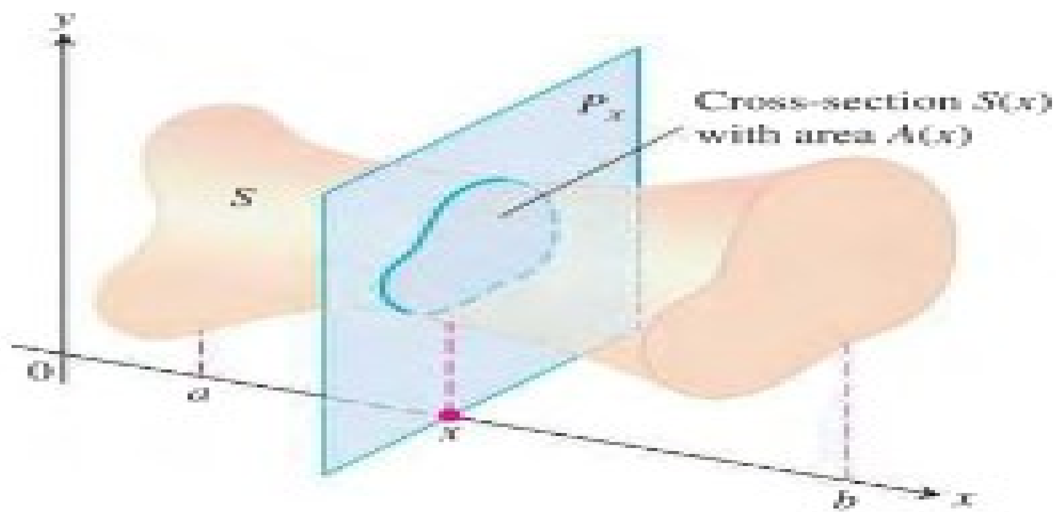
Applications of Definite Integrals

Volumes Using Cross-Sections

DEFINITION The **volume** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\, dx.$$



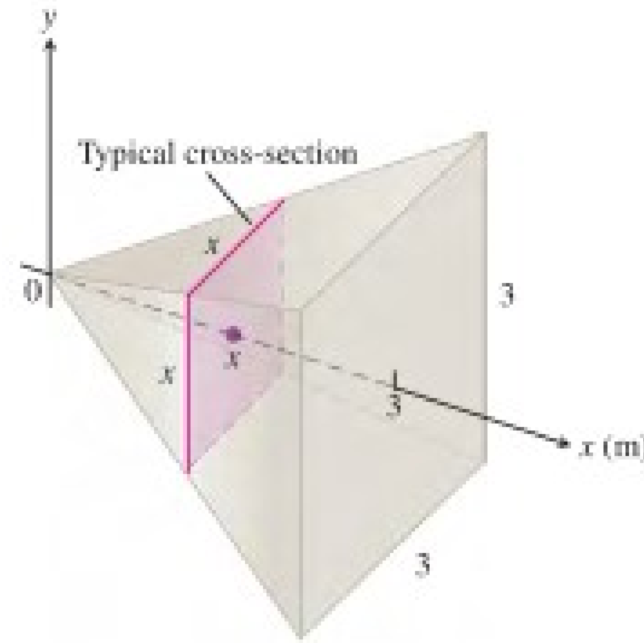
Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for $A(x)$*, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate $A(x)$* to find the volume.

Examples

EXAMPLE 1 A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.

$$V = \int_0^3 A(x)\,dx = \int_0^3 x^2\,dx = \frac{x^3}{3}\bigg]_0^3 = 9\text{ m}^3.$$



Volume by Disks for Rotation About the x-axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.$$

This method for calculating the volume of a solid of revolution is often called the **disk method** because a cross-section is a circular disk of radius $R(x)$.

1) The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

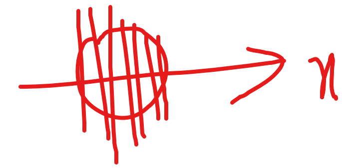
Solution We draw figures showing the region, a typical radius, and the generated solid . The volume is

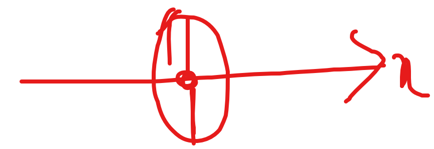
$$\begin{aligned} V &= \int_a^b \pi[R(x)]^2\,dx \\ &= \int_0^4 \pi\left[\sqrt{x}\right]^2 dx \\ &= \pi\int_0^4 x\,dx = \pi\frac{x^2}{2}\bigg]_0^4 = \pi\frac{(4)^2}{2} = 8\pi. \end{aligned}$$

2)

The circle

$$x^2 + y^2 = a^2$$





is rotated about the x-axis to generate a sphere. Find its volume.

Solution We imagine the sphere cut into thin slices by planes perpendicular to the x-axis. The cross-sectional area at a typical point x between $-a$ and a is

$$A(x) = \pi y^2 = \pi(a^2 - x^2).$$

Therefore, the volume is

$$V = \int_{-a}^{a} A(x)\,dx = \int_{-a}^{a} \pi(a^2 - x^2)\,dx = \pi\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3.$$

Volume by Disks for Rotation About the y-axis

$$V = \int_c^d A(y)\,dy = \int_c^d \pi[R(y)]^2\,dy.$$

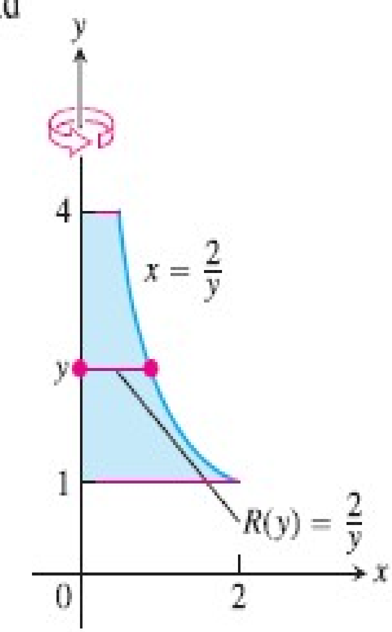
3) Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.

Solution We draw figures showing the region, a typical radius, and the generated solid . The volume is

$$V = \int_1^4 \pi[R(y)]^2\,dy$$

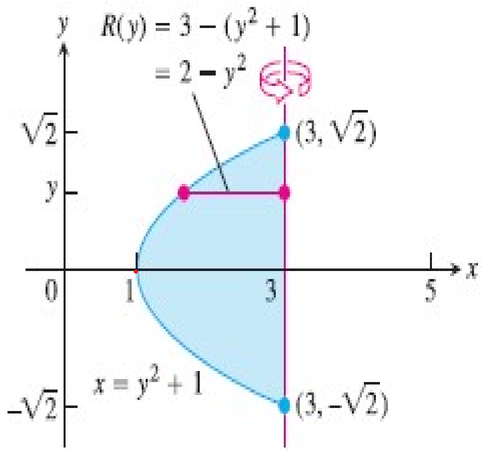
$$= \int_1^4 \pi\left(\frac{2}{y}\right)^2 dy$$

$$= \pi\int_1^4 \frac{4}{y^2}\,dy = 4\pi\left[-\frac{1}{y}\right]_1^4 = 4\pi\left[\frac{3}{4}\right] = 3\pi.$$



Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

Solution We draw figures showing the region, a typical radius, and the generated solid Note that the cross-sections are perpendicular to the line $x = 3$ and have y-coordinates from $y = -\sqrt{2}$ to $y = \sqrt{2}$. The volume is



$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi[R(y)]^2\, dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \pi[2 - y^2]^2\, dy$$

$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4]\, dy$$

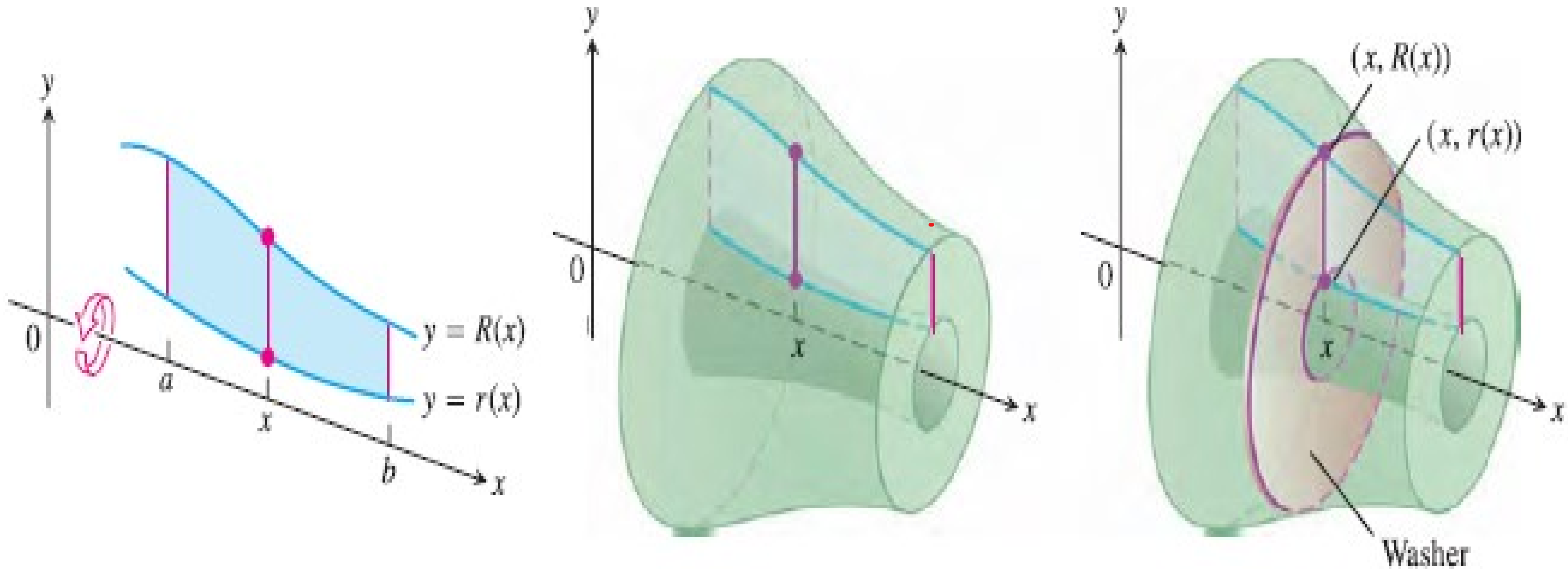
$$= \pi\left[4y - \frac{4}{3}y^3 + \frac{y^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \frac{64\pi\sqrt{2}}{15}.$$

$R(y) = 3 - (y^2+1)$

$= 3 - y^2 - 1$

$= 2 - y^2$



The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)\,dx$ leads to a slightly different formula.

Volume by Washers for Rotation About the *x*-axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx.$$

4) The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Outer radius: $R(x) = -x + 3$

Inner radius: $r(x) = x^2 + 1$

the limits of integration by finding the x-coordinates of the intersection points of the curve and line

$$x^2 + 1 = -x + 3$$

$$x^2 + x - 2 = 0$$

$$(x + 2)(x - 1) = 0$$

$$x = -2, \quad x = 1$$

Evaluate the volume integral.

$$V = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx$$

$$= \int_{-2}^{1} \pi((-x + 3)^2 - (x^2 + 1)^2)\,dx$$

$$= \pi \int_{-2}^{1} (8 - 6x - x^2 - x^4)\,dx$$

$$= \pi \left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5} \right]_{-2}^{1} = \frac{117\pi}{5}$$

5)

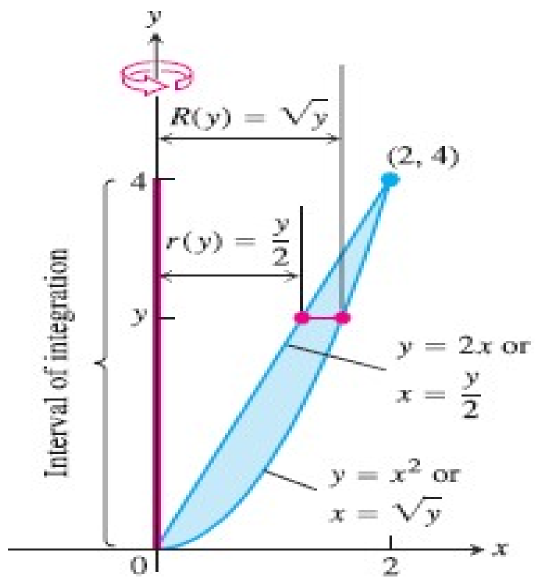
The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

$x = \sqrt{y}$, $x = y/2$

Solution

The radii of the washer swept out by the line segment are $R(y) = \sqrt{y}$, $r(y) = y/2$

The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$. We integrate to find the volume:

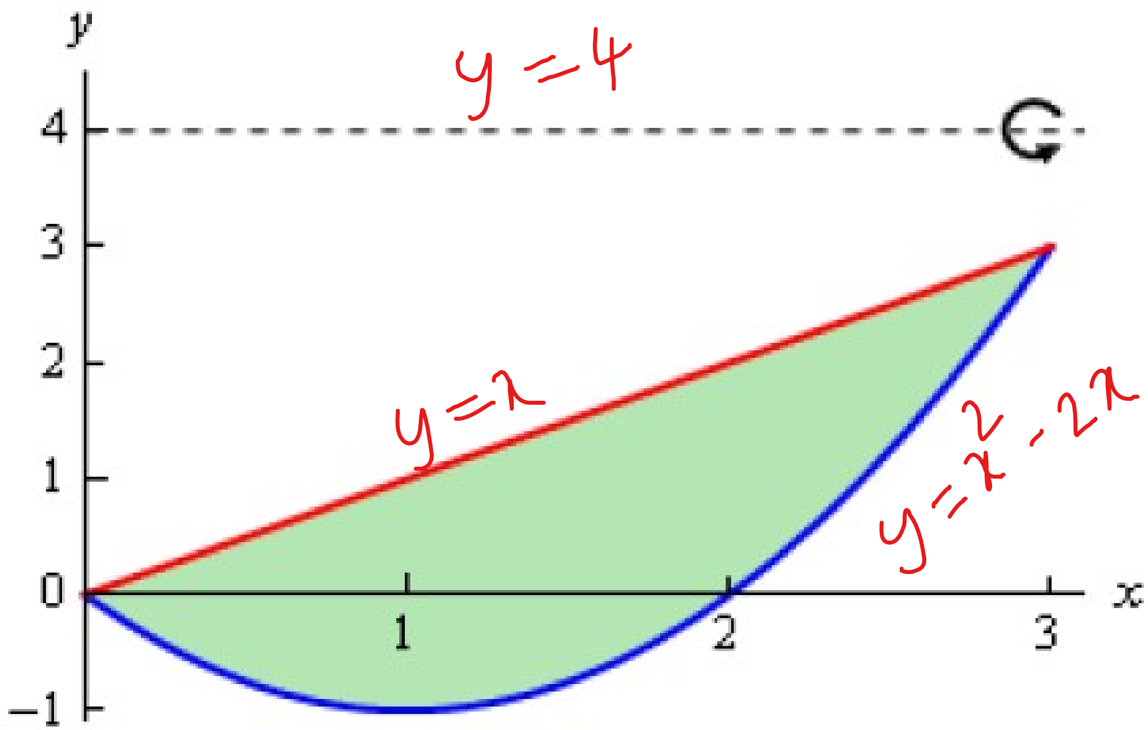


$$V = \int_c^d \pi([R(y)]^2 - [r(y)]^2)\,dy$$

$$= \int_0^4 \pi\left(\left[\sqrt{y}\right]^2 - \left[\frac{y}{2}\right]^2\right)dy$$

$$= \pi \int_0^4 \left(y - \frac{y^2}{4}\right)dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi.$$

Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.



The inner radius must then be the difference between these two

$$\text{inner radius} = 4 - x$$

The outer radius works the same way. The outer radius is,

$$\text{outer radius} = 4 - \left(x^2 - 2x\right) = -x^2 + 2x + 4$$

The cross-sectional area for this case is,

$$A(x) = \pi\left(\left(-x^2 + 2x + 4\right)^2 - \left(4 - x\right)^2\right) = \pi\left(x^4 - 4x^3 - 5x^2 + 24x\right)$$

The first ring will occur at $x = 0$ and the last ring will occur at $x = 3$ and so these are our limits of integration. The volume is then,

$$
\begin{aligned}
V &= \int_a^b A(x)\,dx \\
&= \pi \int_0^3 x^4 - 4x^3 - 5x^2 + 24x\,dx \\
&= \pi \left(\frac{1}{5}x^5 - x^4 - \frac{5}{3}x^3 + 12x^2 \right) \Bigg|_0^3 \\
&= \frac{153\pi}{5}
\end{aligned}
$$