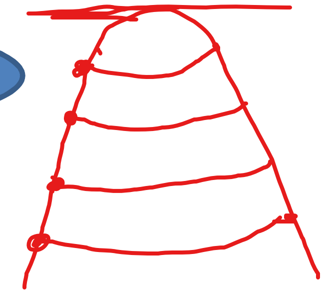
FUNCTIONS OF SEVERAL VARIABLES

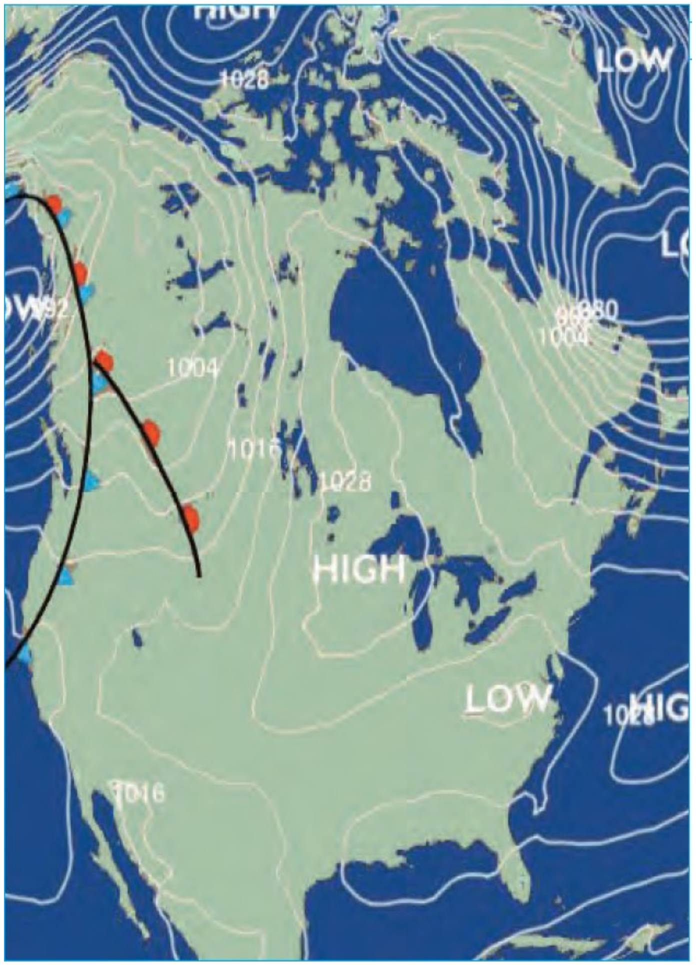
Outline

- Functions of two variables-limits and continuity.
- Partial derivatives & total differential.
- Taylor's expansion for two variables.
- Maxima and minima.
- Constrained maxima and minima(Lagrange's multiplier method)
- Jacobians.

$z = f(x, y)$

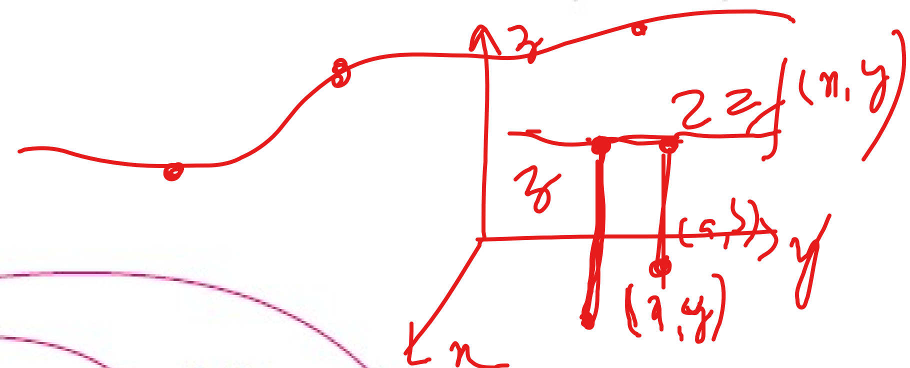
OVERVIEW





Functions of two variables can be visualized by means of level curves, which connect points where the function takes on a given value. Atmospheric pressure at a given time is a function of longitude and latitude and is measured in millibars. Here the level curves are called isobars and those pictured join locations that had the same pressure on March 7, 2007. (The curves labeled 1028, for instance, connect points with pressure 1028 mb.)

A **function** f **of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{ f(x, y) \mid (x, y) \in D\}$.



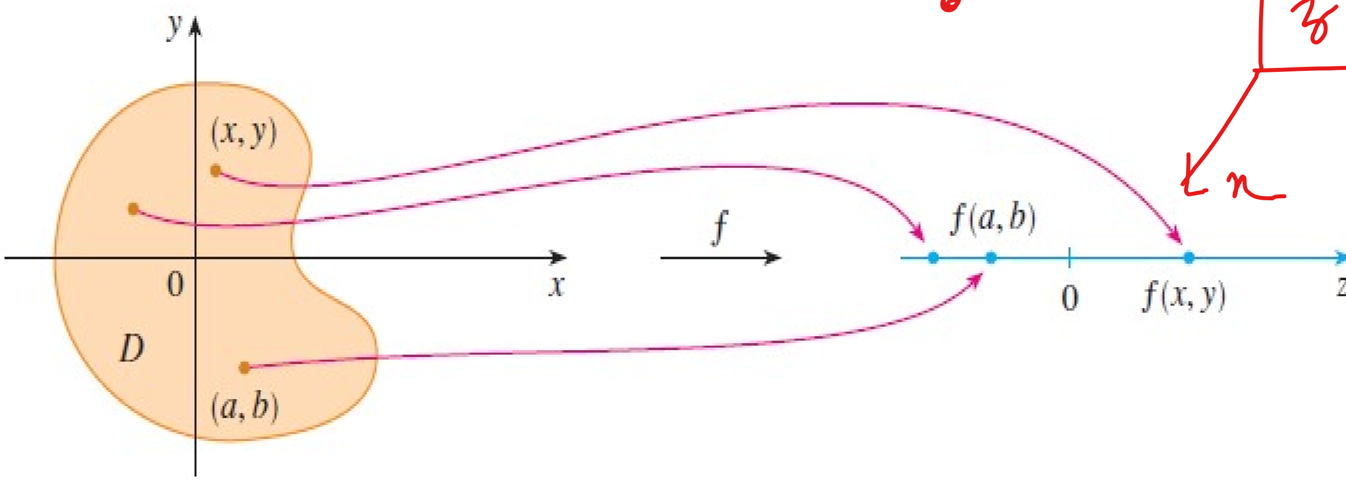


FIGURE 1

NOTE 1:

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). The variables x and y are **independent variables** and z is the **dependent variable.**

NOTE 2:

A function of two variables is just a function whose domain is a subset of $\mathbb{R}^2$ and whose range is a subset of $\mathbb{R}$. One way of visualizing such a function is by means of an arrow diagram (see Figure 1), where the domain D is represented as a subset of the xy-plane.

Definition 1:- Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon$$

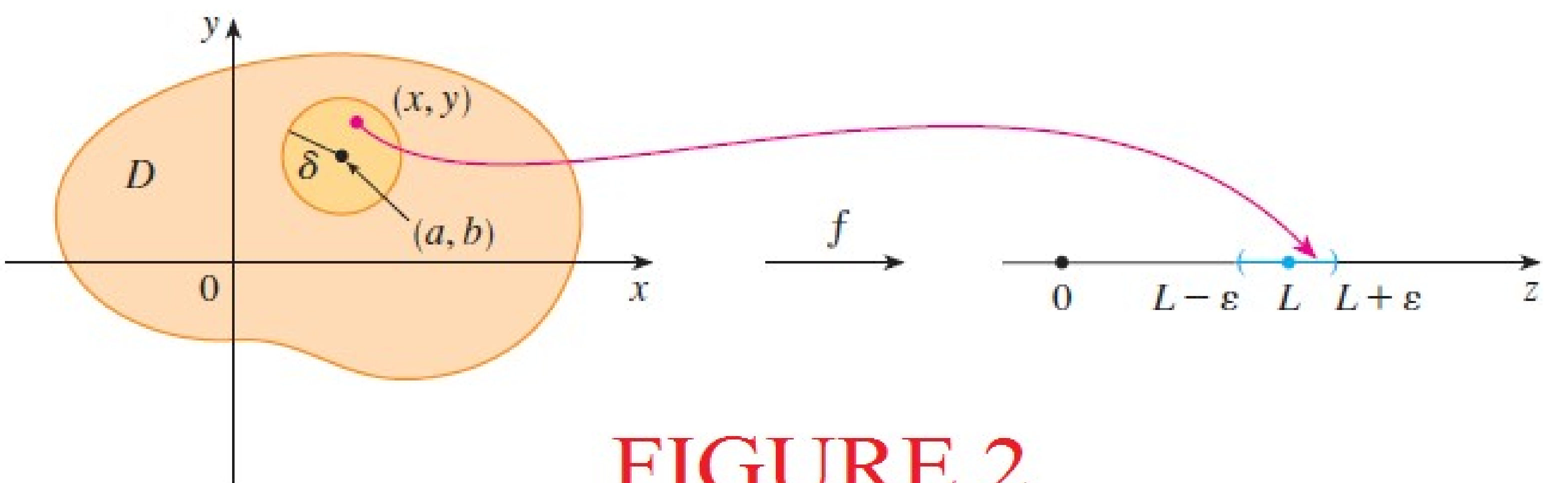


FIGURE 2

NOTE 3:

Other notations for the limit in Definition are

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = L \qquad \text{and} \qquad f(x, y) \to L \text{ as } (x, y) \to (a, b)$$

Notice that $|f(x, y) - L|$ is the distance between the numbers $f(x, y)$ and L, and $\sqrt{(x-a)^2 + (y-b)^2}$ is the distance between the point (x, y) and the point (a, b). Thus Definition says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0). Figure 2 illustrates Definition by means of an arrow diagram. If any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L, then we can find a disk D_δ with center (a, b) and radius $\delta > 0$ such that f maps all the points in D_δ [except possibly (a, b)] into the interval $(L - \varepsilon, L + \varepsilon)$.

Another illustration of Definition is given in Figure 3 where the surface S is the graph of f. If $\varepsilon > 0$ is given, we can find $\delta > 0$ such that if (x, y) is restricted to lie in the disk D_δ and $(x, y) \neq (a, b)$, then the corresponding part of S lies between the horizontal planes $z = L - \varepsilon$ and $z = L + \varepsilon$.

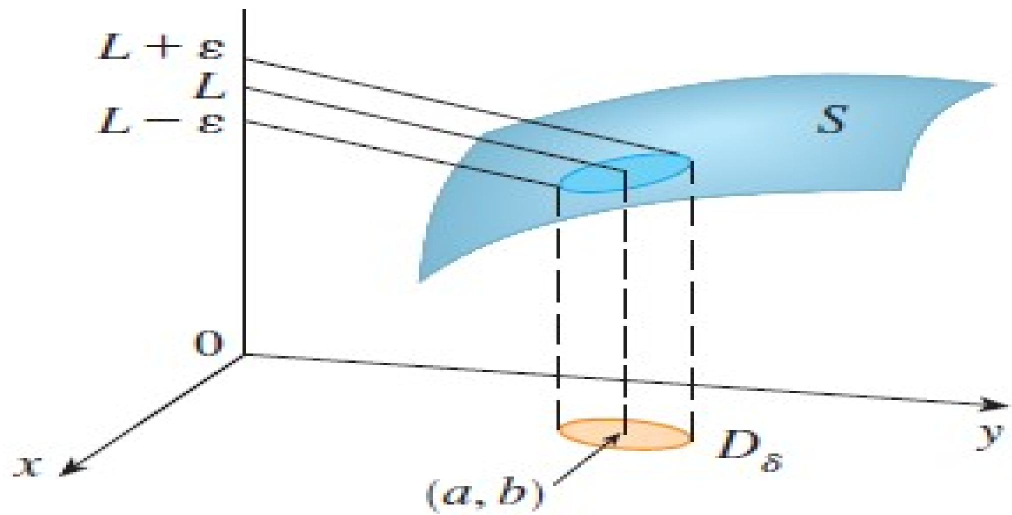


FIGURE 3

NOTE 4:

For functions of two variables the situation is not as simple because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever (see Figure4) as long as (x, y) stays within the domain of f.

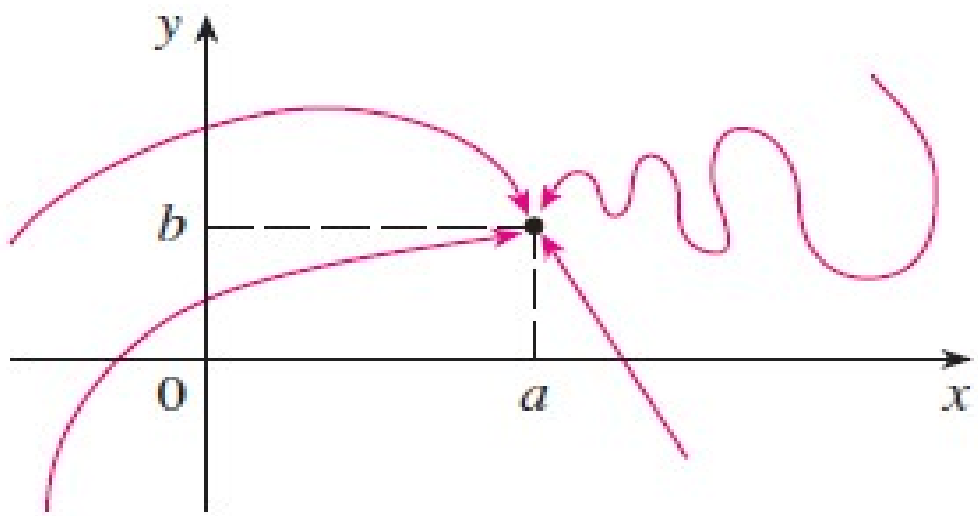


FIGURE 4

Results:- Properties of Limits of Functions of Two Variables

The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $$\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$$
2. *Difference Rule:* $$\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$$
3. *Product Rule:* $$\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$$
4. *Constant Multiple Rule:* $$\lim_{(x,y)\to(x_0,y_0)} (kf(x,y)) = kL \qquad \text{(any number } k\text{)}$$
5. *Quotient Rule:* $$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M} \qquad M \neq 0$$
6. *Power Rule:* If r and s are integers with no common factors, and $s \neq 0$, then

$$\lim_{(x,y)\to(x_0,y_0)} (f(x,y))^{r/s} = L^{r/s}$$

provided $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

NOTE 5:

When we apply the results to polynomials and rational functions, we obtain the useful result that the limits of these functions as $(x, y) \to (x_0, y_0)$ can be calculated by evaluating the functions at (x_0, y_0). The only requirement is that the rational functions be defined at (x_0, y_0).

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x, y)\to(a, b)} f(x, y)$ does not exist.

1) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Sol:-

$f(x,y) = \dfrac{x^2 - y^2}{x^2+y^2}$ Along x-axis, $y=0$

Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$. First let's approach $(0, 0)$ along the x-axis. Then $y = 0$ gives $f(x, 0) = x^2/x^2 = 1$ for all $x \neq 0$, so

$f(x,0) = \dfrac{\cancel{x^2}}{\cancel{x^2}} = 1 \ \forall \ x \neq 0$

$$f(x, y) \to 1 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

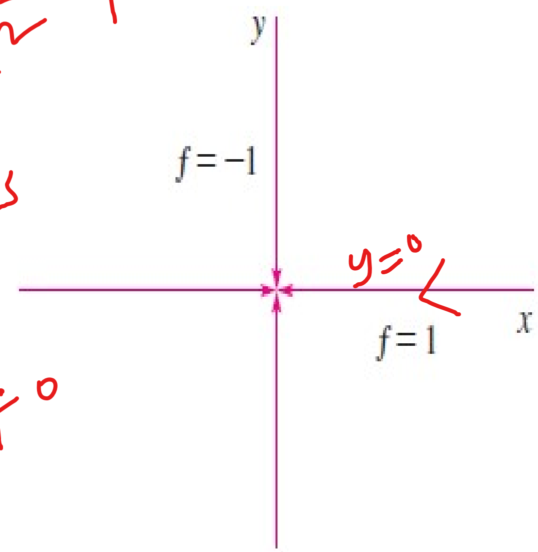
$f(x,y) \to 1$ as $(x,y) \to (0,0)$ along x-axis

We now approach along the y-axis by putting $x = 0$. Then $f(0, y) = \dfrac{-y^2}{y^2} = -1$ for all $y \neq 0$, so

$f(0,y) = -\cancel{y^2}/\cancel{y^2} = -1 \ \forall \ y \neq 0$

$$f(x, y) \to -1 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

$f(x,y) \to -1$ as $(x,y) \to (0,0)$



Since f has two different limits along two different lines, the given limit does not exist.

2) If $f(x, y) = \dfrac{xy}{x^2 + y^2}$, does $\lim\limits_{(x, y) \to (0, 0)} f(x, y)$ exist?

Sol.-

Along x-axis, $y = 0$.

$f(x, 0) = \dfrac{0}{x^2} = 0$

If $y = 0$, then $f(x, 0) = 0/x^2 = 0$. Therefore

as $(x, y) \to (0, 0)$

$f(x, y) \to 0$

$$f(x, y) \to 0 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

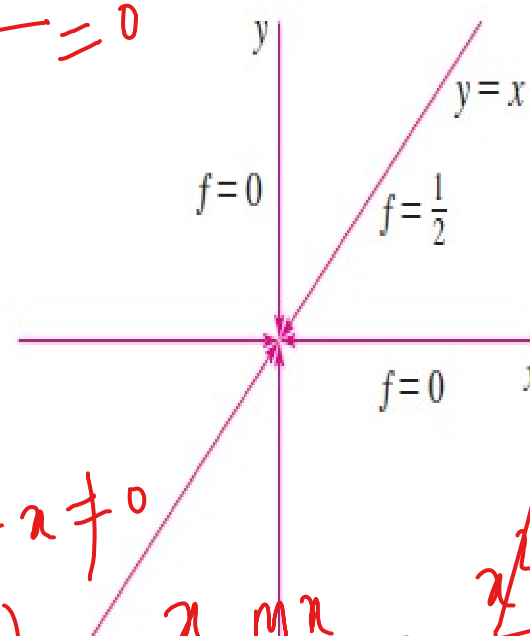
If $x = 0$, then $f(0, y) = 0/y^2 = 0$, so

Along y-axis, $x = 0$

$f(0, y) = \dfrac{0}{y^2} = 0$

$$f(x, y) \to 0 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

$f(x, y) \to 0$ as $(x, y) \to (0, 0)$



Although we have obtained identical limits along the axes, that does not show that the given limit is 0. Let's now approach (0, 0) along another line, say $y = x$. For all $x \neq 0$,

$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

$y = mx \ \forall\ x \neq 0$

$f(x, mx) = \dfrac{x\, mx}{x^2 + m^2 x^2} = \dfrac{\cancel{x^2}\, m}{\cancel{x^2}(1 + m^2)} \neq 0 \ \forall\ m > 0$

Therefore $\quad f(x, y) \to \frac{1}{2} \quad$ as $\quad (x, y) \to (0, 0)$ along $y = x$

Since we have obtained different limits along different paths, the given limit does not exist.

3) If $f(x,y)=\dfrac{xy^2}{x^2+y^4}$, does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist ?

Sol :– $f(x,y)\to 0$ as $(x,y)\to(0,0)$ Along x- axis, $y=0$

If $y=0$, then $f(x,0)=\dfrac{0}{x^2}=0$. $f(x,0)=\dfrac{0}{x^2}=0$

$\therefore\ f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the $x-axis$.

If $x=0$, then $f(0,y)=\dfrac{0}{y^4}=0$, so Along y-axis, $x=0$

$f(0,y)=\dfrac{0}{y^4}=0$ $f(x,y)\to 0$ as $(x,y)\to(0,0)$

$f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the $y-axis$.

Now let $y=mx$, where m is the slope .

$$f(x,y)=f(x,mx)=\frac{x(mx)^2}{x^2+(mx)^4}=\frac{m^2x^3}{x^2+m^4x^4}=\frac{m^2x}{1+x^2m^4}$$

$$\frac{m^2x^{\cancel{3}}}{\cancel{x^2}(1+m^4x^2)}$$

so $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along $y=mx$.

$f(x,y)\to 0$ as $(x,y)\to(0,0)$ along $y=mx$

Thus f *has the same* lim*iting value along every nonverticd line through the origin. But that does not show that the given* lim*it is* 0, *for if we now let* $(x, y) \to (0,0)$ *along the parabola* $x = y^2$, *we have*

Let $y = x^2$

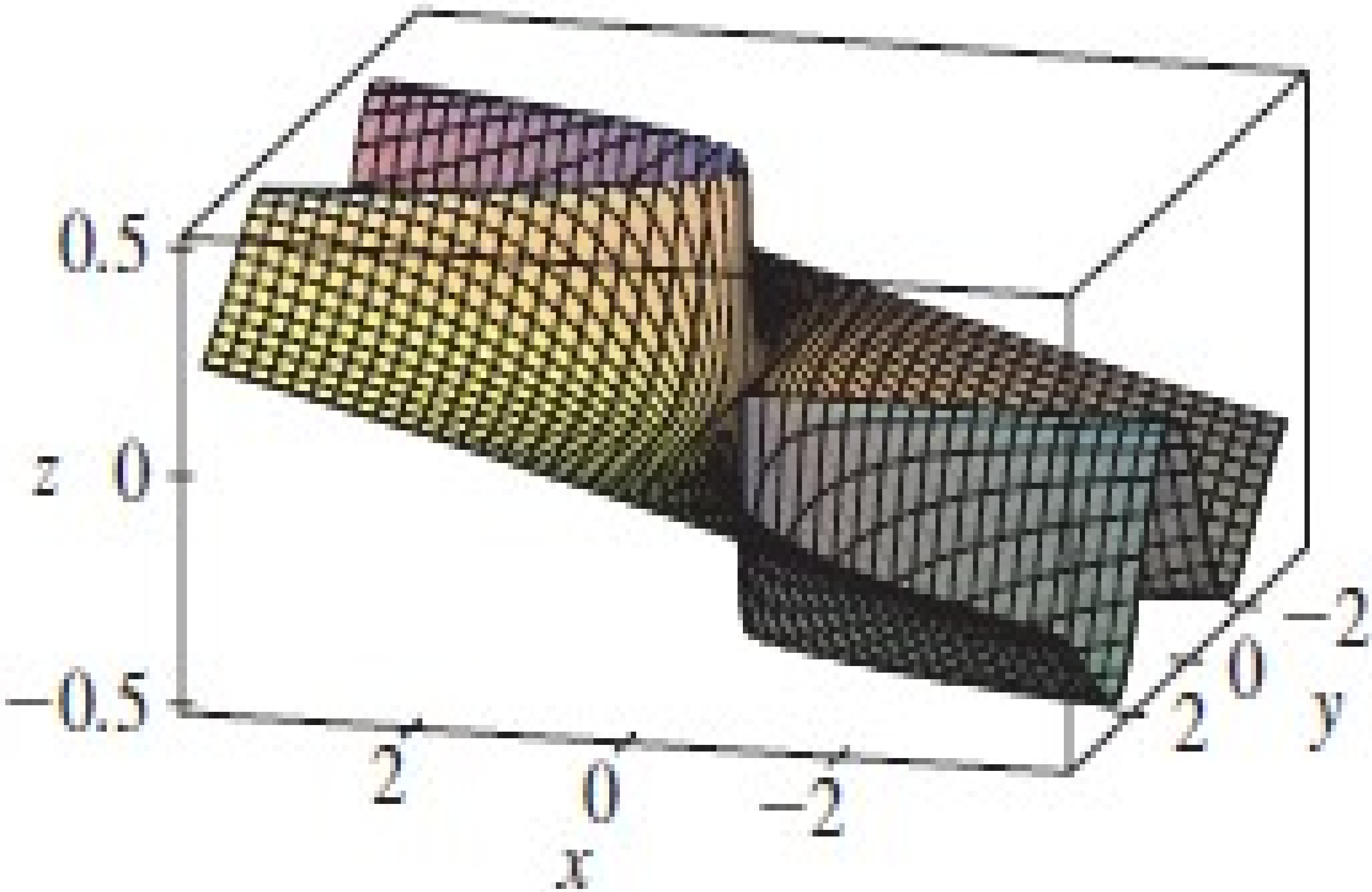
$$f(x, x^2) = \frac{x(x^2)^2}{x^2 + (x^2)^4} = \frac{x^5}{x^2 + x^8} = \frac{x^{\cancel{5}3}}{\cancel{x^2}(1 + x^6)}$$

$$f(x, y) = f(y^2, y) = \frac{y^2 y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

$$\therefore f(x, y) \to \frac{1}{2} \text{ as } (x, y) \to (0,0) \text{ along } x = y^2$$

Since different paths lead to different lim*iting values, the given* lim*it does not exist.*

$f(x,y) \to 0$ as $(x,y) \to (0,0)$ along $y = x^2$


0.5
z 0
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2
0
−2
x
−2
0
2
y

4) *Find* $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ *if it exists* .

Sol :–

Along x-axis, $y=0$

$$If \quad y = 0, \; then \quad f(x,0) = \frac{0}{x^2} = 0$$

$$\therefore \; f(x,y) \to 0 \;\; as \;\; (x,y) \to (0,0) \;\; \underline{along \;\; the \;\; x - axis} \; .$$

$$If \quad x = 0, \; then \quad f(0,y) = \frac{0}{y^2} = 0$$

Along y-axis, $x=0$

$$\therefore \; f(x,y) \to 0 \;\; as \;\; (x,y) \to (0,0) \;\; along \;\; \underline{the \;\; y - axis} \; .$$

Now let $y = mx$ *where* m *is the slope*

$$f(x,y) = f(x,mx) = \frac{3x^2mx}{x^2+(mx)^2} = \frac{3x^3m}{x^2+m^2x^2} = \frac{3mx}{1+m^2}$$

$$\therefore \; f(x,y) \to 0 \;\; as \;\; (x,y) \to (0,0) \;\; along \;\; \underline{y = mx} \; .$$

$$Now\ let\ \ us\ \ consider\ \ y = x^2$$

$$f(x, x^2) = \frac{3x^2 x^2}{x^2 + (x^2)^2} = \frac{3x^4}{x^2 + x^4} = \frac{3x^2}{1 + x^2}$$

$$Thus\ \ f(x, y) \to 0\ \ as\ \ (x, y) \to (0,0)\ \ along\ \ y = x^2$$

$$Consider\ \ x = y^2,\ therefore$$

$$f(y^2, y) = \frac{3(y^2)^2 y}{(y^2)^2 + y^2} = \frac{3y^5}{y^4 + y^2} = \frac{3y^3}{1 + y^2}$$

$$f(x, y) \to 0\ \ as\ \ (x, y) \to (0,0)\ \ along\ \ x = y^2.$$

So we begin to suspect that the lim *it does exist and is equal to* 0.

Let $\varepsilon > 0$, *we want to find* $\delta > 0$ *such that*

$$\text{if} \quad 0 < \sqrt{x^2 + y^2} < \delta, \text{ then } \left| \frac{3x^2 y}{x^2 + y^2} - 0 \right| < \varepsilon$$

$$\text{ie. if} \quad 0 < \sqrt{x^2 + y^2} < \delta, \text{ then } \frac{3x^2 |y|}{x^2 + y^2} < \varepsilon .$$

But $x^2 \le x^2 + y^2$, since $y^2 \ge 0$, *so*

$$x^2 \le x^2 + y^2$$

$$\frac{x^2}{x^2 + y^2} \le 1 \quad \text{and therefore}$$

$$\frac{x^2}{x^2+y^2} \le 1$$

$$\frac{3x^2 |y|}{x^2 + y^2} \le 3|y| = 3\sqrt{y^2} \le 3\sqrt{x^2 + y^2}$$

Thus if we choose $\delta = \dfrac{\varepsilon}{3}$ *and let*

$$0 < \sqrt{x^2 + y^2} < \delta, \text{ then}$$

$$\left| \frac{3x^2 y}{x^2 + y^2} - 0 \right| \le 3\sqrt{x^2 + y^2} < 3\delta = \cancel{3}\frac{\varepsilon}{\cancel{3}} = \varepsilon$$

Hence by definition

$$\lim_{(x,y) \to (0,0)} \frac{3x^2}{x^2 + y^2} = 0 .$$

5) Find $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$ if it exists.

Solution We first observe that along the line $x = 0$, the function always has value 0 when $y \neq 0$. Likewise, along the line $y = 0$, the function has value 0 provided $x \neq 0$. So if the limit does exist as (x, y) approaches $(0, 0)$, the value of the limit must be 0. To see if this is true, we apply the definition of limit.

Let $\epsilon > 0$ be given, but arbitrary. We want to find a $\delta > 0$ such that

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{x^2+y^2} < \delta$$

or

$$\frac{4|x|y^2}{x^2+y^2} < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{x^2+y^2} < \delta.$$

Since $y^2 \leq x^2 + y^2$ we have that

$$\frac{4|x|y^2}{x^2+y^2} \leq 4|x| = 4\sqrt{x^2} \leq 4\sqrt{x^2+y^2}. \qquad \frac{y^2}{x^2+y^2} \leq 1$$

So if we choose $\delta = \epsilon/4$ and let $0 < \sqrt{x^2+y^2} < \delta$, we get

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| \leq 4\sqrt{x^2+y^2} < 4\delta = 4\left(\frac{\epsilon}{4}\right) = \epsilon.$$

It follows from the definition that

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0.$$

Squeeze Theorem or Sandwich Theorem

If $g(x,y) \le f(x,y) \le h(x,y)$ *for all* $(x,y) \ne (x_0,y_0)$ *in a disk centered at* (x_0,y_0) *and if* $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = \lim_{(x,y)\to(x_0,y_0)} h(x,y) = L$ *then* $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$.

Problem :

Does knowing that

$$2|xy| - \frac{x^2y^2}{6} < 4 - 4\cos(\sqrt{|xy|}) < 2|xy|$$

tell you anything about

$$\lim_{(x,y)\to(0,0)} \frac{4 - 4\cos(\sqrt{|xy|})}{|xy|} \ ?$$

Solution :

Dividing each side of the given inequality by $|xy|$ which is positive (hence preserves the inequality) gives us

$$\frac{2\,|xy| - \frac{x^2y^2}{6}}{|xy|} \le \frac{4 - 4\cos\sqrt{|xy|}}{|xy|} \le \frac{2\,\cancel{|xy|}}{\cancel{|xy|}}$$

that is

$$\underbrace{\frac{2\,|xy| - \frac{x^2y^2}{6}}{|xy|}}_{g(x,y)} \le \underbrace{\frac{4 - 4\cos\sqrt{|xy|}}{|xy|}}_{f(x,y) = L} \le \underbrace{2}_{h(x,y)} = L$$

We compute $\lim_{(x,y)\to(0,0)} \frac{2|xy| - \frac{x^2y^2}{6}}{|xy|}$. *We cannot just plug in the point because we get* $\frac{0}{0}$. *We will eliminate the absolute value by considering cases.*

case 1: $xy > 0$. *In this case,* $|xy| = xy$ *hence*

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{2\,|xy| - \frac{x^2y^2}{6}}{|xy|} &= \lim_{(x,y)\to(0,0)} \frac{2xy - \frac{x^2y^2}{6}}{xy} \\
&= \lim_{(x,y)\to(0,0)} \frac{\cancel{xy}\left(2 - \frac{xy}{6}\right)}{\cancel{xy}} \\
&= \lim_{(x,y)\to(0,0)} \left(2 - \underbrace{\frac{xy}{6}}_{=0}\right) \\
&= \underline{\underline{2}}
\end{aligned}$$

case 2: $xy < 0$. *In this case,* $|xy| = -xy$ *hence*

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{2|xy| - \frac{x^2y^2}{6}}{|xy|} &= \lim_{(x,y)\to(0,0)} \frac{-2xy - \frac{x^2y^2}{6}}{-xy} \\
&= \lim_{(x,y)\to(0,0)} \frac{-\cancel{xy}\left(2 + \frac{xy}{6}\right)}{-\cancel{xy}} \\
&= \lim_{(x,y)\to(0,0)} \left(2 + \underset{\overset{\|}{0}}{\frac{xy}{6}}\right) \\
&= 2
\end{aligned}$$

Therefore $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2|xy| - \frac{x^2y^2}{6}}{|xy|} = 2$ *and since* $\displaystyle\lim_{(x,y)\to(0,0)} 2 = 2$, *we are exactly in the situation of the squeeze theorem. We conclude that* $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4 - 4\cos\sqrt{|xy|}}{|xy|} = 2$.

2. For the following, determine whether the limit exists. If yes, compute the limit.

$$\lim_{(x,y)\to(0,0)} \frac{y^2(1-\cos(2x))}{x^4+y^2}$$

Sol:- We can write

$$y^2 \leq y^2 + x^4$$

$$0 \leq \frac{y^2}{y^2+x^4} \leq 1 \quad \text{—— I}$$

Since $(1-\cos(2x))$ is non-negative

(I) becomes

$$0 \le \frac{y^2(1-\cos(2x))}{y^2+x^4} \le (1-\cos(2x))$$

Here $g(x,y)=0$ & $h(x,y)=(1-\cos(2x))$

$$\lim_{(x,y)\to(0,0)} g(x,y) = 0. \quad \& \quad \lim_{(x,y)\to(0,0)} (1-\cos(2x)) = 0$$

By Squeeze theorem $\lim_{(x,y)\to(0,0)} \frac{y^2(1-\cos(2x))}{y^2+x^4}$

$= 0$

3. Use sandwich theorem to find the $\lim_{(x,y)\to(0,0)} f(x,y)$ for

$$f(x,y) = \frac{x^2 y}{x^2+y^2}$$

Sol:-

We have $x^2 \leq x^2+y^2$

ie $0 \leq \dfrac{x^2}{x^2+y^2} \leq 1$ —— I

Now I becomes

$$0\,|y| \leq \frac{x^2 |y|}{x^2+y^2} \leq 1.|y|$$

ie $0 \leq \dfrac{x^2 y}{x^2+y^2} \leq |y|$

$g(x,y)=0$ & $h(x,y) = |y|$

$\lim_{(x,y)\to(0,0)} g(x,y) = 0$ & $\lim_{(x,y)\to(0,0)} h(x,y) = 0$

By sandwich theorem

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0$$

Using polar coordinates, find

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$$

$x = r\cos\theta$

$y = r\sin\theta$

$(x,y) \to (0,0)$

$\simeq r \to 0^+$

Sol:-

WKT $x = r\cos\theta$

$y = r\sin\theta$

$x^3+y^3 = r^3(\cos^3\theta+\sin^3\theta)$

$x^2+y^2 = r^2(\cos^2\theta+\sin^2\theta) = r^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0^+} \frac{r^{\cancel{3}}(\cos^3\theta+\sin^3\theta)}{\cancel{r^2}}$$

$= 0$

2) Use polar co-ordinates to find

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - xy^2}{x^2+y^2}$$

sol:- WKT $x = r\cos\theta$

$y = r\sin\theta$

Now $x^3 - xy^2 = r^3\cos^3\theta - r\cos\theta(r^2\sin^2\theta)$

$= r^3(\cos^3\theta - \cos\theta\sin^2\theta)$

$x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^3-xy^2}{x^2+y^2} = \lim_{r\to 0^+} \frac{r^{\cancel{3}}(\cos^3\theta - \cos\theta\sin^2\theta)}{\cancel{r^2}}$$

$= 0$

CONTINUITY

DEFINITION Continuous Function of Two Variables

A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\lim_{(x, y)\to(x_0, y_0)} f(x, y)$ exists,
3. $\lim_{(x, y)\to(x_0, y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

$\lim_{(x,y)\to(0,0)} f(x,y) = f(x_0, y_0)$

NOTE 1:

The intuitive meaning of continuity is that if the point (x, y) changes by a small amount, then the value of $f(x, y)$ changes by a small amount. This means that a surface that is the graph of a continuous function has no hole or break.

NOTE 2:

It is to be noted that *all polynomials are continuous on* $\mathbb{R}^2$. Likewise, any rational function is continuous on its domain because it is a quotient of continuous functions.

A **polynomial function of two variables** (or polynomial, for short) is a sum of terms of the form cx^my^n, where c is a constant and m and n are nonnegative integers. A **rational function** is a ratio of polynomials. For instance,

$$f(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial, whereas

$$g(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is a rational function.

Problem 1 Evaluate $\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$.

Solution: Since $f(x, y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial, it is continuous everywhere, so we can find the limit by direct substitution:

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 = 11$$

Problem 2. Where is the function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous?

Solution : The function f is discontinuous at $(0, 0)$ because it is not defined there.

Since f is a rational function, it is continuous on its domain, which is the set

$D = \{(x, y) \mid (x, y) \neq (0, 0)\}.$

NOTE 3:

Let

$$g(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Here g is defined at $(0, 0)$ but g is still discontinuous there because $\lim_{(x, y) \to (0, 0)} g(x, y)$ does not exist

NOTE 4:

Let

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

We know f is continuous for $(x, y) \neq (0, 0)$ since it is equal to a rational function there. Also,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0 = f(0,0)$$

Therefore f is continuous at $(0, 0)$, and so it is continuous on $\mathbb{R}^2$.