$f(t) \Rightarrow L[f(t)] = F(s)$ | $F(t) \Rightarrow L[F(t)] = f(s)$

Convolution Theorem

If $L^{-1}[f(s)] = F(t)$ *and* $L^{-1}[g(s)] = G(t)$, *then* $L^{-1}[f(s)]$

$$L^{-1}[f(s)g(s)] = \int_0^t F(u)G(t-u)\,du = F * G$$

$= F(t)$

Proof :-

The required results follows if we can prove that

$$L\left[\int_0^t F(u)G(t-u)\,du\right] = f(s)g(s)$$

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where $f(s) = L[F(t)]$, $g(s) = L[G(t)]$

$t = 0$ to $t \to \infty$

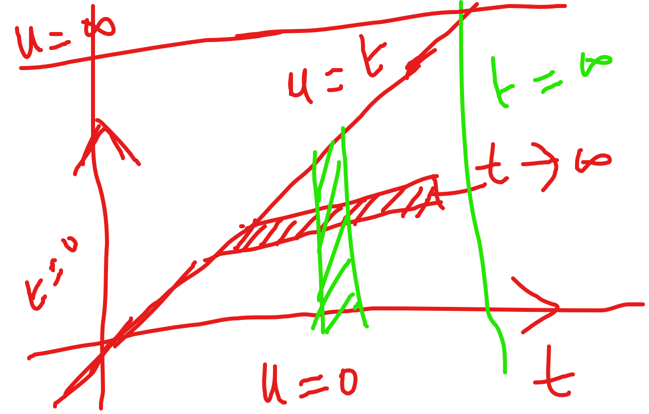
$u = 0$ to $u = t$

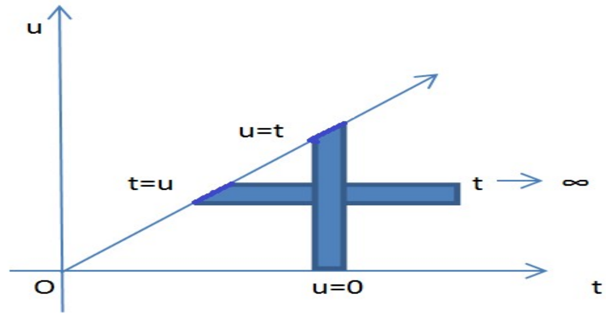
Let $\phi(t) = \int_0^t F(u)G(t-u)\,du$

$$L[\phi(t)] = \int_0^\infty e^{-st}\left(\int_0^t F(u)G(t-u)\,du\right)dt$$

$$= \int_0^\infty \int_0^t e^{-st} F(u)G(t-u)\,du\,dt \qquad (1)$$

$$\int_{u=0}^{u\to\infty}\int_{t=u}^{t\to\infty} e^{-st} F(u)\, G(t-u)\, dt\, du$$





On changing the order of integration, we get

$$L[\phi(t)] = \int_0^\infty \int_u^\infty e^{-st} F(u) G(t-u)\, dt\, du$$

$$e^{-st} \cdot e^{su} = e^{-s(t-u)}$$

$$e^{-su} \cdot e^{su}$$

$$e^{-su+su} = e^0 = 1$$

$$= \int_0^\infty e^{-su} F(u) \left\{ \int_u^\infty e^{-s(t-u)} G(t-u)\, dt \right\} du$$

Let $t - u = v,\ dt = dv$

when $t = u,\ v = 0$ & *when* $t \to \infty,\ v \to \infty$

$$= \int_0^\infty e^{-su} F(u) \left\{ \int_0^\infty e^{-sv} G(v)\, dv \right\} du$$

$$= \int_0^\infty e^{-su} F(u)\, du\ g(s) = f(s).g(s)$$

Evaluate each of the following by use of the convolution theorem.

(a) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$, (b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s+1)^2}\right\}$.

(a) We can write $\dfrac{s}{(s^2+a^2)^2} = \dfrac{s}{s^2+a^2}\cdot\dfrac{1}{s^2+a^2}$. Then since $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at$ and $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+a^2}\right\} = \dfrac{\sin at}{a}$, we have by the convolution theorem,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} &= \int_0^t \cos au \cdot \frac{\sin a(t-u)}{a}\,du \\
&= \frac{1}{a}\int_0^t (\cos au)(\sin at\cos au - \cos at\sin au)\,du \\
&= \frac{1}{a}\sin at\int_0^t \cos^2 au\,du - \frac{1}{a}\cos at\int_0^t \sin au\cos au\,du \\
&= \frac{1}{a}\sin at\int_0^t\left(\frac{1+\cos 2au}{2}\right)du - \frac{1}{a}\cos at\int_0^t \frac{\sin 2au}{2}\,du \\
&= \frac{1}{a}\sin at\left(\frac{t}{2}+\frac{\sin 2at}{4a}\right) - \frac{1}{a}\cos at\left(\frac{1-\cos 2at}{4a}\right) \\
&= \frac{1}{a}\sin at\left(\frac{t}{2}+\frac{\sin at\cos at}{2a}\right) - \frac{1}{a}\cos at\left(\frac{\sin^2 at}{2a}\right) \\
&= \frac{t\sin at}{2a}
\end{aligned}$$

$F(t)$

$\frac{1}{a}$ $\quad$ a $\quad$ $G(t)$

$\sin(A-B)$

$\sin 2A = 2\sin A\cos A$

$\frac{\sin 2A}{2} = \sin A\cos A$

$\int_0^t \left(\frac{1}{2}+\frac{\cos 2au}{2}\right)du$

$\int_0^t \frac{du}{2} + \int_0^t \frac{\cos 2au}{2}\,du$

$\left(\frac{u}{2}\right)_0^t + \left(\frac{\sin 2au}{4a}\right)_0^t$

$\left(\frac{t}{2}-0\right) + \left(\frac{\sin 2at}{4a} - 0\right)$

$\left(-\frac{\cos 2au}{4a}\right)_0^t$

$= -\left(\frac{\cos 2at}{4a} - \frac{1}{4a}\right) = \frac{1-\cos 2at}{4a}$

(b) We have $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$, $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$. Then by the convolution theorem,

$e^{-t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)$

te^{-t}

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t (ue^{-u})(t-u)\,du$$

$$= \int_0^t (ut-u^2)\,e^{-u}\,du$$

$$= (ut-u^2)(-e^{-u}) - (t-2u)(e^{-u}) + (-2)(-e^{-u})\Big|_0^t$$

$= \left(0 + te^{-t} + 2e^{-t}\right)$

$- \left(0 - t + 2\right)$

$$= te^{-t} + 2e^{-t} + t - 2$$

INITIAL AND FINAL VALUE THEOREMS

Prove the *initial-value theorem:* $\lim_{t \to 0} F(t) = \lim_{s \to \infty} s\, f(s)$.

Proof:-

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\, dt = s\, f(s) - F(0) \tag{1}$$

But if $F'(t)$ is sectionally continuous and of exponential order, we have

$$\lim_{s \to \infty} \int_0^\infty e^{-st} F'(t)\, dt = 0 \tag{2}$$

Then taking the limit as $s \to \infty$ in (1), assuming $F(t)$ continuous at $t = 0$, we find that

$$0 = \lim_{s \to \infty} s\, f(s) - F(0) \quad \text{or} \quad \lim_{s \to \infty} s\, f(s) = F(0) = \lim_{t \to 0} F(t)$$

Prove the *final-value theorem*: $\lim\limits_{t \to \infty} F(t) = \lim\limits_{s \to 0} s\,f(s)$.

Proof:-

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\,dt = s\,f(s) - F(0)$$

The limit of the left hand side as $s \to 0$ is

$$\lim_{s \to 0} \int_0^\infty e^{-st} F'(t)\,dt = \int_0^\infty F'(t)\,dt = \lim_{P \to \infty} \int_0^P F'(t)\,dt$$

$$= \lim_{P \to \infty} \{F(P) - F(0)\} = \lim_{t \to \infty} F(t) - F(0)$$

The limit of the right hand side as $s \to 0$ is

$$\lim_{s \to 0} s\,f(s) - F(0)$$

Thus $$\lim_{t \to \infty} F(t) - F(0) = \lim_{s \to 0} s\,f(s) - F(0)$$

or, as required, $$\lim_{t \to \infty} F(t) = \lim_{s \to 0} s\,f(s)$$

Verify the initial value theorem for the voltage function $(5 + 2\cos 3t)$ volts, and state its initial value.

$f(t) = 5 + 2\cos 3t$

By the initial value theorem,

$\mathcal{L}[f(t)] = 5\,\mathcal{L}(1) + 2\,\mathcal{L}(\cos 3t)$

$f(s)$ $F(s)$ $= \left(\frac{5}{s} + 2\frac{s}{s^2+9}\right)$

$$\lim_{t\to 0}[f(t)] = \lim_{s\to\infty}[s\mathcal{L}\{f(t)\}]$$

$$\text{i.e.}\quad \lim_{t\to 0}[5 + 2\cos 3t] = \lim_{s\to\infty}\left[s\left(\frac{5}{s} + \frac{2s}{s^2+9}\right)\right]$$

$$= \lim_{s\to\infty}\left[5 + \frac{2s^2}{s^2+9}\right]$$

$$\frac{2\cancel{s^2}}{\cancel{s^2}(1+9/s^2)}$$

$$\text{i.e.}\quad 5 + 2(1) = 5 + \frac{2\infty^2}{\infty^2+9} = 5 + 2$$

i.e. **7 = 7**, which verifies the theorem in this case.

The initial value of the voltage is thus **7 V**.

$L(\sin 4t) = \frac{4}{s^2+4^2}$ | $L(e^{-2t}\sin 4t) = \frac{4}{(s+2)^2+4^2}$

Verify the final value theorem for the function $(2 + 3e^{-2t} \sin 4t)$ cm, which represents the displacement of a particle. State its final steady value.

$f(t) = 2 + 3e^{-2t}\sin 4t$

$L[f(t)] = 2L(1) + 3L(e^{-2t}\sin 4t)$

By the final value theorem,

$$\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[s\mathcal{L}\{f(t)\}]$$

i.e. $\lim_{t\to\infty}[2 + 3\underset{0}{\underset{\|}{e^{-2t}}} \sin 4t]$

$$= \lim_{s\to 0}\left[s\left(\frac{2}{s} + \frac{12}{(s+2)^2+16}\right)\right]$$

$$= \lim_{s\to 0}\left[2 + \frac{12s}{(s+2)^2+16}\right]$$

i.e. $2 + 0 = 2 + 0$

i.e. $\mathbf{2 = 2}$, which verifies the theorem in this case.

The final value of the displacement is thus 2 cm.