Use the Monotonicity Theorem to find where the given function $H(t) = sint, 0 \le t \le 2\pi$ is increasing and where it is decreasing

Sol:- Given $H(t) = sint, \; 0 \le t \le 2\pi$

$H'(t) = cost$

$H'(t) = 0 \Rightarrow cost = 0$

$t = \pi/2 \;\&\; 3\pi/2$

ie., the split pts are $\pi/2 \;\&\; 3\pi/2$

The intervals are $(0, \pi/2), (\pi/2, 3\pi/2)$

$\&\; (3\pi/2, 2\pi)$

Let $t = \pi/4$ in $(0, \pi/2)$

$H'(t) = H'(\pi/4) = \cos \pi/4 = \frac{1}{\sqrt{2}} > 0$

H is increasing on $[0, \pi/2]$

Let $t = \pi$ in $(\pi/2, 3\pi/2)$

$H'(\pi) = \cos \pi = -1 < 0$

H is decreasing on $[\pi/2, 3\pi/2]$

Let $t = 7\pi/4$ in $(3\pi/2, 2\pi)$

$$H'(7\pi/4) = \cos(7\pi/4) = \cos(2\pi - \pi/4)$$
$$= \cos(-\pi/4) = \cos \pi/4$$
$$= 1/\sqrt{2} > 0$$

$\therefore H$ is increasing on $[3\pi/2, 2\pi]$

$\therefore H$ is increasing on $[0, \pi/2]$ & $[3\pi/2, 2\pi]$ & decreasing on $[\pi/2, 3\pi/2]$.

Concavity Theorem

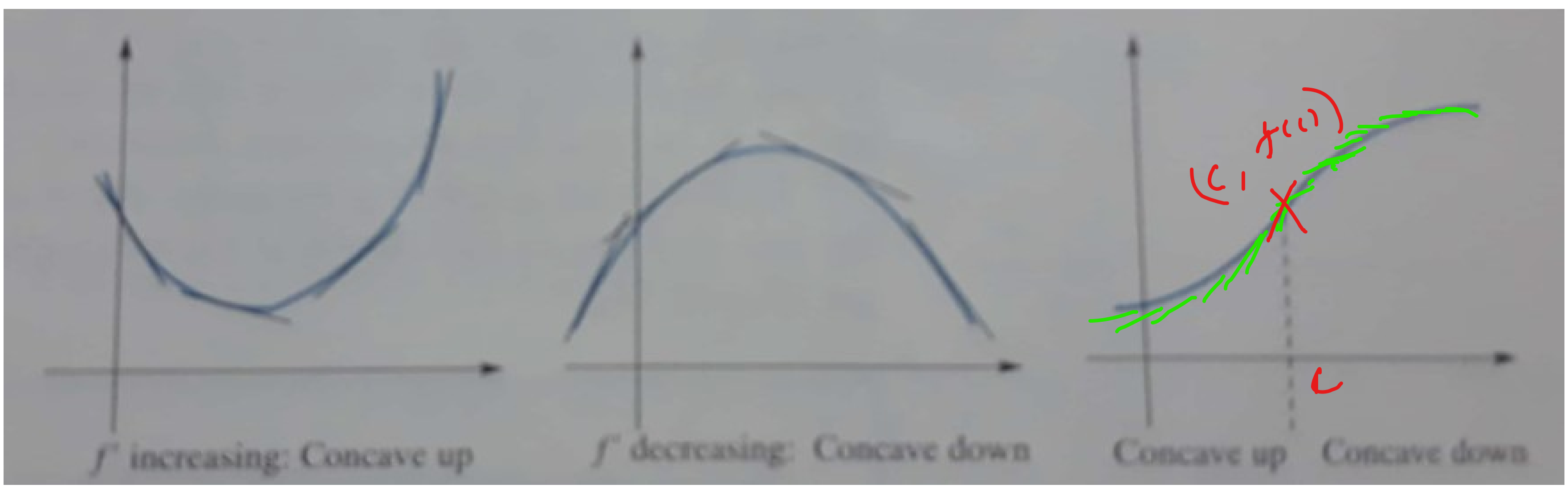
Let f be twice differentiable on the open interval I.

(i) If $f''(x) > 0$ for all x in I, then f is concave up on I.

(ii) If $f''(x) < 0$ for all x in I, then f is concave down on I.

Note:-

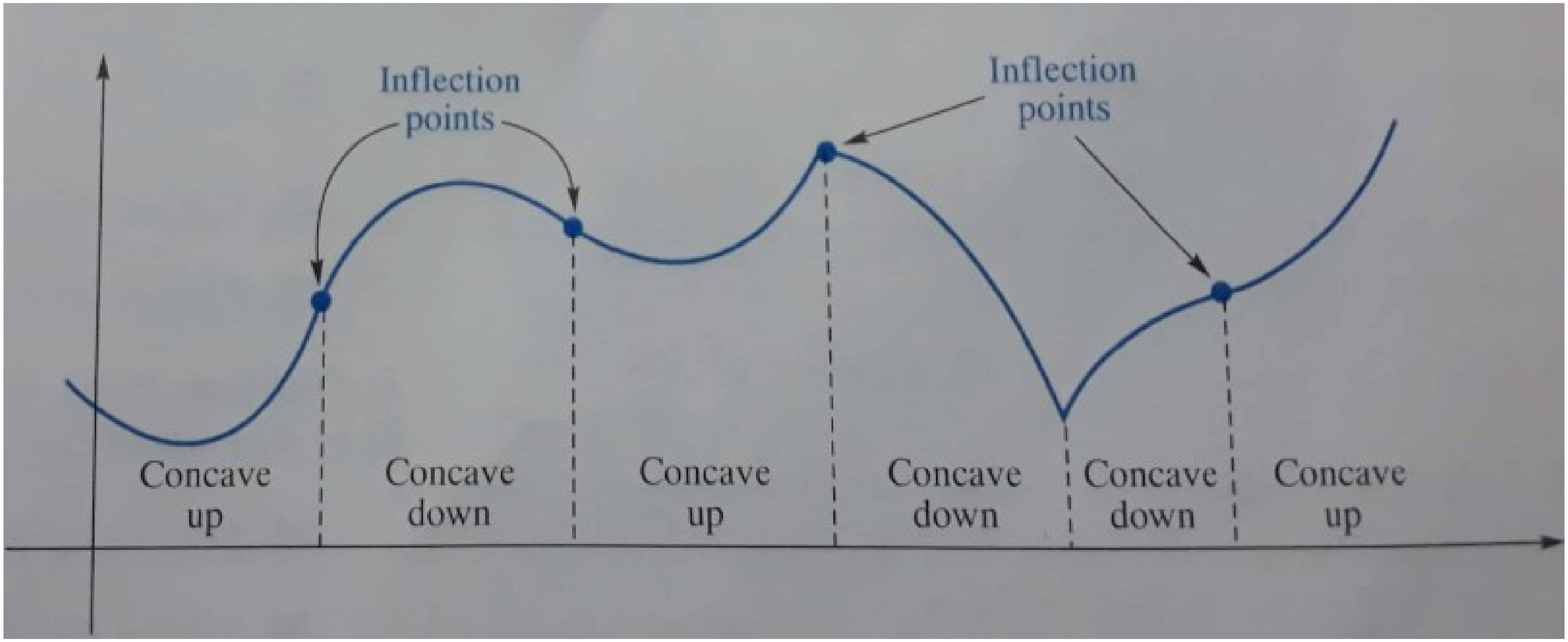
- If the tangent line turns steadily in the *counterclockwise direction*, we say the graph is *concave up*.
- If the tangent turns in the *clockwise direction*, the graph is *concave down*.



Inflection Points:-

Let f be continuous at c. We call $(c, f(c))$ an inflection point of the graph of f if f is concave up on one side of c and concave down on the other side.

Note:- $f''(x) = 0$ or where $f''(x)$ does not exist are the candidates for points of inflection.



Problem 1:- Where is $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$ concave up, and concave down?

Sol:- Given $f(x) = \frac{x^3}{3} - x^2 - 3x + 4$

$f'(x) = x^2 - 2x - 3$

$f''(x) = 2x - 2$

$f''(x) = 0 \Rightarrow 2x - 2 = 0 \Rightarrow x = 1$

The intervals are $(-\infty, 1)$ & $(1, \infty)$

Let $x = 0$ in $(-\infty, 1)$

$f''(0) = -2 < 0$

$\therefore f$ is concave down on $(-\infty, 1)$

Let $x = 2$ in $(1, \infty)$

$f''(2) = 2 > 0$

$\therefore f$ is concave up on $(1, \infty)$

Therefore $f(1) = \frac{1}{3}$,

ie the pt of inflection is $\underline{\underline{(1, 1/3)}}$

Problem 2:- Find all points of inflection of $f(x) = x^{\frac{1}{3}} + 2$.

Sol:- Given $f(x) = x^{1/3} + 2$

$$f'(x) = \frac{1}{3} x^{\frac{1}{3}-1} = \frac{1}{3} x^{-2/3}$$

$$f''(x) = \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right) x^{-\frac{2}{3}-1} = \frac{-2}{9x^{5/3}}$$

$f''(x)$ is never '0' & also it fails to exist at $x = 0$. The intervals are $(-\infty, 0)$ & $(0, \infty)$

Interval	Sample	$f''(x)$	Concavity
$(-\infty, 0)$	-1	$\frac{2}{9}\ (>0)$	CU
$(0, \infty)$	1	$-\frac{2}{9}\ (<0)$	CD

When $x = 0$, $f(0) = 2$

The pt of inflection is $(0, 2)$

Problem 3:-

Use the Concavity Theorem to determine where the given function is concave up and where it is concave down. Also find all inflection points. $T(t) = 3t^3 - 18t$.

Sol:- Given $T(t) = 3t^3 - 18t$

$$T'(t) = 9t^2 - 18, \quad T''(t) = 18t$$

$$T''(t) = 0 \Rightarrow 18t = 0, \text{ ie. } t = 0$$

The intervals are $(-\infty, 0)$ & $(0, \infty)$

Interval	Sample	$T''(t)$	Concavity
$(-\infty, 0)$	-1	$-18 (< 0)$	CD
$(0, \infty)$	1	$18 (> 0)$	CU

At $t = 0$, $T(0) = 0$

$\therefore$ the Pt of inflection is $(0, 0)$.

Problem 4:-

Let $f(x)=\frac{x^6}{30}-\frac{x^5}{20}-x^4+3x+20$. Find all points of inflection and intervals for f.

Sol:- Given $f(x)=\frac{x^6}{30}-\frac{x^5}{20}-x^4+3x+20$

$$f'(x)=\frac{x^5}{5}-\frac{x^4}{4}-4x^3+3$$

$$f''(x)=x^4-x^3-12x^2$$

$$f''(x)=0 \Rightarrow x^4-x^3-12x^2=0$$

$$x^2(x^2-x-12)=0$$

$$x^2(x-4)(x+3)=0$$

$$x=0,\ 4\ \&\ -3$$

Interval	Sample	f''	concavity
$(-\infty, -3)$	-4	$128\ (>0)$	CU
$(-3, 0)$	-1	$-10\ (<0)$	CD
$(0, 4)$	1	$-12\ (<0)$	CD
$(4, \infty)$	5	$200\ (>0)$	CU

$\therefore$ the function is concave up on
$(-\infty, -3) \cup (4, \infty)$
It is concave down on $(-3, 0) \cup (0, 4)$
We see that there is no change in

Concavity at $x=0$. That means that the only inflection points are at $x=-3$ & $x=4$

At $x=-3$, $f(-3) = -33.55$

At $x=4$, $f(4) = -138.67$

Thus the two inflection points for this function are $(-3, -33.55)$ & $(4, -138.67)$