Riemann Sums

The theory of limits of finite approximations was made precise by the German mathematician Bernhard Riemann. We now introduce the notion of a *Riemann sum*, which underlies the theory of the definite integral studied in the next section.

We begin with an arbitrary bounded function f defined on a closed interval $[a, b]$. Like the function pictured in Figure 1, f may have negative as well as positive values. We subdivide the interval $[a, b]$ into subintervals, not necessarily of equal widths (or lengths)

To do so, we choose $n - 1$ points $\{x_1, x_2, x_3, \ldots, x_{n-1}\}$ between a and b and satisfying

$$a < x_1 < x_2 < \cdots < x_{n-1} < b.$$

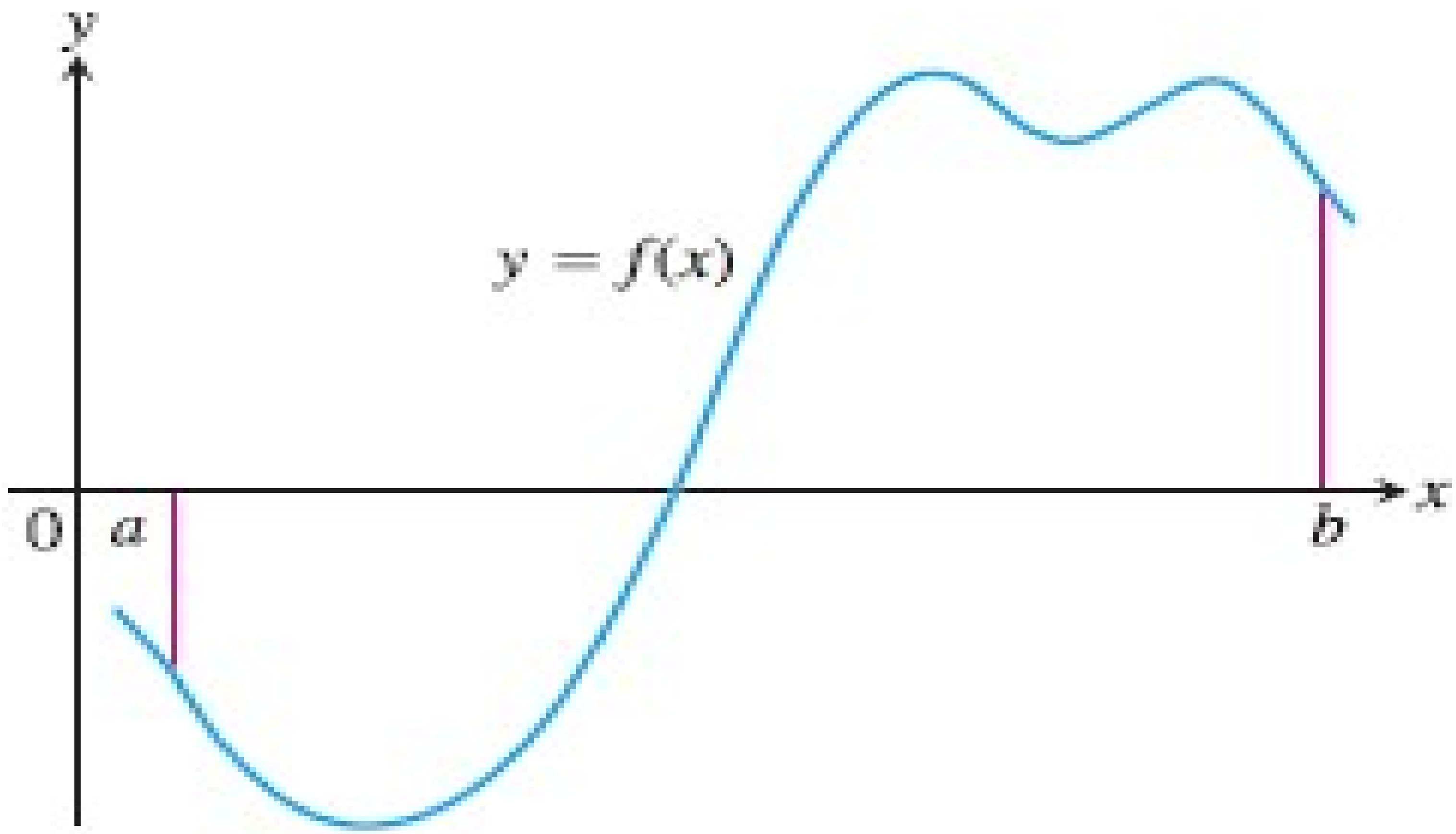


FIGURE 1 A typical continuous function $y = f(x)$ over a closed interval $[a, b]$.

To make the notation consistent, we denote a by x_0 and b by x_n, so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The set

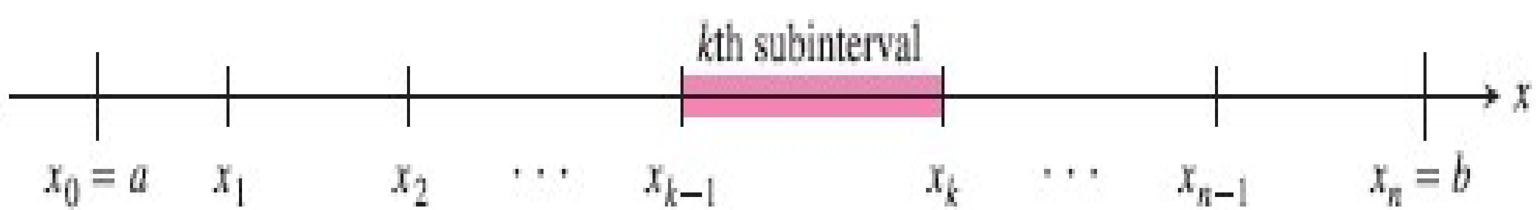
$$P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$$

is called a **partition** of $[a, b]$.

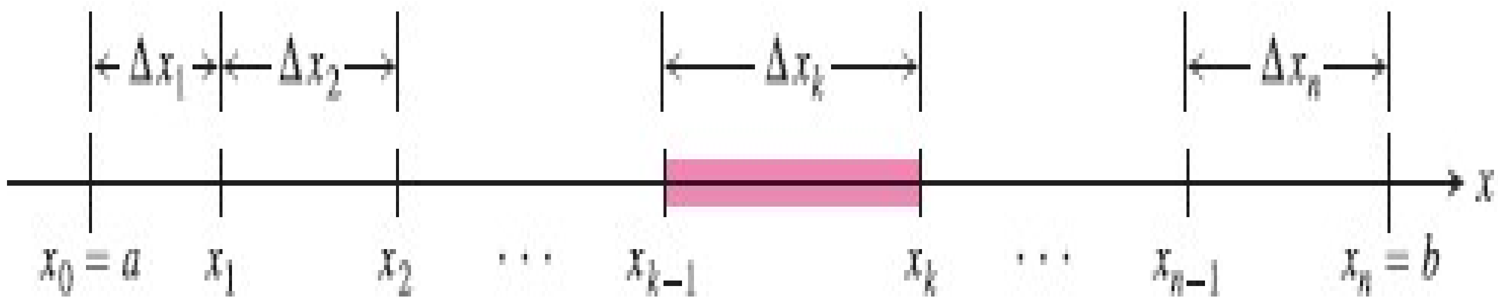
The partition P divides $[a, b]$ into n closed subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

The first of these subintervals is $[x_0, x_1]$, the second is $[x_1, x_2]$, and the ***k*th subinterval of** P is $[x_{k-1}, x_k]$, for k an integer between 1 and n.



The width of the first subinterval $[x_0, x_1]$ is denoted Δx_1, the width of the second $[x_1, x_2]$ is denoted Δx_2, and the width of the kth subinterval is $\Delta x_k = x_k - x_{k-1}$. If all n subintervals have equal width, then the common width Δx is equal to $(b - a)/n$.



In each subinterval we select some point. The point chosen in the kth subinterval $[x_{k-1}, x_k]$ is called c_k. Then on each subinterval we stand a vertical rectangle that stretches from the x-axis to touch the curve at $(c_k, f(c_k))$. These rectangles can be above or below the x-axis, depending on whether $f(c_k)$ is positive or negative, or on the x-axis if $f(c_k) = 0$ (Figure 2).

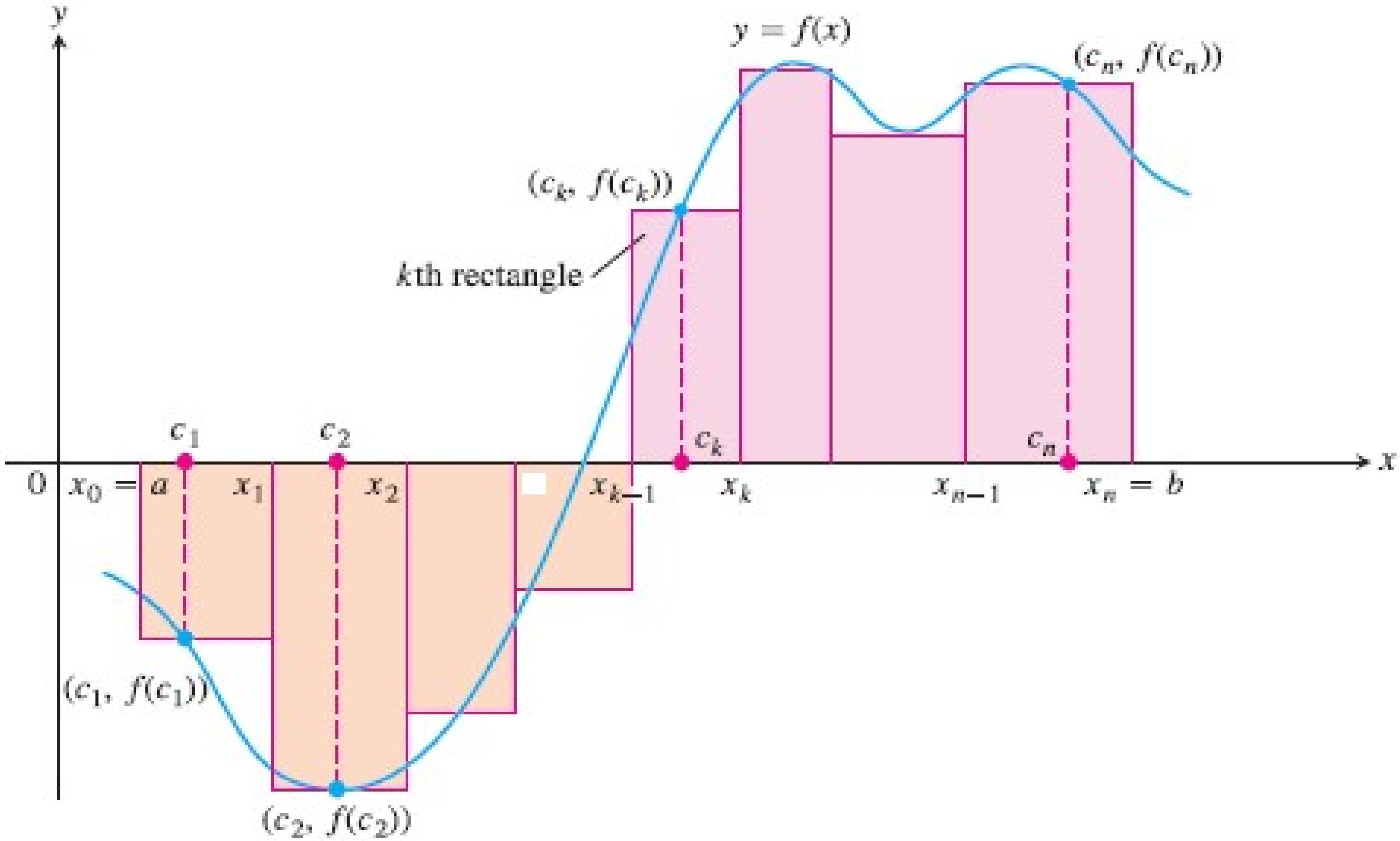


FIGURE 2 The rectangles approximate the region between the graph of the function $y = f(x)$ and the x-axis. Figure 1 has been enlarged to enhance the partition of $[a, b]$ and selection of points c_k that produce the rectangles.

On each subinterval we form the product $f(c_k) \cdot \Delta x_k$. This product is positive, negative, or zero, depending on the sign of $f(c_k)$. When $f(c_k) > 0$, the product $f(c_k) \cdot \Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width Δx_k. When $f(c_k) < 0$, the product $f(c_k) \cdot \Delta x_k$ is a negative number, the negative of the area of a rectangle of width Δx_k that drops from the x-axis to the negative number $f(c_k)$.

Finally we sum all these products to get

$$S_P = \sum_{k=1}^{n} f(c_k)\,\Delta x_k\,.$$

The sum S_P is called a **Riemann sum for *f* on the interval [*a*, *b*]**.

DEFINITION OF A DEFINITE INTEGRAL If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 \, (= a), x_1, x_2, \ldots, x_n \, (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists. If it does exist, we say that f is **integrable** on $[a, b]$.

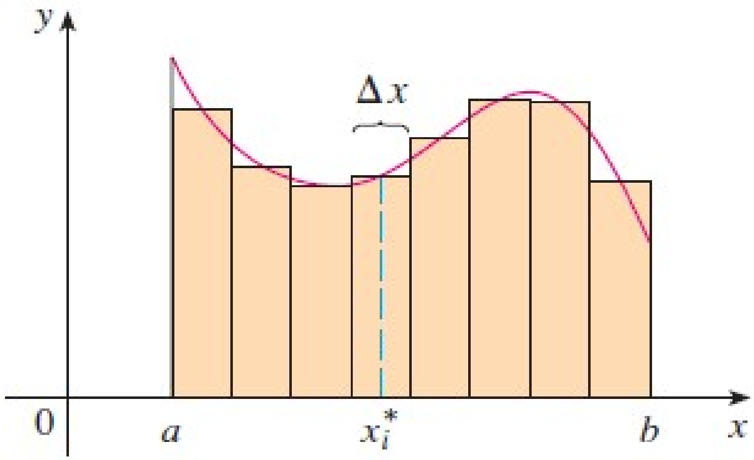


FIGURE 1
If $f(x) \geq 0$, the Riemann sum $\Sigma f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

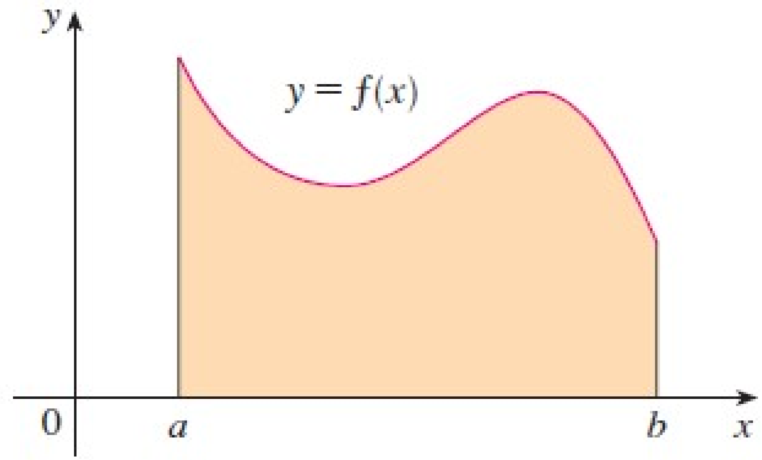


FIGURE 2
If $f(x) \geq 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

Evaluate the Riemann sum for $f(x) = x^3 - 6x$ taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.

With $n = 6$ the interval width is

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

$x_0 = 0$

$x_1 = 0.5$

$x_2 = 1.0$

$x_3 = 1.5$

and the right endpoints are $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$, $x_4 = 2.0$, $x_5 = 2.5$, and $x_6 = 3.0$. So the Riemann sum is

$x_4 = 2.0$

$x_5 = 2.5$

$x_6 = b = 3.0$

$$R_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x$$

$$= f(0.5)\,\Delta x + f(1.0)\,\Delta x + f(1.5)\,\Delta x + f(2.0)\,\Delta x + f(2.5)\,\Delta x + f(3.0)\,\Delta x$$

$$= \tfrac{1}{2}(-2.875 - 5 - 5.625 - 4 + 0.625 + 9)$$

$$= -3.9375$$

Average Value of a Function

Main applications of definite integrals is to find the average value of a function $y = f(x)$ over a specific interval $[a, b]$.

To find this average value we must integrate the function by using the Fundamental Theorem of Calculus and divide the answer by the length of the interval.

The average (or the mean) value of $f(x)$ on $[a, b]$ is defined by

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\ dx$$

Mean Value Theorem for Definite Integrals:-

Let $y = f(x)$ be a continuous function on the closed interval $[a, b]$. The mean value theorem for integrals states that there exists a point $'c'$ in that interval such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\ dx$$

1) The average value of a function $y = f(x)$ over the interval $x \in [1, 5]$ is 2. What is the value of $\int_1^5 f(x)\,dx$?

Sol:-

Given $\bar{f} = 2$ on $[1, 5]$

By defn:-

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\Rightarrow \int_a^b f(x)\,dx = \bar{f}\,(b-a)$$

$$\therefore \int_1^5 f(x)\,dx = 2(5-1) = 8$$

2. Find the average value of the cubic function $f(x) = x^3$ on the interval $[0, 1]$.

Sol:-

WKT

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

$$\bar{f} = \frac{1}{1-0} \int_0^1 x^3\, dx = \left[\frac{x^4}{4}\right]_0^1$$

$$= 1/4$$