}$$

$$F(s) = \frac{1}{s+2} \quad , \quad f(t) = L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$$

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\,dt$$

$$L^{-1}\left\{\frac{1}{s(s+1)}\right\} = \int_0^t e^{-2t}\,dt = \left(\frac{e^{-2t}}{-2}\right)_0^t = -\frac{1}{2}\left(e^{-2t}-1\right)$$

$$= \frac{1-e^{-2t}}{2}$$

→ find $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$

$$\frac{cs+D}{(s^2+a^2)^2}$$

$$L^{-1}\left\{\frac{1}{s}\frac{s}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s}F(s)\right\}$$

$$\frac{s-1}{(s^2+a^2)^2}$$

$$F(s) = \frac{s}{(s^2+a^2)^2}$$

$$\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

$$(s^2+a^2)^2 \qquad \left\{(s^2+a^2)\right\}$$

$$f(t) = L^{-1}\{F(s)\} = tL^{-1}\left\{\int_s^\infty F(s)\,ds\right\}$$

$$= L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2}L^{-1}\left\{\int_s^\infty \frac{2s}{(s^2+a^2)^2}\,ds\right\}$$

$$f(t) = \frac{t}{2}L^{-1}\left\{\left(-\frac{1}{s^2+a^2}\right)_s^\infty\right\}$$

$$f(t) = \frac{t}{2}L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{t}{2}\cdot\frac{1}{a}\sin at$$

$$f(t) = \frac{t}{2a}\sin at$$

$$L^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(t)\,dt$$

$$L^{-1}\left\{\frac{1}{s}\cdot\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \frac{t}{2a}\sin at\,dt$$

$$= \frac{1}{2a}\left[t\left(\frac{-\cos at}{a}\right) + \frac{\sin at}{a^2}\right]_0^t$$

$$= \frac{1}{2a}\left[-\frac{t\cos at}{a} + \frac{\sin at}{a^2} - 0 - 0\right]$$

$$= \frac{1}{2a}\left[-\frac{t\cos at}{a} + \frac{\sin at}{a^2}\right]$$

→ find $L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\}$

$L^{-1}\left\{\frac{s+1}{s^2(s^2+1)}\right\}$

→ Convolution:

let $f(t)$ & $g(t)$ be two functions defined for $t>0$ then the convolution of $f(t)$ & $g(t)$ is denoted by $f(t) * g(t)$ and it is defined as

$$\boxed{f(t) * g(t) = \int_0^t f(u)\,g(t-u)\,du.}$$

$f(t) * g(t) = g(t) * f(t)$

$$\boxed{f(t)*g(t) = \int_0 f(u)\,g(t-u)\ \ldots}$$

→ Convolution is always commutative ⇒ $f(t)*g(t) = g(t)*f(t)$

→ convolution is always Associative ⇒ $f(t)*[g(t)*h(t)] = [f(t)*g(t)]*h(t)$.

→ $f(t)*0 = 0*f(t) = 0$.

→ $f(t)*1 \neq f(t)$

Ex: $f(t) = e^t$, $g(t) = 1$, find $f(t)*g(t)$

sol: $f(u) = e^u$, $g(t-u) = 1$

$$f(t)*g(t) = \int_0^t f(u)\,g(t-u)\,du = \int_0^t e^u \cdot 1\,du = (e^u)_0^t$$

$$e^t * 1 = (e^t - 1)$$

→ find $f(t)*g(t)$ for the following.

1) $f(t) = e^t$, $g(t) = \sin at$

2) $f(t) = \cos at$, $g(t) = e^t$

3) $f(t) = e^t$, $g(t) = e^t$

4) $f(t) = \sin at$ $g(t) = \sin at$

5) $f(t) = \cos at$ $g(t) = \cos at$.

1 $f(t) = e^t$, $g(t) = \sin at$.

$$f(t)*g(t) = \int_0^t f(u)\,g(t-u)\,du = \int_0^t e^u \cdot \sin a(t-u)\,du$$

$$= \int_0^t e^u \sin(at - au)\,du$$

$$= \int_0^t e^u[\sin at \cos au - \cos at \sin au]\,du$$

$$= \sin at \int_0^t e^u \cos au\,du - \cos at \int_0^t e^u \sin au\,du.$$

$\int e^{ax} \ldots = \frac{e^{ax}}{\ldots}(a \sin bx - b \cos bx)$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$= \sin at\left[\frac{e^u}{1+a^2}(\cos au + a\sin au)\right]_0^t -$$
$$\cos at\left[\frac{e^u}{1+a^2}(\sin au - a\cos au)\right]_0^t$$

$$= \frac{\sin at}{1+a^2}\left[e^t(\cos at - a\sin at) - (1-0)\right]$$
$$-\frac{\cos at}{1+a^2}\left[e^t(\sin at - a\cos at) - (0-a)\right]$$

→ Convolution theorem:

If $L^{-1}\{F(s)\} = f(t)$ & $L^{-1}\{G(s)\} = g(t)$ then

$$L^{-1}\{F(s)G(s)\} = f(t) * g(t)$$

→ Find the inverse Laplace transform of the following using convolution theorem.

i) $L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = L^{-1}\{F(s)G(s)\}$

Sol: $F(s) = \frac{1}{s+a}$ $\quad G(s) = \frac{1}{s+b}$

$f(t) = L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$, $\quad g(t) = L^{-1}\left\{\frac{1}{s+b}\right\} = e^{-bt}$

$f(u) = e^{-au}$, $\quad g(t-u) = e^{-b(t-u)}$

By convolution theorem $L^{-1}\{F(s)G(s)\} = f(t) * g(t) = \int_0^t f(u)g(t-u)\,du$

$$\Rightarrow L^{-1}\left\{\frac{1}{s+a}\cdot\frac{1}{s+b}\right\} = \int_0^t e^{-au}\cdot e^{-b(t-u)}\,du$$

$$= e^{-bt}\int_0^t e^{(b-a)u}\,du$$

$$= e^{-bt}\left[\frac{e^{(b-a)u}}{\ }\right]_0^t$$

$$= e^{-bt}\left[\frac{e^{(b-a)u}}{b-a}\right]_0^t$$

$$= \frac{e^{-bt}}{b-a}\left[e^{(b-a)t} - 1\right].$$

$\rightarrow$ find $L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\}$

sol: $L^{-1}\left\{\frac{1}{s^2+2s+2}\cdot\frac{1}{s}\right\} = L^{-1}\{F(s)\,G(s)\}$

$F(s) = \frac{1}{s^2+2s+2}$, $G(s) = \frac{1}{s}$

$f(t) = L^{-1}\left\{\frac{1}{s^2+2s+2}\right\} = L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-t}L^{-1}\left\{\frac{1}{s^2+1}\right\} = e^{-t}\sin t$

$g(t) = L^{-1}\left\{\frac{1}{s}\right\} = 1$

$f(u) = e^{-u}\sin u$, $g(t-u) = 1$

By convolution , $L^{-1}\{F(s)\,G(s)\} = \int_0^t f(u)\,g(t-u)\,du$

$\Rightarrow L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\} = \int_0^t e^{-u}\sin u\cdot 1\,du$

$$= \left\{\frac{e^{-u}}{1^2+(-1)^2}\left[-1\sin u - 1\cos u\right]\right\}_0^t$$

$$= \frac{1}{2}\left[e^{-t}(-\sin t-\cos t) - 1(0-1)\right]$$

$$= \frac{1}{2}\left[1 - e^{-t}(\sin t+\cos t)\right].$$

$\rightarrow$ find $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$

$\frac{As+B}{s^2+a^2} + \frac{Cs+D}{(s^2+a^2)^2}$

$\frac{s+1}{s^2+a^2} + \frac{s-1}{(s^2+a^2)^2}$

sol: $L^{-1}\left\{\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\} = L^{-1}\{F(s)\,G(s)\}$

$F(s) = \frac{1}{s^2+a^2}$. $G(s) = \frac{1}{s^2+a^2}$

$\frac{1}{a}\sin at$

$F(s) = \frac{1}{s^2+a^2}$ $\qquad$ s^2+a^2

$f(t) = \frac{1}{a}\sin at \quad , \quad g(t) = \frac{1}{a}\sin at$

$f(u) = \frac{1}{a}\sin au, \quad g(t-u) = \frac{1}{a}\sin a(t-u)$

By convolution $L^{-1}\{F(s)G(s)\} = \int_0^t f(u)\, g(t-u)\,du$

$$\Rightarrow L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \int_0^t \frac{1}{a}\sin au \,\frac{1}{a}\sin a(t-u)\,du$$

$$= \frac{1}{2a^2}\int_0^t 2\sin au \sin(at-au)\,du$$

$$= \frac{1}{2a^2}\int_0^t \left[\cos(au-(at-au)) - \cos(\cancel{au}+at-\cancel{au})\right]du$$

$$= \frac{1}{2a^2}\int_0^t \left[\cos(2au-at) - \cos at\right]du$$

$$= \frac{1}{2a^2}\left[\int_0^t \cos(2au-at)\,du - \int_0^t \cos at\,du\right]$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2au-at)}{2a} - \cos at\,(u)\right]_0^t$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2at-at)}{2a} - t\cos at - \left(\frac{\sin(-at)}{2a} - 0\right)\right]$$

$$= \frac{1}{2a^2}\left[\frac{1}{2a}\sin at - t\cos at + \frac{1}{2a}\sin at\right]$$

$$= \frac{1}{2a^2}\left[\frac{1}{a}\sin at - t\cos at\right]$$

$$\rightarrow L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2a}\sin at$$

$$\rightarrow L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{a\sin at - b\sin bt}{a^2-b^2}$$

$\rightarrow \quad L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\}$

Sol: $L^{-1}\left\{\frac{1}{s(s+1)} \cdot \frac{1}{s+2}\right\} = L^{-1}\{F(s)\,G(s)\} \longrightarrow (1)$

$F(s) = \frac{1}{s(s+1)} \quad , \quad G(s) = \frac{1}{s+2}$

$f(t) = L^{-1}\left\{\frac{1}{s(s+1)}\right\} = L^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s+1}\right\} = L^{-1}\{H(s) \cdot J(s)\} \longrightarrow (2)$

$H(s) = \frac{1}{s} \quad , \quad J(s) = \frac{1}{s+1}$

$h(t) = L^{-1}\left\{\frac{1}{s}\right\} = 1 \quad , \quad j(t) = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$

$h(u) = 1 \quad , \quad j(t-u) = e^{-(t-u)}$.

$(2) \Rightarrow f(t) = L^{-1}\{H(s) \cdot J(s)\} = \int_0^t h(u)\, j(t-u)\, du$

$= \int_0^t 1 \cdot e^{-(t-u)}\, du = e^{-t} \int_0^t e^{u}\, du$

$f(t) = e^{-t}\left[e^{u}\right]_0^t = e^{-t}\left[e^{t} - 1\right] = \left[1 - e^{-t}\right]$

$f(t) = 1 - e^{-t}$.

$g(t) = L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$

$f(u) = 1 - e^{-u} \quad , \quad g(t-u) = e^{-2(t-u)}$.

$(1) \Rightarrow L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\} = L^{-1}\{F(s)G(s)\} = \int_0^t f(u)\, g(t-u)\, du$

$= \int_0^t (1 - e^{-u})\, e^{-2(t-u)}\, du$

$= e^{-2t} \int_0^t (1 - e^{-u})\, e^{2u}\, du$

$= e^{-2t} \int_0^t (e^{2u} - e^{u})\, du$

$$= e^{-2t}\int_0^t \left(e^{2u} - e^{u}\right) du$$

$$= e^{-2t}\left[\frac{e^{2u}}{2} - e^{u}\right]_0^t$$

$$= e^{-2t}\left[\frac{e^{2t}}{2} - e^{t} - \left(\frac{1}{2} - 1\right)\right]$$

$$= e^{-2t}\left[\frac{e^{2t}}{2} - e^{t} + \frac{1}{2}\right]$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

→ find $L^{-1}\left\{\frac{1}{(s^2+4)(s+1)^2}\right\}$