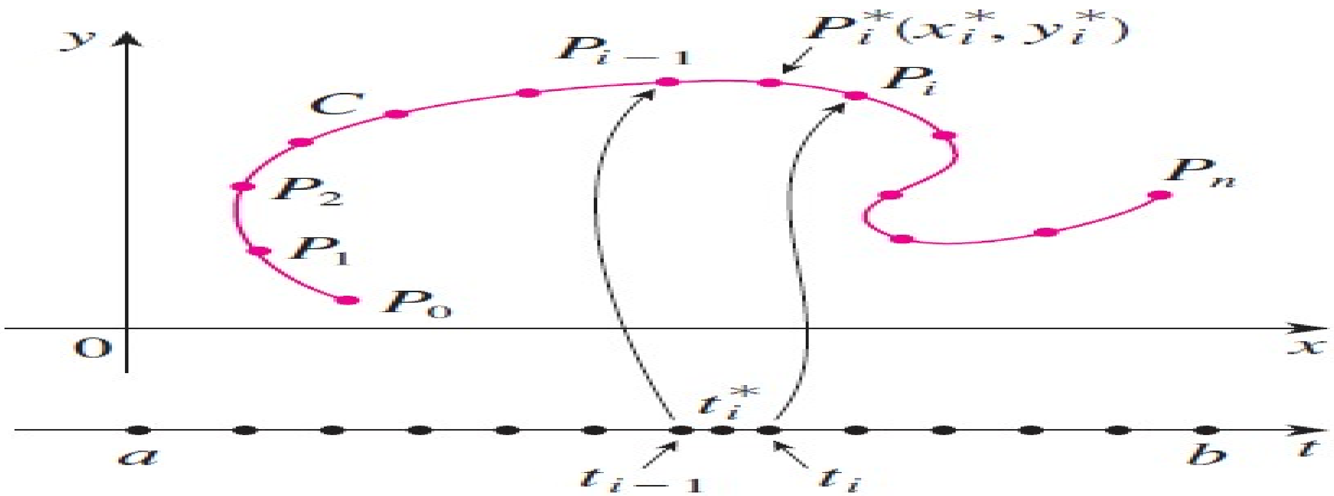
Module VII

LINE INTEGRALS

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval $[a, b]$, we integrate over a curve C. Such integrals are called *line integrals*, although "curve integrals" would be better terminology. They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism.



We start with a plane curve C given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad a \leqslant t \leqslant b$$

or, equivalently, by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, and we assume that C is a smooth curve. [This means that $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$. If we divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width and we let $x_i = x(t_i)$ and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. (See Figure .) We choose any point $P_i^*(x_i^*, y_i^*)$ in the ith subarc. (This corresponds to a point t_i^* in $[t_{i-1}, t_i]$.) Now if f is any function of two variables whose domain includes the curve C, we evaluate f at the point (x_i^*, y_i^*), multiply by the length Δs_i of the subarc, and form the sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

DEFINITION If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

if this limit exists.

NOTE :

$$\int_C f(x, y)\, ds = \int_a^b f\big(x(t), y(t)\big) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

DEFINITION Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

Problem: Evaluate $\int_C xy^4\, ds$ where C is the right half of the circle, $x^2+y^2=16$ rotated in the counter clockwise direction.

$x^2+y^2=4^2$

$x=4\cos t,\ y=4\sin t$

Solution

We first need a parameterization of the circle. This is given by,

$$x=4\cos t \qquad y=4\sin t$$

We now need a range of t's that will give the right half of the circle. The following range of t's will do this.

$$-\frac{\pi}{2}\le t\le\frac{\pi}{2}$$

Now, we need the derivatives of the parametric equations and let's compute ds.

$$\frac{dx}{dt}=-4\sin t \qquad \frac{dy}{dt}=4\cos t$$

$$ds=\sqrt{16\sin^2 t+16\cos^2 t}\,dt=4\,dt$$

The line integral is then,

$$\int_C xy^4\,ds=\int_{-\pi/2}^{\pi/2}4\cos t\,(4\sin t)^4\,(4)\,dt$$

$$=4096\int_{-\pi/2}^{\pi/2}\cos t\,\sin^4 t\,dt$$

$$=\frac{4096}{5}\sin^5 t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=\frac{8192}{5}$$

$u=\sin t$

$du=\cos t\,dt$

$t=\pi/2$

$u=1$

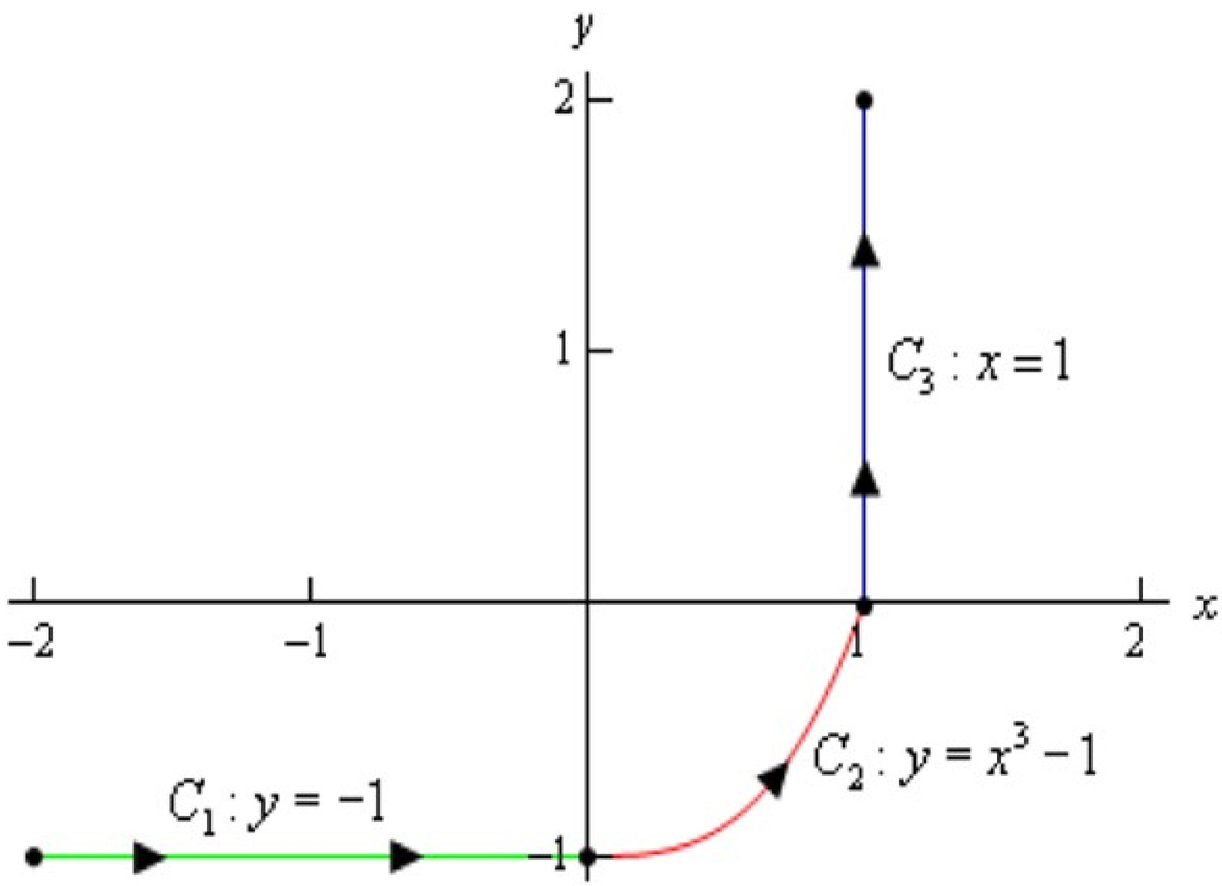
when $t=-\pi/2$

$u=-1$

$=4096\int_{-1}^{1}u^4\,du$

$=\frac{4096}{5}\left(u^5\right)\Big|_{-1}^{1}=\frac{4096}{5}\left[1-(-1)\right]$

Problem :- Evaluate $\int_C 4x^3 \, ds$ where C is the curve shown below.



Solution

So, first we need to parameterize each of the curves.

$$C_1 : \; x = t,\; y = -1, \qquad -2 \le t \le 0$$
$$C_2 : \; x = t,\; y = t^3 - 1, \qquad 0 \le t \le 1$$
$$C_3 : \; x = 1,\; y = t, \qquad 0 \le t \le 2$$

$\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 0$

$\frac{dx}{dt} = 0,\ \frac{dy}{dt} = 1$

Now let's do the line integral over each of these curves.

$$\int_{C_1} 4x^3\, ds = \int_{-2}^{0} 4t^3 \sqrt{(1)^2 + (0)^2}\, dt = \int_{-2}^{0} 4t^3\, dt = t^4 \Big|_{-2}^{0} = -16$$

$\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 3t^2$

$t = 0,\ u = 1$

$t = 1,\ u = 10$

$$\int_{C_2} 4x^3\, ds = \int_{0}^{1} 4t^3 \sqrt{(1)^2 + \left(3t^2\right)^2}\, dt$$
$$= \int_0^1 4t^3 \sqrt{1 + 9t^4}\, dt$$
$$= \frac{1}{9}\left(\frac{2}{3}\right)\left(1 + 9t^4\right)^{\frac{3}{2}}\Big|_0^1 = \frac{2}{27}\left(10^{\frac{3}{2}} - 1\right) = 2.268$$

$u = 1 + 9t^4$

$du = 36t^3\, dt$

$$\int_{C_3} 4x^3\, ds = \int_0^2 4(1)^3 \sqrt{(0)^2 + (1)^2}\, dt = \int_0^2 4\, dt = 8$$

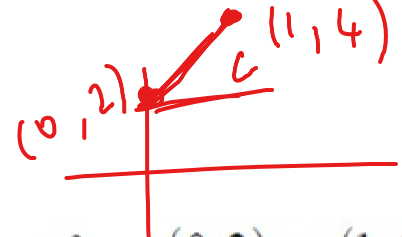
Finally, the line integral that we were asked to compute is,

$$\int_C 4x^3\, ds = \int_{C_1} 4x^3\, ds + \int_{C_2} 4x^3\, ds + \int_{C_3} 4x^3\, ds$$
$$= -16 + 2.268 + 8$$
$$= -5.732$$

$x = (1-t)x_1 + tx_2$

$y = (1-t)y_1 + ty_2$

$(0,2)$ (x_1, y_1) $(1,4)$ (x_2, y_2)



Problem : Evaluate $\int_C \sin(\pi y)\,dy + yx^2\,dx$ where C is the line segment from $(0,2)$ to $(1,4)$.

$x = (1-t)(0) + t(1) \Rightarrow x = t$

$y = (1-t)2 + 4t = 2 - 2t + 4t = 2 + 2t$

$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$

Solution

Here is the parameterization of the curve.

$$\vec{r}(t) = (1-t)\langle 0,2\rangle + t\langle 1,4\rangle = \langle t, 2+2t\rangle \qquad 0 \le t \le 1$$

$\vec{r}(t) = t\,\vec{i} + (2+2t)\,\vec{j}$

$x = t$, $y = 2+2t$

dy

The line integral is,

$\vec{r}$

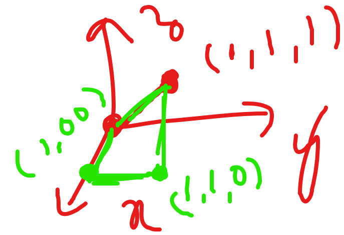
$\langle x, y\rangle = (1-t)\langle x_1, y_1\rangle + t\langle x_2, y_2\rangle$

$$\int_C \sin(\pi y)\,dy + yx^2\,dx = \int_C \sin(\pi y)\,dy + \int_C yx^2\,dx$$

$$= \int_0^1 \sin(\pi(2+2t))(2)\,dt + \int_0^1 (2+2t)(t)^2(1)\,dt$$

$$= -\frac{1}{\pi}\cos(2\pi + 2\pi t)\Big|_0^1 + \left(\frac{2}{3}t^3 + \frac{1}{2}t^4\right)\Big|_0^1$$

$$= \frac{7}{6}$$



Problem :-

If $\mathbf{A} = (3x^2+6y)\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$, evaluate $\int_C \mathbf{A}\cdot d\mathbf{r}$ from (0,0,0) to (1,1,1) along the following paths C:

(a) $x = t,\ y = t^2,\ z = t^3$.

(b) the straight lines from (0,0,0) to (1,0,0), then to (1,1,0), and then to (1,1,1).

(c) the straight line joining (0,0,0) and (1,1,1).

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_C \left[(3x^2+6y)\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\mathbf{k}\right]\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

(a) If $x=t,\ y=t^2,\ z=t^3$, points (0,0,0) and (1,1,1) correspond to $t=0$ and $t=1$ respectively. Then

$dx = dt$

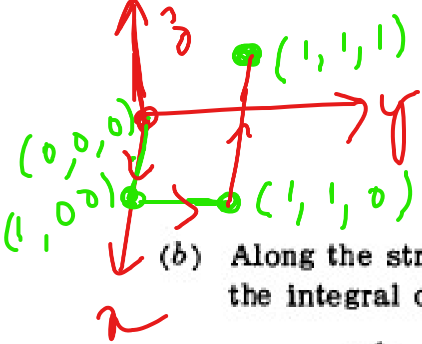
$dy = 2t\,dt$

$dz = 3t^2 dt$

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t=0}^{1} (3t^2+6t^2)\,dt - 14(t^2)(t^3)\,\underbrace{d(t^2)}_{2t\,dt} + 20(t)(t^3)^2\,\underbrace{d(t^3)}_{3t^2dt}$$

$$= \int_{t=0}^{1} 9t^2\,dt - 28t^6\,dt + 60t^9\,dt$$

$$= \int_{t=0}^{1} (9t^2 - 28t^6 + 60t^9)\,dt = 3t^3 - 4t^7 + 6t^{10}\Big|_0^1 = 5$$



$x = t \quad dx = dt$

$y = 0 \quad dy = 0$

$z = 0 \quad dz = 0$

$$\int_{t=0}^{t=1} 3t^2\, dt = 3\left(\frac{t^3}{3}\right)\Big|_0^1 = 1$$

(*b*) Along the straight line from (0,0,0) to (1,0,0) $y = 0$, $z = 0$, $dy = 0$, $dz = 0$ while x varies from 0 to 1. Then the integral over this part of the path is

$$\int_{x=0}^{1} \left(3x^2 + 6(0)\right) dx \;-\; 14(0)(0)(0) \;+\; 20x(0)^2(0) \;=\; \int_{x=0}^{1} 3x^2\, dx \;=\; x^3\Big|_0^1 \;=\; 1$$

Along the straight line from (1,0,0) to (1,1,0) $x = 1$, $z = 0$, $dx = 0$, $dz = 0$ while y varies from 0 to 1. Then the integral over this part of the path is

$x = 1 \quad dx = 0$

$y = t \quad dy = dt$

$z = 0$

$dz = 0$

$$\int_{y=0}^{1} (3(1)^2 + 6y)\,0 \;-\; 14y(0)\,dy \;+\; 20(1)(0)^2\,0 \;=\; 0$$

$x = 1$

$y = 1 \quad z = t$

Along the straight line from (1,1,0) to (1,1,1) $x = 1$, $y = 1$, $dx = 0$, $dy = 0$ while z varies from 0 to 1. Then the integral over this part of the path is

$$\int_{z=0}^{1} \left(3(1)^2 + 6(1)\right) 0 \;-\; 14(1)\,z(0) \;+\; 20(1)\,z^2\,dz \;=\; \int_{z=0}^{1} 20z^2\,dz \;=\; \frac{20z^3}{3}\Big|_0^1 \;=\; \frac{20}{3}$$

Adding,
$$\int_C \mathbf{A}\cdot d\mathbf{r} \;=\; 1 + 0 + \frac{20}{3} \;=\; \frac{23}{3}$$

(*c*) The straight line joining (0,0,0) and (1,1,1) is given in parametric form by $x = t$, $y = t$, $z = t$. Then

$dx = dt \quad dy = dt$

$dz = dt$

$$\int_C \mathbf{A}\cdot d\mathbf{r} \;=\; \int_{t=0}^{1} (3t^2 + 6t)\,dt \;-\; 14(t)(t)\,dt \;+\; 20(t)(t)^2\,dt$$

$$= \int_{t=0}^{1} (3t^2 + 6t - 14t^2 + 20t^3)\,dt \;=\; \int_{t=0}^{1} (6t - 11t^2 + 20t^3)\,dt \;=\; \frac{13}{3}$$

Find the total work done in moving a particle in a force field given by $\mathbf{F} = 3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t=1$ to $t=2$.

$dx = 2t\,dt$

$dy = 4t\,dt$

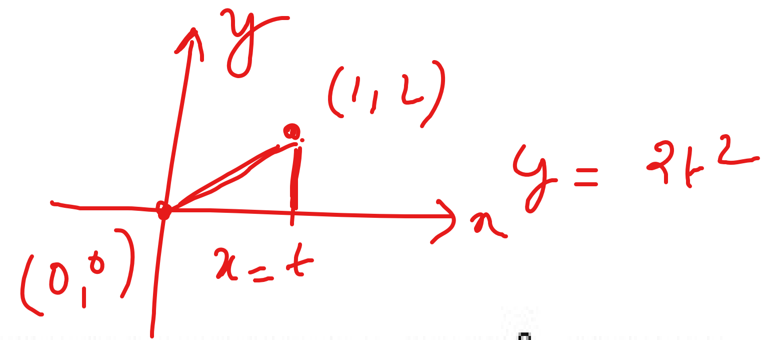
$dz = 3t^2\,dt$

$$\text{Total work} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C 3xy\,dx - 5z\,dy + 10x\,dz$$

$$= \int_{t=1}^{2} 3(t^2+1)(2t^2)\,\underline{d(t^2+1)} - 5(t^3)\,\underline{d(2t^2)} + 10(t^2+1)\,\underline{d(t^3)}$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt = \underline{\underline{303}}$$



Problem :-

If $\mathbf{F} = 3xy\,\mathbf{i} - y^2\,\mathbf{j}$, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve in the xy plane, $y = 2x^2$, from (0,0) to (1,2).

$\vec{r} = x\vec{i} + y\vec{j}$

Since the integration is performed in the xy plane ($z=0$), we can take $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (3xy\,\mathbf{i} - y^2\,\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= \int_C \underline{3xy\,dx - y^2\,dy}$$

Let $x = t$ in $y = 2x^2$. Then the parametric equations of C are $x = t,\ y = 2t^2$. Points (0,0) and (1,2) correspond to $t = 0$ and $t = 1$ respectively. Then

$dx = dt \qquad dy = 4t\,dt$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{t=0}^{1} 3(t)(2t^2)\,dt - (2t^2)^2\,d(2t^2) = \int_{t=0}^{1} \underline{(6t^3 - 16t^5)}\,dt = -\frac{7}{6}$$

$$\nabla \times \vec{F} = \vec{0}$$

Problem :-

(*a*) Show that $\mathbf{F} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$ is a conservative force field.

(b) Find the work done in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$.

Solution :-

(*a*) The necessary and sufficient condition that a force will be conservative is that curl $\mathbf{F} = \nabla\times\mathbf{F} = \mathbf{0}$.

Now
$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix} = \mathbf{0}.$$

$$\vec{i}\left(\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(x^2)\right) \quad \| \; 0$$

$$-\vec{j}\left(\cancel{\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(2xy+z^3)}\right)$$

$$+\vec{k}\left(\cancel{\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy+z^3)}\right)$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}$$

Thus $\mathbf{F}$ is a conservative force field.

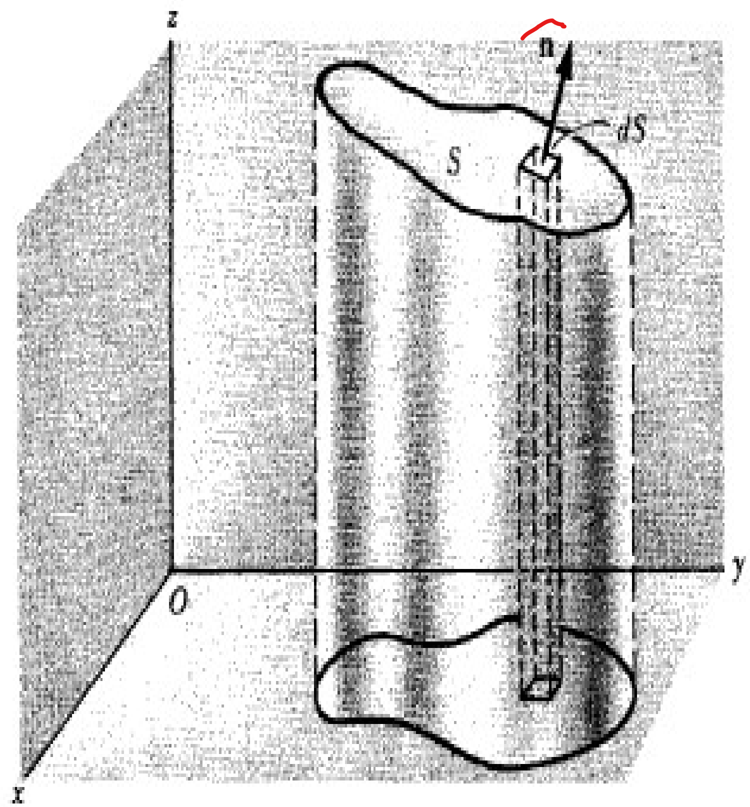
(*c*) Work done
$$= \int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r}$$
$$= \int_{P_1}^{P_2} \underbrace{(2xy + z^3)\,dx + x^2\,dy + 3xz^2\,dz}$$
$$= \int_{P_1}^{P_2} d(x^2y + xz^3) = x^2y + xz^3\Big|_{P_1}^{P_2} = x^2y + xz^3\Big|_{(1,-2,1)}^{(3,1,4)} = 202$$

$$\vec{F}\cdot d\vec{r} = \left[(2xy+z^3)\vec{i} + x^2\vec{j} + (3xz^2)\vec{k}\right]\cdot\left[dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}\right]$$

$$= (2xy+z^3)\,dx + x^2\,dy + 3xz^2\,dz$$

$$d(x^2y + xz^3) = 2xy\,dx + x^2\,dy + 3xz^2\,dz + z^3\,dx$$

SURFACE INTEGRALS. Let S be a two-sided surface, such as shown in the figure below. Let one side of S be considered arbitrarily as the positive side (if S is a closed surface this is taken as the outer side). A unit normal $\mathbf{n}$ to any point of th positive side of S is called a *positive* or *outward drawn* unit normal.



Associate with the differential of surface area dS a vector $d\mathbf{S}$ whose magnitude is dS and whose direction is that of $\mathbf{n}$. Then $d\mathbf{S} = \mathbf{n}\,dS$.

$d\vec{S} = \hat{n}\,ds$

The integral

$$\iint_S \mathbf{A}\cdot d\mathbf{S} = \iint_S \mathbf{A}\cdot\mathbf{n}\,dS$$

NOTE 1:-

The notation $\oiint_S$ is sometimes used to indicate integration over the closed surface S. Where no confusion can arise the notation $\oint_S$ may also be used.

NOTE 2:-

To evaluate surface integrals, it is convenient to express them as double integrals taken over the projected area of the surface S on one of the coordinate planes. This is possible if any line perpendicular to the coordinate plane chosen meets the surface in no more than one point. However, this does not pose any real problem since we can generally subdivide S into surfaces which do satisfy this restriction.

$Note \quad 3:-$

The $\iint\limits_S \overline{F} \cdot \hat{n}\, dS$ denotes the total mass flux of fluid through the surface S, when $\overline{F}$ is the velocity of the fluid.

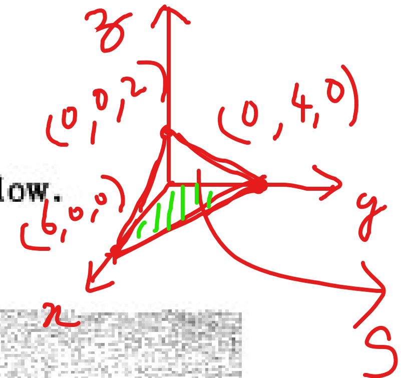
$Note \quad 4:-$

$$dS = \frac{dxdy}{\left|\hat{n} \cdot \overline{k}\right|};\ dS = \frac{dydz}{\left|\hat{n} \cdot \overline{i}\right|};\ dS = \frac{dzdx}{\left|\hat{n} \cdot \overline{j}\right|}.$$

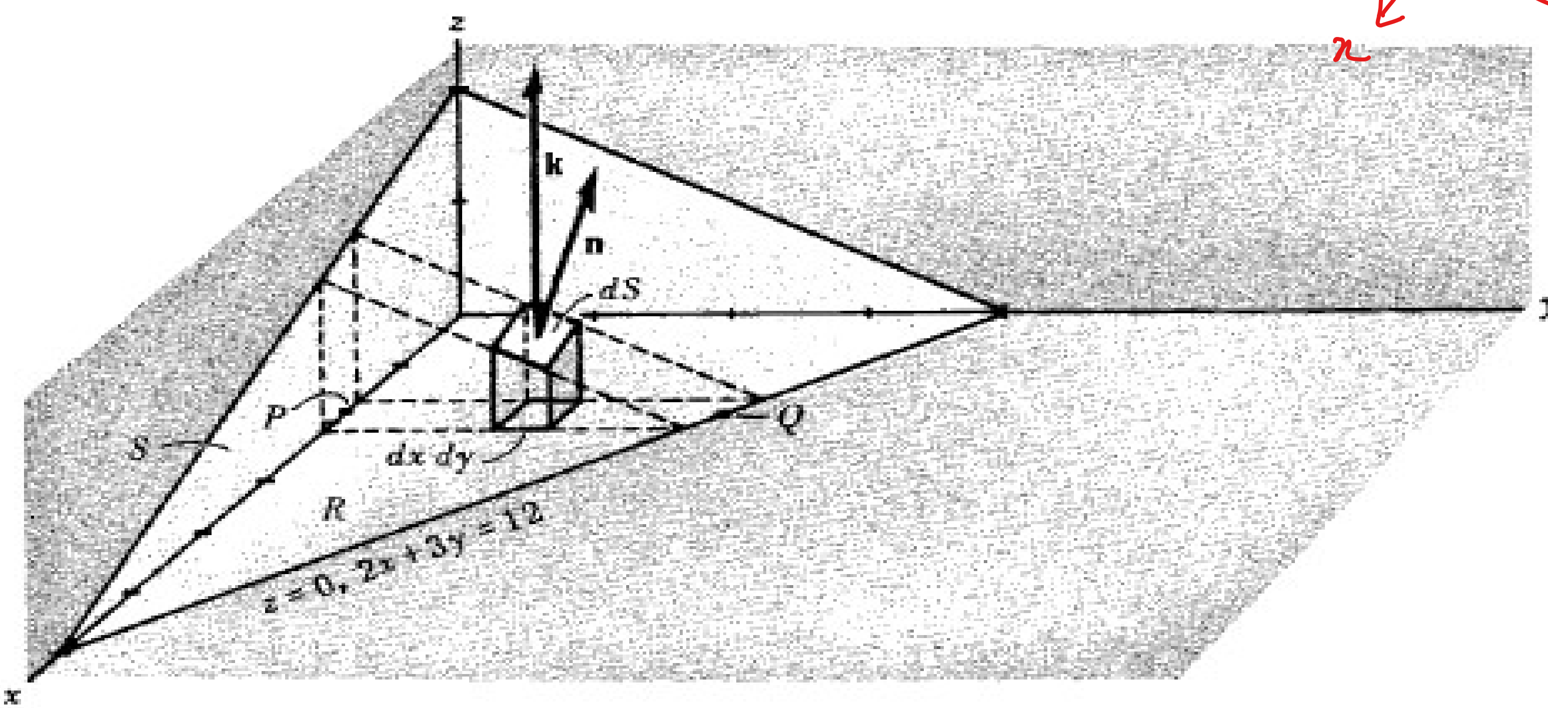
Problem :-

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

Evaluate $\iint_S \mathbf{A}\cdot\mathbf{n}\,dS$, where $\mathbf{A} = 18z\,\mathbf{i} - 12\mathbf{j} + 3y\,\mathbf{k}$ and S is that part of the plane $2x + 3y + 6z = 12$ which is located in the first octant.



The surface S and its projection R on the xy plane are shown in the figure below.



$$\phi = 2x + 3y + 6z - 12 = 0$$

$\nabla\phi = 2\vec{i} + 3\vec{j} + 6\vec{k}$

$\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2\vec{i}+3\vec{j}+6\vec{k}}{\sqrt{2^2+3^2+6^2}}$

$$\iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \iint_R \mathbf{A}\cdot\mathbf{n}\,\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$$

$\nabla\phi = \dfrac{\partial}{\partial x}(2x+3y+6z)\vec{i} + \dfrac{\partial}{\partial y}(2x+3y+6z)\vec{j} + \partial/\partial z(2x+3y+6z)\vec{k}$

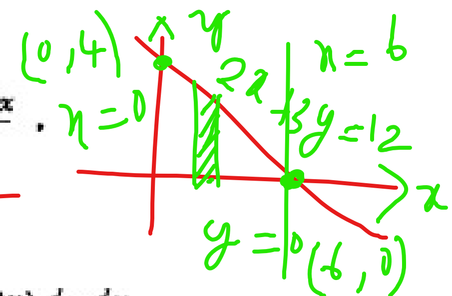
To obtain $\mathbf{n}$ note that a vector perpendicular to the surface $2x+3y+6z = 12$ is given by $\nabla(2x+3y+6z) = 2\mathbf{i}+3\mathbf{j}+6\mathbf{k}$. Then a unit normal to any point of S (see figure above) is

$$\mathbf{n} = \frac{2\mathbf{i}+3\mathbf{j}+6\mathbf{k}}{\sqrt{2^2+3^2+6^2}} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$$

$\dfrac{dx\,dy}{|\hat{n}\cdot\vec{k}|} = \dfrac{7}{6}dx\,dy$

Thus $\mathbf{n}\cdot\mathbf{k} = (\frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k})\cdot\mathbf{k} = \frac{6}{7}$ and so $\dfrac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|} = \frac{7}{6}\,dx\,dy$.

Also $\mathbf{A}\cdot\mathbf{n} = (18z\,\mathbf{i} - 12\mathbf{j} + 3y\,\mathbf{k})\cdot(\frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}) = \dfrac{36z - 36 + 18y}{7} = \dfrac{36-12x}{7}$, using the fact that $z = \dfrac{12-2x-3y}{6}$ from the equation of S. Then



$$\iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \iint_R \mathbf{A}\cdot\mathbf{n}\,\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|} = \iint_R \left(\frac{36-12x}{7}\right)\frac{7}{6}\,dx\,dy = \iint_R (6-2x)\,dx\,dy$$

To evaluate this double integral over R, keep x fixed and integrate with respect to y from $y=0$ (P in the figure above) to $y = \dfrac{12-2x}{3}$ (Q in the figure above); then integrate with respect to x from $x=0$ to $x=6$. In this manner R is completely covered. The integral becomes

$$\int_{x=0}^{6}\int_{y=0}^{(12-2x)/3} (6-2x)\,dy\,dx = \int_{x=0}^{6}\left(24 - 12x + \frac{4x^2}{3}\right)dx = \underline{\underline{24}}$$

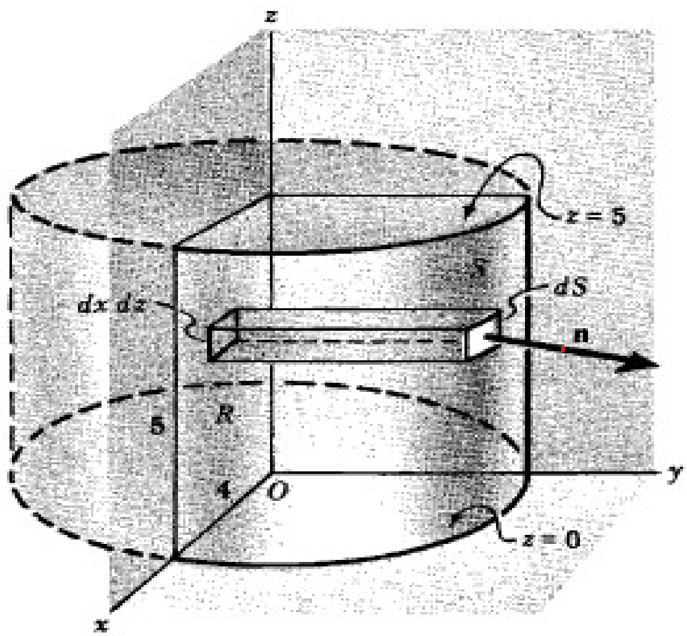
Evaluate $\iint_S \mathbf{A}\cdot\mathbf{n}\, dS$, where $\mathbf{A} = z\,\mathbf{i} + x\,\mathbf{j} - 3y^2 z\,\mathbf{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

$\phi = x^2+y^2-16=0$

Project S on the xz plane as in the figure below and call the projection R. Note that the projection of S on the xy plane cannot be used here. Then

$$\iint_S \mathbf{A}\cdot\mathbf{n}\, dS = \iint_R \mathbf{A}\cdot\mathbf{n}\,\frac{dx\,dz}{|\mathbf{n}\cdot\mathbf{j}|}$$

A normal to $x^2+y^2=16$ is $\nabla(x^2+y^2) = 2x\mathbf{i}+2y\mathbf{j}$. Thus the unit normal to S as shown in the adjoining figure, is



$$\hat{\mathbf{n}} = \frac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{\sqrt{(2x)^2+(2y)^2}} = \frac{x\,\mathbf{i}+y\,\mathbf{j}}{4}$$

since $x^2+y^2=16$ on S.

$y = \sqrt{16-x^2}$

$$\mathbf{A}\cdot\mathbf{n} = (z\,\mathbf{i}+x\,\mathbf{j}-3y^2z\,\mathbf{k})\cdot\left(\frac{x\,\mathbf{i}+y\,\mathbf{j}}{4}\right) = \frac{1}{4}(xz+xy)$$

$$\mathbf{n}\cdot\mathbf{j} = \frac{x\,\mathbf{i}+y\,\mathbf{j}}{4}\cdot\mathbf{j} = \frac{y}{4}.$$

Then the surface integral equals

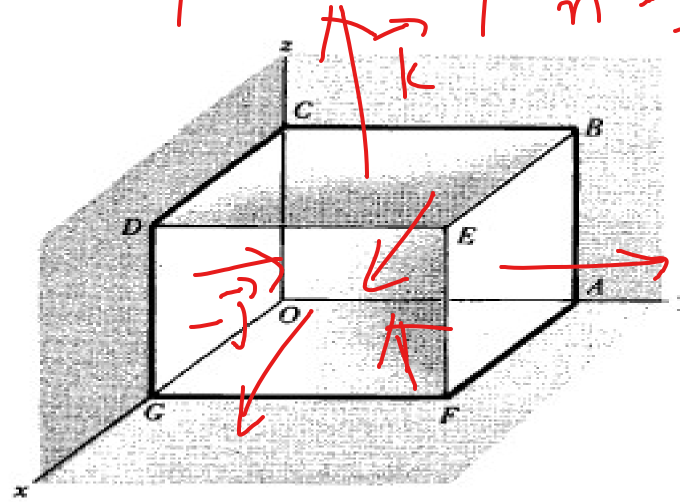
$$\iint_R \frac{xz+xy}{y}\,dx\,dz = \int_{z=0}^{5}\int_{x=0}^{4}\underbrace{\left(\frac{xz}{\sqrt{16-x^2}}+x\right)}\,dx\,dz = \int_{z=0}^{5}\underline{(4z+8)}\,dz = \underline{90}$$

$\phi = x - 1 = 0$

$\nabla\phi = \vec{i}$

$\hat{n} = \dfrac{\vec{i}}{|\vec{i}|}$

If $\mathbf{F} = 4xz\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k}$, evaluate $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ where S is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.



Face DEFG: $\hat{\mathbf{n}} = \mathbf{i}$, $x = 1$. Then

$$\iint_{DEFG} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (4z\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k})\cdot\mathbf{i}\,dy\,dz$$

$|\hat{n}\cdot\vec{i}|$

$$= \int_0^1\int_0^1 4z\,dy\,dz = 2$$

Face ABCO: $\mathbf{n} = -\mathbf{i}$, $x = 0$. Then

$$\iint_{ABCO} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (-y^2\,\mathbf{j} + yz\,\mathbf{k})\cdot(-\mathbf{i})\,dy\,dz = 0$$

Face ABEF: $\mathbf{n} = \mathbf{j}$, $y = 1$. Then

$$\iint_{ABEF} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (4xz\,\mathbf{i} - \mathbf{j} + z\,\mathbf{k})\cdot\mathbf{j}\,dx\,dz = \int_0^1\int_0^1 -dx\,dz = -1$$

$|\hat{n}\cdot\vec{j}|$

Face OGDC: $\mathbf{n} = -\mathbf{j}$, $y = 0$. Then

$$\iint_{OGDC} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (4xz\,\mathbf{i})\cdot(-\mathbf{j})\,dx\,dz = 0$$

Face BCDE: $\mathbf{n} = \mathbf{k}$, $z = 1$. Then

$$\iint_{BCDE} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (4x\,\mathbf{i} - y^2\,\mathbf{j} + y\,\mathbf{k})\cdot\mathbf{k}\,dx\,dy = \int_0^1\int_0^1 y\,dx\,dy = \tfrac{1}{2}$$

$|\vec{n}\cdot\vec{k}|$

Face AFGO: $\mathbf{n} = -\mathbf{k}$, $z = 0$. Then

$$\iint_{AFGO} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (-y^2\,\mathbf{j})\cdot(-\mathbf{k})\,dx\,dy = 0$$

Adding, $$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 2 + 0 + (-1) + 0 + \tfrac{1}{2} + 0 = \frac{3}{2}.$$

GREEN'S THEOREM

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C. (See Figure 1. We assume that D consists of all points inside C as well as all points on C.) In stating Green's Theorem we use the convention that the **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C. Thus if C is given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C. (See Figure 2.)

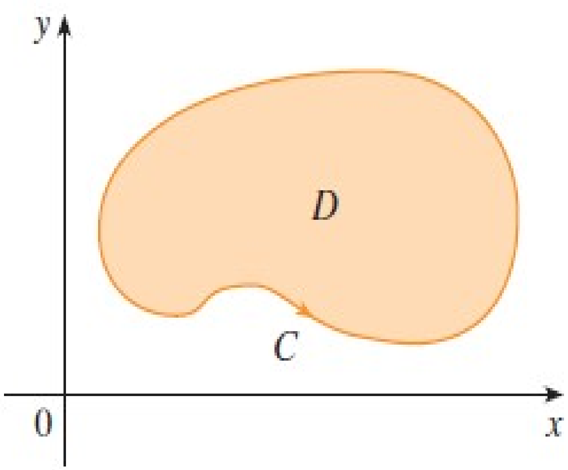


FIGURE 1

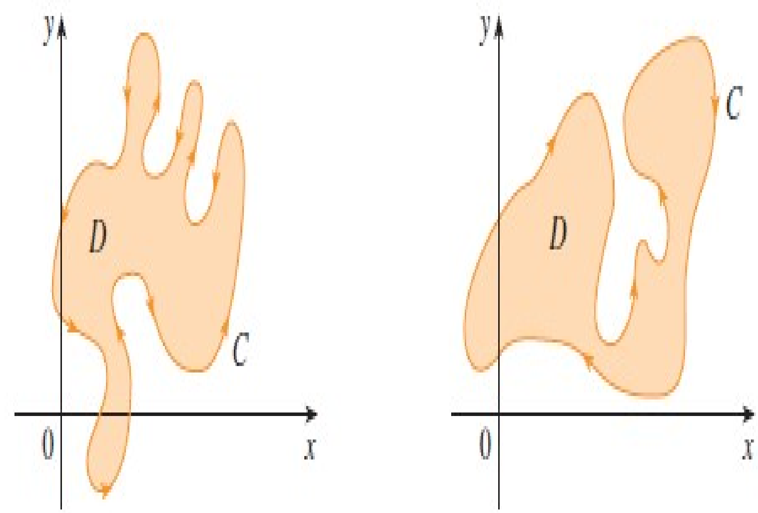


FIGURE 2

(a) Positive orientation

(b) Negative orientation

GREEN'S THEOREM IN THE PLANE. If R is a closed region of the xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

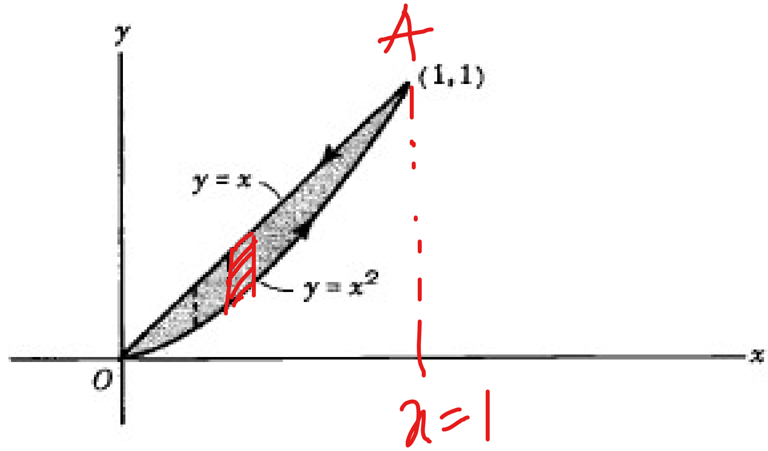
LHS = RHS

where C is traversed in the positive (counterclockwise) direction. Unless otherwise stated we shall always assume $\oint$ to mean that the integral is described in the positive sense.

$M = xy + y^2 \qquad N = x^2 \qquad 2x - (x+2y) \Rightarrow x - 2y$

$\frac{\partial M}{\partial y} = x + 2y \qquad \frac{\partial N}{\partial x} = 2x$

Verify Green's theorem in the plane for $\oint_C (xy + y^2)\,dx + x^2\,dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.



$y = x$ and $y = x^2$ intersect at $(0,0)$ and $(1,1)$. The positive direction in traversing C is as shown in the adjacent diagram.

Along $y = x^2$, the line integral equals

$dy = 2x\,dx$

$$\int_0^1 ((x)(x^2) + x^4)\,dx + (x^2)(2x)\,dx = \int_0^1 (3x^3 + x^4)\,dx = \frac{19}{20}$$

Along $y = x$ from $(1,1)$ to $(0,0)$ the line integral equals

$y = x \Rightarrow dy = dx$

$$\int_1^0 ((x)(x) + x^2)\,dx + x^2\,dx = \int_1^0 3x^2\,dx = -1$$

Then the required line integral $= \frac{19}{20} - 1 = -\frac{1}{20}$. LHS

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dx\,dy$$

$$= \iint_R (x - 2y)\,dx\,dy = \int_{x=0}^{1} \int_{y=x^2}^{y=x} (x - 2y)\,dy\,dx$$

$$= \int_0^1 \left[\int_{x^2}^{x} (x - 2y)\,dy\right] dx = \int_0^1 (xy - y^2)\Big|_{x^2}^{x}\,dx$$

$$= \int_0^1 (x^4 - x^3)\,dx = -\frac{1}{20}$$ RHS

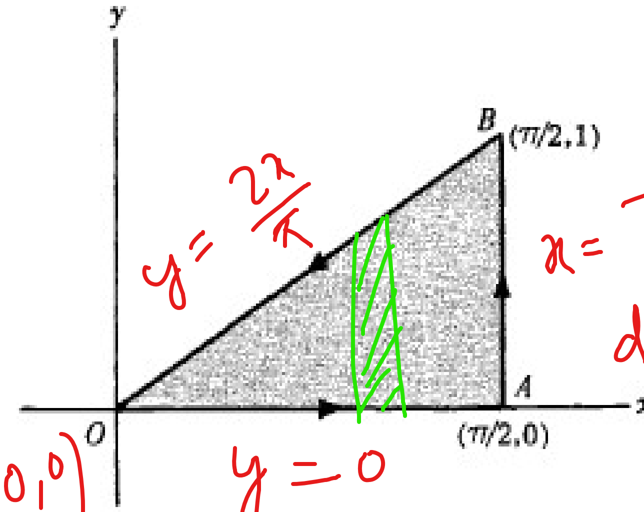
so that the theorem is verified.

$y - 0 = \frac{1 - 0}{\pi/2 - 0}(x - 0) \Rightarrow y = \frac{x}{\pi/2} = \frac{2}{\pi}x$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$(0,0)$; (x_1, y_1) $(\pi/2, 1)$ (x_2, y_2)

Evaluate $\oint_C (y - \sin x)\,dx + \cos x\, dy$, where C is the triangle of the adjoining figure:
(a) directly,
(b) by using Green's theorem in the plane.



(a) Along OA, $y = 0$, $dy = 0$ and the integral equals

$$\int_0^{\pi/2} (0 - \sin x)\,dx + (\cos x)(0) = \int_0^{\pi/2} -\sin x\, dx$$

$$= \cos x \Big|_0^{\pi/2} = -1$$

Along AB, $x = \frac{\pi}{2}$, $dx = 0$ and the integral equals

$$\int_0^1 (y-1)0 + 0\,dy = 0$$

$dy = \frac{2}{\pi} dx$

Along BO, $y = \frac{2x}{\pi}$, $dy = \frac{2}{\pi}dx$ and the integral equals

$$\int_{\pi/2}^{0} \left(\frac{2x}{\pi} - \sin x\right)dx + \frac{2}{\pi}\cos x\, dx = \left(\frac{x^2}{\pi} + \cos x + \frac{2}{\pi}\sin x\right)\Big|_{\pi/2}^{0} = 1 - \frac{\pi}{4} - \frac{2}{\pi}$$

Then the integral along C $= -1 + 0 + 1 - \frac{\pi}{4} - \frac{2}{\pi} = -\frac{\pi}{4} - \frac{2}{\pi}$.

$M = y - \sin x$, $N = \cos x$

(b) $M = y - \sin x$, $N = \cos x$, $\frac{\partial N}{\partial x} = -\sin x$, $\frac{\partial M}{\partial y} = 1$ and

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (-\sin x - 1)\,dy\,dx$$

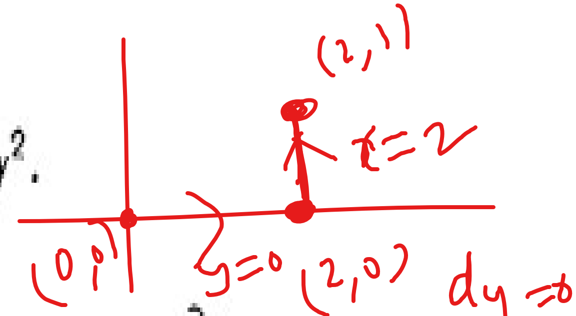
$$= \int_{x=0}^{\pi/2} \left[\int_{y=0}^{2x/\pi} (-\sin x - 1)\,dy\right] dx = \int_{x=0}^{\pi/2} (-y\sin x - y)\Big|_0^{2x/\pi} dx$$

$$= \int_0^{\pi/2} \left(-\frac{2x}{\pi}\sin x - \frac{2x}{\pi}\right) dx = -\frac{2}{\pi}(-x\cos x + \sin x) - \frac{x^2}{\pi}\Big|_0^{\pi/2} = -\frac{2}{\pi} - \frac{\pi}{4}$$

in agreement with part (a).

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Evaluate $\displaystyle\int_{(0,0)}^{(2,1)} (10x^4 - \underset{=0}{2xy^3})\,dx - 3x^2y^2\,dy$ along the path $x^4 - 6xy^3 = 4y^2$.



A direct evaluation is difficult. However, noting that $M = 10x^4 - 2xy^3$, $N = -3x^2y^2$ and $\frac{\partial M}{\partial y} = -6xy^2$ $= \frac{\partial N}{\partial x}$, it follows that the integral is independent of the path. Then we can use any path, for example the path consisting of straight line segments from (0,0) to (2,0) and then from (2,0) to (2,1).

Along the straight line path from (0,0) to (2,0), $y = 0$, $dy = 0$ and the integral equals $\displaystyle\int_{x=0}^{2} 10x^4\,dx = 64$.

Along the straight line path from (2,0) to (2,1), $x = 2$, $dx = 0$ and the integral equals $\displaystyle\int_{y=0}^{1} -12y^2\,dy = -4$.

Then the required value of the line integral $= 64 - 4 = 60$.

STOKES' THEOREM

Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem. Whereas Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve, Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (which is a space curve). Figure 1 shows an oriented surface with unit normal vector $\mathbf{n}$. The orientation of S induces the **positive orientation of the boundary curve** C shown in the figure. This means that if you walk in the positive direction around C with your head pointing in the direction of $\mathbf{n}$, then the surface will always be on your left.

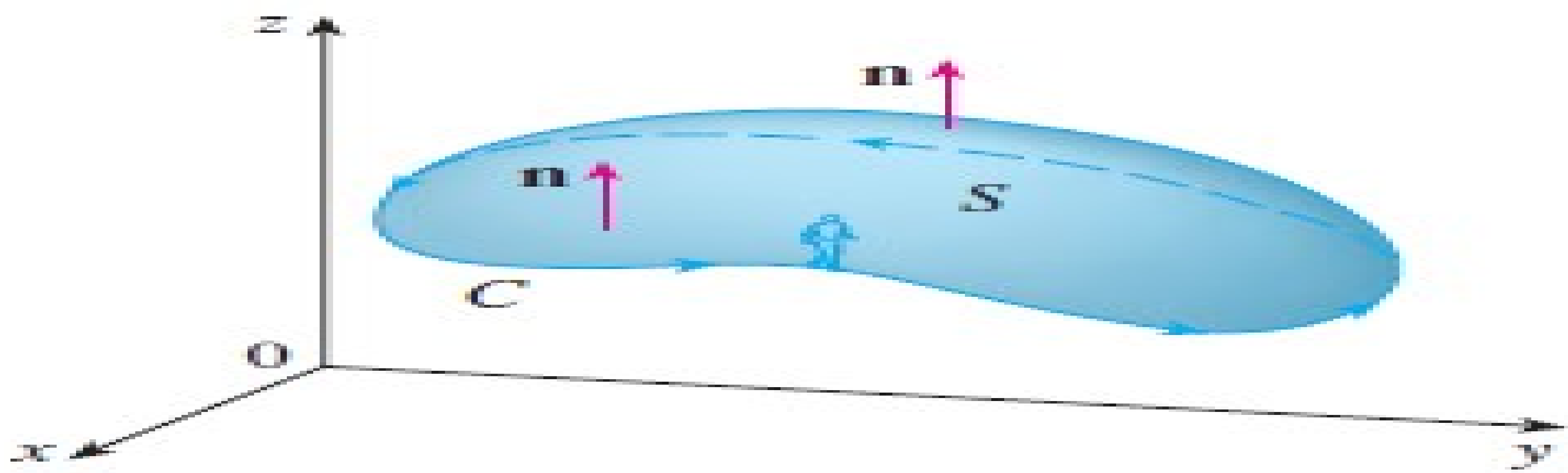


FIGURE 1

STOKES' THEOREM Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

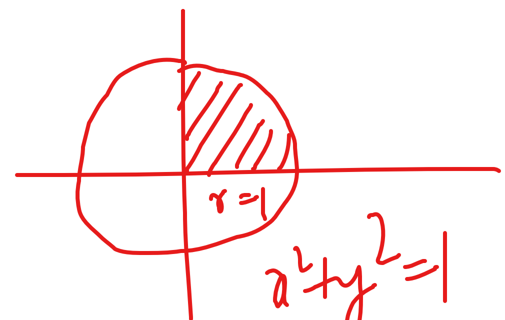
Verify Stokes' theorem for $\mathbf{A} = (2x-y)\mathbf{i} - yz^2\mathbf{j} - y^2z\mathbf{k}$, where S is the upper half surface of the sphere $x^2+y^2+z^2 = 1$ and C is its boundary.

The boundary C of S is a circle in the xy plane of radius one and center at the origin. Let $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t \le 2\pi$ be parametric equations of C. Then

$dx = -\sin t\, dt$

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \oint_C (2x-y)\,dx - yz^2\,dy - y^2z\,dz$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\,dt = \pi$$



LHS

$x^2+y^2=1$

Also,

$$\nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \mathbf{k}$$

$$\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

$x = \pm 1$

Then

$$\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS = \iint_S \mathbf{k}\cdot\mathbf{n}\,dS = \iint_R dx\,dy$$

since $\mathbf{n}\cdot\mathbf{k}\,dS = dx\,dy$ and R is the projection of S on the xy plane. This last integral equals

$y^2 = 1-x^2$

$$\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx = 4\int_0^1\int_0^{\sqrt{1-x^2}} dy\,dx = 4\int_0^1 \sqrt{1-x^2}\,dx = \pi$$

$y = \pm\sqrt{1-x^2}$

and Stokes' theorem is verified.

RHS

THE DIVERGENCE THEOREM OF GAUSS states that if V is the volume bounded by a closed surface S and $\mathbf{A}$ is a vector function of position with continuous derivatives, then

$$\iiint_V \nabla\cdot\mathbf{A}\,dV = \iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \oiint_S \mathbf{A}\cdot d\mathbf{S}$$

where $\mathbf{n}$ is the positive (outward drawn) normal to S.

Evaluate $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{F} = 4xz\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k}$ and S is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

By the divergence theorem, the required integral is equal to

$$\iiint_V \nabla\cdot\mathbf{F}\,dV = \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right]dV$$

$$= \iiint_V (4z-y)\,dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(4z-y)\,dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} 2z^2 - yz\Big|_{z=0}^{1}\,dy\,dx = \int_{x=0}^{1}\int_{y=0}^{1}(2-y)\,dy\,dx = \frac{3}{2}$$

Verify the divergence theorem for $\mathbf{A} = 4x\,\mathbf{i} - 2y^2\,\mathbf{j} + z^2\,\mathbf{k}$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$.

$$\text{Volume integral} = \iiint_V \nabla\cdot\mathbf{A}\,dV = \iiint_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dV$$

$$= \iiint_V (4 - 4y + 2z)\,dV = \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{3} (4 - 4y + 2z)\,dz\,dy\,dx = 84\pi$$

The surface S of the cylinder consists of a base S_1 $(z = 0)$, the top S_2 $(z = 3)$ and the convex portion S_3 $(x^2 + y^2 = 4)$. Then

$$\text{Surface integral} = \iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \iint_{S_1} \mathbf{A}\cdot\mathbf{n}\,dS_1 + \iint_{S_2} \mathbf{A}\cdot\mathbf{n}\,dS_2 + \iint_{S_3} \mathbf{A}\cdot\mathbf{n}\,dS_3$$

<u>On S_1 $(z=0)$</u>, $\mathbf{n}=-\mathbf{k}$, $\mathbf{A} = 4x\,\mathbf{i} - 2y^2\mathbf{j}$ and $\mathbf{A}\cdot\mathbf{n} = 0$, so that $\iint_{S_1} \mathbf{A}\cdot\mathbf{n}\,dS_1 = 0$.

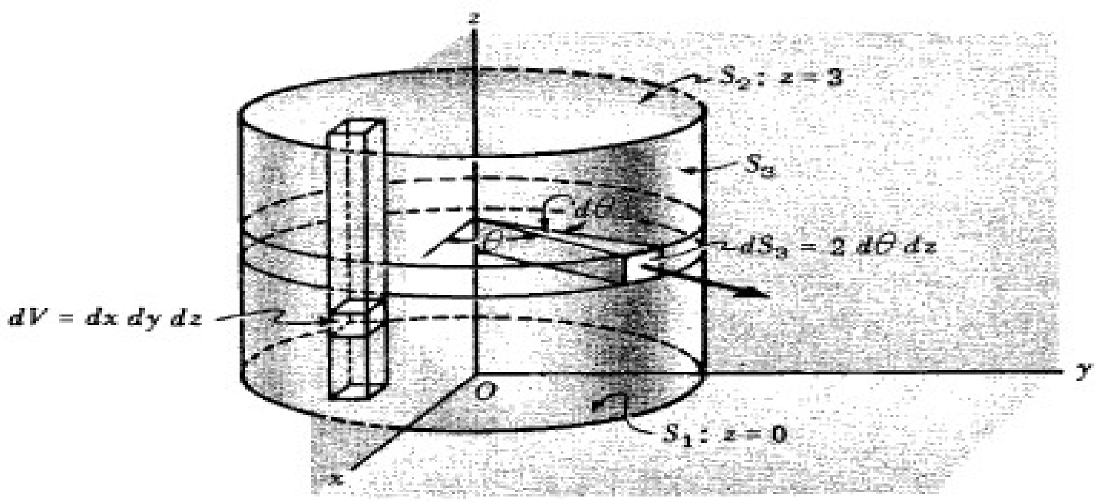
<u>On S_2 $(z=3)$</u>, $\mathbf{n}=\mathbf{k}$, $\mathbf{A} = 4x\,\mathbf{i} - 2y^2\mathbf{j} + 9\mathbf{k}$ and $\mathbf{A}\cdot\mathbf{n} = 9$, so that

$$\iint_{S_2} \mathbf{A}\cdot\mathbf{n}\,dS_2 = 9\iint_{S_2} dS_2 = 36\pi, \quad \text{since area of } S_2 = 4\pi$$

<u>On S_3 $(x^2+y^2=4)$.</u> A perpendicular to $x^2+y^2=4$ has the direction $\nabla(x^2+y^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$.

Then a unit normal is $\mathbf{n} = \dfrac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{\sqrt{4x^2+4y^2}} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{2}$ since $x^2+y^2=4$.

$$\mathbf{A}\cdot\mathbf{n} = (4x\,\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k})\cdot\left(\frac{x\,\mathbf{i} + y\,\mathbf{j}}{2}\right) = 2x^2 - y^3$$



From the figure above, $x = 2\cos\theta$, $y = 2\sin\theta$, $dS_3 = 2\,d\theta\,dz$ and so

$$\iint_{S_3} \mathbf{A}\cdot\mathbf{n}\,dS_3 = \int_{\theta=0}^{2\pi}\int_{z=0}^{3}\left[2(2\cos\theta)^2 - (2\sin\theta)^3\right]2\,dz\,d\theta$$

$$= \int_{\theta=0}^{2\pi}(48\cos^2\theta - 48\sin^3\theta)\,d\theta = \int_{\theta=0}^{2\pi} 48\cos^2\theta\,d\theta = 48\pi$$

Then the surface integral $= 0 + 36\pi + 48\pi = 84\pi$, agreeing with the volume integral and verifying the divergence theorem.