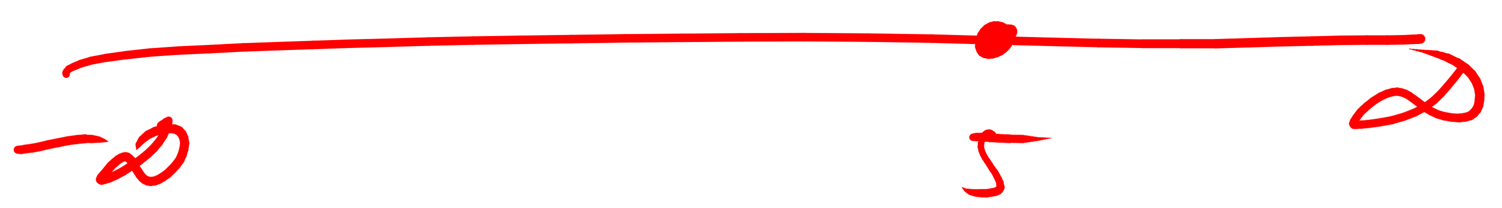
CALCULUS

BMAT101L

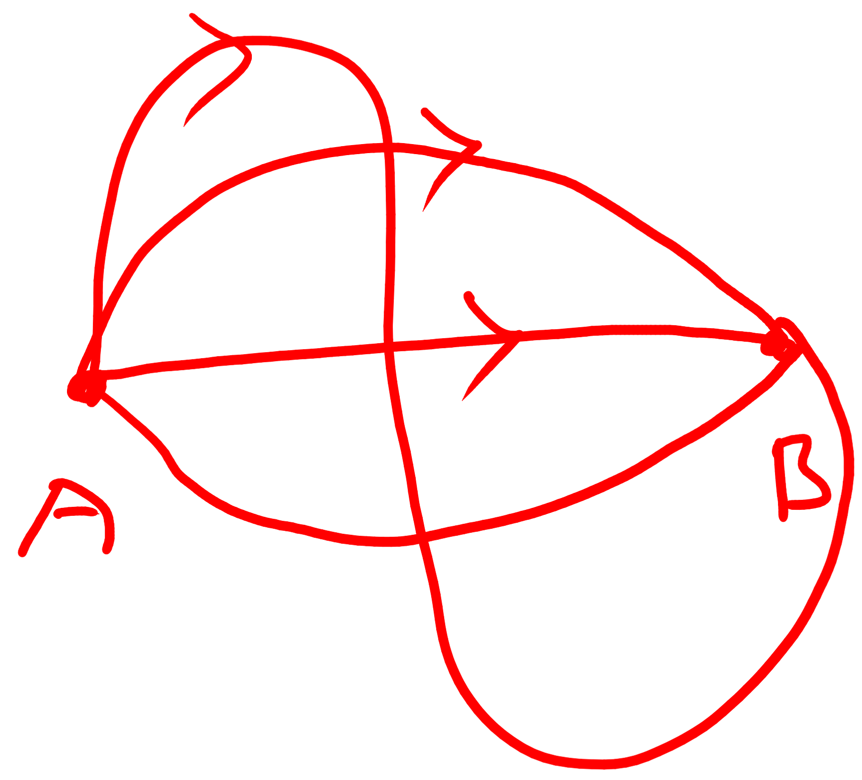
Module - 6

S



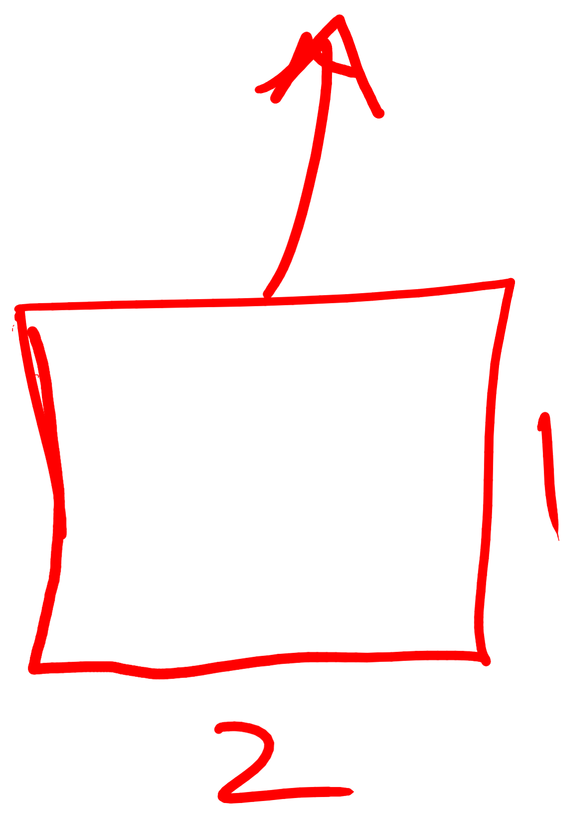
$\mathbb{R}$

vectors: $3\vec{i} + 4\vec{j} + 5\vec{k}$

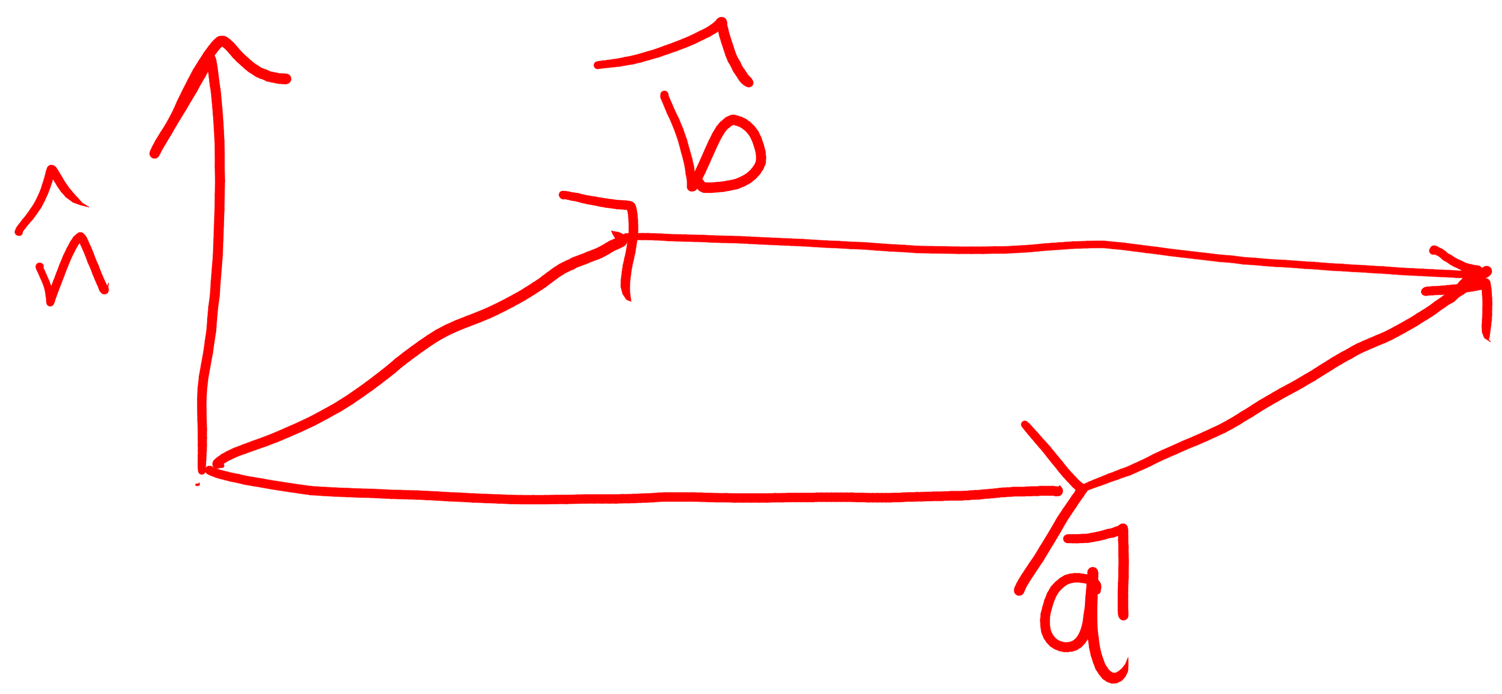


scalar	vector
distance	displacement
mass	weight
Speed	velocity
density	Force
Area	

5 units



$\overline{dA}$



$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$

area of

$\int \vec{a} \; \vec{b}$

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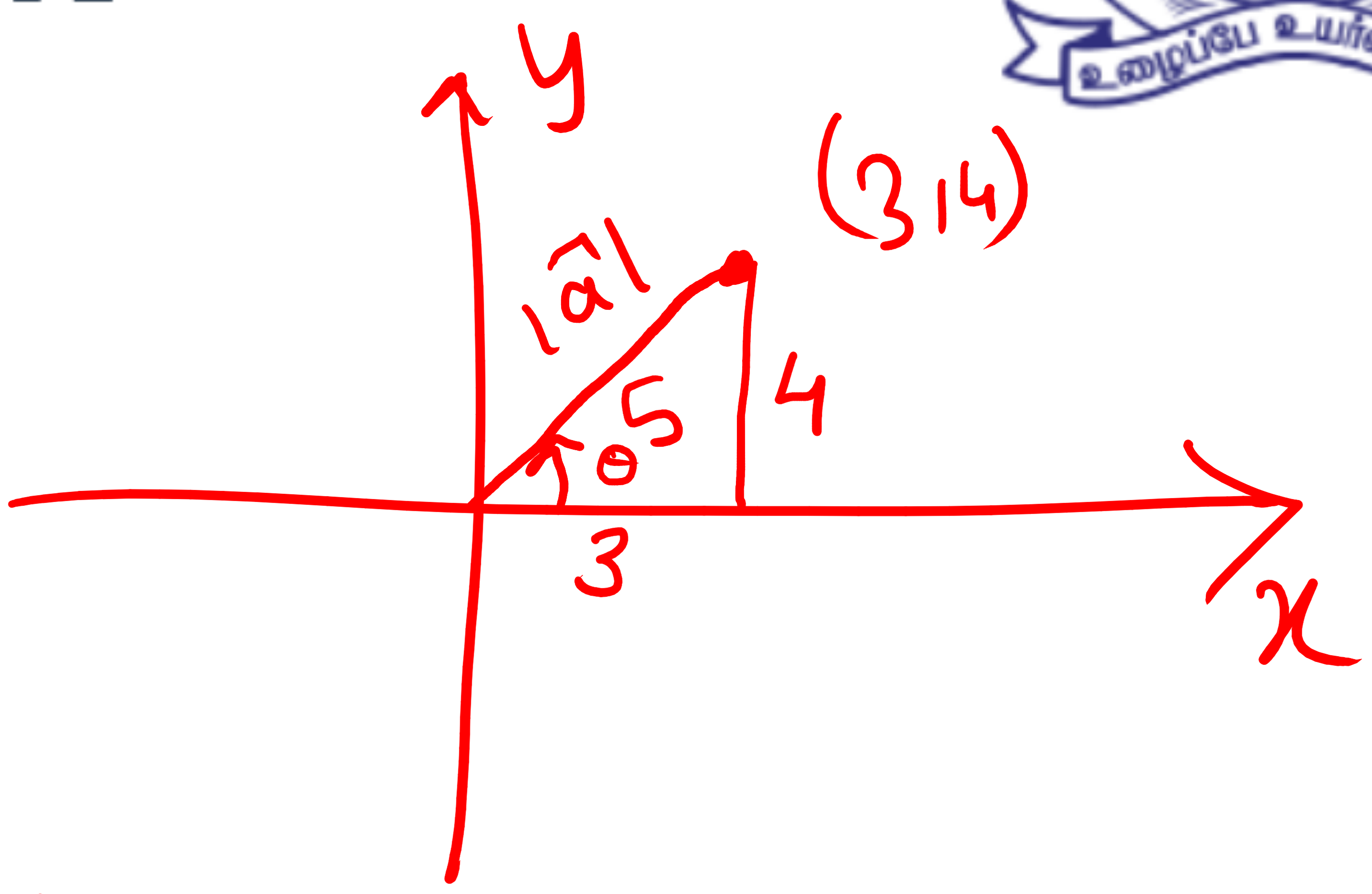
$\vec{a} = 3\vec{i} + 4\vec{j}$

$|\vec{a}| = \sqrt{3^2+4^2}$

$= 5 \qquad \tan\theta = \frac{4}{3}$

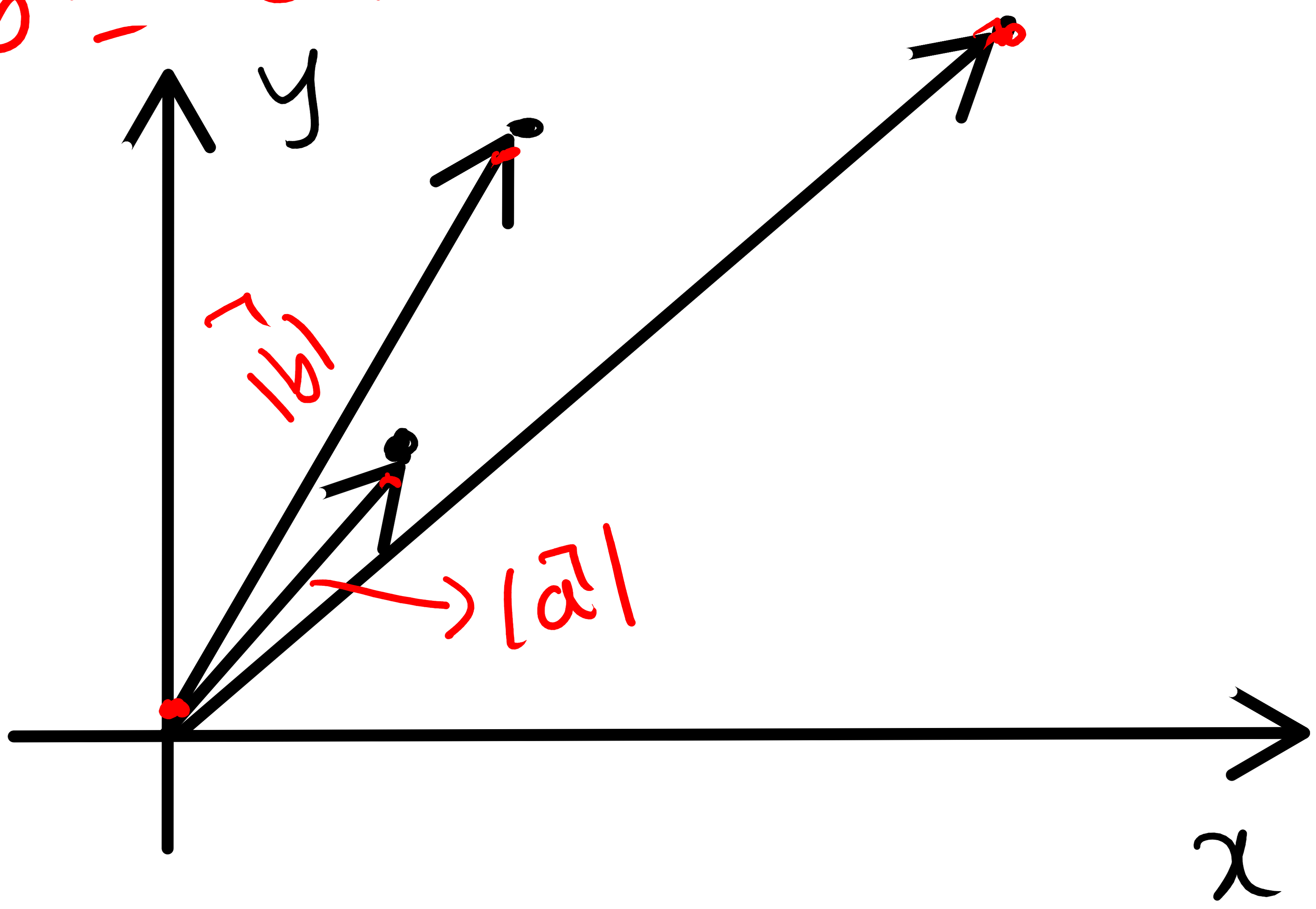


$\theta = \tan^{-1}\frac{4}{3} = 53^\circ$

$\vec{a} = 3\vec{i} + 4\vec{j}$

$\vec{b} = 5\vec{i} + 6\vec{j}$

$\vec{a} + \vec{b} = 8\vec{i} + 10\vec{j}$



$|\vec{a}| = 5$

$|\vec{b}| = \sqrt{61} = 7.8106$

$|\vec{a}+\vec{b}| = \sqrt{164} = 12.806$

$\underline{|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|}$

$\vec{a} - \vec{b}$

+, −, dot, cross

$a \in \mathbb{R}, \; b \in \mathbb{R}$

$a+b \in \mathbb{R}$

$a-b \in \mathbb{R}$

$a \times b \in \mathbb{R}$

$\underline{3}\vec{i} + \underline{4}\vec{j} \in \underline{\mathbb{R}^2}$

$(3,4) \in \mathbb{R}\times\mathbb{R}$

$\vec{a} \in \mathbb{R}^2$

$\vec{b} \in \mathbb{R}^2$

$\vec{a}+\vec{b} \in \mathbb{R}^2$

$\vec{a}-\vec{b} \in \mathbb{R}^2$

$3\vec{i} + 4\vec{j} + 5\vec{k} \in \mathbb{R}^3$

$\vec{a} \cdot \vec{b} \in \mathbb{R}$

$4I + 8J + 12K$ ✓

$(4, 8, 12)$ ✓

$4\vec{i} + 8\vec{j} + 12\vec{k}$ ✓

3 $\quad (9, 12)$

$3x\,\vec{i} + 4x\,\vec{j}$

$\cdot : \mathbb{R}^2 \to \mathbb{R}$

$\times : \mathbb{R}^2 \to \mathbb{R}^2$

$+ : \mathbb{R}^2 \to \mathbb{R}^2$

$- : \mathbb{R}^2 \to \mathbb{R}^2$

$f : \mathbb{R} \to \mathbb{R}^2$

$\mathbb{R} \to \mathbb{R}^2$

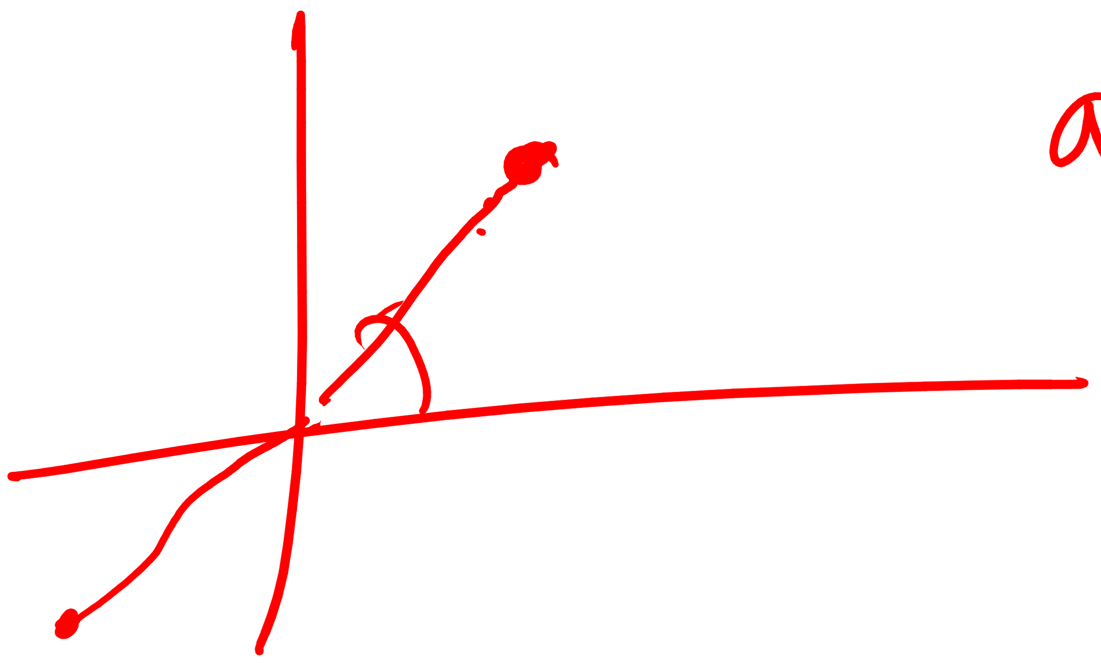
$f : \mathbb{R} \to \mathbb{R}$

$2 \to 4$

$3 \to 6$

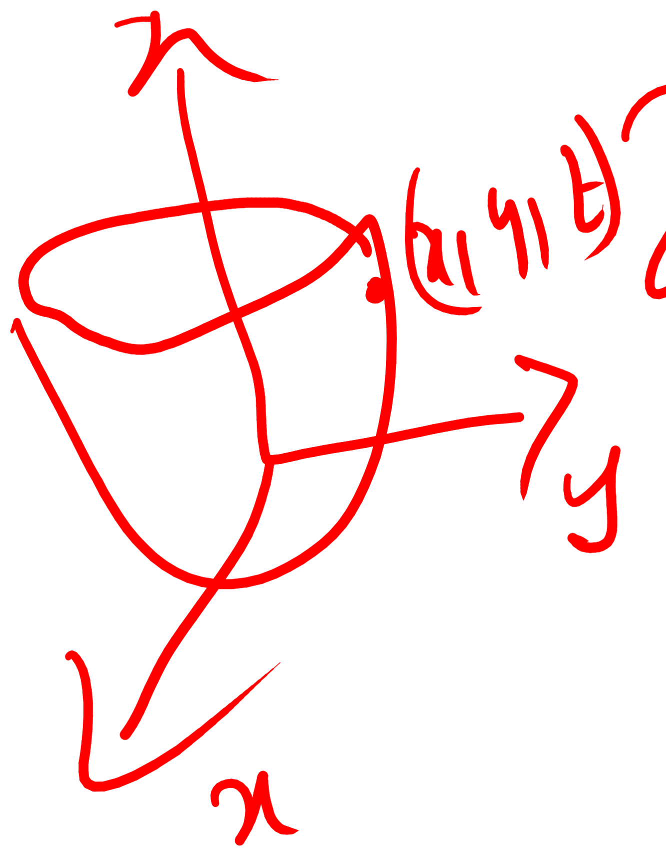
$4 \to 8$

$f = 2x$



$a\vec{i} + b\vec{j}$ $\quad (3, 4)$

$\mathbb{R}^n \to \mathbb{R}^m$

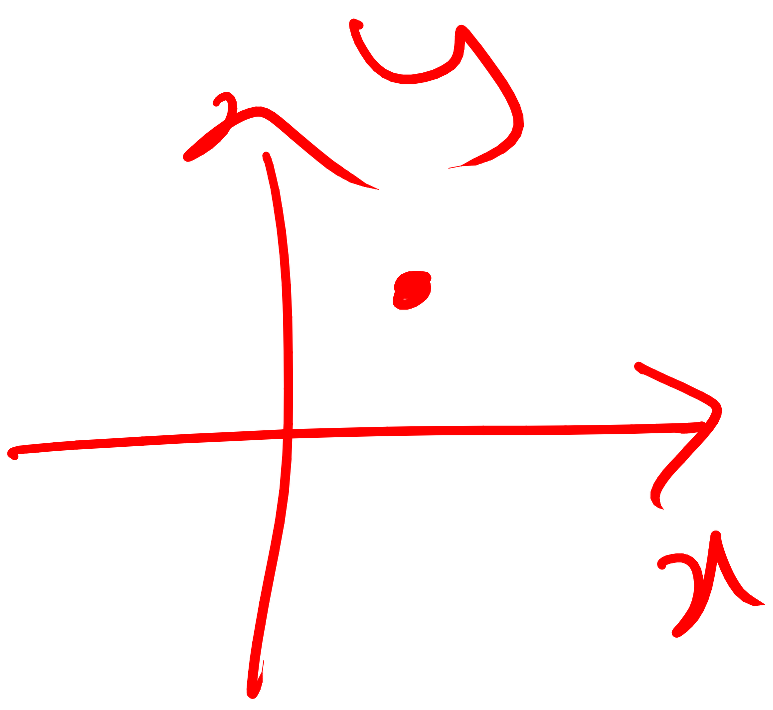


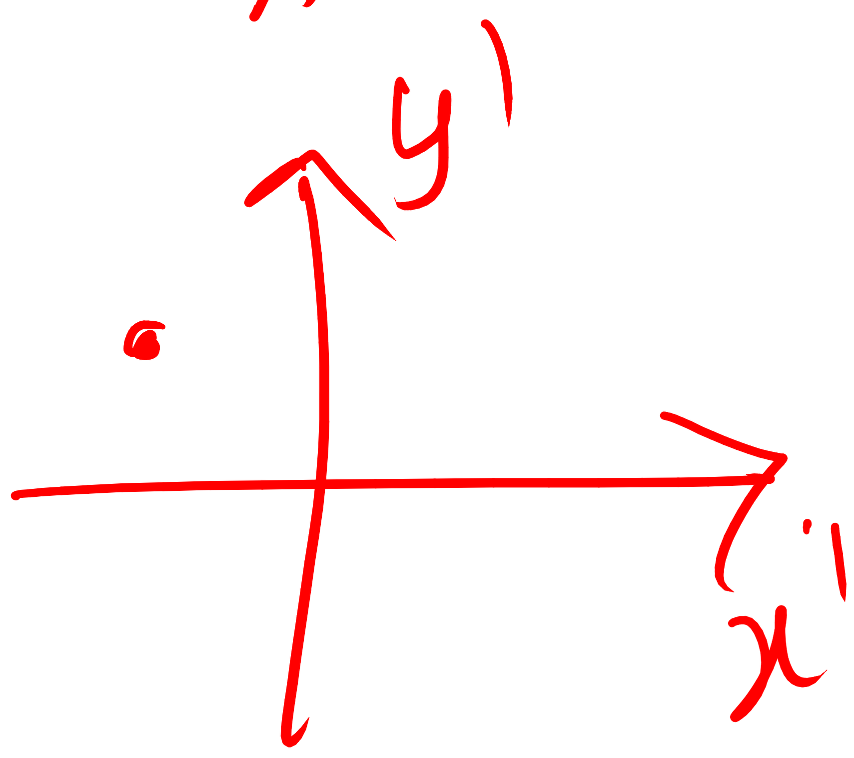
(x, y, z) $\quad f(x, y) = x^2 + y^2$ $\quad \mathbb{R}^2 \to \mathbb{R}$

$z = x^2 + y^2$ $\quad \mathbb{R}^3 \to \mathbb{R}$

$f(x, y, z) = x^2 + y^2 + z^2$ $\quad \mathbb{R}^3 \to \mathbb{R}^2$

$\mathbb{R}^2 \to \mathbb{R}^2$





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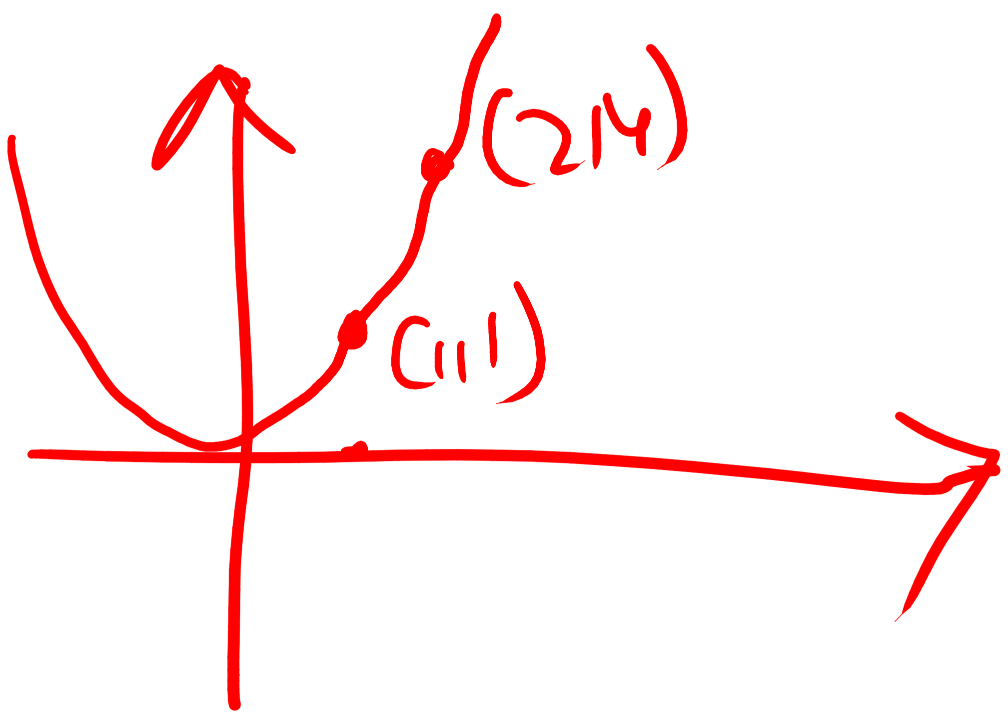
$(4x+3y)\vec{i} + (x^2+y^2)\vec{j} \to$ Vector valued fn.

$\mathbb{R}^2 \to \mathbb{R}^2$

$(1,2) \to (10,5)$

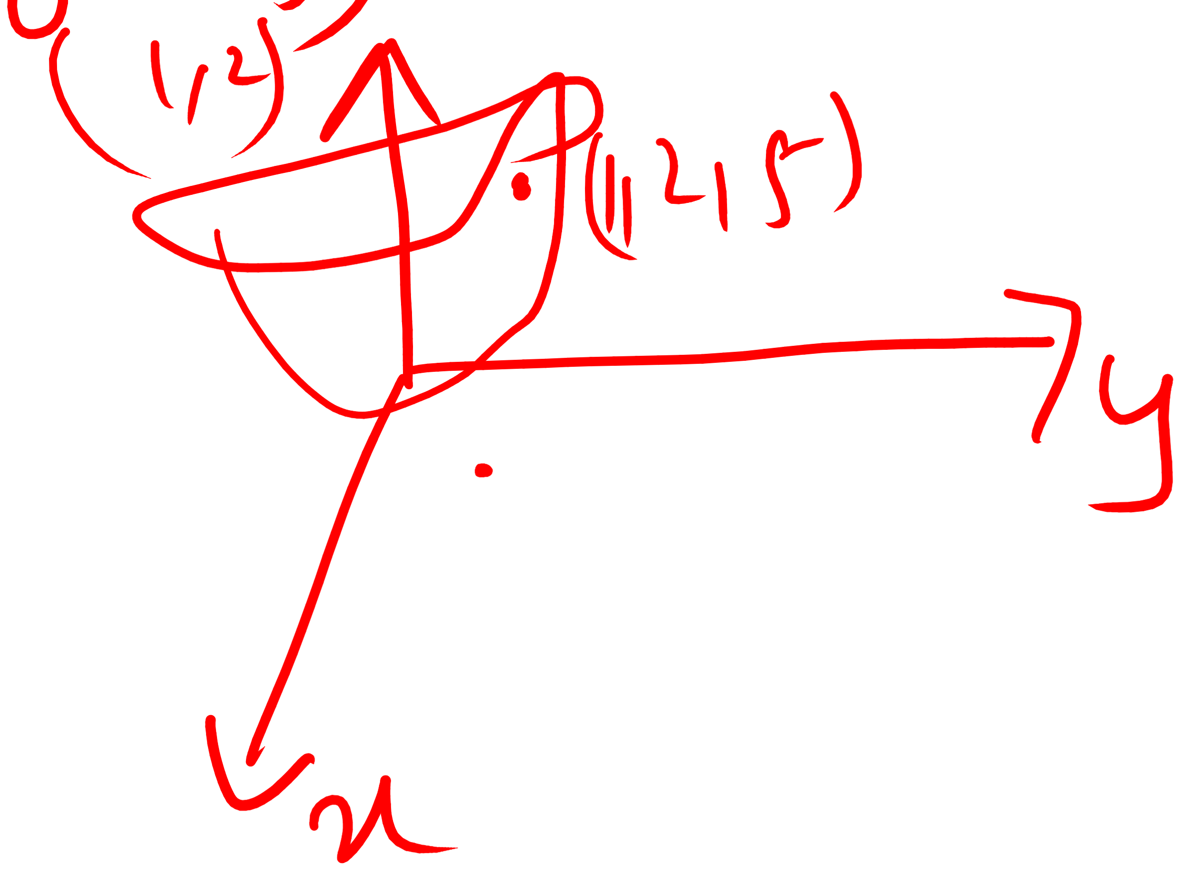
$f:\mathbb{R}\to\mathbb{R}$

$f(x)=x^2$



$f:\mathbb{R}^2\to\mathbb{R}$

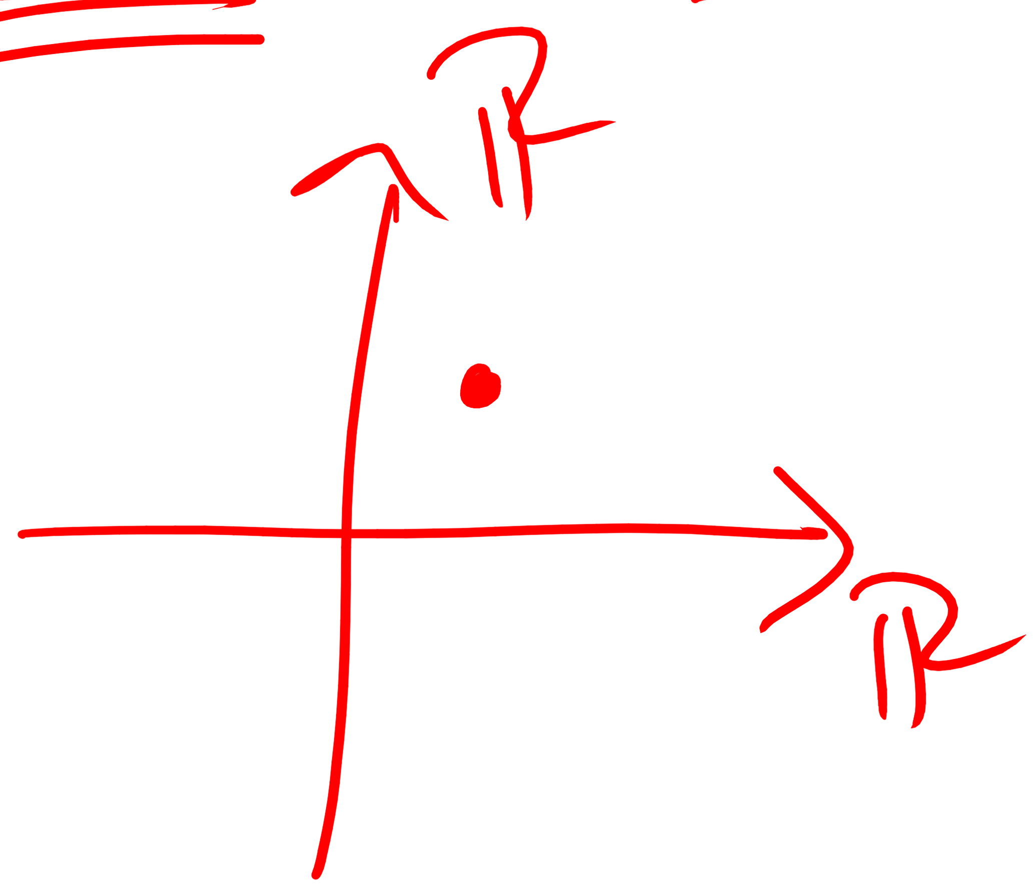
$f(x,y) = x^2+y^2$

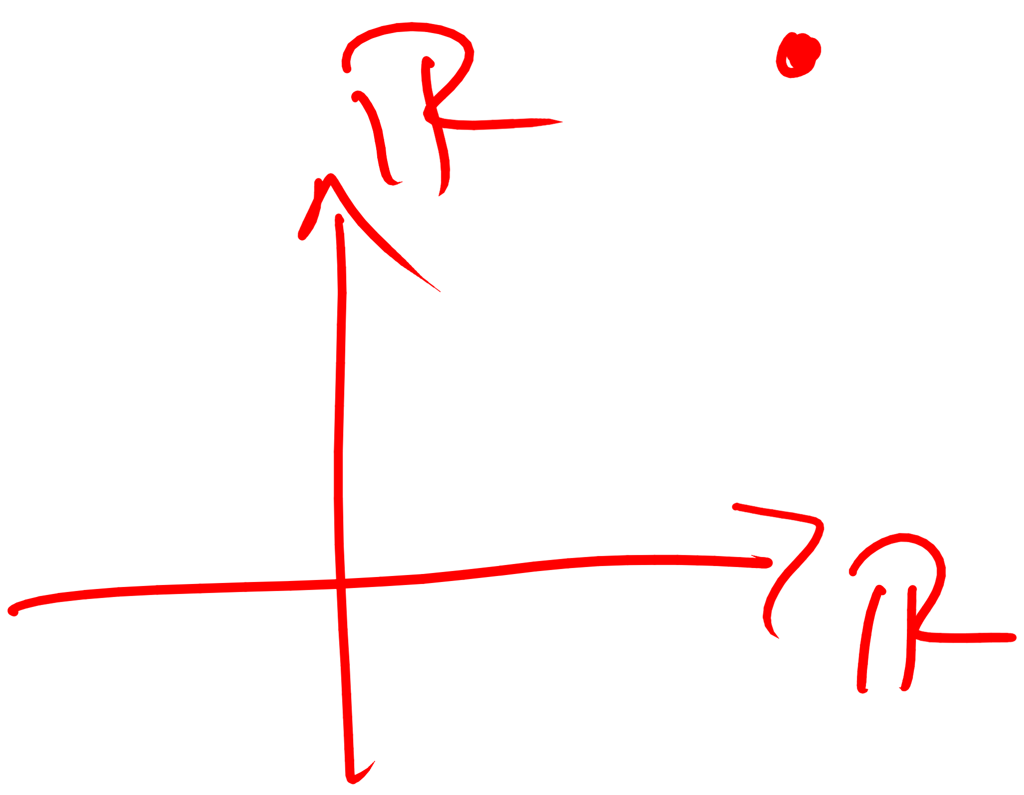


$f(x,y) : (3x+4y)\vec{i} + (x^2+y^2)\vec{j}$

$f(1,2) = (11,5)$

$\underline{\underline{(1,2)}} \to \underline{\underline{(11,5)}}$





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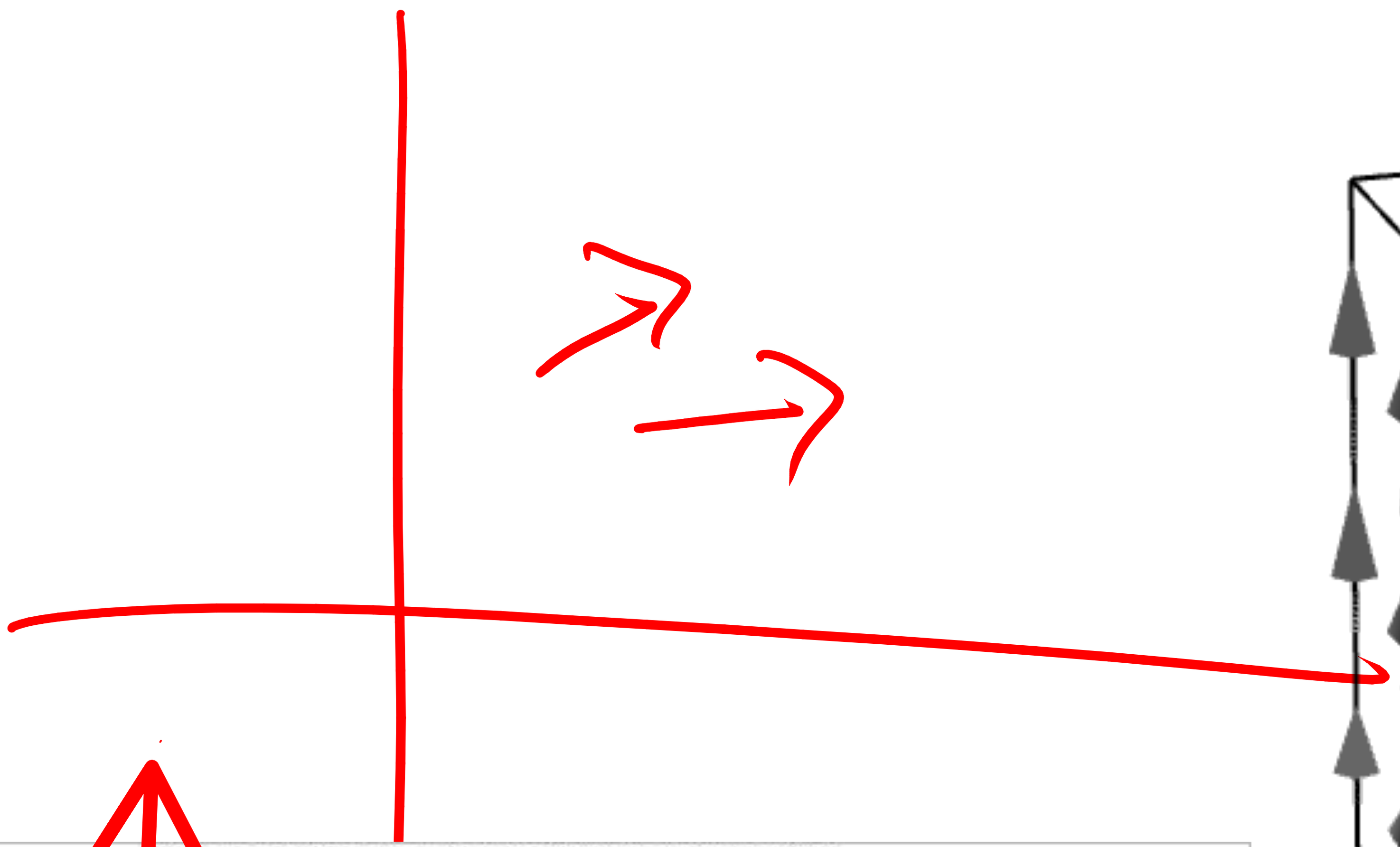
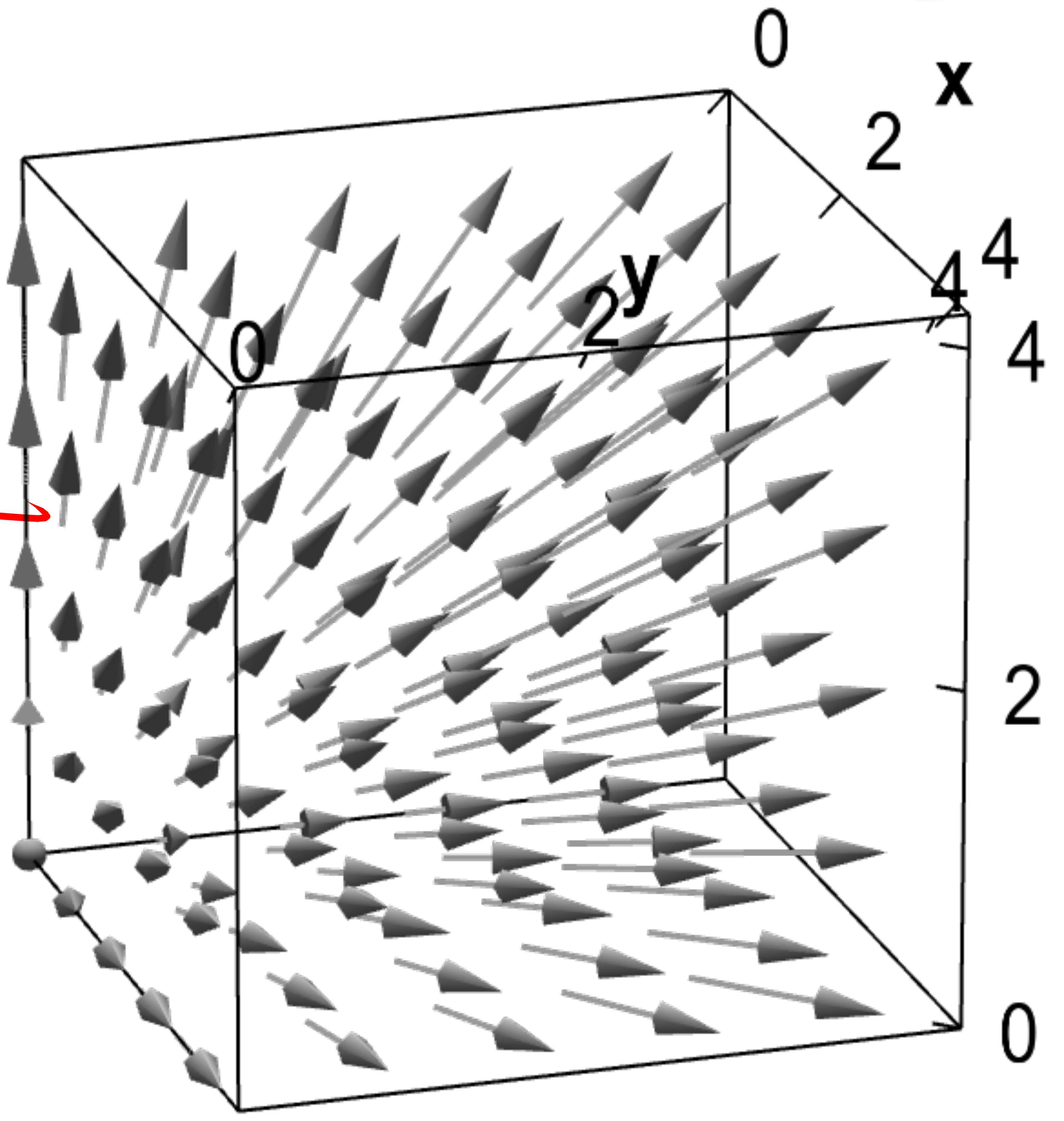
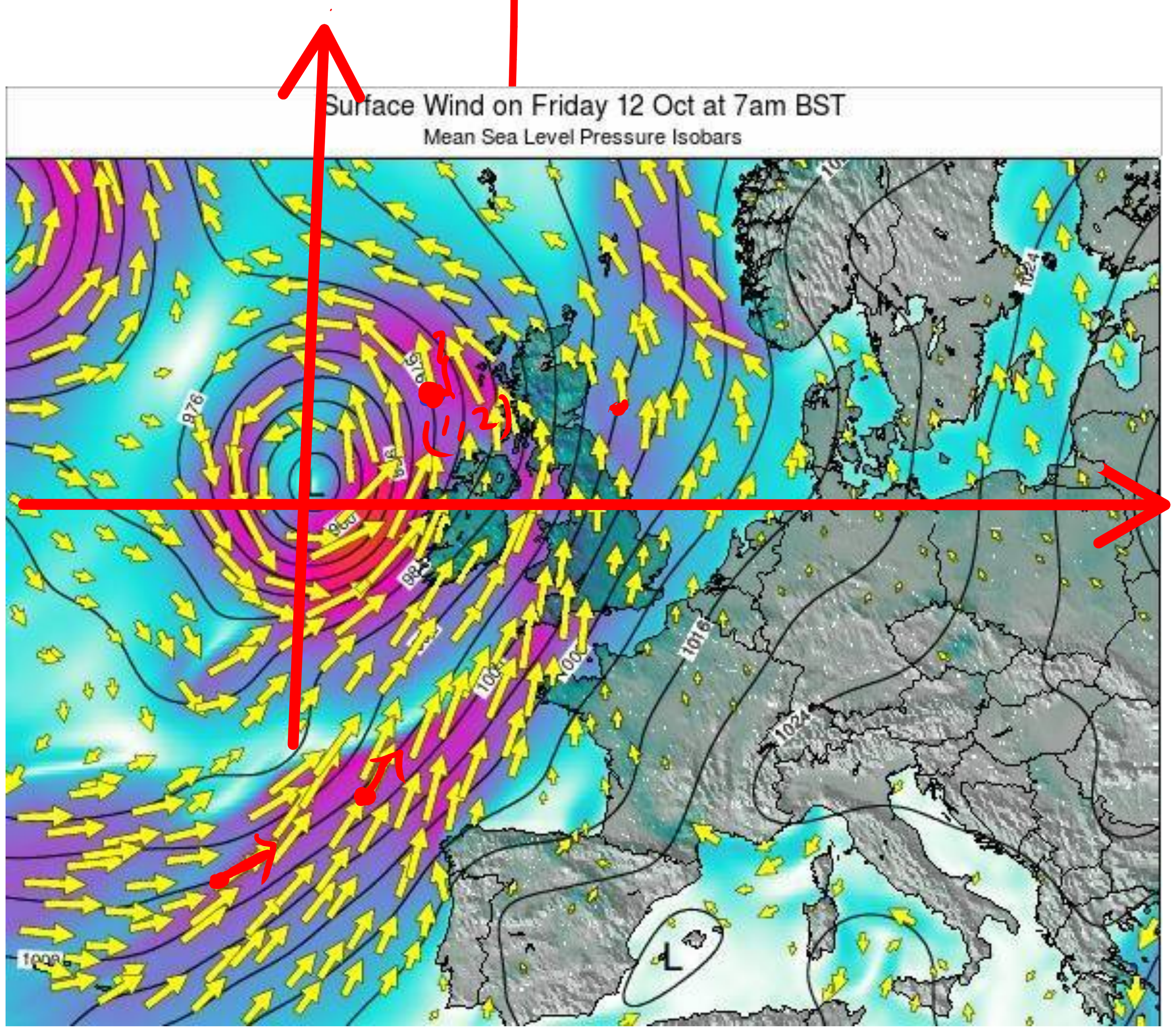
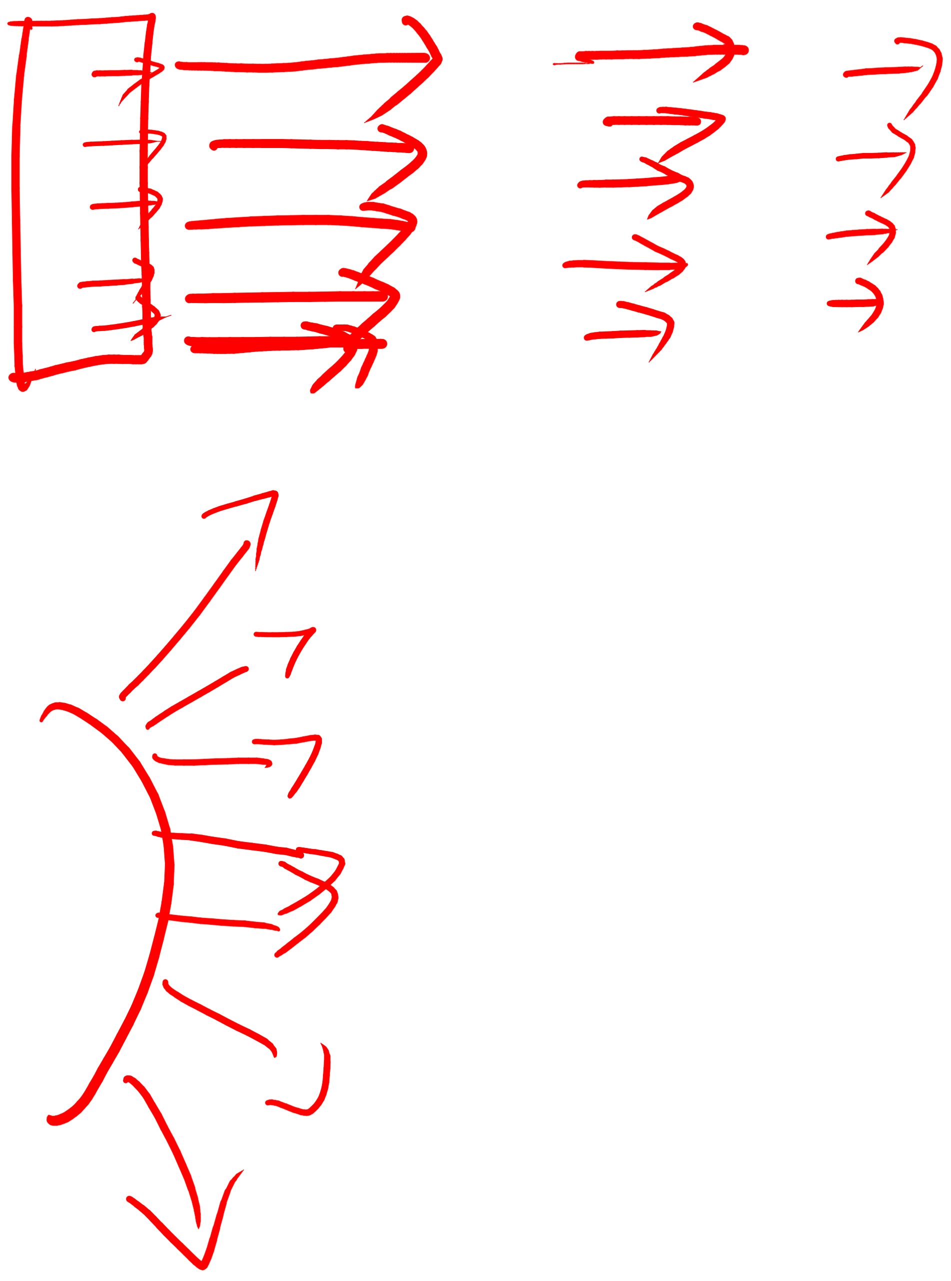


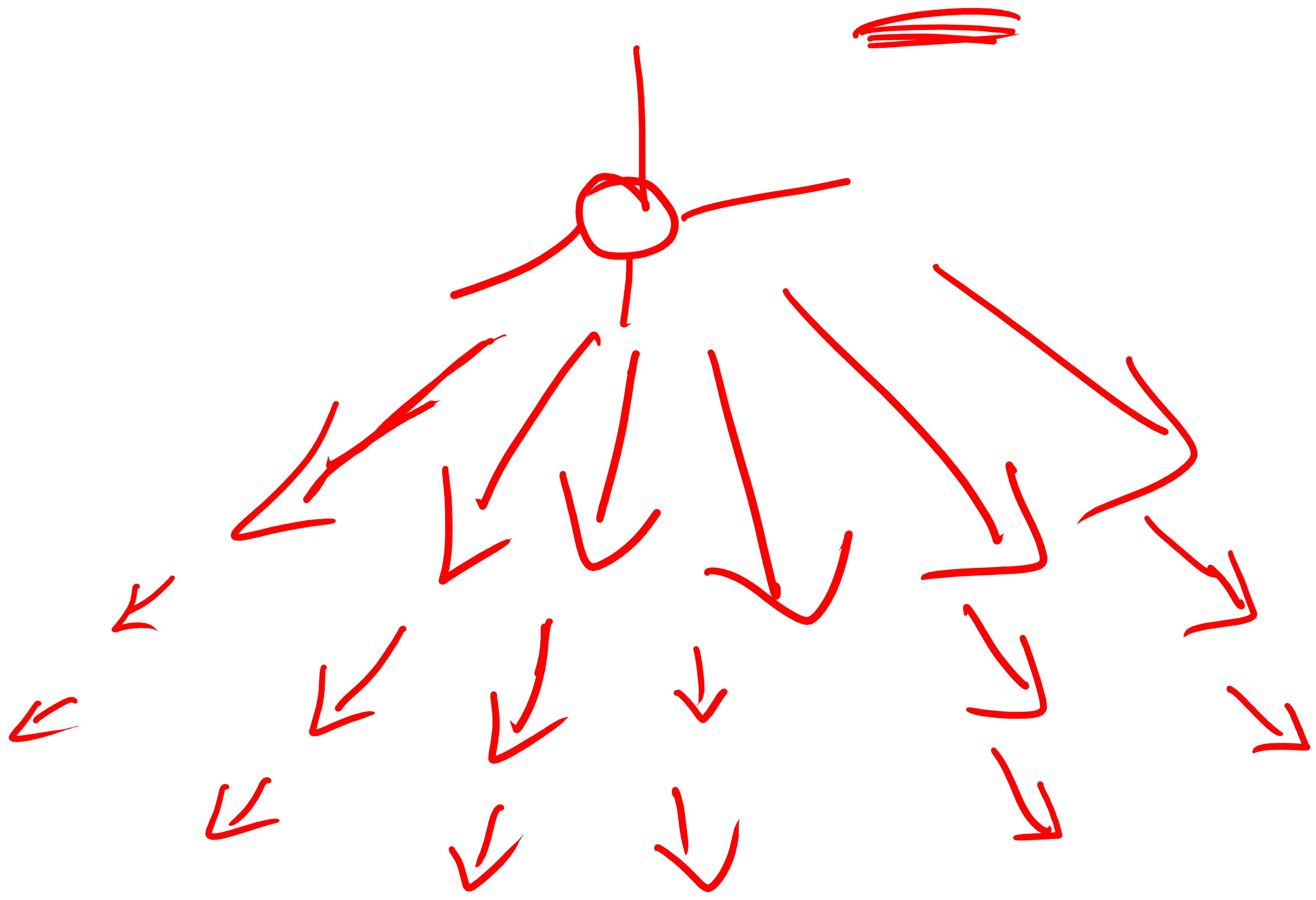




$(4x+3y)\,i + (x^2+y^2)\,j$

$(1,2) \to (10,5)$ $\sqrt{125}$





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Find the area enclosed by $x^4+y^4=1$

$$\text{Area} = \iint_D dy\,dx$$

$$\iint_D x^{l-1}\,y^{m-1}\,dx\,dy = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}h^{l+m}$$ *where D is the domain* $x \geq 0,\ y \geq 0$ *and* $x+y \leq h$.

$$\iint_D dx\,dy = \frac{1}{16}\iint_R u^{-3/4}\,v^{-3/4}\,du\,dv$$

$x^4 = u$

$y^4 = v$

$u+v \leq 1$

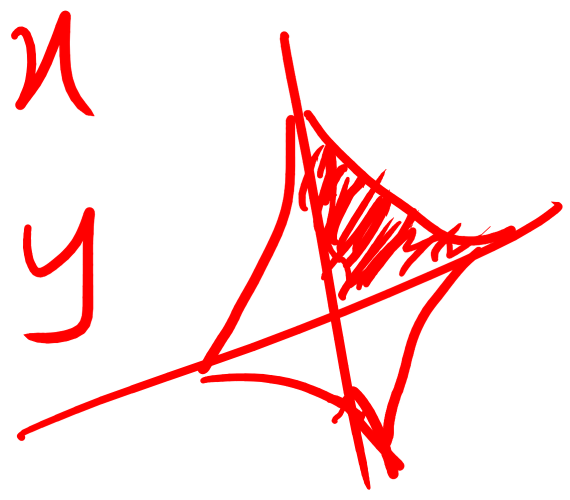
$x^4+y^4=1$

$dx = \frac{1}{4x^3}du$

$= \frac{1}{4}u^{-3/4}du$

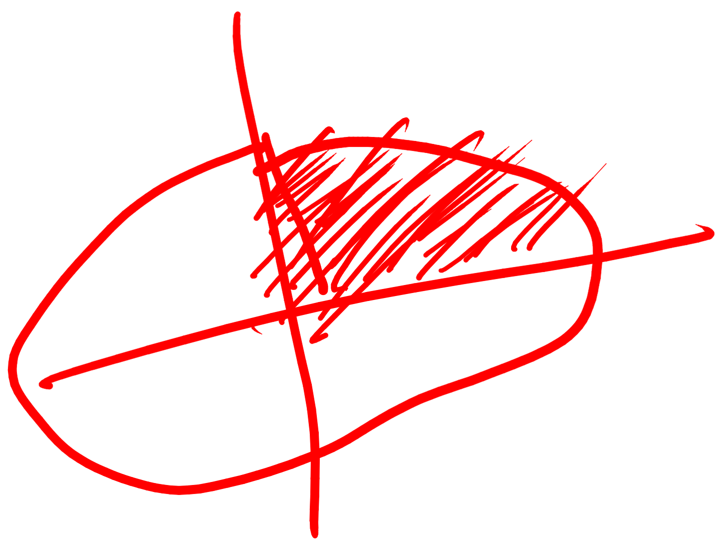
$du = 4x^3\,dx$

$dv = 4y^3\,dy$



$x^2+y^2=1$

$dy = \frac{1}{4}v^{-3/4}dv$



$$= \frac{1}{16}\,\frac{\Gamma(1/4)\,\Gamma(1/4)}{\Gamma(3/2)} = \frac{[\Gamma(1/4)]^2}{8\,\frac{\sqrt{\pi}}{2}} \times 4$$

$\Gamma(\tfrac{3}{2}) = \Gamma(1+\tfrac{1}{2}) = \tfrac{1}{2}\Gamma(1/2)$

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$$z = 10xye^{-x-y} = 10xye^{-x}e^{-y}$$

$$z_x = 10ye^{-y}\left[-xe^{-x} + e^{-x}\right] = 0$$

$$z_y = 10xe^{-x}\left[-ye^{-y} + e^{-y}\right] = 0$$

$$-10xye^{-x-y} + 10ye^{-x-y} = 0$$

$$-10xye^{-x-y} + 10xe^{-x-y} = 0$$

$\underline{x = y}$

$$-10x^2e^{-2x} + 10xe^{-2x} = 0$$

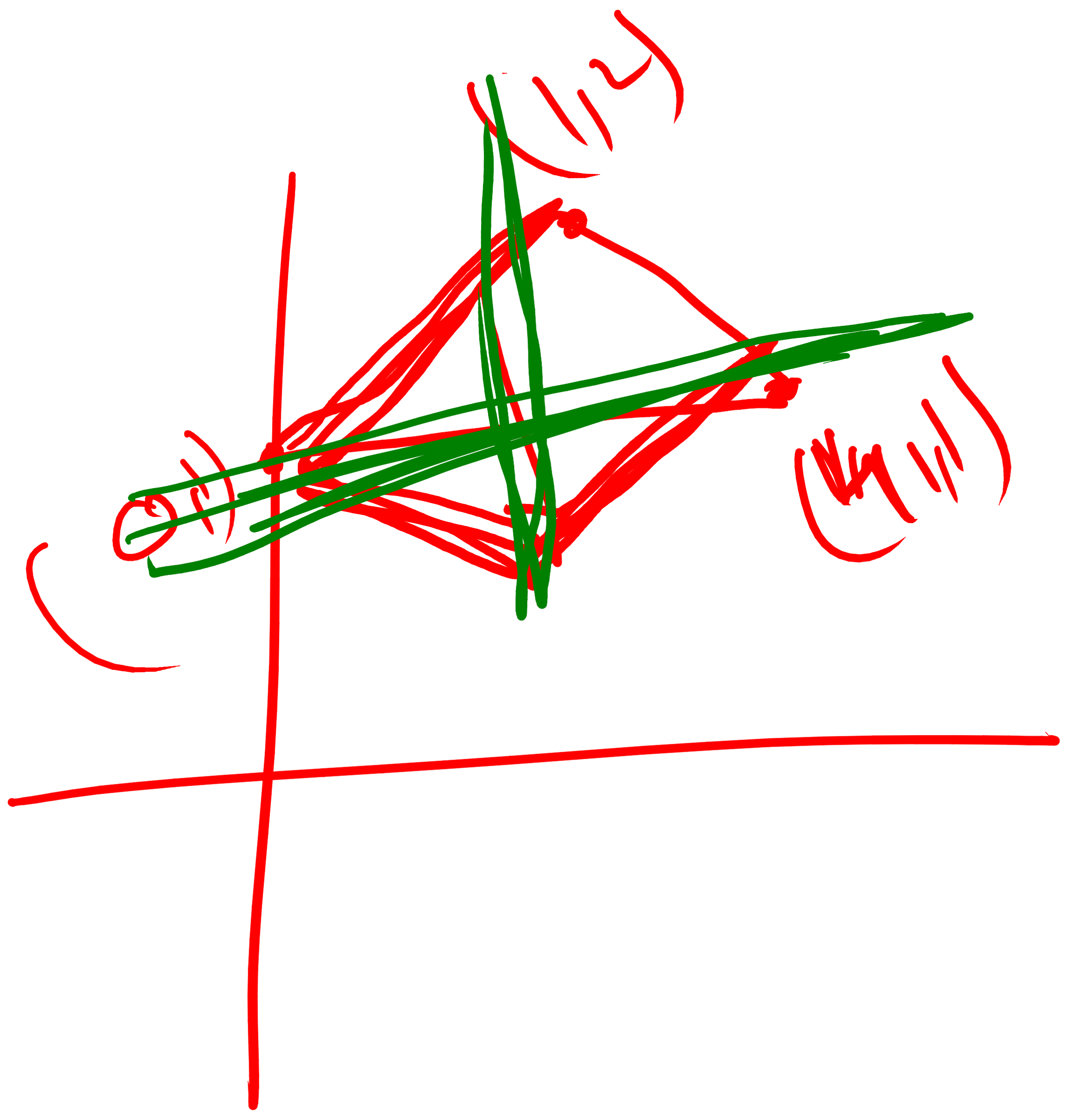
$$10e^{-2x}\left[-x^2 + x\right] = 0$$

$$-x^2 + x = 0$$

$x = 0$ or $x = 1$

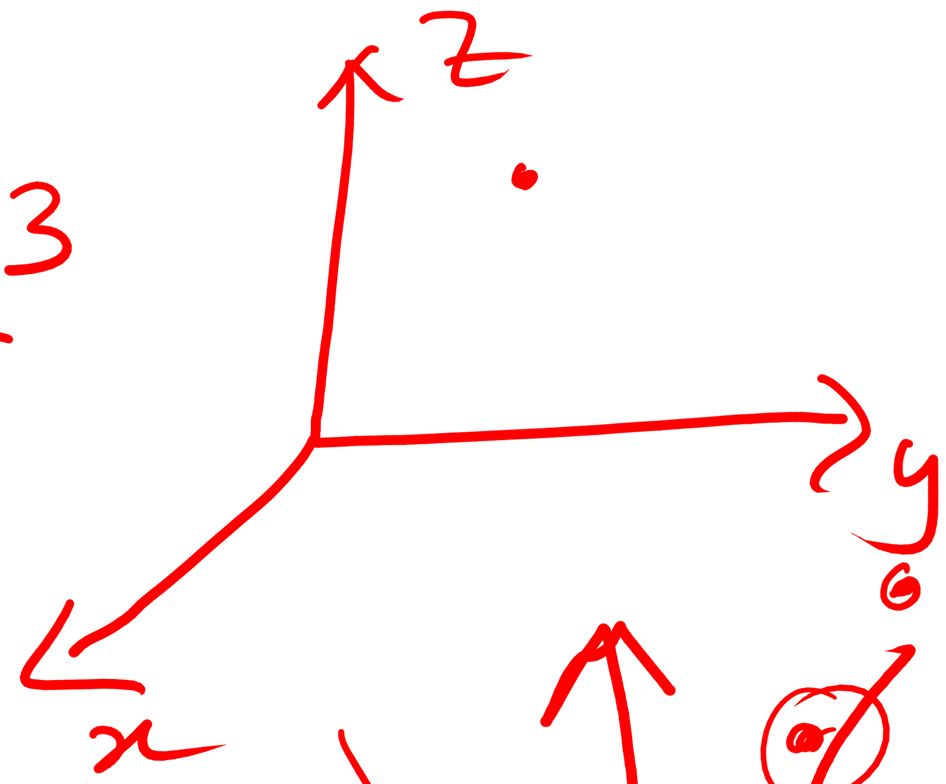
$y = 0$ $\quad y = 1$

$(0,0)$ $(1,1)$



$3\vec{i} + 4\vec{j} + 5\vec{k}$

$(3,4,5) \in \mathbb{R}^3$



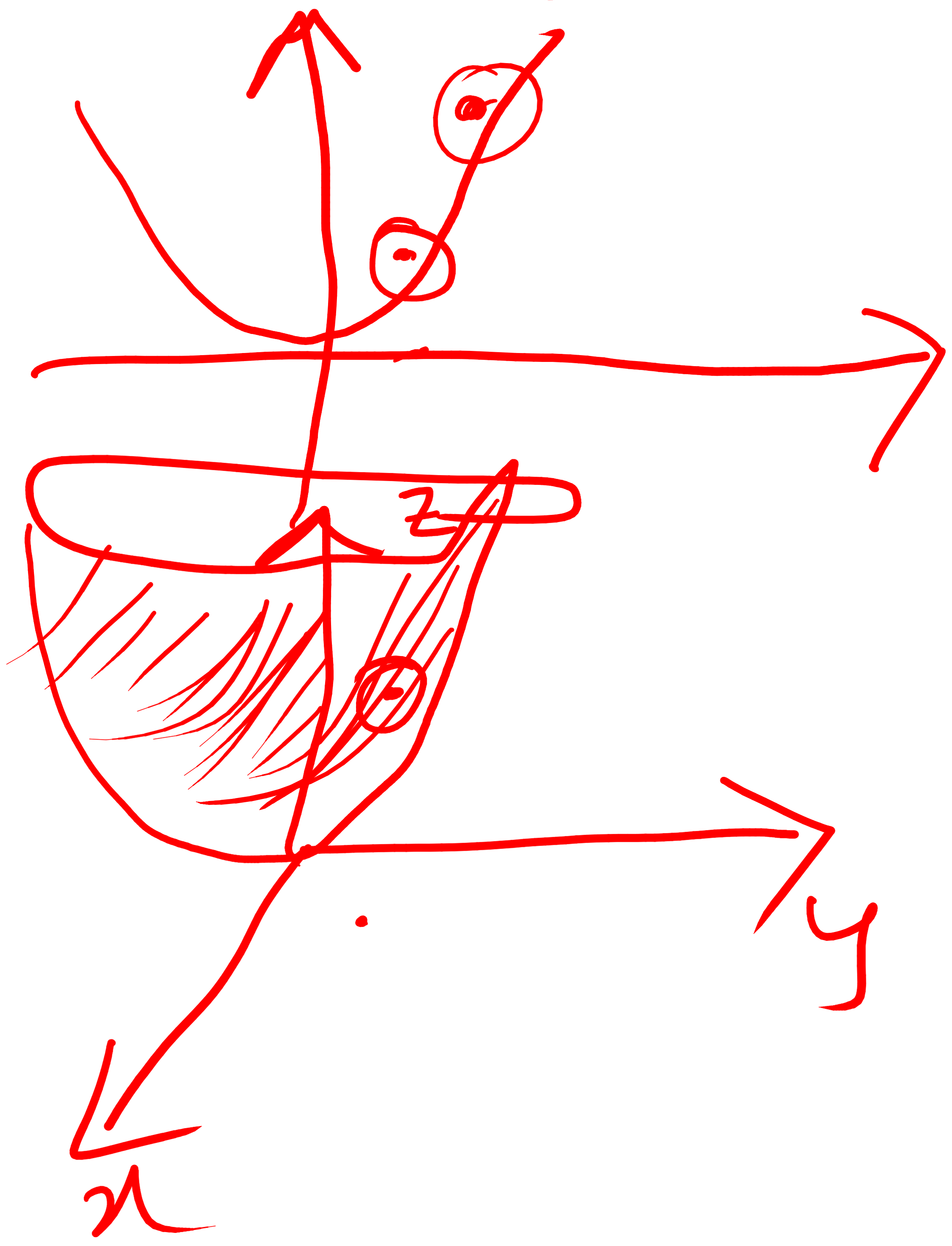
$f(x) = x^2$

$f : \mathbb{R} \to \mathbb{R}$

$f(x,y) = x^2 + y^2$

$f : \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \to z$



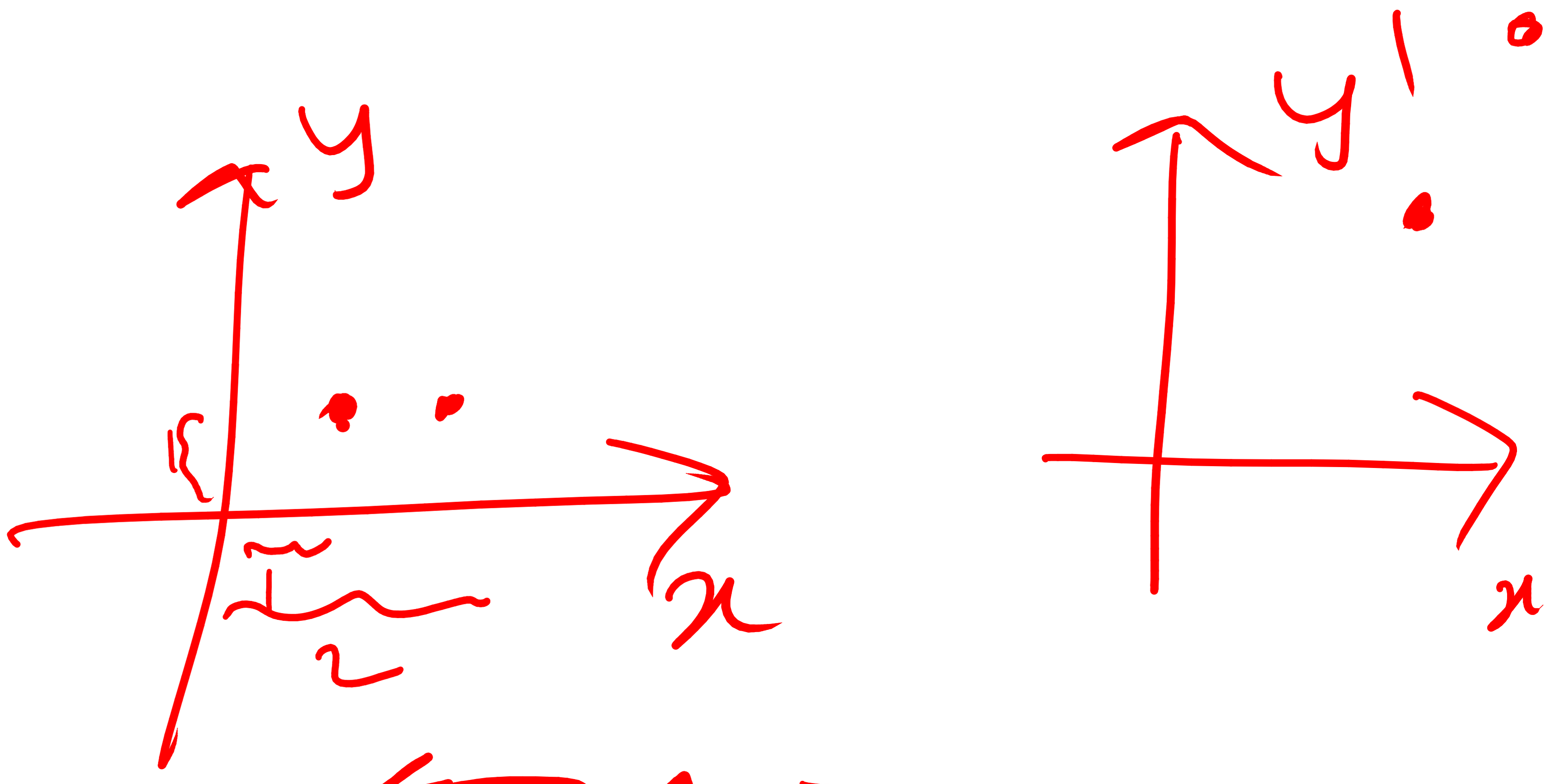
$\vec{F} = (x+y)\vec{i} + (x^2+y^2)\vec{j}$

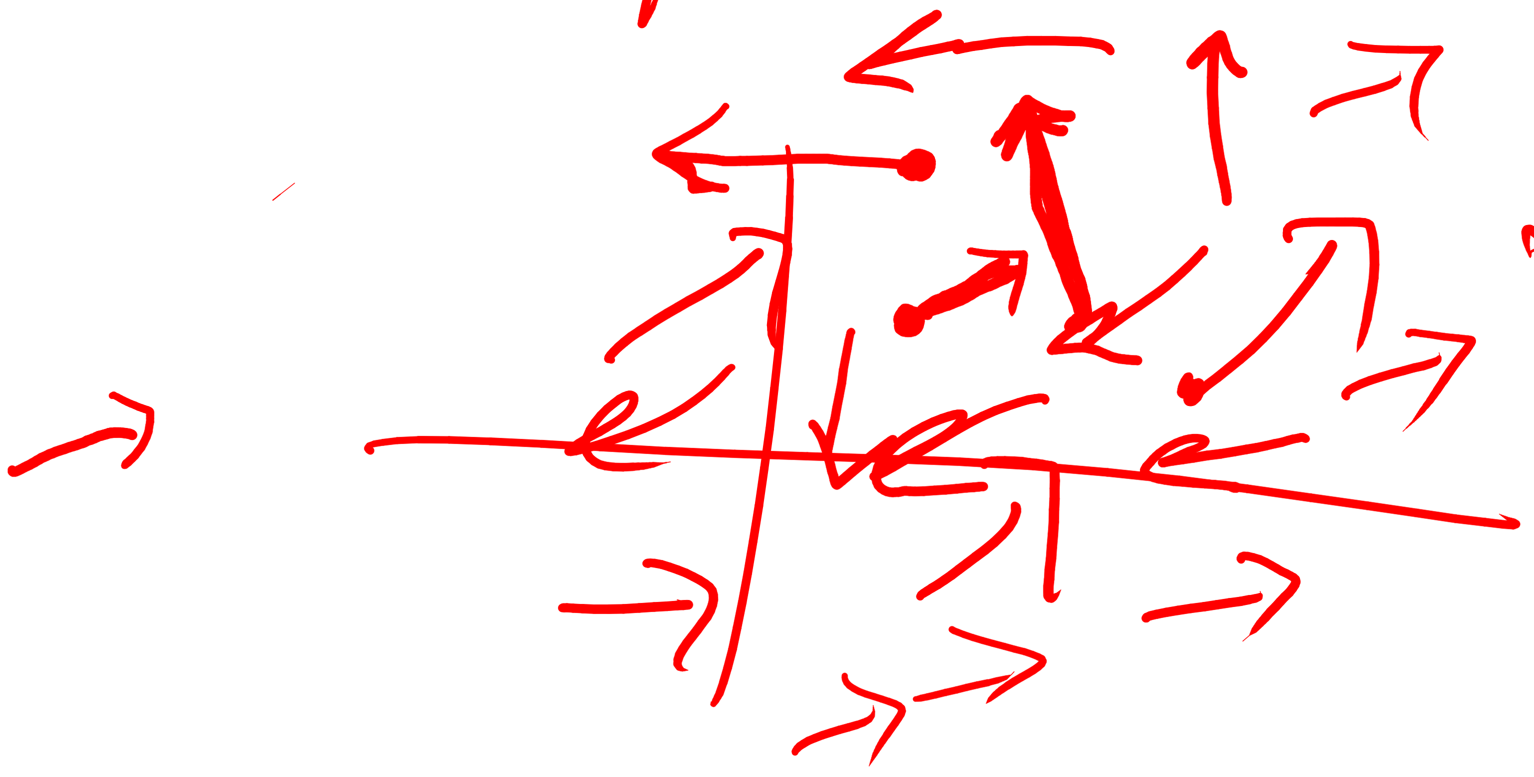
$\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$

$(1,1) \quad 2\vec{i} + 2\vec{j}$

$(2,2)$

$(2,1) \quad (3,5)$





vector field

$x\vec{i} + y\vec{j}$

$\vec{f}: \mathbb{R}^2 \to \mathbb{R}^2$

$(1,1)$ $\quad \vec{i}+\vec{j}$

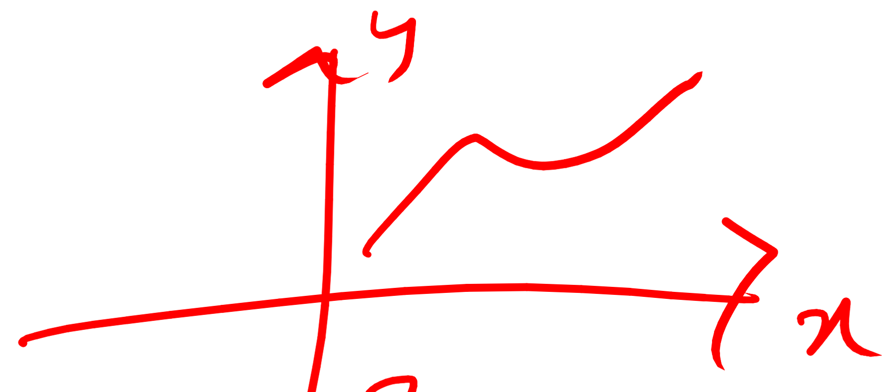
$(1,1)$

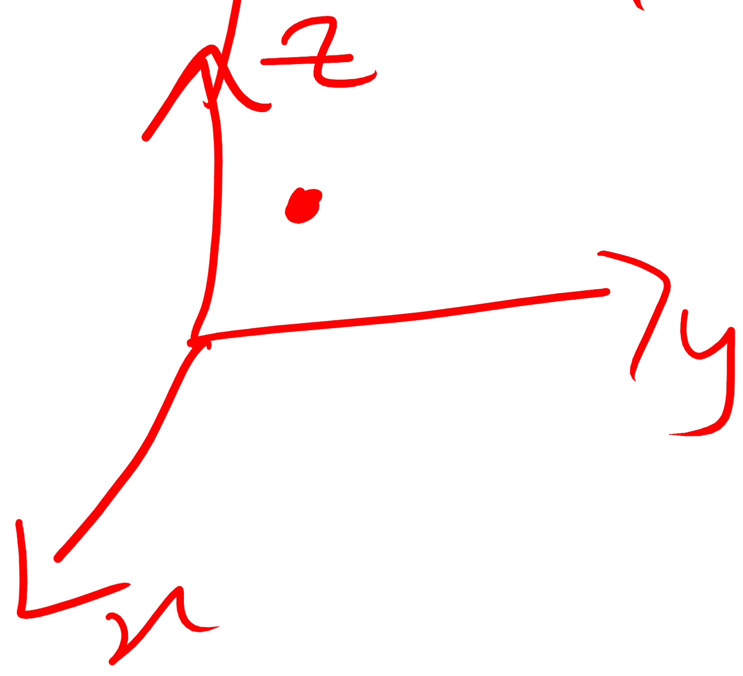
$(5,6)$ $\quad (5,6)$

$f: \mathbb{R} \to \mathbb{R}$

$f: \mathbb{R}^2 \to \mathbb{R}$

$\underline{f = x^2 + y^2}$



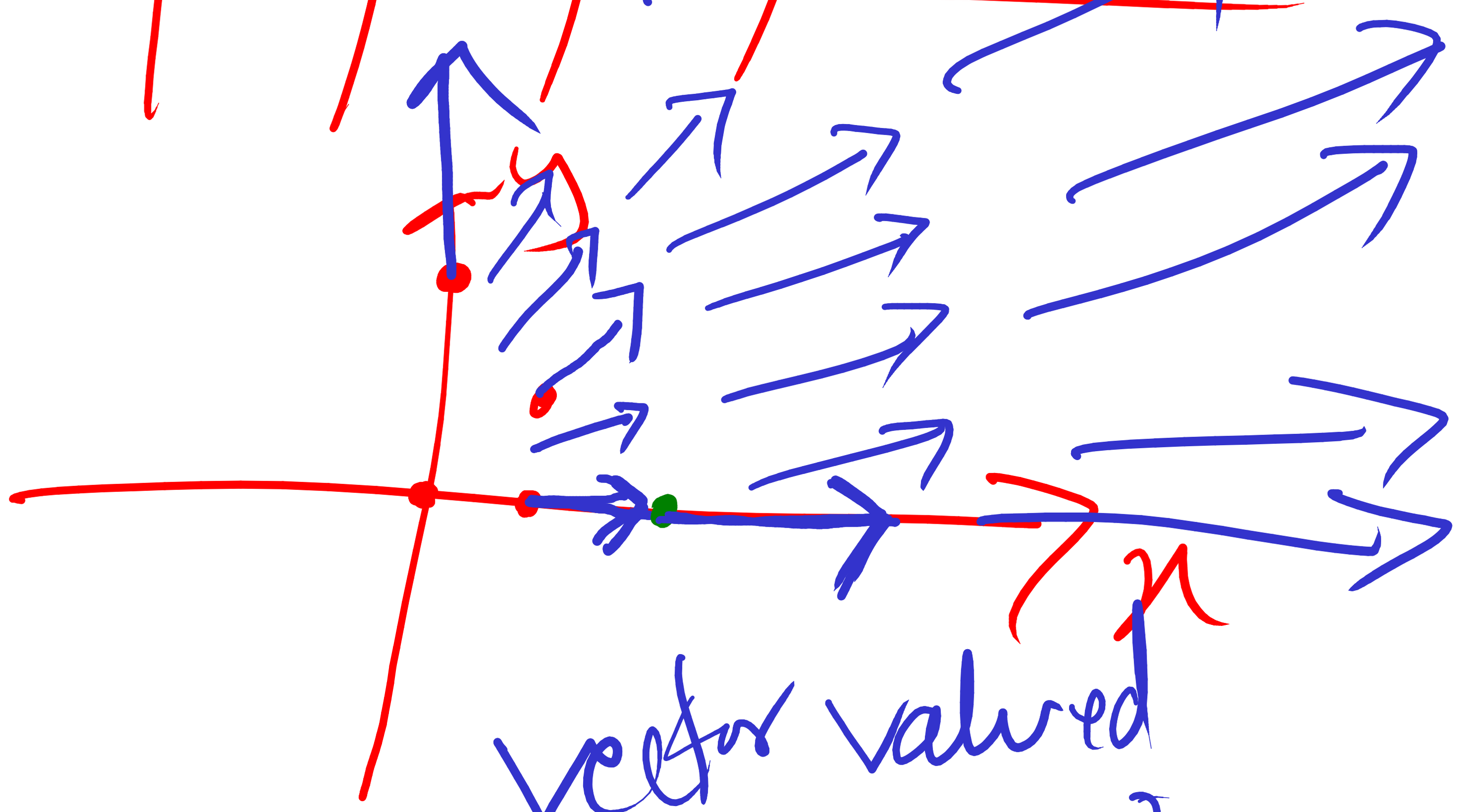


$f(x) = x^2$

y \ x	0	1	2
0	(0,0)	(1,0)	(2,0)
1	(0,1)	(1,1)	(2,1)
2	(0,2)	(1,2)	(2,2)

x	0	1	2
z	0	1	4

y \ x	0	1	2
0	0	1	4
1	1	2	5
2	4	5	8



vector valued

$\vec{F}$

$\mathbb{R}^2 \to \mathbb{R}^2$

$\mathbb{R}^3 \to \mathbb{R}^3$

$\mathbb{R}^2 \to \mathbb{R}^2$

$\mathbb{R}^3 \to \mathbb{R}^3$

$\tan^{-1}\left(\frac{y}{x}\right)$

$2\vec{j}$

$(0,2)$

scalar valued

$(1,0)$

$\vec{F} =$

$\mathbb{R} \to \mathbb{R}$

$\mathbb{R}^2 \to \mathbb{R}$

$\mathbb{R}^3 \to \mathbb{R}$

$\underline{y = 2x^2}$

$y = 8\vec{j}$

$\mathbb{R} \to \mathbb{R}$

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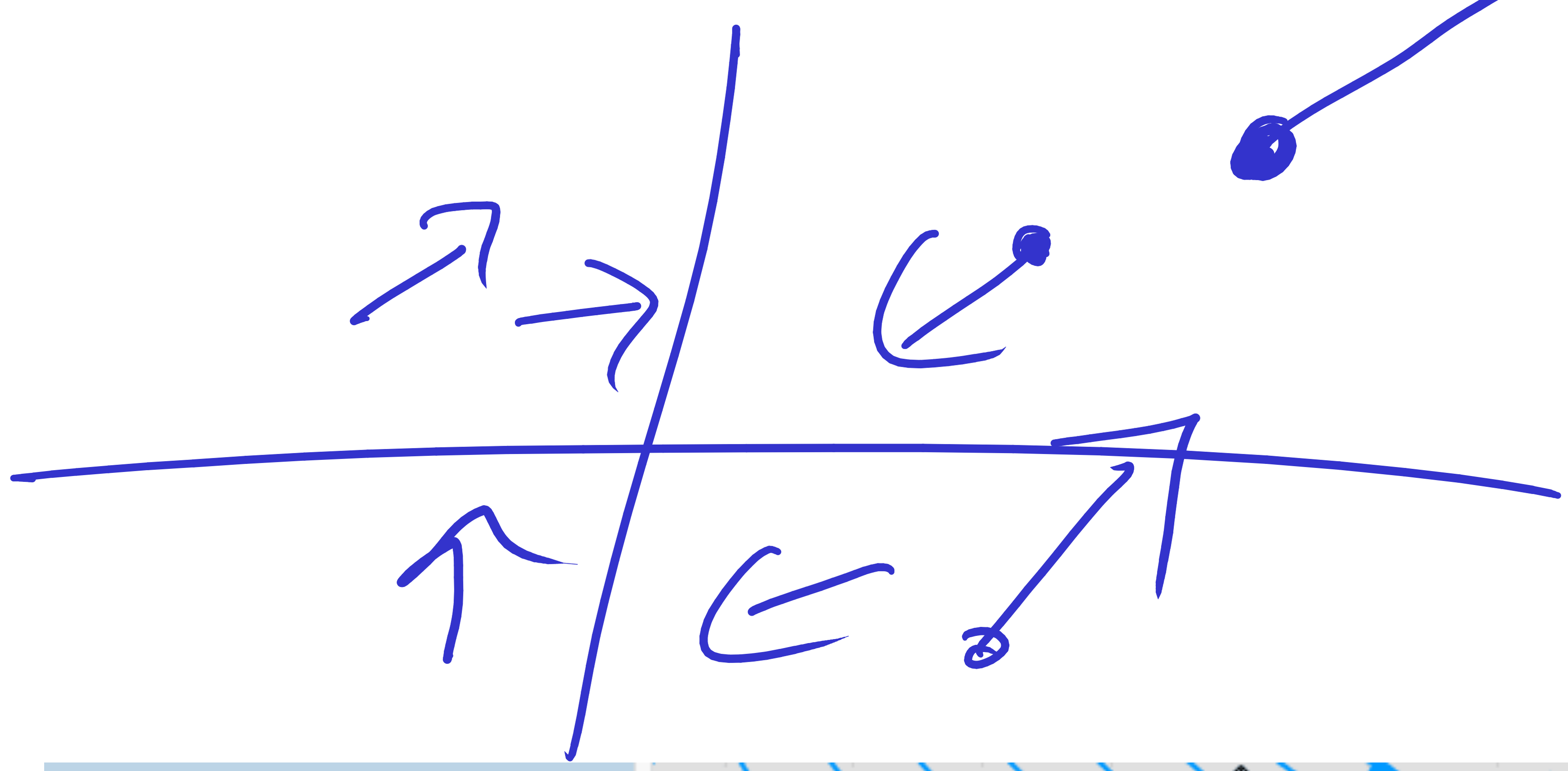
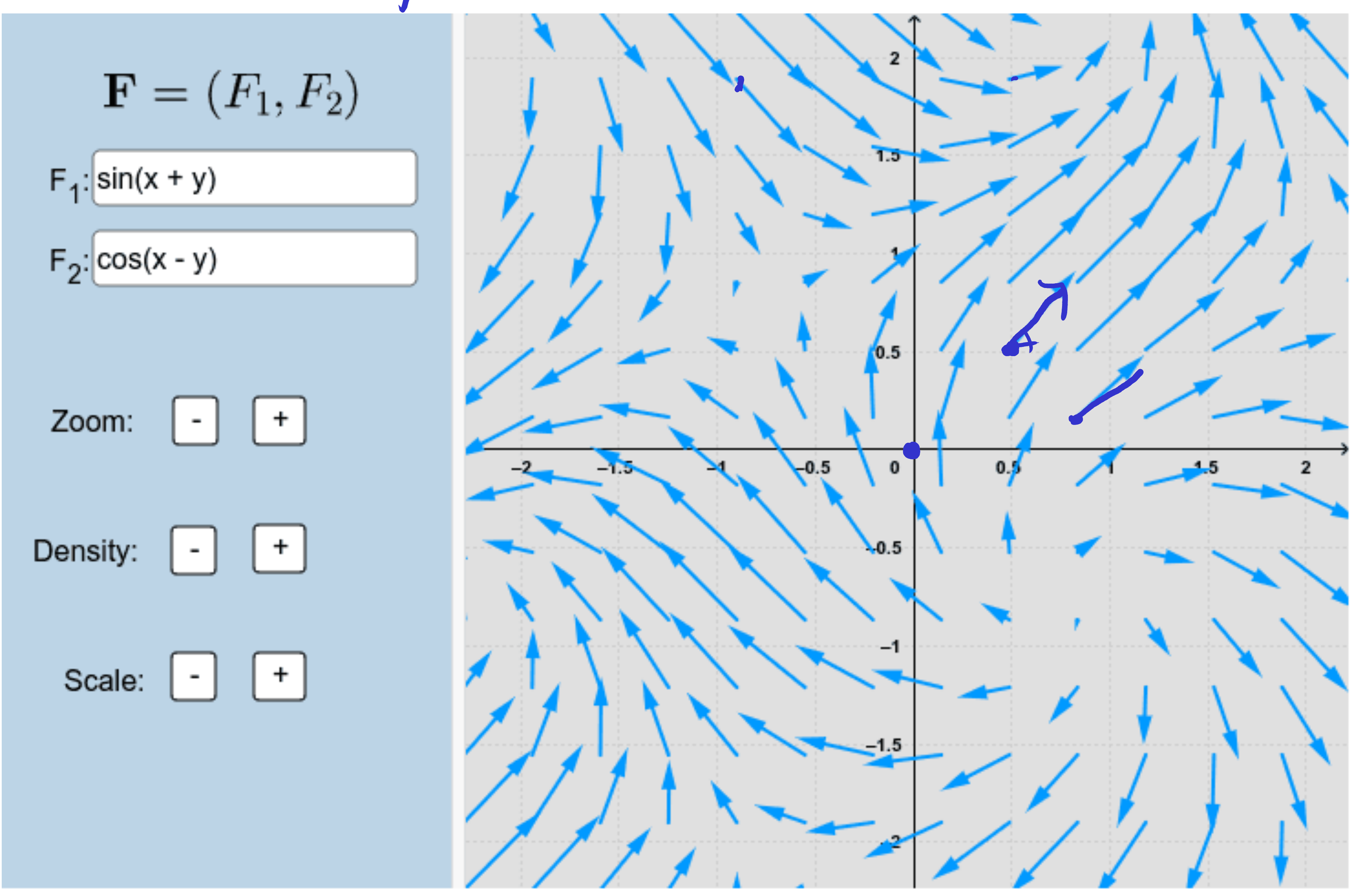
$$\vec{F} = (x+y)\vec{i} + \sin(x)\cos(y)\vec{j} + z\vec{k}$$



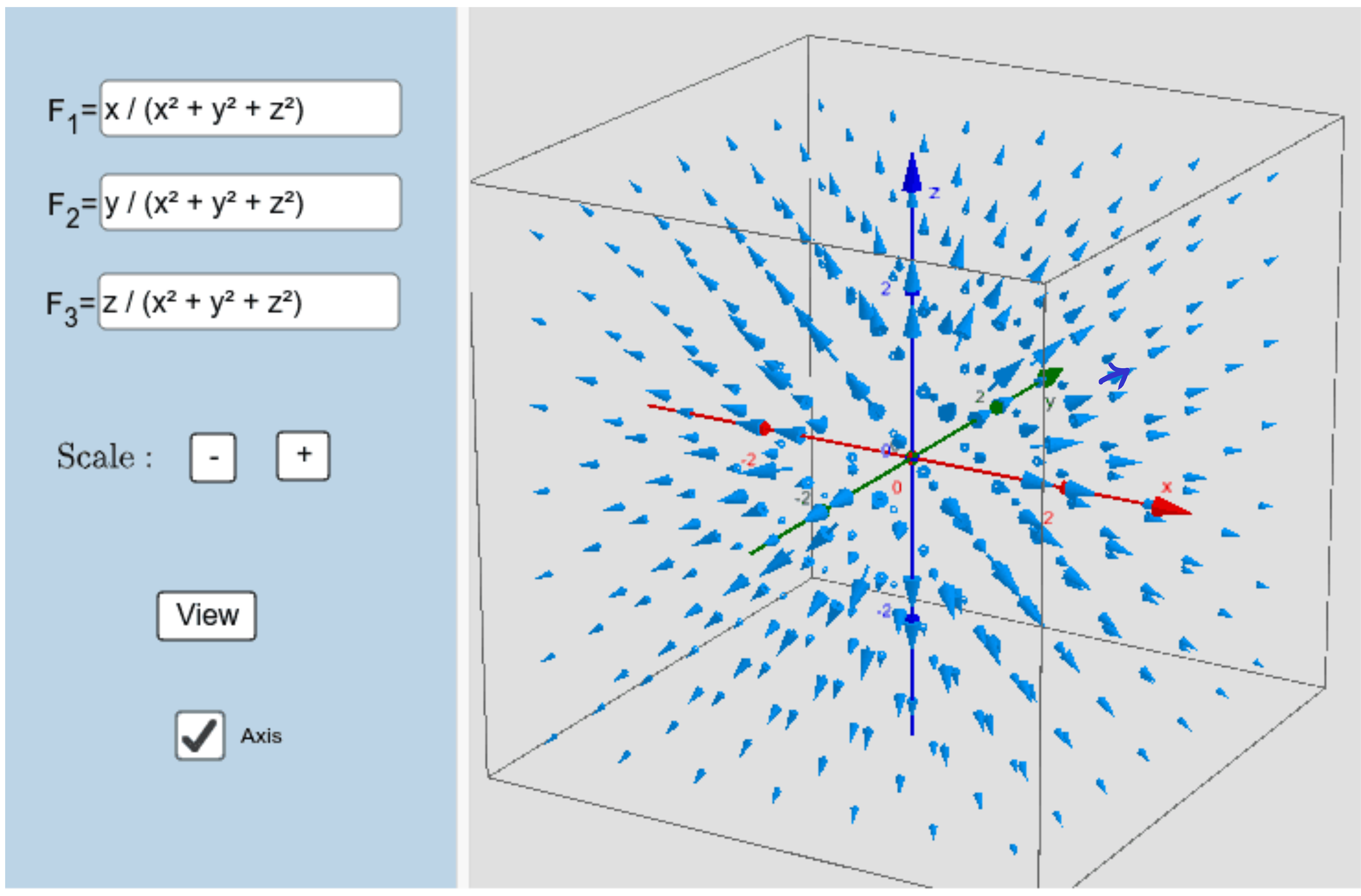


$$\boxed{\sin(x+y)\vec{i} + \cos(x-y)\vec{j}}$$

$0\vec{i} + \vec{j}$

$(0,1)$

$\sin(1)\vec{i} + \vec{j}$



$(x,y,z) \to (r,\theta,\phi)$

$r = \sqrt{x^2+y^2+z^2}$

$$\frac{x}{x^2+y^2+z^2}\vec{i} + \frac{y}{x^2+y^2+z^2}\vec{j} + \frac{z}{x^2+y^2+z^2}\vec{k}$$

$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$

$\frac{1}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{1}{3}\vec{k}$

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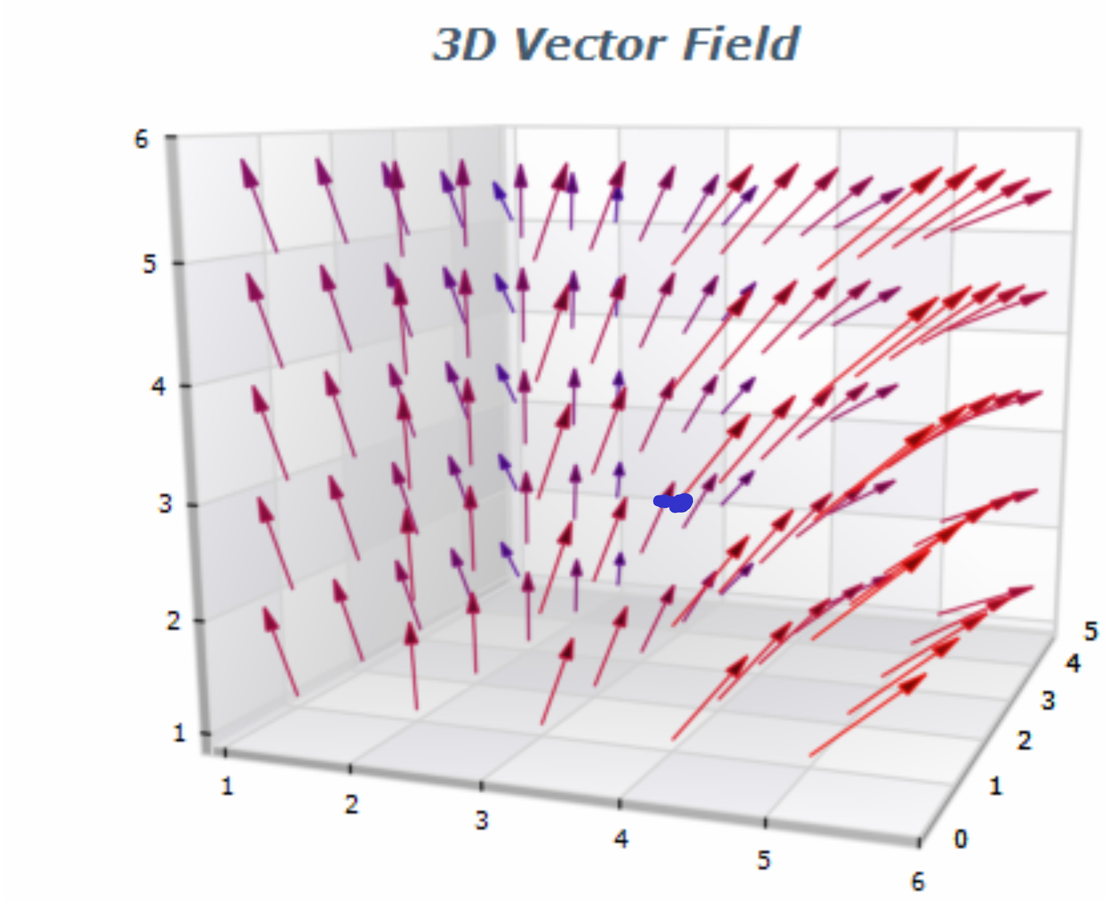
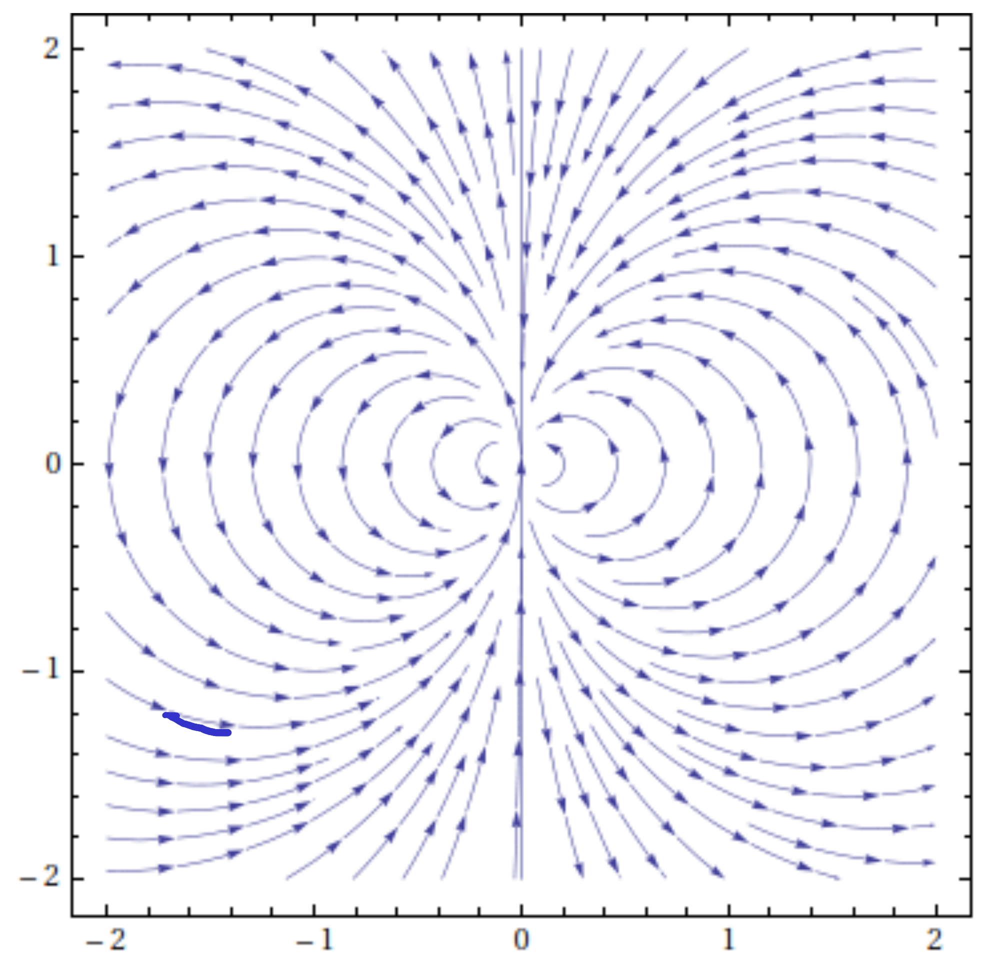
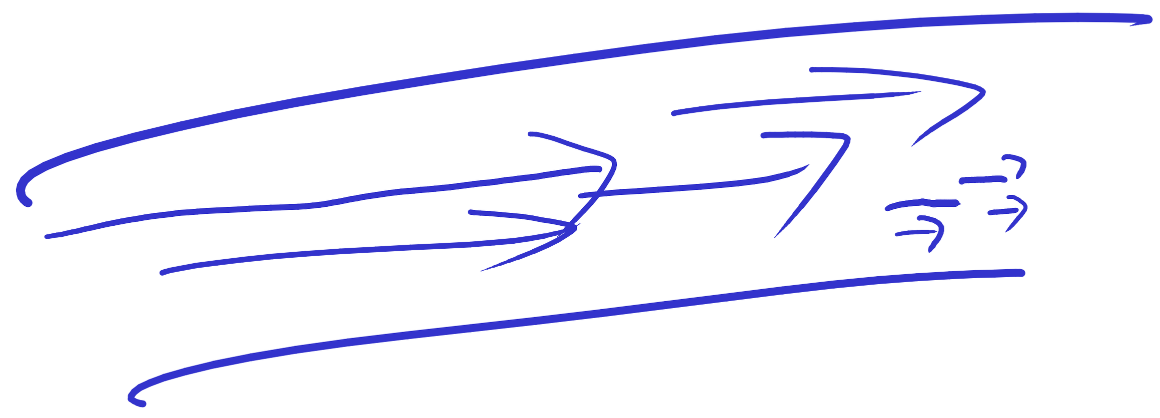


$\vec{F}$



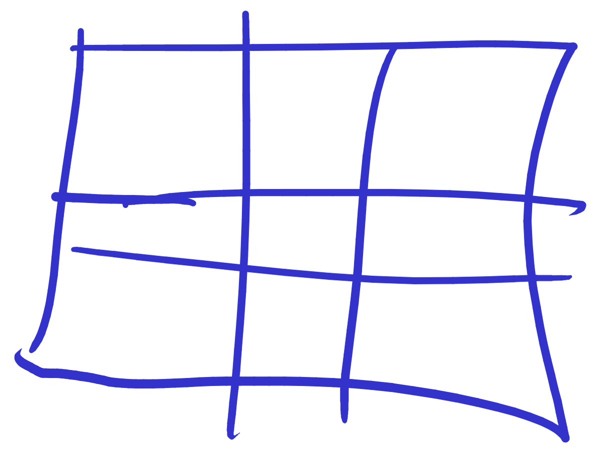


$$\boxed{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}} = 2}$$

$$2+x = x^2 \qquad x^2-x-2=0$$

$$(x-2)(x+1)=0$$

$$x=2 \quad x=-1$$



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$$x^n+y^n = z^n \qquad n>2 \quad \text{fermat}$$

$$3^2+4^2=5^2 \qquad x^3+y^3=z^3$$

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$\mathbb{R} \to \mathbb{R}^2$

$\mathbb{R}^2 \to \mathbb{R}$

$\mathbb{R}^3 \to \mathbb{R}$

$\vec{F} = xy\,\vec{i} + x^2y^2\,\vec{j}$ ✓

vector valued function

$\vec{F} = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + O(x,y,z)\,\vec{k}$

del $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$

$\nabla f \to$ gradient

$f : x^2 y \sin(z)$

$$\nabla f = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) f$$

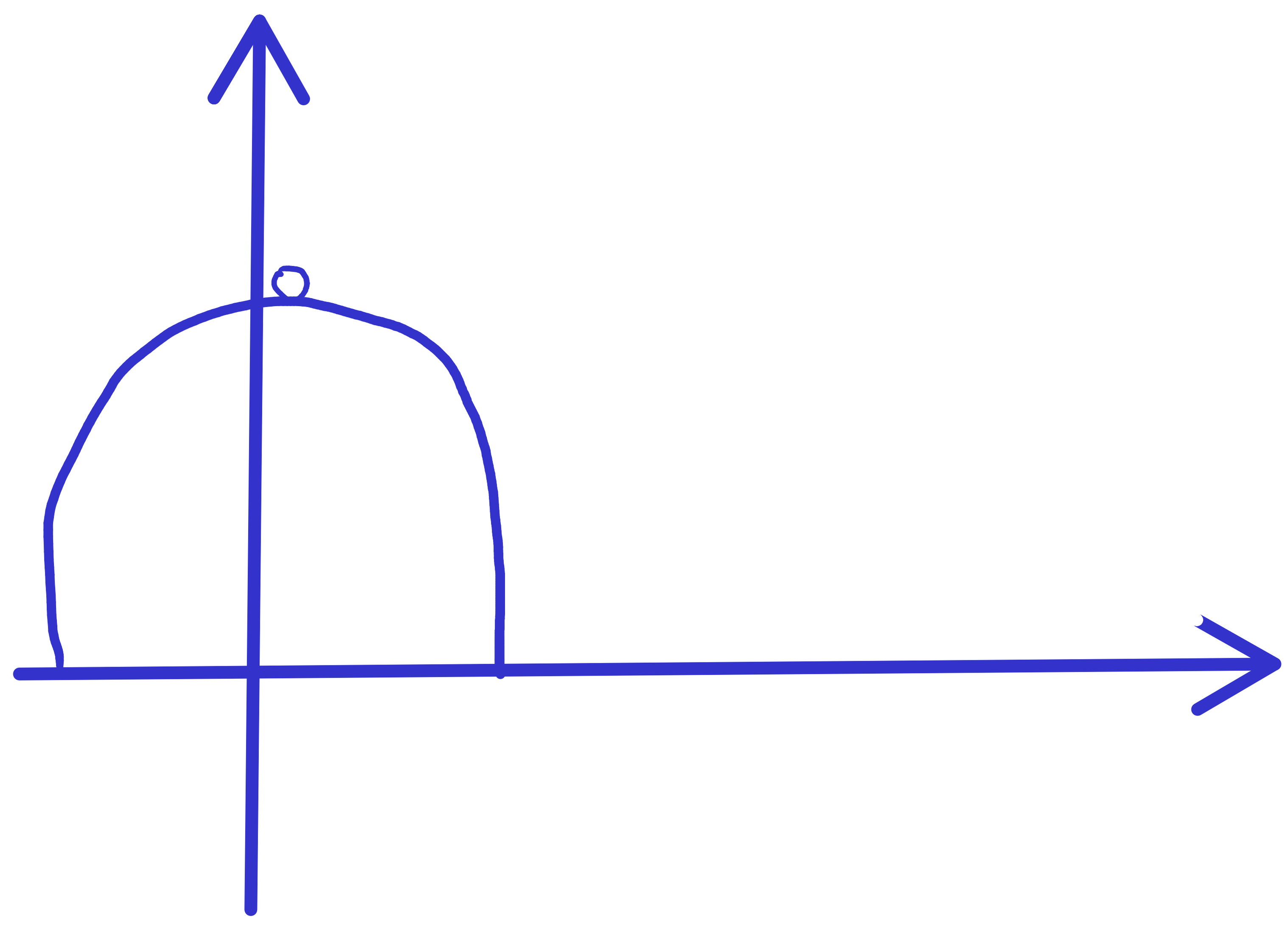
$$= \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right)$$

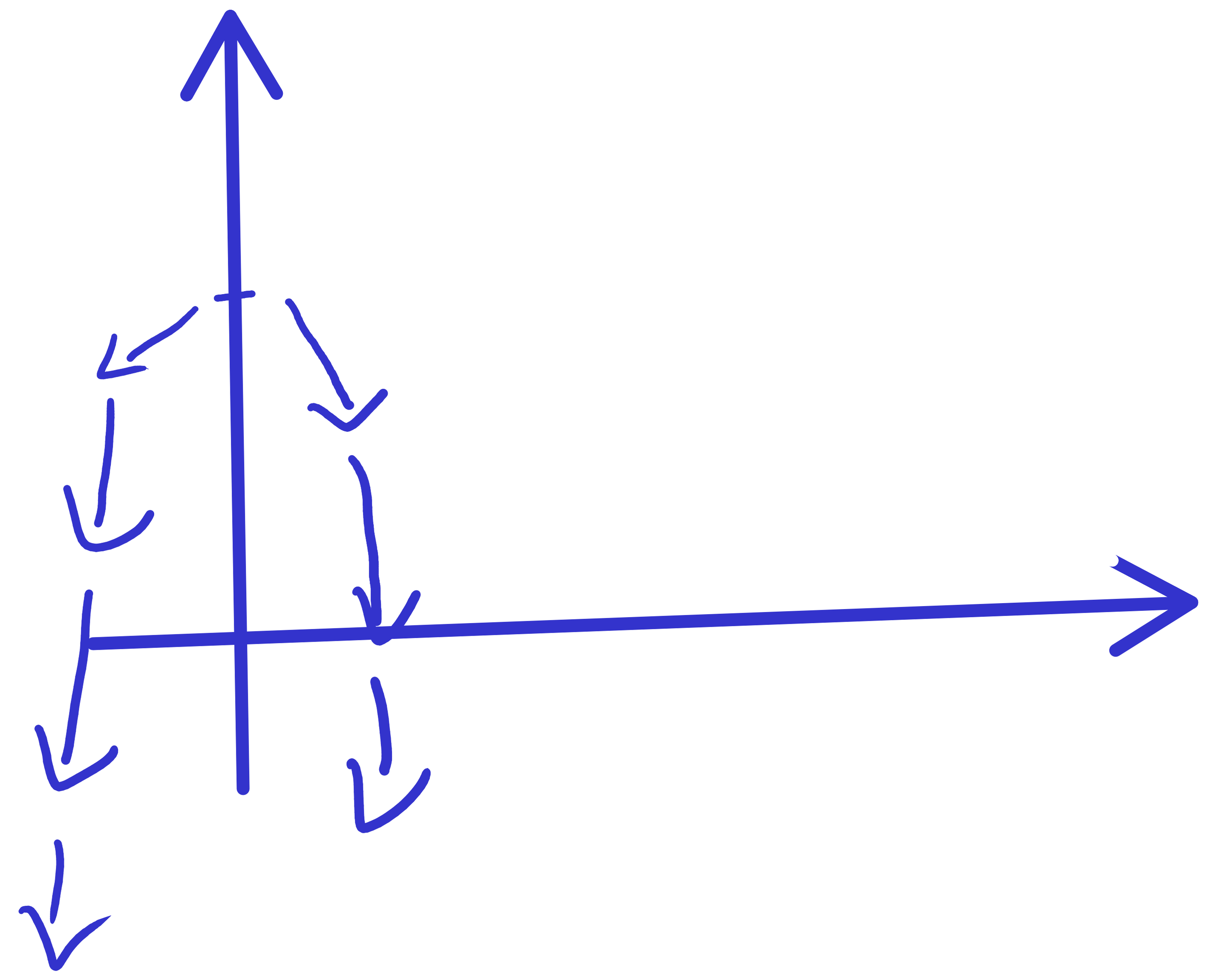
$$\nabla f = 2xy\sin(z)\,\vec{i} + x^2\sin(z)\,\vec{j} + x^2y\cos(z)\,\vec{k}$$

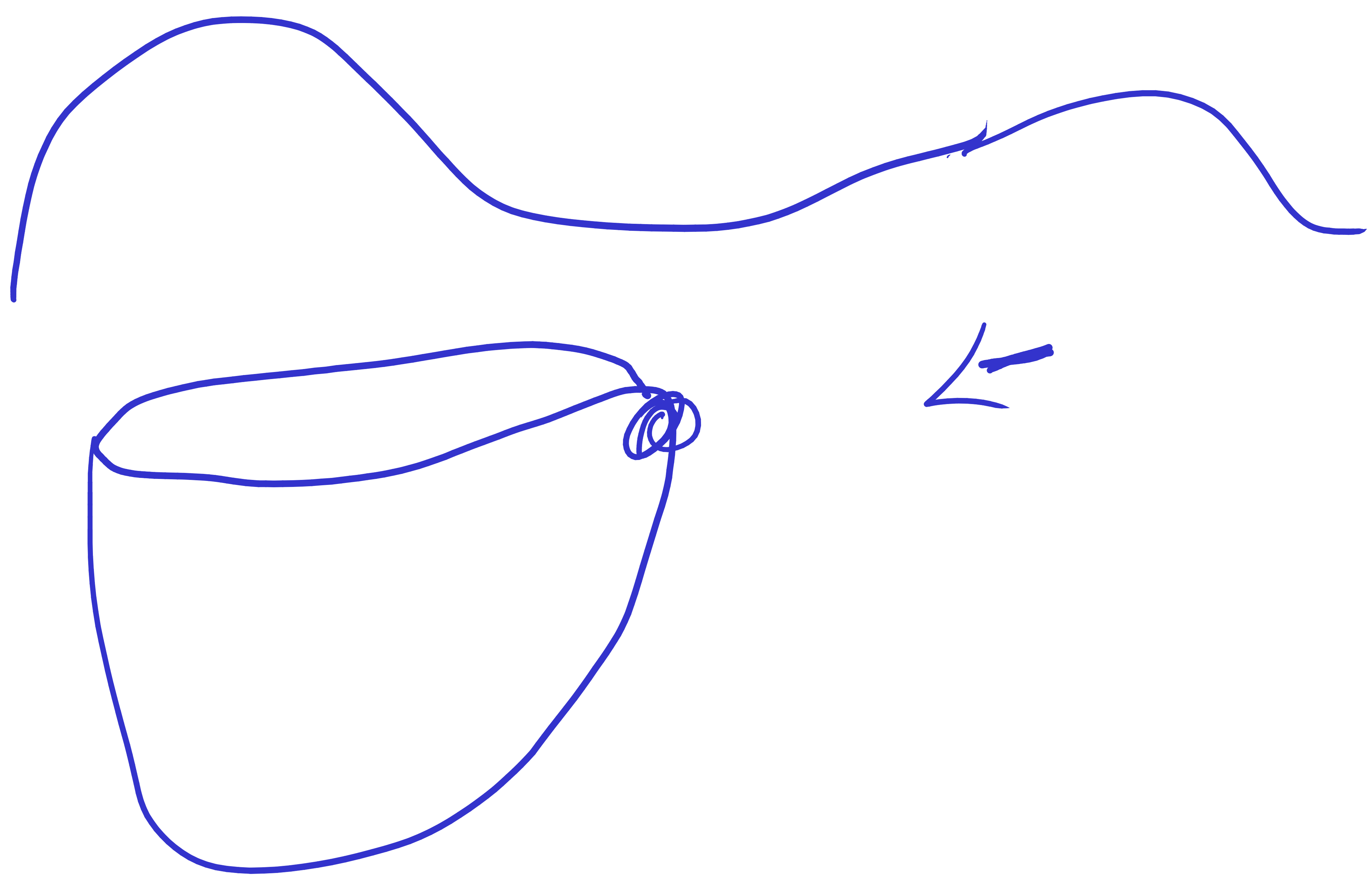
$\nabla f : \mathbb{R} \to \mathbb{R}^3$

$f = x+y$

$f : \mathbb{R}^2 \to \mathbb{R}$







$\nabla \cdot \vec{F} \rightarrow$ Divergence

$$\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$$

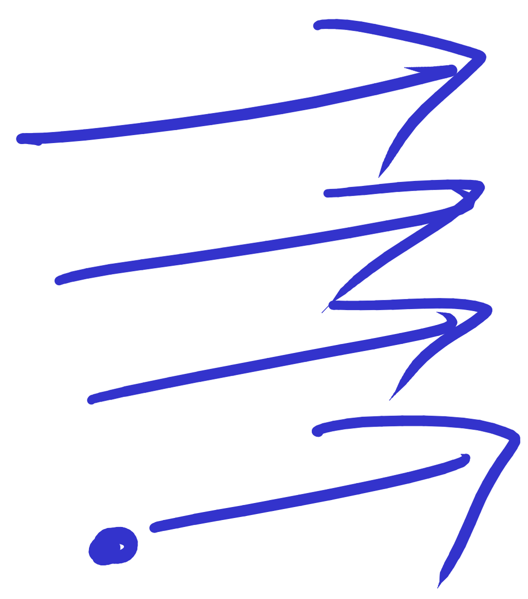
$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

$$\underline{\underline{\vec{F}}} = x^2 y \sin(z)\,\vec{i} + yz\,\vec{j} + xyz\,\vec{k}$$

$$\nabla\cdot\vec{F} = \boxed{2xy\sin(z) + z + xy}$$

$$\nabla\cdot\vec{F} : \mathbb{R}^3 \to \mathbb{R}$$



$$\nabla\times\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$$

<u>Curl</u>

$$\nabla \quad \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

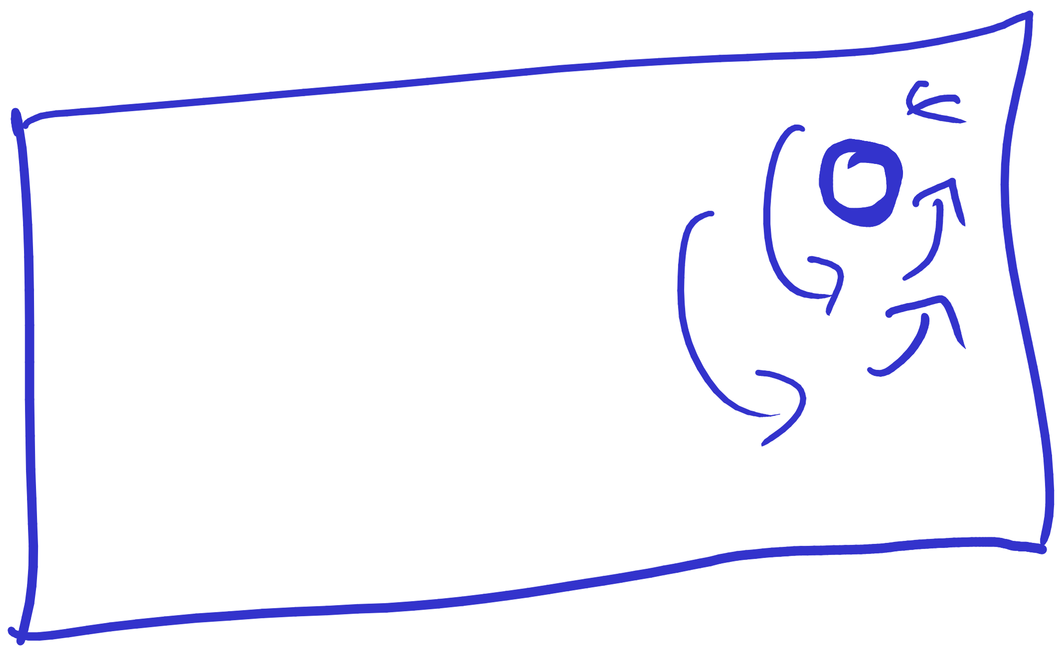
$$\vec{F} = yx\,\vec{i} + zy\,\vec{j} + xz\,\vec{k}$$

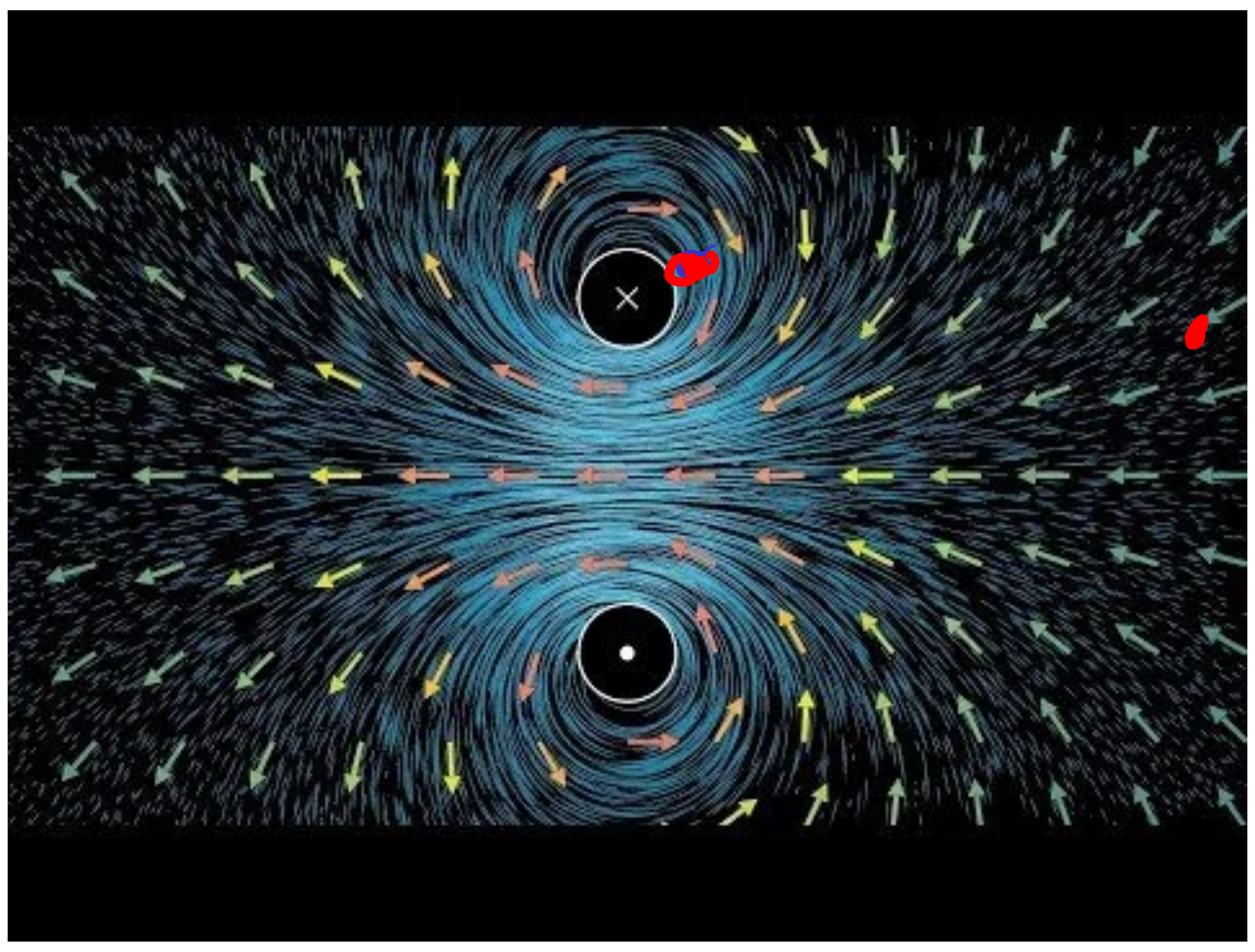
$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix}$$

$$= \vec{i}(0-y) - \vec{j}(z-0) + \vec{k}(0-x)$$

$$\underset{\text{Curl}}{\nabla\times\vec{F}}\Big|_{(1,4,7)} = -y\,\vec{i} - z\,\vec{j} - x\,\vec{k}$$

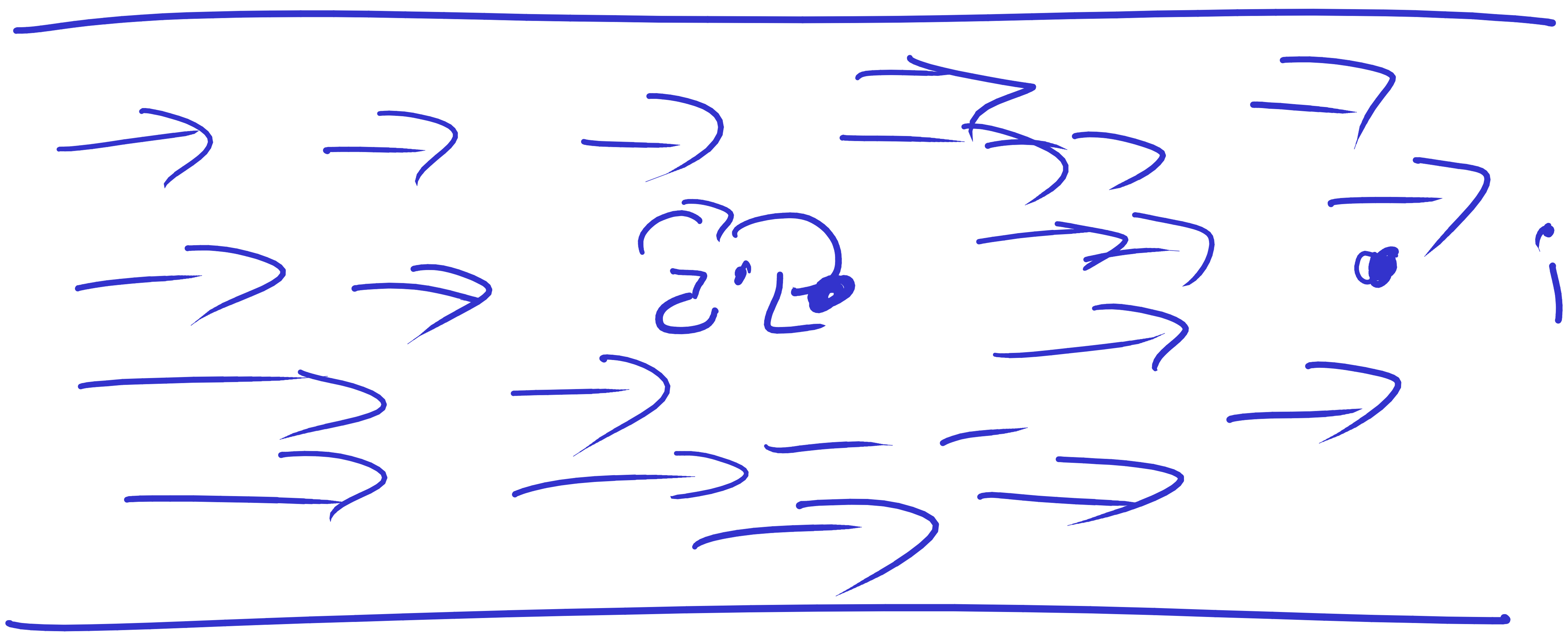
$$-2\vec{i} - 4\vec{j} - \vec{k}$$





$|\nabla \times \vec{F}|$

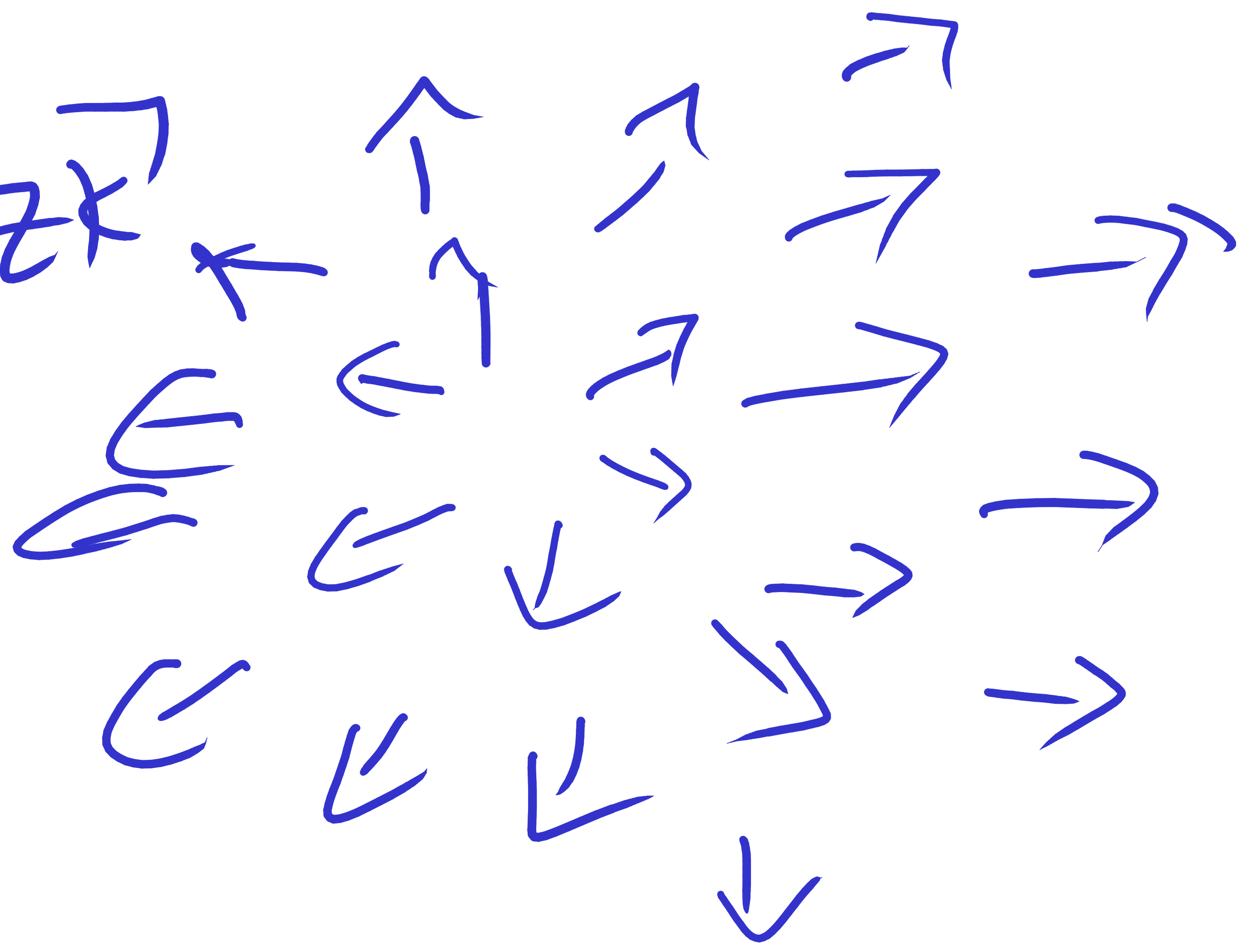
irrotational

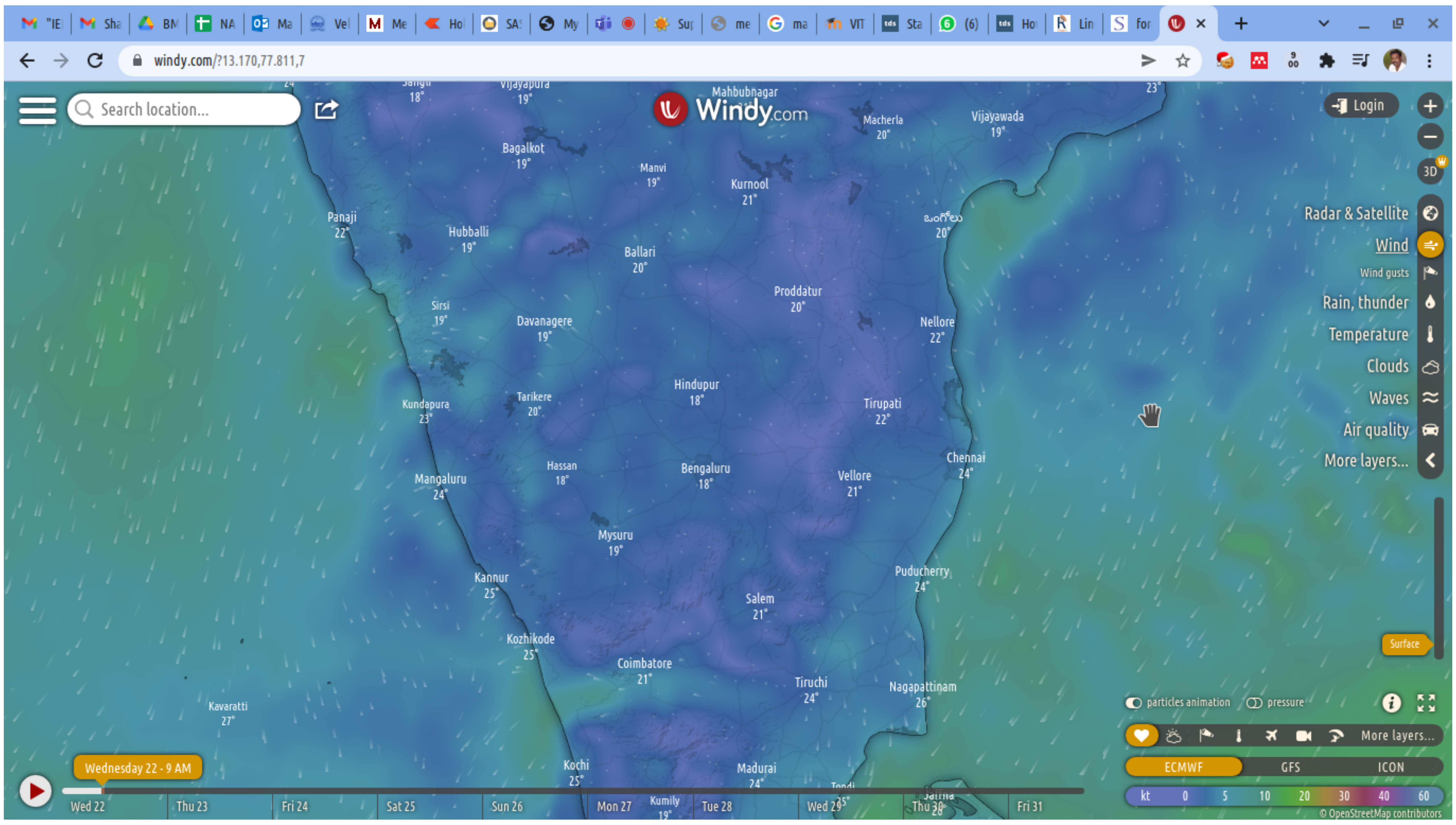


irrotational
$\nabla \times \vec{F} = 0$

$\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$

$\nabla \times \vec{F} = 0$





$$\text{scalar} \rightarrow \nabla f \rightarrow \underline{\text{grad}(f)} \rightarrow \text{vector}$$

$$\text{vector} \rightarrow \nabla \cdot \bar{f} \rightarrow \underline{\text{div}(\bar{f})} \rightarrow \text{scalar}$$

$$\text{vector} \rightarrow \nabla \times \bar{f} \rightarrow \underline{\text{curl}(\bar{f})} \rightarrow \text{vector}$$

Prove that $\nabla r^n = n r^{n-2} \vec{R}$ where $\vec{R} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{R}|$

$$r = |\vec{R}| = \sqrt{x^2+y^2+z^2}$$

$$r^n = (x^2+y^2+z^2)^{n/2}$$

$$r = (x^2+y^2+z^2)^{1/2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2} 2x$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x}{r}$$

$$\nabla r^n = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) r^n$$

$$= \vec{i}\frac{\partial r^n}{\partial x} + \vec{j}\frac{\partial r^n}{\partial y} + \vec{k}\frac{\partial r^n}{\partial z}$$

$$= \vec{i}\, n r^{n-1}\frac{\partial r}{\partial x} + \vec{j}\, n r^{n-1}\frac{\partial r}{\partial y} + \vec{k}\, n r^{n-1}\frac{\partial r}{\partial z}$$

$$= n r^{n-1}\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right]$$

$$= n r^{n-2}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$

$$\nabla r^n = n r^{n-2}\vec{R}$$

If $\vec{R} = x\vec{i} + y\vec{j} + z\vec{k}$ → position vector

$\nabla \cdot \vec{R} = 3$

$\nabla \times \vec{R} = 0$

$\nabla \cdot \vec{R} =$

Find div $\overline{\mathbf{F}}$ *and curl* $\overline{\mathbf{F}}$, *where* $\overline{\mathbf{F}} = grad\,(x^3 + y^3 + z^3 - 3xyz)$.

$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

$\vec{F} = \vec{i}(3x^2 - 3yz) + \vec{j}(3y^2 - 3xz) + \vec{k}(3z^2 - 3xy)$

Gradient $\nabla\phi : \mathbb{R} \to \mathbb{R}^3$

$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

divergence $\nabla \cdot \vec{F} : \mathbb{R}^3 \to \mathbb{R}$

$\nabla \cdot \vec{F} = 6x + 6y + 6z = 6(x+y+z)$

Curl $\nabla \times \vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix} = \vec{0}$$

$\nabla \times \vec{F} = 0 \qquad \vec{F} = \nabla \phi$

If $\vec{F} = \nabla \phi \Rightarrow \nabla \times \vec{F} = 0$

$$\boxed{\nabla \times \nabla \phi = 0}$$

$$\nabla \times \nabla f = \nabla \times \left(\mathbf{I} \frac{\partial f}{\partial x} + \mathbf{J} \frac{\partial f}{\partial y} + \mathbf{K} \frac{\partial f}{\partial z} \right) = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \Sigma \mathbf{I} \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) = \mathbf{0}$$

Find a unit vector normal to the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$.

unit normal vector $= \dfrac{\overline{\nabla \phi}}{|\nabla \phi|}$

$\phi = xy^3z^2 - 4$

$\nabla \phi = \vec{i}(y^3z^2) + \vec{j}(x3y^2z^2) + \vec{k}(xy^3 2z)$

$\nabla \phi \big|_{(-1,-1,2)} = -4\vec{i} - 12\vec{j} + 4\vec{k}$

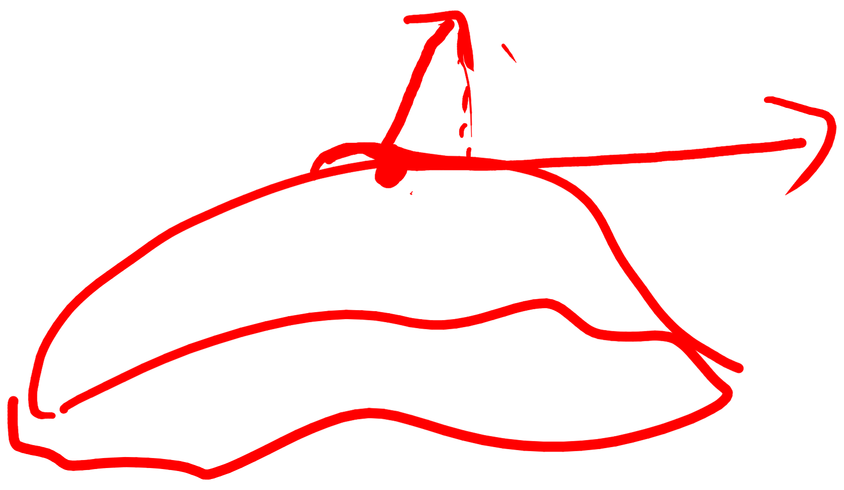
$$= \frac{-4\vec{i} - 12\vec{j} + 4\vec{k}}{\sqrt{(-4)^2 + (-12)^2 + (4)^2}} = -\frac{(\vec{i} + 3\vec{j} - \vec{k})}{\sqrt{11}}$$

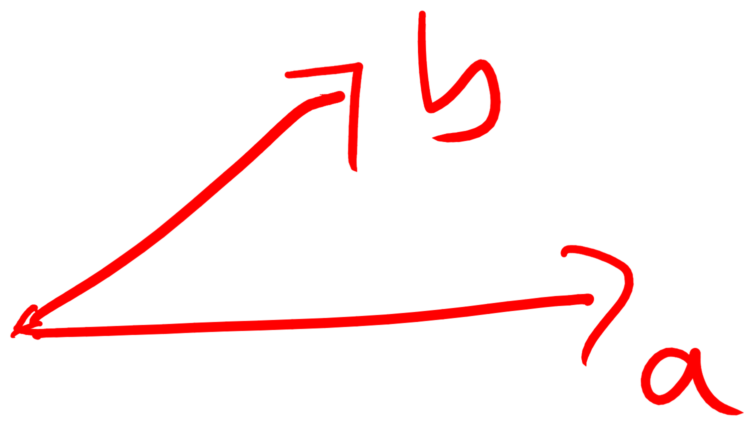
Example 8.13. *Find the directional derivative of $f(x, y, z) = xy^3 + yz^3$ at the point $(2, -1, 1)$ in the direction of vector $\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}$.* (*Bhopal, 2008 ; Kurukshetra, 2006 ; Rohtak, 2003*)

$\underbrace{\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}}_{\vec{a}}$

$$\nabla f \cdot \frac{\vec{a}}{|\vec{a}|}$$

$\nabla f|_{(2,-1,1)}$





$\vec{b} \cdot \vec{a} =$

$$\nabla f = y^3\vec{i} + (3xy^2 + z^3)\vec{j} + (3yz^2)\vec{k}$$

$$\nabla f|_{(2,-1,1)} = -\vec{i} + 7\vec{j} - 3\vec{k}$$

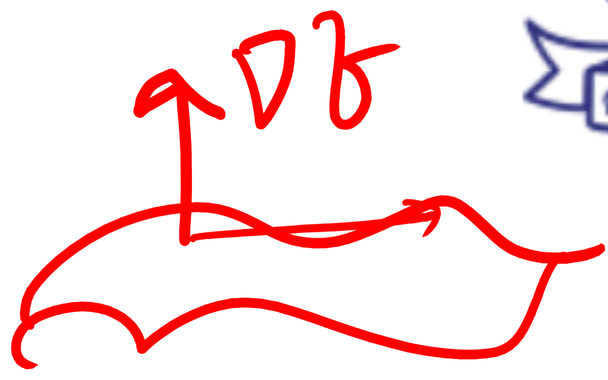
$$\frac{(-\vec{i} + 7\vec{j} - 3\vec{k}) \cdot (\vec{i} + 2\vec{j} + 2\vec{k})}{\sqrt{1+4+4}}$$

$$= \frac{-1+14-6}{3} = \frac{7}{3}$$

Example 8.14. *Find the directional derivative of $f = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ where Q is the point $(5, 0, 4)$. Also calculate the magnitude of the maximum directional derivative.*

$$\nabla f \cdot \frac{\vec{PQ}}{|\vec{PQ}|}$$

$$\vec{PQ} = \vec{OQ} - \vec{OP}$$
$$= (5\vec{i} + 0\vec{j} + 4\vec{k}) - (\vec{i} + 2\vec{j} + 3\vec{k})$$
$$= 4\vec{i} - 2\vec{j} + \vec{k}$$



Directional derivative $= \dfrac{28}{\sqrt{21}}$

$|\nabla \phi| =$ maximum directional derivative

$= \sqrt{164}$

Example 8.15. *Find the directional derivative of* $\phi = 5x^2y - 5y^2z + 2.5z^2x$ *at the point P(1, 1, 1) in the direction of the line* $\dfrac{x-1}{2} = \dfrac{y-3}{-2} = \dfrac{z}{1}$. (*Bhopal, 2008 ; U.P.T.U., 2004*)

$$D.D = \nabla\phi\big|_{(1,1,1)} \cdot \frac{\vec{a}}{|\vec{a}|}$$

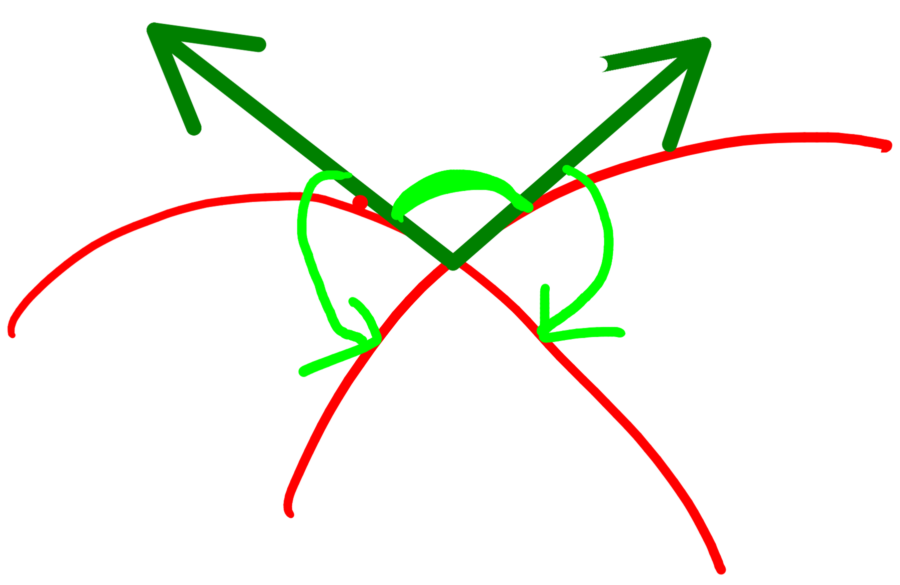
$$= 11\tfrac{2}{3}$$

$$\vec{a} = 2\vec{i} - 2\vec{j} + \vec{k}$$

$$|\vec{a}| = \sqrt{9}$$

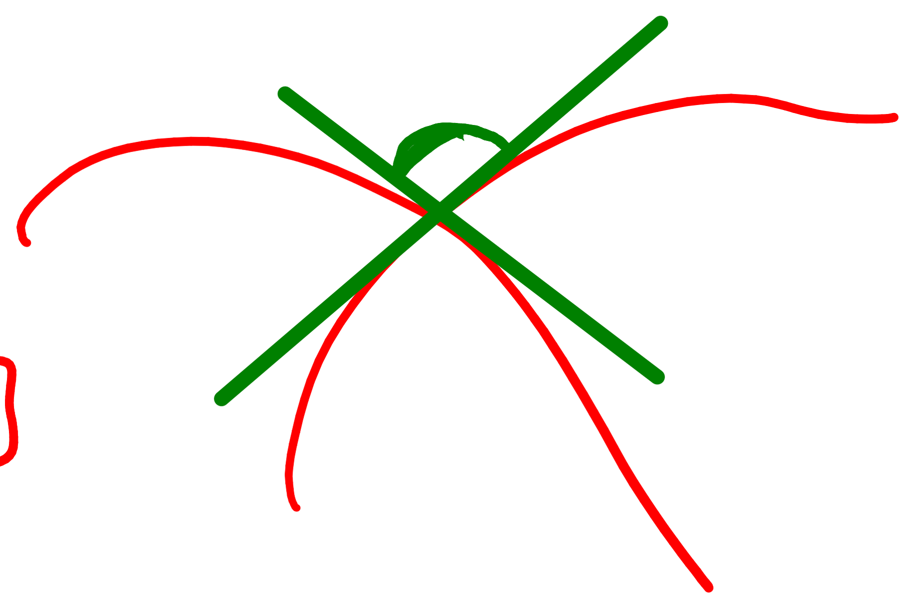
$$= 3$$

Example 8.16. *Find the angle between the surfaces* $x^2 + y^2 + z^2 = 9$ *and* $z = x^2 + y^2 - 3$ *at the point (2, – 1, 2).* (*V.T.U., 2010 ; Kottayam, 2005 ; U.P.T.U., 2003*)



$$\phi_1 = x^2 + y^2 + z^2 - 9$$

$$\phi_2 = z - x^2 - y^2 + 3$$



$\nabla\phi_1$ $\quad$ $\nabla\phi_2$ $\quad$ $\vec{a} + \vec{b}$

$$\theta = \cos^{-1}\left[\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right]$$

$$\vec{a} = \nabla\phi_1\big|_{(\ )} \qquad \theta = \cos^{-1}\left[\frac{8}{3\sqrt{21}}\right]$$

$$\vec{b} = \nabla\phi_2\big|_{(\ )}$$

Example 8.17. *Find the values of a and b such that the surface $ax^2 - byz = (a + 2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at (1, – 1, 2).* (Madras, 2004)

ϕ_1 $\qquad$ ϕ_2

$$\theta = 90^\circ \qquad \frac{\nabla\phi_1\cdot\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = 0$$

$$\cos\theta = 0 \qquad \nabla\phi_1\cdot\nabla\phi_2 = 0$$

$$\nabla\phi_1 = (a-2)\vec{i} - 2b\vec{j} + b\vec{k}$$

$$\nabla\phi_2 = -8\vec{i} + 4\vec{j} + 12\vec{k}$$

$$-8(a-2) - 8b + 12b = 0$$

$$-2a + b - 4 = 0$$

$$a + 2b - (a+2) = 0$$

$$b = 1$$

$$\Rightarrow a = 5/2$$

The temperature of points in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at (1, 1, 2) desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move ?

Find the values of a and b so that the surface $5x^2 - 2yz - 9z = 0$ may cut the surface $ax^2 + by^3 = 4$ orthogonally at $(1, -1, 2)$. (*Nagpur, 2009*)

If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at the point (1, 1, 1) has maximum magnitude 15 in the direction parallel to the line $\boxed{\frac{x-1}{2} = \frac{y-3}{-2} = z}$, find the values of a, b and c. (*U.P.T.U., 2002*)

$$\nabla\phi = (2xya + cz^2)\vec{i} + (2yzb + ax^2)\vec{j} + (2cxz + by^2)\vec{k}$$

$$\nabla\phi\big|_{(1,1,1)} = (2a+c)\vec{i} + (2b+a)\vec{j} + (2c+b)\vec{k}$$

$$\vec{u} = 2\vec{i} - 2\vec{j} + \vec{k}$$

$$\boxed{|\nabla\phi| = 15}$$

$$\frac{\nabla\phi\cdot\vec{u}}{|\vec{u}|} \quad \frac{(2a+c)2 + (2b+a)(-2) + (2c+b)}{3} = 15$$

$$\vec{\nabla\phi} \times \vec{u} = \vec{0}$$

$$\vec{a}\cdot\vec{b} = 0$$
$$\vec{a}\times\vec{b} = \vec{0}$$

$$\begin{cases} a = 6 \\ b = 24 \\ c = \end{cases}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2a+c & 2b+a & 2c+b \\ 2 & -2 & 1 \end{vmatrix} = \vec{0}$$

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Prove that div $(r^n \mathbf{R}) = (n+3)\, r^n$. Hence show that $\mathbf{R}/r^3$ is solenoidal.

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \qquad R = (\vec{r})$$

$$r = \sqrt{x^2+y^2+z^2} \Rightarrow r^n = (x^2+y^2+z^2)^{n/2}$$

$$r^n \vec{r} = (x^2+y^2+z^2)^{n/2}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\operatorname{div}(r^n \vec{r}) = \underline{(x^2+y^2+z^2)^{n/2}}(1) + x\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}\, 2x$$

$$+ (\qquad\qquad)$$

$$+ (\qquad\qquad)$$

$$= 3r^n + nx^2(r^2)^{n/2-1} + ny^2(r^2)^{n/2-1} + nz^2(r^2)^{n/2-1}$$

$$= 3r^n + n(r^2)^{n/2-1}\left[x^2+y^2+z^2\right]$$

$$= 3r^n + n\, r^{n-2}\, r^2$$

$$= 3r^n + n r^n = (3+n)r^n$$

$\nabla \times \vec{F} = 0 \Rightarrow \vec{F}$ is irrotational

$\nabla \cdot \vec{F} = 0 \Rightarrow \vec{F}$ is solenoidal

$\operatorname{div}(r^n \vec{r}) = (n+3) r^n$, if $n = -3$, $\operatorname{div}\left(\frac{\vec{r}}{r^3}\right) = 0$

To prove $\operatorname{div}\left(\frac{\vec{r}}{r^3}\right) = 0 \Rightarrow \frac{\vec{r}}{r^3}$ is solenoidal

Example 8.21. *Show that $r^{\alpha}\mathbf{R}$ is any irrotational vector for any value of α but is solenoidal if $\alpha + 3 = 0$ where $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ and r is the magnitude of $\mathbf{R}$.* (V.T.U., 2006 ; Kottayam, 2005)

$\vec{F} = r^{\alpha} \vec{r}$

$= (x^2 + y^2 + z^2)^{\alpha/2} (x\vec{i} + y\vec{j} + z\vec{k})$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$r = |\vec{r}|$

$= \sqrt{x^2 + y^2 + z^2}$

$\nabla \cdot \vec{F} = 0$ iff $\alpha = -3$ ✓

$\nabla \times \vec{F} = 0 \quad \forall \alpha$

$\nabla \cdot \vec{F} = (\alpha + 3) r^{\alpha} = 0$ iff $\alpha = -3$

$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2} 2y$

$= \frac{y}{\sqrt{x^2+y^2+z^2}}$

$$\nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x r^{\alpha} & y r^{\alpha} & z r^{\alpha} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y} z r^{\alpha} - \frac{\partial}{\partial z} y r^{\alpha}\right) - \vec{j}\left(\frac{\partial}{\partial x} z r^{\alpha} - \frac{\partial}{\partial z} x r^{\alpha}\right) + \vec{k}\left(\frac{\partial}{\partial x} y r^{\alpha} - \frac{\partial}{\partial y} x r^{\alpha}\right)$$

$$= \vec{i}\left(z \alpha r^{\alpha-1} \frac{\partial r}{\partial y} - y \alpha r^{\alpha-1} \frac{\partial r}{\partial z}\right) - \vec{j}\left(z \alpha r^{\alpha-1} \frac{\partial r}{\partial x} - x \alpha r^{\alpha-1} \frac{\partial r}{\partial z}\right) + \vec{k}\left(y \alpha r^{\alpha-1} \frac{\partial r}{\partial x} - x \alpha r^{\alpha-1} \frac{\partial r}{\partial y}\right)$$

$$= \vec{i}\left[\alpha r^{\alpha-1}\left(\frac{z y}{\sqrt{x^2+y^2+z^2}} - \frac{y z}{\sqrt{x^2+y^2+z^2}}\right)\right] + \vec{j}(0) + \vec{k}(0) = \vec{0}$$

$$\nabla \times \nabla f = \mathbf{0}$$

$$\nabla \cdot \nabla \times \mathbf{F} = 0$$

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$$

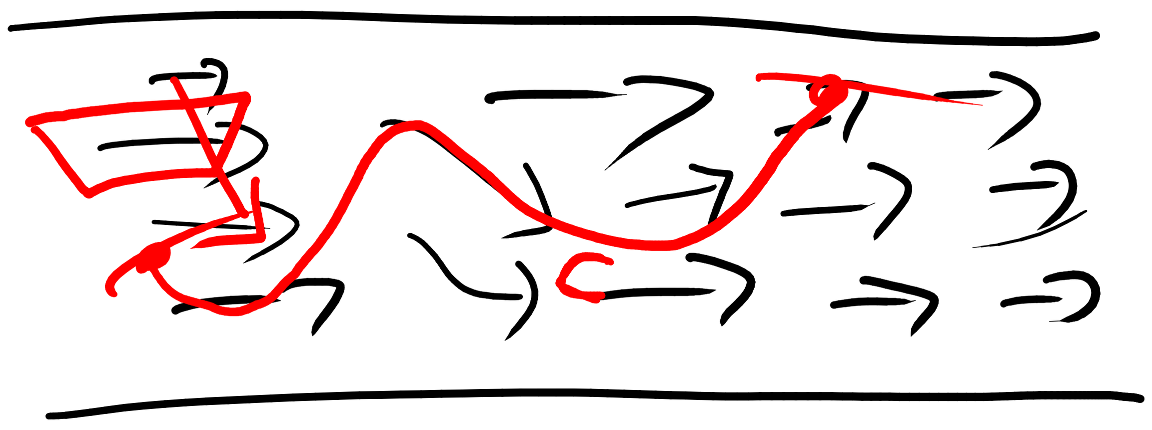
$$div\ grad\ f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla(\nabla \cdot \mathbf{F}) = \nabla \times (\nabla \times \mathbf{F}) + \nabla^2 \mathbf{F}.$$

Line Integral

$$\boxed{\int_C \vec{F}\cdot d\vec{r}}$$

$$\int_a^b g(u)\,du$$



$\vec{F}$

Evaluating the Line Integral of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ Along C: $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$

1. Express the vector field $\mathbf{F}$ in terms of the parametrized curve C as $\mathbf{F}(\mathbf{r}(t))$ by substituting the components $x = g(t)$, $y = h(t)$, $z = k(t)$ of $\mathbf{r}$ into the scalar components $M(x, y, z)$, $N(x, y, z)$, $P(x, y, z)$ of $\mathbf{F}$.
2. Find the derivative (velocity) vector $d\mathbf{r}/dt$.
3. Evaluate the line integral with respect to the parameter t, $a \le t \le b$, to obtain

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt.$$

EXAMPLE 2 Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = z\mathbf{i} + xy\mathbf{j} - y^2\mathbf{k}$ along the curve C given by $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \sqrt{t}\,\mathbf{k}$, $0 \le t \le 1$.

$$\vec{F}(r(t)) = \sqrt{t}\,\vec{i} + t^3\vec{j} - t^2\vec{k}$$

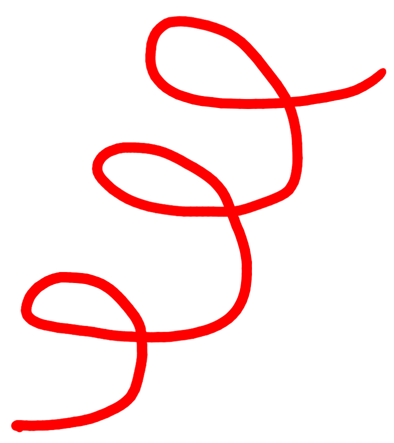
$x = t^2$
$y = t$
$z = \sqrt{t}$

$$\frac{d\vec{r}}{dt} = 2t\,\vec{i} + \vec{j} + \frac{1}{2\sqrt{t}}\vec{k}$$

$$\vec{F}(r(t))\cdot\frac{d\vec{r}}{dt} = 2t^{3/2} + t^3 - \frac{1}{2}t^{3/2} = \frac{3}{2}t^{3/2} + t^3$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left(\frac{3}{2}t^{3/2} + t^3\right) dt = \frac{17}{20}$$

EXAMPLE 3 Evaluate the line integral $\int_C -y\,dx + z\,dy + 2x\,dz$, where C is the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le 2\pi$.



$x = \cos t$
$y = \sin t$
$z = t$

$\vec{F} = (-y\vec{i} + z\vec{j} + 2x\vec{k})$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

$\vec{F}\cdot d\vec{r} = -y\,dx + z\,dy + 2x\,dz$

$$\vec{F}(r(t)) = -\sin t\,\vec{i} + t\,\vec{j} + 2\cos t\,\vec{k}$$

$$\frac{d\vec{r}}{dt} = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}$$

$$\vec{F}(r(t))\cdot\frac{d\vec{r}}{dt} = +\sin^2 t + t\cos t + 2\cos t$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \sin^2 t + t\cos t + 2\cos t\,dt = \pi$$

$$\int_C -y\,dx + z\,dy + 2x\,dz$$

$x = \cos t \Rightarrow dx = -\sin t\,dt$

$y = +\sin t \Rightarrow dy = +\cos t\,dt$

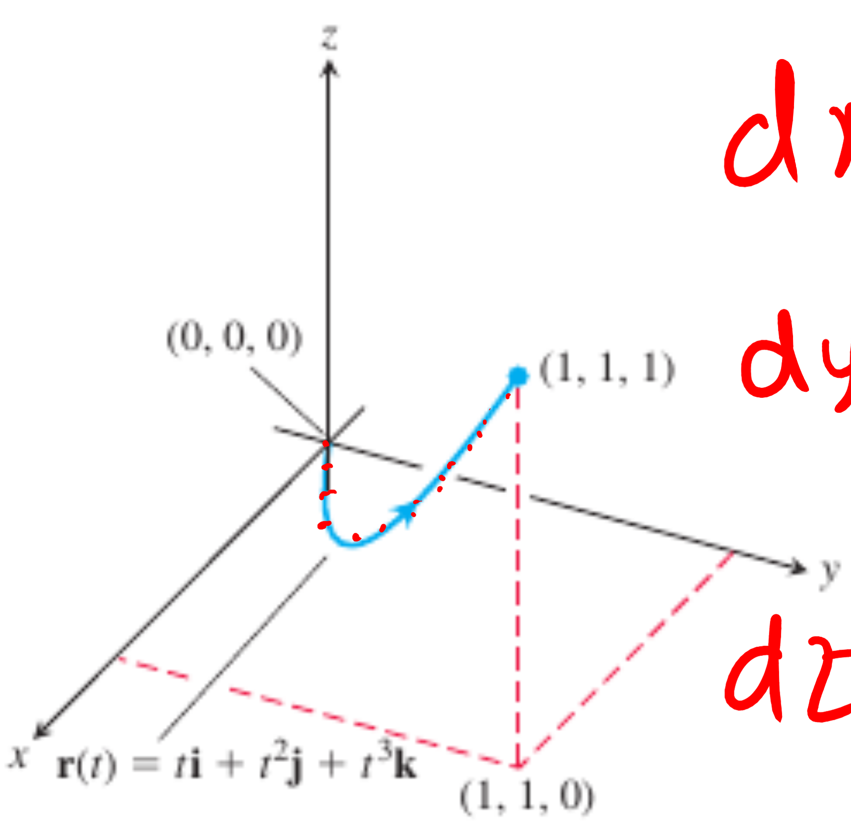
$z = t \Rightarrow dz = dt$

$$\int_0^{2\pi} (-\sin(t))(-\sin t)\,dt + t(+\cos t)\,dt + 2\cos t\,dt$$

$$= \int_0^{2\pi} (\sin^2 t + t\cos t + 2\cos t)\,dt$$

EXAMPLE 4 Find the work done by the force field $\mathbf{F} = (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k}$ along the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \le t \le 1$, from (0, 0, 0) to (1, 1, 1) (Figure 16.18).

$t=0 \quad t=1$



$dx = 1\,dt \Leftarrow x = t$

$dy = 2t\,dt \Leftarrow y = t^2$

$\Leftarrow z = t^3$

$dz = 3t^2\,dt$

$$\text{Work done} = \int_C \vec{F}\cdot\vec{dr}$$

$$\vec{F}\cdot\vec{dr} = (y - x^2)\,dx + (z - y^2)\,dy + (x - z^2)\,dz$$

$$W.D = \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8)\,dt = \frac{29}{60}$$

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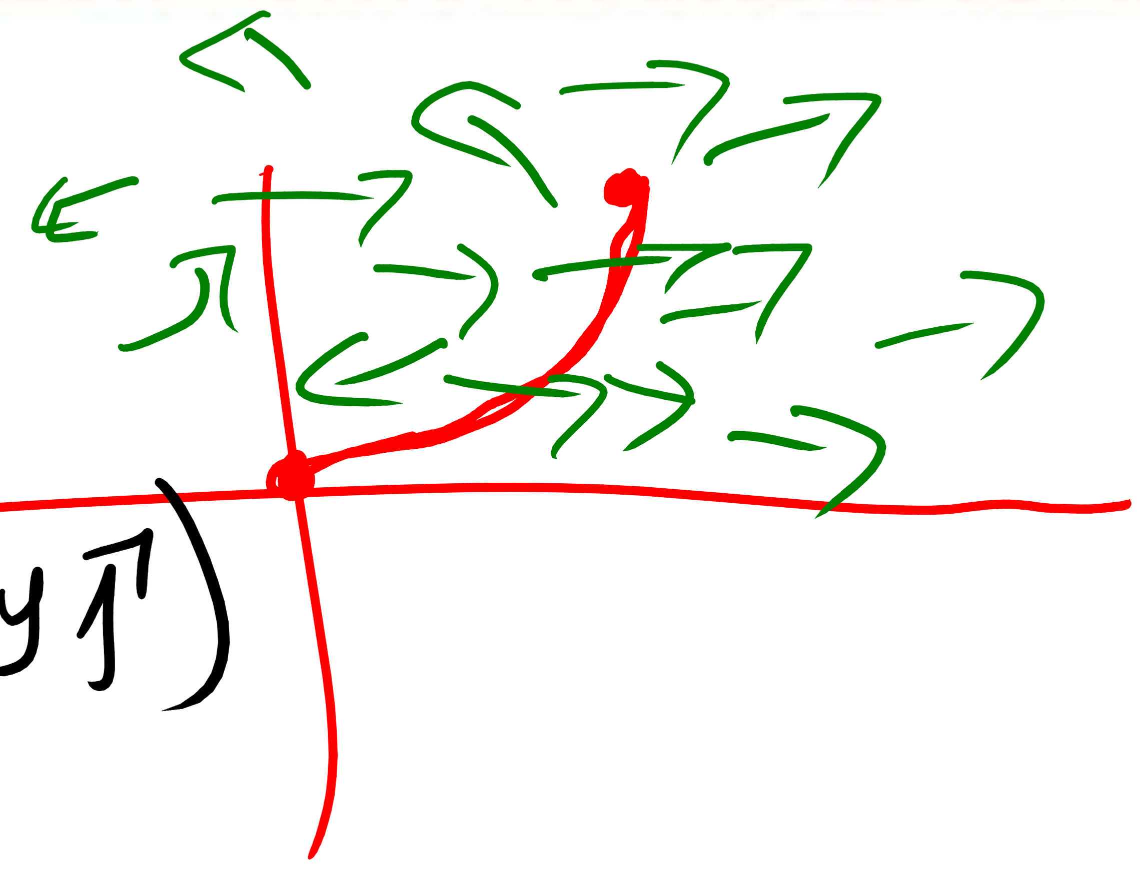
Example 8.28. *If* $\mathbf{F} = 3xy\mathbf{I} - y^2\mathbf{J}$, *evaluate* $\int_C \mathbf{F} \cdot d\mathbf{R}$, *where C is the curve in the xy-plane* $y = 2x^2$ *from* (0, 0) *to* (1, 2). (V.T.U., 2010)



$$\int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C (3xy\,\vec{i} - y^2\,\vec{j}) \cdot (dx\,\vec{i} + dy\,\vec{j})$$

$$= \int_C 3xy\,dx - y^2\,dy$$

$y = 2x^2$

$dy = 4x\,dx$

$(0,0) \to (1,2)$

$r(t) = t\,\vec{i} + 2t^2\,\vec{j}$

$x = t \quad y = 2t^2$

$$= \int_C 3x(2x^2)\,dx - (2x^2)^2\, 4x\,dx$$

$(t, 2t^2)$

$(0,0) \to (1,2)$

$t = 0, 1$

$$= \int_C 6x^3 - 16x^5\,dx = \int_0^1 (6x^3 - 16x^5)\,dx = \frac{-7}{6}$$

$(0,0) \quad (2,8)$

$x = t/4$

$y = t^2/8$

$y = 2x^2$

$\frac{t^2}{8} = 2\frac{t^2}{16}$

$(0,0)$ to $(1,2)$

$t \to 0$ to 4

$$= \int_C 3(t)(2t^2)\,dx - 4t^4\, 4t\,dt$$

$$= \int_0^1 (6t^3 - 16t^5)\,dt = -7/6$$

Example 8.30. *Find the work done in moving a particle in the force field* $\mathbf{F} = 3x^2\mathbf{I} + (2xz - y)\mathbf{J} + z\mathbf{K}$, *along*
(a) *the straight line from (0, 0, 0) to (2, 1, 3).* (S.V.T.U., 2007 ; J.N.T.U., 2002)
(b) *the curve defined by* $x^2 = 4y$, $3x^3 = 8z$ *from* $x = 0$ *to* $x = 2$. (Delhi, 2002)

b) $x = t$ $\quad t \to 0$ to 2

$y = t^2/4$

$z = 3\,t^3/8$

a) $r(t) = 2t\,\vec{i} + \vec{j} + 3t\,\vec{k}$, $t \to 0$ to 1

(x_0, y_0, z_0) $\quad (x_1, y_1, z_1)$

$(0,0,0)$ $\quad (2,1,3)$

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

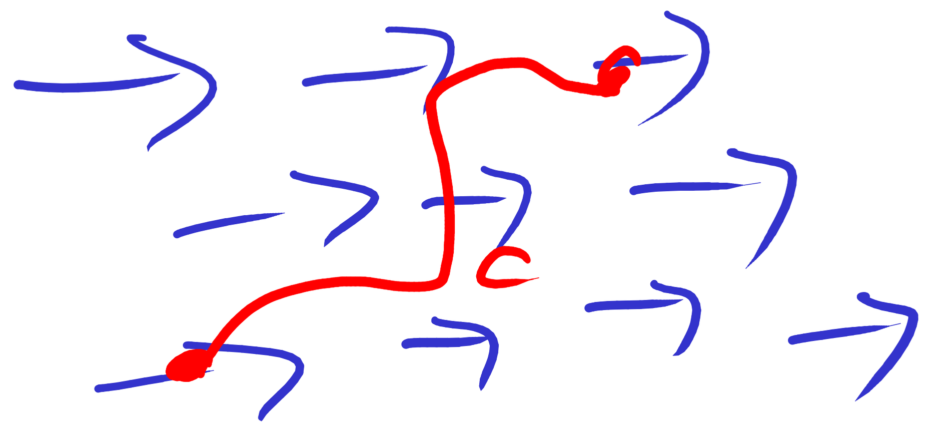
$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$$

$x = 2t$ $\quad t \to 0$ to 1

$y = t$

$z = 3t$

$$WD = \int_C \vec{F} \cdot d\vec{r} = \int \vec{F}(r(t)) \cdot d\vec{r}$$



flux across surface $\left\{ \int_S \vec{F} \cdot d\vec{s} \right.$

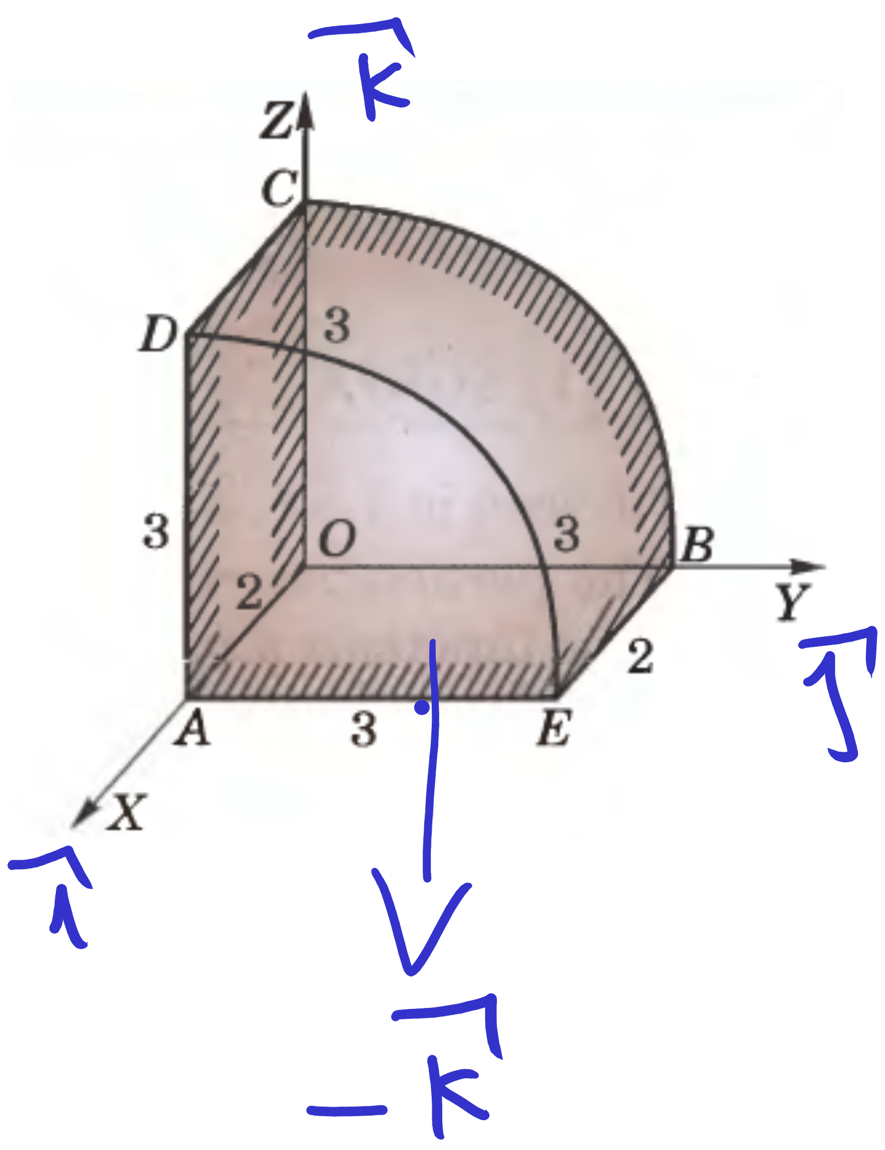
$$= \int_S \vec{F} \cdot \hat{n} \, ds$$

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

Example 8.31. *Evaluate* $\int_S \mathbf{F} \cdot \mathbf{N} ds$ *where* $\mathbf{F} = 2x^2y\mathbf{I} - y^2\mathbf{J} + 4xz^2\mathbf{K}$ *and S is the closed surface of the region in the first octant bounded by the cylinder* $y^2 + z^2 = 9$ *and the planes* $x = 0, x = 2, y = 0$ *and* $z = 0$.

$$\int_S \vec{F} \cdot \hat{n} \, ds = \int_{S_1} + \int_{S_2} + \int_{S_3} + \int_{S_4} + \int_{S_5}$$

S_1 OAED, S_2 OADC, S_3 AED, S_4 OBC, S_5 BEDC



S_3 $\vec{i}$ $ds = dy\,dz$

$x = 2$

$y \to 0$ to 3

$z \to 0$ to 3

S_4 $-\vec{i}$ $ds = dy\,dz$

$x = 0$

S_2 $-\vec{j}$ $y = 0$

$ds = dx\,dz$

$x \to 0$ to 2

$z \to 0$ to 3

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$$\int_{S_1}\vec{F}\cdot\hat{n}\,ds = \iint \left(2x^2y\,\vec{i} - y^2\,\vec{j} + 4xz^2\,\vec{k}\right)\cdot\frac{(-\vec{k})}{|\vec{k}|}\,ds$$

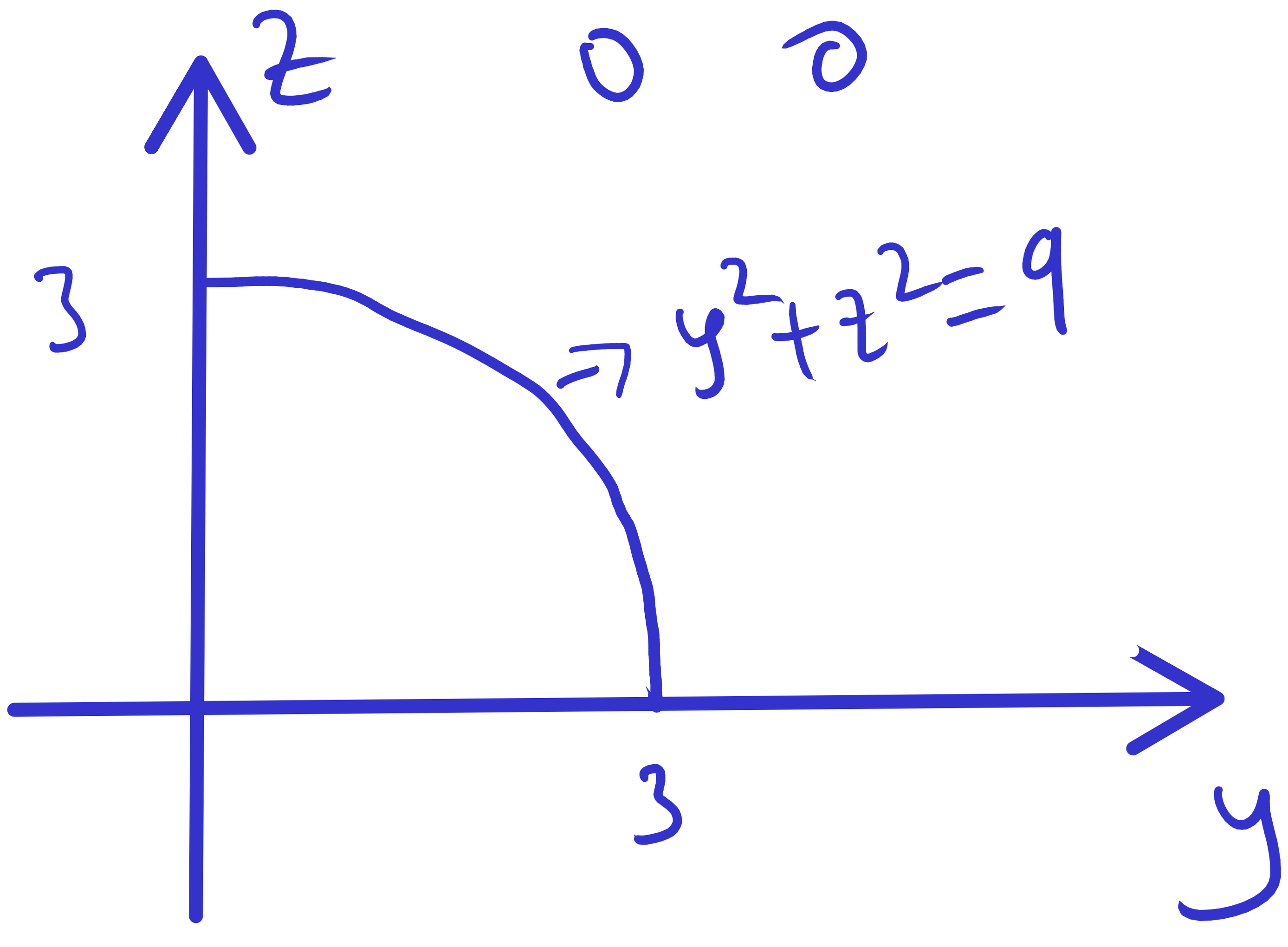
$$= \int_{x=0}^{2}\int_{y=0}^{3} -4xz^2\,dy\,dx \qquad \text{on } S_1 \quad z=0$$

$$= 0$$

$$\int_{S_2}\vec{F}\cdot\hat{n}\,ds = \int_{z=0}^{3}\int_{x=0}^{2}\left(2x^2y\,\vec{i} - y^2\,\vec{j} + 4xz^2\,\vec{k}\right)\cdot\frac{(-\vec{j})}{1}\,dx\,dz$$

$$= \int_0^3\int_0^2 y^2\,dx\,dz \qquad = 0$$

$$\int_{S_3}\vec{F}\cdot\hat{n}\,ds = \int_0^3\int_0^{\sqrt{9-y^2}} 2x^2y\,dz\,dy = \int_0^3\int_0^{\sqrt{9-y^2}} 8y\,dz\,dy$$



$$= \int_0^3 8y\sqrt{9-y^2}\,dy$$

$$= 72$$

$y \to 0$ to 3

$z \to 0$ to $\sqrt{9-y^2}$

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$S \to xy \Rightarrow ds = \dfrac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$ $\qquad \hat{n} = \nabla\phi$

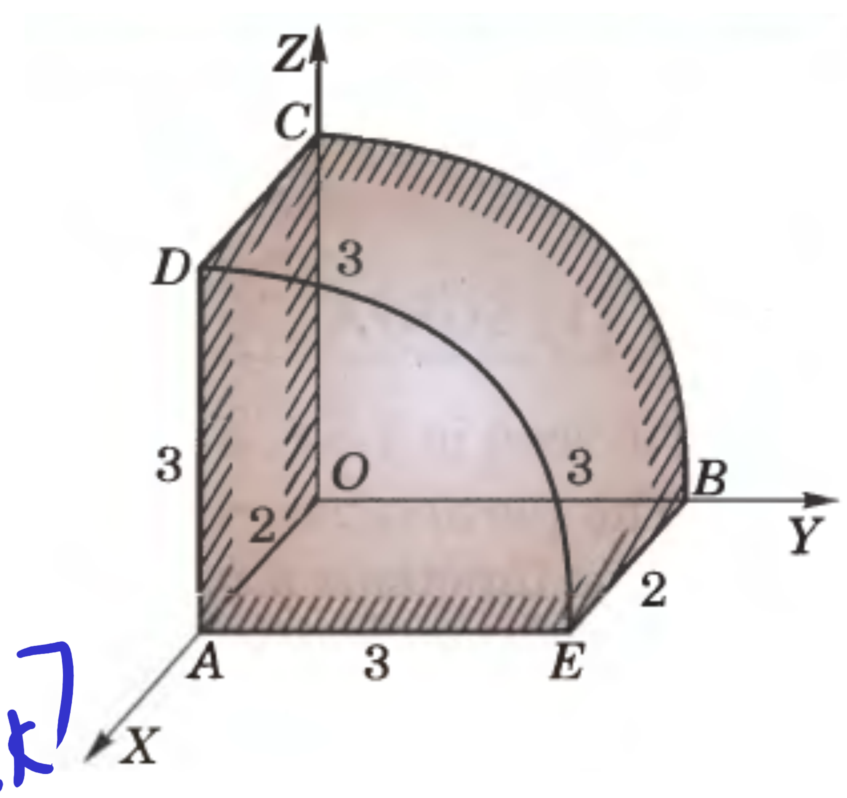
$S \to yz \Rightarrow ds = \dfrac{dy\,dz}{|\hat{n}\cdot\vec{i}|}$

$S \to xz \Rightarrow ds = \dfrac{dx\,dz}{|\hat{n}\cdot\vec{j}|}$

$$\int_{S_4} \vec{F}\cdot\vec{n}\,ds = \iint -2x^2y\,dx\,dy$$

$$= 0$$

on S_4
$x = 0$



$\int_{S_5}$

$$\hat{n} = \nabla\phi$$

$$= \frac{2y\vec{j}+2z\vec{k}}{\sqrt{(2y)^2+(2z)^2}} = \frac{y\vec{j}+z\vec{k}}{3}$$

S_5 $\quad \underline{\underline{EBCD}}$

$y^2+z^2=9$

y^2+z^2-9

$$\vec{F}\cdot\vec{n}\,ds = (2x^2y\,\vec{i} - y^2\vec{j} + 4xz^2\vec{k})\cdot \frac{(2y\vec{j}+2z\vec{k})}{3}\,\frac{dx\,dy}{|2y\vec{j}+2z\vec{k}\cdot\vec{k}|}$$

$$\vec{F}\cdot\vec{n}\,ds = (-2y^3+8xz^3)\frac{dx\,dy}{2z}$$

$$\int_S \vec{F}\cdot\hat{n}\,ds$$

$$S \Rightarrow xy \Rightarrow ds = \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$S \Rightarrow yz \Rightarrow ds = \frac{dy\,dz}{|\hat{n}\cdot\vec{i}|}$$

$$\phi$$

$$S \Rightarrow xz \Rightarrow ds = \frac{dx\,dz}{|\hat{n}\cdot\vec{j}|} \qquad \hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

S_3 $\quad y^2+z^2=9$

$$\phi = y^2+z^2-9$$

$$\nabla\phi = 2y\vec{j}+2z\vec{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2y\vec{j}+2z\vec{k}}{\sqrt{4y^2+4z^2}}$$

$$\hat{n} = \frac{2y\vec{j} + 2z\vec{k}}{2\sqrt{y^2+z^2}} = \frac{y\vec{j} + z\vec{k}}{3}$$

$$ds = \frac{dx\,dy}{|\vec{n}\cdot\vec{k}|} = \frac{dx\,dy}{\left|\frac{y\vec{j}+z\vec{k}}{3}\cdot\vec{k}\right|}$$

$$= \frac{dx\,dy}{z/3}$$

$$\vec{F}\cdot\vec{n}\,ds = (2x^2y\vec{i} - y^2\vec{j} + 4xz^2\vec{k})\cdot\frac{(y\vec{j}+z\vec{k})}{3}\;\frac{dx\,dy}{z/3}$$

$$= \frac{-y^3+4xz^3}{3}\cdot\frac{1}{z/3}\,dx\,dy$$

$$\vec{F}\cdot\vec{n}\,ds = \left(-\frac{y^3}{z} + 4xz^2\right)dx\,dy$$

$$\iint \vec{F}\cdot\vec{n}\,ds = \iint\left(-\frac{y^3}{z} + 4xz^2\right)dx\,dy$$

$$\int_S \vec{F}\cdot d\vec{s} = \int_S \vec{F}\cdot\hat{n}\, ds$$

$$\phi$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$S \to xy \Rightarrow ds = \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

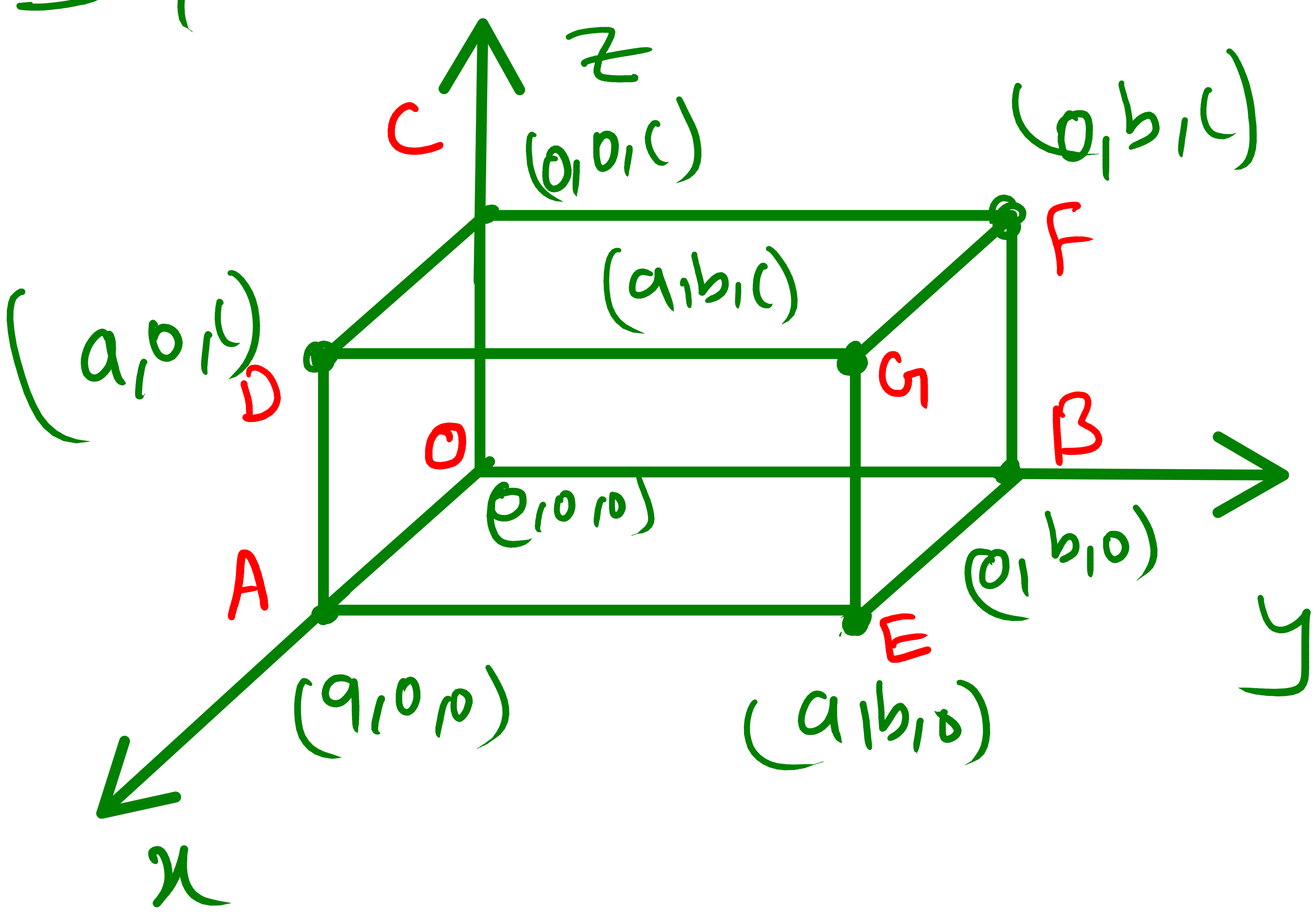
$$S \to yz \Rightarrow ds = \frac{dy\,dz}{|\hat{n}\cdot\vec{i}|}$$

$$S \to xz \Rightarrow ds = \frac{dx\,dz}{|\hat{n}\cdot\vec{j}|}$$

$$\vec{F} = (x^2 - yz)\,\vec{i} + (y^2 - xz)\,\vec{j} + (z^2 - xy)\,\vec{k}$$

$$0 \le x \le a, \quad 0 \le y \le b, \quad 0 \le z \le c$$

$$\iint_S \vec{F}\cdot d\vec{s}$$



$\vec{F} = (x^2-yz)\vec{i} + (y^2-xz)\vec{j} + (z^2-xy)\vec{k}$

CALCULUS

BMAT101L

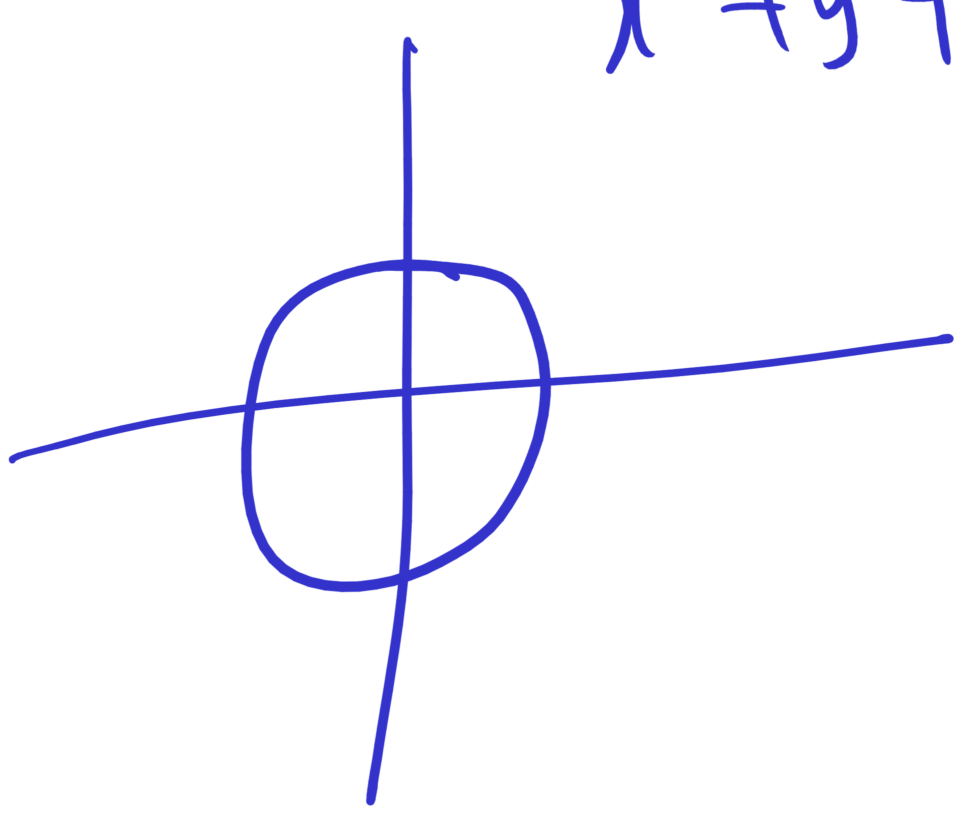
surface	$\hat{n}$	$\vec{F}\cdot\hat{n}$	ds	limits	
OAEB	$-\vec{k}$	$-(z^2-xy)$	$\frac{dx\,dy}{\lvert\hat{n}\cdot\vec{k}\rvert}$	$x\to 0$ to a, $y\to 0$ to b, $z=0$	$\int_0^a\int_0^b xy\,dy\,dx$
CDGF	$\vec{k}$	(z^2-xy)	$\frac{dx\,dy}{1}$	$x\to 0$ to a, $y\to 0$ to b, $z=c$	$\int_0^a\int_0^b (c^2-xy)\,dy\,dx$
OADC	$-\vec{j}$	$-(y^2-xz)$	$dx\,dz$	$x\to 0$ to a, $z\to 0$ to c, $y=0$	$\int_0^a\int_0^c xz\,dz\,dx$
EBFG	$\vec{j}$	(y^2-xz)	$dx\,dz$	$x\to 0$ to a, $z\to 0$ to c, $y=b$	$\int_0^a\int_0^c (b^2-xz)\,dz\,dx$
OBFC	$-\vec{i}$	$-(x^2-yz)$	$dy\,dz$	$y\to 0$ to b, $z\to 0$ to c, $x=0$	$\int_0^b\int_0^c yz\,dz\,dy$
AEGD	$\vec{i}$	(x^2-yz)	$dy\,dz$	$y\to 0$ to b, $z\to 0$ to c, $x=a$	

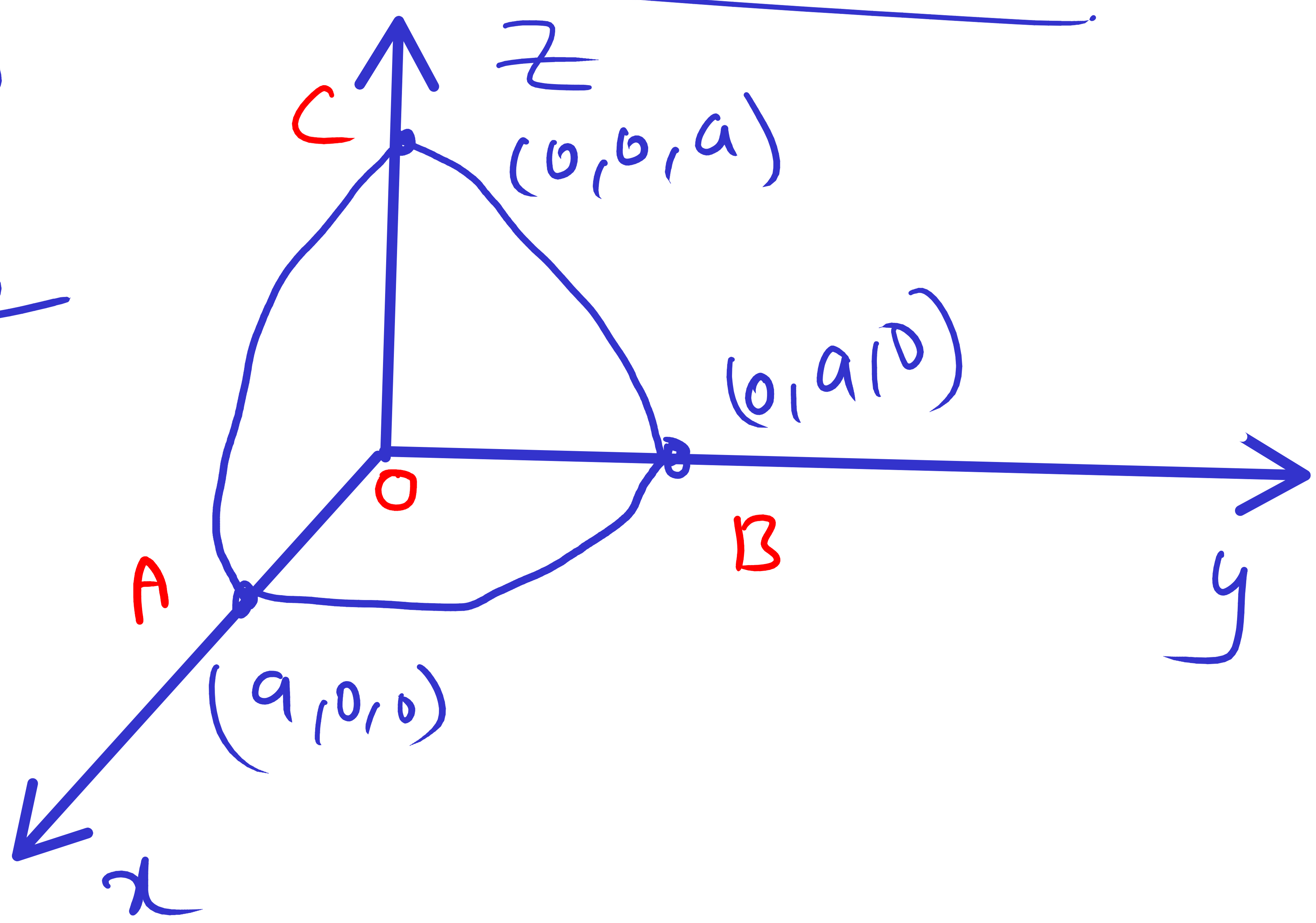
$$\int_0^b\int_0^c (a^2-yz)\,dz\,dy$$

$$\int_S \vec{F}\cdot\hat{n}\,ds = abc(a+b+c)$$

$\vec{F} = yz\vec{i} + xz\vec{j} + xy\vec{k}$

$x^2+y^2+z^2=a^2$





Surfaces : $\underset{S_1}{OAB}$, $\underset{S_2}{OBC}$, $\underset{S_3}{OAC}$, $\underset{S_4}{ABC}$

OAB $\quad \hat{n} = -\vec{k}$

$$\vec{F}\cdot\hat{n} = -xy$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|} = \frac{dx\,dy}{1} = dx\,dy$$

$x \to 0$ to a

$y \to 0$ to $\sqrt{a^2-x^2}$

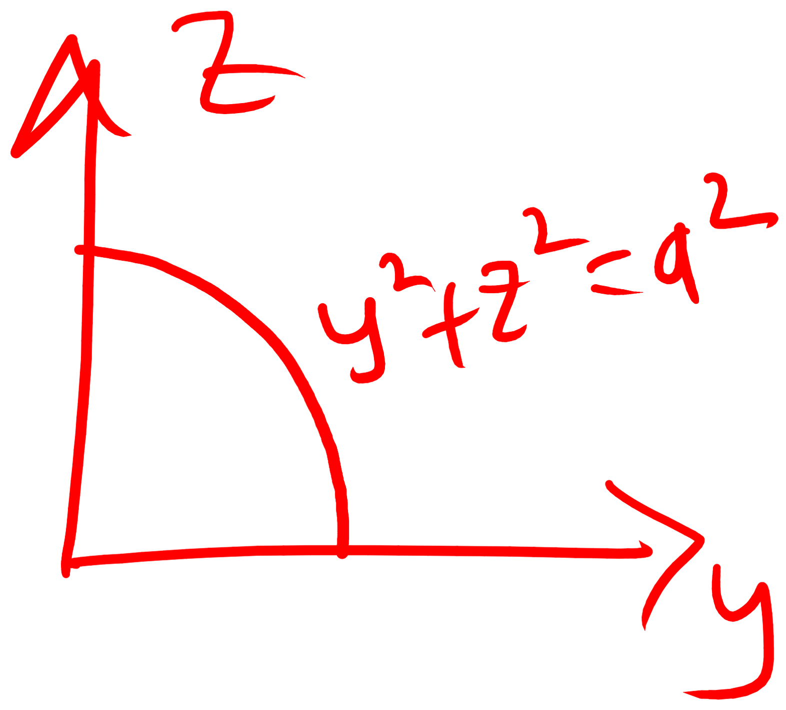
$$\int_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_0^a \int_0^{\sqrt{a^2-x^2}} -xy\,dy\,dx = \frac{-a^4}{8}$$

OBC

$$\hat{n} = -\vec{i}$$

$$\vec{F}\cdot\hat{n} = -yz$$

$$ds = \frac{dy\,dz}{|\hat{n}\cdot\vec{i}|} = dy\,dz$$



$y \to 0$ to a

$z \to 0$ to $\sqrt{a^2-y^2}$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^a \int_0^{\sqrt{a^2-y^2}} -yz \, dz \, dy = \frac{-a^4}{8}$$

OAC

$$\hat{n} = -\vec{j}$$

$$\vec{F}\cdot\hat{n} = -xz$$

$$ds = dx\,dz$$

$x \to 0$ to a

$z \to 0$ to $\sqrt{a^2-x^2}$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^a \int_0^{\sqrt{a^2-x^2}} -xz \, dz \, dx = \frac{-a^4}{8}$$

ABC

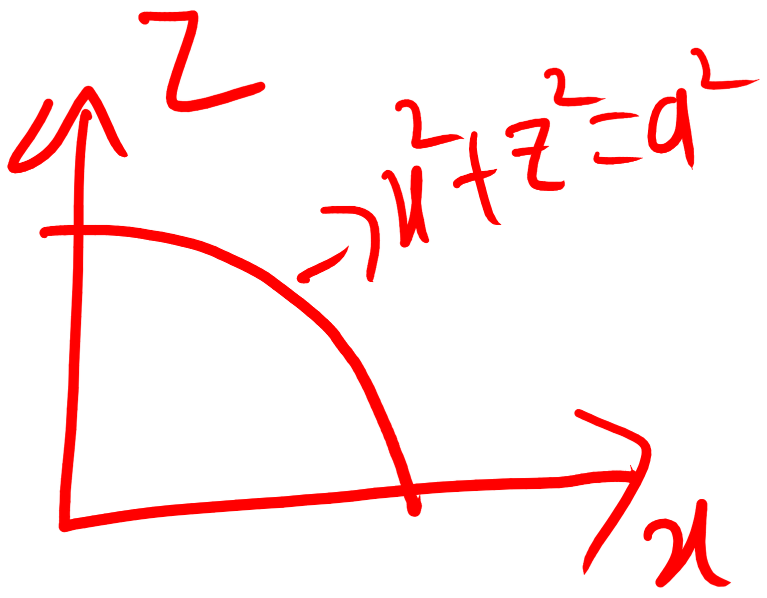
$$\phi = x^2+y^2+z^2-a^2$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\hat{n} = \frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\vec{i}+y\vec{j}+z\vec{k}}{a}$$

$$ds = \frac{dx\,dz}{|\hat{n}\cdot\vec{j}|} = \frac{dx\,dz}{\left|\frac{x\vec{i}+y\vec{j}+z\vec{k}}{a}\cdot\vec{j}\right|}$$

$$ds = \frac{dx\,dz}{y/a}$$



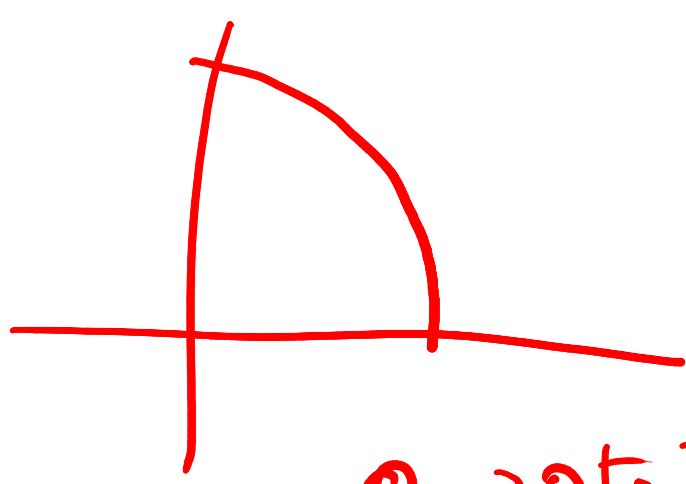
$$\left.\begin{array}{l} x \to 0 \text{ to } a \\ z \to 0 \text{ to } \sqrt{a^2-x^2} \end{array}\right\} \quad y = \sqrt{a^2-x^2-z^2}$$

$\vec{F} = yz\vec{i} + xz\vec{j} + xy\vec{k}$

$\hat{n} = \dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{a}$

$\vec{F}\cdot\hat{n} = (xyz + xyz + xyz)/a = \dfrac{3xyz}{a}$

$$\int_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{3xyz}{a} \frac{dz\,dx}{y/a}$$



$\theta \to 0$ to $\pi/2$

$r \to 0$ to a

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} 3xz\,dz\,dx$$

Take $x = a\cos\theta$

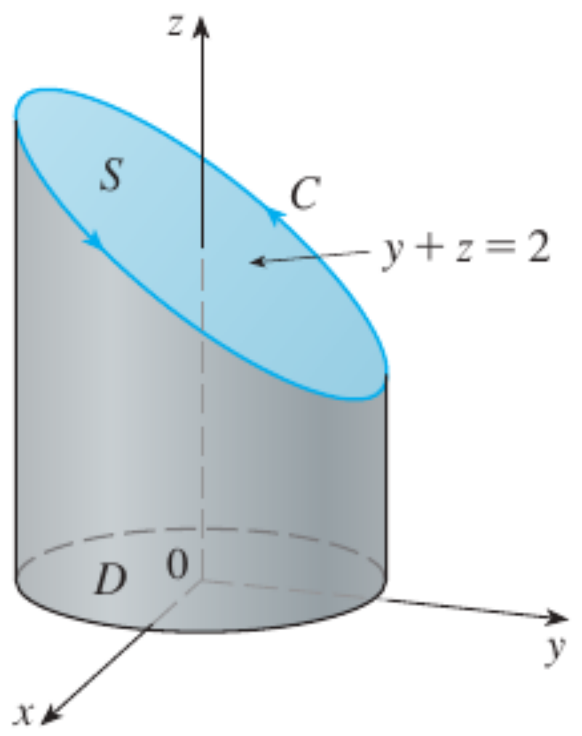
$z = a\sin\theta$

$$= \frac{3a^4}{8}$$

Stokes' Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

EXAMPLE 1 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above.)



$$\Leftarrow \quad \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix}$$

$$= (1+2y)\,\vec{k}$$

$$\iint_S (1+2y)\vec{k} \cdot d\vec{S} = \iint_S (1+2y)\vec{k} \cdot \hat{n} \, \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

$$\phi = y + z - 2 \qquad \hat{n} = \frac{\vec{j}+\vec{k}}{\sqrt{2}}$$

$$\nabla\phi = \vec{j} + \vec{k} \qquad \hat{n}\cdot\vec{k} = 1/\sqrt{2}$$

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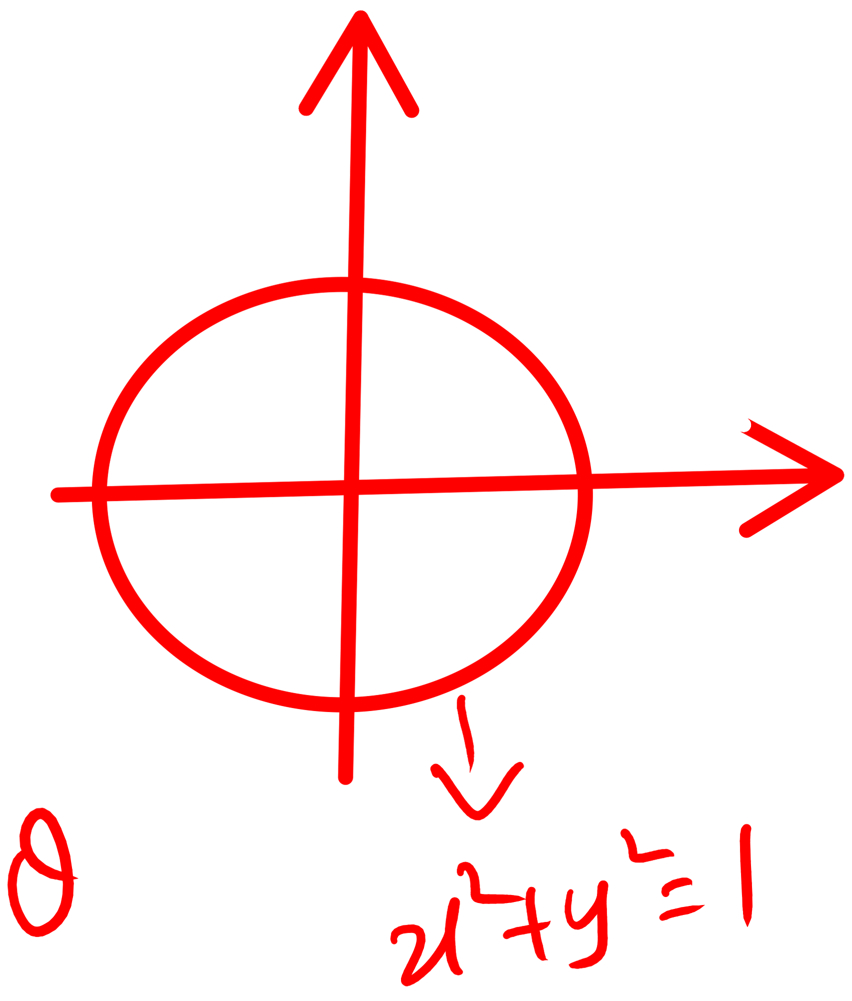
$$\iint_S (1+2y)\,\bar{k}\cdot\frac{(\bar{j}+\bar{k})}{\sqrt{2}}\;\frac{dx\,dy}{1/\sqrt{2}}$$

$$=\iint_S (1+2y)\,dx\,dy$$



$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{1}(1+2r\sin\theta)\,r\,dr\,d\theta$$

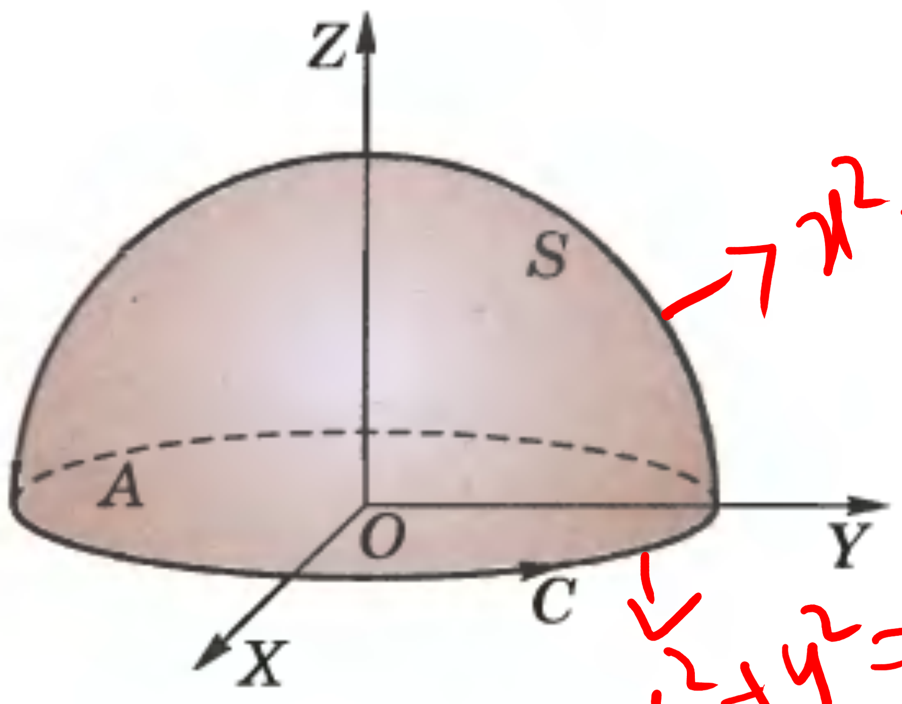
$$=\int_{0}^{2\pi}\int_{0}^{1}(r+2r^2\sin\theta)\,dr\,d\theta$$

$$=\int_{0}^{2\pi}\left.\frac{r^2}{2}+\frac{2r^3}{3}\sin\theta\right|_0^1 d\theta$$

$$=\pi$$

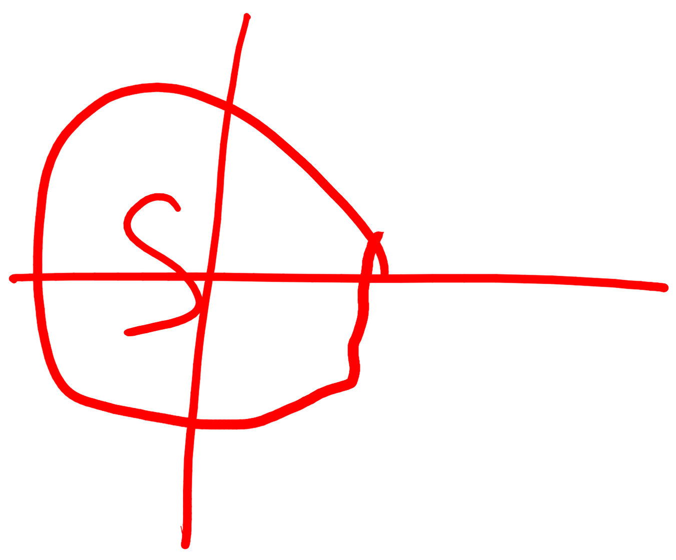
Example 8.37. *Verify Stoke's theorem for the vector field* $F = (2x - y)\,\mathbf{I} - yz^2\,\mathbf{J} - y^2z\,\mathbf{K}$ *over the upper half surface of* $x^2 + y^2 + z^2 = 1$, *bounded by its projection on the xy-plane.*

(Bhopal, 2008 ; Madras, 2006 ; S.V.T.U., 2006)



$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot d\vec{S}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (2x-y)\,dx - yz^2\,dy - y^2z\,dz$$



$$= \int_C (2x-y)\,dx$$

$x = \cos\theta$
$y = \sin\theta$

$$= \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)\,d\theta$$

$$= \int_0^{2\pi} -\sin 2\theta + \sin^2\theta \; d\theta = \pi$$

<u>RHS</u>

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \vec{k}$$

$$\phi = x^2 + y^2 + z^2 - 1$$

$$\nabla\phi = 2x\,\vec{i} + 2y\,\vec{j} + 2z\,\vec{k}$$

$$\hat{n} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

$$\nabla\times\vec{F}\cdot\hat{n} = z \qquad ds = \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|} = \frac{dx\,dy}{z}$$

$$\iint_S \nabla\times\vec{F}\cdot\vec{ds} = \iint_S z\,\frac{dx\,dy}{z} = \iint_S dx\,dy$$

$$= \text{Area of circle}$$

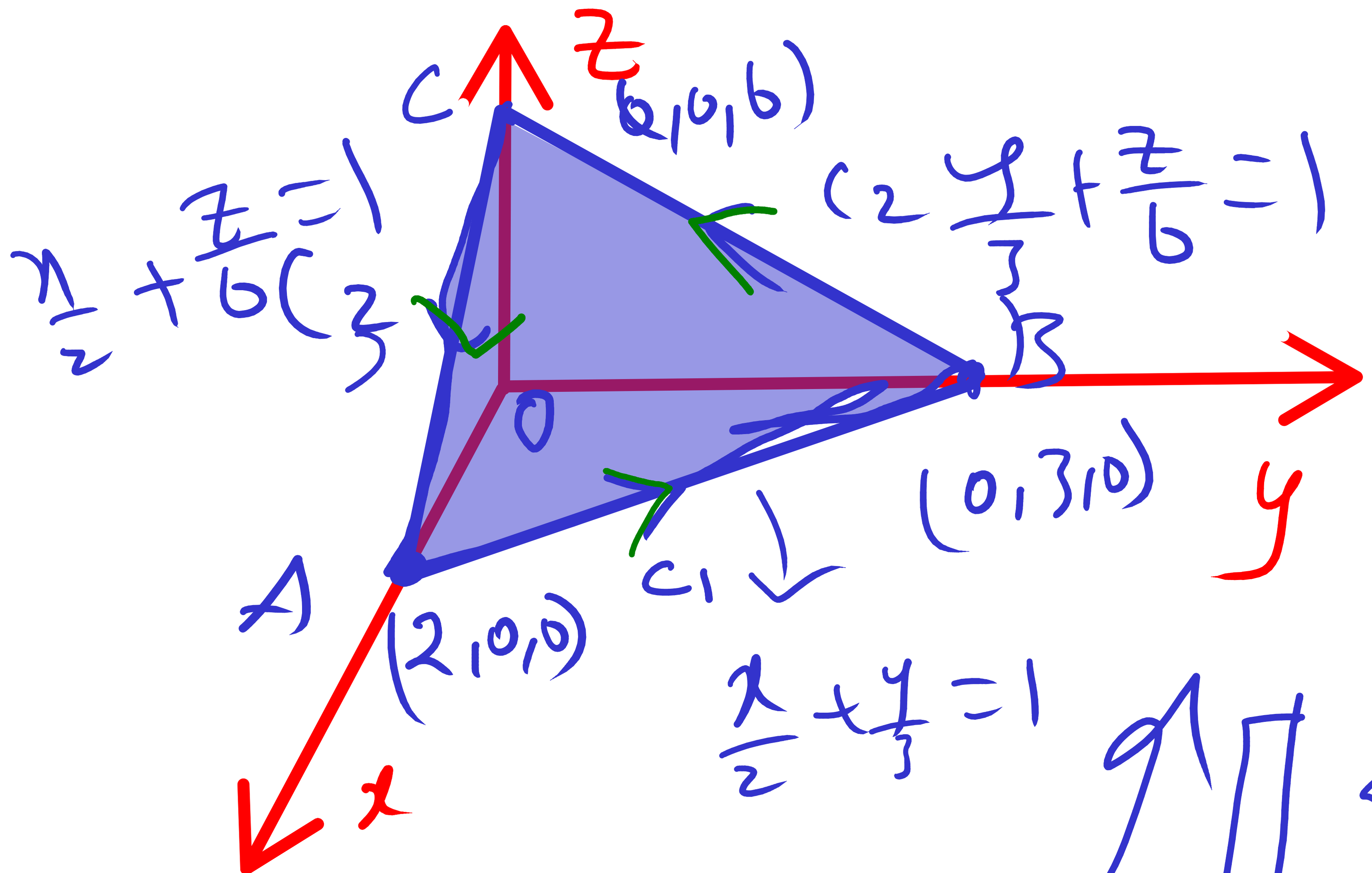
$$= \pi r^2 = \pi$$

Example 8.38. *Uses Stoke's theorem evaluate* $\int_C [(x + y)dx + (2x - z)dy + (y + z)dz]$ *where C is the boundary of the triangle with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6).*

(*Nagpur, 2009 ; Kurukshetra, 2009 S ; Kerala, 2005*)

$$\vec{F} = (x+y)\,\vec{i} + (2x-z)\,\vec{j} + (y+z)\,\vec{k}$$

$$\int_C \vec{F}\cdot\vec{dr} = \iint_S \nabla\times\vec{F}\cdot\vec{ds}$$



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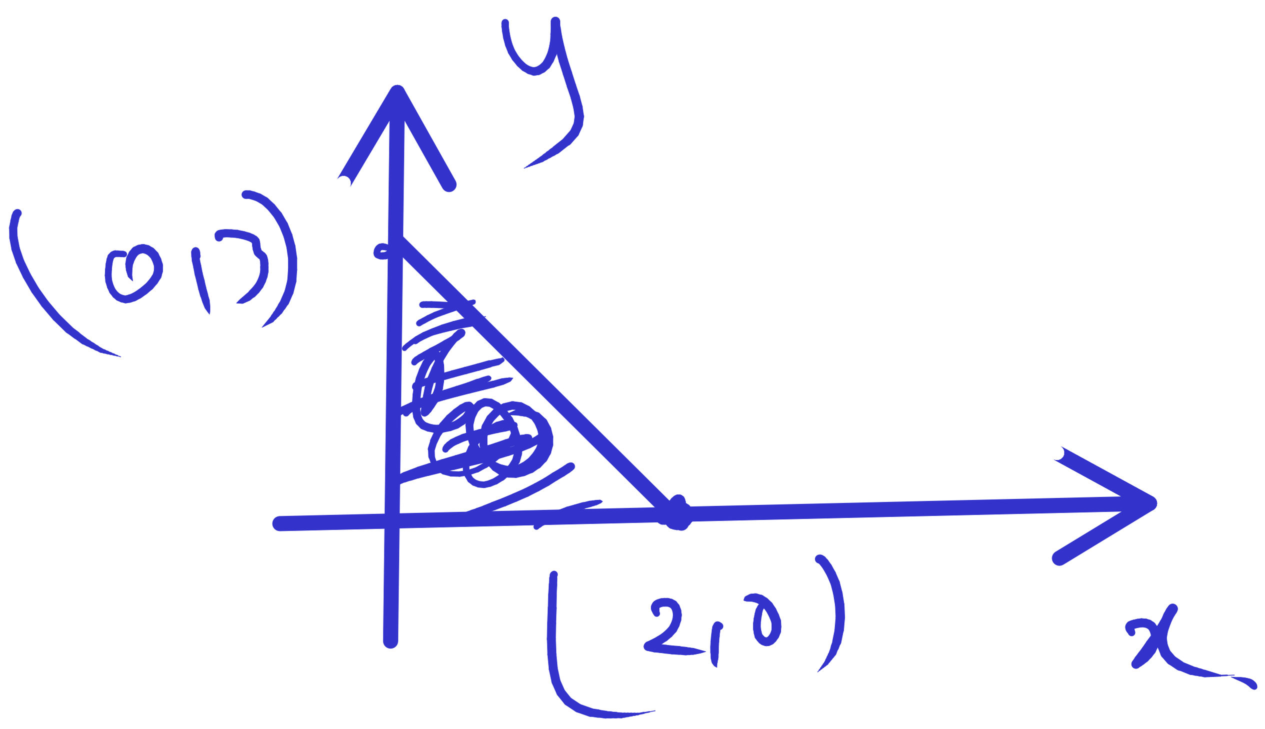
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y & 2x-z & y+z \end{vmatrix} = 2\vec{i} + \vec{k}$$

$\hat{n}$ $\quad \phi = \frac{x}{2} + \frac{y}{3} + \frac{z}{6} - 1 \quad \phi = 3x + 2y + z - 6$

$$\nabla \phi = 3\vec{i} + 2\vec{j} + \vec{k}$$

$$\hat{n} = \frac{3\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{14}}$$

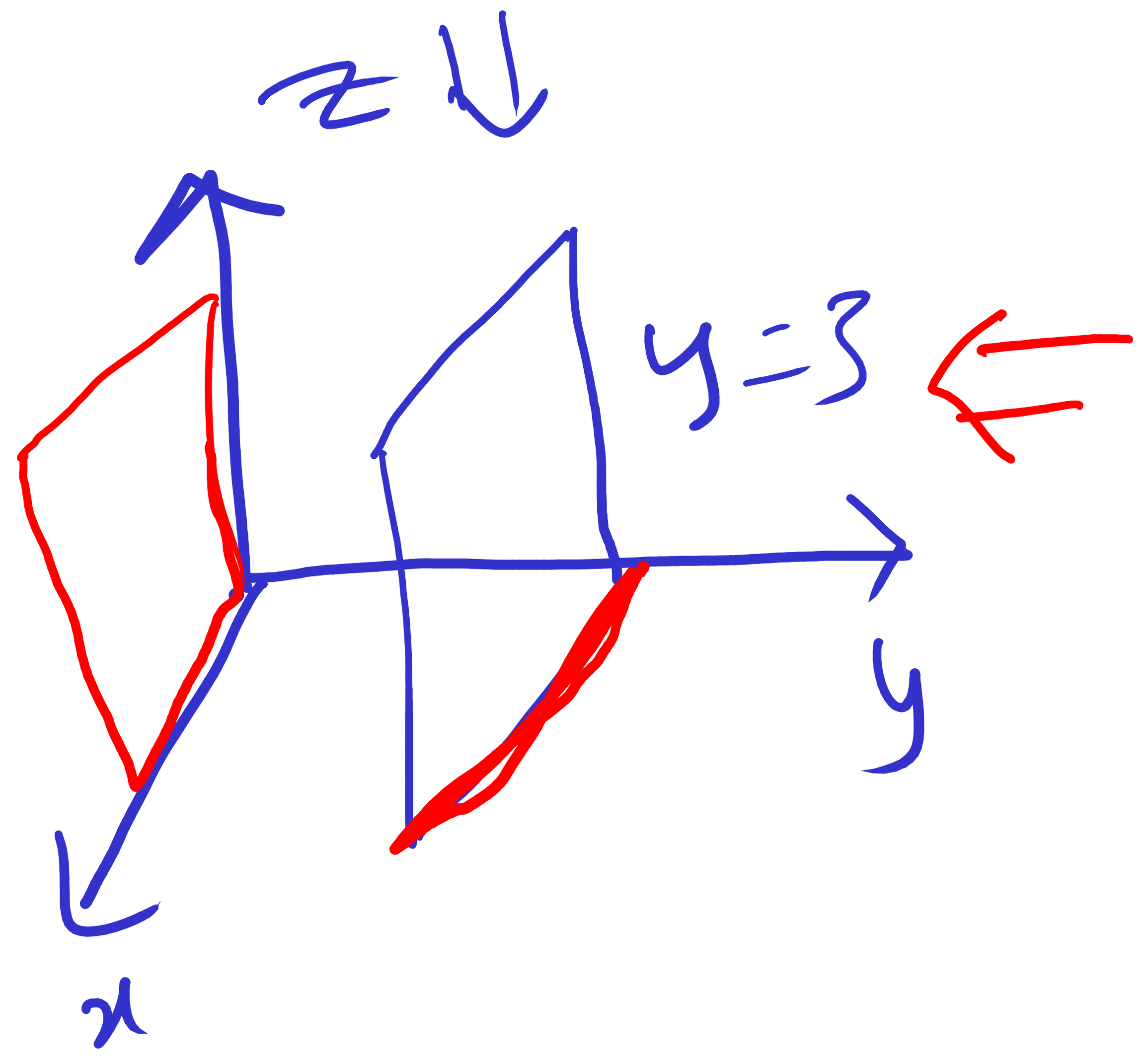


$$\nabla \times \vec{F} \cdot \hat{n} = \frac{7}{\sqrt{14}}$$

$$ds = \frac{dx\,dy}{|\hat{n} \cdot \vec{k}|} = \frac{dx\,dy}{1/\sqrt{14}}$$

$$\iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds = \iint \frac{7}{\sqrt{14}} \frac{dx\,dy}{1/\sqrt{14}} = 7 \iint_S dx\,dy$$

$$= 7 \times 3 = 21$$

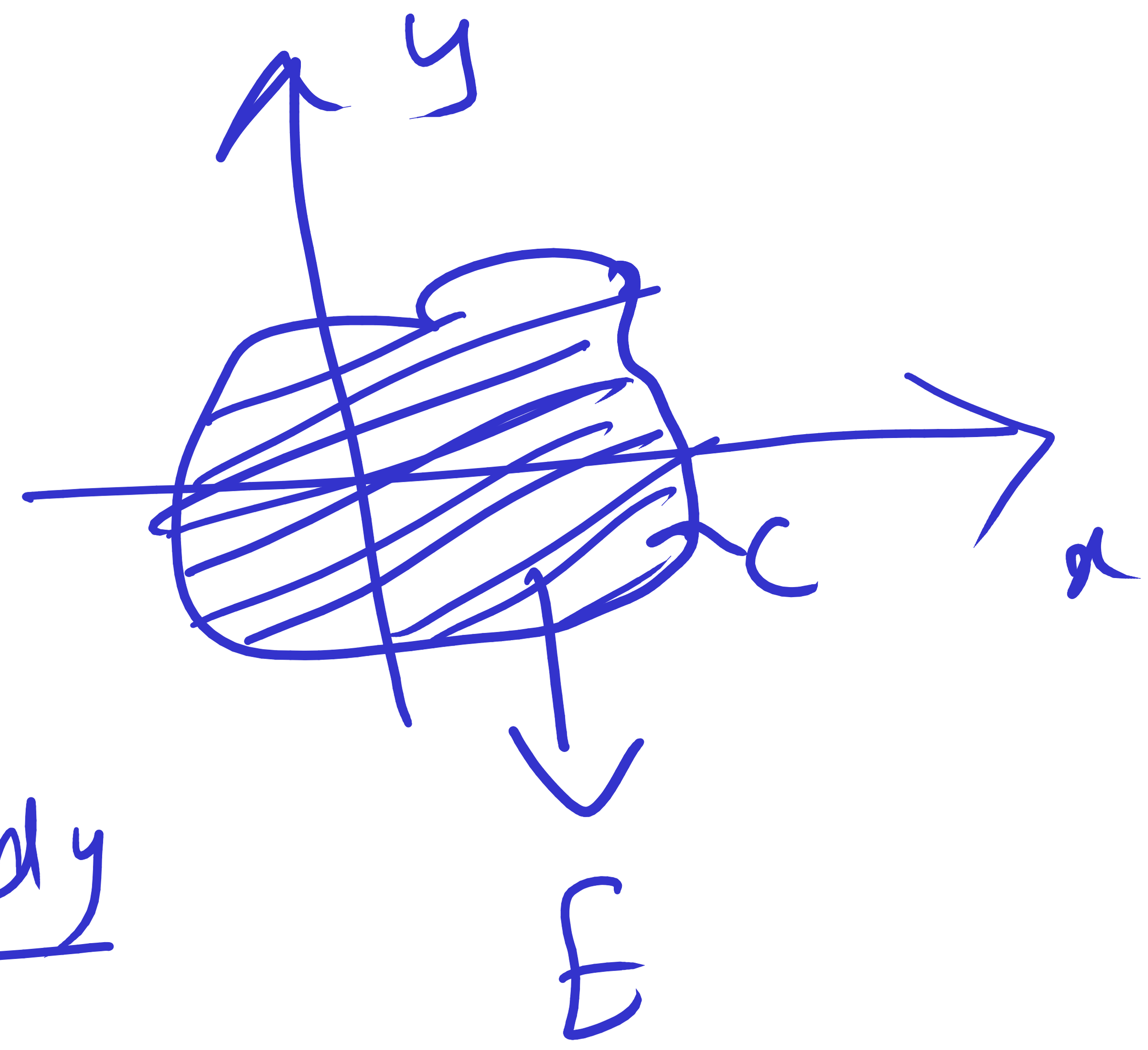


Stokes theorem

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot d\vec{S}$$

Green's Theorem

$$\vec{F} = f(x,y)\,\vec{i} + g(x,y)\,\vec{j}$$



$$\int_C f\,dx + g\,dy = \iint_E \nabla\times\vec{F}\cdot\hat{n}\;\underline{dx\,dy}$$

$$= \iint_E \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\; dx\,dy$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f & g & 0 \end{vmatrix} = \vec{i}\,(0-0) - \vec{j}\,(0-0) + \vec{k}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)$$

$$= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\vec{k}$$

$$d\vec{S} = \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|} = dx\,dy$$

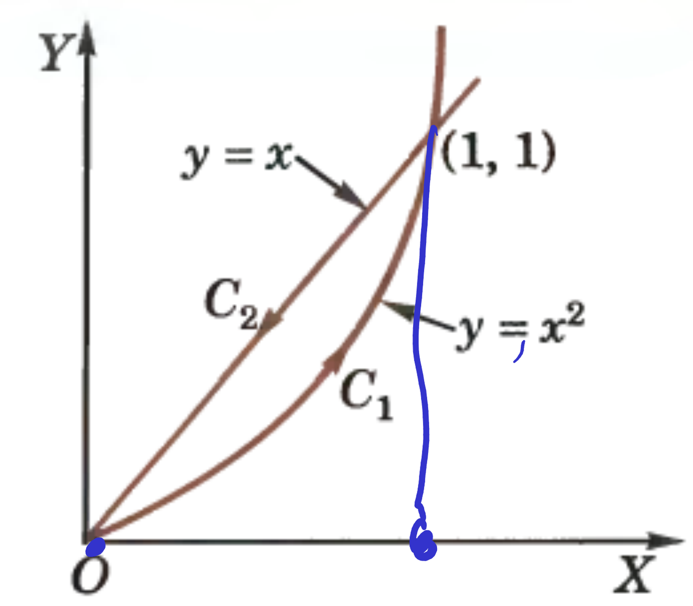
G.T | $\oint_C f\,dx + g\,dy = \iint_E \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$ Green's Theorem

S.T | $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot d\vec{S}$

Example 8.32. *Verify Green's theorem for* $\int_C [(xy+y^2)\,dx + x^2\,dy]$, *where C is bounded by* $y = x$ *and* $y = x^2$.

(V.T.U., 2011 ; S.V.T.U., 2009 ; Rohtak, 2003)

f g



$\int_C (xy+y^2)\,dx + x^2\,dy$

$= \int_{C_1} \qquad + \int_{C_2}$

Along C_1

$\int_{x=0}^{1} \left(x(x^2) + x^4\right)dx + x^2(2x\,dx)$

$y = x^2$

$dy = 2x\,dx$

$\frac{dy}{2x} = \frac{dy}{2\sqrt{y}}$

$\int_{y=0}^{1} (\sqrt{y}\,y + y^2)\frac{dy}{2\sqrt{y}} + y\,dy = \frac{19}{20}$

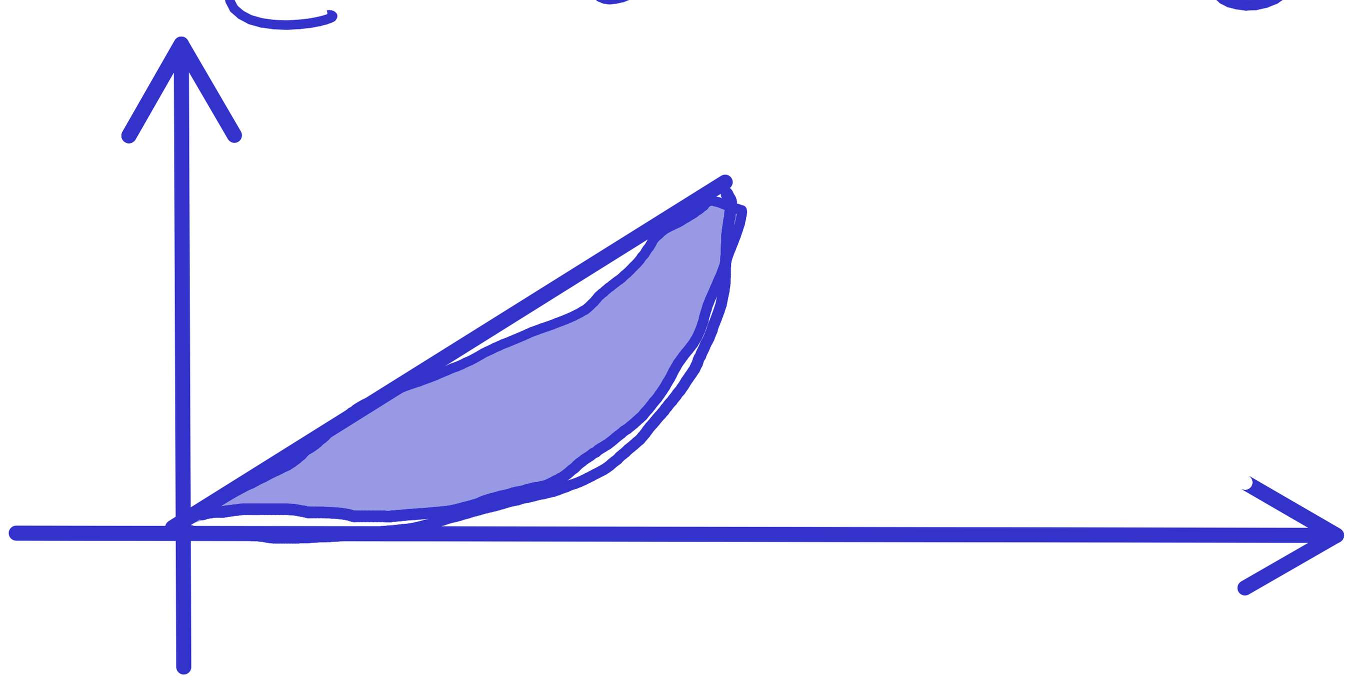
Along C_2 $\displaystyle\int_{x=1}^{0} (x(x)+x^2)\,dx + x^2\,dx$

$$= \int_1^0 3x^2\,dx = -1$$

$$\int_C f\,dx + g\,dy = \frac{19}{20} - 1 = \frac{-1}{20}$$

$$\iint_E \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\,dx\,dy = \iint_E 2x - (x+2y)\,dx\,dy$$

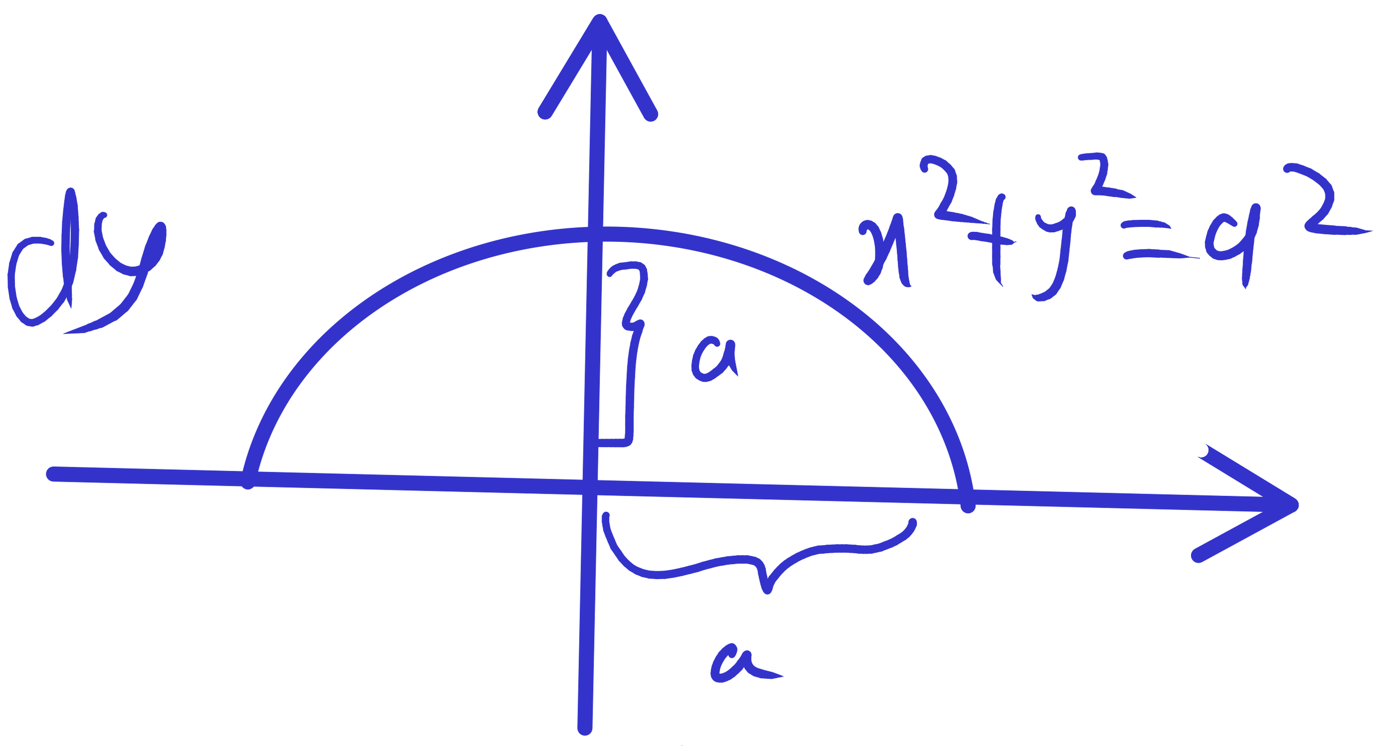
$$\int_C \underbrace{(xy+y^2)}_{f}\,dx + \underbrace{x^2}_{g}\,dy \qquad = \int_0^1 \int_{x^2}^{x} (x-2y)\,dy\,dx$$



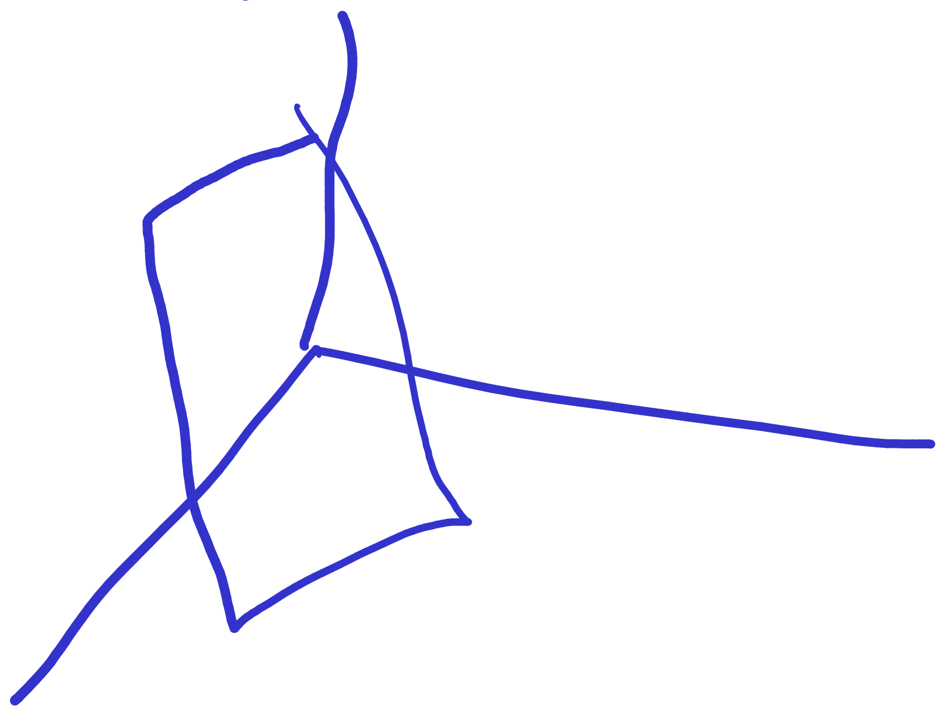
$$= \frac{-1}{20}$$

Example 8.35. *Apply Green's theorem to evaluate* $\int_C [\underbrace{(2x^2 - y^2)}_{f}dx + \underbrace{(x^2 + y^2)}_{g}dy]$*, where C is the boundary of the area enclosed by the x-axis and the upper-half of the circle* $x^2 + y^2 = a^2$. (U.P.T.U., 2005)

$$\int_C f\,dx + g\,dy = \iint_E \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$



$$= \iint_E (2x + 2y)\,dx\,dy$$



$$= \int_{-a}^{a} \int_0^{\sqrt{a^2-x^2}} 2(x+y)\,dy\,dx$$

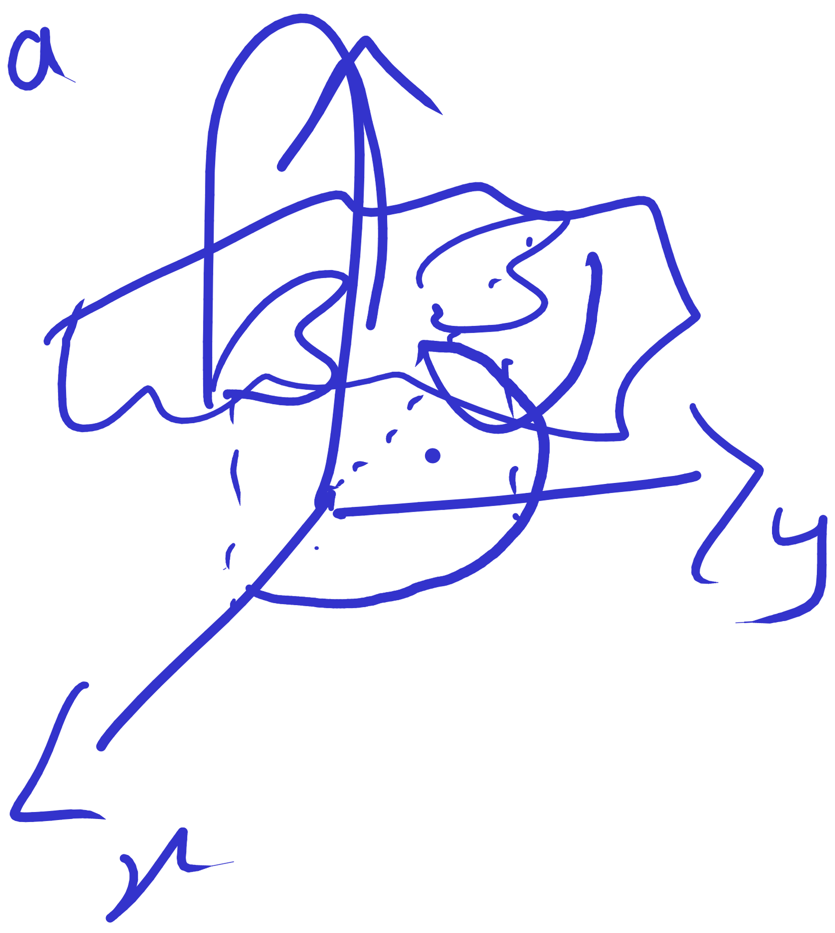
$$= \int_{\theta=0}^{\pi} \int_{r=0}^{a} 2(r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$$

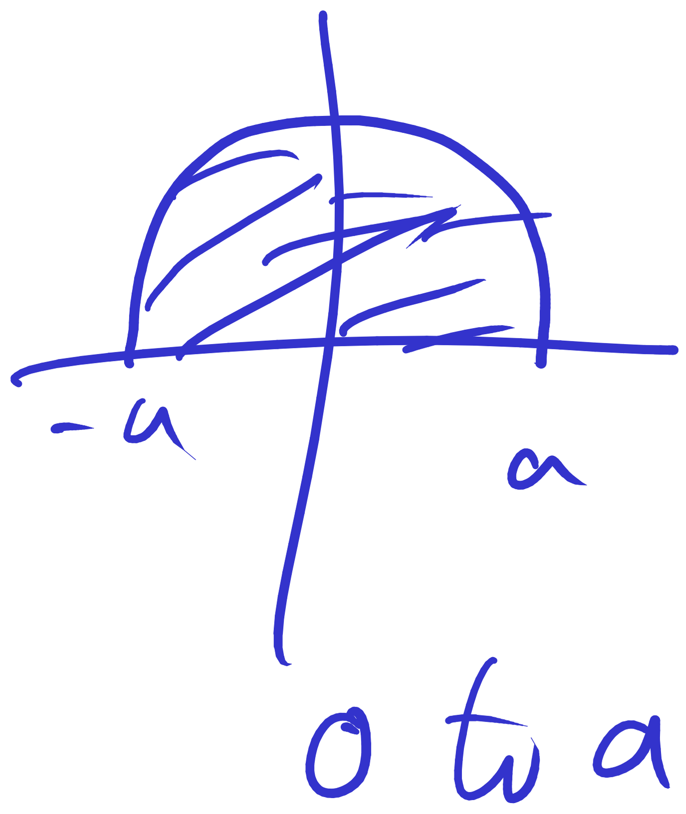
$$= \frac{4a^3}{3}$$

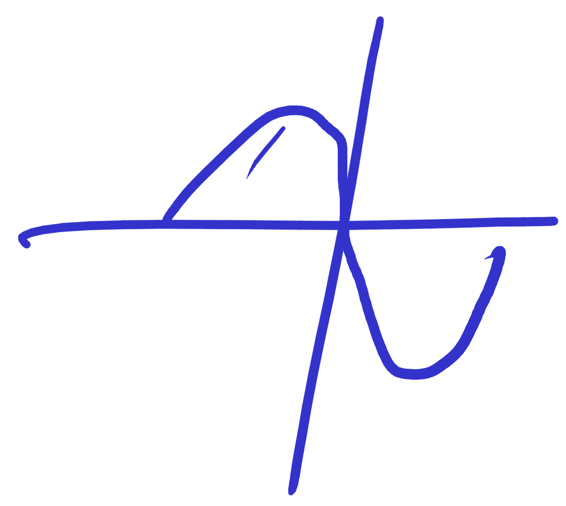
$$\int_{-a}^{a}\int (\quad)\,dx\,dy = \text{volume} \qquad = 2\int_{0}^{a}$$

$$\int_{-a}^{a}\int dx\,dy = \text{Area}$$

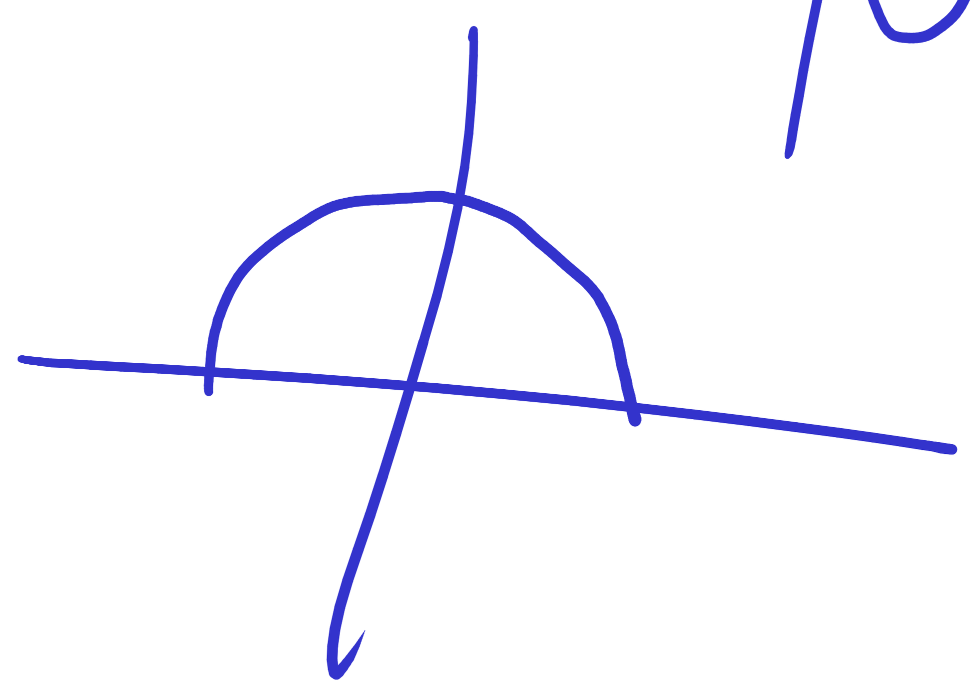
$$2\int_{0}^{a}$$





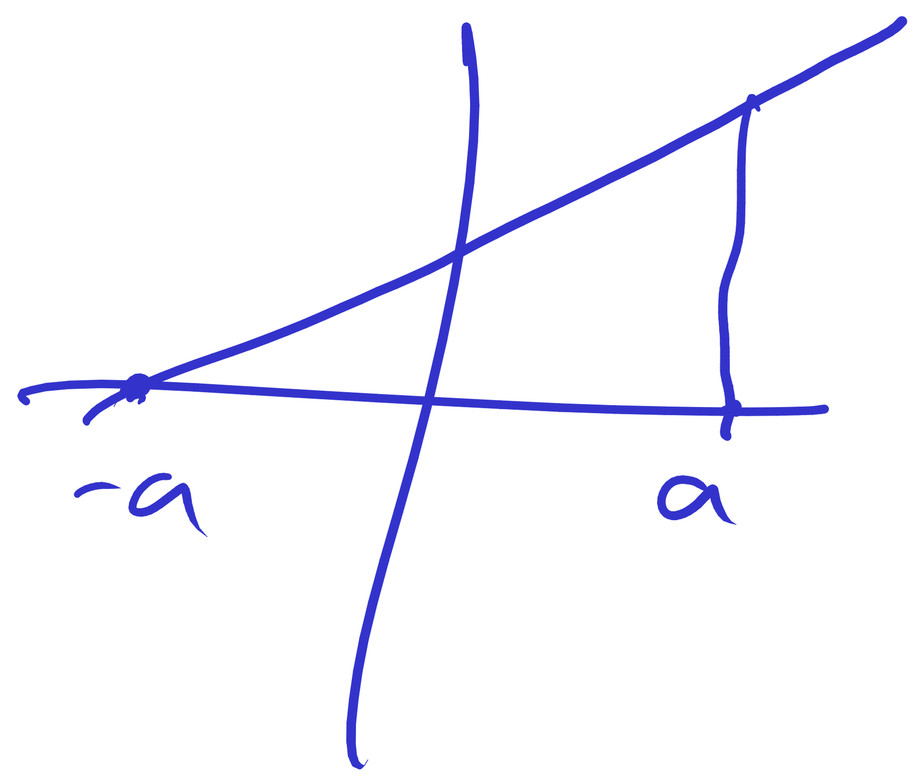


$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} \\ 0 \\ 0 \end{cases}$$



$f(x) = f(-x)$ even

$-f(x) = f(-x)$ odd

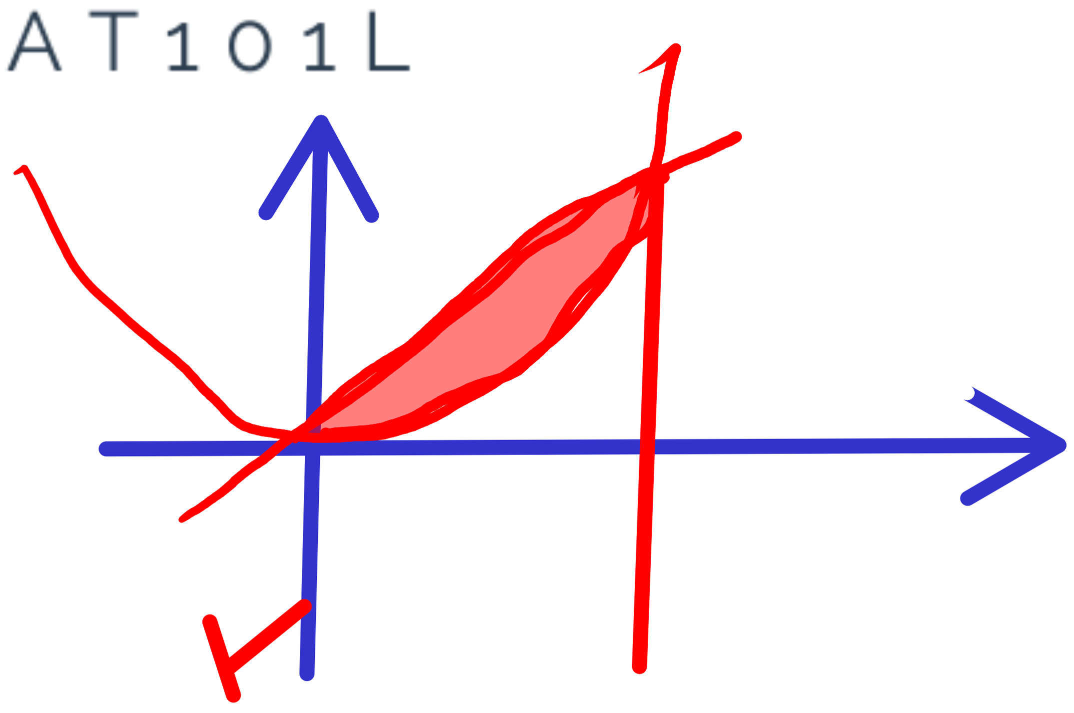


$$\left.\begin{array}{ll} f(x,y) = f(-x,-y) & \text{even} \\ -f(x,y) = f(-x,-y) & \text{odd} \end{array}\right\} \text{I have to check}$$

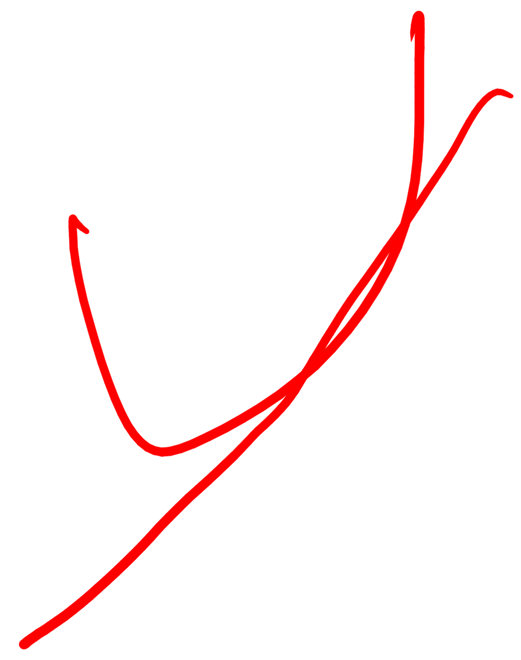
$$\int_0^2\int_{x^2}^{2x}(4x+2)\,dy\,dx$$

$y \to x^2$ to $2x$

$x \to 0$ to 2



$x=0$ $\quad$ $x=2$



$y \to 0$ to 4

$x \to \frac{y}{2}$ to $\sqrt{y}$