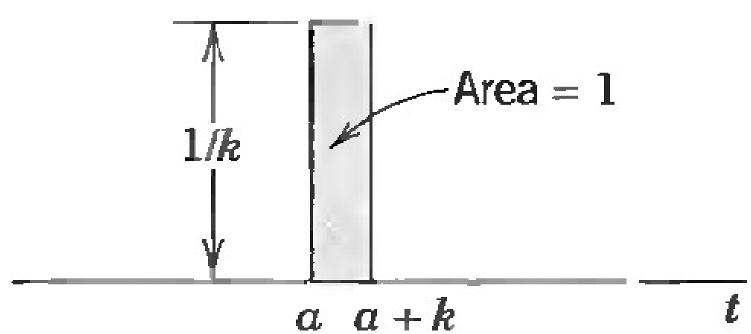
Dirac Delta Function

Phenomena of an impulsive nature, such as the action of forces or voltages over short intervals of time, arise in various applications, for instance, if a mechanical system is hit by a hammerblow, an airplane makes a "hard" landing, a ship is hit by a single high wave, or we hit a tennisball by a racket, and so on. Our goal is to show how such problems are modeled by "Dirac's delta function" and can be solved very efficiently by the Laplace transform.

$$f_k(t-a) = \begin{cases} 1/k & \text{if } a \leqq t \leqq a+k \\ 0 & \text{otherwise} \end{cases}$$



To find out what will happen if k becomes smaller and smaller, we take the limit of f_k as $k \to 0$ ($k > 0$). This limit is denoted by $\delta(t - a)$, that is,

$$\delta(t - a) = \lim_{k \to 0} f_k(t - a).$$

$\delta(t - a)$ is called the **Dirac delta function**[2] or the **unit impulse function.**

Laplace transform of $\delta(t - a)$

$$\mathscr{L}\{\delta(t - a)\} = e^{-as}.$$

the fact that $\mathscr{L}\{\delta(t)\} = 1$.

[2]PAUL DIRAC (1902–1984), English physicist, was awarded the Nobel Prize [jointly with the Austrian ERWIN SCHRÖDINGER (1887–1961)] in 1933 for his work in quantum mechanics.

Given $f(t)$

DEFINITION OF INVERSE LAPLACE TRANSFORM

$f(t)$ $F(s)$

ie $L[f(t)] = F(s)$

If the Laplace transform of a function $F(t)$ is $f(s)$, i.e. if $\mathcal{L}\{F(t)\} = f(s)$, then $F(t)$ is called an *inverse Laplace transform* of $f(s)$ and we write symbolically $F(t) = \mathcal{L}^{-1}\{f(s)\}$ where $\mathcal{L}^{-1}$ is called the *inverse Laplace transformation operator.*

$f(t) = L^{-1}[F(s)]$

Example. Since $\mathcal{L}\{e^{-3t}\} = \dfrac{1}{s+3}$ we can write

$$L(e^{-3t}) = \frac{1}{s+3}$$

$$L^{-1}\left[\frac{1}{s+3}\right] = \underline{e^{-3t}}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$$

$L(t^n) = \frac{n!}{s^{n+1}}$ $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$

$L^{-1}[F(s)] = f(t)$

Table of Inverse Laplace Transforms

	$f(s)$ $F(s)$	$\mathcal{L}^{-1}\{f(s)\} = F(t)$
1.	$\frac{1}{s}$	1 ✓
2.	$\frac{1}{s^2}$	t
3.	$\frac{1}{s^{n+1}} \quad n = 0, 1, 2, \ldots$	$\frac{t^n}{n!}$
4.	$\frac{1}{s-a}$	e^{at}
5.	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
6.	$\frac{s}{s^2+a^2}$	$\cos at$
7.	$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
8.	$\frac{s}{s^2-a^2}$	$\cosh at$

SOME IMPORTANT PROPERTIES OF INVERSE LAPLACE TRANSFORMS

1. Linearity property.

$F_1(s) + F_2(s)$

If c_1 and c_2 are any constants while $f_1(s)$ and $f_2(s)$ are the Laplace transforms of $F_1(t)$ and $F_2(t)$ respectively, then

$f_1(t) + f_2(t)$

$$\mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} = c_1 \mathcal{L}^{-1}\{f_1(s)\} + c_2 \mathcal{L}^{-1}\{f_2(s)\} \tag{1}$$

$$= c_1 F_1(t) + c_2 F_2(t)$$

The result is easily extended to more than two functions. $L^{-1}[c_1 F_1(s) + c_2 F_2(s)]$

$= c_1 f_1(t) + c_2 f_2(t)$

Example.

$$\mathcal{L}^{-1}\left\{\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$4 L^{-1}\left[\frac{1}{s-2}\right] - 3 L^{-1}\left[\frac{s}{s^2+16}\right]$

$4e^{2t} - 3\cos 4t + \frac{5}{2} L^{-1}\left[\frac{2}{s^2+4}\right]$

$$= 4e^{2t} - 3\cos 4t + \frac{5}{2}\sin 2t$$

Because of this property we can say that $\mathcal{L}^{-1}$ is a *linear operator* or that it has the *linearity property.*

$+\frac{5}{2}\sin 2t$

$$\frac{5s+4}{s^3} = \frac{5\cancel{s}}{s^{\cancel{3}2}} + \frac{4}{s^3} = \frac{5}{s^2} + \frac{4}{s^3}$$

$$\frac{2s-18}{s^2+9} = \frac{2s}{s^2+9} - \frac{18}{s^2+9}$$

Find (a) $\mathcal{L}^{-1}\left\{\frac{5s+4}{s^3} - \frac{2s-18}{s^2+9} + \frac{24-30\sqrt{s}}{s^4}\right\}$

$$\frac{24-30s^{1/2}}{s^4} = \frac{24}{s^4} - \frac{30}{s^{7/2}}$$

$$\mathcal{L}^{-1}\left\{\frac{5s+4}{s^3} - \frac{2s-18}{s^2+9} + \frac{24-30\sqrt{s}}{s^4}\right\}$$

$n+1 = 7/2$

$n = 7/2 - 1 = 5/2$

$$= \mathcal{L}^{-1}\left\{\frac{5}{s^2} + \frac{4}{s^3} - \frac{2s}{s^2+9} + \frac{18}{s^2+9} + \frac{24}{s^4} - \frac{30}{s^{7/2}}\right\}$$

$$\frac{-\cancel{30}^{2}}{\frac{\cancel{15}}{8}\sqrt{\pi}} = \frac{-16}{\sqrt{\pi}}$$

$$= 5t + 4(t^2/2!) - 2\cos 3t + 18(\tfrac{1}{3}\sin 3t) + 24(t^3/3!) - 30\{t^{5/2}/\Gamma(7/2)\}$$

$$-30\,\frac{1}{\Gamma(7/2)}\,\overbrace{L^{-1}\left(\frac{\Gamma(7/2)}{s^{7/2}}\right)}^{t^{5/2}}$$

$$= 5t + 2t^2 - 2\cos 3t + 6\sin 3t + 4t^3 - 16t^{5/2}/\sqrt{\pi}$$

since $\Gamma(7/2) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{15}{8}\sqrt{\pi}$.

$$5L^{-1}\left(\frac{1}{s^2}\right) + 2L^{-1}\left(\frac{2!}{s^3}\right) - 2L^{-1}\left(\frac{s}{s^2+3^2}\right) + 6L^{-1}\left(\frac{3}{s^2+3^2}\right) + 4L^{-1}\left(\frac{3!}{s^4}\right)$$

$\Gamma(n+1) = n\Gamma(n)$ $\Gamma(7/2) = \Gamma(5/2+1) = 5/2\,\Gamma(5/2) = \frac{5}{2}\Gamma(\frac{3}{2}+1)$

$= 5/2 \cdot 3/2\,\Gamma(3/2) = 5/2 \cdot 3/2\,\Gamma(\frac{1}{2}+1)$

Prove the *first translation or shifting property:* If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then $= \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\,\underline{\Gamma(1/2)}$ $\sqrt{\pi}$

$$\mathcal{L}^{-1}\{f(s-a)\} \quad = \quad e^{at}F(t)$$

$L^{-1}[F(s)] = f(t)$

$\underline{\text{Rem.}}$

$L[e^{at}f(t)]$

$L[f(t)] = (F(s))$

$s \to s-a$

Since $f(s) = \int_0^\infty e^{-st}F(t)\,dt$, we have

$L^{-1}[\underline{F(s-a)}] = e^{at}f(t)$

$$f(s-a) \quad = \quad \int_0^\infty e^{-(s-a)t}F(t)\,dt \quad = \quad \int_0^\infty e^{-st}\{e^{at}F(t)\}\,dt \quad = \quad \mathcal{L}\{e^{at}F(t)\}$$

Then $$\mathcal{L}^{-1}\{f(s-a)\} \quad = \quad e^{at}F(t)$$

$6s-4 = 6s-12+8$

$= 6(s-2)+8$

Find each of the following:

(a) $\mathcal{L}^{-1}\left\{\dfrac{6s-4}{s^2-4s+20}\right\}$ (b) $\mathcal{L}^{-1}\left\{\dfrac{3s+7}{s^2-2s-3}\right\}$

$s^2-4s+20$

(a) $$\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\} = \mathcal{L}^{-1}\left\{\frac{6s-4}{(s-2)^2+16}\right\} = \mathcal{L}^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+16}\right\}$$

$s^2-4s+4+16$

$$= 6\,\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+16}\right\} + 2\,\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2+16}\right\}$$

$(s-2)^2+16$

$\dfrac{6(s-2)}{(s-2)^2+16}$

$$= 6\,e^{2t}\cos 4t + 2\,e^{2t}\sin 4t = 2\,e^{2t}(3\cos 4t + \sin 4t)$$

$6\,L^{-1}\left[\dfrac{(s-2)}{(s-2)^2+16}\right]$

$= 2\,L^{-1}\left[\dfrac{4}{(s-2)^2+4^2}\right] = 2e^{2t}L^{-1}\left(\dfrac{4}{s^2+4^2}\right)$

(b) $$\mathcal{L}^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{3s+7}{(s-1)^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{3(s-1)+10}{(s-1)^2-4}\right\}$$

$= 2e^{2t}\sin 4t$

$6e^{2t}L^{-1}\left(\dfrac{s}{s^2+4^2}\right)$

$$= 3\,\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2-4}\right\} + 5\,\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2-4}\right\}$$

$6e^{2t}\cos 4t$

$$= 3\,e^{t}\cosh 2t + 5\,e^{t}\sinh 2t = e^{t}(3\cosh 2t + 5\sinh 2t)$$

$$= 4\,e^{3t} - e^{-t}$$

$\dfrac{3s+7}{s^2-2s-3} = \dfrac{3s-3+10}{s^2-2s+1-4} = \dfrac{3(s-1)+10}{(s-1)^2-4}$

$3L^{-1}\left[\dfrac{s-1}{(s-1)^2-4}\right] + 5L^{-1}\left[\dfrac{2}{(s-1)^2-4}\right]$

$3e^t L^{-1}\left(\frac{s}{s^2-2^2}\right)$ $\rightarrow$ $5e^t L^{-1}\left[\frac{2}{s^2-2^2}\right]_t$

$= 3e^t \cosh 2t$ $= 5e^t \sinh 2t$

The *second translation or shifting property:*

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then $\mathcal{L}^{-1}\{e^{-as}f(s)\} = G(t)$ where

$$G(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$$

Find each of the following:

$$\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{(s-2)^4}\right\},$$

$L^{-1}\left[\frac{1}{(s-2)^4}\right] = e^{2t} L^{-1}\left(\frac{1}{s^4}\right) = \frac{e^{2t}}{6} L^{-1}\left(\frac{3!}{s^4}\right)$

$= e^{2t}/6 \; t^3$

$$L^{-1}\left[\frac{e^{-5s}}{(s-2)^4}\right] = \begin{cases} \frac{1}{6}(t-5)^3 e^{2(t-5)}, & t>5 \\ 0, & t<5 \end{cases}$$

(a) Since $\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^4}\right\} = e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{t^3 e^{2t}}{3!} = \frac{1}{6}t^3 e^{2t}$,

$$\frac{1}{6}(t-5)^3 e^{2(t-5)} u(t-5)$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{(s-2)^4}\right\} = \begin{cases} \frac{1}{6}(t-5)^3 e^{2(t-5)} & t>5 \\ 0 & t<5 \end{cases}$$

$$= \frac{1}{6}(t-5)^3 e^{2(t-5)}\,\mathcal{U}(t-5)$$