Monotonicity and Concavity

Monotonicity Theorem

Let f be continuous on an interval I and differentiable at every interior point of I.

(i) If $f'(x) > 0$ for all x interior to I, then f is increasing on I.

(ii) If $f'(x) < 0$ for all x interior to I, then f is decreasing on I.

Problems:-

1) If $f(x) = 2x^3 - 3x^2 - 12x + 7$, find where f is increasing and where it is decreasing.

sol:- Given $f(x) = 2x^3 - 3x^2 - 12x + 7$

$f'(x) = 6x^2 - 6x - 12$

$f'(x) = 0 \Rightarrow 6x^2 - 6x - 12 = 0$

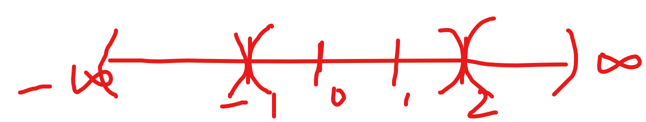
ie., $x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

-2

$-2 \quad 1$

The split pts are -1 & 2



The three intervals are $(-\infty,-1)$, $(-1,2)$ & $(2,\infty)$

Let the test points be -2, 0 & 3

when $x=-2$

$$f'(x) = f'(-2) = 6(-2)^2 - 6(-2) - 12 > 0$$

$\therefore$ f is increasing in $(-\infty,-1)$

When $x=0$

$$f'(x) = f'(0) = -12 < 0$$

f is decreasing in $(-1,2)$

When $x=3$, $f'(x) = f'(3) = 6(3)^2 - 6(3) - 12 > 0$

f is increasing in $(2,\infty)$

Therefore f is increasing on $(-\infty, -1]$ & $[2, \infty)$.

f is decreasing on $[-1, 2]$.

Problem 2:-

Determine where $f(x) = \frac{x}{(1+x^2)}$ is increasing and where it is decreasing.

Sol:- Given $f(x) = \frac{x}{(1+x^2)}$

$$f'(x) = \frac{(1+x^2)(1) - x(2x)}{(1+x^2)^2} = \frac{1+x^2-2x^2}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2} = \frac{(1-x)(1+x)}{(1+x^2)^2}$$

$f'(x) = 0 \Rightarrow (1-x)(1+x) = 0$

The split pts are $x = -1$ & 1

The intervals are $(-\infty, -1)$, $(-1, 1)$ & $(1, \infty)$

Let $x = -2$ in $(-\infty, -1)$

$$f'(-2) = \frac{(1-(-2))(1-2)}{(1+(-2)^2)^2} < 0$$

$\therefore f$ is decreasing on $(-\infty, -1)$

Let $x = 0$ in $(-1, 1)$

$$f'(0) = 1 > 0$$

$\therefore f$ is increasing on $(-1, 1)$

Let $x = 2$ in $(1, \infty)$

$$f'(2) = \frac{(1-2)(1+2)}{(1+2^2)^2} < 0$$

$\therefore f$ is decreasing on $(1, \infty)$

We conclude that

f is increasing on $[-1, 1]$ &

decreasing on $(-\infty, -1]$ & $[1, \infty)$