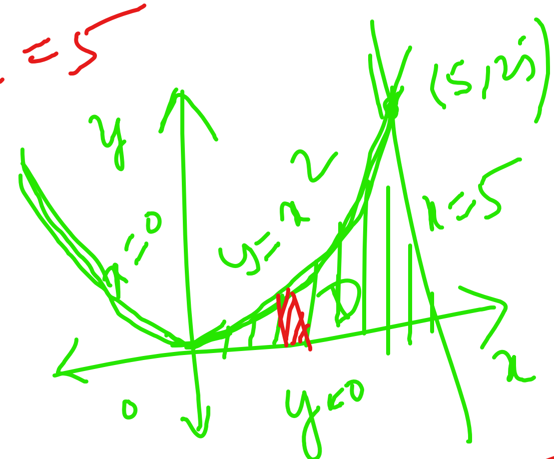
1. Evaluate $\int_0^5 \int_0^{x^2} x(x^2+y^2)dx\,dy$

Sol:- Given $I = \int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy$

Rewriting $I = \int_0^5 \int_0^{x^2} x(x^2+y^2)\,dy\,dx$

Region is bounded by $x=0$ to $x=5$

& $y=0$ to $y=x^2$



$$= \int_0^5 \left(x^3 y + x\frac{y^3}{3}\right)_0^{x^2} dx = \int_0^5 \left(x^5 + \frac{x^7}{3}\right)dx = \left(\frac{x^6}{6} + \frac{x^8}{24}\right)_0^5$$

$$= 5^6\left(\frac{29}{24}\right)$$

Change of order of Integration

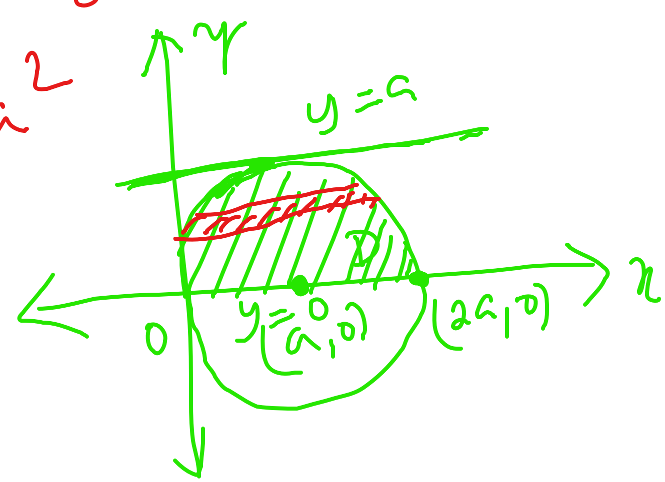
1. Change the order of integration in $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy\, dx\, dy$ and evaluate it.

Sol:- The region of integration is bounded by $y=0$ to $y=a$ & $x = a-\sqrt{a^2-y^2}$ to $x = a+\sqrt{a^2-y^2}$

Let $x = a+\sqrt{a^2-y^2}$

$x-a = \sqrt{a^2-y^2}$, Squaring both sides

$(x-a)^2 = a^2-y^2 \Rightarrow (x-a)^2+y^2 = a^2$

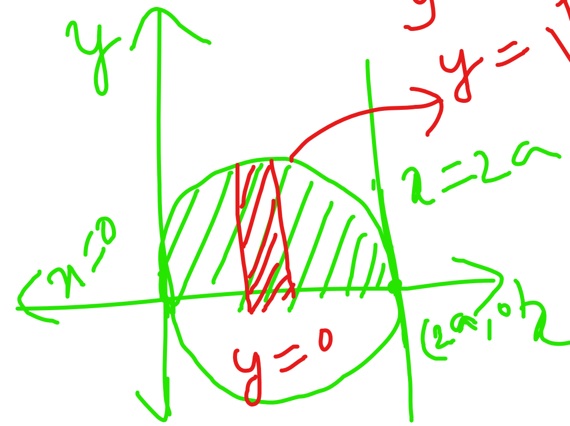


Changing the order of integration

$(x-a)^2+y^2=a^2$

$y^2=a^2-(x-a)^2$

$y=\sqrt{a^2-(x-a)^2}$



ie $$\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy\,dx\,dy$$

$$= \int_{x=0}^{x=2a} \int_{y=0}^{y=\sqrt{a^2-(x-a)^2}} xy\,dy\,dx$$

$$= \int_0^{2a} \left(\frac{xy^2}{2}\right)_0^{\sqrt{a^2-(x-a)^2}} dx$$

$$= \frac{1}{2}\int_0^{2a} x\left(a^2 - (x-a)^2\right) dx$$

$$= \frac{1}{2}\int_0^{2a} x\left(a^2 - x^2 - a^2 + 2ax\right) dx$$

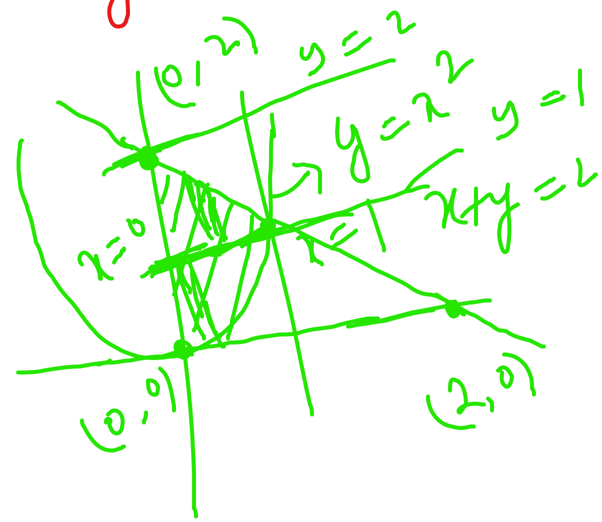
$$= \frac{1}{2}\int_0^{2a} \left(\cancel{xa^2} - x^3 - \cancel{xa^2} + 2ax^2\right) dx$$

$$= \frac{1}{2}\int_0^{2a} \left(-x^3 + 2ax^2\right) dx$$

$$= \frac{1}{2}\left[\frac{-x^4}{4} + 2a\frac{x^3}{3}\right]_0^{2a} = \frac{2}{3}a^4 \quad //$$

2. Change the order of integration and hence evaluate
$\int_0^1 \int_{x^2}^{2-x} xy\, dy dx.$

Sol:- The region of integration is bounded by $x=0$ to $x=1$ & $y=x^2$ to $y=2-x$

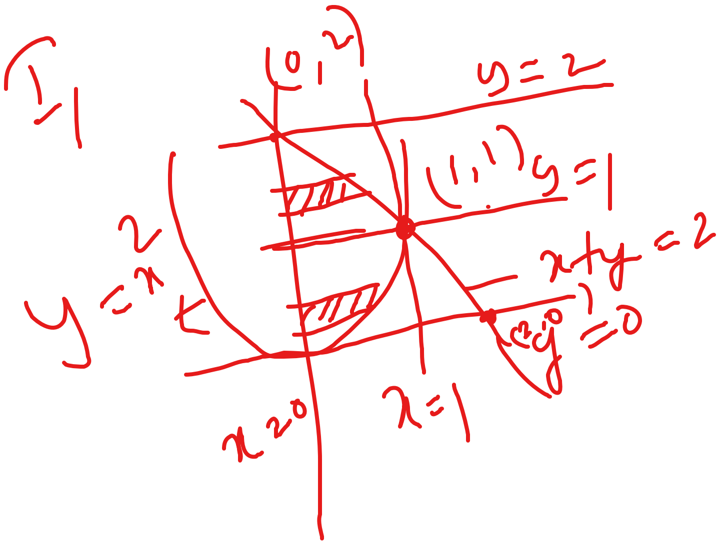


Now we divide the region into

$$I = I_1 + I_2$$

$$= \int_{x=0}^{x=1}\int_{y=x^2}^{y=1} xy\, dy\, dx + \int_{x=0}^{x=1}\int_{y=1}^{y=2-x} xy\, dy\, dx$$

By changing the order in I_1



$$\int_0^1 \int_{x^2}^1 xy\,dy\,dx$$

$$= \int_{y=0}^{y=1} \int_{x=0}^{x=\sqrt{y}} xy\,dx\,dy$$

$$= \int_0^1 \left(y\frac{x^2}{2}\right)_0^{\sqrt{y}} dy$$

$$= \int_0^1 \frac{y^2}{2}\,dy = \frac{1}{2}\left(\frac{y^3}{3}\right)_0^1$$

$$= \frac{1}{6}$$

By changing the order in I_2

$$\int_0^1 \int_1^{2-x} xy\,dy\,dx = \int_{y=1}^{y=2} \int_{x=0}^{x=2-y} xy\,dx\,dy$$

$$= \int_1^2 \left(y\frac{x^2}{2}\right)_0^{2-y} dy$$

$$= \int_1^2 \frac{y(2-y)^2}{2}\,dy$$

$$= \frac{1}{2}\int_1^2 (4+y^2-4y)\,y\,dy$$

$$= \frac{1}{2}\int_1^2 (4y+y^3-4y^2)\,dy$$

$$= \frac{1}{2}\left[\frac{4y^2}{2}+\frac{y^4}{4}-4\frac{y^3}{3}\right]_1^2$$

$$= 5/24$$

$$I = I_1 + I_2 = \frac{1}{6}+\frac{5}{24} = \frac{\cancel{9}}{\cancel{24}}\quad 3/8$$

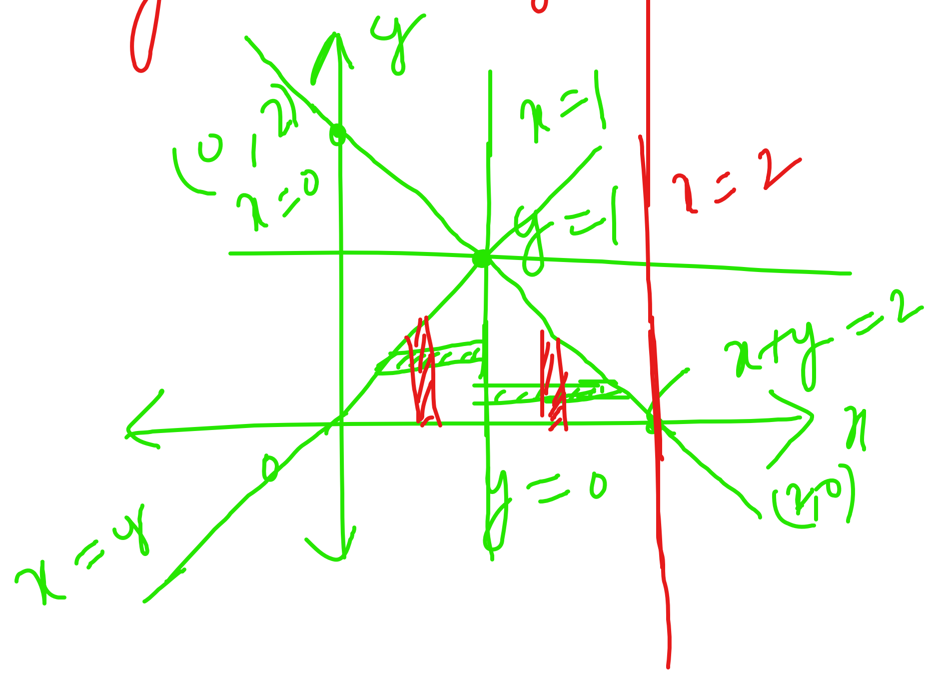
3. Change the order of integration in $\int_0^1 \int_y^{2-y} xy\, dxdy$ and hence evaluate it.

Sol:-

The region of integration is bounded by $y=0$ to $y=1$ & $x=y$ to $x=2-y$

$$I = I_1 + I_2$$

$$= \int_{y=0}^{y=1} \int_{x=y}^{x=1} xy\, dx\, dy + \int_{0}^{1} \int_{x=1}^{x=2-y} xy\, dx\, dy$$



Changing the order in I_1

$$\int_0^1 \int_y^1 xy \, dx \, dy = \int_{x=0}^{x=1} \int_{y=0}^{y=x} xy \, dy \, dx$$

$$= \int_0^1 \left(x \frac{y^2}{2} \right)_0^x dx = \frac{1}{2} \int_0^1 x^3 \, dx = \frac{1}{2} \left(\frac{x^4}{4} \right)_0^1$$

$$= 1/8$$

Changing the order of integration in I_2

$$\int_0^1 \int_1^{2-y} xy \, dx \, dy = \int_{x=1}^{x=2} \int_{y=0}^{y=2-x} xy \, dy \, dx$$

$$= \int_1^2 \left(\frac{x y^2}{2} \right)_0^{2-x} dx$$

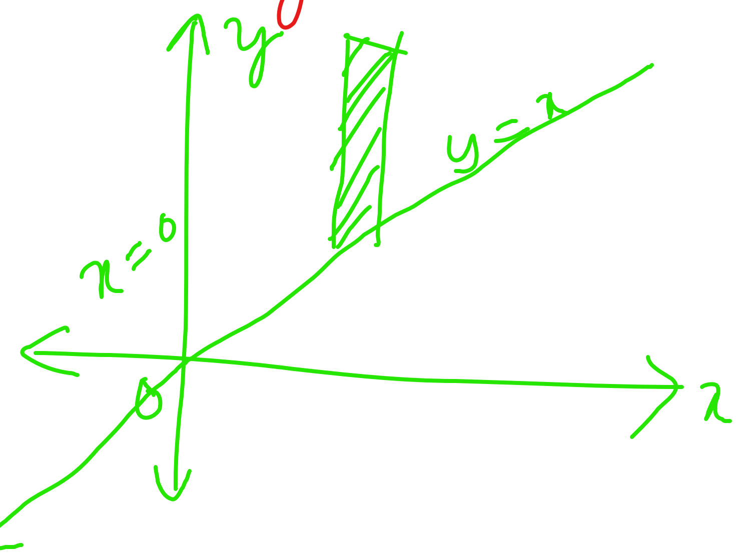
$$= \frac{1}{2} \int_1^2 \left(4x + x^3 - 4x^2 \right) dx$$

$$= \frac{1}{2} \left(\frac{5}{12} \right) = 5/24$$

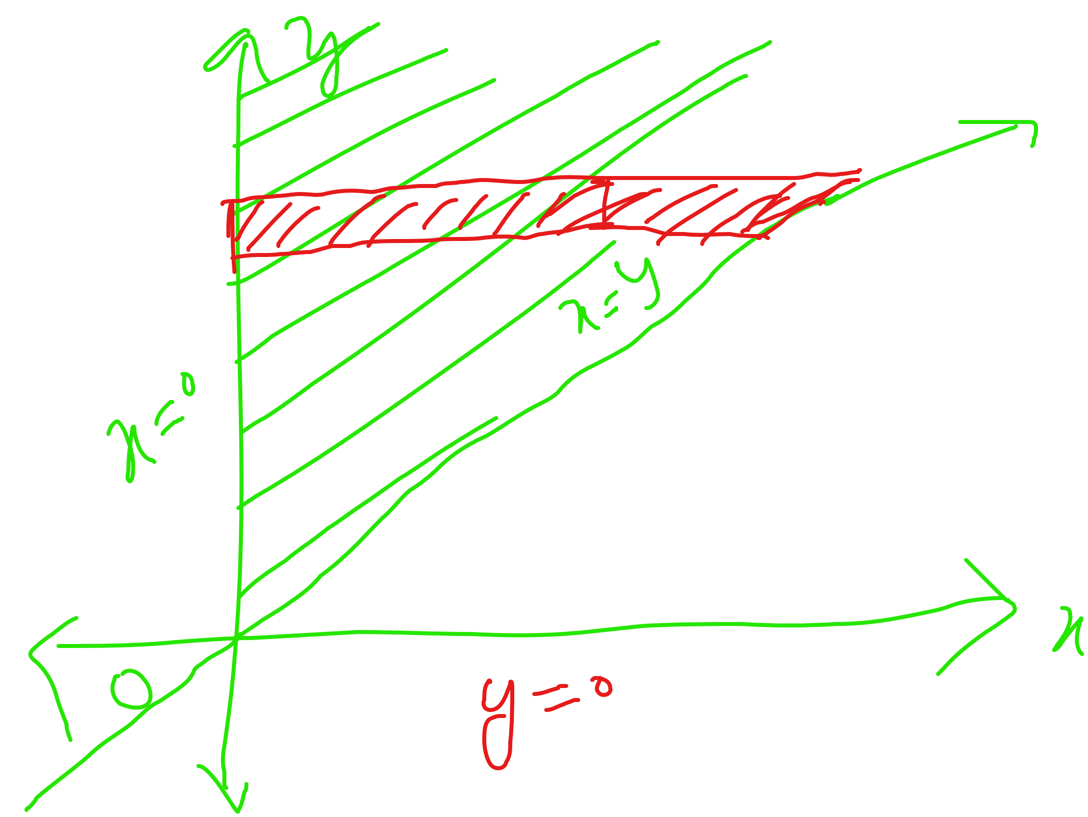
Hence $I = \frac{1}{8} + 5/24 = 8/24 = 1/3$

4. Change the order of integration $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$ and hence evaluate it.

Sol:- The region of integration is bounded by $x=0$ to $x \to \infty$ & $y=x$ to $y \to \infty$



After changing the order



$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$$

$$= \int_{y=0}^{y=\infty} \int_{x=0}^{x=y} \frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\,(x)_0^y\,dy = \int_0^\infty \frac{e^{-y}}{\cancel{y}}\,\cancel{y}\,dy$$

$$= \left(-e^{-y}\right)_0^\infty = -\left(e^{-\infty} - e^{-0}\right)$$

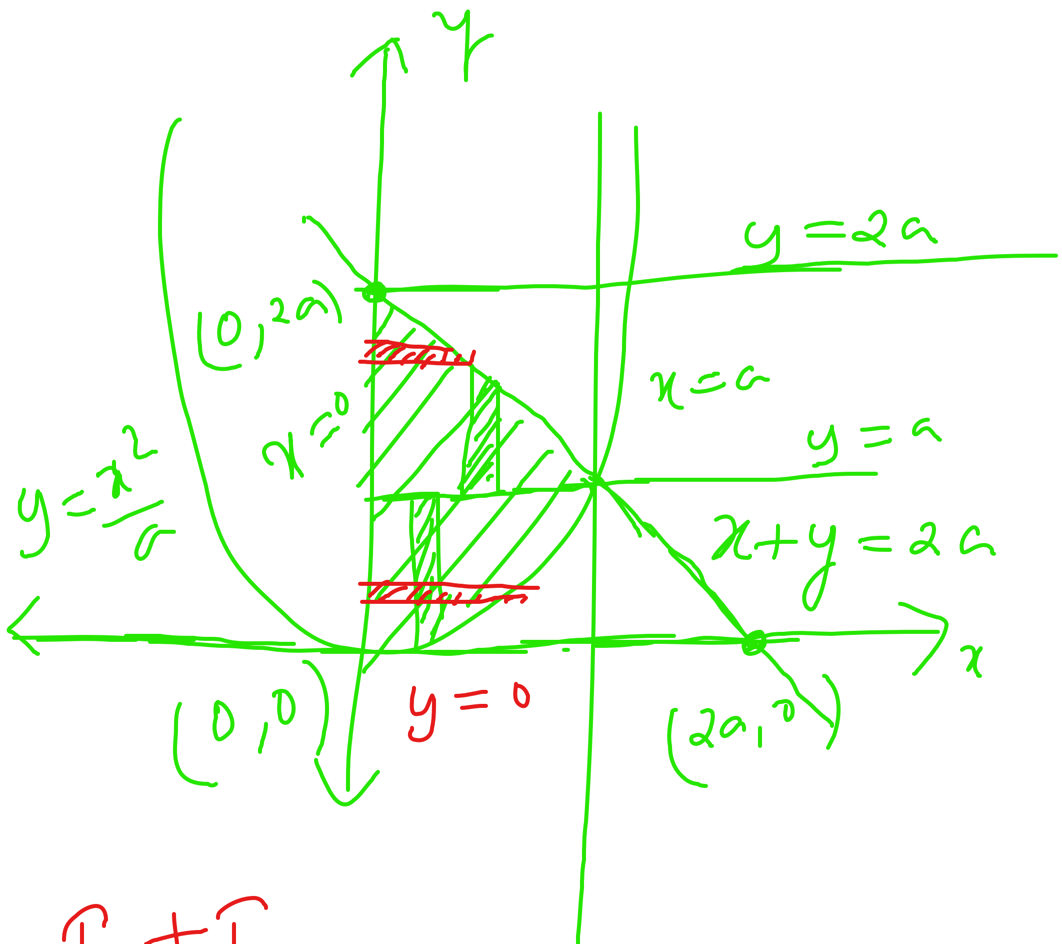
$$= -(0-1) = \underline{\underline{1}}$$

5. Change the order of integration in $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\, dxdy$ and hence evaluate the same.

Sol:-

Given $I = \int_0^a \int_{x^2/a}^{2a-x} xy\, dy dx$

The region of integration is bounded by $x=0$ to $x=a$ & $y=\frac{x^2}{a}$ to $y=2a-x$



Now

$$I = I_1 + I_2$$

$$= \int_{x=0}^{x=a} \int_{y=\frac{x^2}{a}}^{y=a} xy \, dy \, dx + \int_{x=0}^{x=a} \int_{y=a}^{y=2a-x} xy \, dy \, dx \qquad (1)$$

Changing the order of integration in I_1

$$\int_0^a \int_{x^2/a}^{a} xy\,dy\,dx = \int_{y=0}^{y=a} \int_{x=0}^{x=\sqrt{ay}} xy\,dx\,dy$$

$$= \int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{ay}} dy = \frac{a}{2}\int_0^a y^2\,dy$$

$$= \frac{a}{2}\left(\frac{y^3}{3}\right)_0^a = \frac{a^4}{6} \; //$$

Changing the order of integration in I_2

$$\int_{x=0}^{x=a}\int_{y=a}^{y=2a-x} xy\,dy\,dx = \int_{y=a}^{y=2a}\int_{x=0}^{x=2a-y} xy\,dx\,dy$$

$$= \int_{a}^{2a} y\left(\frac{x^2}{2}\right)_0^{2a-y} dy$$

$$= \frac{1}{2}\int_{a}^{2a} y(2a-y)^2\,dy$$

$$= \frac{1}{2}\int_{a}^{2a} y(4a^2+y^2-4ay)\,dy$$

$$= \frac{1}{2}\int_{a}^{2a} \left(4a^2y + y^3 - 4ay^2\right) dy$$

$$= \frac{1}{2}\left[\cancel{4}^{2}a^2\frac{y^2}{\cancel{2}} + \frac{y^4}{4} - 4a\frac{y^3}{3}\right]_a^{2a}$$

$$= \frac{1}{2}\left[\left(8a^4 + \frac{\cancel{16}^{4}a^4}{\cancel{4}} - \frac{48}{3}a^4\right) - \left(2a^4 + \frac{a^4}{4} - \frac{4}{3}a^4\right)\right]$$

$$= \frac{5a^4}{24}$$

$\therefore$ ①, becomes $I = \frac{a^4}{6} + \frac{5a^4}{24} = \frac{9a^4}{24}$

$$= 3/8\, a^4$$