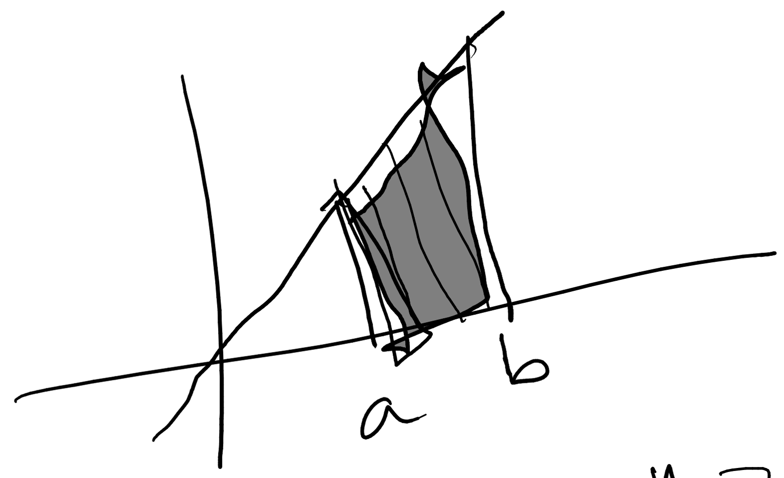
CALCULUS

BMAT101L

Derivatives and Its Applications

$y = x^2 \quad \frac{dy}{dx} = 2x$

$\int_a^b x\,dx = \frac{x^2}{2}\Big|_a^b = \frac{b^2-a^2}{2}$



a to b

n

$n \to \infty$ area of each piece

Integration → Anti derivative

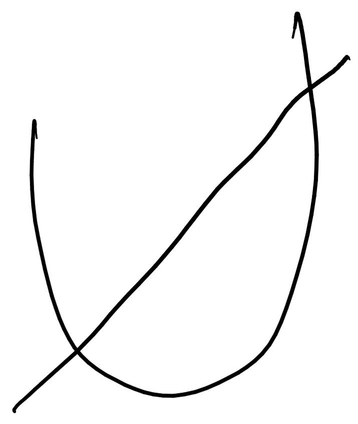
$\frac{d}{dx} x^2 = \int 2x$

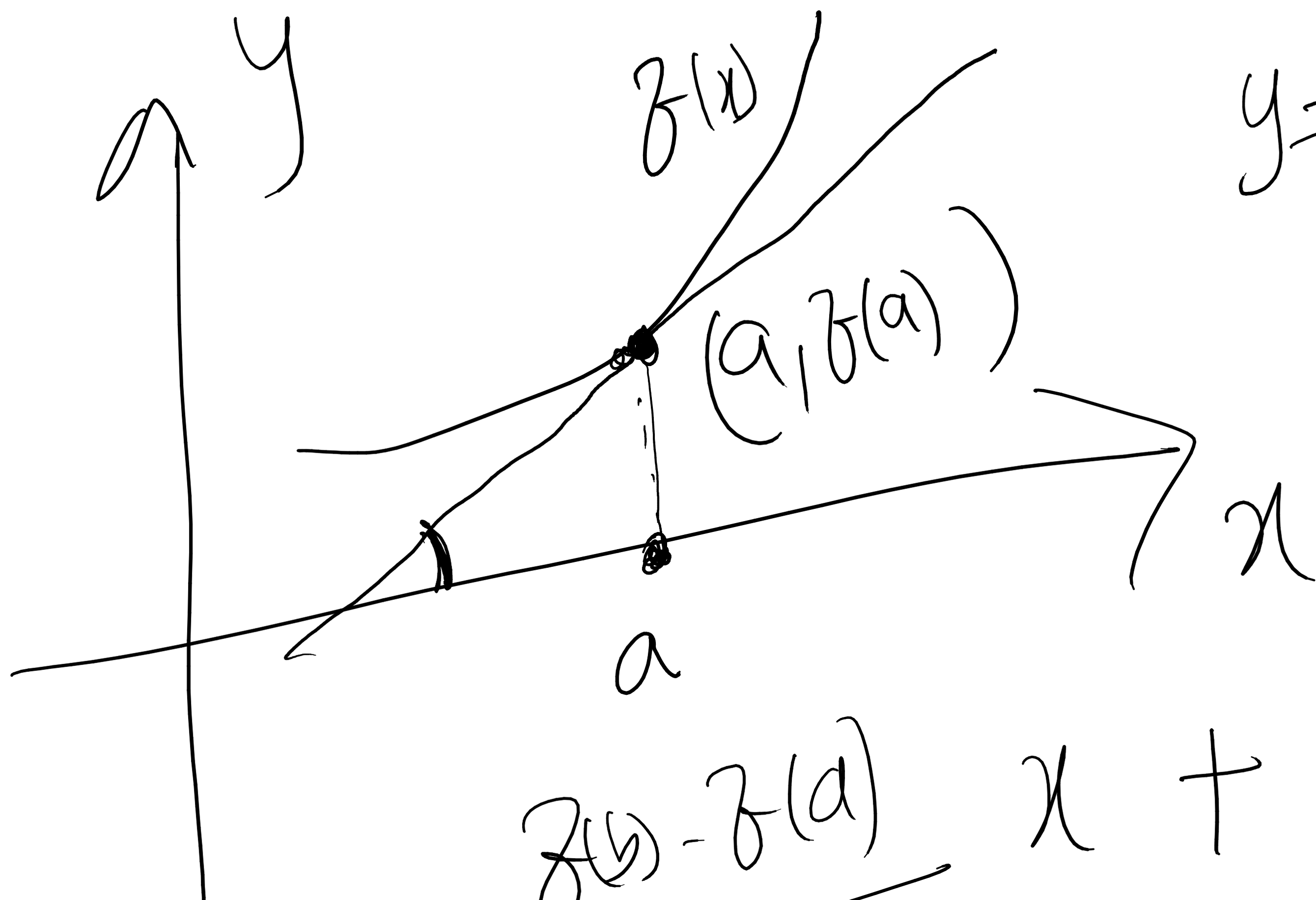
$= x^2 + 100$

$= x^2 + 3$

$\frac{dy}{dx} = ? \quad \frac{d}{dx}(x^2) = 2x$

$\frac{d f(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$





$y = mx + c$

$y = \frac{f(b) - f(a)}{b-a} x +$

$\frac{y - f(b)}{f(a) - f(b)} = \frac{x-b}{a-b}$

$y = \frac{f(a) - f(b)}{a-b}(x-b) + f(b)$

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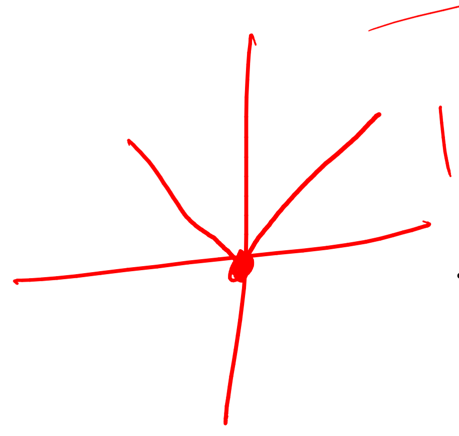
When $\lim_{x\to a} f(x)$ exists

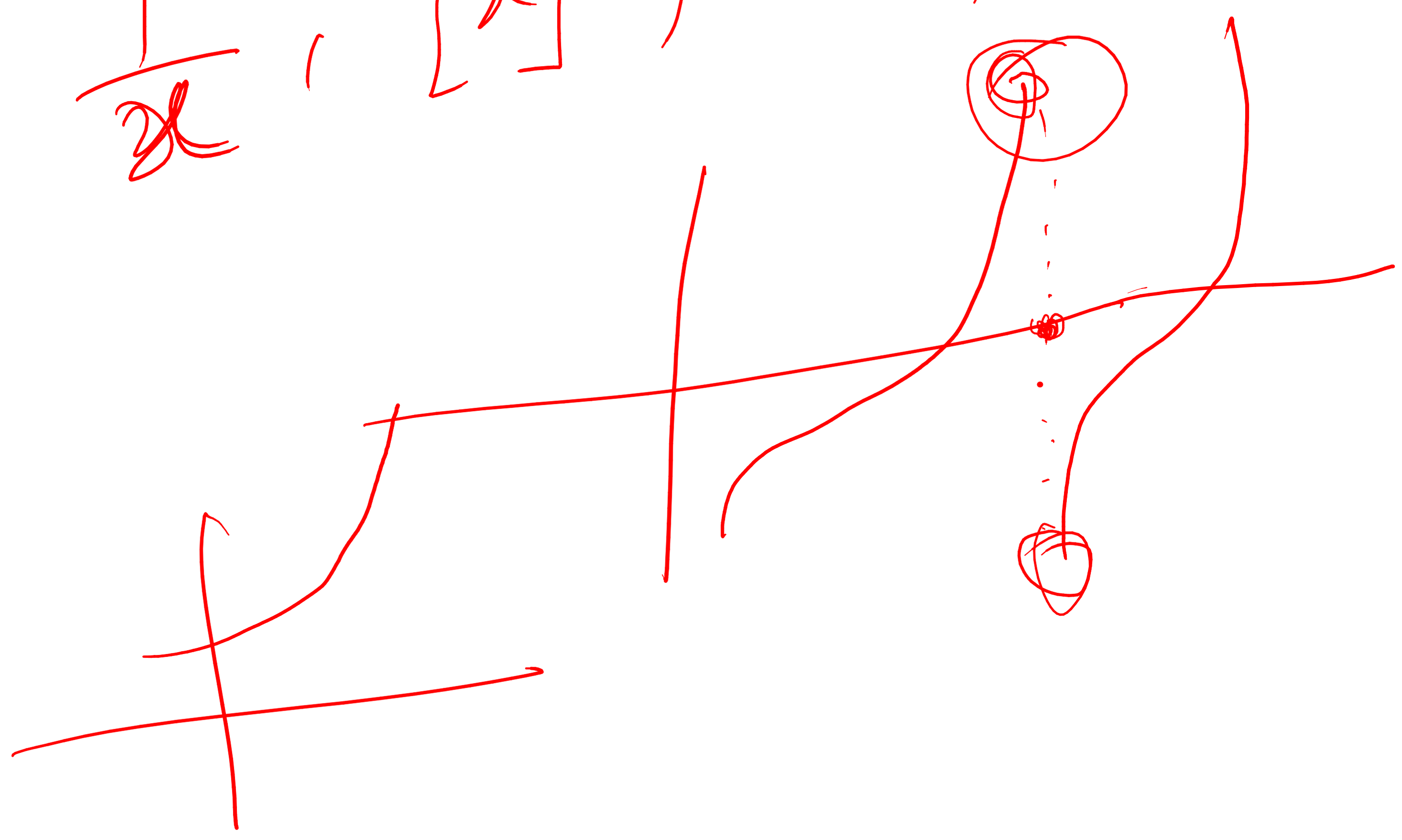
$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$$

$f(x) = |x|$



$\tan x$, $\frac{1}{x}$, $[x]$, $\lfloor x \rfloor$,

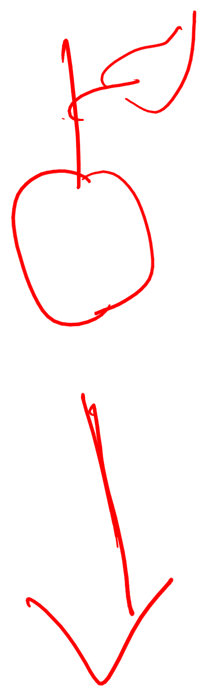
tan 135° tan 45°
-1 1



$$\lim_{x\to\infty} e^x = \infty$$

Newton , Leibniz

$\frac{dy}{dx}$, $y'(x)$, Dy



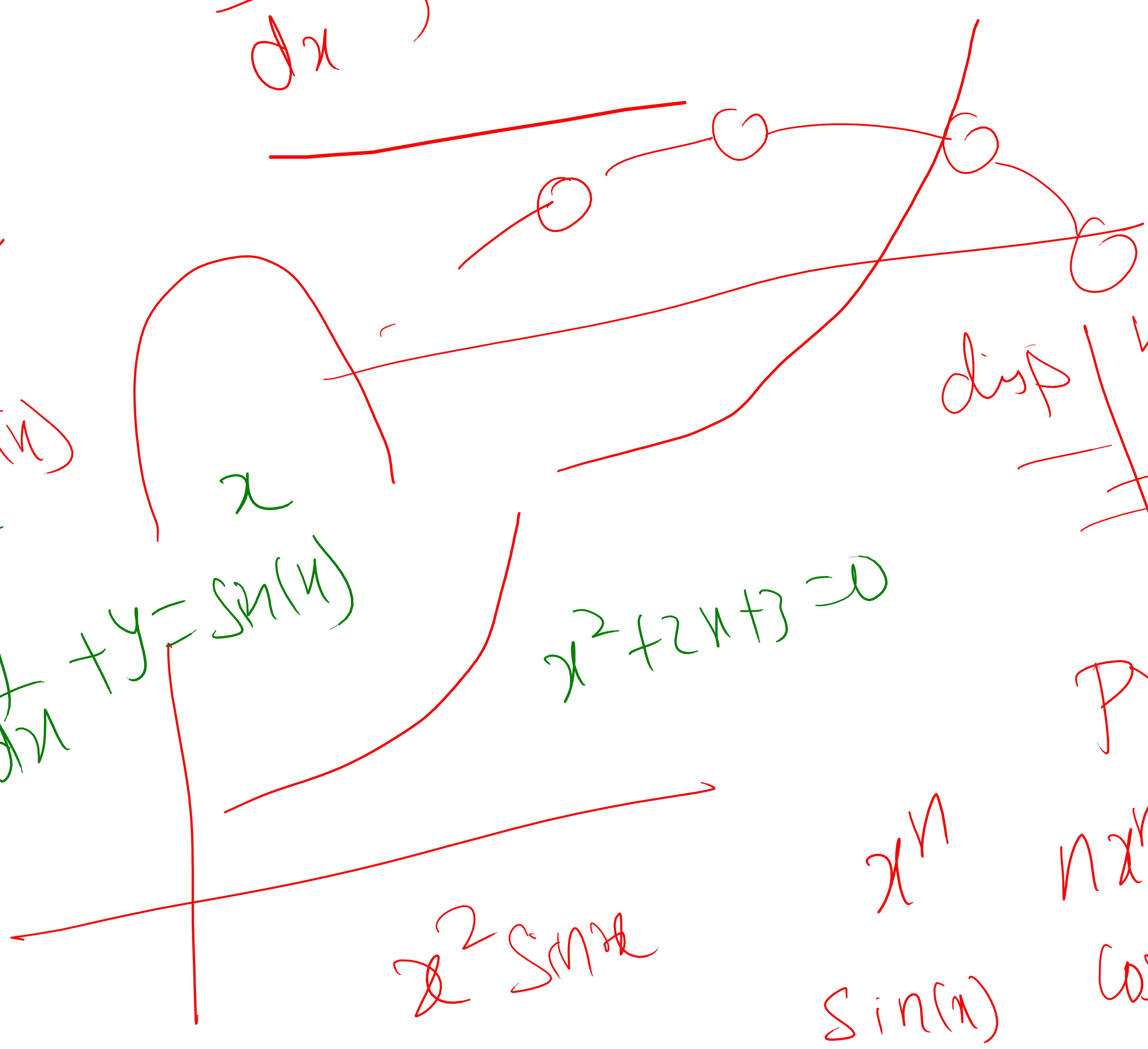
$t=0$	10
$t=1$	8
$t=2$	6
$t=3$	4

$F = ma$

$\sin(x)$

$m\frac{d^2x}{dt^2} = F$

$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin(x)$



$x^2 + 2x + 3 = 0$

disp velo Accel

Principia mathematica

x^n $\quad nx^{n-1}$ $\quad n(n-1)x^{n-2}$

$\sin(x)$ $\quad \cos(x)$ $\quad -\sin(x)$

$x^2 \sin x$

$f(x) \leq M \qquad x \in (-\infty, \infty)$

$f(x) \geq m \qquad x \in D$

absolute / Global

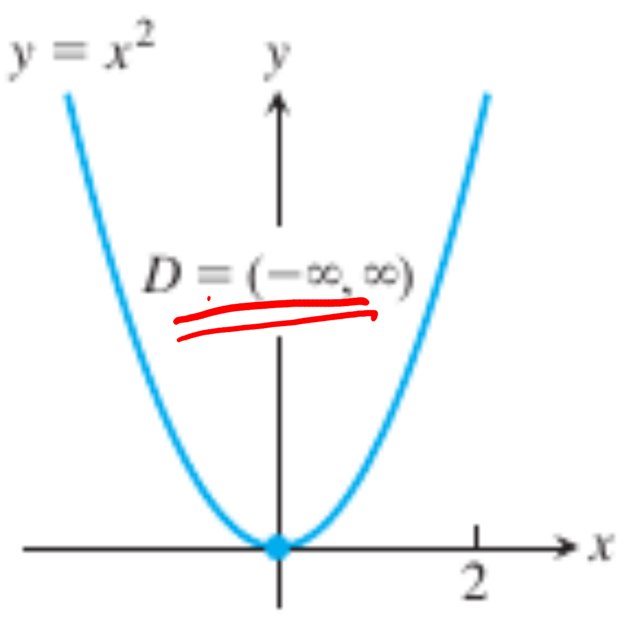
Local

DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) = M \quad \text{for all } x \text{ in } D$$

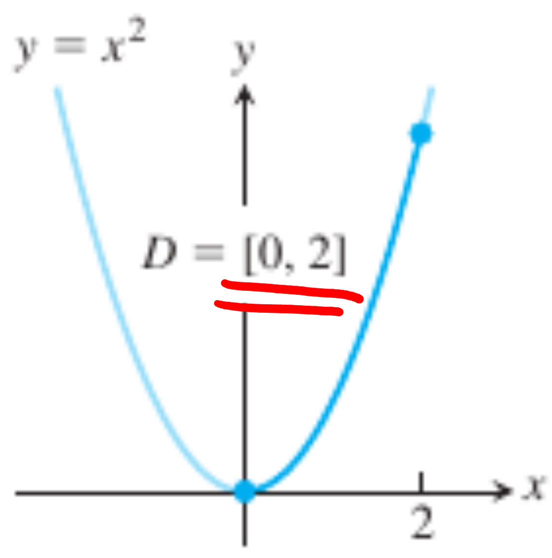
and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) = m \quad \text{for all } x \text{ in } D.$$



Absolute minimum at $x=0$

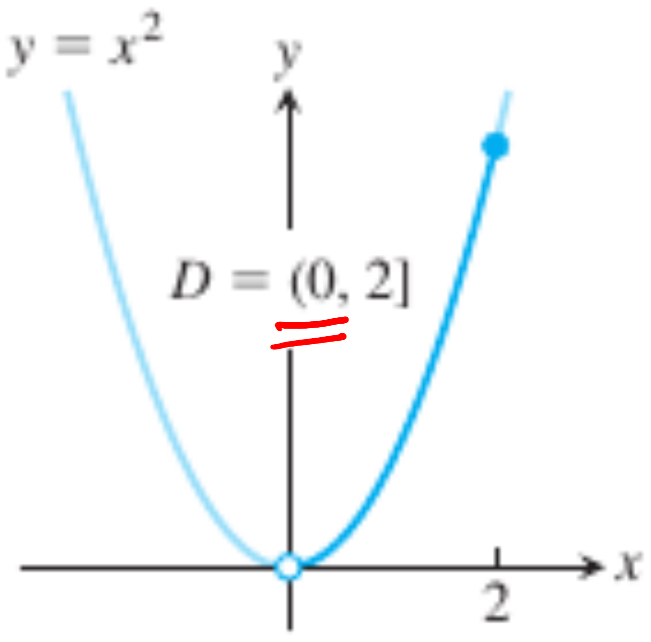
min $=0$



(b) abs max and min

max at $x=2$, min at $x=0$

max $=4$

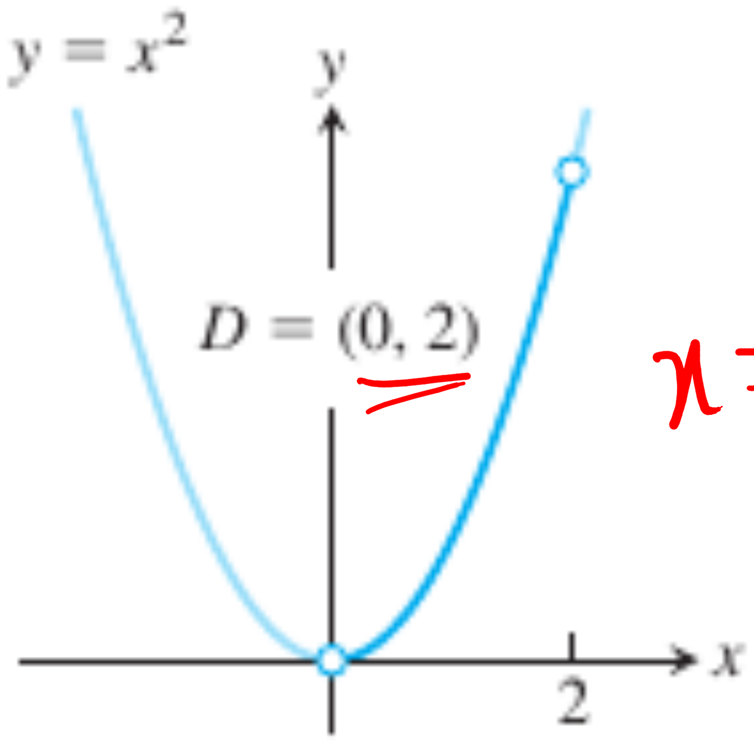


Abs max at $x=2$

min

$x = 0.00001$

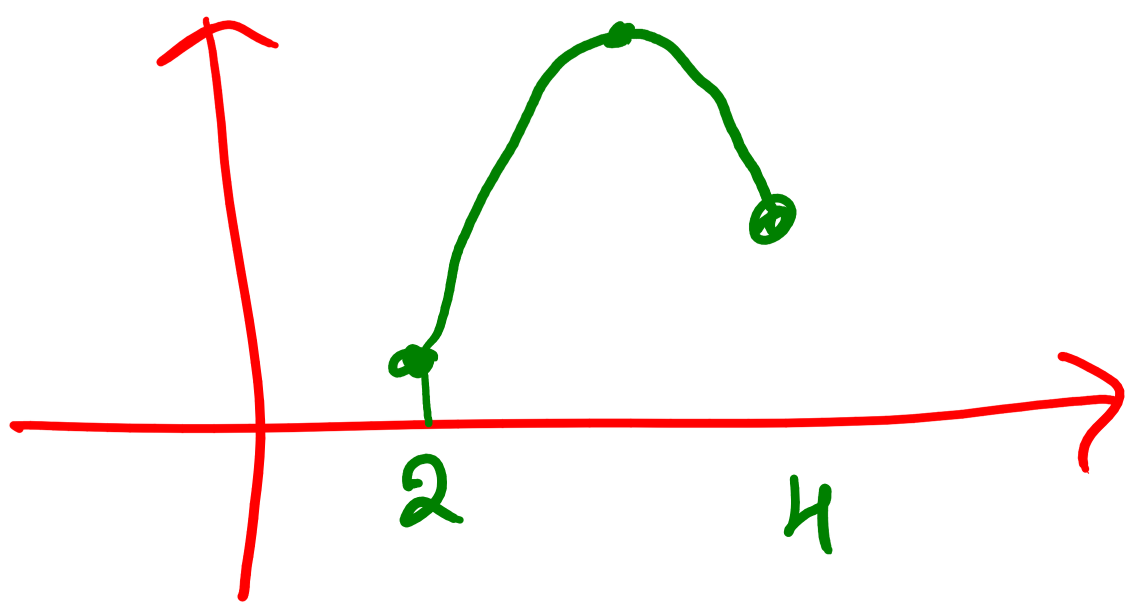
$f(x) = 0.0000000001$



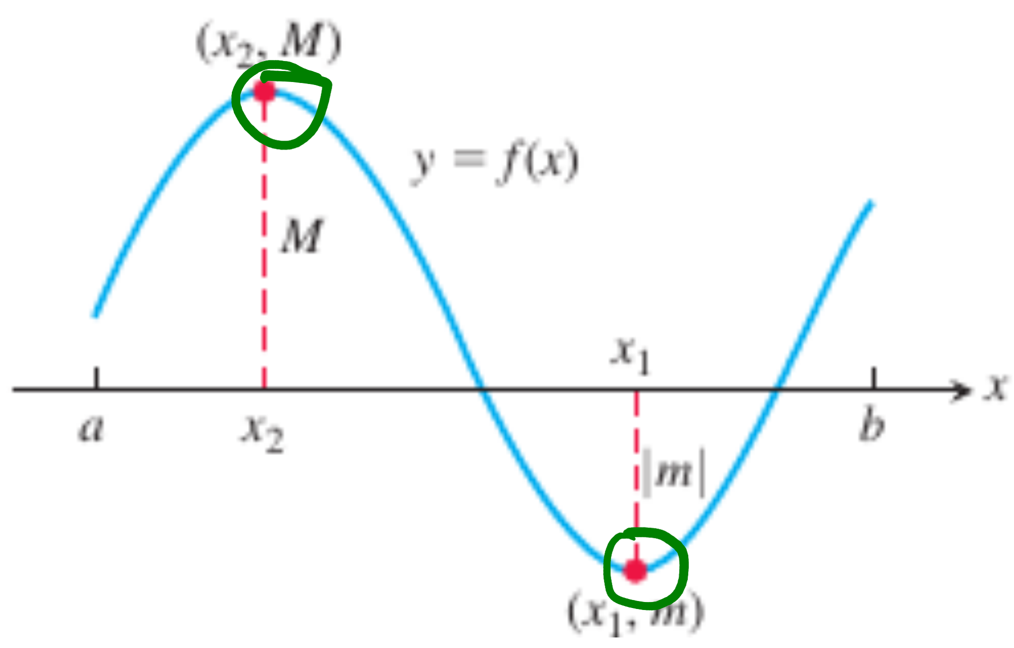
$x = 1.99999$

No max / No min

THEOREM 1—The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.



$(2,4)$

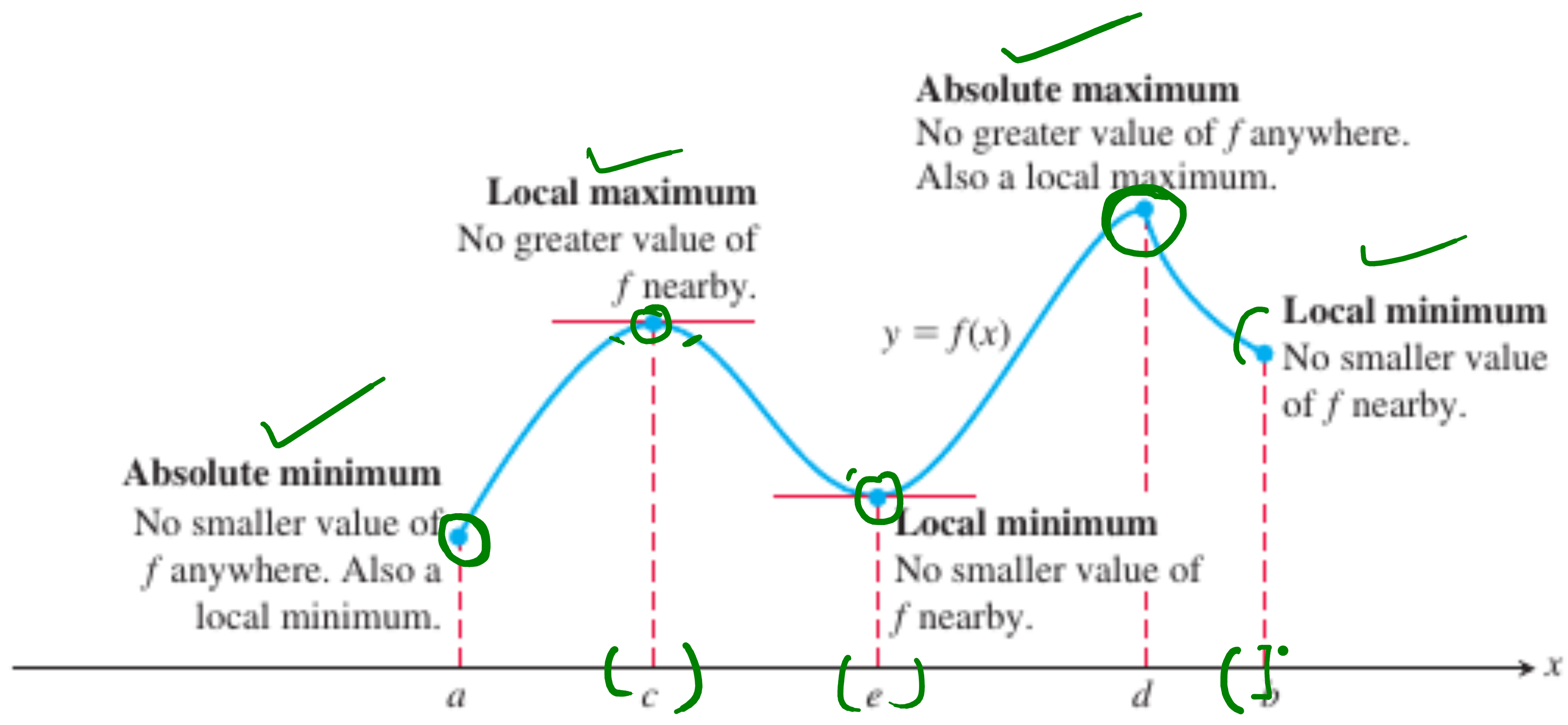


$f(x)$, $x \in (a,b)$

DEFINITIONS A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.

$f(x)$,
$x \in [a,b]$



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THEOREM 2—The First Derivative Theorem for Local Extreme Values If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

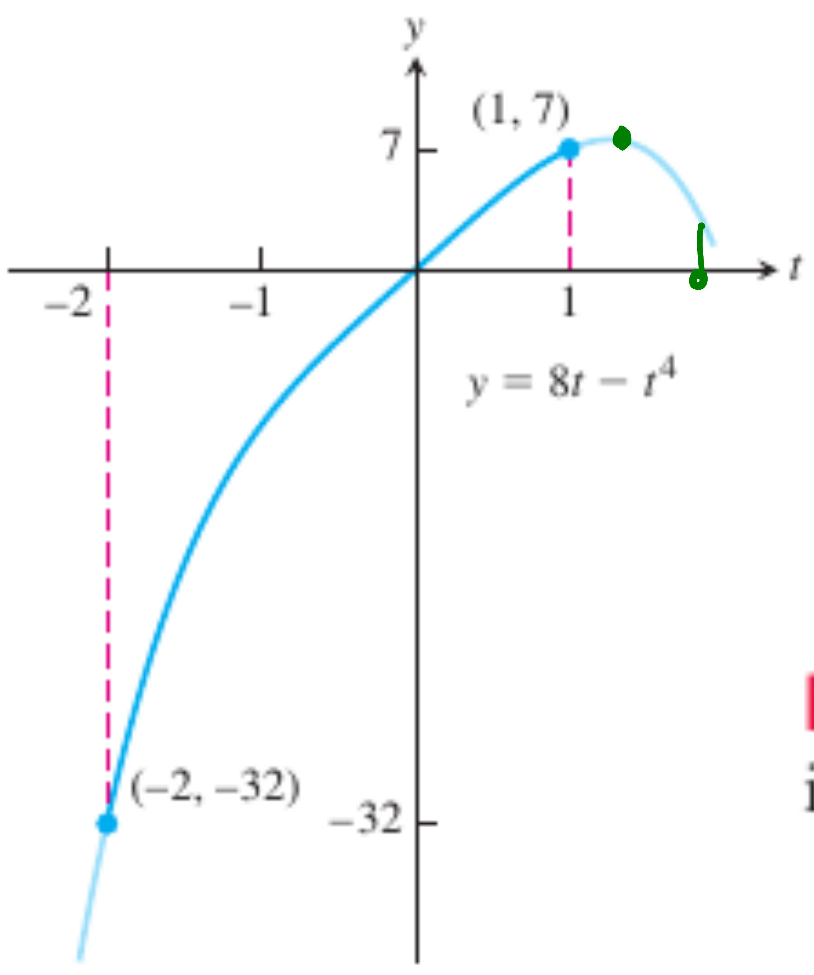
$$f'(c) = 0.$$

1. interior points where $f' = 0$,
2. interior points where f' is undefined,
3. endpoints of the domain of f.

Global / Local extremum

DEFINITION An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

EXAMPLE 3 Find the absolute maximum and minimum values of $g(t) = 8t - t^4$ on $[-2, 1]$.



Global/local min at $t=-2$ $= -32$

max at $t = 1$ $= 7$

1.23 $t > 1$

$g'(t) = 8 - 4t^3$

$g'(t) = 0$

$4t^3 = 8$

$t^3 = 2$

$t = \sqrt[3]{2}$

EXAMPLE 4 Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

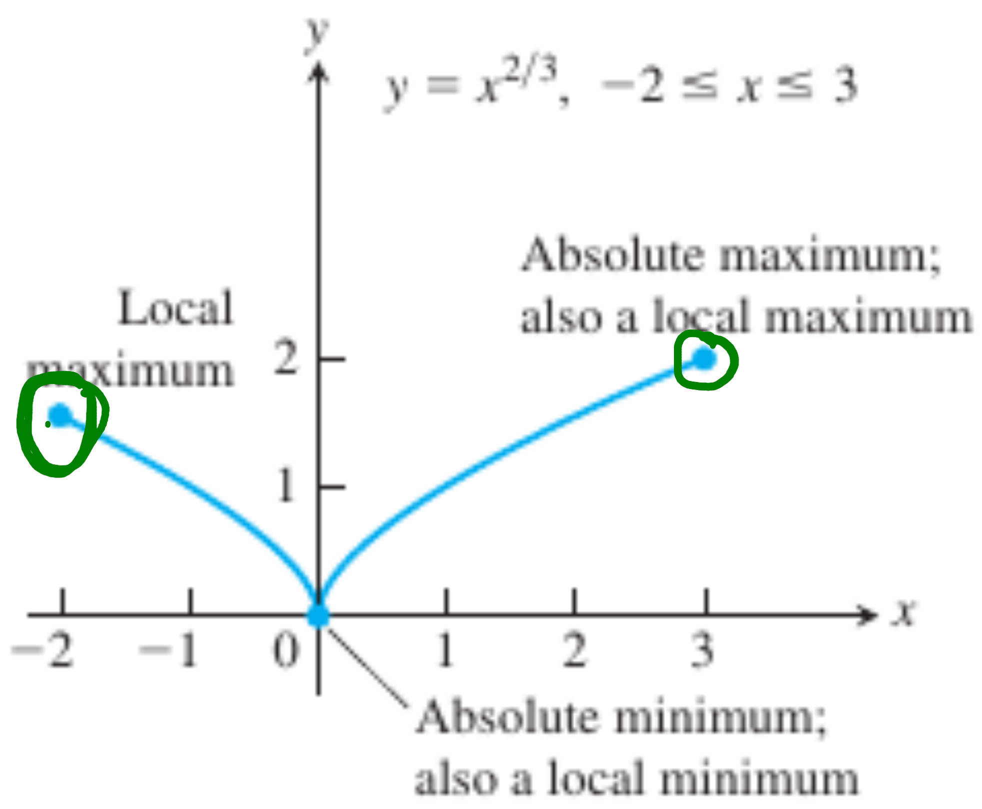
$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}} = 0$$

$x \to \infty$ out of domain

f' is undefined at $x=0$

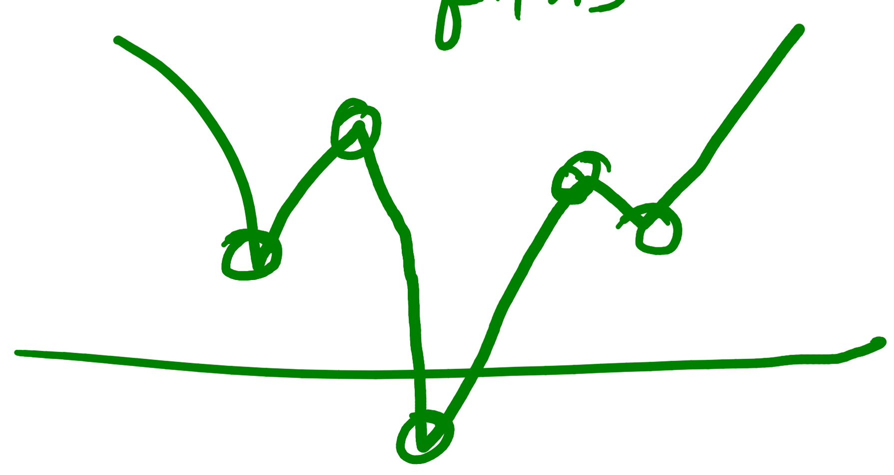
	$f(x)$	
$x=0$	0	$\rightarrow$ min $\rightarrow$ interior $x^{2/3}$
$x=-2$	$\sqrt[3]{4}$	
$x=3$	$\sqrt[3]{9}$	$\rightarrow$ max $\rightarrow$ endpoint

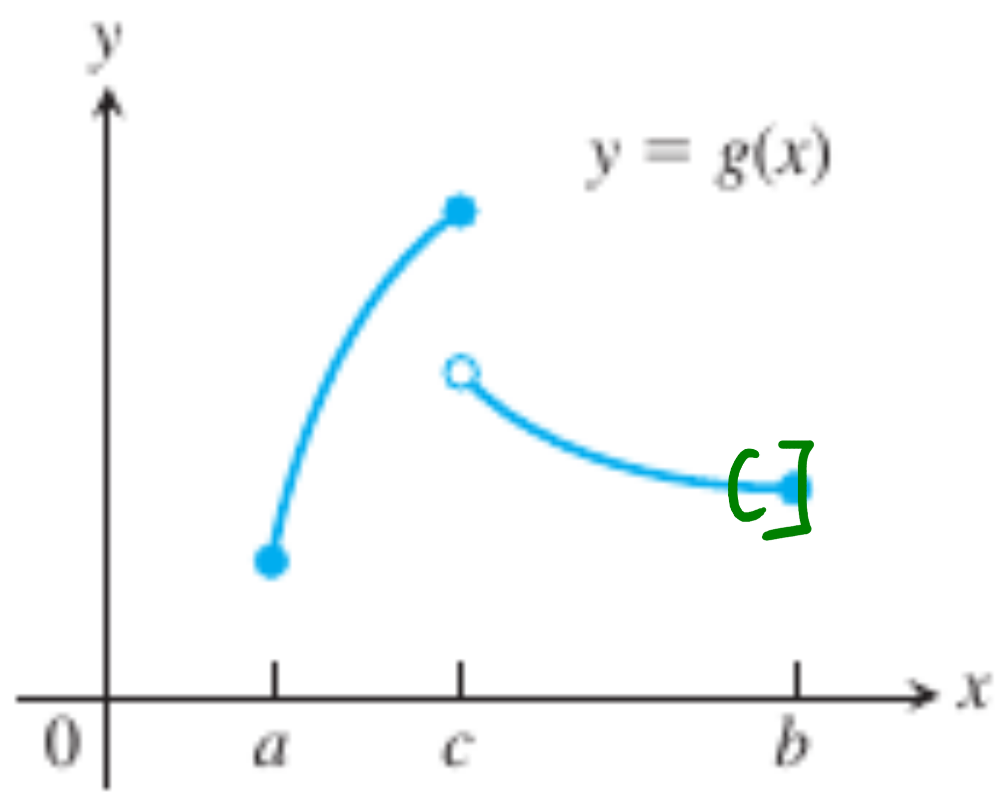
$$\left.\begin{array}{l} f'(x)=0 \\ f'(x)=\infty \end{array}\right\}$$



$f'(x)=0$

$f'(x)$

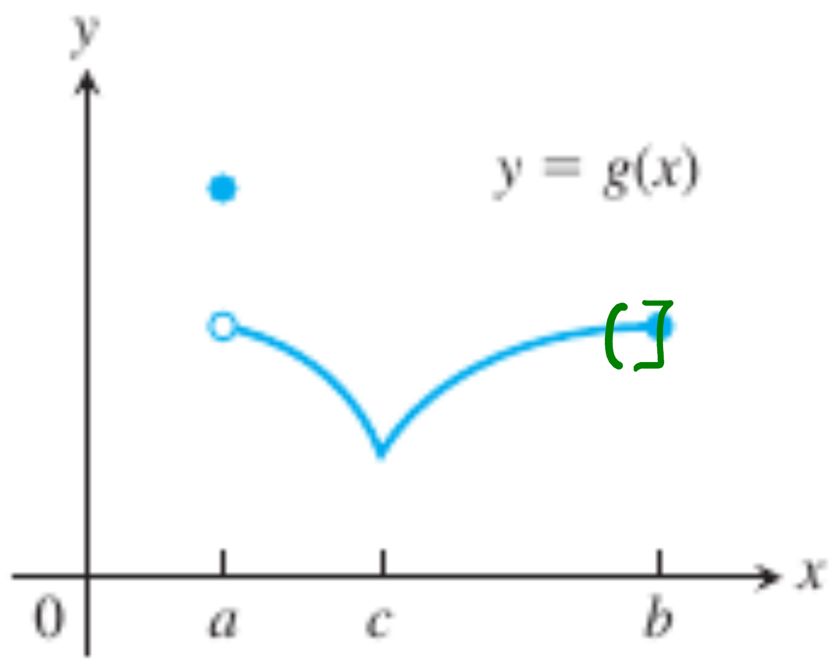




G Max at c

G min at a

L min at b



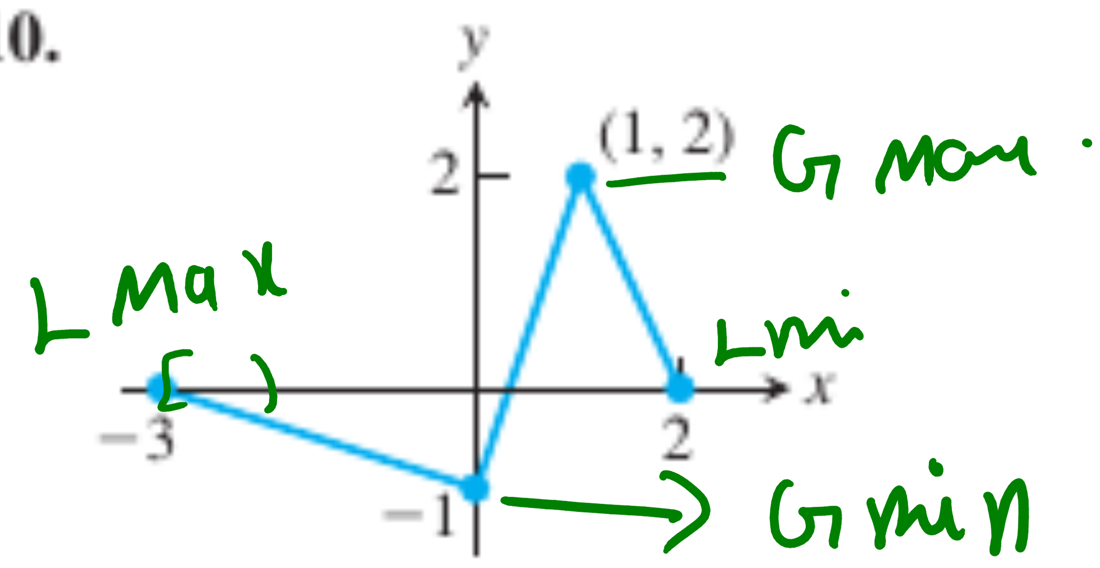
G max at a

L/G min at c

L max at b

$\lim_{x \to a} g(x)$ ~~do~~ does not exist

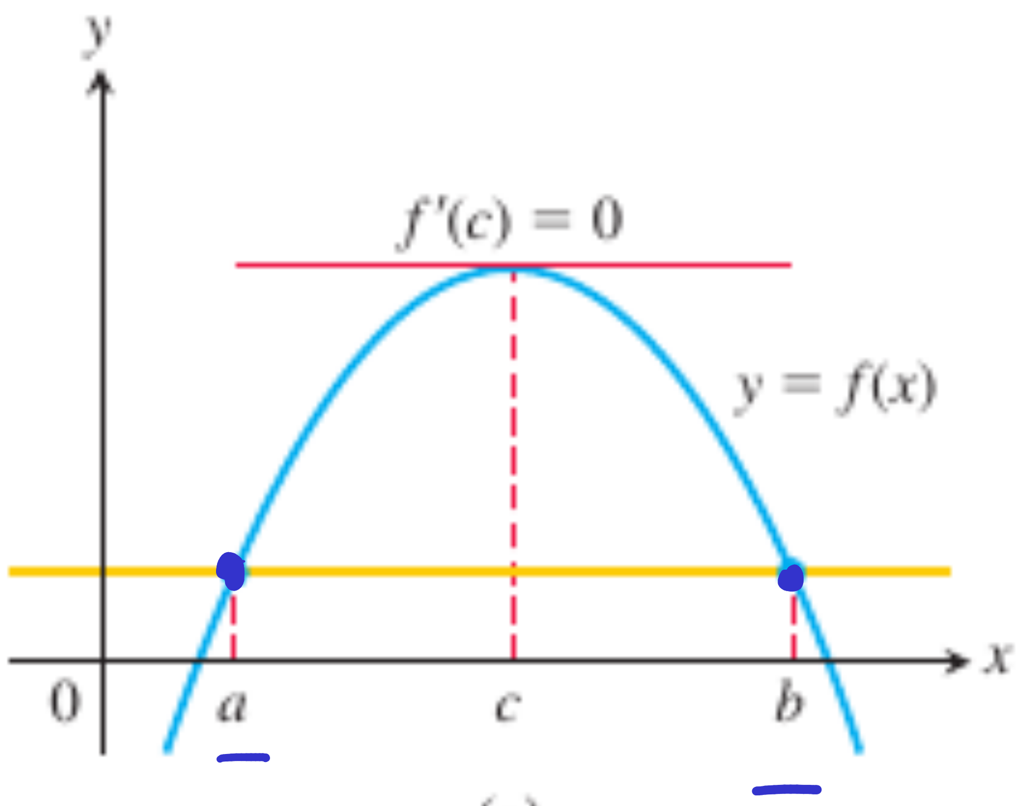
10.



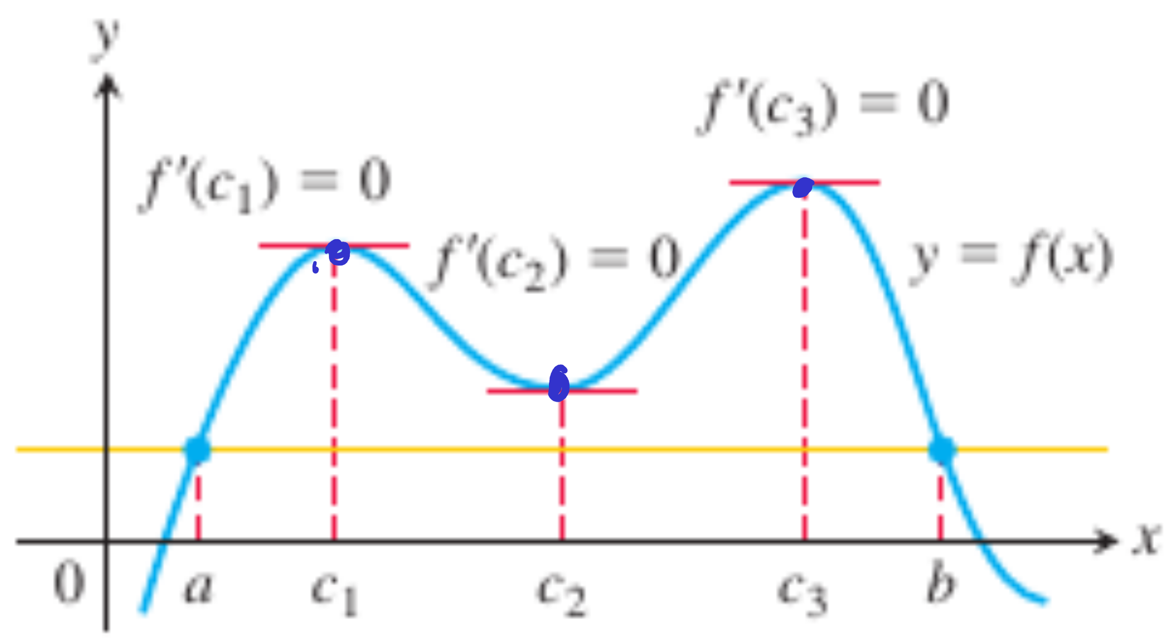
Pg 191 at Thomas calculus

Rolle's Theorem

THEOREM 3—Rolle's Theorem Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.



(a)



EXAMPLE 1 Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

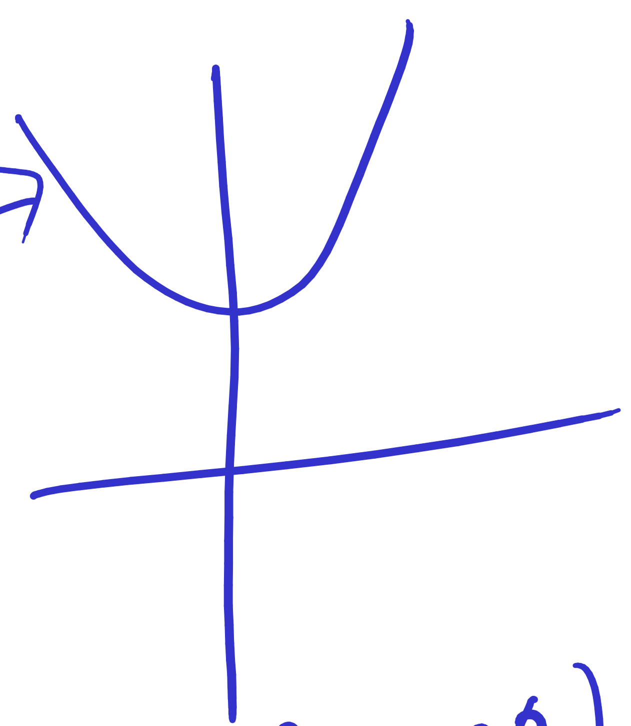
$f(x) = x^2$

$f(x) = 0$

$f(x) = x^3 + 3x + 1$

$f'(x) = 3x^2 + 3 > 0$

$f'(x) \neq 0$



range $(0, \infty)$

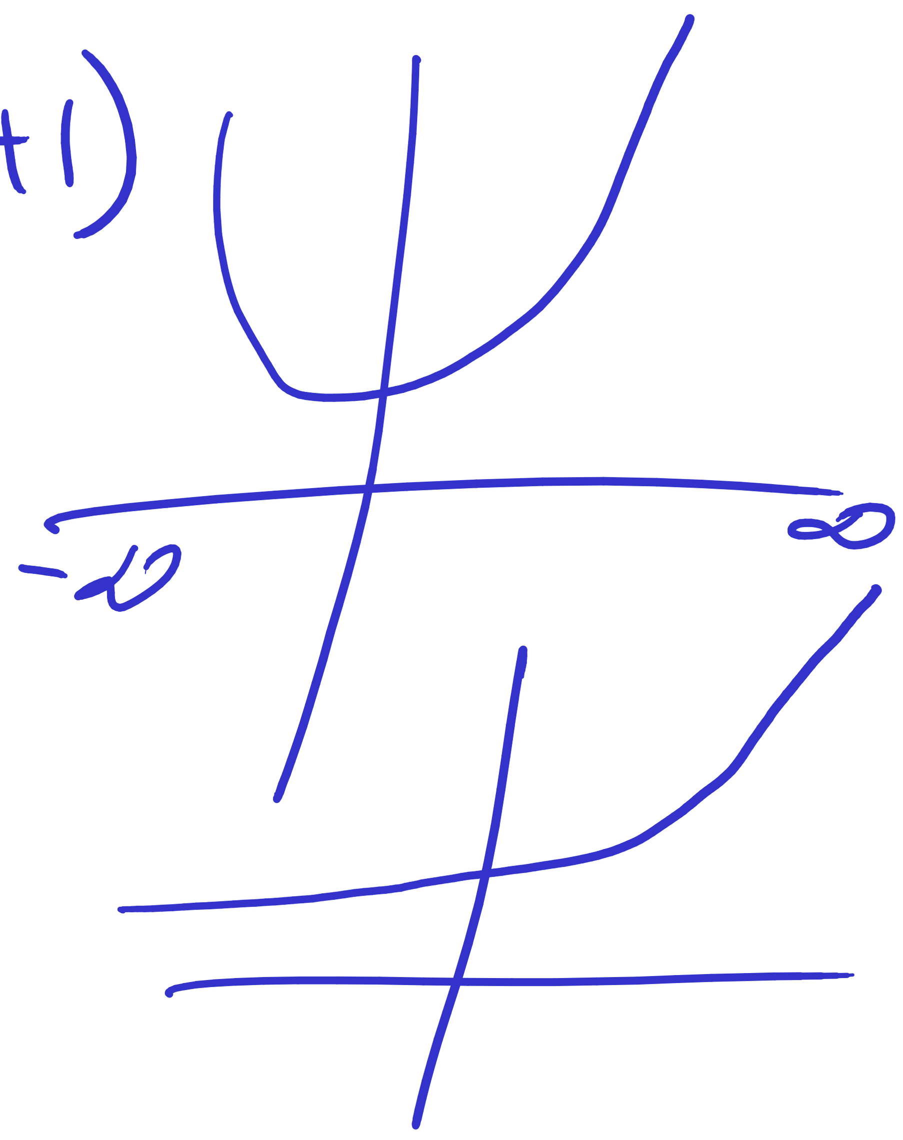
range $(-\infty, \infty)$ //

$f'(x) = 3(x^2+1)$

$f(0) = 1$

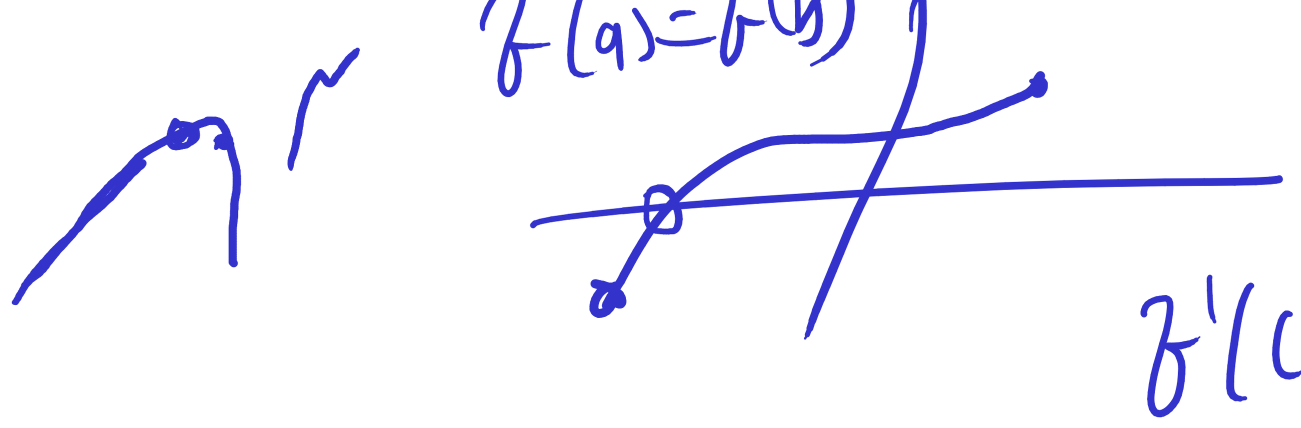
$f(1) = 5$

$f(-1) = -3$



$f(x) = x^4 + 10x^3 + x^2$

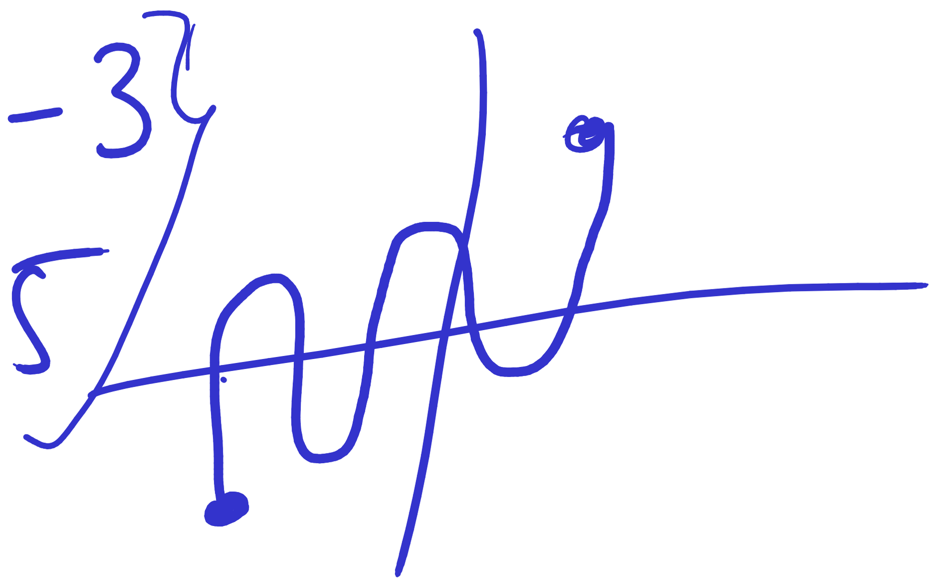
$f(a) = f(b)$



$f'(c) \neq 0$

$f(-1) = -3$

$f(1) = 5$



CALCULUS

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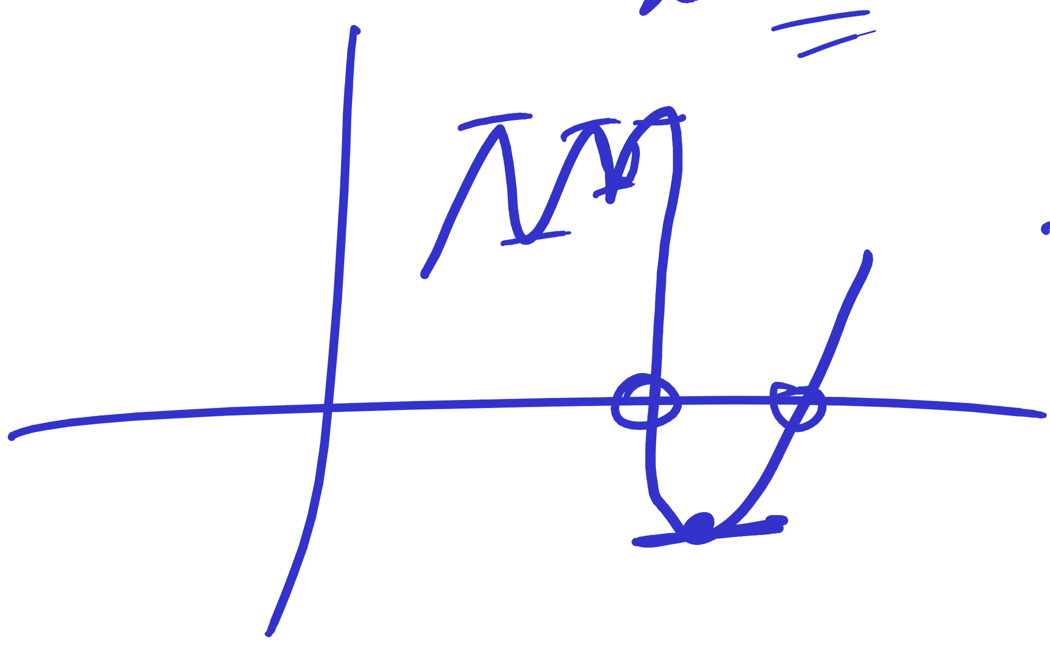
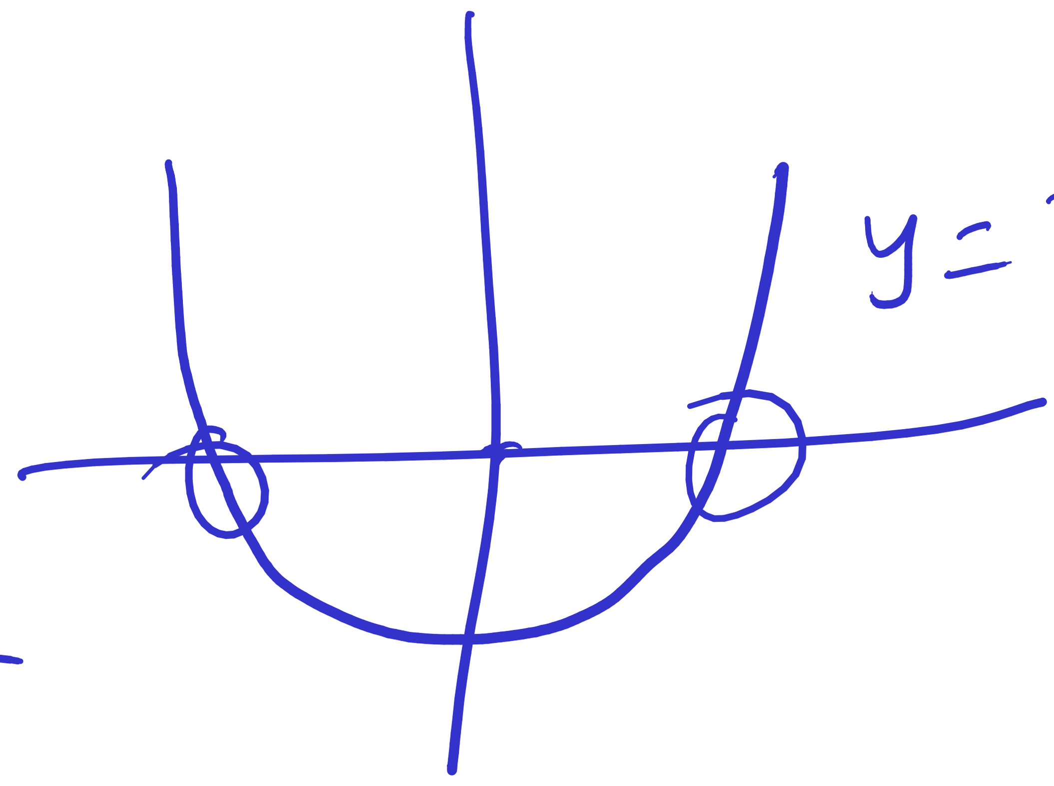
$f(x) = x^2 - 1$ $\quad f'(x) = 2x$

$x = 0$

$f(0) = -1$
$f(2) = 3$



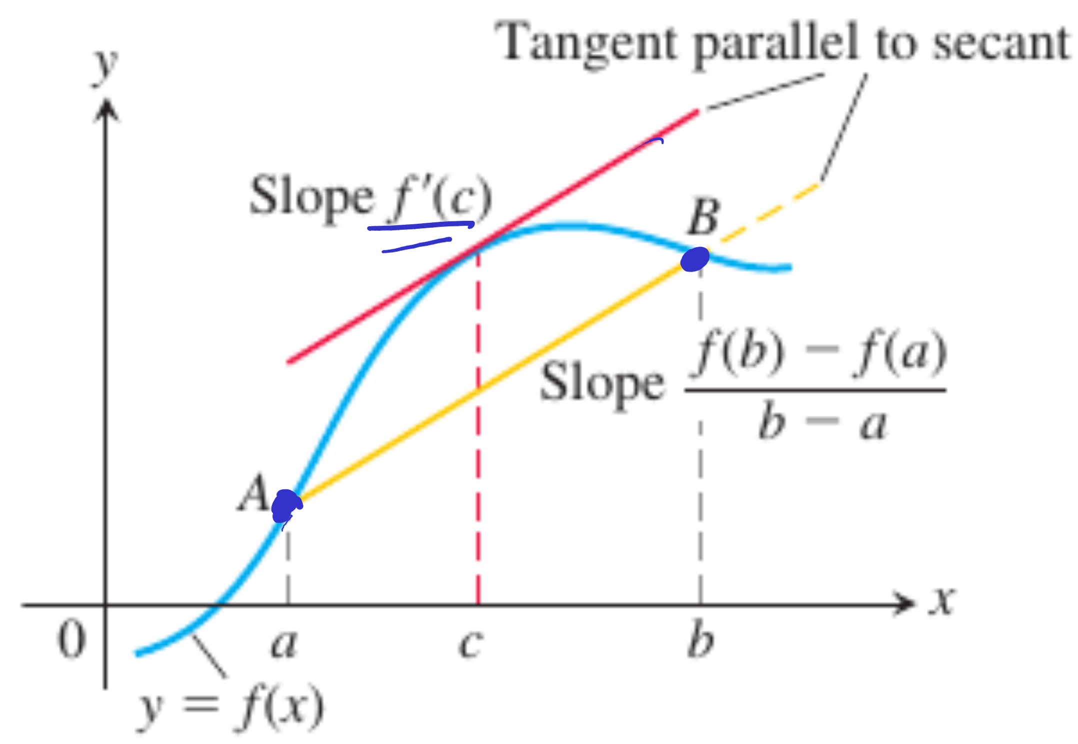
$y = x^2 - 1$

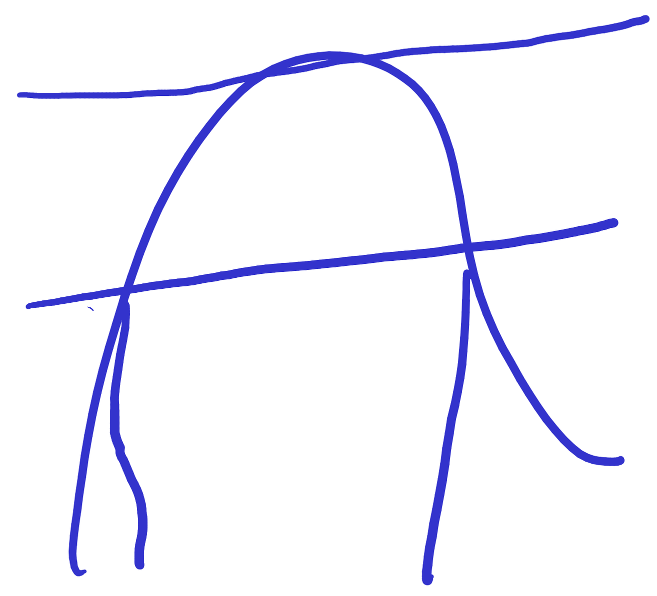


THEOREM 4—The Mean Value Theorem Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$f(a) \neq f(b)$

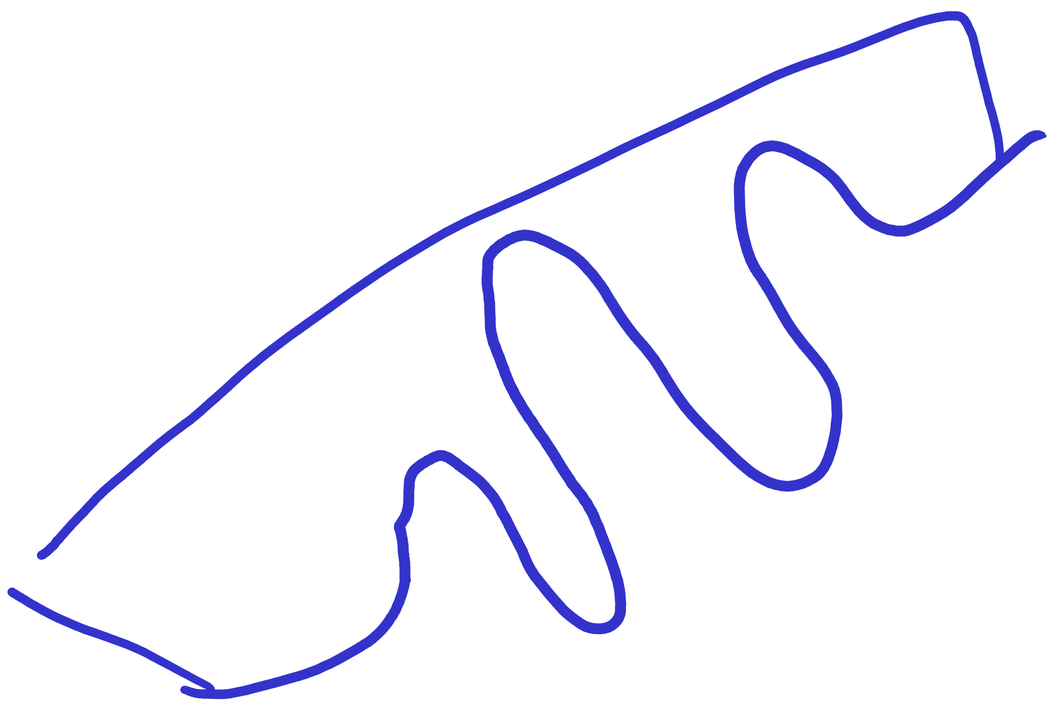
$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

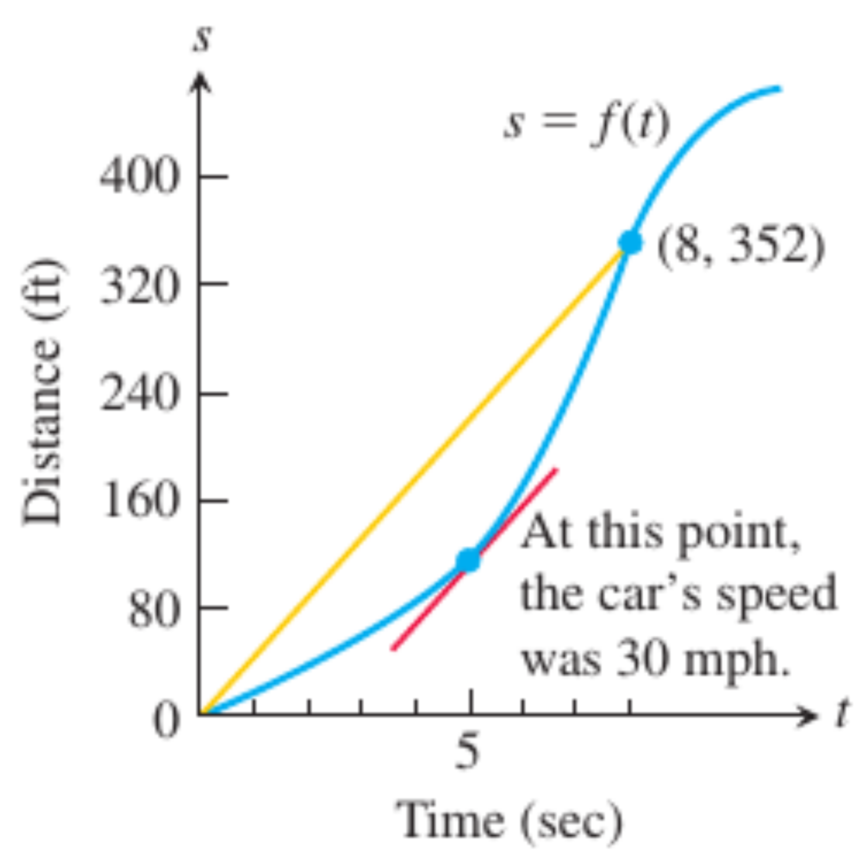




a b

EXAMPLE 3 If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is $352/8 = 44$ ft/sec. The Mean Value Theorem says that at some point during the acceleration the speedometer must read exactly 30 mph (44 ft/sec) (Figure 4.18). ■





$a = 0$ $\quad b = 8$

$f(a) = 0$ $\quad f(b) = 352$

$$f'(c) = \frac{352 - 0}{8 - 0}$$

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$f(x)$, $x = 1, 2, 3, \ldots$

$f(x) > 0$

$x_1 = 1$ $x_2 = 2$

$f(x_1) = a$ $f(2) = b$

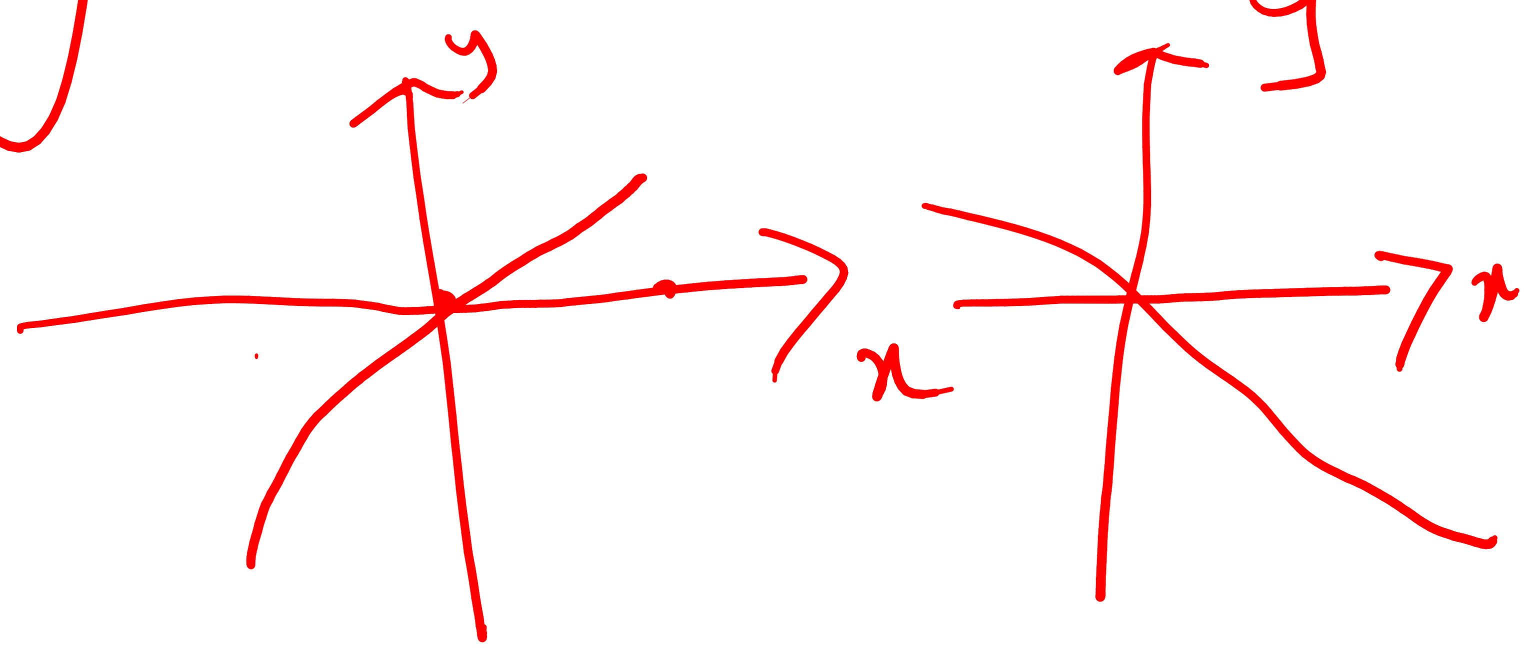
slope > 0 $x_2 > x_1$

$f(x_2) > f(x_1)$ $(1, 2)$

$\left.\begin{array}{ll} x \text{ increases} & y \text{ increases} \\ x \text{ decreases} & y \text{ decrease} \end{array}\right\}$ increasing function

$f(x)$ $f'(x) > 0$

$f'(x) < 0$



COROLLARY 3 Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Critical point

$f'(x) = 0$ $f'(x)$ not defined

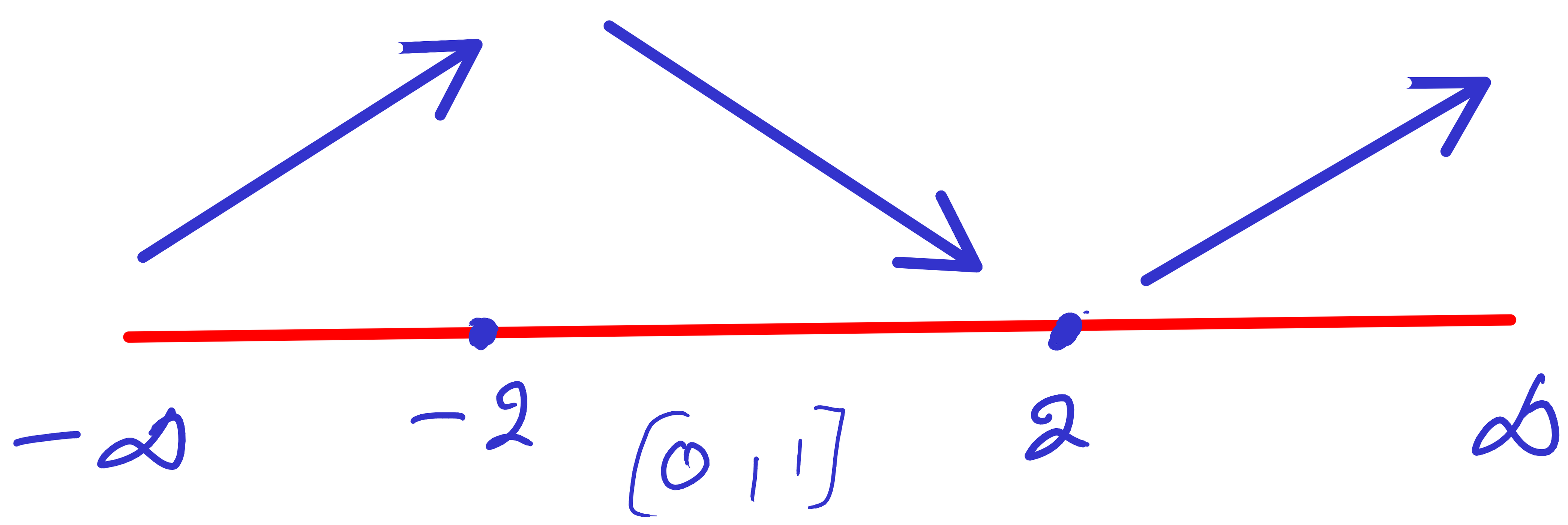
$\left.\begin{array}{l} f'(x) = \infty \\ f'(x) = -\infty \end{array}\right\}$ defined

EXAMPLE 1 Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and on which f is decreasing.

$f(x) = x^3 - 12x - 5$

$f'(x) = 3x^2 - 12$

$f'(x) = 0 \Rightarrow 3x^2 - 12 = 0 \Rightarrow x = +2, -2$



$(-\infty, -2)$ $(-2, 2)$ $(2, \infty)$

Concavity

Concave upward
Convex downward

Concave downward
Convex upward

Inflection points

$f''(x) > 0$	$f(x)$ Concave upward	$f'(x)$ is increasing
$f''(x) < 0$	Concave downward	$f'(x)$ is decreasing

Inflection Points

$f''(x) = 0$ or $f''(x)$ is undefined

DEFINITION The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I; $f''(x) > 0$

(b) **concave down** on an open interval I if f' is decreasing on I. $f''(x) < 0$

EXAMPLE 2 Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

$y = 3 + \sin(x)$

$\frac{dy}{dx} = \cos(x)$

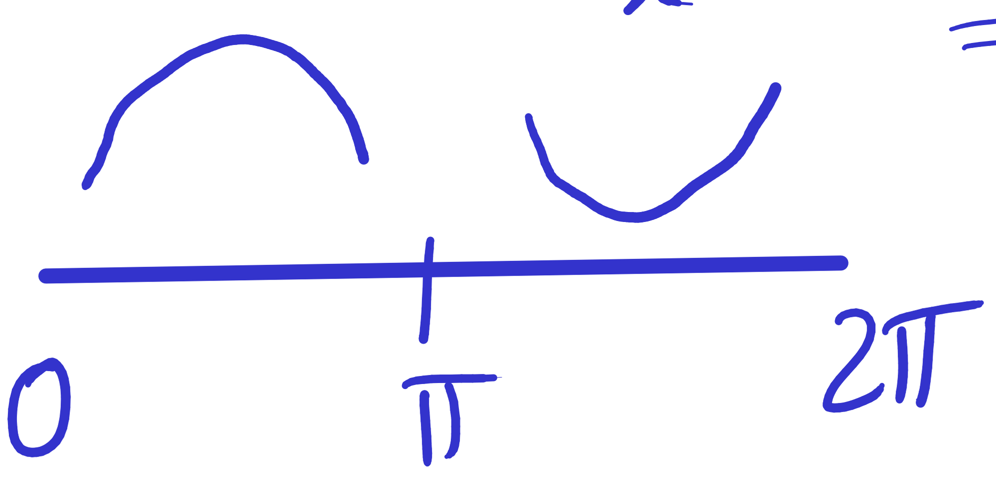
$\frac{d^2y}{dx^2} = -\sin(x)$

$y''\big|_{x=\pi/2} = -1 < 0$

$y'' = 0$

$\Rightarrow -\sin(x) = 0$

$x = \pi$



$(0, \pi)$ Concave down

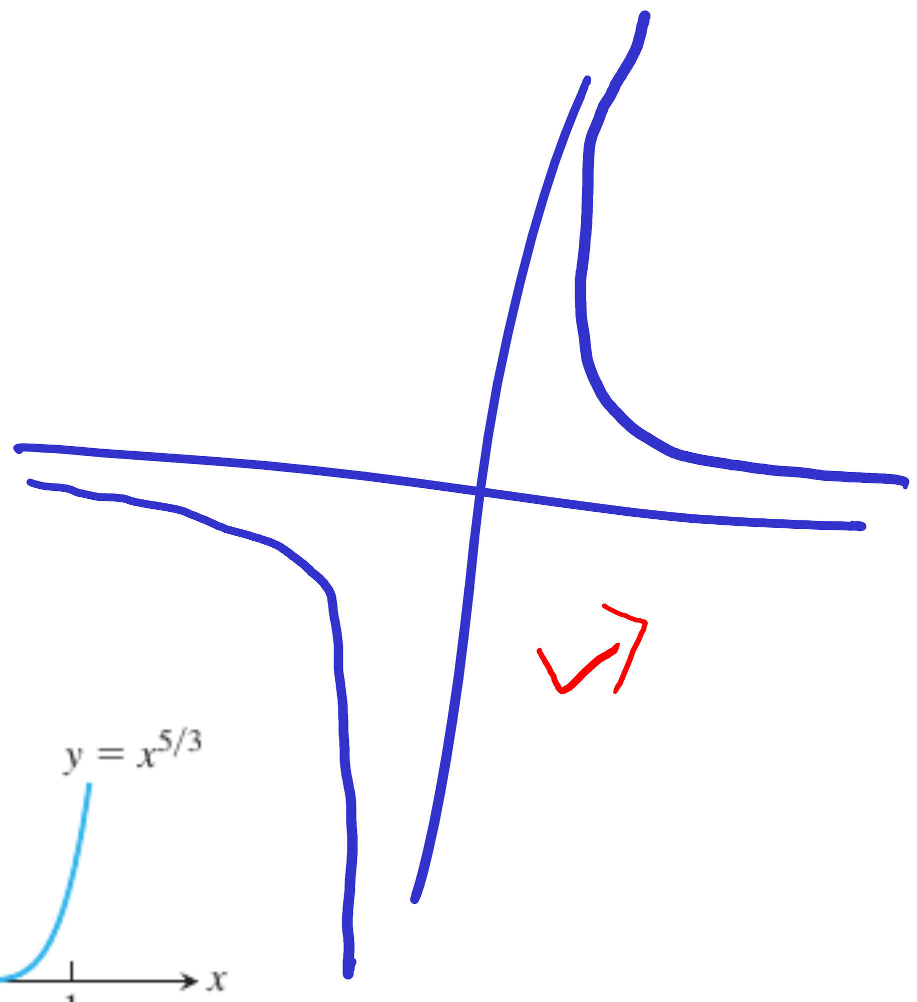
$(\pi, 2\pi)$ concave upward

EXAMPLE 3 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f'(x) = (5/3)x^{2/3} = 0$ when $x = 0$. However, the second derivative

$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{2/3}\right) = \frac{10}{9}x^{-1/3}$$

$x=0$ inflection

$\frac{10}{9x^{1/3}}$

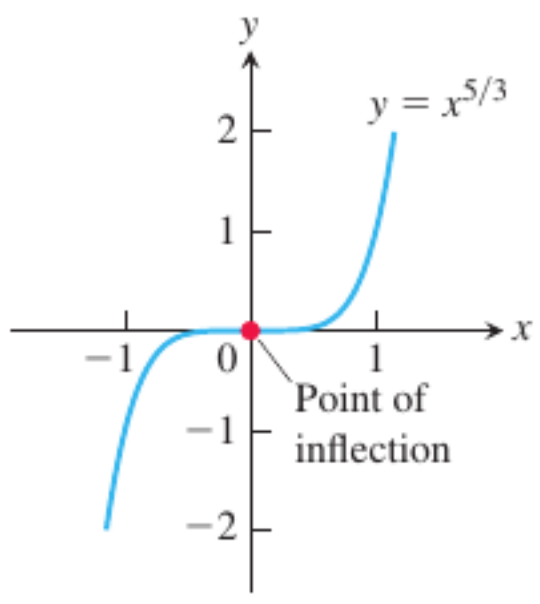


$(-\infty, 0)$ $(0, \infty)$

concave down concave upward

$f = x$

at $x = \infty$



$y = x^4$

$y' = 4x^3$

$y'' = 12x^2$

$y''(x) = 0$ at $x = 0$

$(-\infty, 0)$ $(0, \infty)$

Concave upwards

$(-\infty, \infty)$

$y = x^{1/3}$

$y' = \frac{1}{3}x^{-2/3}$

$y'' = -\frac{2}{9}x^{-5/3} = \frac{-2}{9x^{5/3}} = \frac{-2}{9[(x)^{1/3}]^5} > 0$

> 0 $<$

$(-\infty, 0)$ $(0, \infty)$

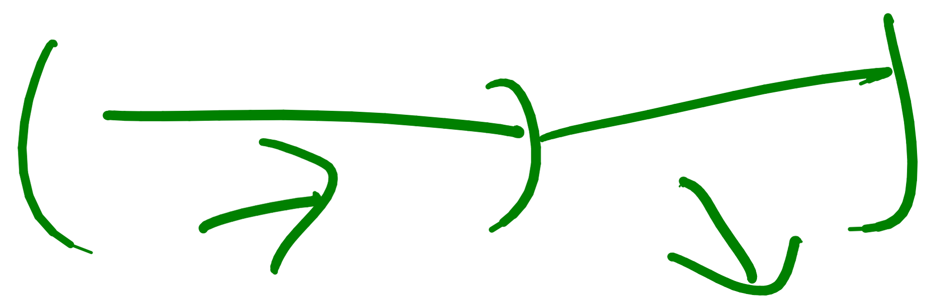
X

$y''=0$ $\quad$ y'' is undefined

$y'=0$ $\quad$ y' is undefined $\Longleftrightarrow$ Critical point

$y''=0$ $\quad$ y'' is undefined $\not\Rightarrow$ Inflection point

$f''(x)$



$y=x$

$y'=1$

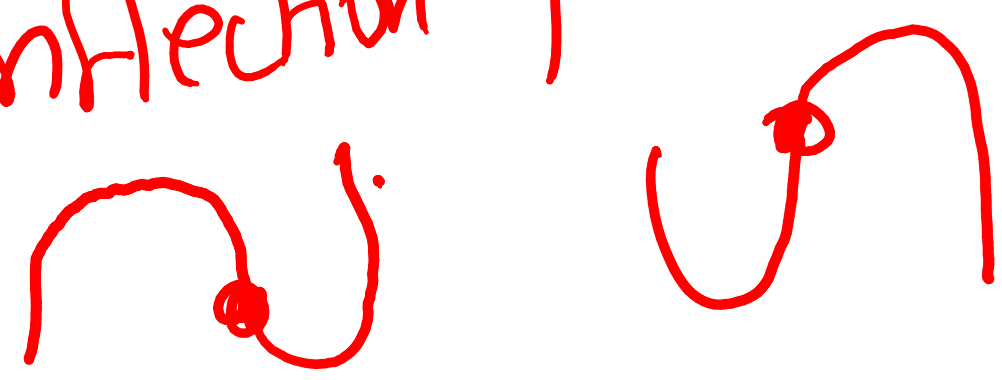
$y''=0$

Critical point

$y'=0$

y'

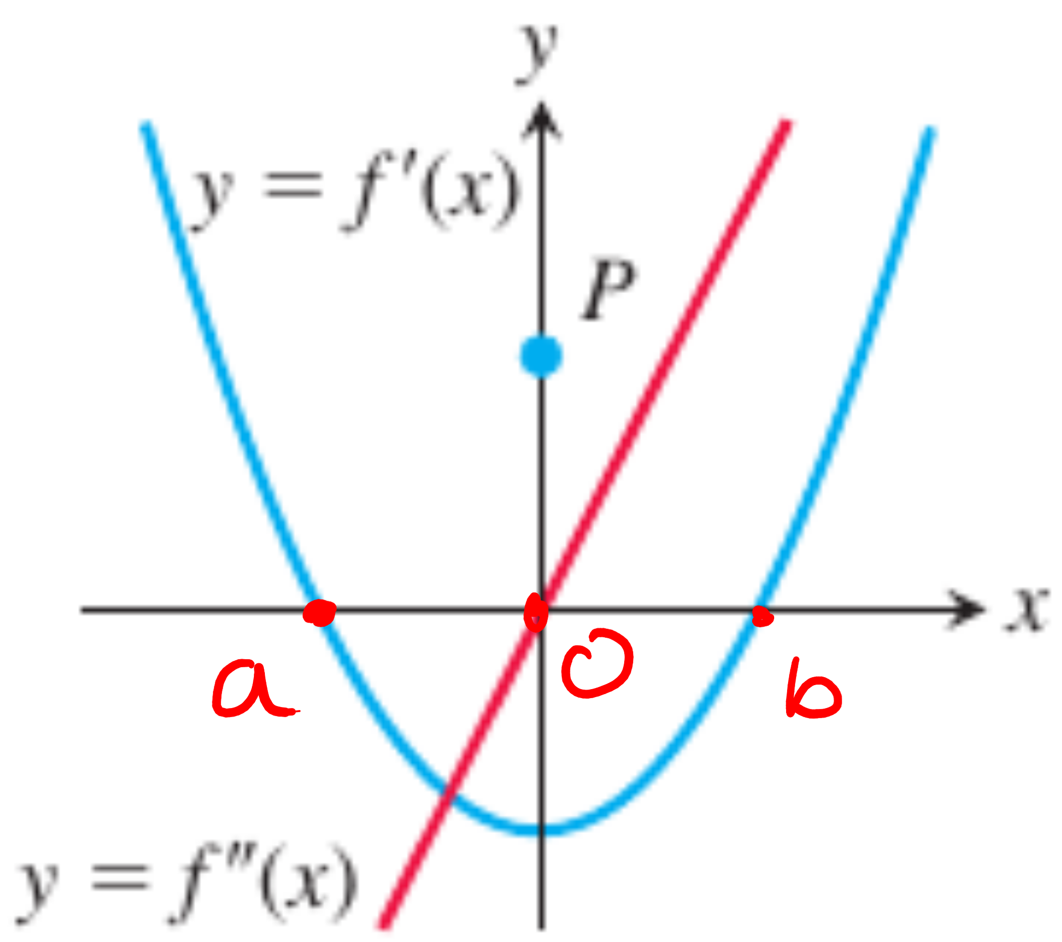
Inflection point



$y''=0$

y'' undef

geogebra



$(-\infty, a)$ $\nearrow$

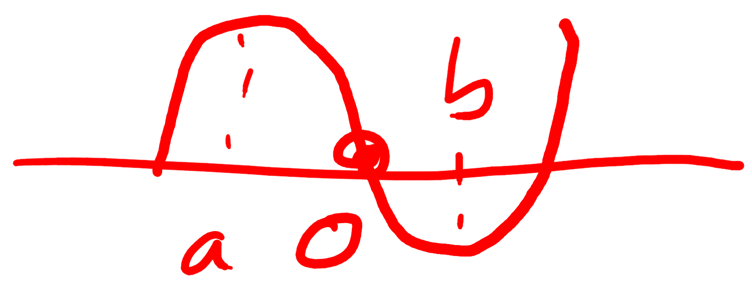
(a,b) $\searrow$

(b,∞) $\nearrow$

$f''(x) = ax+b$

$(-\infty, 0)$ $\cap$

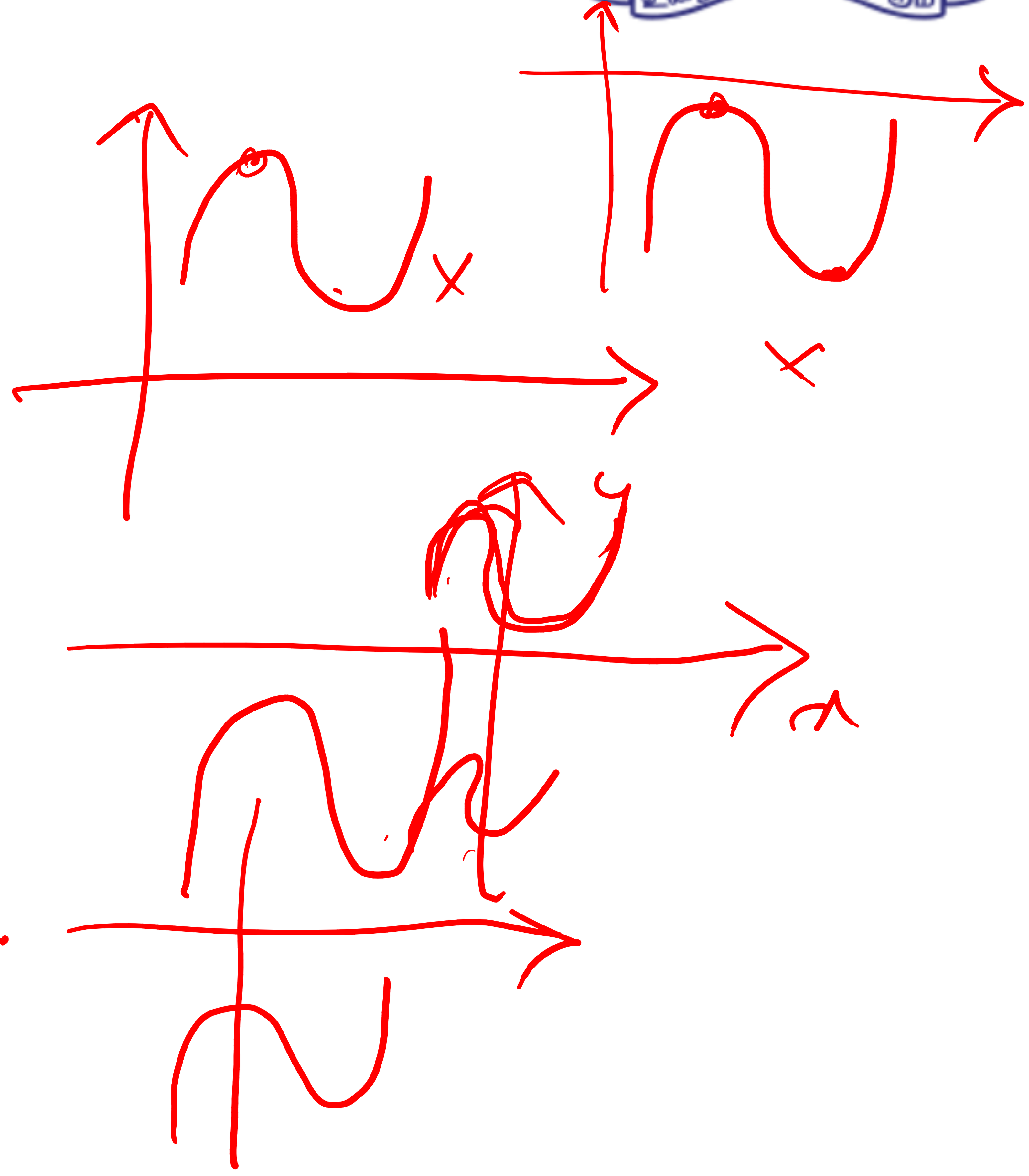
$(0,\infty)$ $\cup$

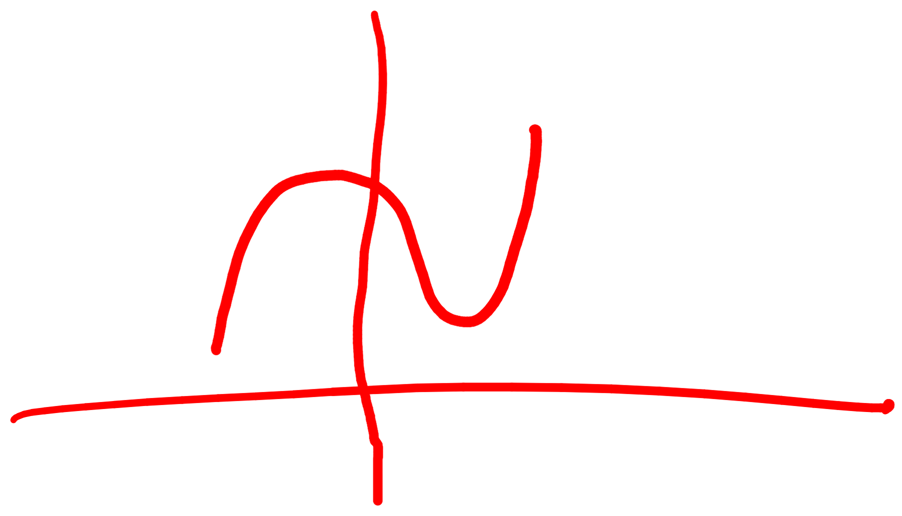


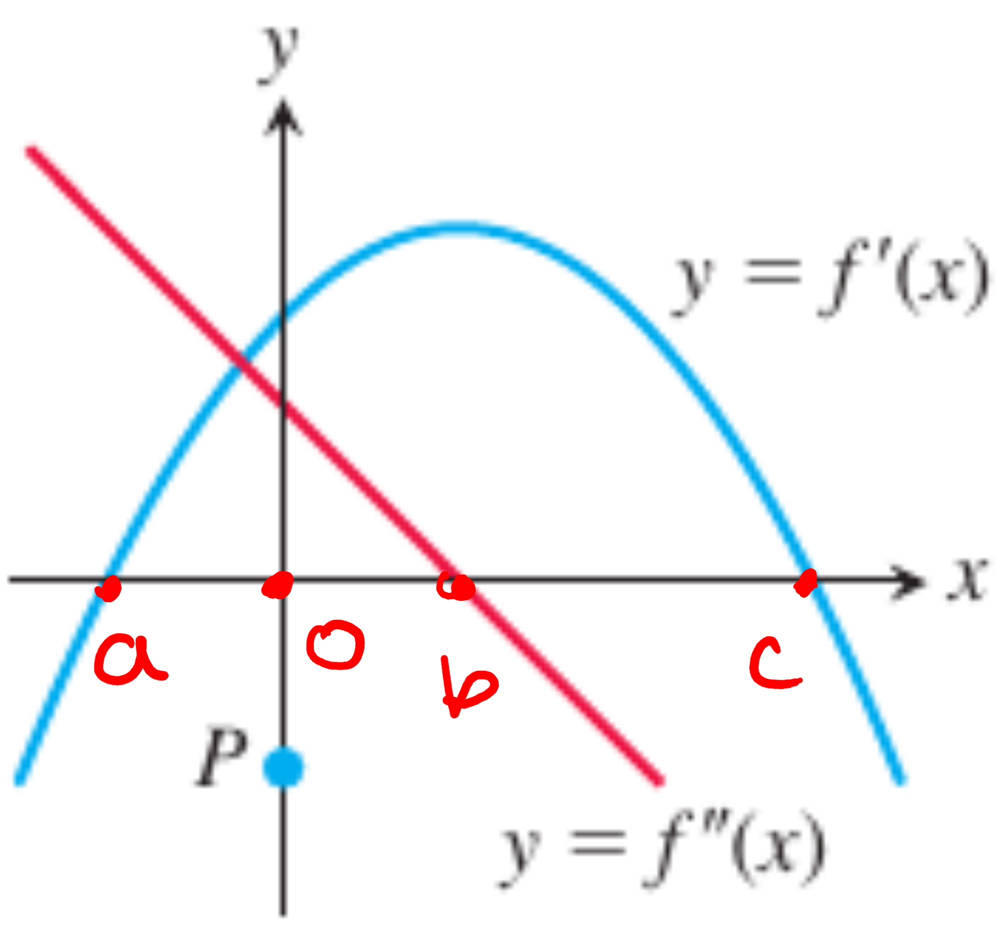
$f''(x) = ax$

$f'(x) = \frac{ax^2}{2} + b$

$f(x) = \frac{ax^3}{6} + bx + c$







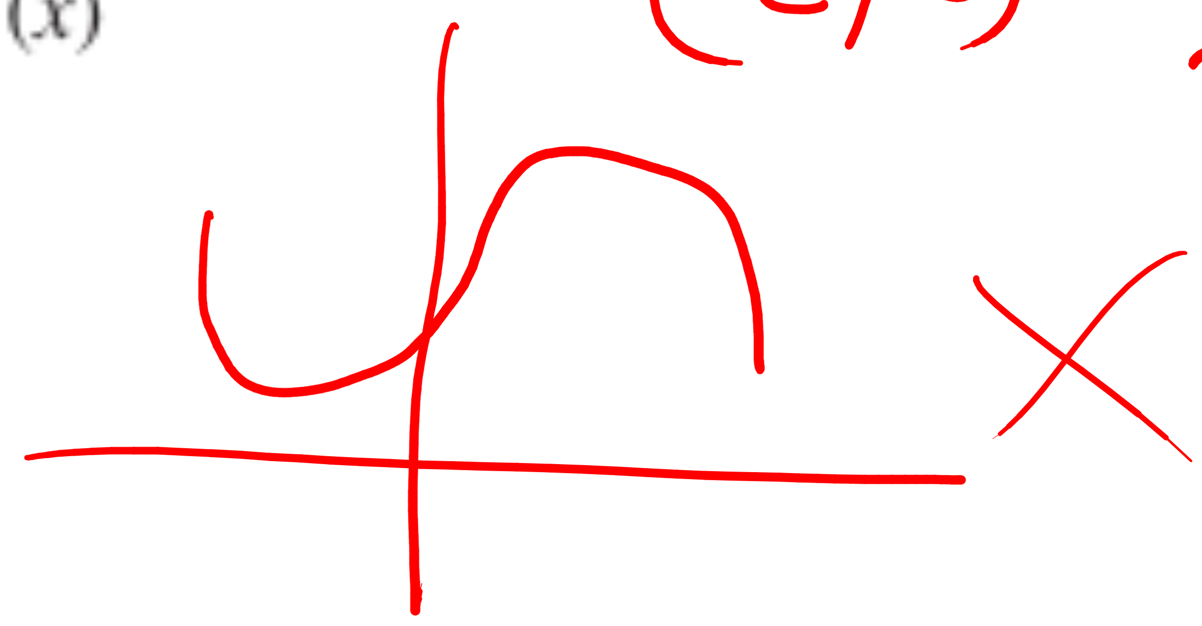
$(-\infty, a)$ ↘

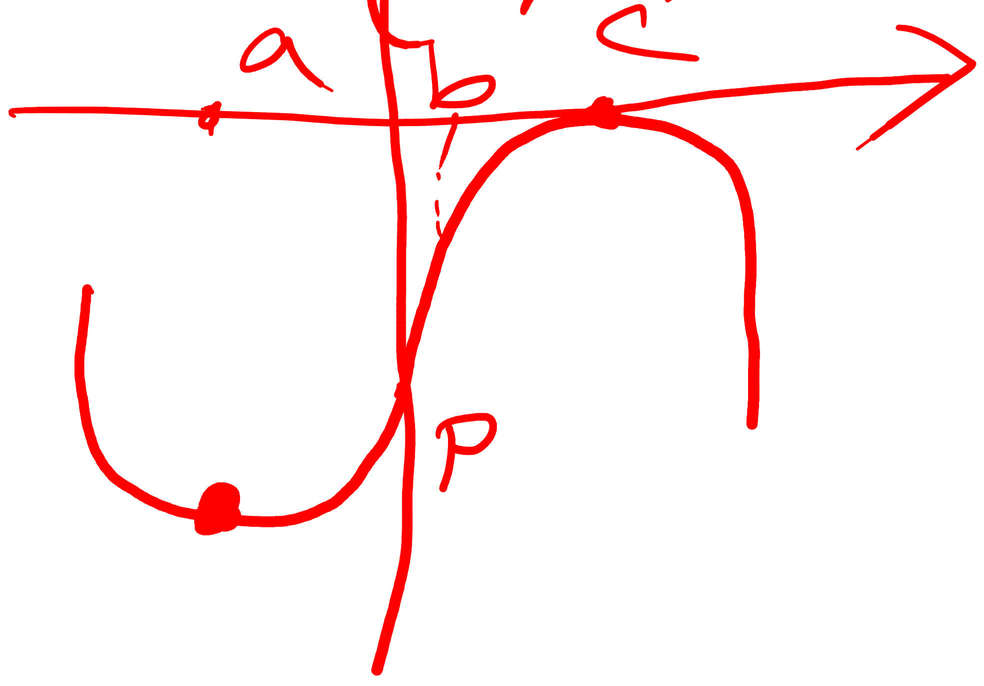
(a, c) ↗

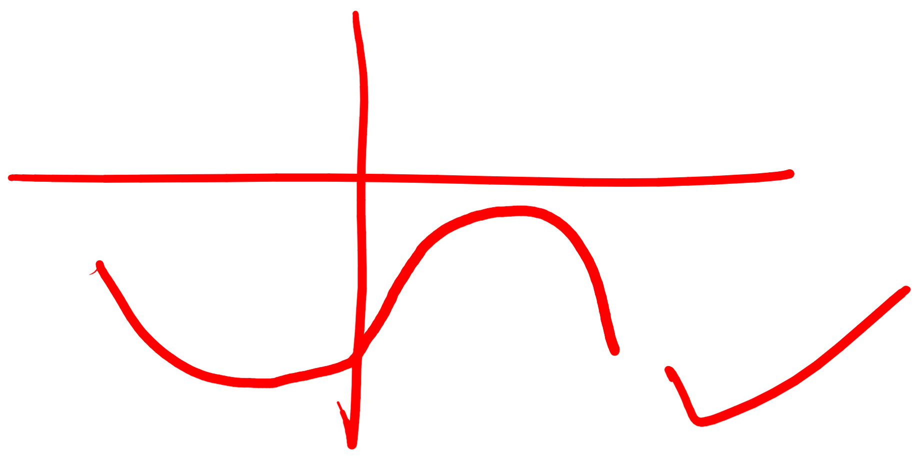
(c, ∞) ↘

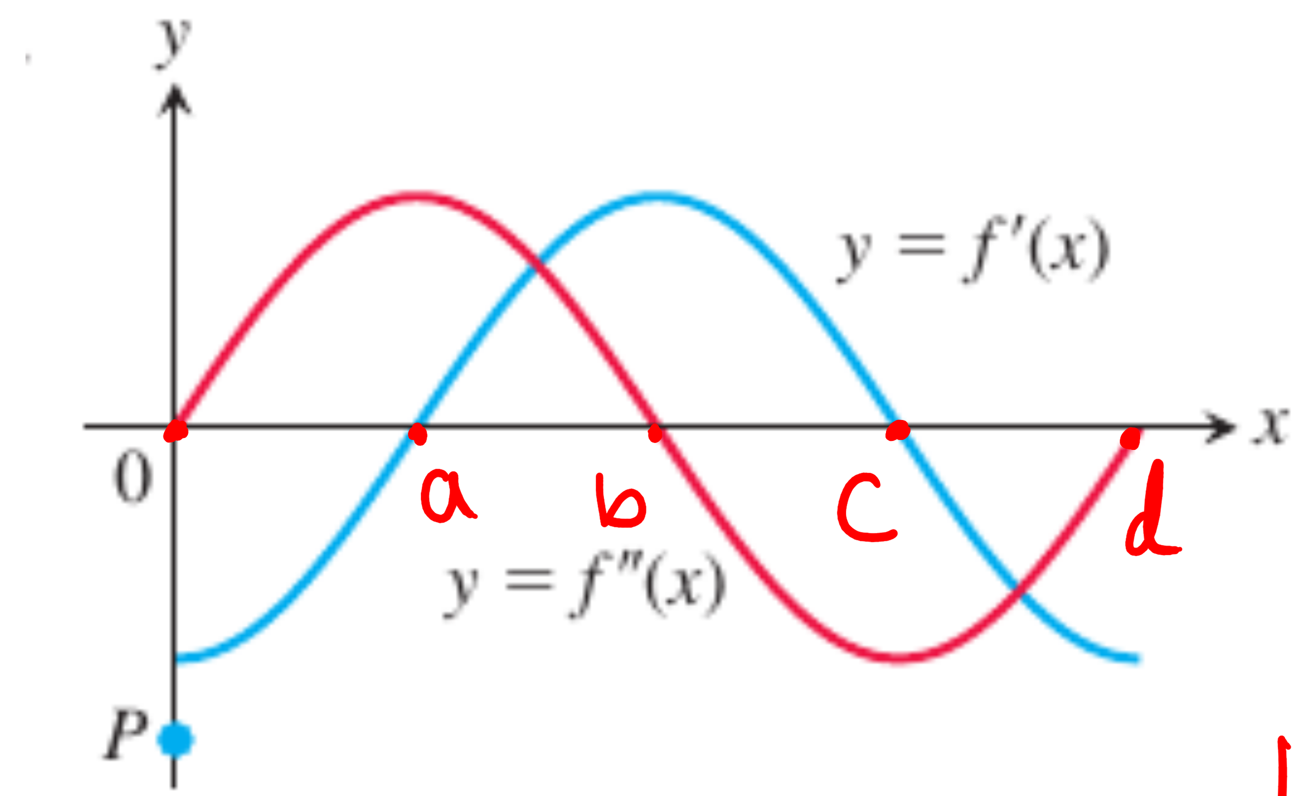
$(-\infty, b)$ ∪

(b, ∞) ∩





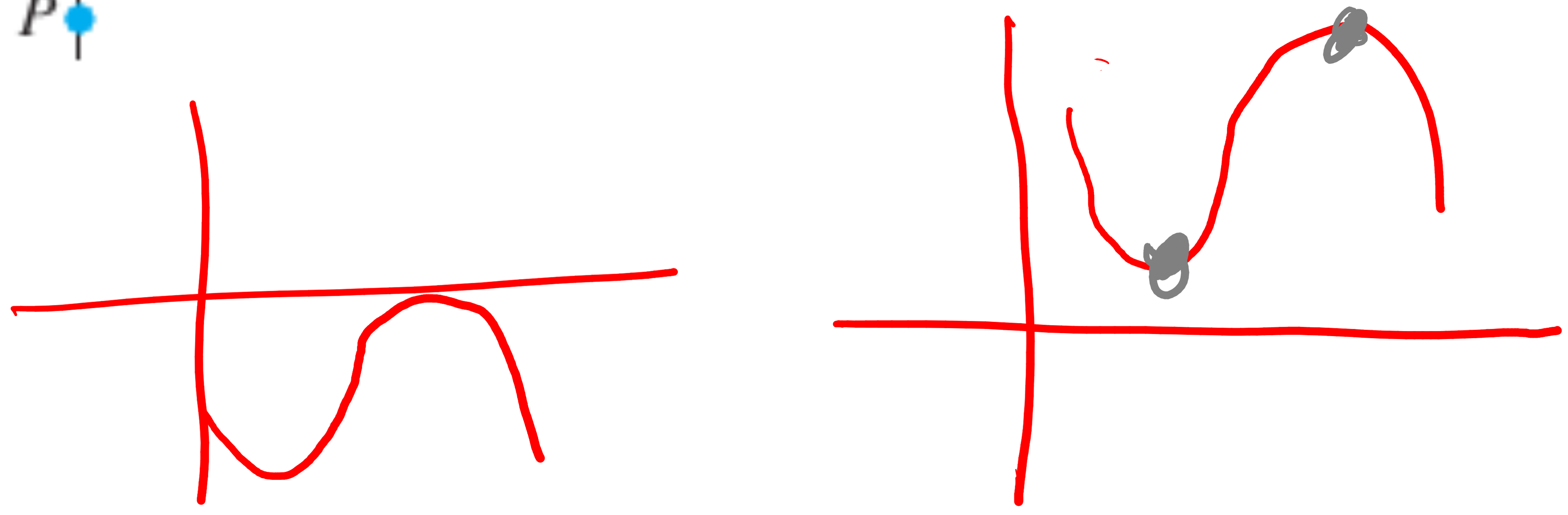


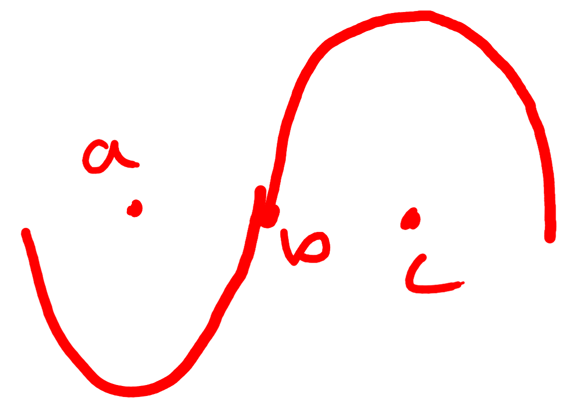


$(0,a) \searrow$ $\quad$ $(0,b)$ con. up

$(a,c) \nearrow$ $\quad$ (b,d) con. down

$(c,d) \searrow$





maxima/minima $\qquad f''(x) > 0$

If c.p lies in con.up minimum

If c.p lies in con.dow maxim

$f''(x) < 0$

Find the values of constants a, b, and c so that the graph of $y = (x^2 + a)/(bx + c)$ has a local minimum at $x = 3$ and a local maximum at $(-1, -2)$.

$$y = \frac{x^2+a}{bx+c} \Rightarrow y' = \frac{(bx+c)2x - (x^2+a)b}{(bx+c)^2} = 0$$

$$2bx^2 + 2cx - bx^2 - ab = 0$$

$$bx^2 + 2cx - ab = 0$$

$9b+bc-ab=0$

$b-2c-ab=0$

$y=\dfrac{x^2+a}{bx+c}$

$x=-1,\ y=-2$

$-2=\dfrac{1+a}{-b+c}$

$1+a=2b-2c$

$a=3$
$b=1$
$c=-1$

$3-2-2=-1$

$a-2b+2c=-1$
$9b+bc-ab=0$
$b-2c-ab=0$

$a=-1$
$b=\frac{9}{7}$
$c=\frac{9}{7}$

$a=-3/5$
$b=1$
$c=\frac{4}{5}$

$9b+bc=b-2c$

$8b=-8c$

$b=-c$

$-9c+bc=-ac$

$-c-2c=-ac$

$x-y=5$

$x=5+k$

$(5+k,k)$

$y=\dfrac{x^2+a}{bx+c}$

$bx^2+2cx-ab=0$

$x=3$
$x=-1$
$x=-2$

$9b+bc-ab=0$ — ①

$b-2c-ab=0$ — ②

$4b-4c-ab=0$ — ③

$9b+bc=b-2c \Rightarrow 8b+8c=0 \Rightarrow b=-c$

$b-2c=4b-4c \Rightarrow -3b+2c=0 \Rightarrow$

Step 1 1 of 2

Since the graph of $y = \dfrac{x^2+a}{bx+c}$ passes its local maximum point $(-1,-2)$, we have:

$$-2 = \frac{(-1)^2+a}{b(-1)+c} \Rightarrow -2 = \frac{1+a}{c-b} \Rightarrow 2b-2c = 1+a$$

Next, we need to find the first derivative, which is $y' = \dfrac{bx^2+2cx-ab}{(bx+c)^2}$. Since the graph has a local minimum at $x=3$ and a local maximum at $x=-1$ (and recall that the derivative at the x-value of a local extremum is 0), we have:

$$\frac{b(3)^2+2c(3)-ab}{(b(3)+c)^2} = 0 \Rightarrow \frac{9b+6c-ab}{(3b+c)^2} = 0 \Rightarrow 9b+6c-ab = 0$$

and

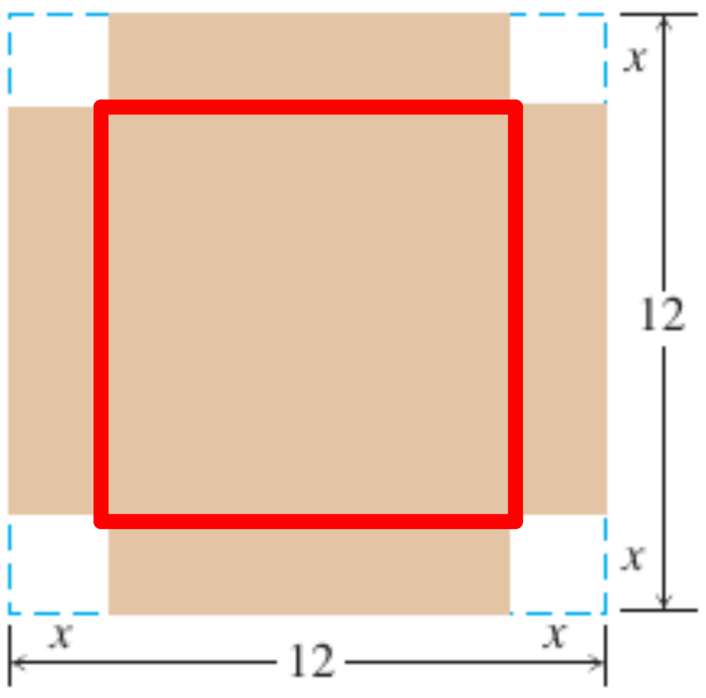
$$\frac{b(-1)^2+2c(-1)-ab}{(b(-1)+c)^2} = 0 \Rightarrow \frac{b-2c-ab}{(c-b)^2} = 0 \Rightarrow b-2c-ab = 0$$

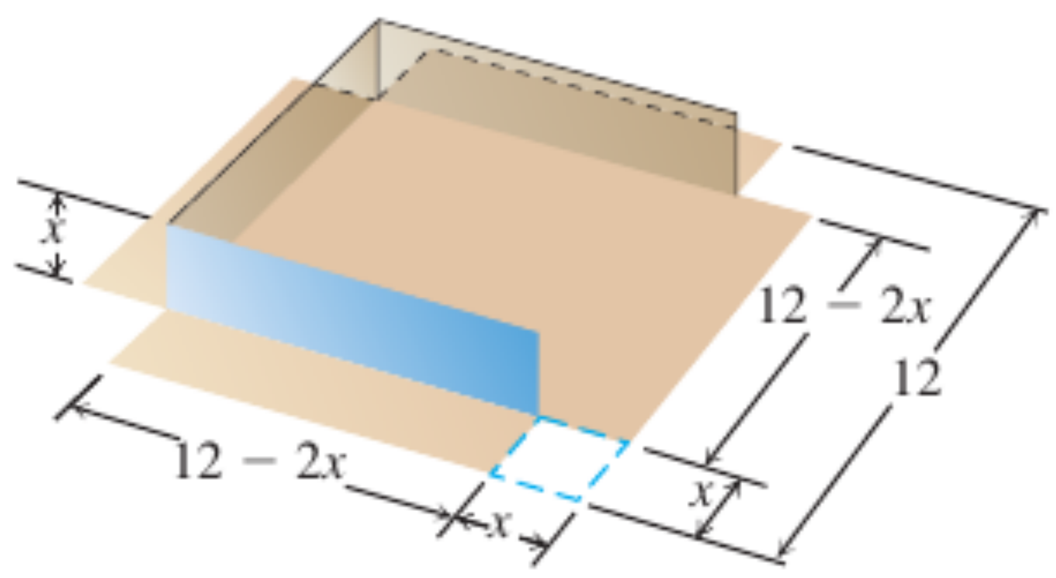
So we have three equations: (1) $2b-2c = 1+a$, (2) $9b+6c-ab = 0$, and (3) $b-2c-ab = 0$, and three variables a, b and c to solve. Since the number of equations we have is equal to the number of variables, we have

$$\frac{b(-1)^2+2c(-1)-ab}{(b(-1)+c)^2} = 0 \Rightarrow \frac{b-2c-ab}{(c-b)^2} = 0 \Rightarrow b-2c-ab = 0$$

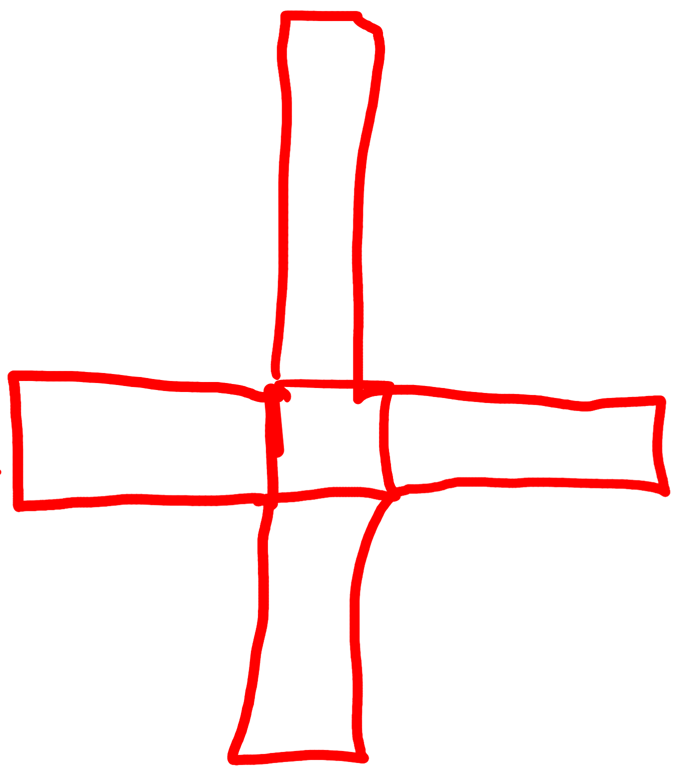
So we have three equations: (1) $2b-2c = 1+a$, (2) $9b+6c-ab = 0$, and (3) $b-2c-ab = 0$, and three variables a, b and c to solve. Since the number of equations we have is equal to the number of variables, we have sufficient information to solve the variables. For example, we subtract Eq. (3) from Eq. (2) to obtain $8b+8c=0$. This means $c=-b$. Now substitute this in Eq. (1), we have $2b-2(-b) = 1+a \Rightarrow 4b = 1-a \Rightarrow a = 4b-1$. Substitute $c=-b$ and $a=4b-1$ into Eq. (3), we have $b-2(-b)-(4b-1)b = 0 \Rightarrow 4b-4b^2 = 0 \Rightarrow 4b(1-b) = 0$, which yields $b=0$ or $b=1$. However, b can't be 0. Otherwise, $y = \dfrac{x^2+a}{c}$ will be a quadratic function, which will not have both maximum and minimum. Therefore, $b=1$. Since $c=-b$, $c=-1$. Finally, since $a=4b-1$, $a=4(1)-1=3$.

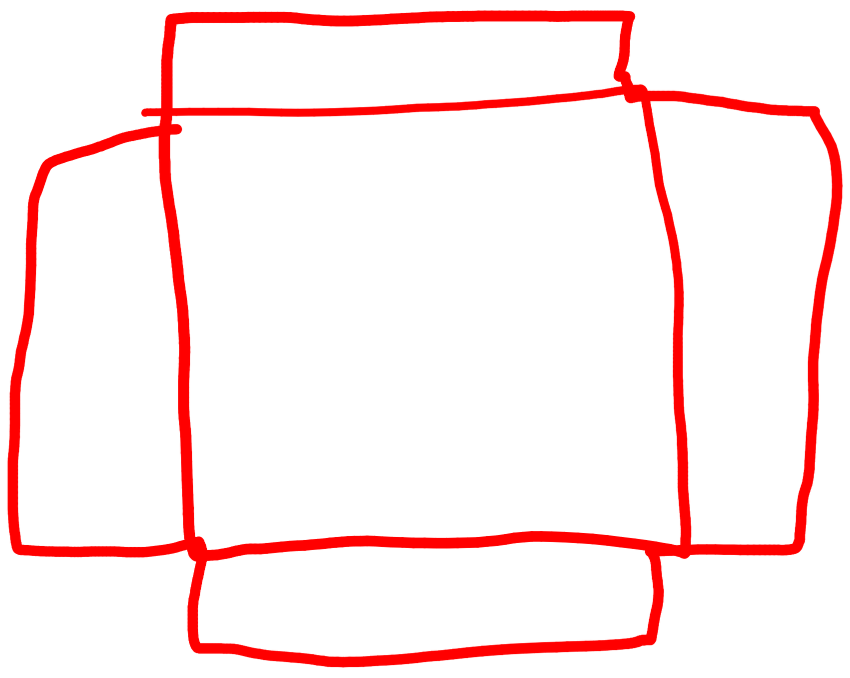
EXAMPLE 1 An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?





$$V = \underbrace{(12-2x)}_{l}\,\underbrace{(12-2x)}_{b}\,\underbrace{x}_{h}$$





$x = 1$

x	V
1	100
$\boxed{2}$	$\boxed{128}$
3	108
4	64
5	20
6	0

$$V = (12-2x)^2 x$$

$$V' = (12-2x)^2(1) + x(2)(12-2x)(-2)$$

$$= 12(2-x)(6-x)$$

For CP $V' = 0 \Rightarrow x = 2, 6$

$$V'' = 12[(2-x)(-1) + (6-x)(-1)]$$

$$V''(2) = -ve$$

$$V''(6) = +ve$$

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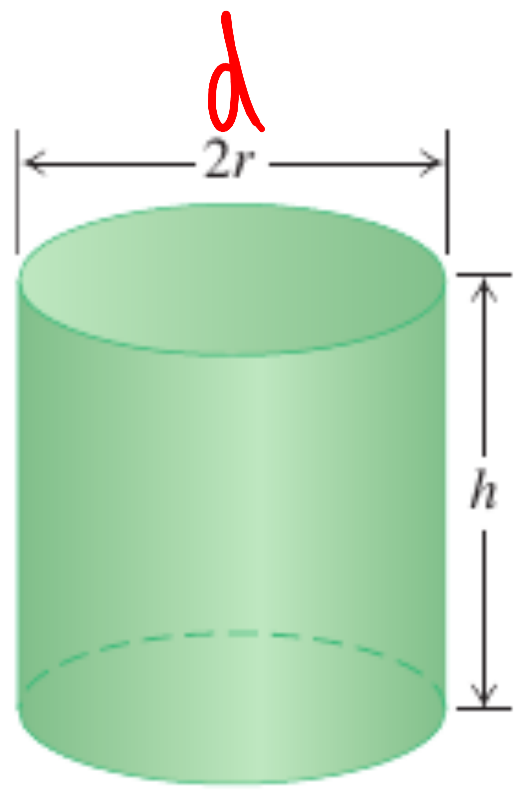
EXAMPLE 2 You have been asked to design a one-liter can shaped like a right circular cylinder (Figure 4.36). What dimensions will use the least material?



$$\pi r^2 h = 1000 \text{ cm}^3 \Rightarrow h = \frac{1000}{\pi r^2}$$

$$\pi r^2 = \frac{1000}{h}$$

TSA $\quad 2\pi r h + \pi r^2 + \pi r^2$

$$TSA = 2\pi r \frac{1000}{\pi r^2} + 2\pi r^2$$

$$A = \frac{2000}{r} + 2\pi r^2$$

$$A'(r) = -\frac{2000}{r^2} + 4\pi r = 0$$

$$\Rightarrow 4\pi r = \frac{2000}{r^2}$$

$$r^3 = \frac{500}{\pi}$$

$$r = \sqrt[3]{500/\pi} \simeq 5.42 \text{ cm}$$

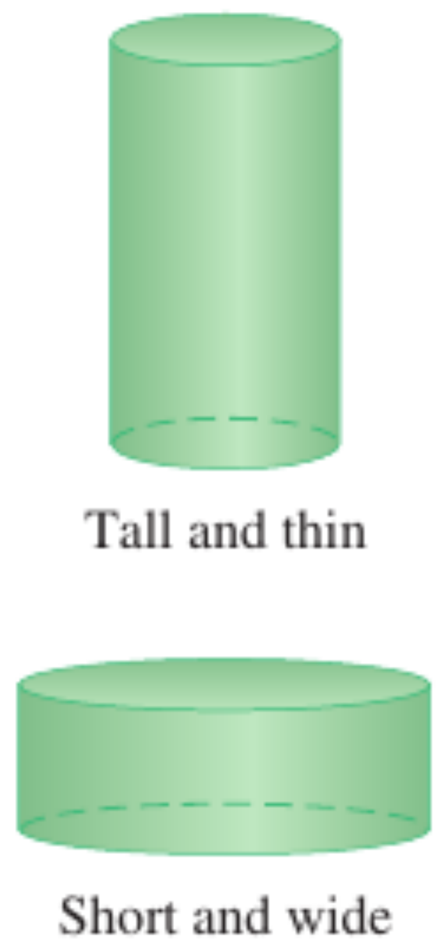
$$A''(r) = \frac{4000}{r^3} + 4\pi$$

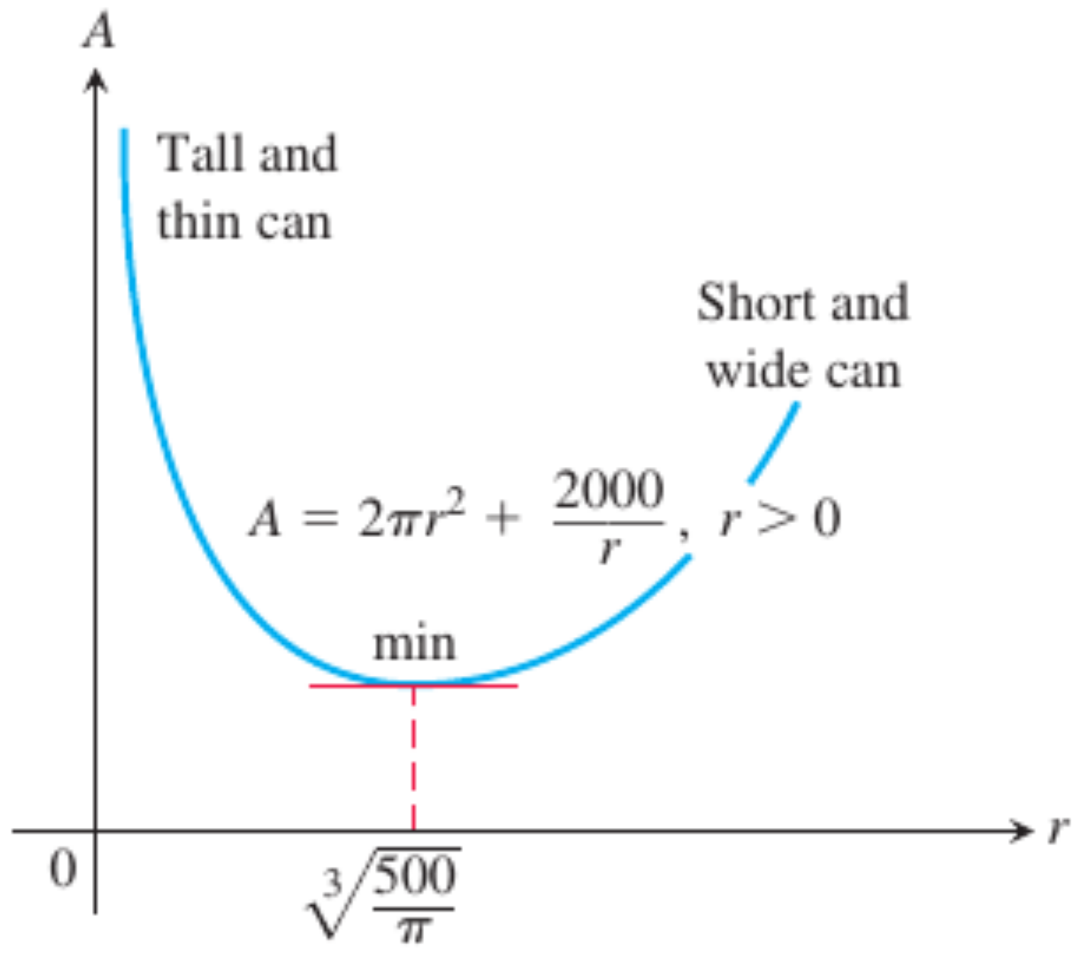
$$A''\left(\sqrt[3]{500/\pi}\right) > 0$$

$$\Rightarrow h = \frac{1000}{\pi r^2}$$

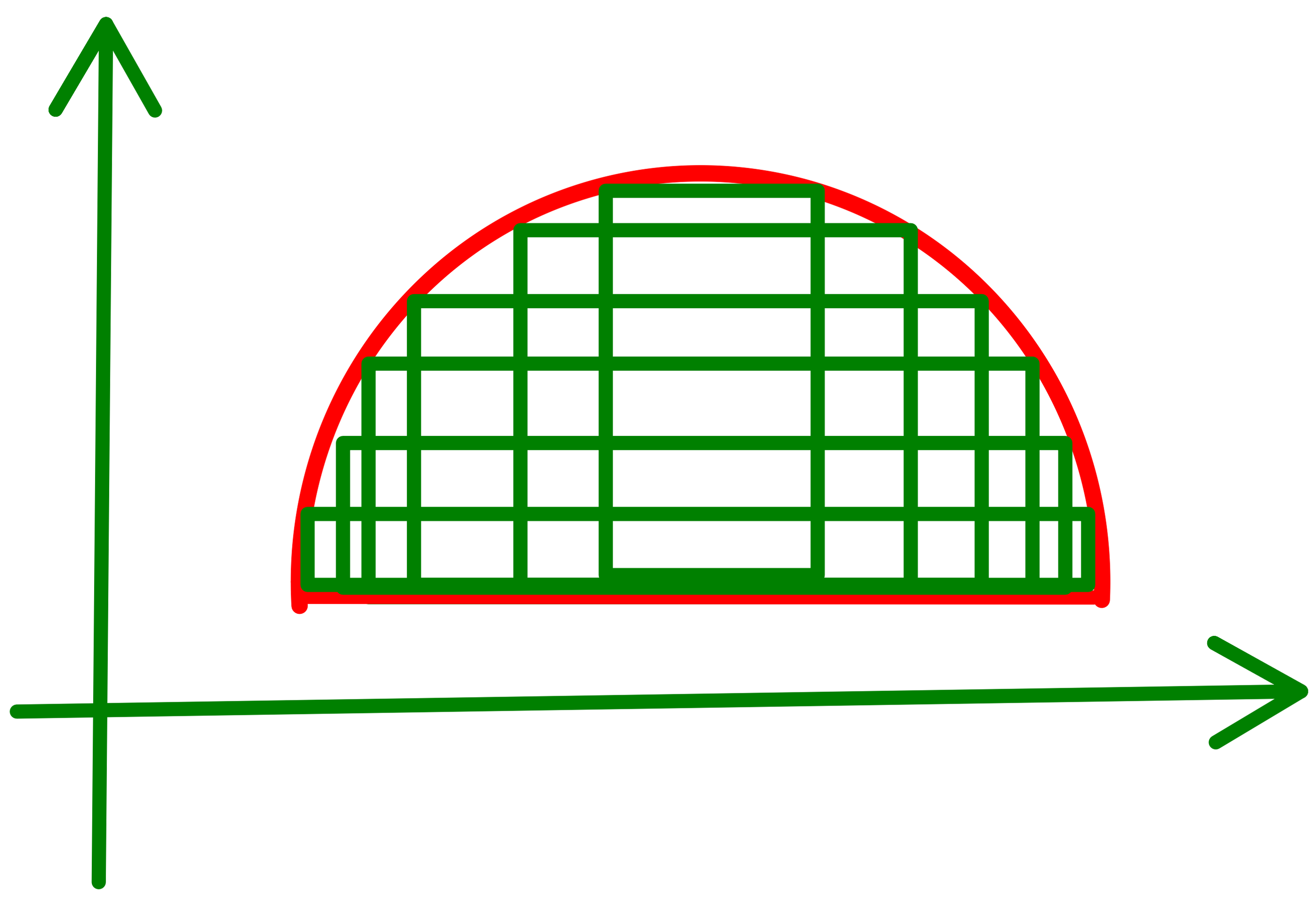
$$h = \frac{1000}{\pi (500/\pi)^{2/3}}$$

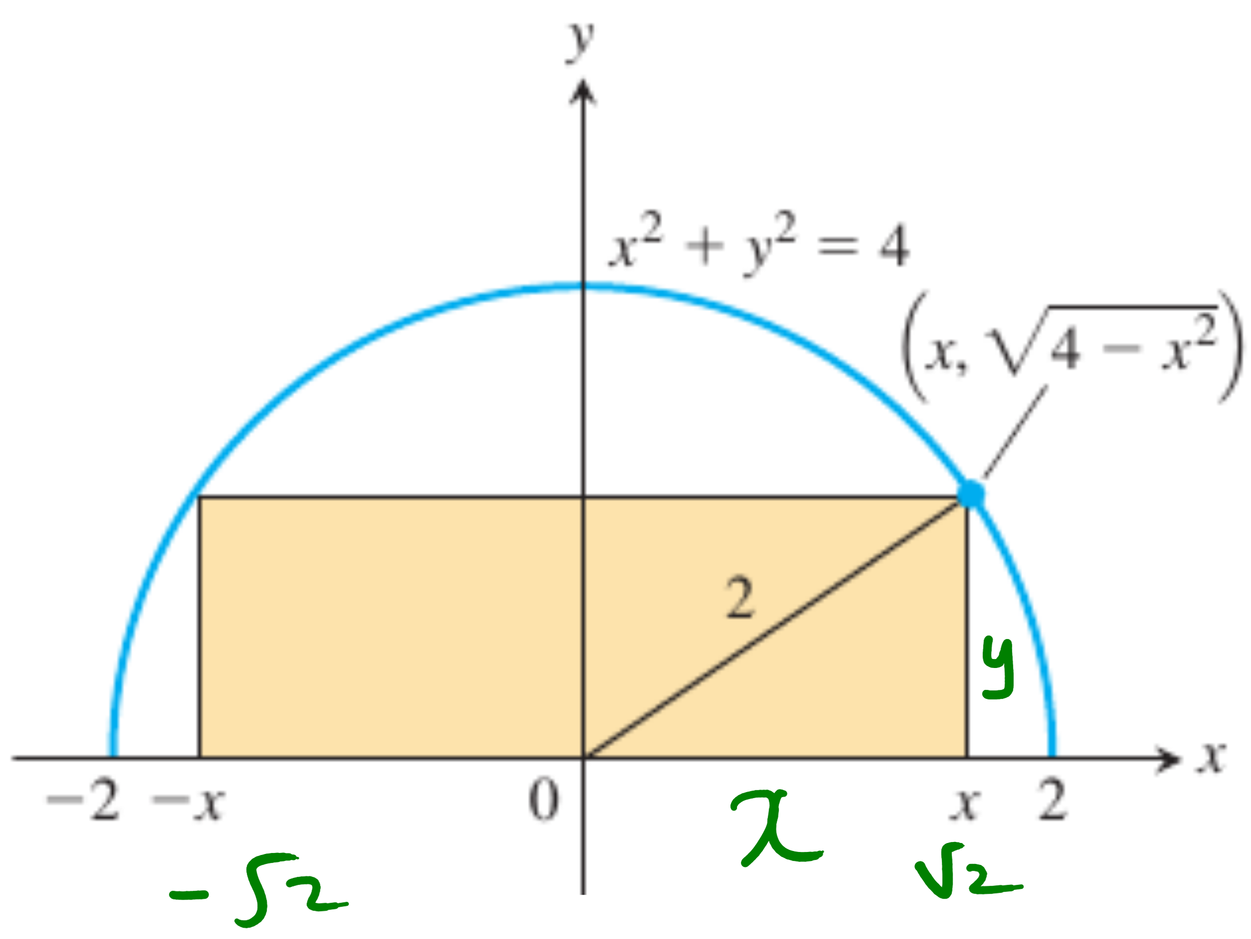
$$\simeq 10.84 \text{ cm}$$





EXAMPLE 3 A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?





Length $= 2x$

breadth $= \sqrt{4-x^2}$

$x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$y = \sqrt{4-x^2}$

$A =$ Area $= 2x\sqrt{4-x^2}$

$$A' = 2x \frac{1(-2x)}{2\sqrt{4-x^2}} + \sqrt{4-x^2}\,(2)$$

$$A' = \frac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2}$$

$A' = 0 \Rightarrow -2x^2 + 2(4-x^2) = 0$

$\Rightarrow 8 - 4x^2 = 0$

$\Rightarrow x^2 = 2 \Rightarrow x = \sqrt{2}, -\sqrt{2}$

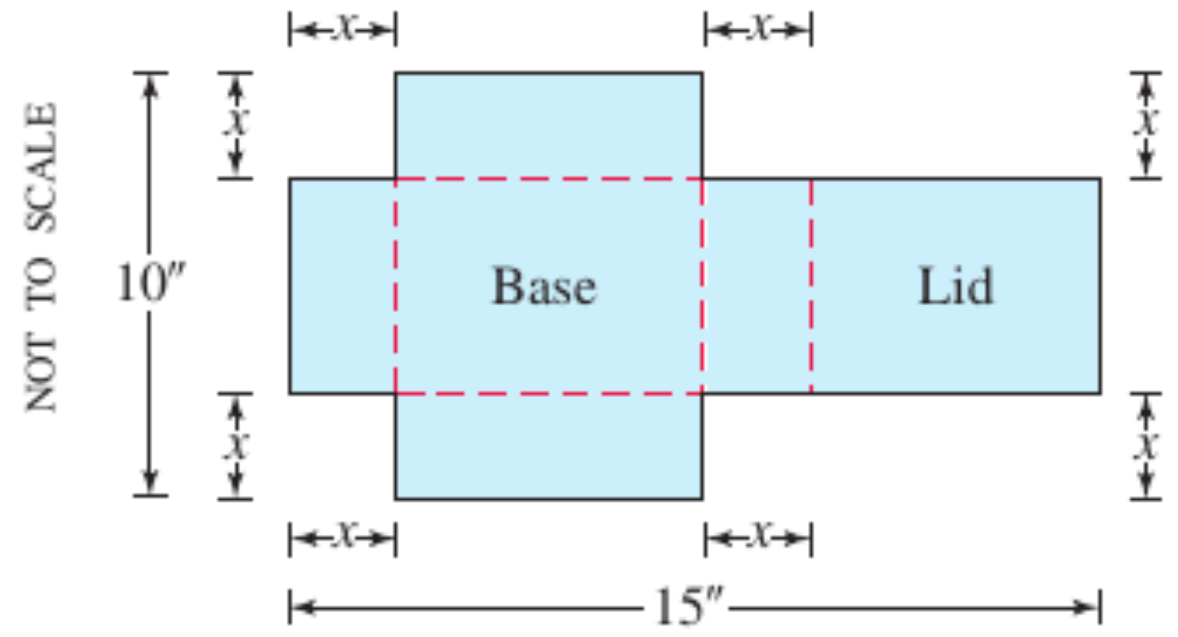
$A'' = -ve \Rightarrow x = \pm\sqrt{2}$ is maximum

$y = \sqrt{4-2} = \sqrt{2}$

Length $= 2\sqrt{2}$, Breadth $= \sqrt{2}$ //

Designing a can What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cm^3? Compare the result here with the result in Example 2.

Designing a box with a lid A piece of cardboard measures 10 in. by 15 in. Two equal squares are removed from the corners of a 10-in. side as shown in the figure. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.



a. Write a formula $V(x)$ for the volume of the box.

b. Find the domain of V for the problem situation and graph V over this domain.

c. Use a graphical method to find the maximum volume and the value of x that gives it.

d. Confirm your result in part (c) analytically.

Integrals

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad f(x)$$

$$\int f(x)\,dx = F(x) \Rightarrow F'(x) = f(x)$$

$$\int x^2\,dx = \frac{x^3}{3} \Rightarrow \frac{d}{dx}\left(\frac{x^3}{3}\right) = \frac{3x^2}{3} = x^2$$

$$= \frac{x^3}{3} + C \qquad \frac{d}{dx}\left(\frac{x^3}{3} + 4\right) = x^2$$

$$\int \log(x)\,dx = x\log(x) - x + C \qquad \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

$$\frac{d}{dx}\left(x\log(x) - x\right) = x\cancel{\frac{1}{x}} + \log(x) - \cancel{x}$$

$$= \log(x)$$

$$(fg)' = fg' + gf'$$

$$\frac{d}{dx}(f)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

Basic Forms

1. $\displaystyle\int k\,dx = kx + C$ (any number k)

2. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

3. $\displaystyle\int \frac{dx}{x} = \ln|x| + C$

4. $\displaystyle\int e^x\,dx = e^x + C$

5. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$

6. $\displaystyle\int \sin x\,dx = -\cos x + C$

7. $\displaystyle\int \cos x\,dx = \sin x + C$

8. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

9. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$

10. $\displaystyle\int \sec x \tan x\,dx = \sec x + C$

11. $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$

12. $\displaystyle\int \tan x\,dx = \ln|\sec x| + C$

13. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$

14. $\displaystyle\int \sinh x\,dx = \cosh x + C$

15. $\displaystyle\int \cosh x\,dx = \sinh x + C$

16. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$

17. $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

18. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

19. $\displaystyle\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\frac{x}{a} + C \quad (a > 0)$

20. $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + C \quad (x > a > 0)$

Cut here and keep for reference

TABLE OF INTEGRALS

FORMS INVOLVING $a + bu$

47. $\displaystyle\int \frac{u\,du}{a + bu} = \frac{1}{b^2}(a + bu - a\ln|a + bu|) + C$

48. $\displaystyle\int \frac{u^2\,du}{a + bu} = \frac{1}{2b^3}\left[(a + bu)^2 - 4a(a + bu) + 2a^2\ln|a + bu|\right] + C$

49. $\displaystyle\int \frac{du}{u(a + bu)} = \frac{1}{a}\ln\left|\frac{u}{a + bu}\right| + C$

50. $\displaystyle\int \frac{du}{u^2(a + bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a + bu}{u}\right| + C$

51. $\displaystyle\int \frac{u\,du}{(a + bu)^2} = \frac{a}{b^2(a + bu)} + \frac{1}{b^2}\ln|a + bu| + C$

52. $\displaystyle\int \frac{du}{u(a + bu)^2} = \frac{1}{a(a + bu)} - \frac{1}{a^2}\ln\left|\frac{a + bu}{u}\right| + C$

53. $\displaystyle\int \frac{u^2\,du}{(a + bu)^2} = \frac{1}{b^3}\left(a + bu - \frac{a^2}{a + bu} - 2a\ln|a + bu|\right) + C$

54. $\displaystyle\int u\sqrt{a + bu}\,du = \frac{2}{15b^2}(3bu - 2a)(a + bu)^{3/2} + C$

55. $\displaystyle\int \frac{u\,du}{\sqrt{a + bu}} = \frac{2}{3b^2}(bu - 2a)\sqrt{a + bu} + C$

56. $\displaystyle\int \frac{u^2\,du}{\sqrt{a + bu}} = \frac{2}{15b^3}(8a^2 + 3b^2u^2 - 4abu)\sqrt{a + bu} + C$

57. $\displaystyle\int \frac{du}{u\sqrt{a + bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a + bu} - \sqrt{a}}{\sqrt{a + bu} + \sqrt{a}}\right| + C, \quad \text{if } a > 0$
$\displaystyle\qquad = \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a + bu}{-a}} + C, \quad \text{if } a < 0$

58. $\displaystyle\int \frac{\sqrt{a + bu}}{u}\,du = 2\sqrt{a + bu} + a\int \frac{du}{u\sqrt{a + bu}}$

59. $\displaystyle\int \frac{\sqrt{a + bu}}{u^2}\,du = -\frac{\sqrt{a + bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a + bu}}$

60. $\displaystyle\int u^n\sqrt{a + bu}\,du = \frac{2}{b(2n + 3)}\left[u^n(a + bu)^{3/2} - na\int u^{n-1}\sqrt{a + bu}\,du\right]$

61. $\displaystyle\int \frac{u^n\,du}{\sqrt{a + bu}} = \frac{2u^n\sqrt{a + bu}}{b(2n + 1)} - \frac{2na}{b(2n + 1)}\int \frac{u^{n-1}\,du}{\sqrt{a + bu}}$

$\int u\,dv$ (or) $\int uv\,dx$ $\qquad \int \frac{u}{v}\,dx = \int u\left(\frac{1}{v}\right)dx$

$||$

$u\int v\,dx - \int u'\int v\,dx\;dx$

$$\int u\,dv = uv - \int v\,du \qquad v = \int dv,\quad du = u'$$

$\int \underset{u}{x}\,\underset{dv}{\sin(x)\,dx}$ $\qquad$ ILATE

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

u', u'', u'''

$v_1 \quad v_2 \quad v_3$

$v_1 = \int v\,dx$

$v_2 = \int v_1\,dx$

$v_3 = \int v_2\,dx$

$\int x\sin(x)\,dx$

$u = x \qquad v = \sin(x)$

$u' = 1 \qquad v_1 = -\cos(x)$

$u'' = 0 \qquad v_2 = -\sin(x)$

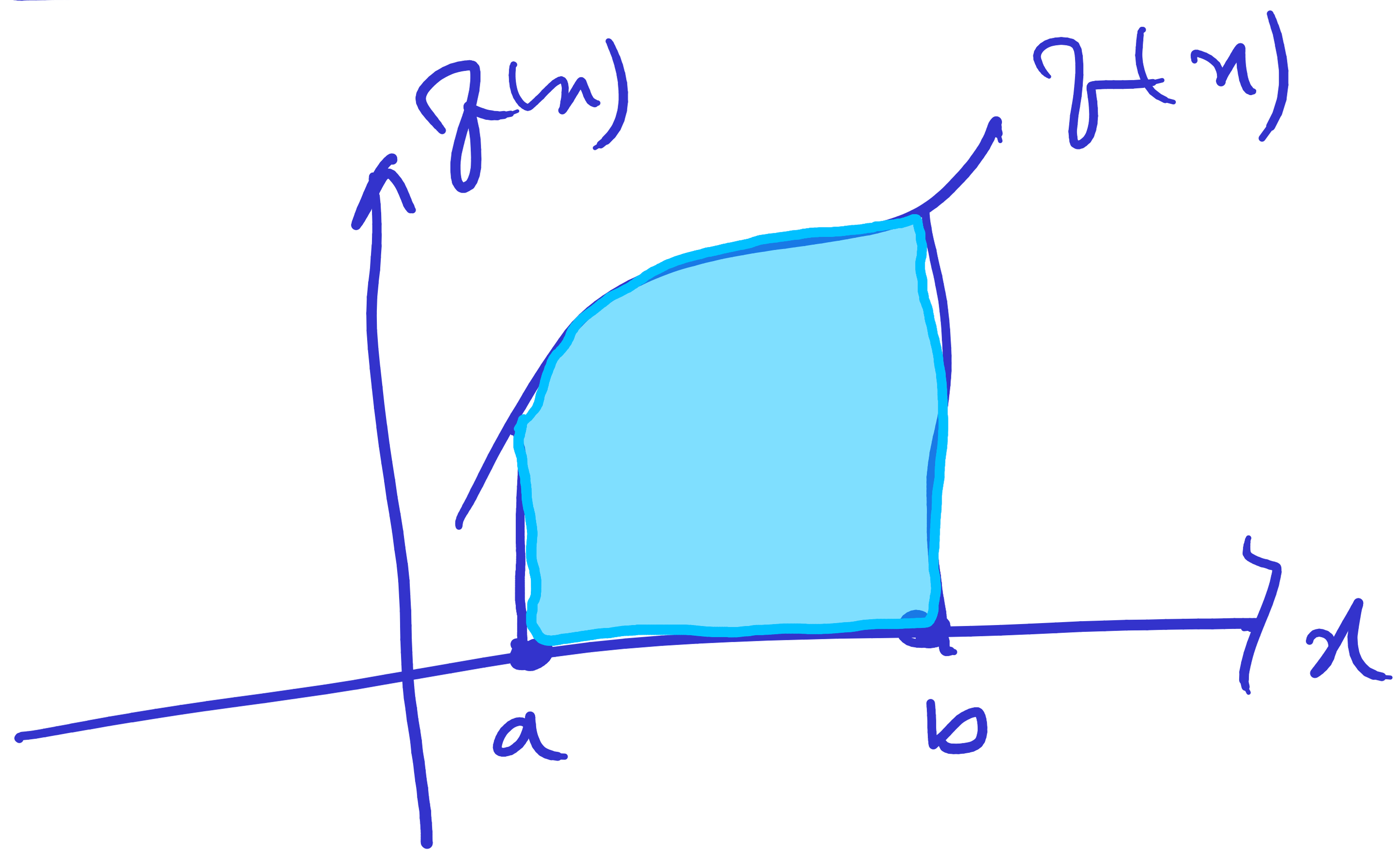
$v_3 = \cos(x)$

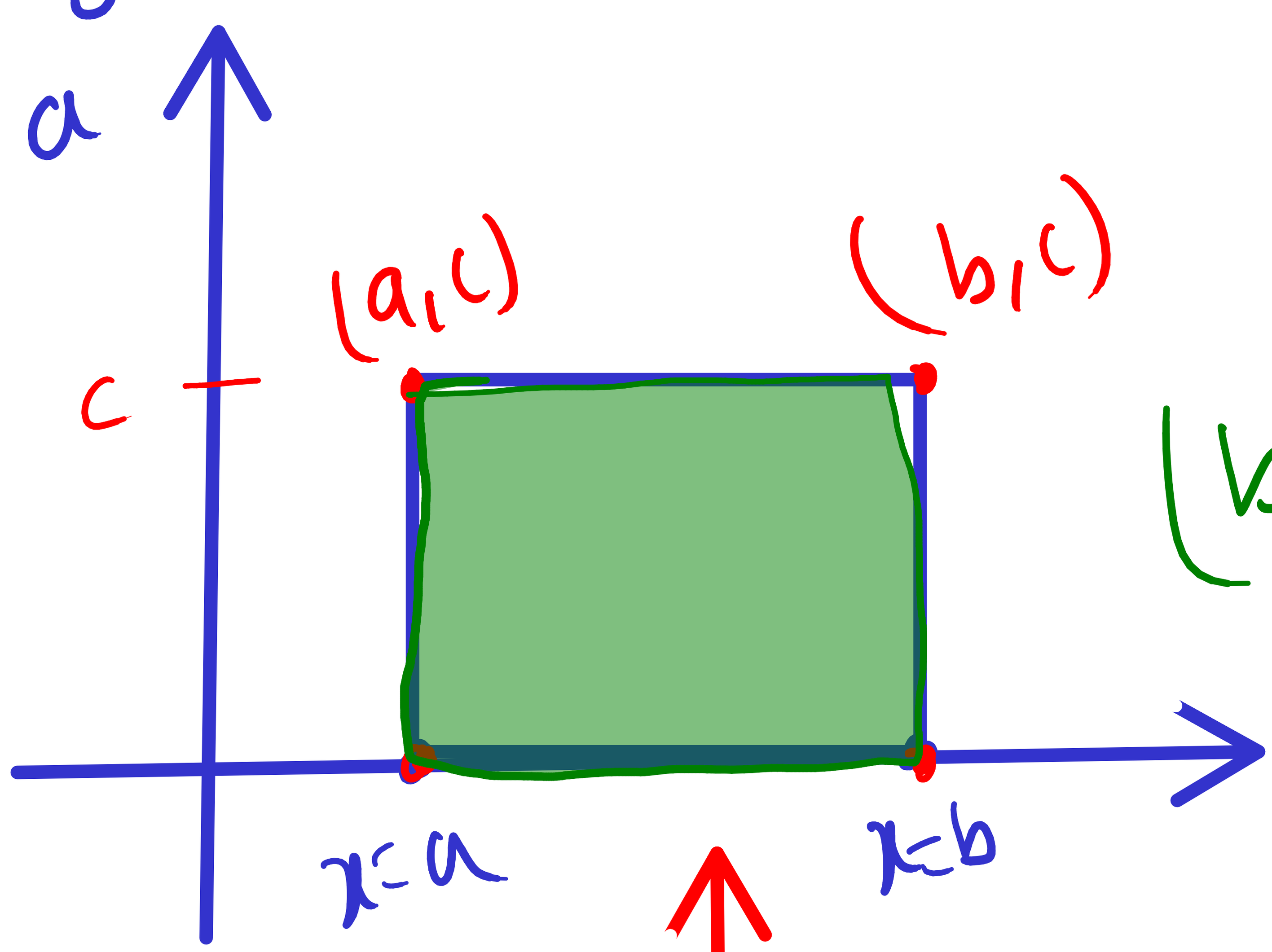
$$\int x\sin(x)\,dx = x(-\cos(x)) - 1(-\sin(x)) + 0$$
$$= -x\cos x + \sin(x) + C$$

Definite Integral

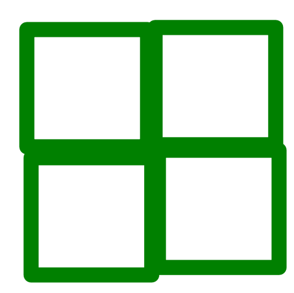
$$\int_1^2 x^2\,dx = \left[\frac{x^3}{3}+c\right]_1^2 = \left(\frac{8}{3}+c\right) - \left(\frac{1}{3}+c\right)$$
$$= \frac{7}{3}$$

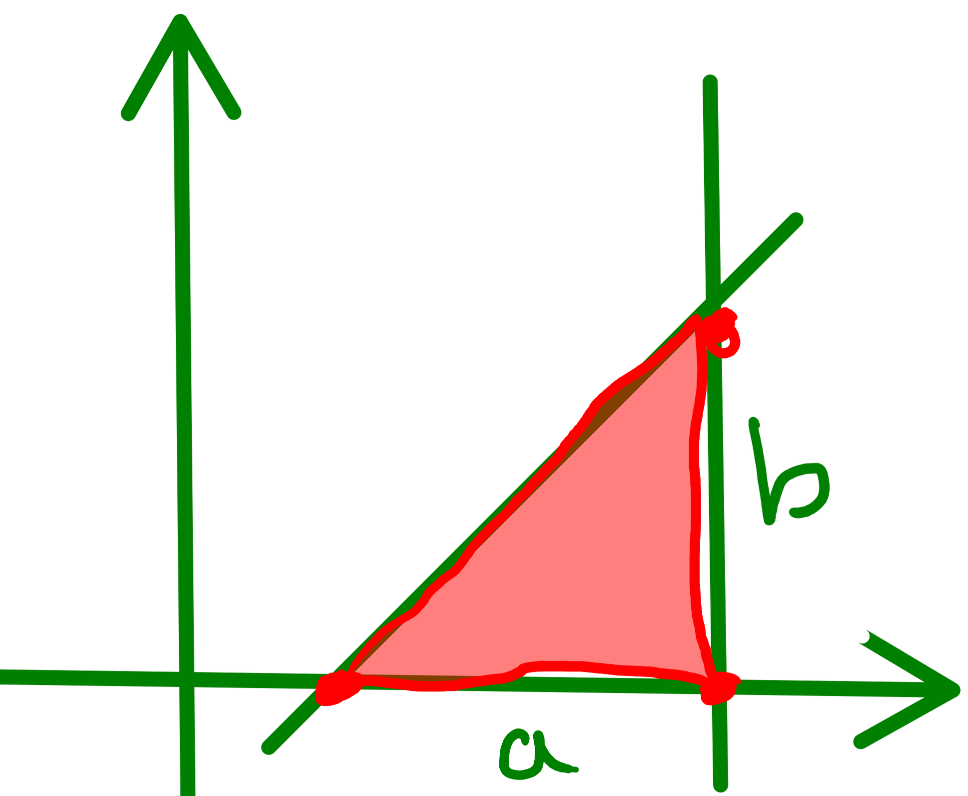
$$\int_a^b f(x)\,dx$$



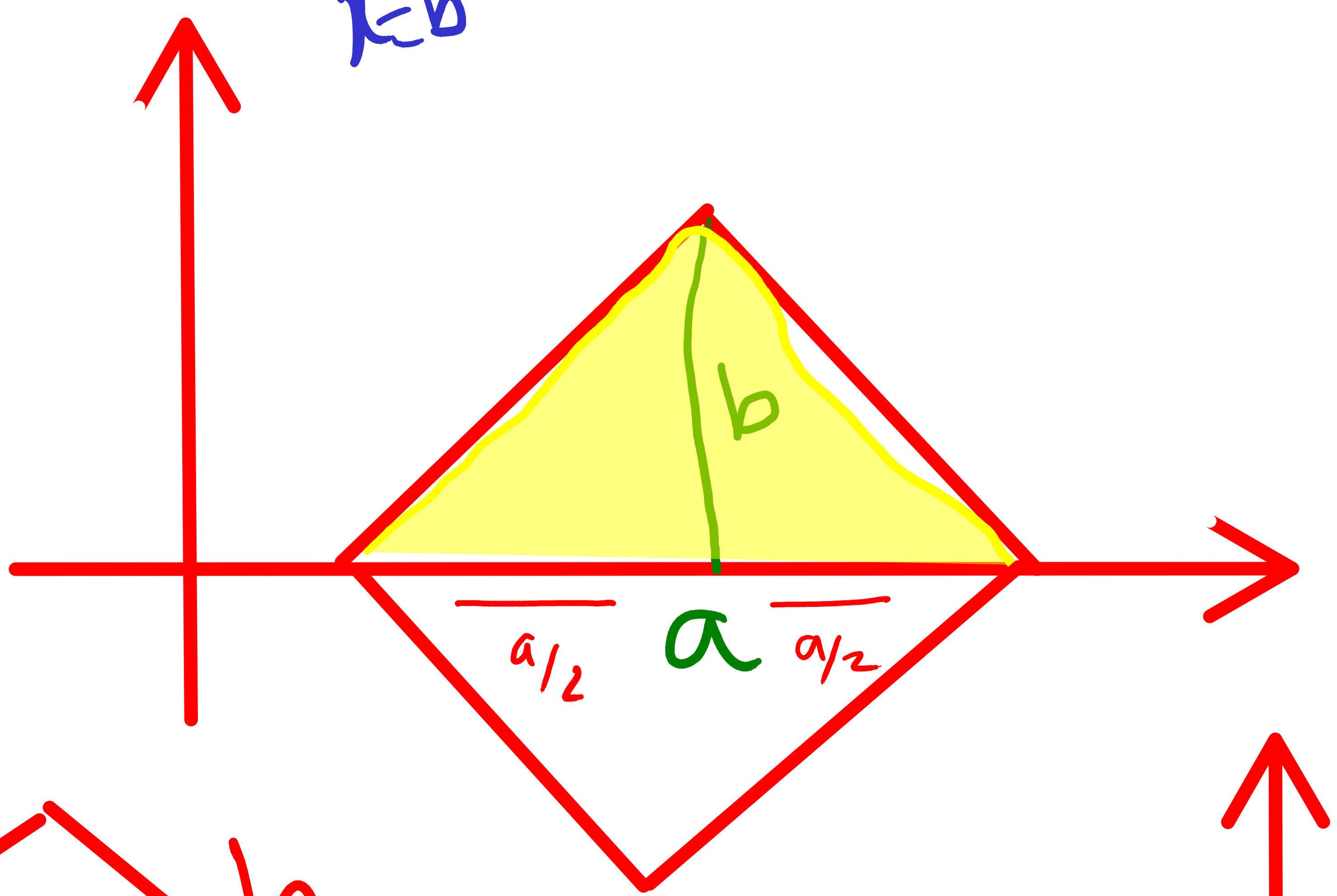


$(b-a)c$

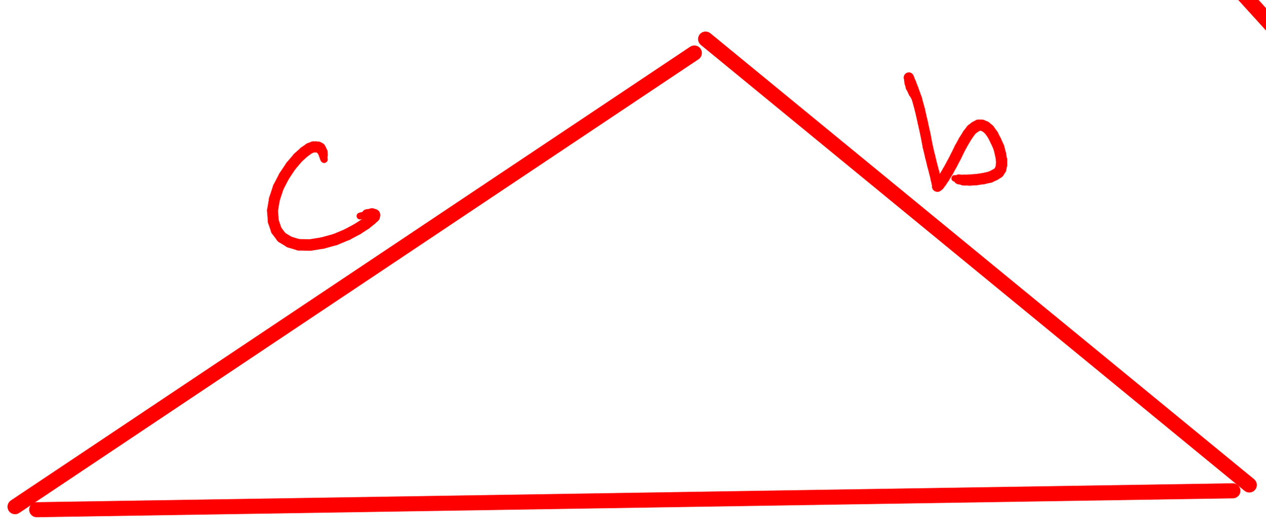




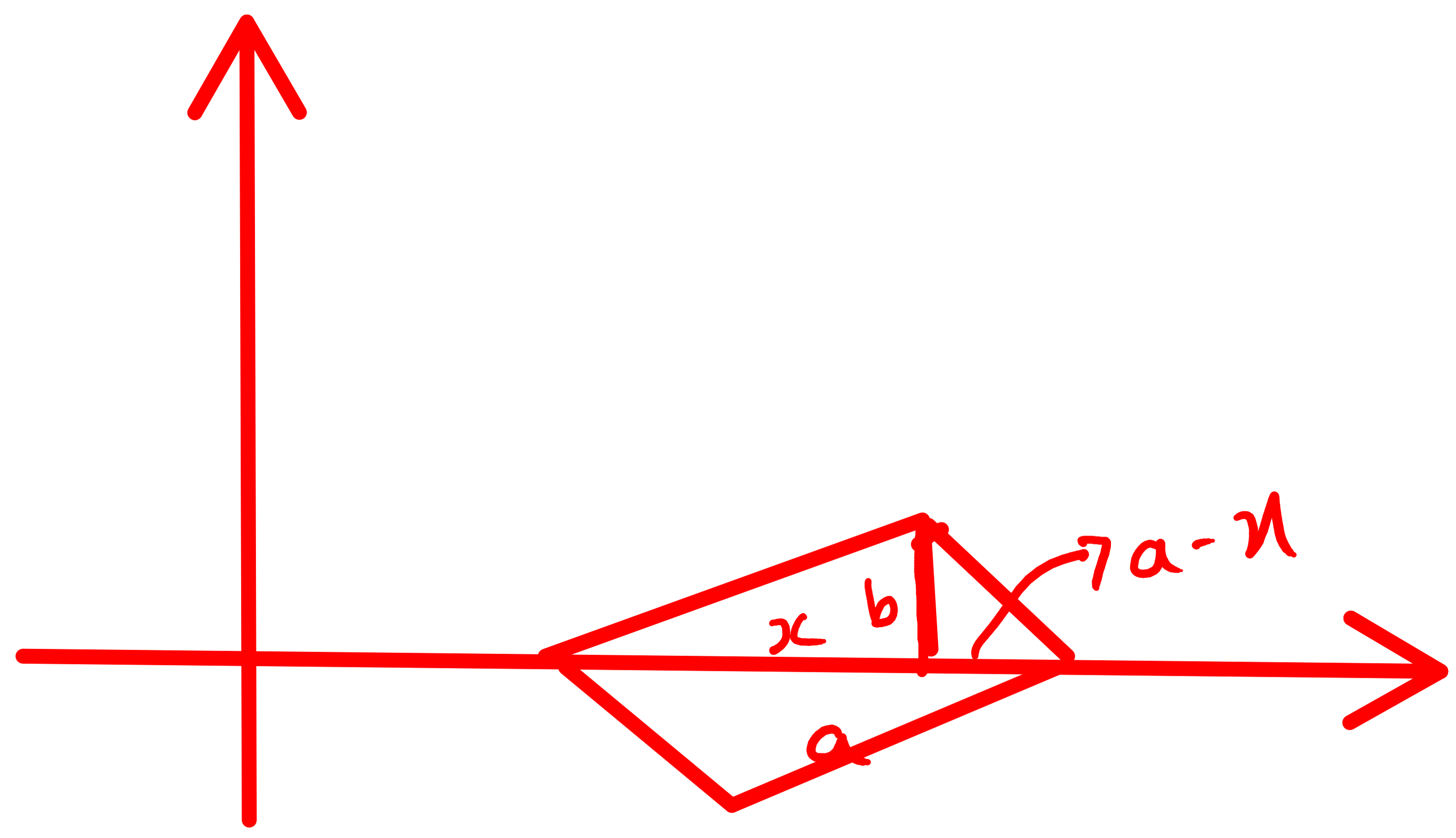
$= \frac{1}{2}ab$



$\frac{1}{2}ab$

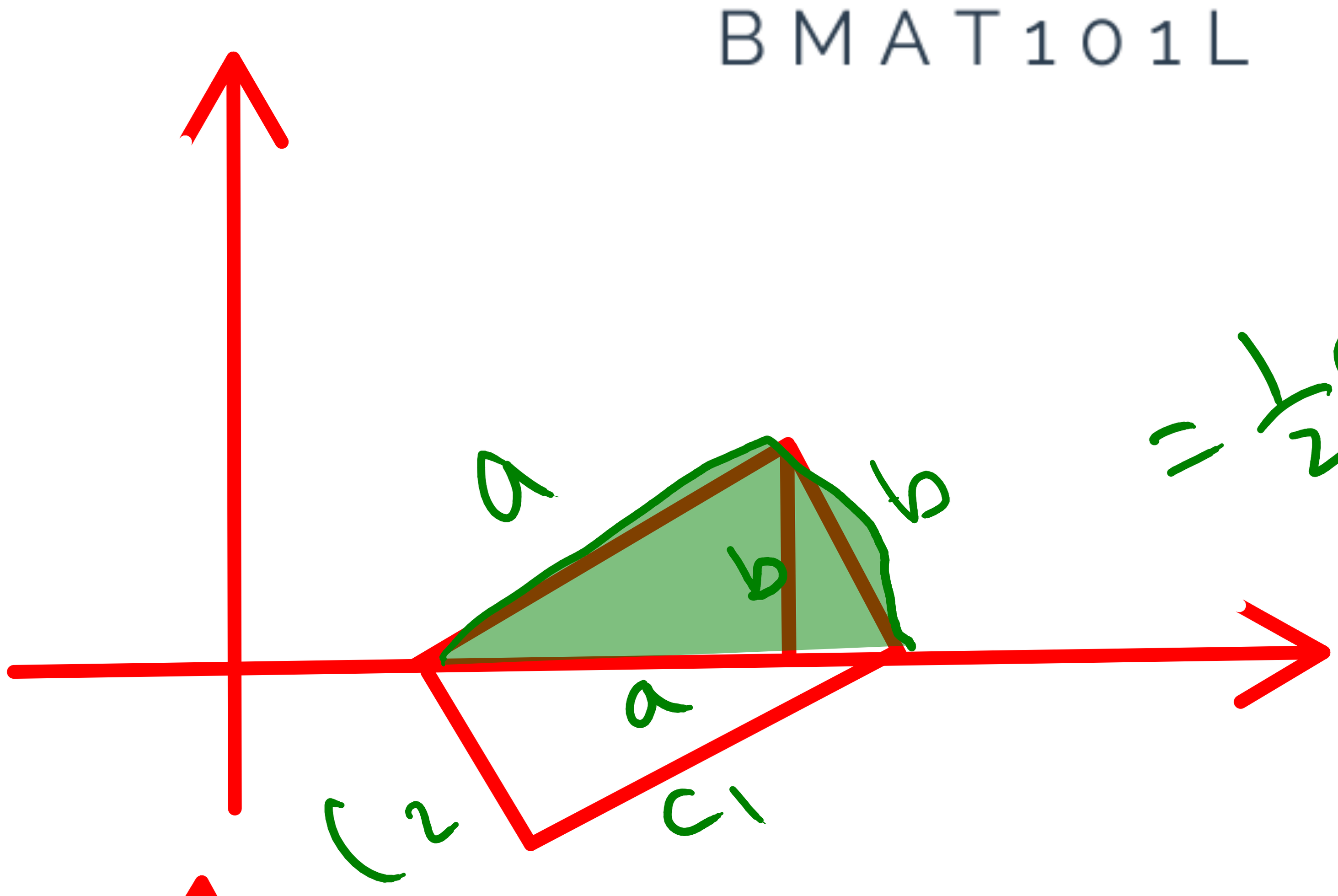


$s = \frac{a+b+c}{2}$ $\quad \sqrt{(s-a)(s-b)(s-c)s}$



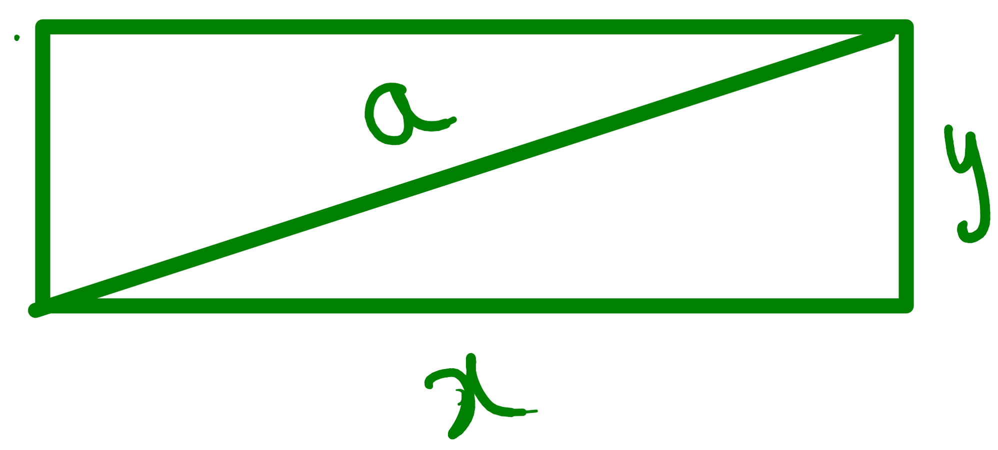
CALCULUS

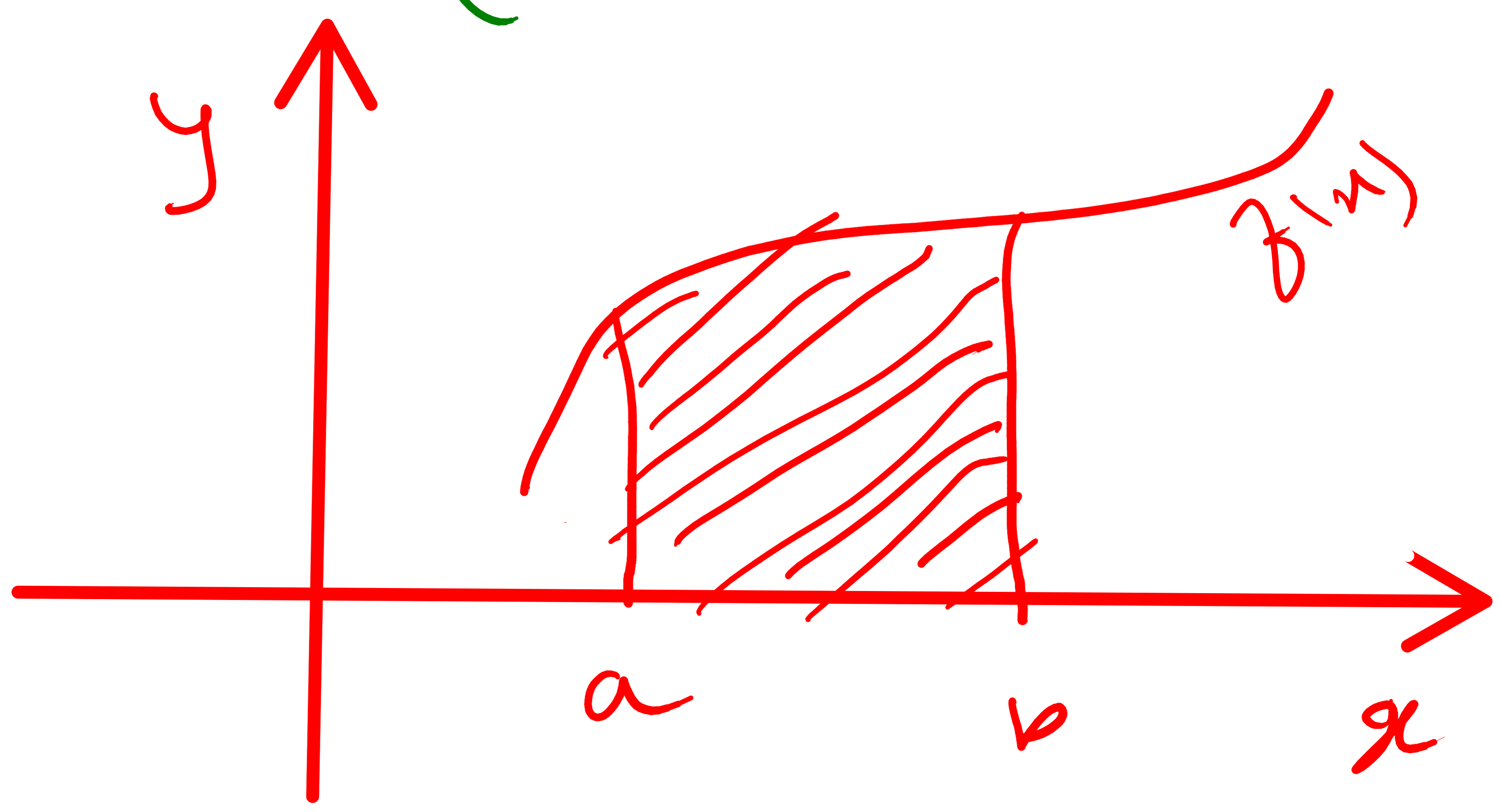
BMAT101L



$= \frac{1}{2}ab$

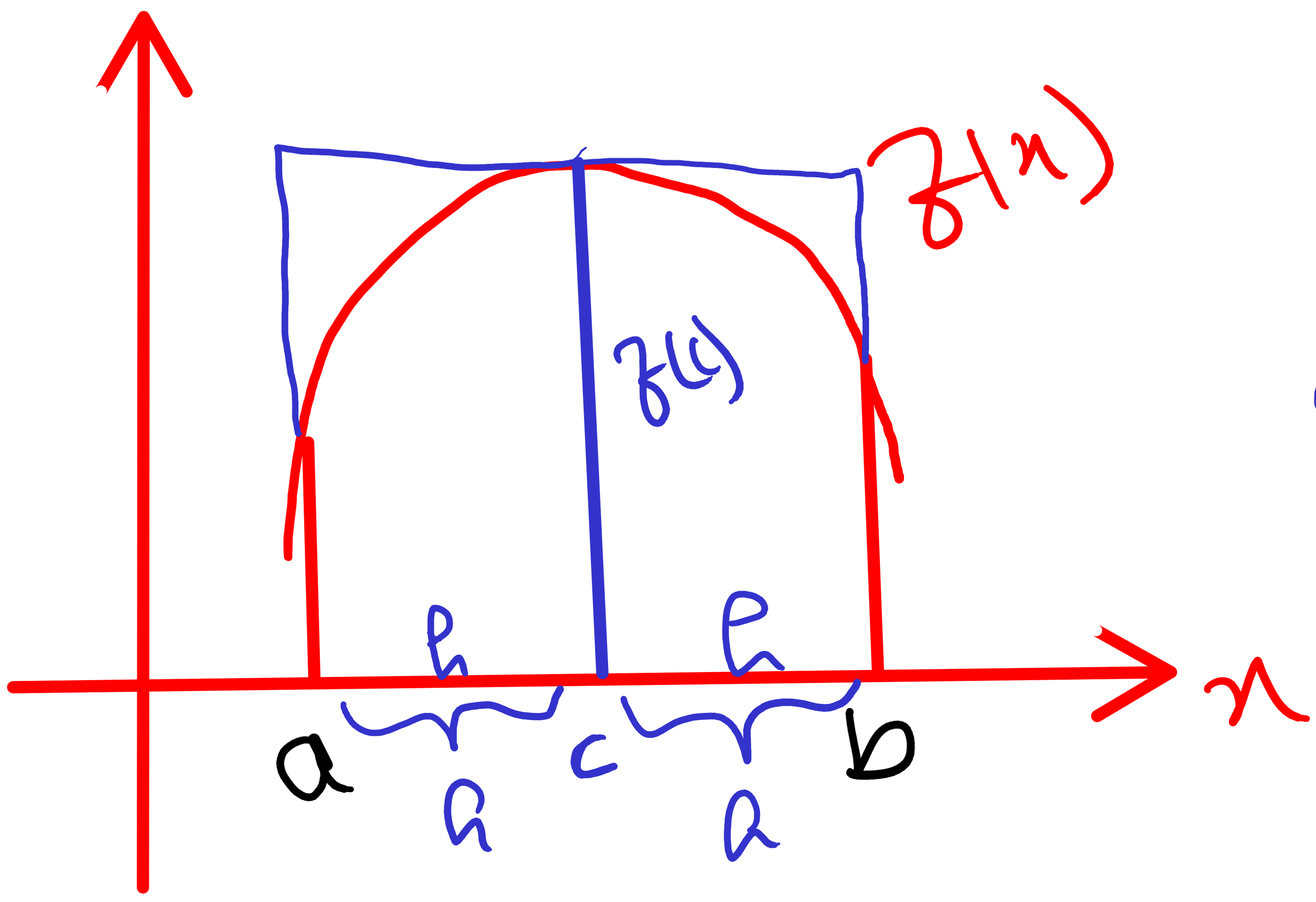
$a = \sqrt{x^2+y^2}$





$$\int_a^b f(x)\,dx =$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

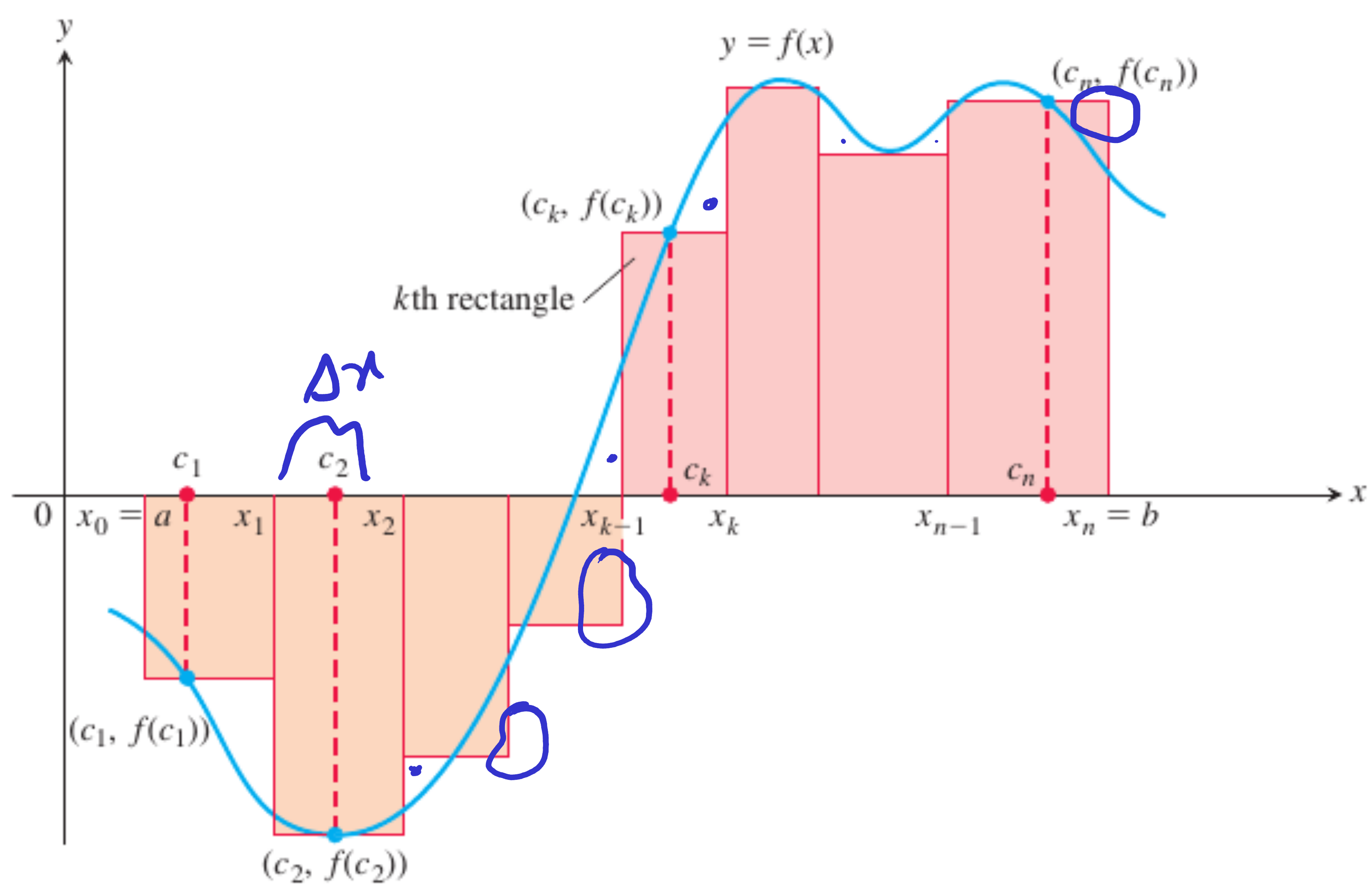


Area $= h\,f(c) + h\,f(c)$

$f'(a)$

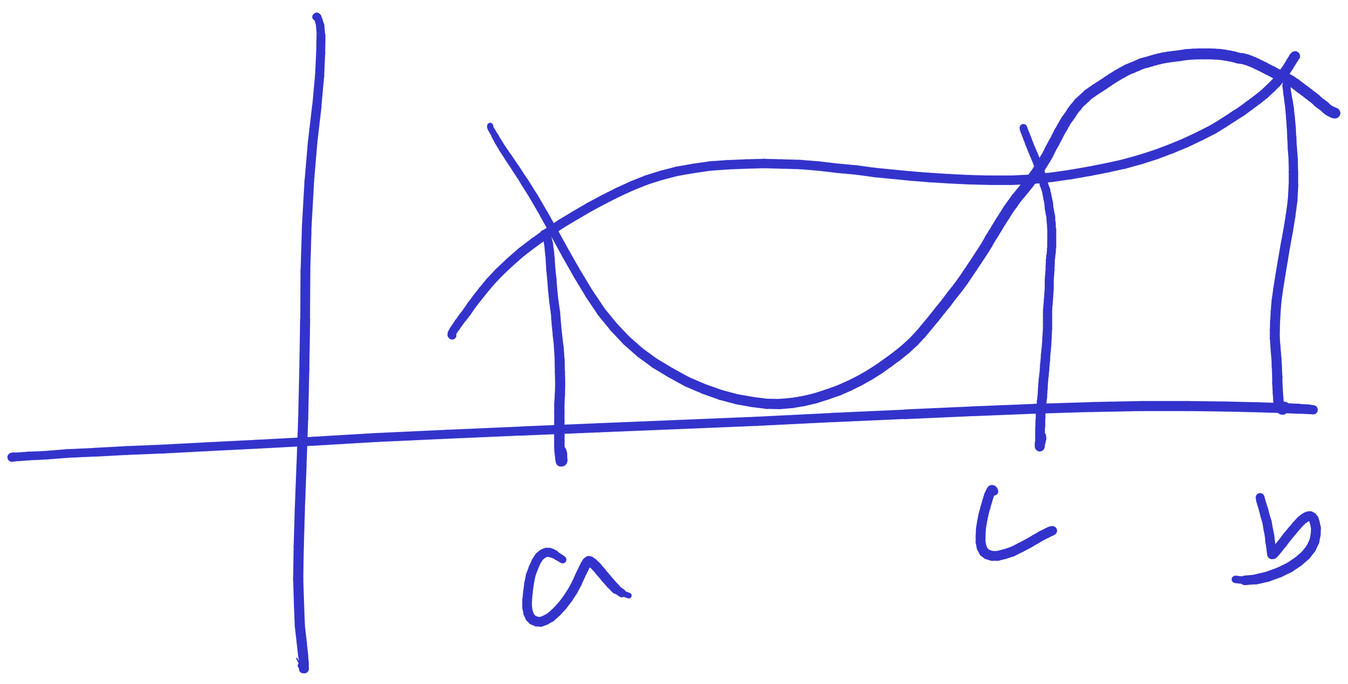
$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\,\Delta x$$

$$= \int_a^b f(x)\,dx$$

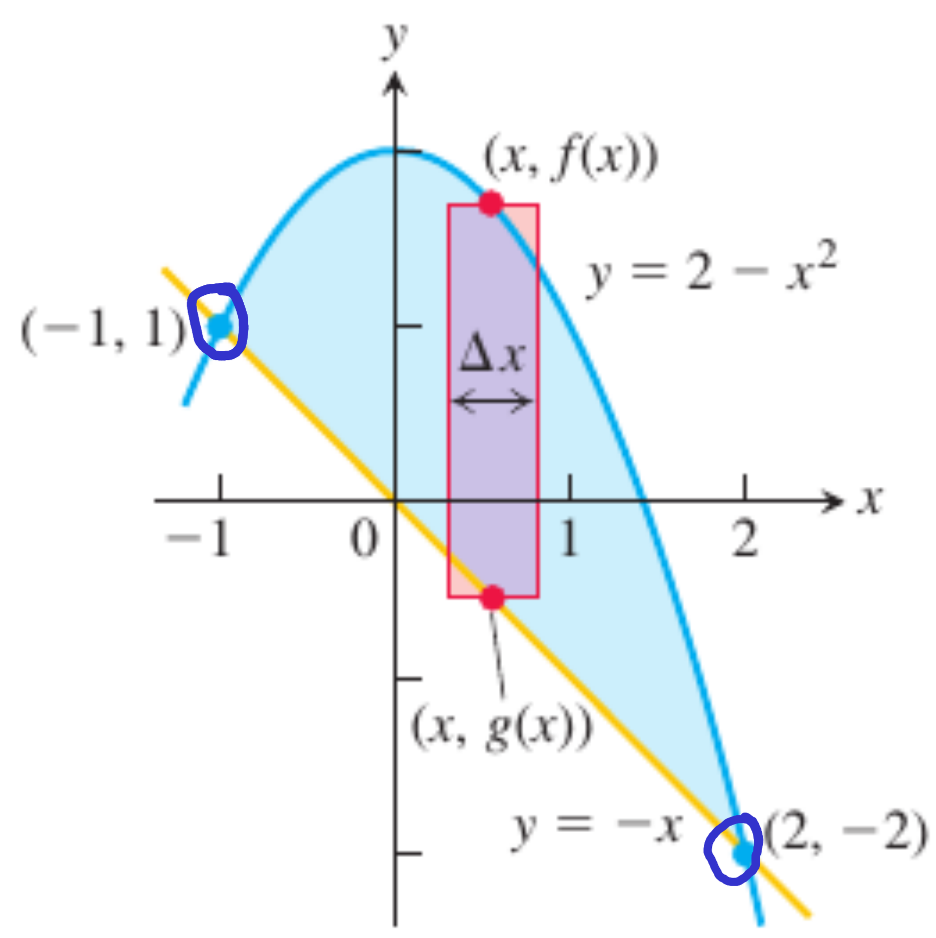


DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\,dx.$$



EXAMPLE 4 Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.



$$\left.\begin{array}{l} y = 2 - x^2 \\ y = -x \end{array}\right\} \Rightarrow -x = 2 - x^2$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = -1, 2$$

$$y = 1 \quad -2$$

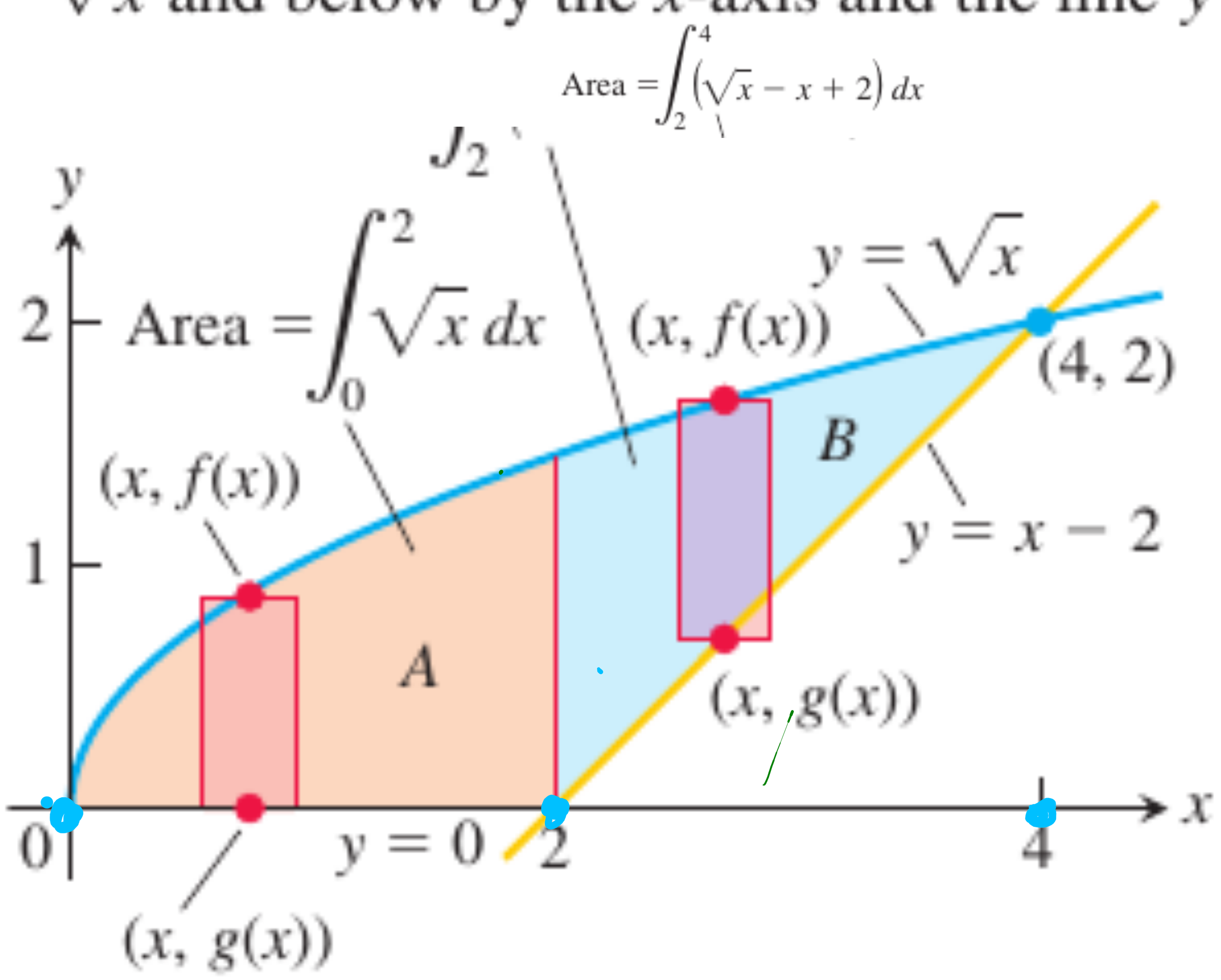
$$(-1, 1) \quad (2, -2)$$

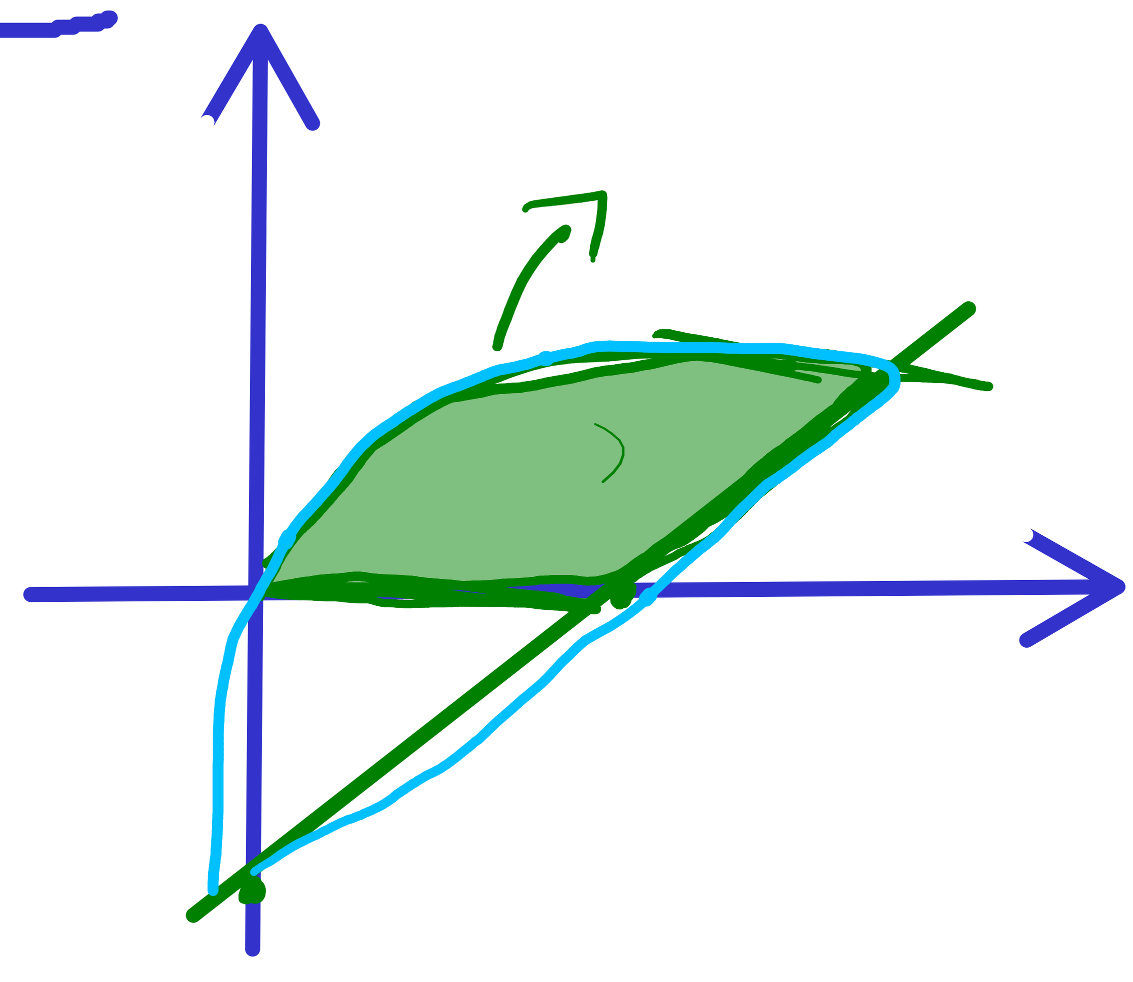
$$\text{Area} = \int_{-1}^{2} (f - g)\,dx = \int_{-1}^{2} (2 - x^2) - (-x)\,dx$$

$$= 2x - \frac{x^3}{3} + \frac{x^2}{2}\Big|_{-1}^{2}$$

$$= 9/2$$

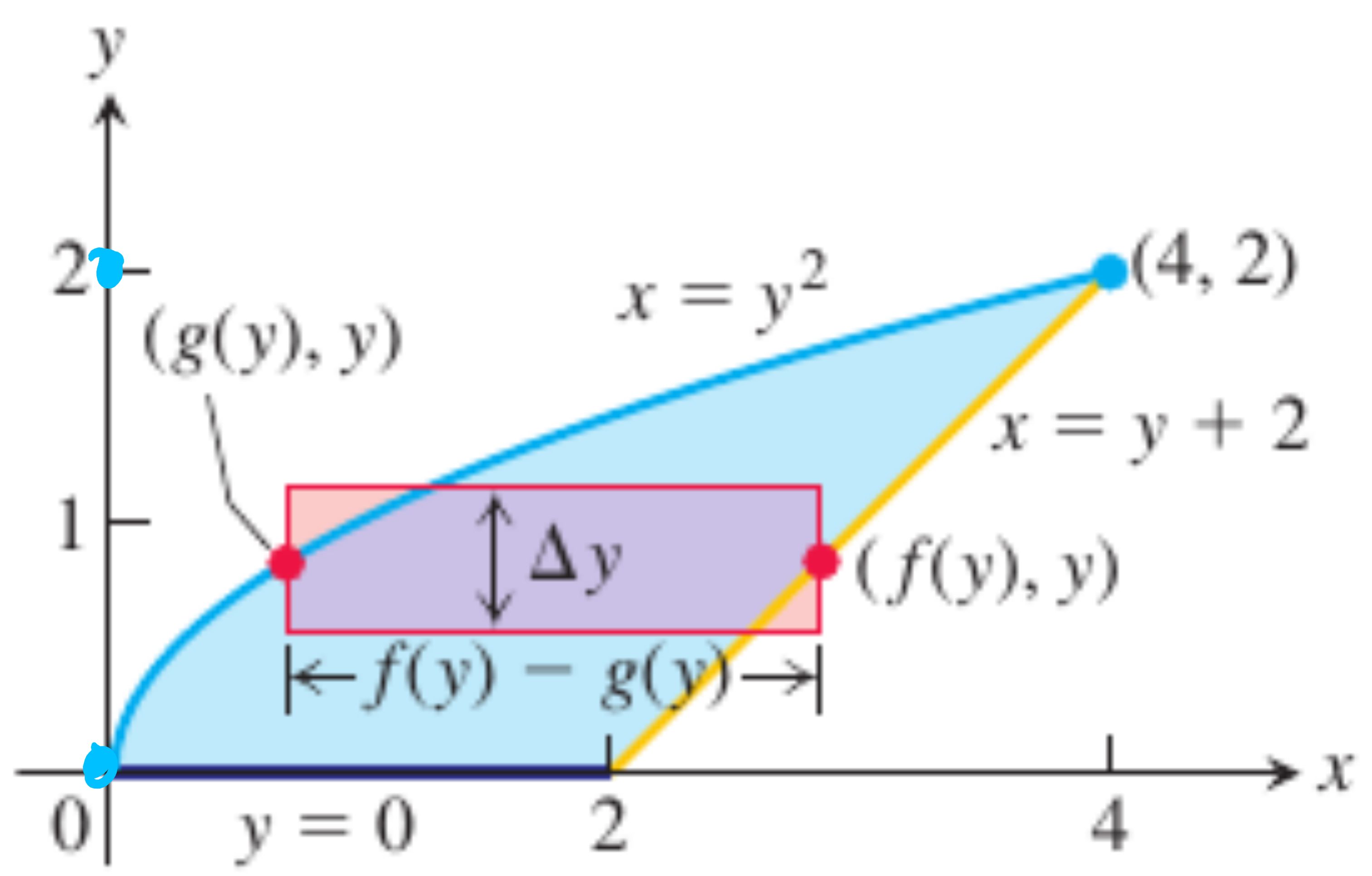
EXAMPLE 5 Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.





Ans = 10/3

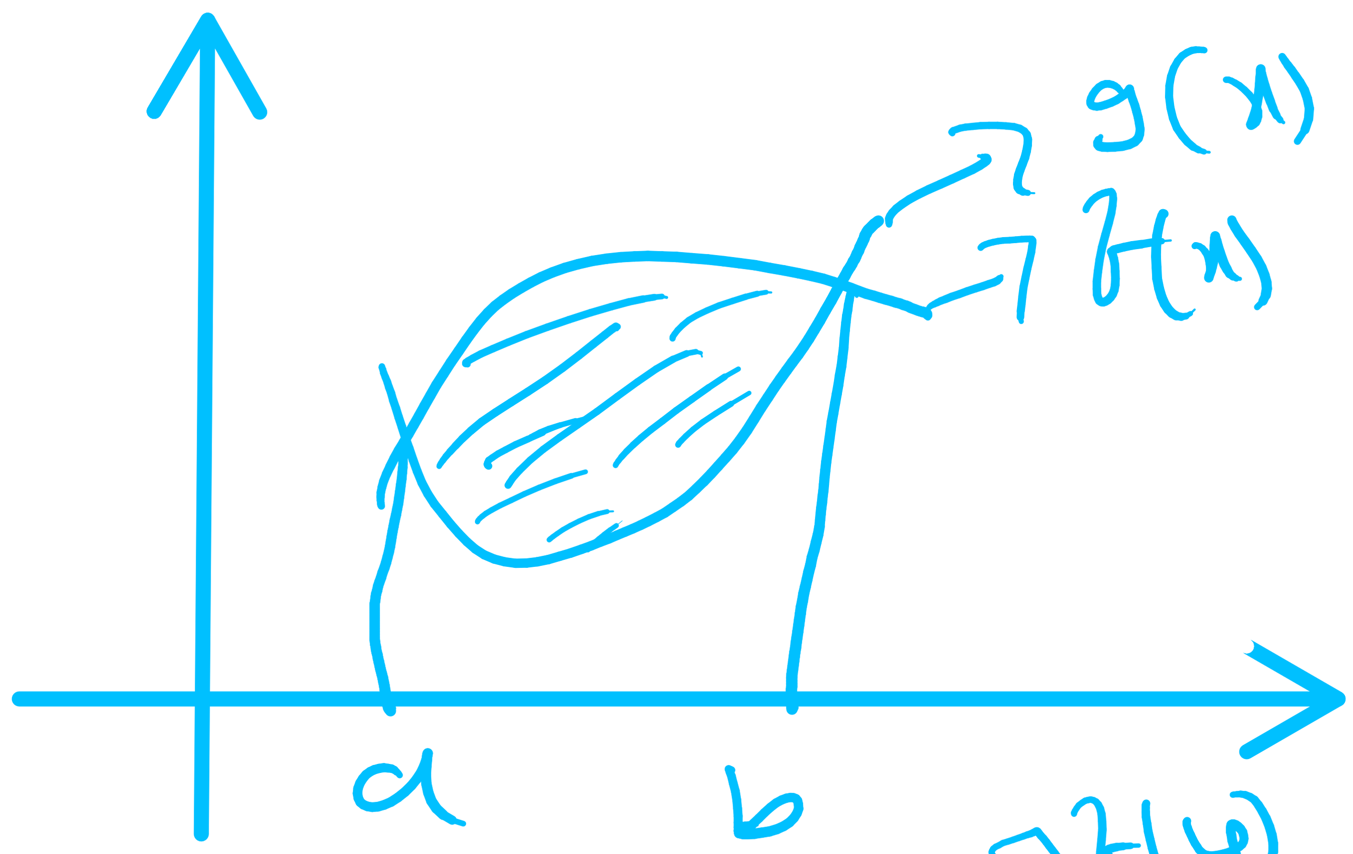
$\int_0^4 \sqrt{x} - (x-2)\,dx \quad \int_0^4 (\sqrt{x} - 0)\,dx$



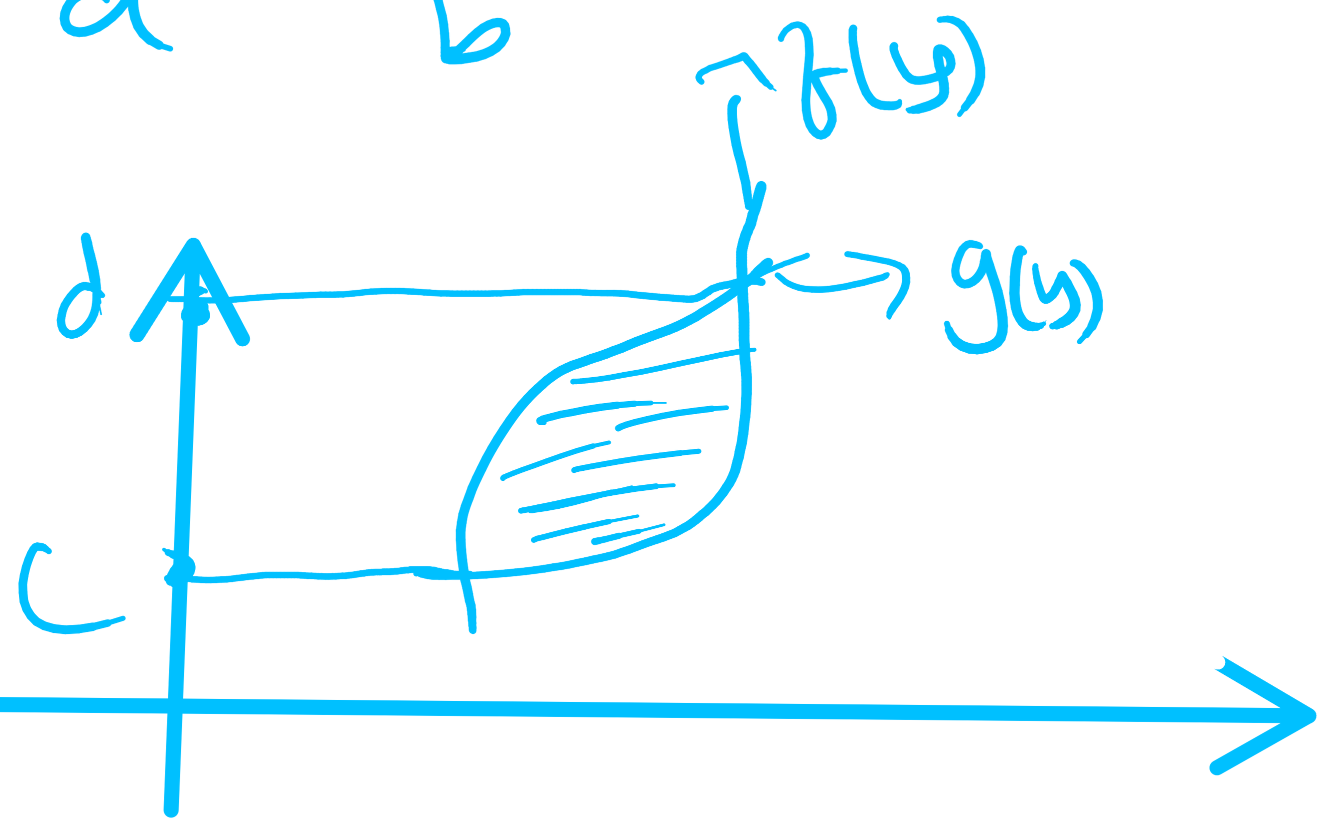
$\int_0^2 g(y) - f(y)\, dy$

$\int_0^2 [(y+2) - y^2]\, dy$

$= 10/3$

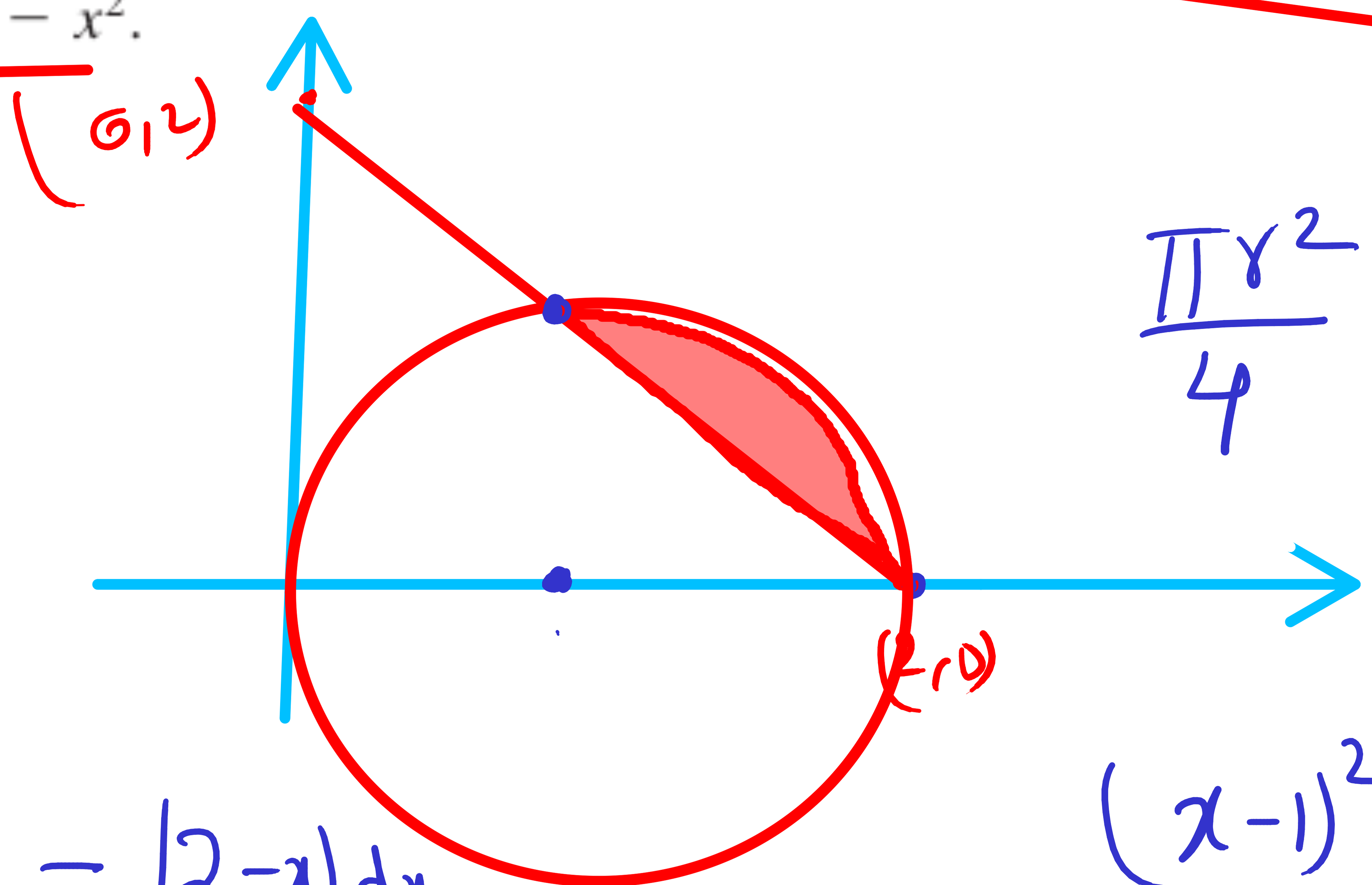


$$\int_a^b (f-g)\,dx$$



$$\int_c^d f(y)-g(y)\,dy$$

EXAMPLE 7 Find the area of the region bounded below by the line $y = 2 - x$ and above by the curve $y = \sqrt{2x - x^2}$.



$$y^2 + x^2 - 2x = 0$$

$$(x-1)^2 + y^2 = 1$$

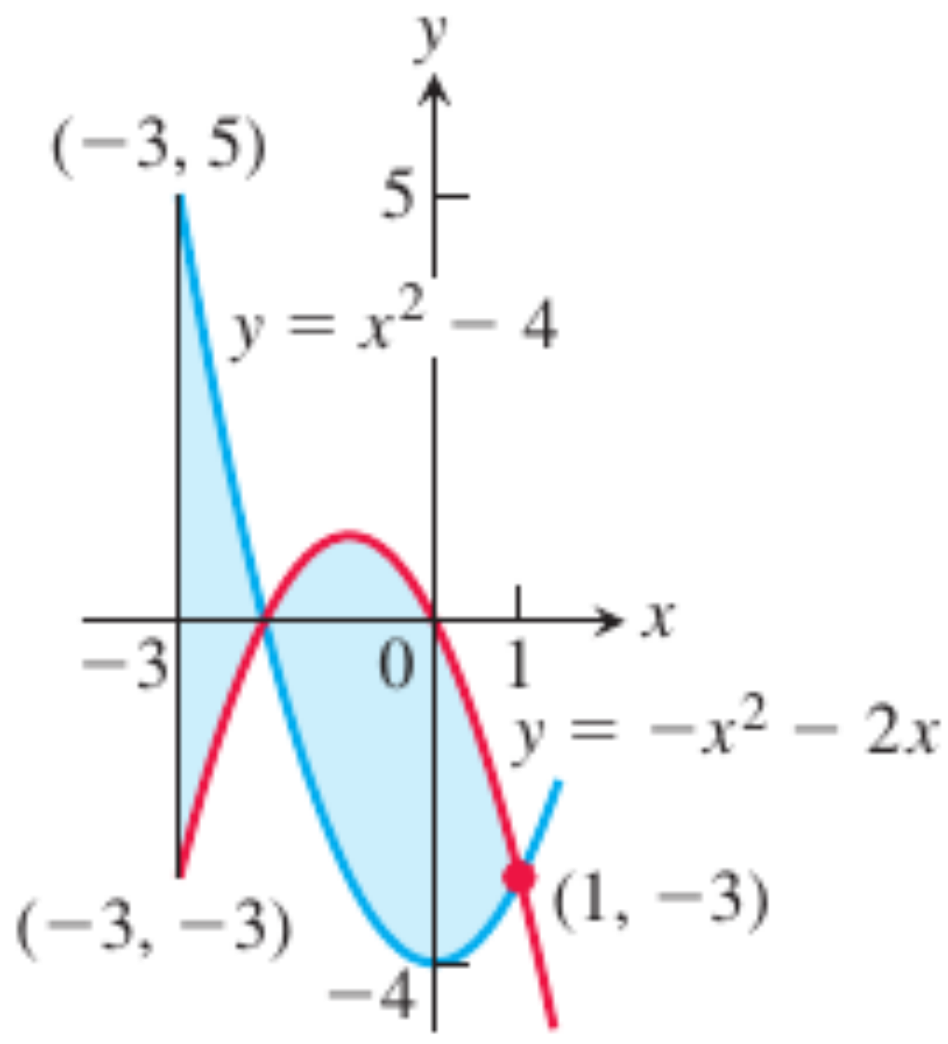
$$\frac{\pi r^2}{4} -$$

$$\int_1^2 \left(\sqrt{2x - x^2}\right) - (2-x)\,dx = 0.285$$

$$(x-1)^2 = 1 - y^2$$

$$x = \sqrt{1-y^2} + 1$$

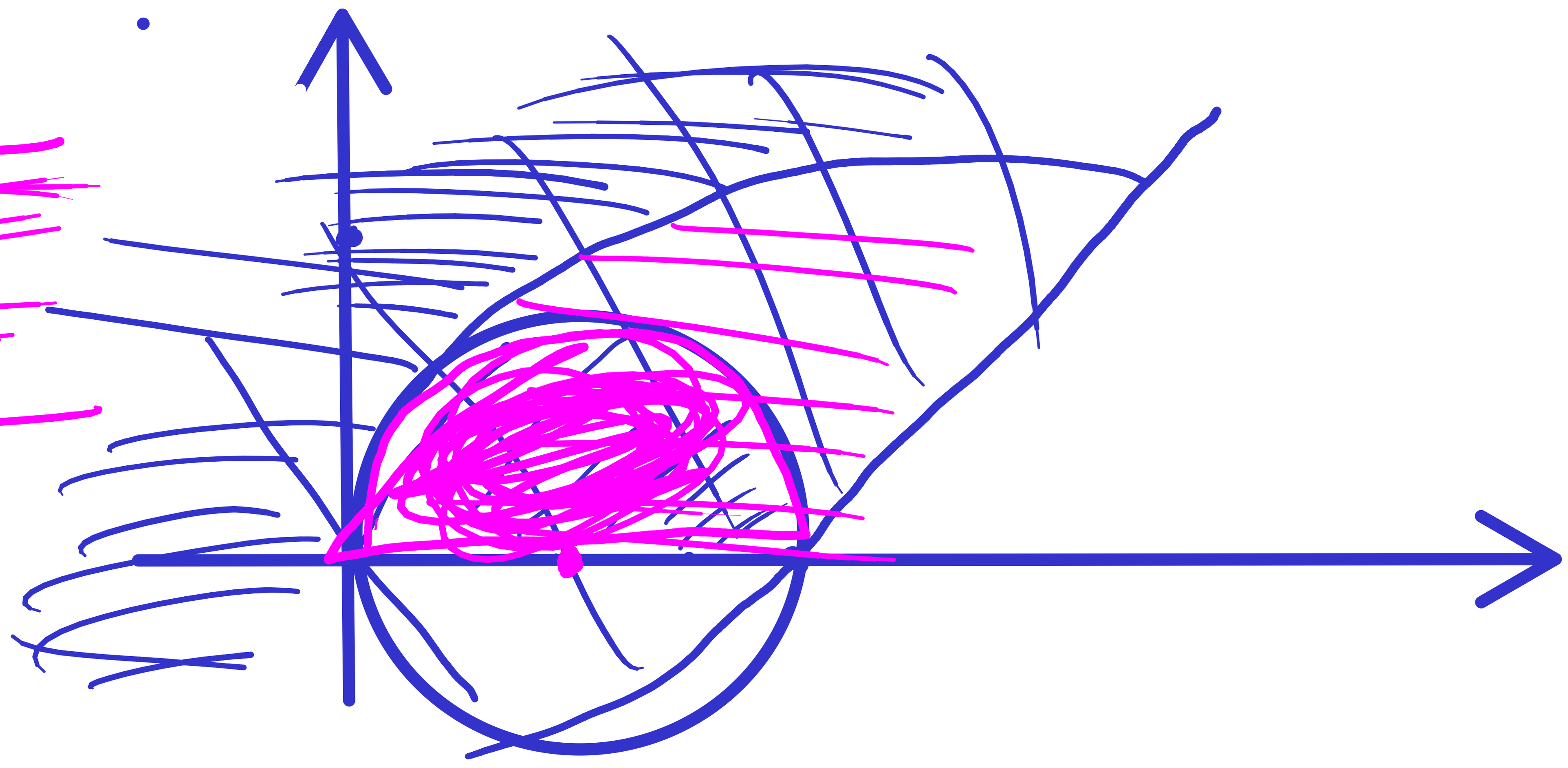
$$\int_0^1 \left(1 + \sqrt{1-y^2}\right) - (2-y)\,dy = \frac{\pi}{4} - \frac{1}{2} \simeq 0.285$$

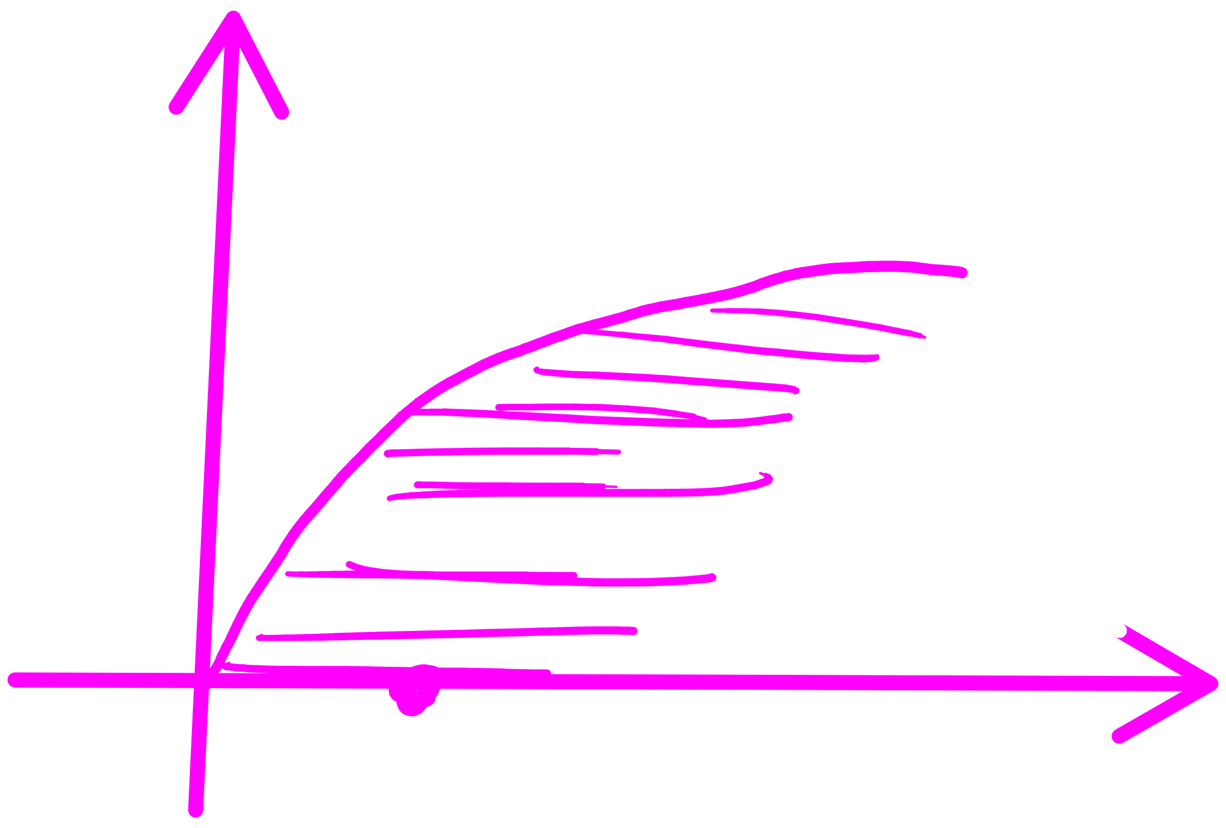


$\boxed{y^2 \le 4x}$ (2,0)

$x^2+y^2 \ge 2x$

$x \le y+2 \Rightarrow x-y \le 2$





$y^2 \le 4x$

$x^2+y^2 \ge 2x$

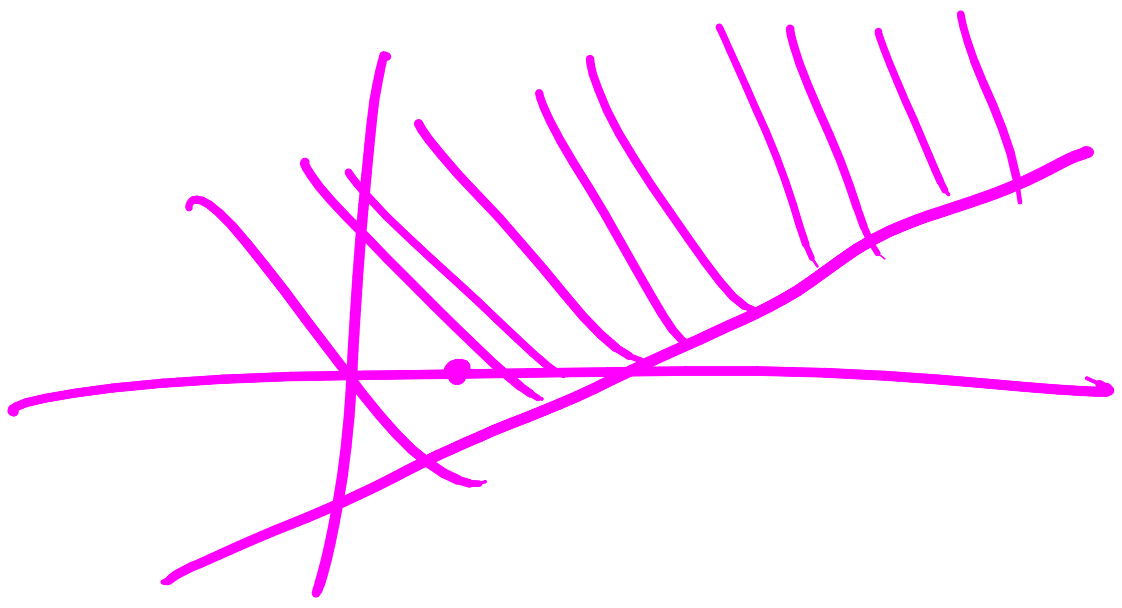
$1+0 \ge 2$

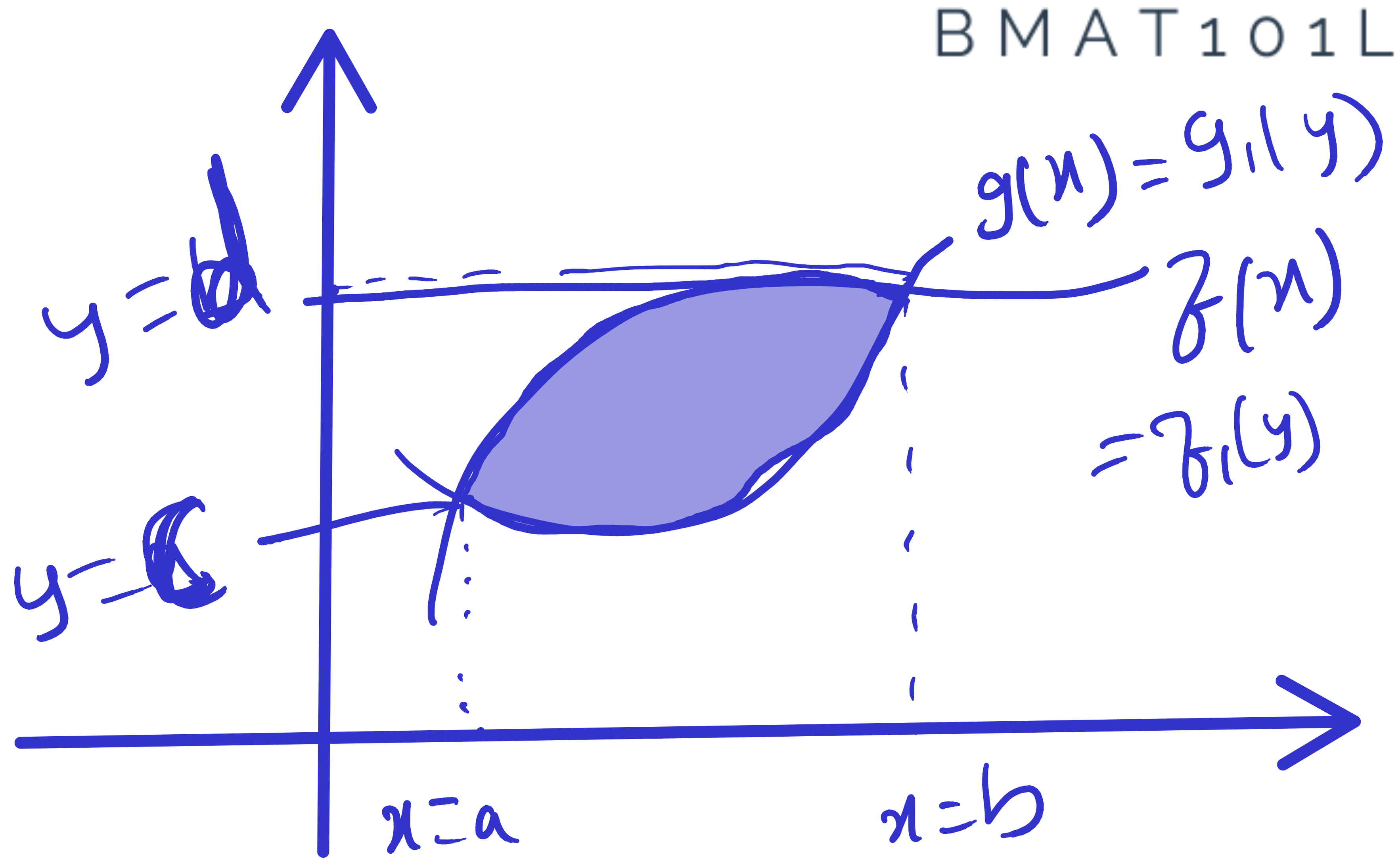
$1 \ge 2$

$x \le y+2$

$1 \le 0+2$

$1 \le 2$



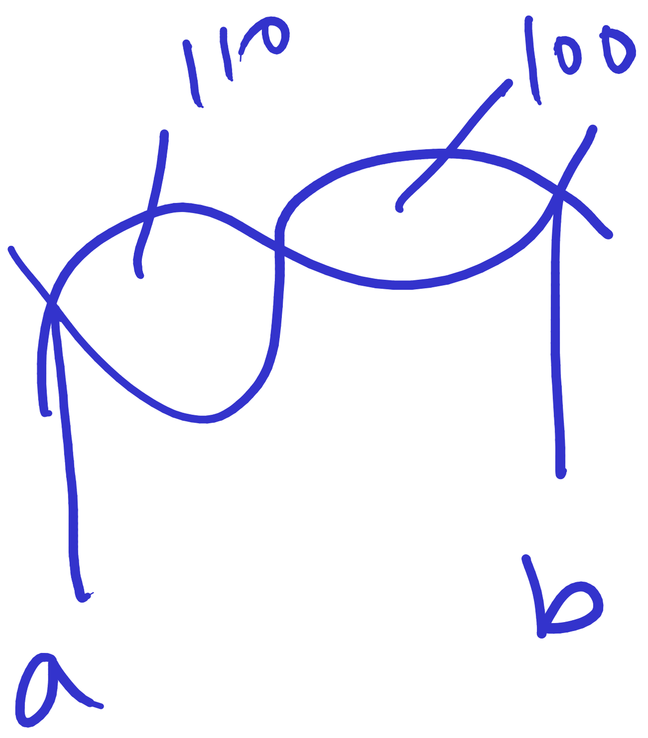


$x + y = 10$

$$\int_a^b (f - g)\, dx$$

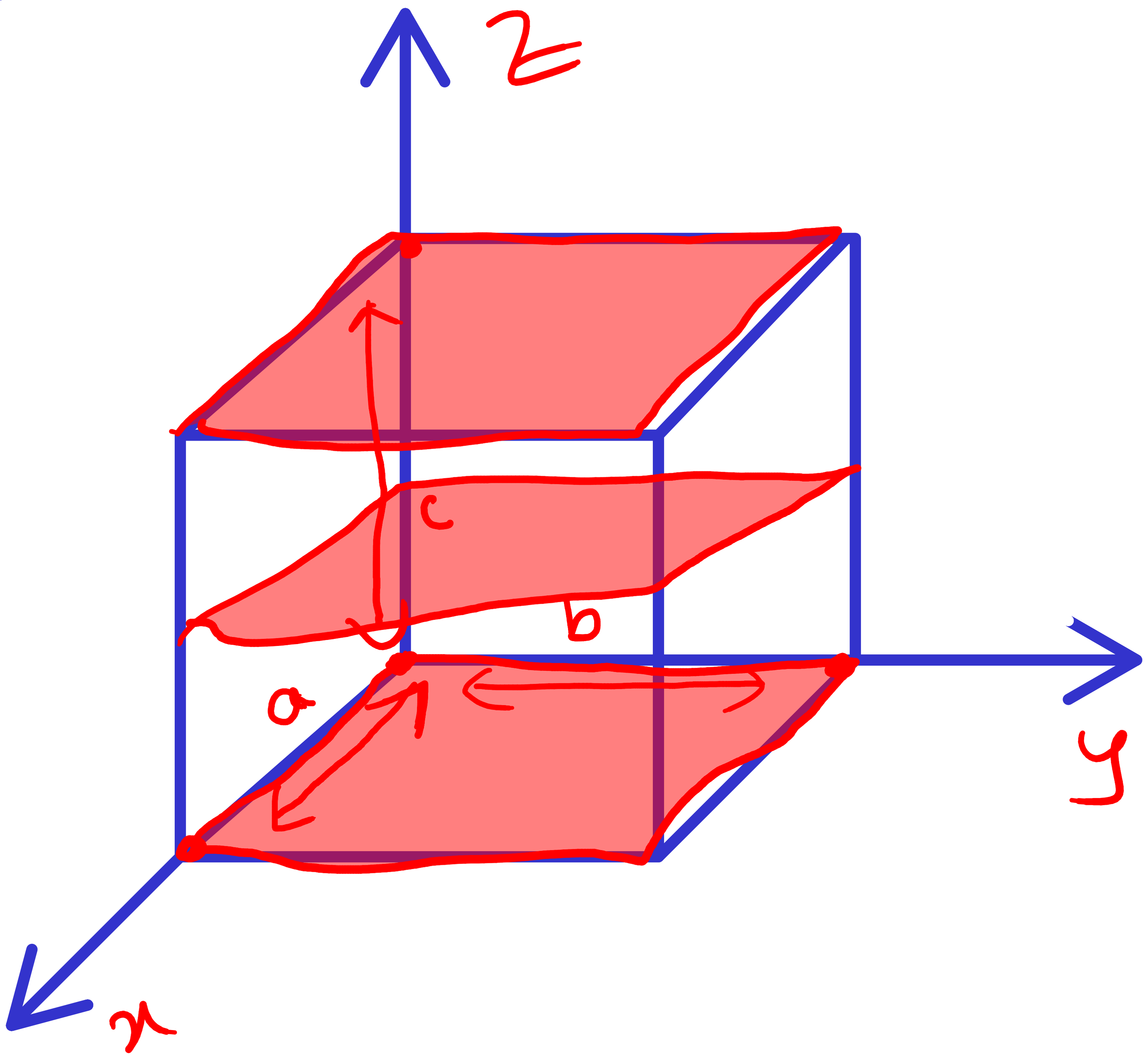
$f(x) \geq g(x)$

$$\int_c^d [g_1(y) - f_1(y)]\, dy$$

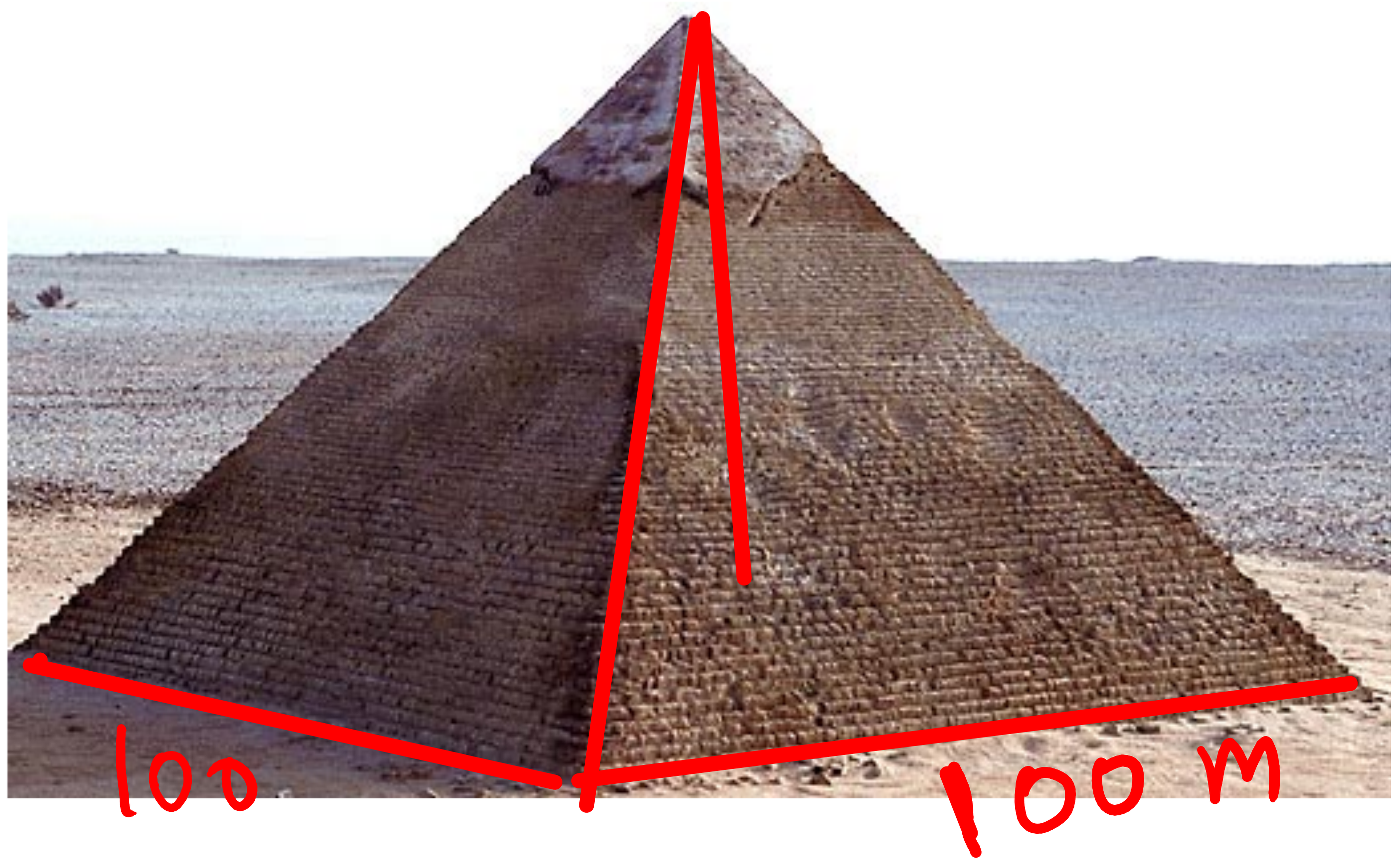


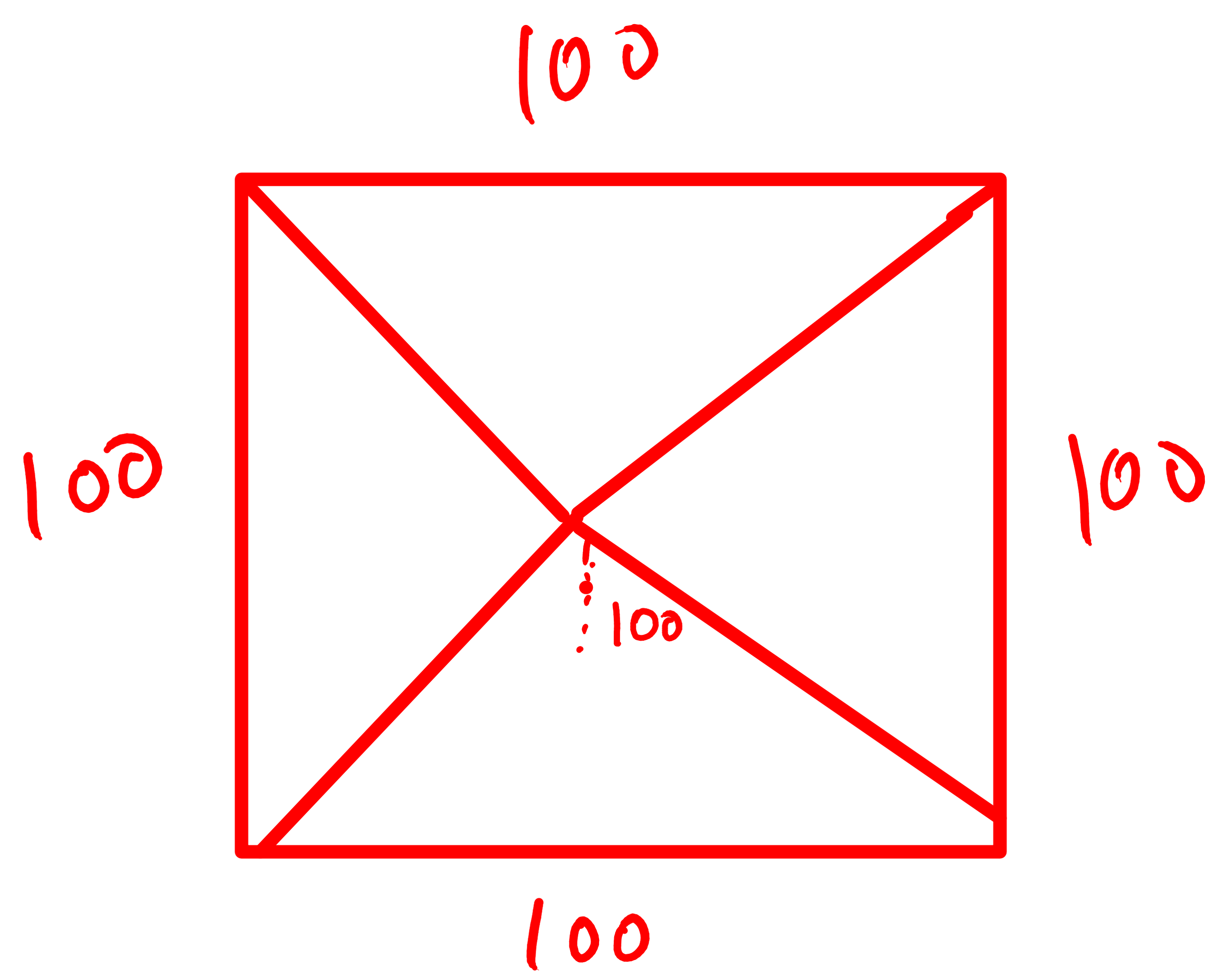
Volume = $\underline{abc}$?

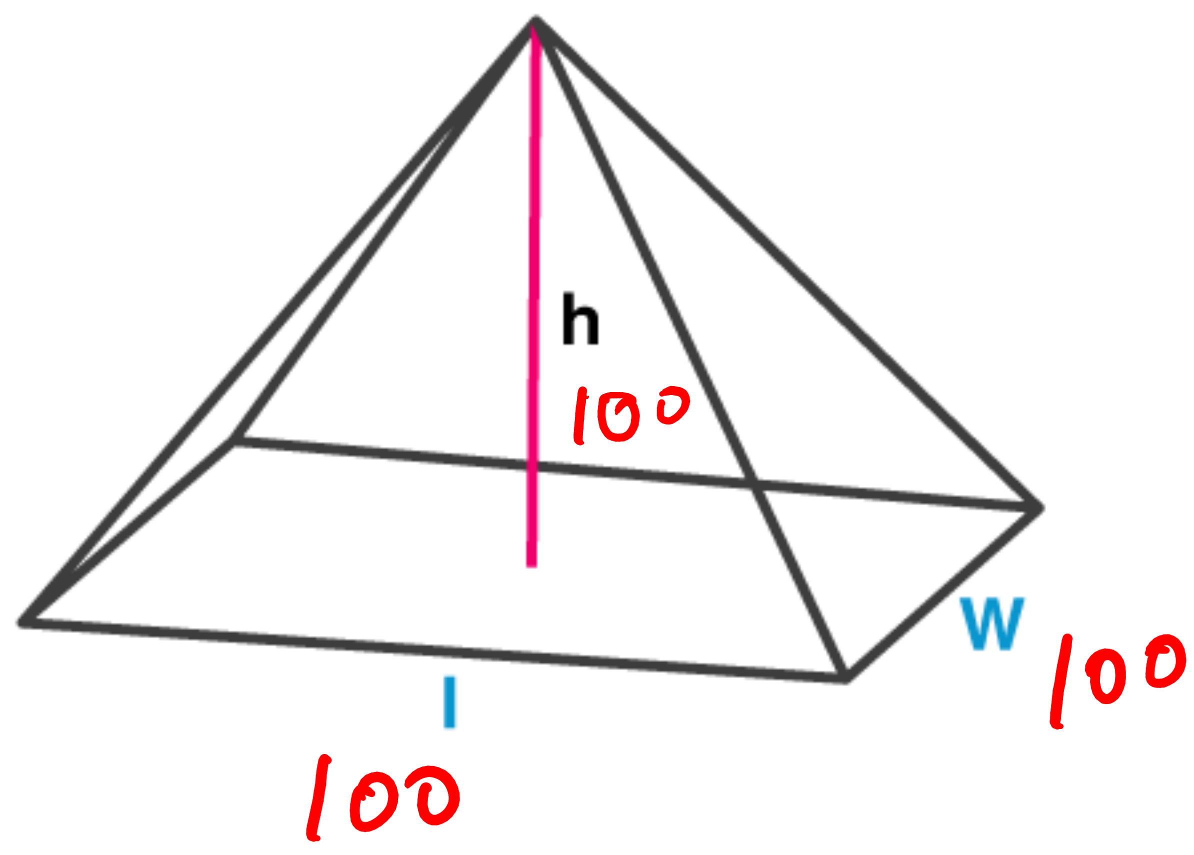
=



$$\int_0^c \text{Area of region } dx$$

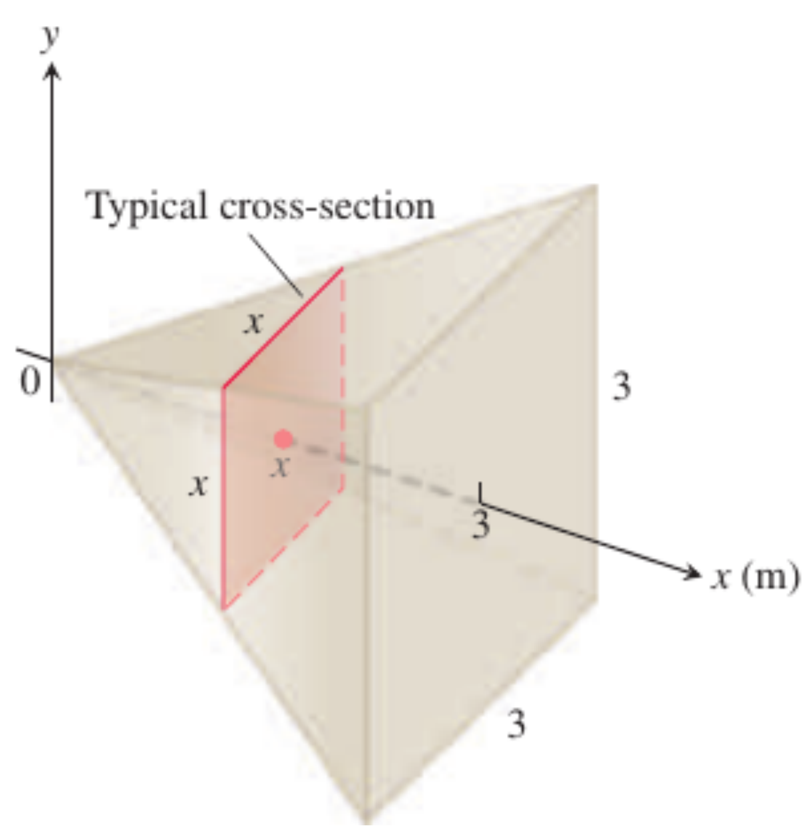






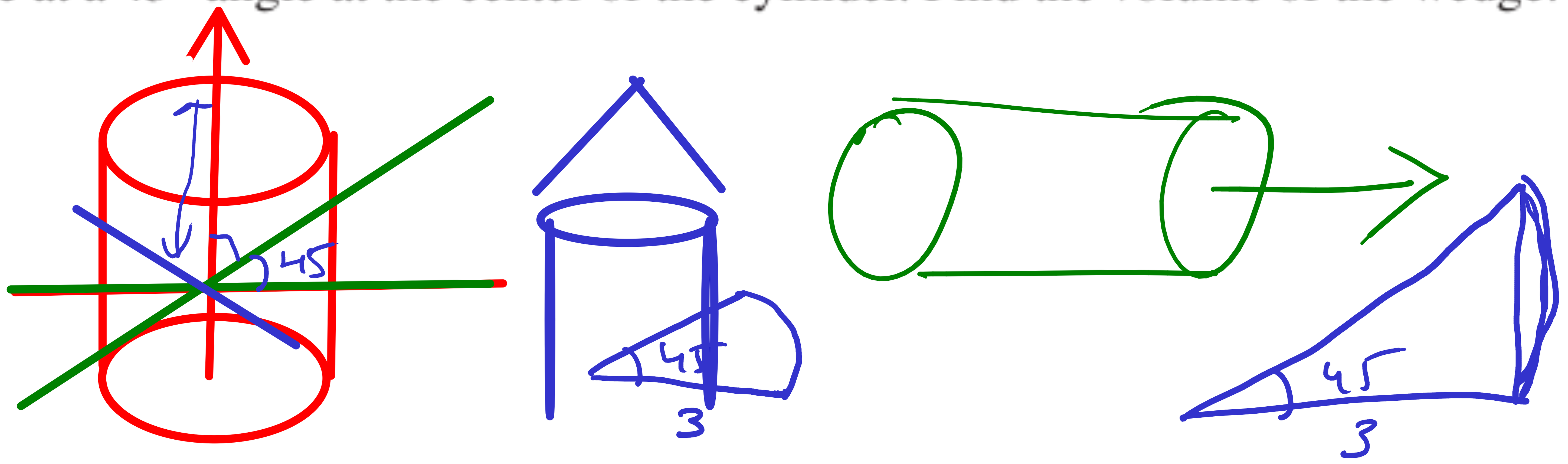
$$= \int_0^{100}$$

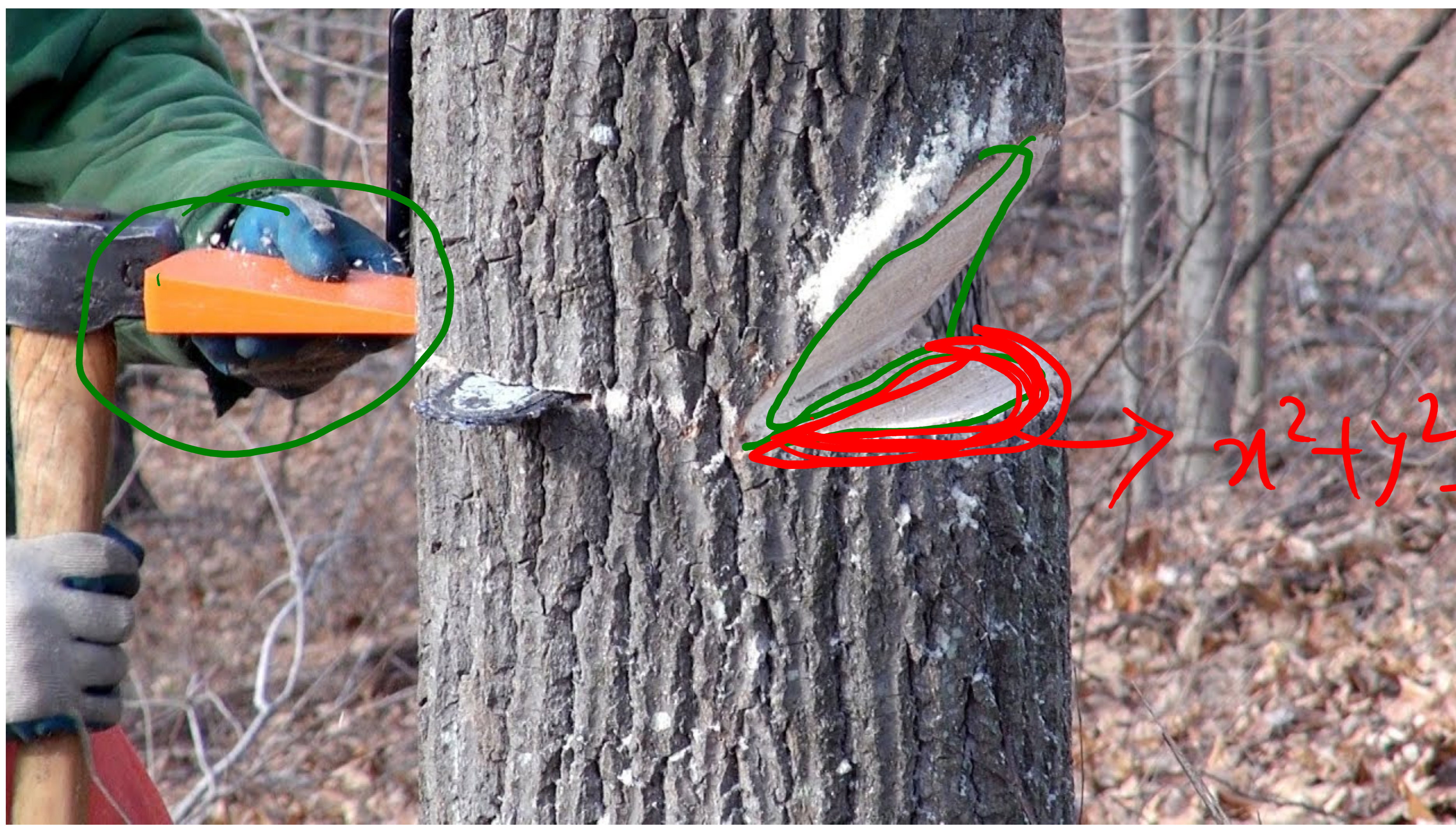
EXAMPLE 1 A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.



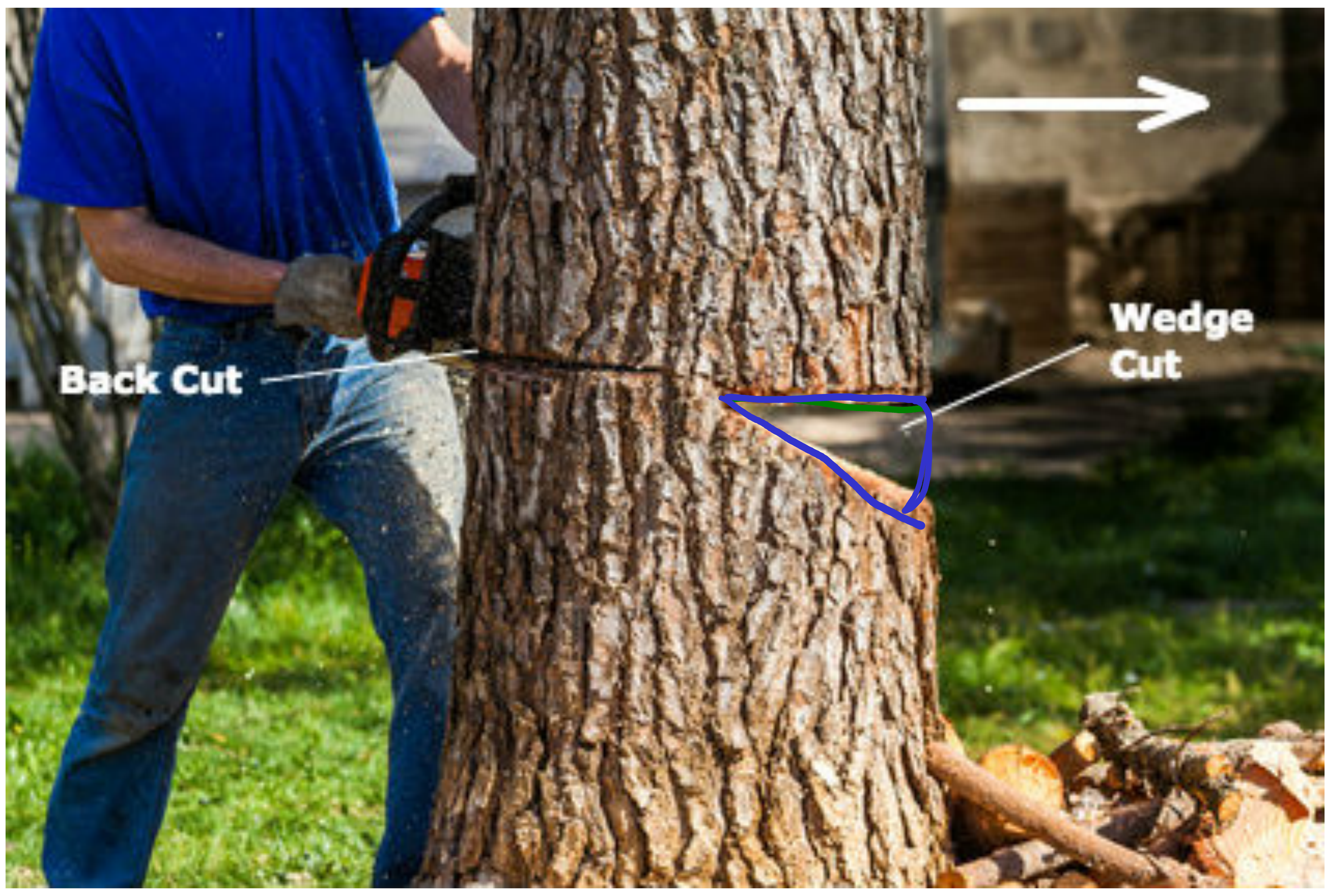
$$\text{Volume} = \int_0^3 x^2\,dx = \left.\frac{x^3}{3}\right|_0^3 = 9$$

EXAMPLE 2 A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

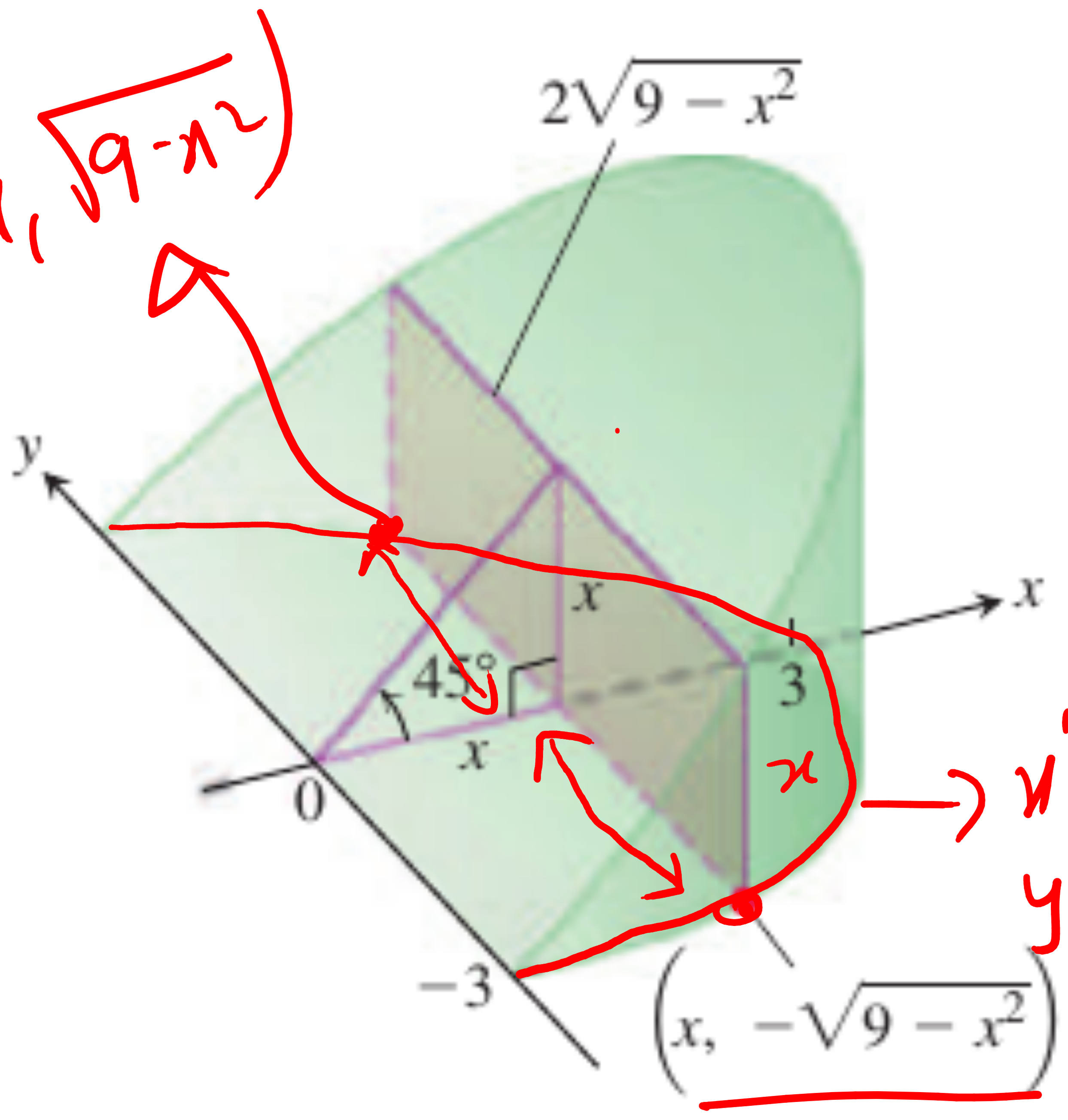




$\rightarrow x^2+y^2=9$



$(x, \sqrt{9-x^2})$



$\rightarrow x^2+y^2=9$

$y^2 = 9-x^2 =$

$$\int_0^3 \left(\text{Area of that rectangle}\right) dx$$

$$\int_0^3 \underbrace{2\sqrt{9-x^2}}_{\text{length}} \; \underbrace{x}_{\text{breadth}} \; dx$$

$$2\int_0^3 x\sqrt{9-x^2}\,dx = 18$$

$x = 3\cos(t)$

$dx = -3\sin(t)\,dt$

x	0	3
t	$\pi/2$	0

$$= -2\int_{\pi/2}^{0} 3\cos(t)\,3\sin(t)\,(-3\sin(t))\,dt$$

$3\cos(t) = 0$

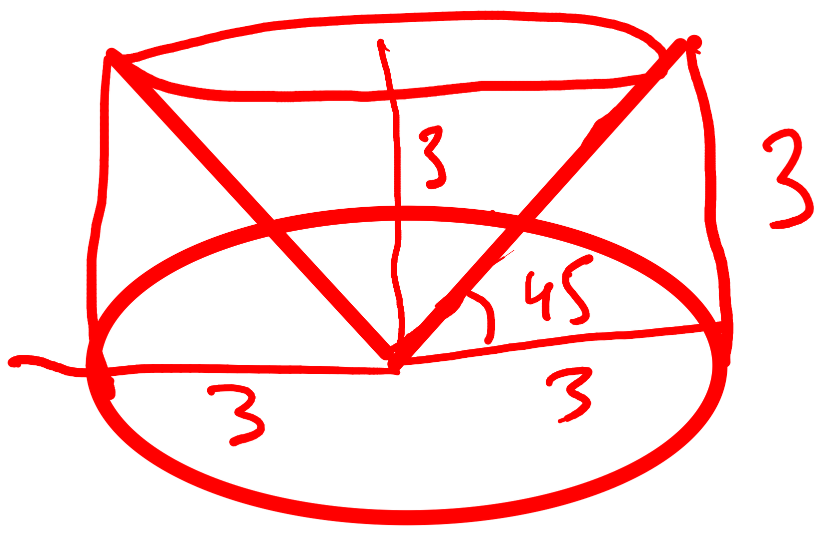
$t = \pi/2$

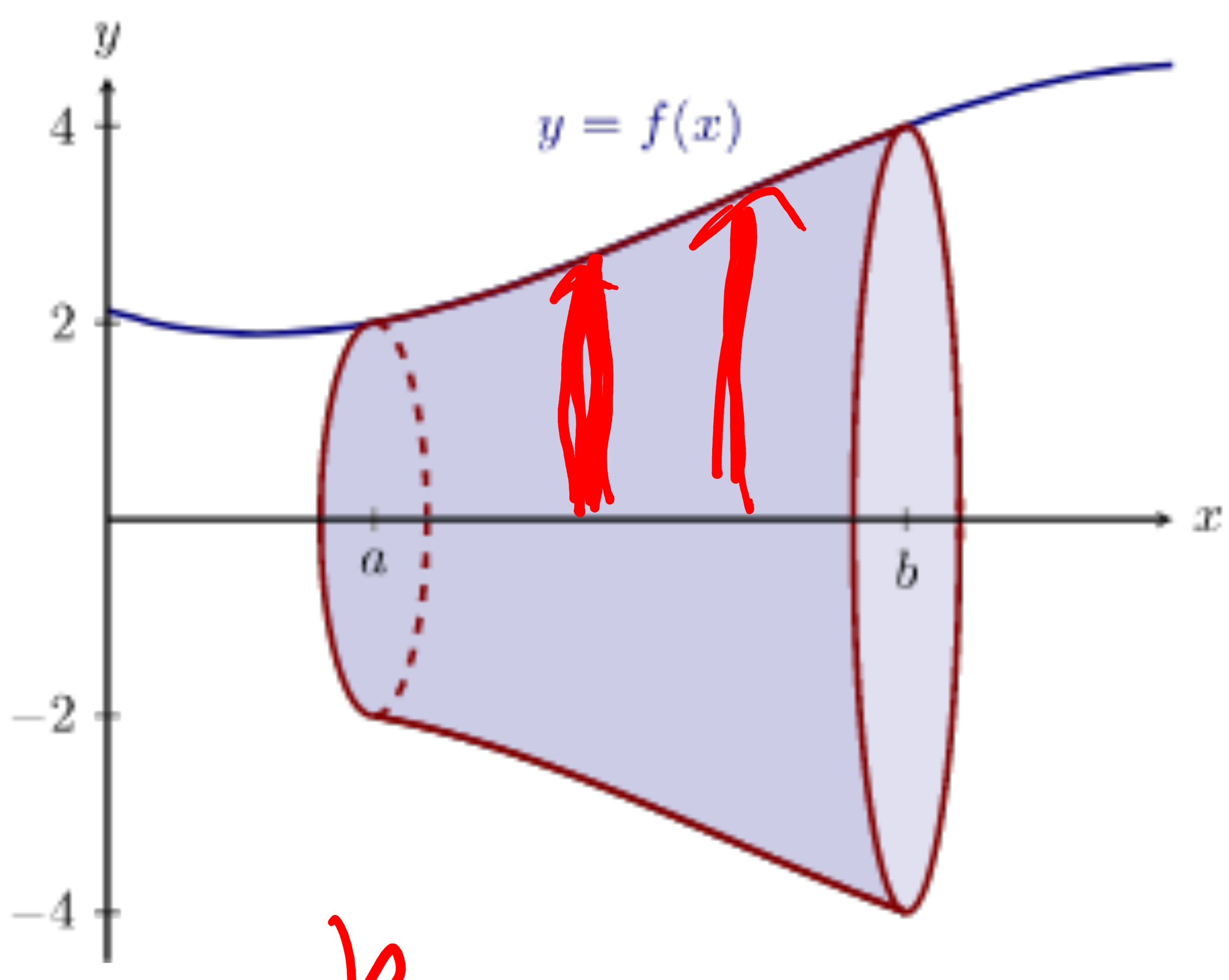
$3\cos(t) = 3$

$\cos(t) = 1$

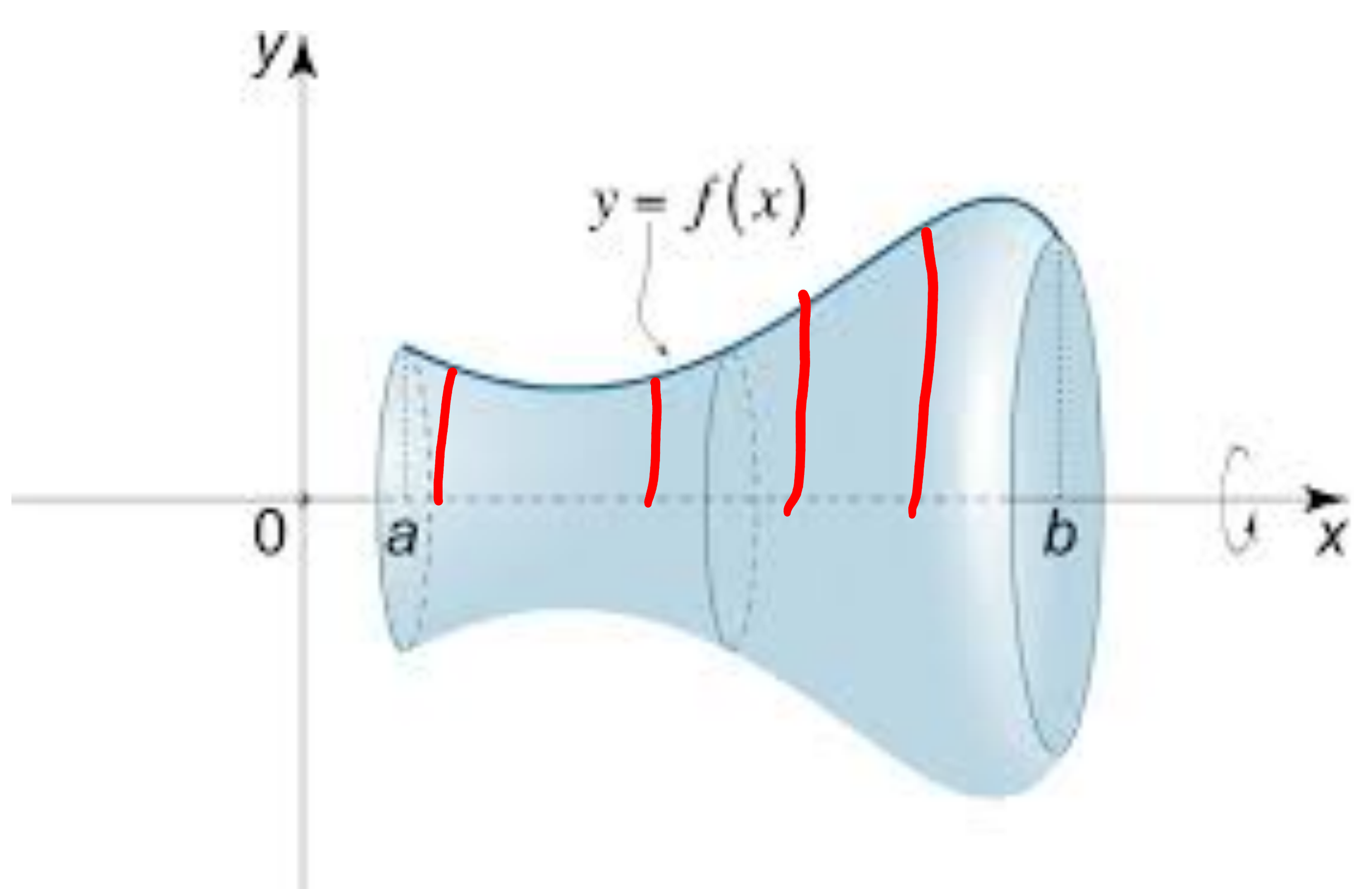
$t = 0$

$$= 2\int_0^{\pi/2} 27\cos(t)\sin^2(t)\,dt$$



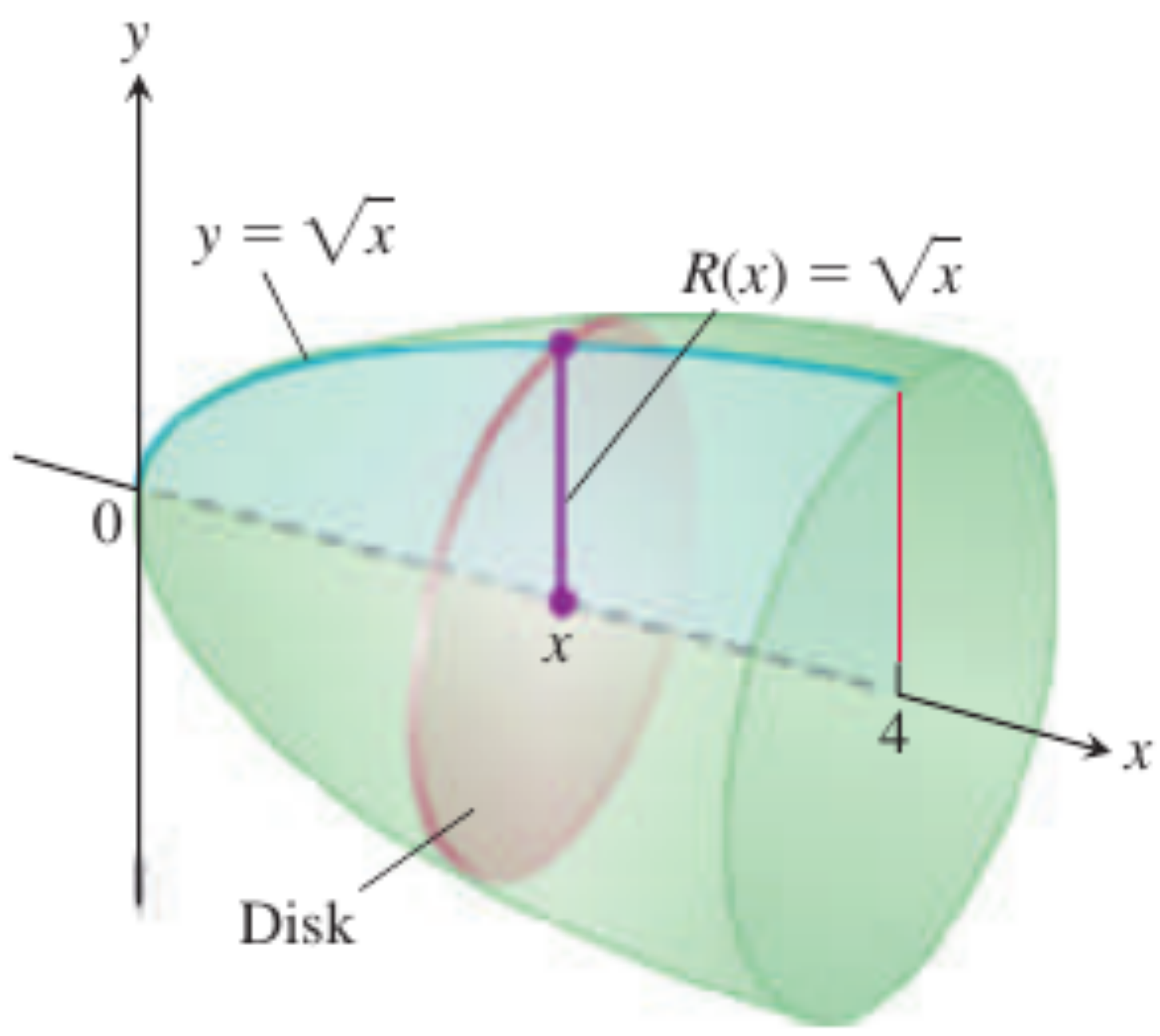


$$\int_a^b \pi\,[f(x)]^2\,dx$$



$$\int_a^b \pi\,[f(x)]^2\,dx$$

EXAMPLE 4 The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.



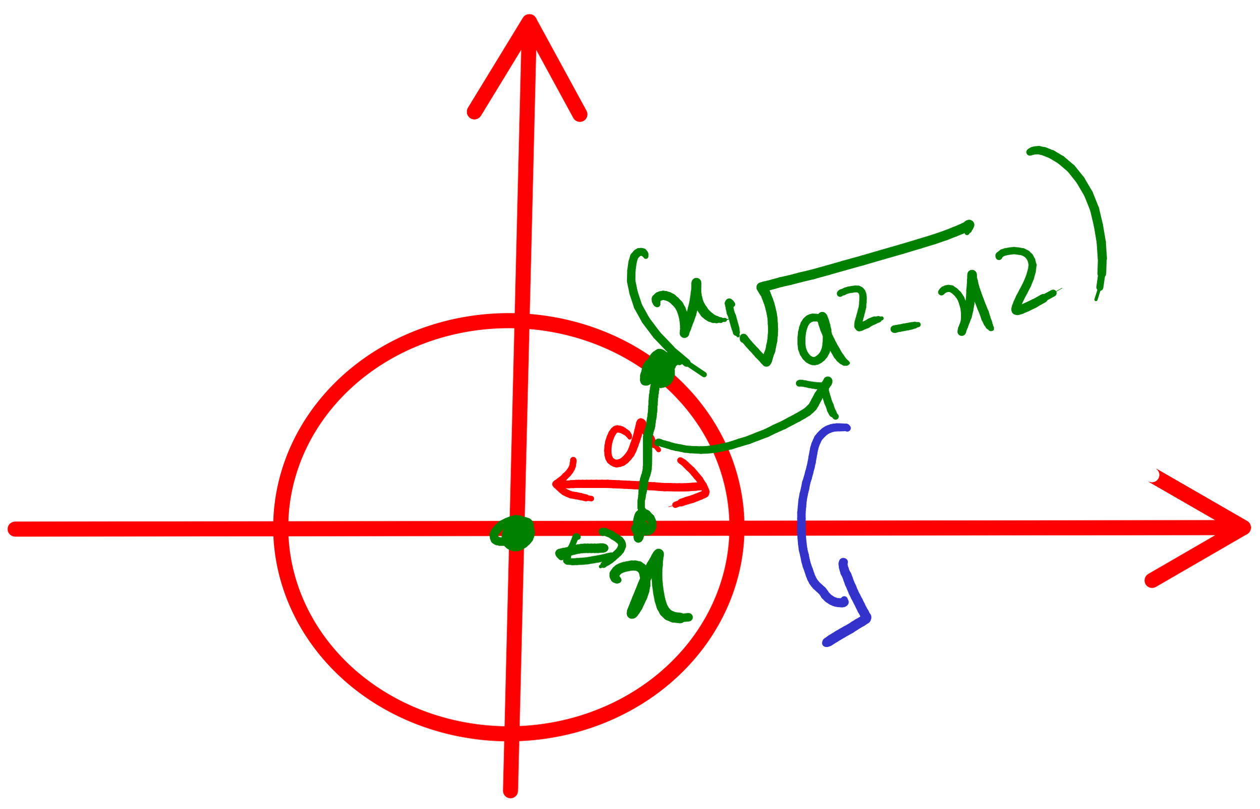
$$\text{volume} = \int_0^4 \pi (\sqrt{x})^2 dx$$

$$= 8\pi$$

EXAMPLE 5 The circle

$$x^2 + y^2 = a^2$$

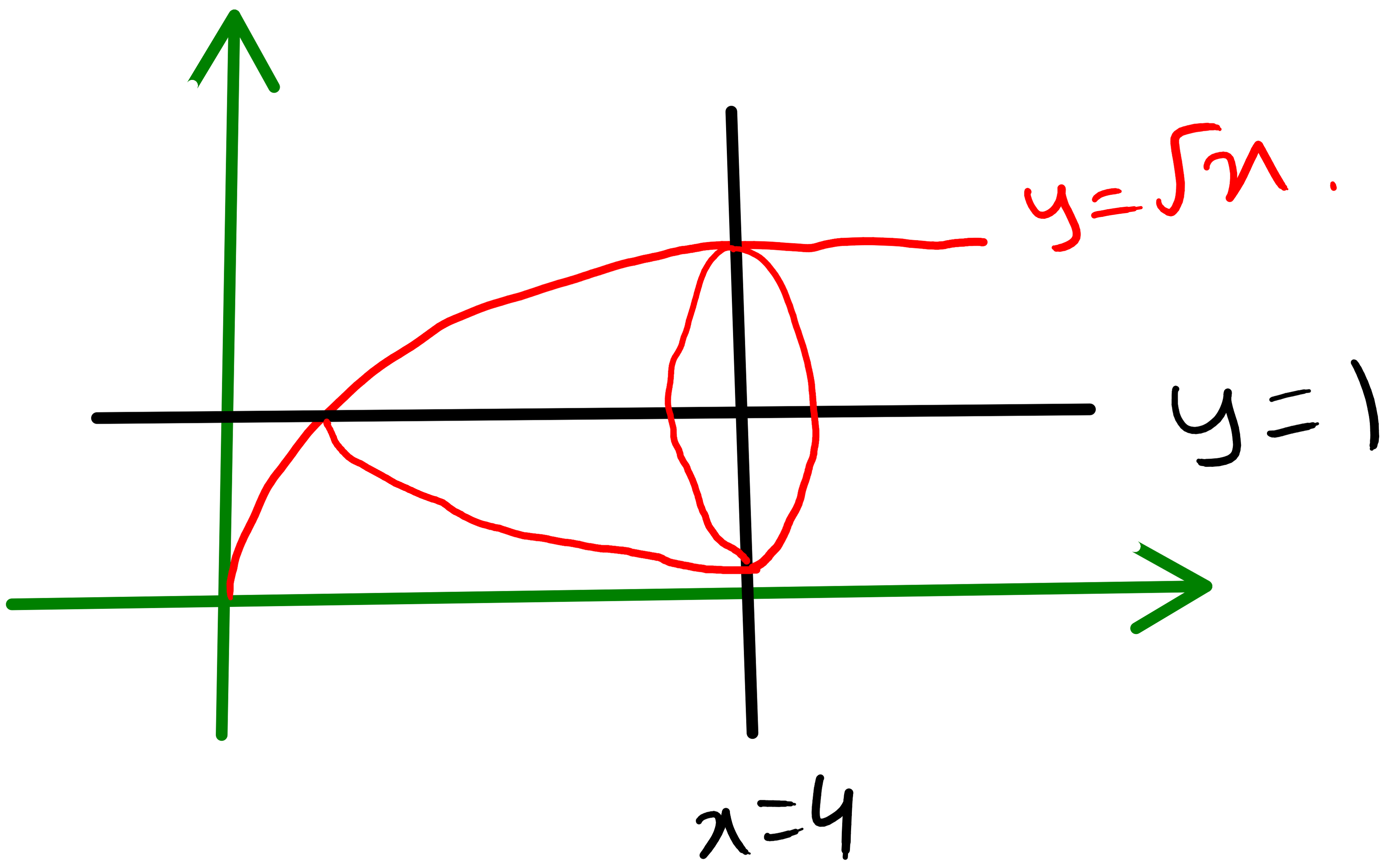
is rotated about the x-axis to generate a sphere. Find its volume.

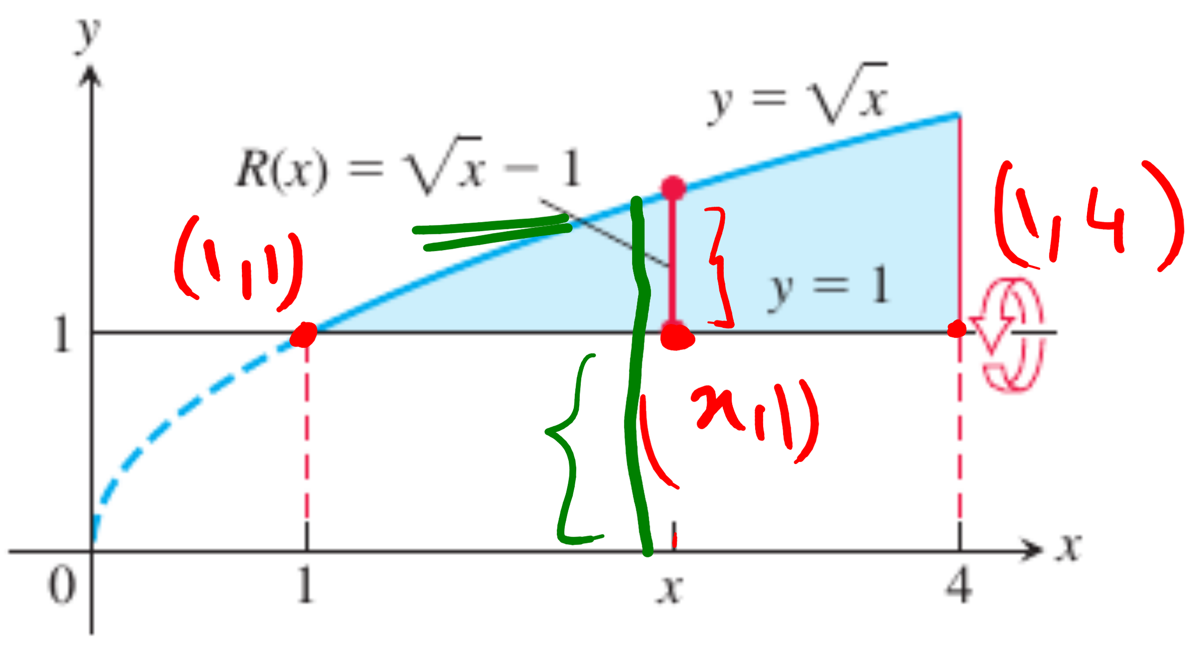


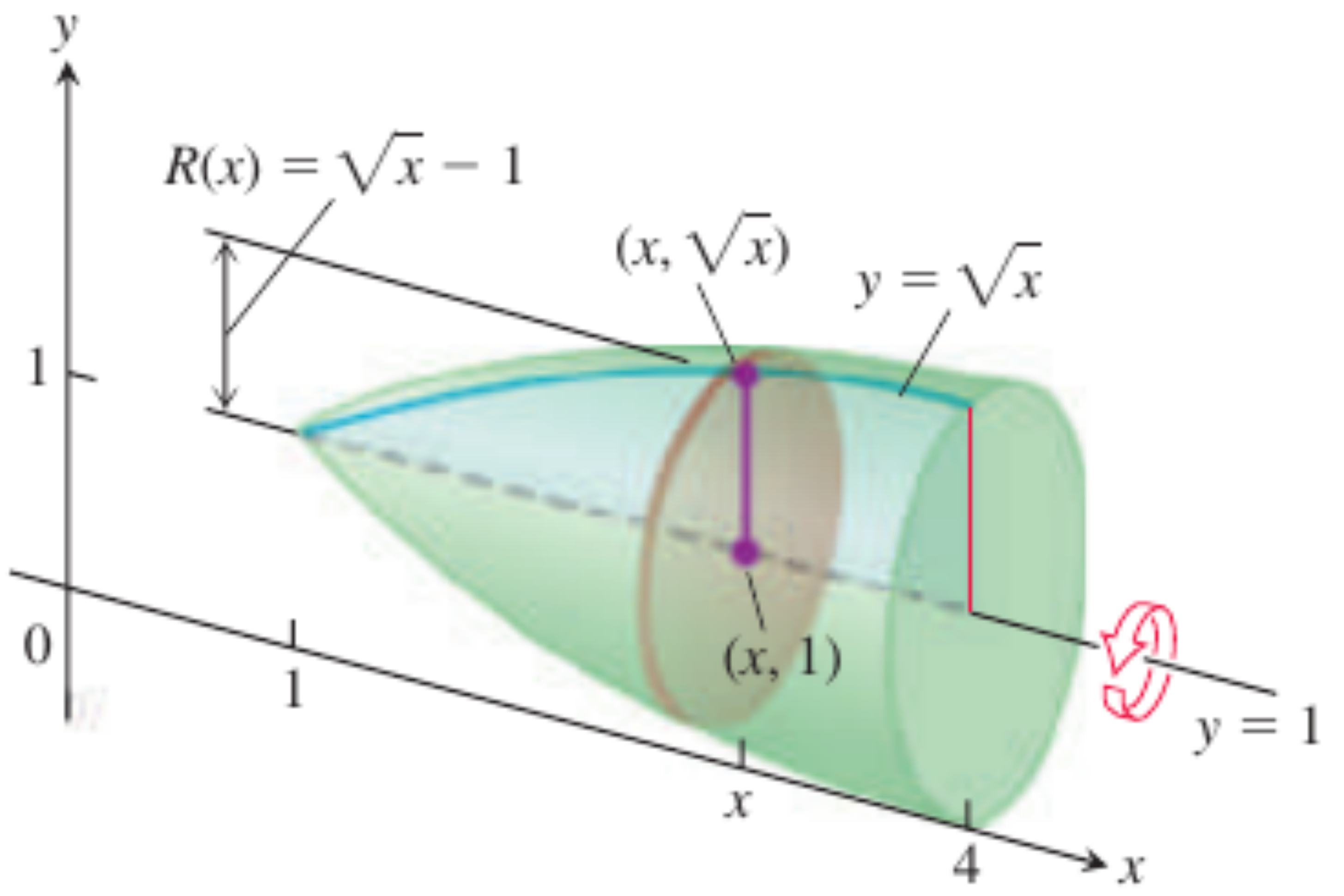
$$\text{volume} = \int_{-a}^{a} \pi (\sqrt{a^2 - x^2})^2 dx$$

$$= \frac{4}{3}\pi a^3$$

EXAMPLE 6 Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.







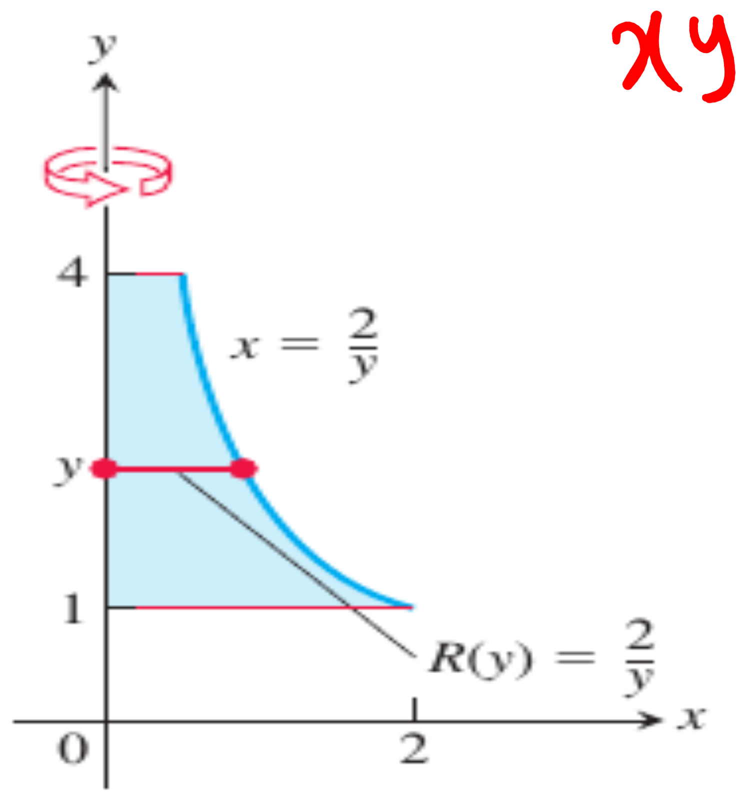
(b)

$$\text{volume} = \int_1^4 \pi(\sqrt{x}-1)^2\,dx$$

$$= \frac{7\pi}{6}$$

EXAMPLE 7 Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.

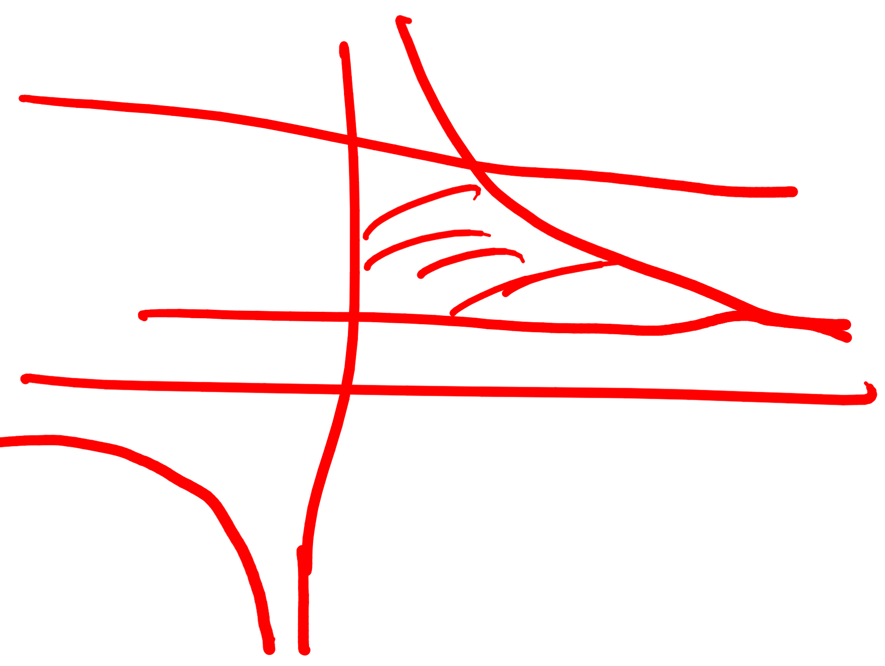
$xy=2$

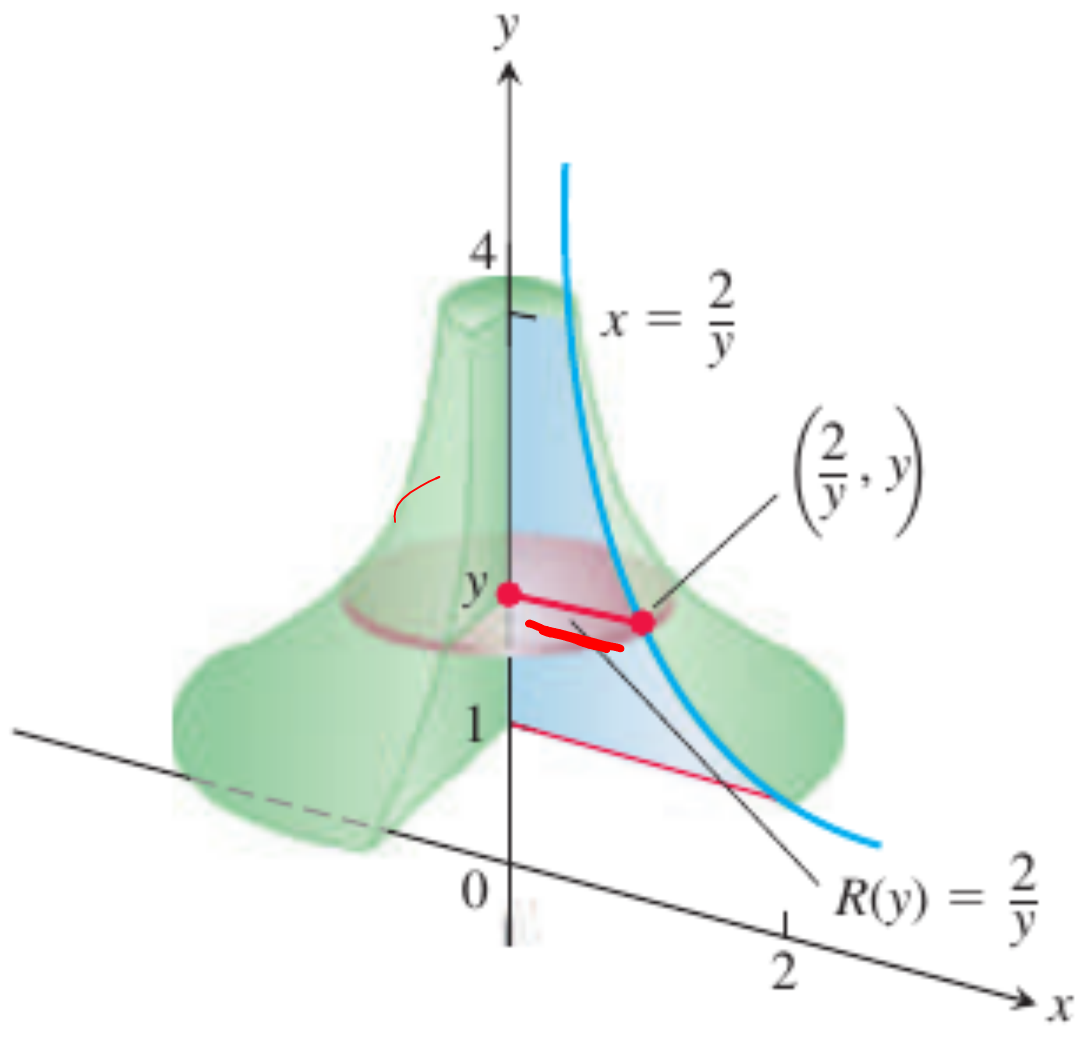


(a)

$$\text{volume} = \int_a^b \pi\,[R(y)]^2\,dy$$

$$= \int_1^4 \pi\left(\frac{2}{y}\right)^2 dy = 3\pi$$





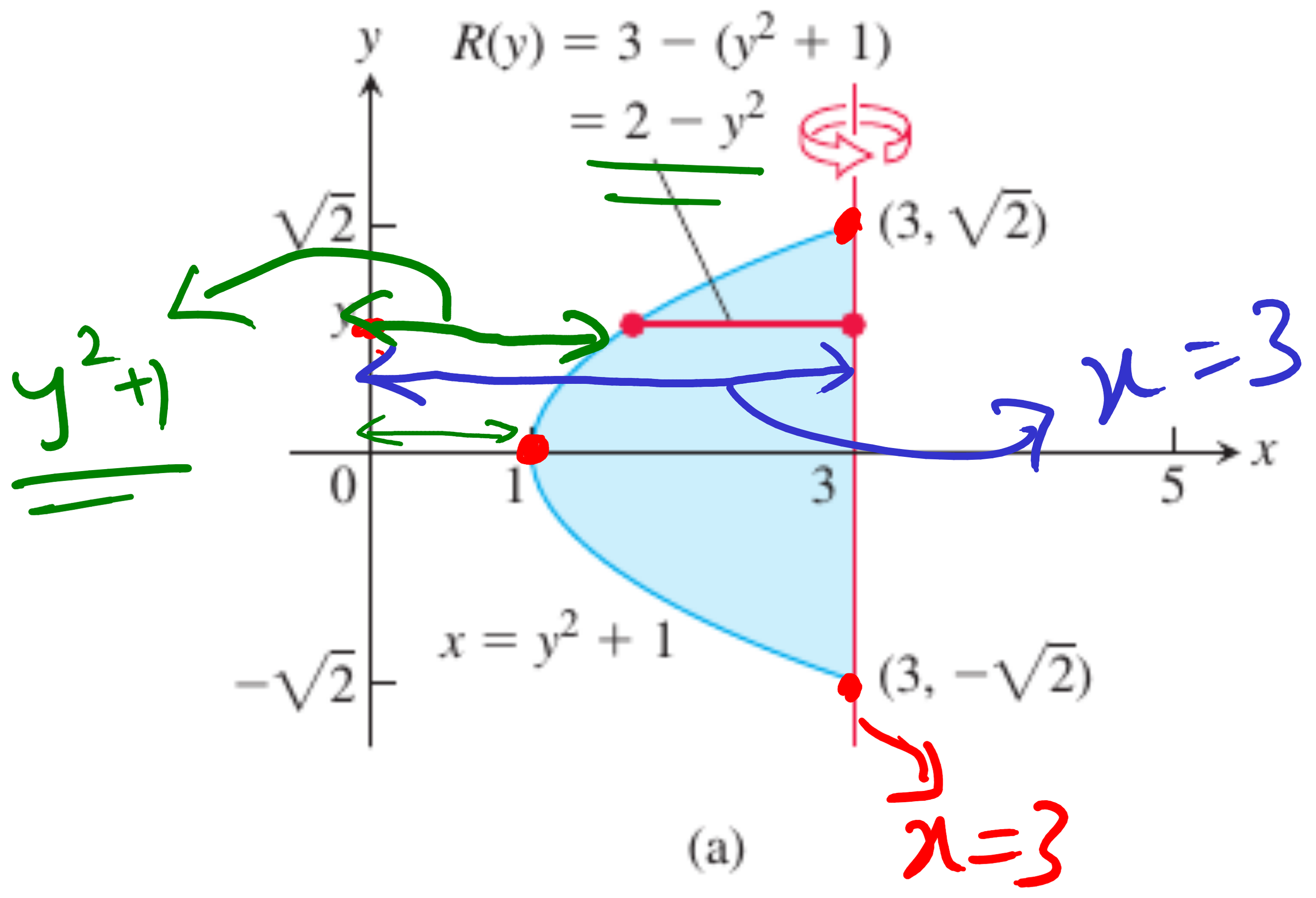
(b)

EXAMPLE 8 Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

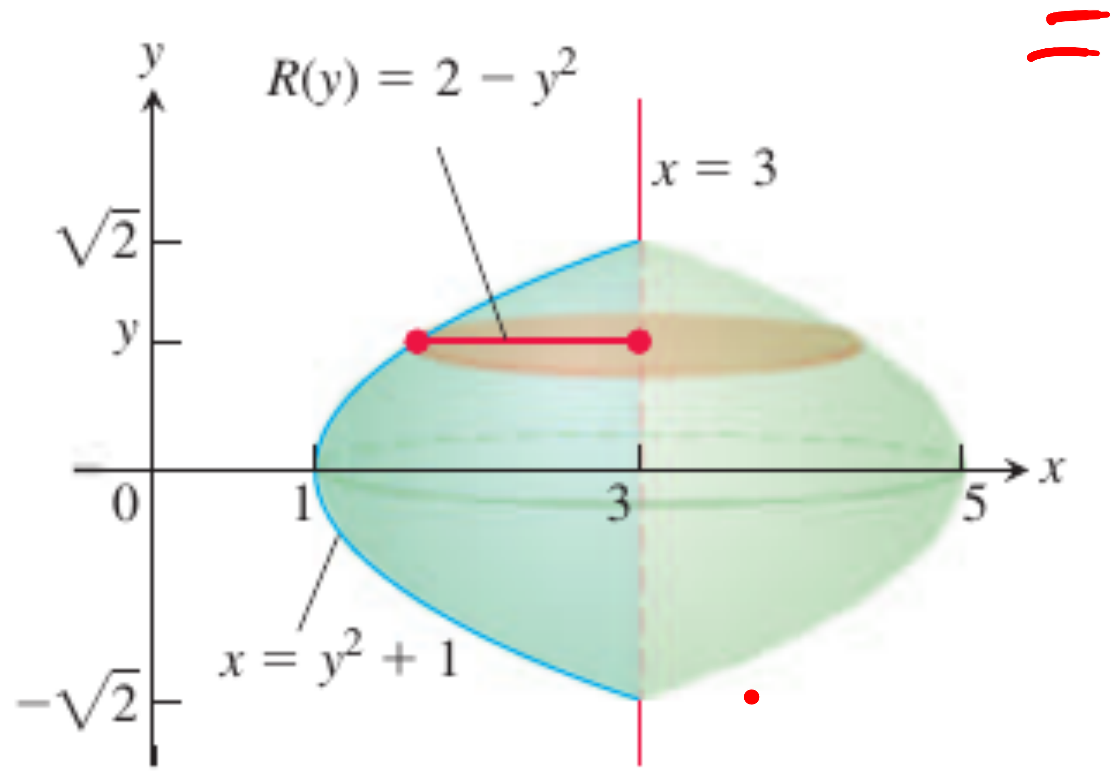
$y^2 = x - 1$

$$\text{Volume} = \int_{-\sqrt{2}}^{\sqrt{2}} \pi (2 - y^2)^2 \, dy$$

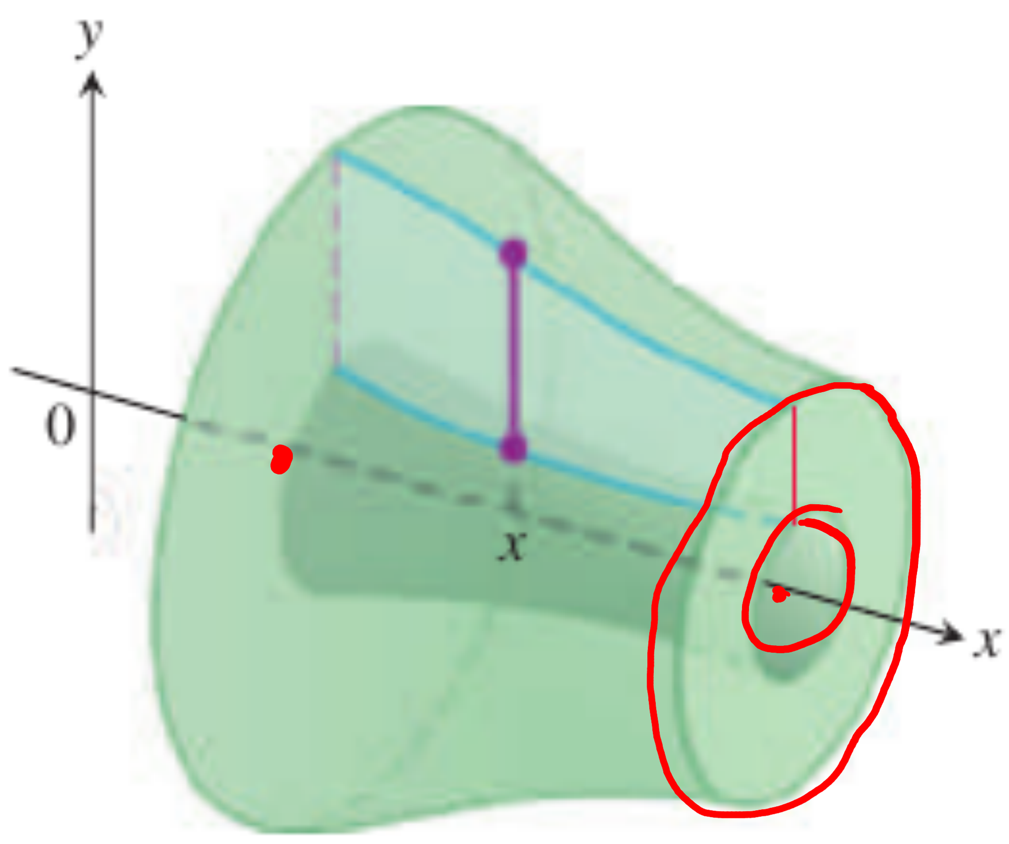
$$= \frac{64\pi\sqrt{2}}{15}$$

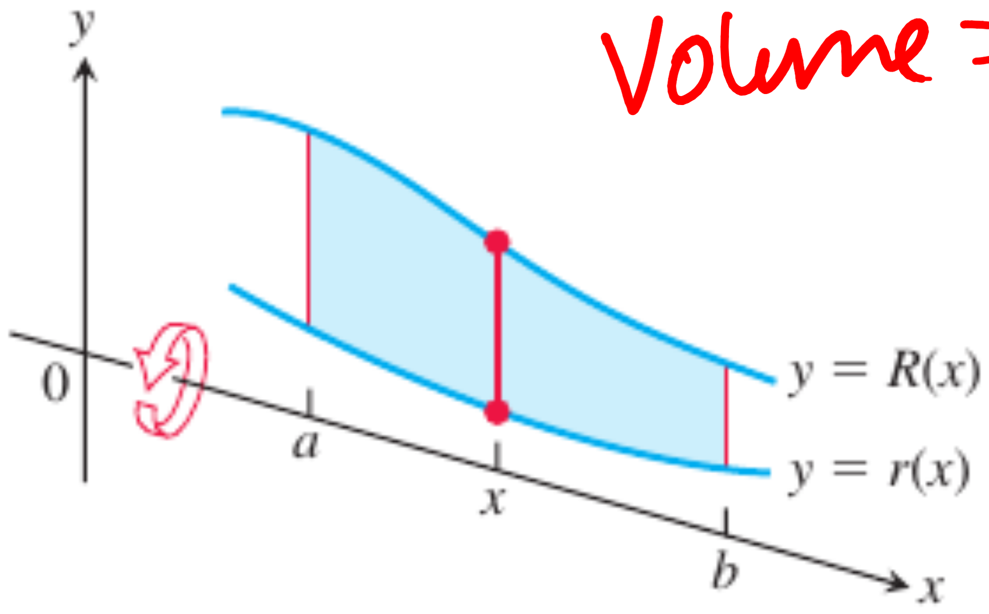


(a)



(b)

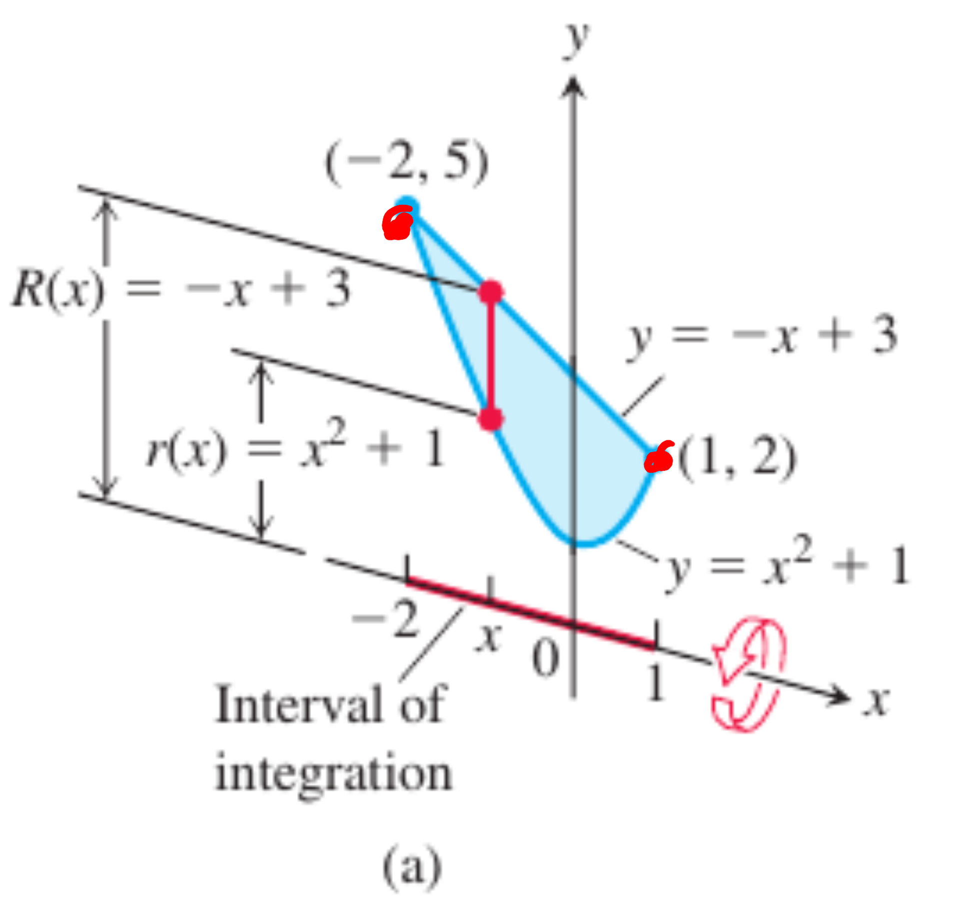


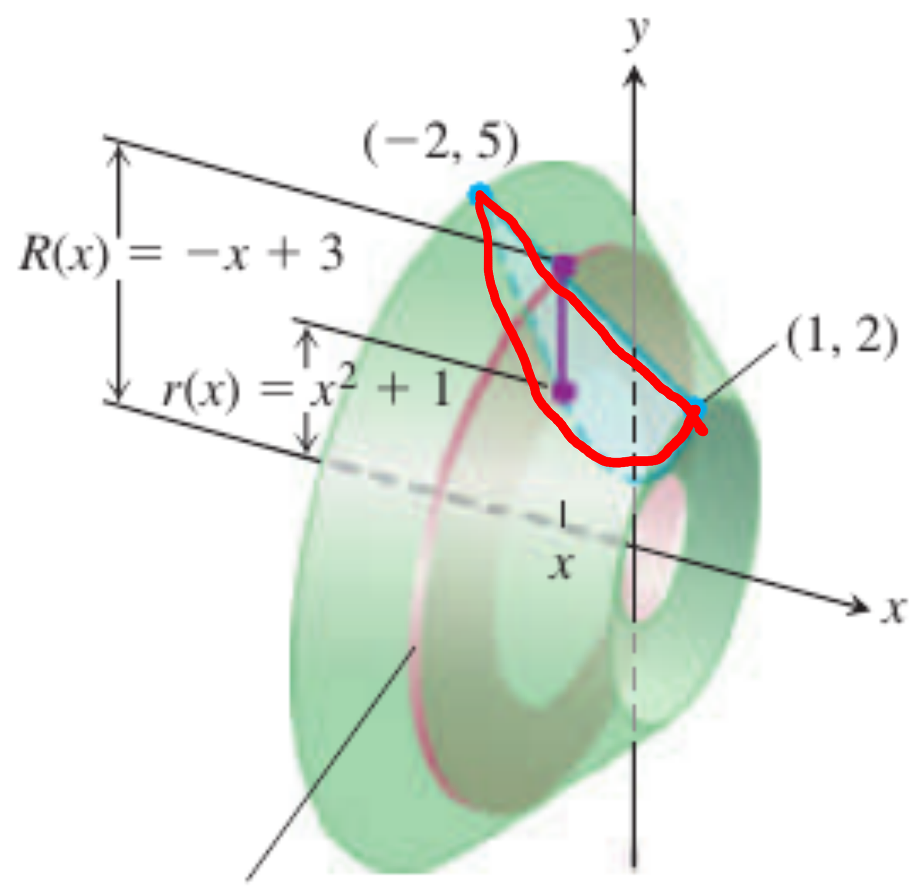


$$\text{Volume} = \int_a^b \pi [R^2 - r^2] \, dx$$

EXAMPLE 9 The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

$x^2+1 = -x+3$





Washer cross-section
Outer radius: $R(x) = -x + 3$
Inner radius: $r(x) = x^2 + 1$

$$V = \pi \int_{-2}^{1} (-x+3)^2 - (x^2+1)^2 \, dx$$

EXAMPLE 10 The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

$$V = \pi \int_0^2 (\sqrt{y})^2 - \left(\frac{y}{2}\right)^2 dy$$