PARTIAL DERIVATIVES

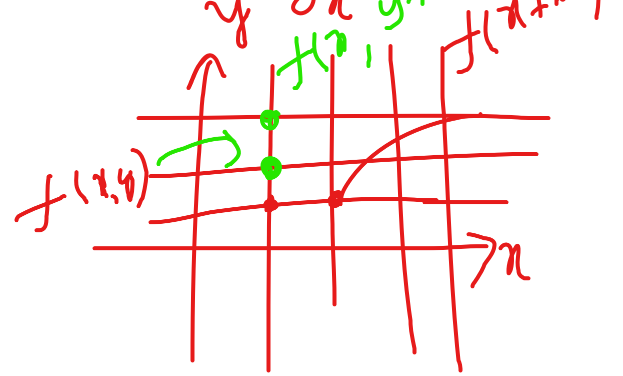
DEFINITION :

If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

$$f_x = \frac{\partial f(x,y)}{\partial x}$$



$z = f(x, y)$

NOTATIONS FOR PARTIAL DERIVATIVES If $z = f(x, y)$, we write

$\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

f_x f_y

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

RULE FOR FINDING PARTIAL DERIVATIVES OF $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Problem 1 If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

SOLUTION Holding y constant and differentiating with respect to x, we get

$f(x,y) = x^3 + x^2y^3 - 2y^2$

$f_x = 3x^2 + 2xy^3$

$$f_x(x, y) = 3x^2 + 2xy^3$$

$f_x(2,1) = 3(2)^2 + 2(2)(1)^3$

and so

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Holding x constant and differentiating with respect to y, we get

$f_y = 3x^2y^2 - 4y$

$$f_y(x, y) = 3x^2y^2 - 4y$$

$f_y(2,1) =$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

INTERPRETATIONS OF PARTIAL DERIVATIVES

$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}$ $z = f(x, y)$

In the case of the heat index function, partial derivatives can also be interpreted as *rates of change*. If $z = f(x, y)$, then $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\partial z/\partial y$ represents the rate of change of z with respect to y when x is fixed.

$\frac{\partial z}{\partial x}$

$\frac{\partial z}{\partial y}$

Problem 2) If $f(x, y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

SOLUTION Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right) \cdot \frac{x}{(1+y)^2}$$

$x^3 + y^3 + z^3 + 6xyz = 1$ $\qquad z = f(x, y)$

$3x^2 + 0 + 3z^2 \frac{\partial z}{\partial x} + 6y\left(z + x\frac{\partial z}{\partial x}\right) = 0$

Problem 3: Find $\partial z/\partial x$ and $\partial z/\partial y$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

SOLUTION To find $\partial z/\partial x$, we differentiate implicitly with respect to x, being careful to treat y as a constant:

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

$\frac{\partial z}{\partial x}(3z^2 + 6xy) = -3x^2 - 6yz = -3(x^2+2yz)$

Solving this equation for $\partial z/\partial x$, we obtain

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$\frac{\partial z}{\partial x} = -\frac{\cancel{3}(x^2+2yz)}{\cancel{3}(z^2+2xy)}$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

$0 + 3y^2 + 3z^2\frac{\partial z}{\partial y} + 6x\left(z + \frac{\partial z}{\partial y}y\right) = 0$

$3y^2 + 3z^2\frac{\partial z}{\partial y} + 6xz + 6xy\frac{\partial z}{\partial y} = 0$

$$\frac{\partial z}{\partial y}(3z^2 + 6xy) = -3y^2 - 6xz \Rightarrow \frac{\partial z}{\partial y} = -\frac{\cancel{3}(y^2 + 2xz)}{\cancel{3}(z^2 + 2xy)}$$

HIGHER DERIVATIVES

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial y\, \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial x\, \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

$f_x = 3x^2 + 2xy^3$

Problem 1: Find the second partial derivatives of

$f_{xx} = 6x + 2y^3$

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

$f_y = 3x^2y^2 - 4y$

SOLUTION we found that

$f_{yy} = 6x^2y - 4$

$$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2y^2 - 4y$$

Therefore $f_{xy} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial y}\left(\underset{=0}{3x^2} + 2xy^3\right) = 6xy^2$

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 + 2xy^3) = 6x + 2y^3 \qquad f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 - 4y) = 6xy^2 \qquad f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 - 4y) = 6x^2y - 4$$

$f_{yx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(3x^2y^2 - \underset{=0}{4y}\right) = 6xy^2$

CLAIRAUT'S THEOREM Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Problem 2: Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$.

SOLUTION

$$f_x = 3\cos(3x + yz)$$

$$f_{xx} = -9\sin(3x + yz)$$

$-9\cos(3x+yz)$
$(0+z)$

$$f_{xxy} = -9z\cos(3x + yz)$$

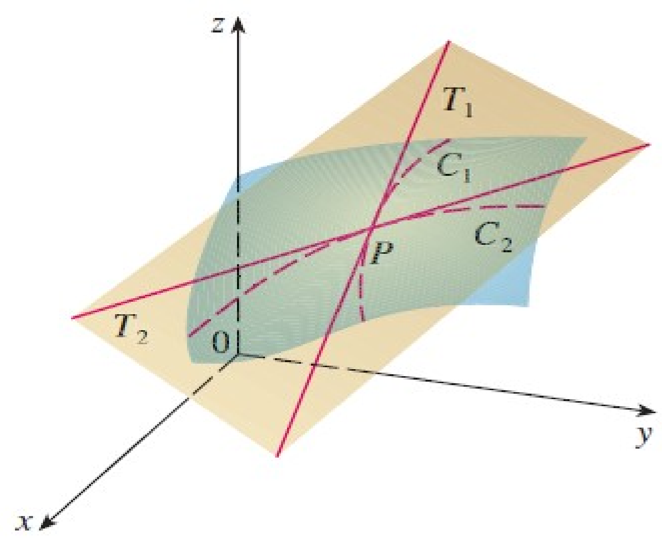
$\frac{\partial}{\partial z}(f_{xxy})$

$$f_{xxyz} = -9\cos(3x + yz) + 9yz\sin(3x + yz)$$

$-9\left[\cos(3x+yz) + z\sin(3x+yz)\cdot y\right]$

TANGENT PLANES

Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S. As in the preceding section, let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Then the point P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P. Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2. (See Figure)



Defn: Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

Problem : Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point (1, 1, 3).

SOLUTION Let $f(x, y) = 2x^2 + y^2$. Then

$f(x,y) = 2x^2 + y^2$

$f_x(x,y) = 4x$

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$

$f_x(1,1) = 4$

$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

$f_y(x,y) = 2y$

$f_y(1,1) = 2$

Therefore the equation of the tangent plane at (1, 1, 3) as

$z - 3 = 4(x-1) + 2(y-1)$

$$z - 3 = 4(x - 1) + 2(y - 1)$$

or

$$z = 4x + 2y - 3$$

$z - 3 = 4x - 4 + 2y - 2 \qquad \underline{z = 4x + 2y - 3}$

Figure 2(a) shows the elliptic paraboloid and its tangent plane at (1, 1, 3) that we found in Example 1. In parts (b) and (c) we zoom in toward the point (1, 1, 3) by restricting the domain of the function $f(x, y) = 2x^2 + y^2$. Notice that the more we zoom in, the flatter the graph appears and the more it resembles its tangent plane.

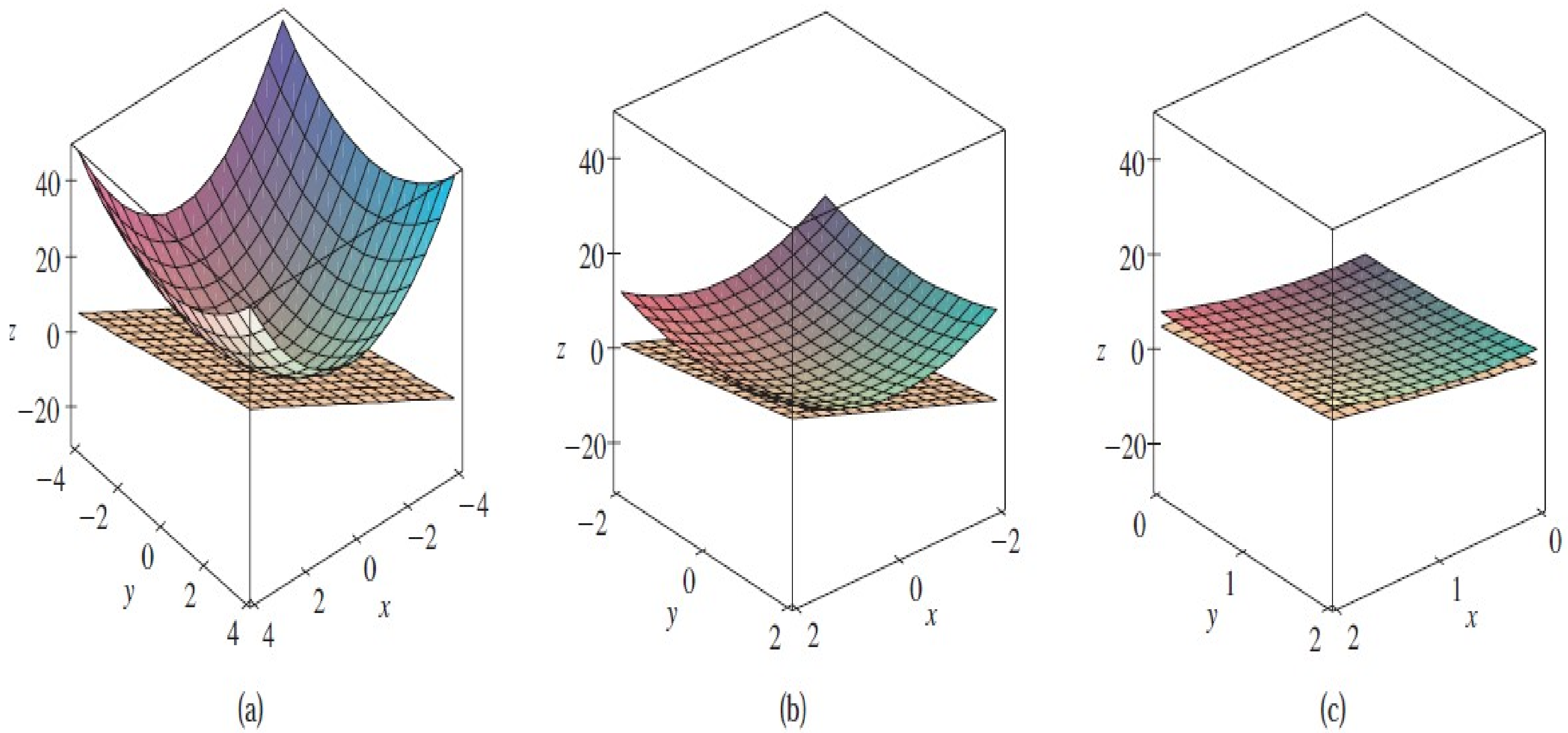
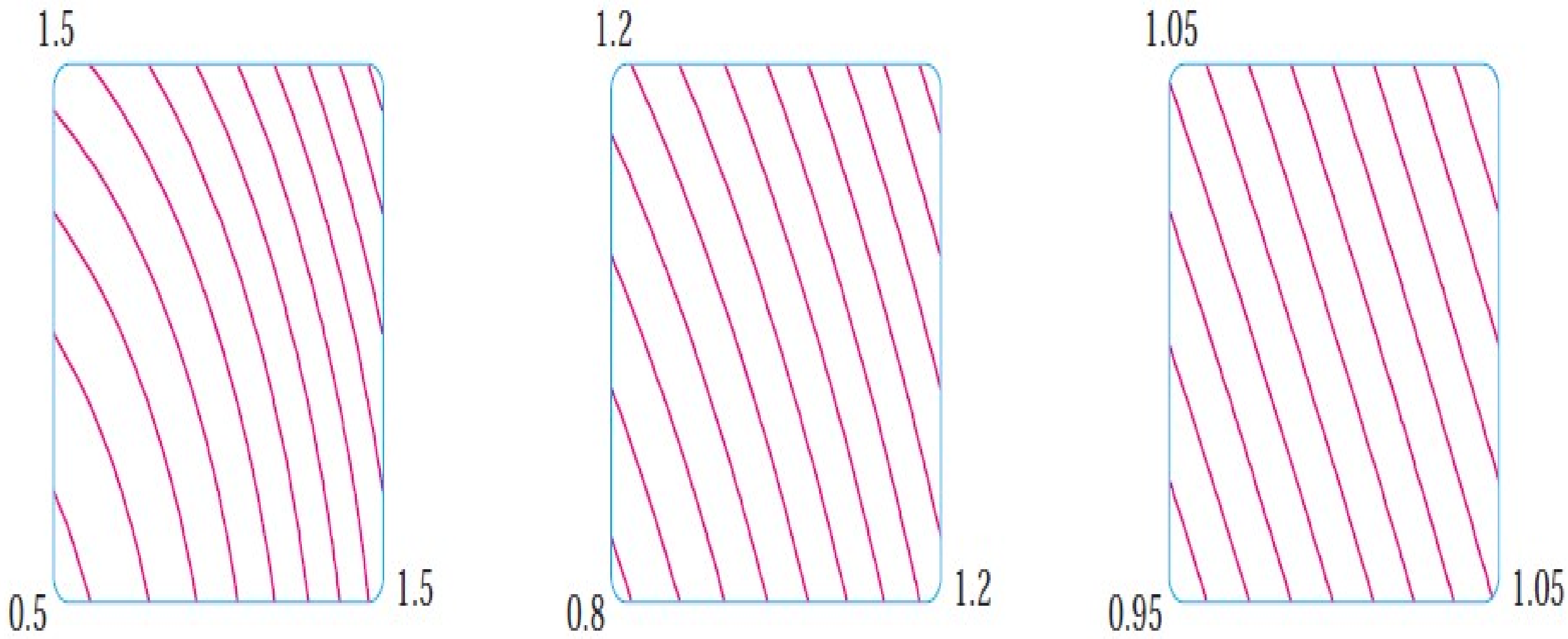


FIGURE 2 The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward (1, 1, 3).

In Figure 3 we corroborate this impression by zooming in toward the point (1, 1) on a contour map of the function $f(x, y) = 2x^2 + y^2$. Notice that the more we zoom in, the more the level curves look like equally spaced parallel lines, which is characteristic of a plane.



$f(x,y) = x\cos y - ye^x$

Find the plane tangent to the surface $z = x\cos y - ye^x$ at $(0, 0, 0)$.

Solution We calculate the partial derivatives of $f(x, y) = x\cos y - ye^x$

$f_x(x,y) = \cos y - ye^x \Rightarrow f_x(0,0) = 1 - 0\cdot 1 = 1$

$$f_x(0, 0) = (\cos y - ye^x)_{(0,0)} = 1 - 0 \cdot 1 = 1$$

$f_y(x,y) = -x\sin y - e^x \Rightarrow f_y(0,0) = 0 - 1 = -1$

$$f_y(0, 0) = (-x\sin y - e^x)_{(0,0)} = 0 - 1 = -1.$$

The tangent plane is therefore $z - 0 = 1(x-0) - 1(y-0)$

$$1 \cdot (x - 0) - 1 \cdot (y - 0) - (z - 0) = 0,$$

or $x - y = z \Rightarrow x - y - z = 0$

$$x - y - z = 0.$$

$z^2 = 14 - x^2 - y^2$ at $(1,2,3)$ $= (14 - x^2 - y^2)^{1/2}$

$z = f(x,y) = \sqrt{14 - x^2 - y^2}$ at $x_0 = 1$, $y_0 = 2$

$z_0 = 3$

Problem: Find the tangent plane to the sphere $z^2 = 14 - x^2 - y^2$ at $(1, 2, 3)$.

Solution Instead of $z^2 = 14 - x^2 - y^2$ we have $z = \sqrt{14 - x^2 - y^2}$. At $x_0 = 1$, $y_0 = 2$ the height is now $z_0 = 3$. The surface is a sphere with radius $\sqrt{14}$.

$\frac{\partial z}{\partial x} = \frac{1}{2}(14 - x^2 - y^2)^{-1/2}(0 - 2x - 0) = \frac{-x}{\sqrt{14 - x^2 - y^2}}$ | $\frac{\partial z}{\partial x}(1,2) = \frac{-1}{\sqrt{14 - 1^2 - 2^2}} = -1/3$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\sqrt{14 - x^2 - y^2} = \frac{\frac{1}{2}(-2x)}{\sqrt{14 - x^2 - y^2}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{\frac{1}{2}(-2y)}{\sqrt{14 - x^2 - y^2}}$$

$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{14 - x^2 - y^2}}$ | $\partial z/\partial y\,(1,2) = \frac{-2}{3}$

At $(1, 2)$ those slopes are $-\frac{1}{3}$ and $-\frac{2}{3}$. The equation of the tangent plane is linear: $z - 3 = -\frac{1}{3}(x - 1) - \frac{2}{3}(y - 2)$. I cannot resist improving the equation, by multiplying through by 3 and moving all terms to the left side:

$$\textit{tangent plane to sphere:} \quad 1(x - 1) + 2(y - 2) + 3(z - 3) = 0.$$

$z - 3 = -1/3(x - 1) - 2/3(y - 2) \Rightarrow (x - 1) + 2(y - 2) + 3(z - 3) = 0$

$x - 1 + 2y - 4 + 3z - 9 = 0$

$x + 2y + 3z = 14$

Normal Line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \qquad y = y_0 + f_y(P_0)t, \qquad z = z_0 + f_z(P_0)t$$

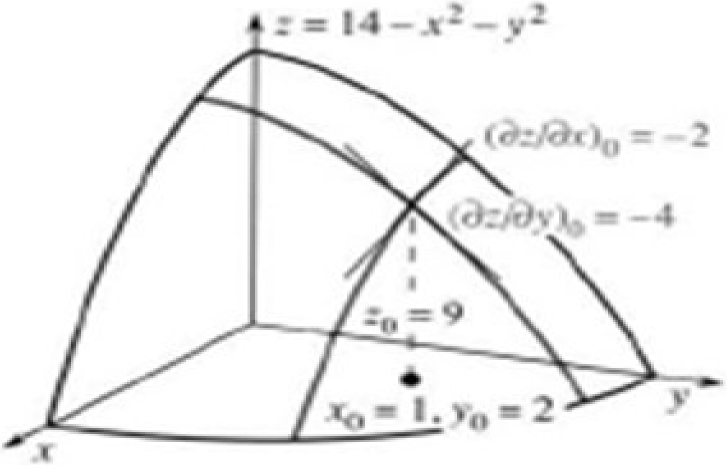
Problem : Find the tangent plane to $z = 14 - x^2 - y^2$ at $(x_0, y_0, z_0) = (1, 2, 9)$.

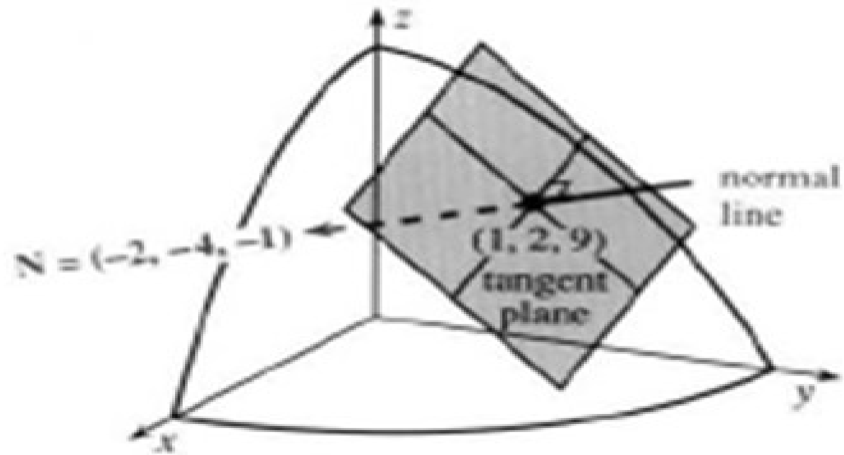
Solution The derivatives are $\partial f/\partial x = -2x$ and $\partial f/\partial y = -2y$. When $x = 1$ and $y = 2$ those are $(\partial f/\partial x)_0 = -2$ and $(\partial f/\partial y)_0 = -4$. The equation of the tangent plane is

$f_x(1,2) = -2$ $\quad$ $f_y(1,2) = -4$

$$z - 9 = -2(x - 1) - 4(y - 2) \qquad \text{or} \qquad z + 2x + 4y = 19.$$

This $z(x, y)$ has derivatives -2 and -4, just like the surface. So the plane is tangent. The normal vector $\mathbf{N}$ has components -2, -4, -1. ***The equation of the normal line is*** $(x, y, z) = (1, 2, 9) + t(-2, -4, -1)$. Starting from $(1, 2, 9)$ the line goes out along $\mathbf{N}$—perpendicular to the plane and the surface.





LINEAR APPROXIMATIONS

$z = f(x, y)$

We found that an equation of the tangent plane to the graph of the function $f(x, y) = 2x^2 + y^2$ at the point $(1, 1, 3)$ is $z = 4x + 2y - 3$. Therefore, in view of the visual evidence in Figures 2 and 3, the linear function of two variables

$f(x, y) = 2x^2 + y^2$

$z = 4x + 2y - 3$

$$L(x, y) = 4x + 2y - 3$$

is a good approximation to $f(x, y)$ when (x, y) is near $(1, 1)$. The function L is called the *linearization* of f at $(1, 1)$ and the approximation

$$f(x, y) \approx 4x + 2y - 3$$

is called the *linear approximation* or *tangent plane approximation* of f at $(1, 1)$.

THEOREM If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

$z = f(x,y) = xe^{xy}$

Problem : Show that $f(x, y) = xe^{xy}$ is differentiable at (1, 0) and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

$L(x,y)$

$=$

SOLUTION The partial derivatives are

$$f_x(x, y) = e^{xy} + xye^{xy} \qquad f_y(x, y) = x^2e^{xy}$$

$$f_x(1, 0) = 1 \qquad f_y(1, 0) = 1$$

$z - z_0 = f_x(x_0, y_0)(x - x_0)$

Both f_x and f_y are continuous functions, so f is differentiable by Theorem . The linearization is

$$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$$

$+ f_y(x_0, y_0)(y - y_0)$

$$= 1 + 1(x - 1) + 1 \cdot y = x + y$$

$xe^{xy} \simeq x + y$

The corresponding linear approximation is

$z - 1 = 1(x - 1) + 1(y - 0)$

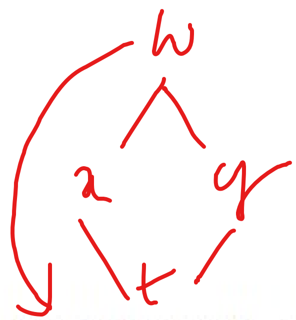
$$xe^{xy} \approx x + y$$

so $$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$$

$z = 1 + (x - 1) + y$

Compare this with the actual value of $f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542.$

$z = \cancel{1} + x \cancel{-1} + y$



Total derivative :

If $w = f(x, y)$ has continuous partial derivatives f_x and f_y and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

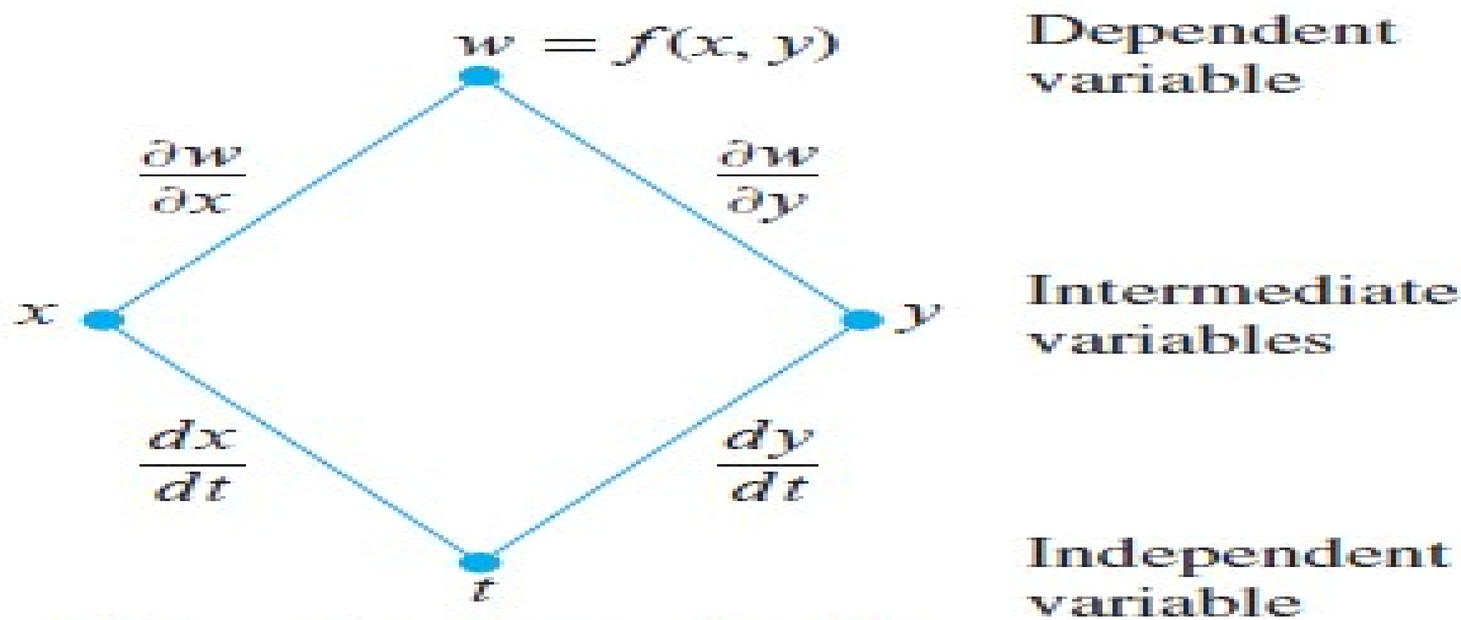
$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Thus we can find the ordinary derivative $\frac{dw}{dt}$ which is called the total derivative of w to distinguish it from the partial derivatives.

To remember the Chain Rule picture the diagram below. To find dw/dt, start at w and read down each route to t, multiplying derivatives along the way. Then add the products.

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Problem :

Use the Chain Rule to find the derivative of

$$w = xy$$

with respect to t along the path $x = \cos t, y = \sin t$. What is the derivative's value at $t = \pi/2$?

$w = xy$

$x = \cos t, \; y = \sin t$

Solution We apply the Chain Rule to find dw/dt as follows:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial(xy)}{\partial x} \cdot \frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y} \cdot \frac{d}{dt}(\sin t) \\
&= (y)(-\sin t) + (x)(\cos t) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) \\
&= -\sin^2 t + \cos^2 t \\
&= \cos 2t.
\end{aligned}$$

$\cos n\pi = (-1)^n$

$= -1$

$$\left(\frac{dw}{dt}\right)_{t=\pi/2} = \cos\left(2 \cdot \frac{\pi}{2}\right) = \cos \pi = -1 .$$

THEOREM **Chain Rule for Functions of Three Independent Variables**

If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t and

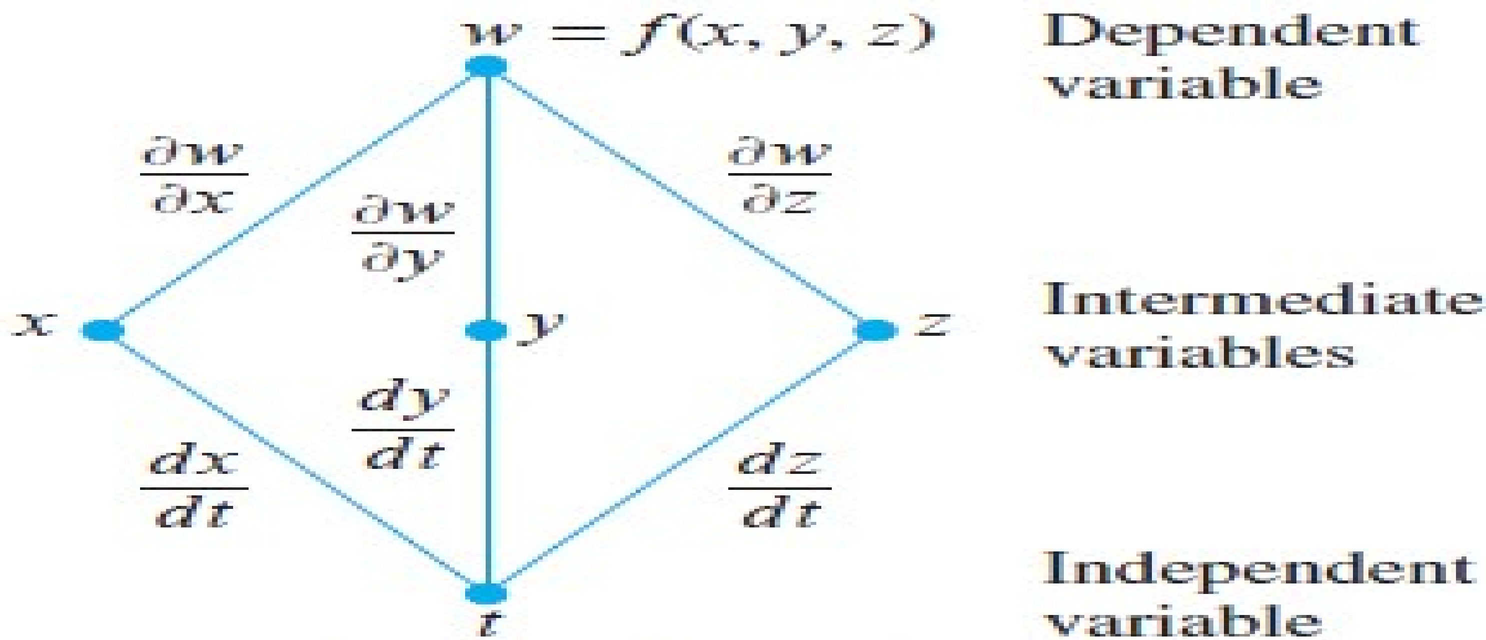
$w = f(x, y, z)$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Here we have three routes from w to t instead of two, but finding dw/dt is still the same. Read down each route, multiplying derivatives along the way; then add.

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$w = f(x, y, z)$

Find dw/dt if

$$w = xy + z, \qquad x = \cos t, \qquad y = \sin t, \qquad z = t.$$

What is the derivative's value at t = 0?

Solution

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\
&= (y)(-\sin t) + (x)(\cos t) + (1)(1) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 \\
&= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t.
\end{aligned}$$

$$\left(\frac{dw}{dt}\right)_{t=0} = 1 + \cos(0) = 2.$$

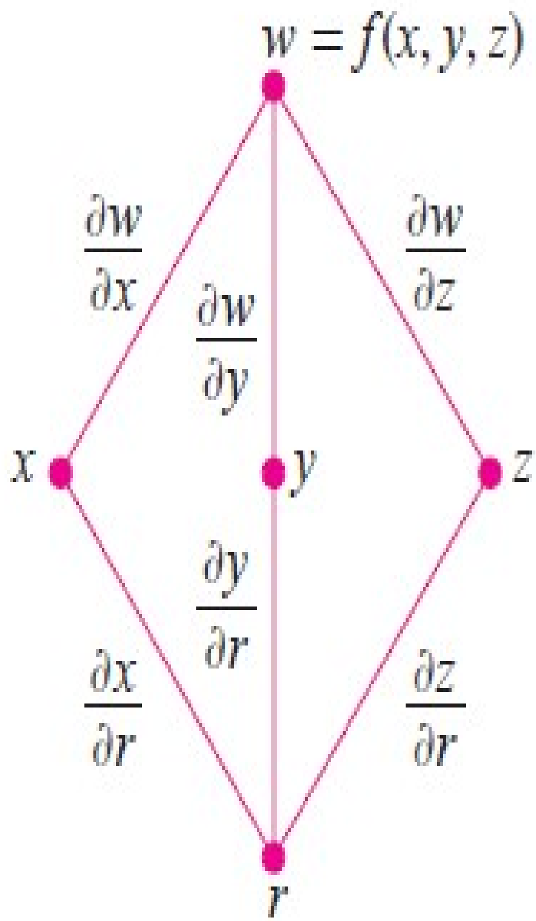
THEOREM **Chain Rule for Two Independent Variables and Three Intermediate Variables**

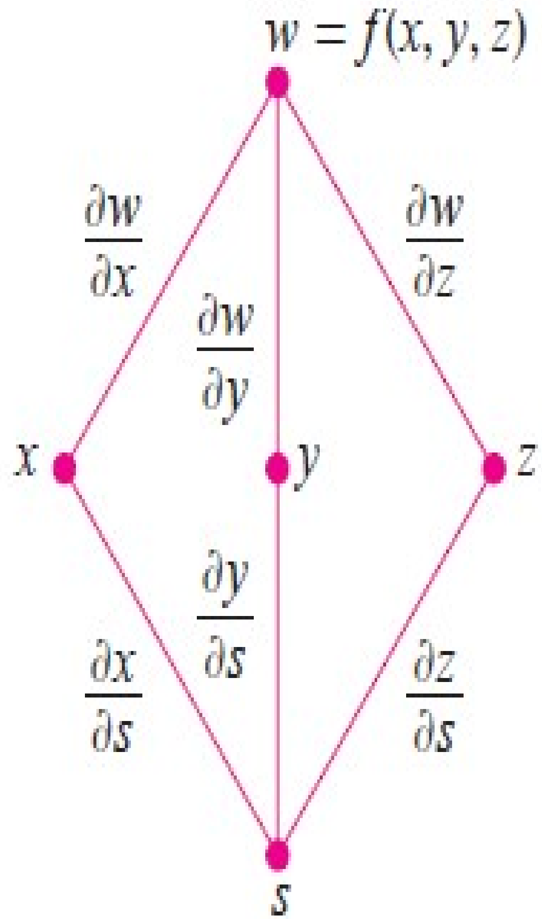
Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$\frac{\partial w}{\partial r}$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \qquad \frac{\partial w}{\partial s}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$





$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x + 2y + z^2, \qquad x = \frac{r}{s}, \qquad y = r^2 + \ln s, \qquad z = 2r.$$

Solution

$w = f(x, y, z) \quad x = f(r, s) \quad y = f(r, s), \quad z = f(r, s)$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2)$$

$$= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r$$

Substitute for intermediate variable z.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}$$

Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x^2 + y^2, \quad x = r - s, \quad y = r + s.$$

$$w = f(x, y), \quad x = f(r, s), \quad y = f(r, s)$$

Solution

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$
$$= (2x)(1) + (2y)(1)$$
$$= 2(r \cancel{- s}) + 2(r \cancel{+ s})$$
$$= 4r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$
$$= (2x)(-1) + (2y)(1)$$
$$= -2(r - s) + 2(r + s)$$
$$= 4s$$

Substitute for the intermediate variables.

IMPLICIT DIFFERENTIATION

The Chain Rule can be used to give a more complete description of the process of implicit differentiation that was introduced in Sections earlier . We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f. If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x. Since both x and y are functions of x, we obtain

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

But $dx/dx = 1$, so if $\partial F/\partial y \neq 0$ we solve for dy/dx and obtain

$$\boxed{\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}}$$

$F(x, y) = 0$

$x^2 + y^2 + 6xy + 7 = 0$

$2x + 2y\underbrace{\frac{dy}{dx}}_{} + 6\left(x\underbrace{\frac{dy}{dx}}_{} + y(1)\right) = 0$

$\frac{dy}{dx}(2y + 6x) = -2x - 6y$

$\frac{dy}{dx} = \frac{-2(x + 3y)}{2(y + 3x)}$

$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$

$\frac{dy}{dx} = \frac{-2(x+3y)}{2(y+3x)}$ $F_x = 2x + 6y$

$F_y = 2y + 6x$

Problem : Find y' if $x^3 + y^3 = 6xy$.

SOLUTION The given equation can be written as

$$\frac{dy}{dx} = \frac{-F_x}{F_y}$$

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

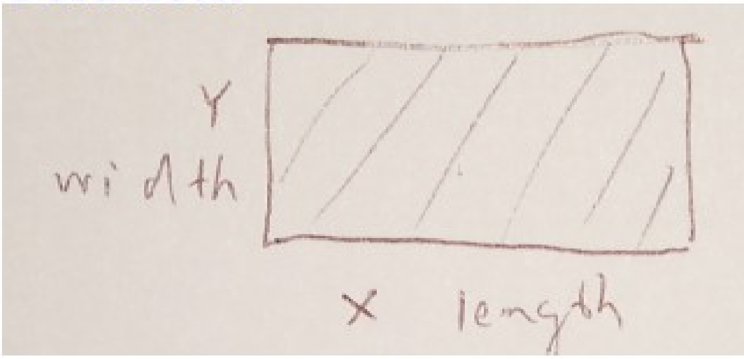
$$F_x = 3x^2 - 6y$$

$$F_y = 3y^2 - 6x$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

Example: A rectangle is changing in such a manner that its length is increasing 5 ft/sec and its width is decreasing 2 ft/sec. At what rate is the area changing at the instant when the length equals 10 feet and the width equals 8 feet?

Solution:



A area of triangle

Given: $\frac{dx}{dt} = 5$ ft/sec $\frac{dy}{dt} = -2$ ft/sec

Find: $\frac{dA}{dt}$ when $x = 100$ ft.

$A = xy$ and so $\frac{dA}{dt} = \frac{d}{dt}(xy) = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}$

$$\frac{dA}{dt} = 5 \cdot 8 + 10 \cdot (-2) = 40 - 20 = 20.$$

The area is increasing by a rate of 20 ft^2/sec.

$A = xy$

$x = f(t)$, $y = f(t)$

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt}$$

$= y\,5 + x(-2)$

$= 8 \cdot 5 + 10(-2)$

$= 40 - 20$

$= 20$ ft^2/sec

1) If x increases at the rate of 2 cm / sec at the instant when $x = 3\ cm, and\ y = 1\ cm$, at what rate must y be changing in order that the function $2xy - 3x^2y$ shall be neither increasing nor decreasing?

$u = f(x, y)$

Sol:-

$x = f(t),\ y = f(t)$

Let $u = 2xy - 3x^2y,$ so that

$u_x = 2y - 6xy$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} = (2y - 6xy)\frac{dx}{dt} + (2x - 3x^2)\frac{dy}{dt} \qquad (1)$$

$u_y = 2x - 3x^2$

Given u is neither increasing nor decreasing i.e., $\frac{du}{dt} = 0$ and

$\frac{du}{dt} = 0$

When $x = 3\ and\ y = 1, \frac{dx}{dt} = 2.$

Therefore (1) becomes

$$0 = \big(2 - (6)(3)\big)(2) + ((2)(3) - (3)(9))\frac{dy}{dt}$$

$$\frac{dy}{dt} = -\frac{32}{21}\ cm/sec$$

Thus y is decreasing at the rate of $\frac{32}{21}\frac{cm}{sec}$.

3. Your company manufactures right circular cylindrical molasses storage tanks that are 25 ft high with a radius of 5 ft. How sensitive are the tanks' volumes to small variations in height and radius?

Sol:-

Volume $V = \pi r^2 h$ $\quad V = f(r, h)$

$$dv = \frac{\partial v}{\partial r} dr + \frac{\partial v}{\partial h} dh \qquad \frac{\partial v}{\partial r} = 2\pi r h$$

$$\frac{\partial v}{\partial h} = \pi r^2$$

$$= (2\pi r h)\, dr + (\pi r^2)\, dh$$

Given $r = 5$ ft, $h = 25$ ft

$$dv = 250\pi\, dr + 25\pi\, dh$$

1 unit change in 'r' will change V by about 250π units

1 unit change in 'h' will change V by about 25π units

The tank's volume is 10 times more sensitive to a small change in 'r' than it is to a small change of equal size in 'h'.