Laplace Transforms

In mathematics, a transform is usually a device that converts one type of problem into another type, presumably easier to solve. The strategy is to solve the transformed problem, then transform back the other way to obtain the solution of the original problem. In the case of the Laplace transform, initial value problems are often converted to algebra problems, a process we can diagram as follows:

initial value problem

$\Downarrow$

algebra problem

$\Downarrow$

solution of the algebra problem

$\Downarrow$

solution of the initial value problem.

If $f(t)$ is a function defined for all $t \geqq 0$, its **Laplace transform**[1] is the integral of $f(t)$ times e^{-st} from $t = 0$ to ∞. It is a function of s, say, $F(s)$, and is denoted by $\mathcal{L}(f)$; thus

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

(1) $$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt.$$

Here we must assume that $f(t)$ is such that the integral exists

Not only is the result $F(s)$ called the Laplace transform, but the operation just described, which yields $F(s)$ from a given $f(t)$, is also called the **Laplace transform.** It is an **"integral transform"**

$$F(s) = \int_0^{\infty} k(s, t) f(t)\, dt$$

with **"kernel"** $k(s, t) = e^{-st}$.

Furthermore, the given function $f(t)$ in (1) is called the **inverse transform** of $F(s)$ and is denoted by $\mathcal{L}^{-1}(F)$; that is, we shall write

$$L^{-1}[F(s)] = f(t)$$

(1*) $$f(t) = \mathcal{L}^{-1}(F). \qquad L^{-1}[L[f(t)]] = f(t)$$

Note that (1) and (1*) together imply $\mathcal{L}^{-1}(\mathcal{L}(f)) = f$ and $\mathcal{L}(\mathcal{L}^{-1}(F)) = F$.

Existence of the Transform. The Laplace integral $\int_0^\infty e^{-st} f(t)\,dt$ is known to exist in the sense of the improper integral definition[1]

$$\int_0^\infty g(t)dt = \lim_{N\to\infty} \int_0^N g(t)dt$$

provided $f(t)$ belongs to a class of functions known in the literature as functions of **exponential order**. For this class of functions the relation

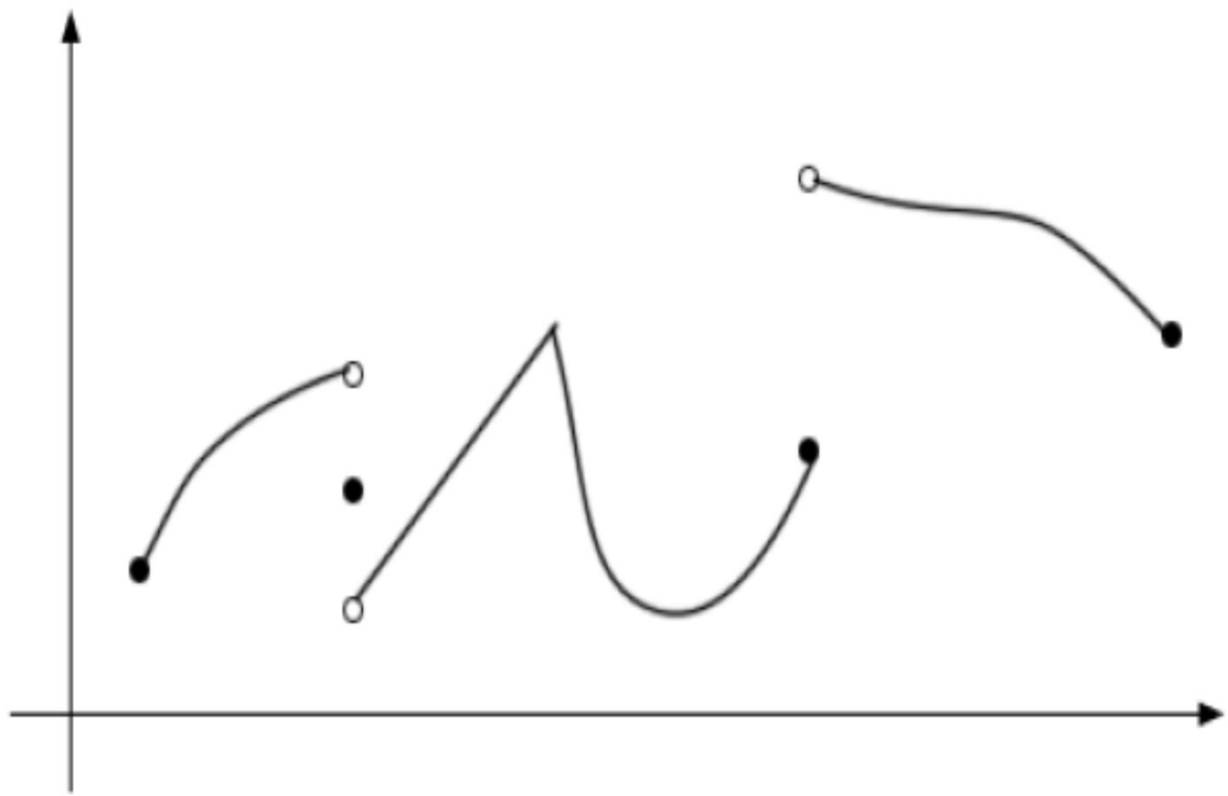
$$\lim_{t\to\infty} \frac{f(t)}{e^{at}} = 0 \tag{2}$$

is required to hold for some real number a, or equivalently, for some constants M and α,

$$|f(t)| \le Me^{\alpha t}. \tag{3}$$

In addition, $f(t)$ is required to be **piecewise continuous** on each finite subinterval of $0 \le t < \infty$, a term defined as follows.

A function is called **piecewise continuous** on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval (*i.e.* the subinterval without it's endpoints) and has a finite limit at the endpoints of each subinterval. Below is a sketch of a piecewise continuous function.



In other words, a piecewise continuous function is a function that has a finite number of breaks in it and doesn't blow up to infinity anywhere.

Laplace Transform

Let $f(t) = 1$ when $t \geqq 0$. Find $F(s)$.

Solution. From (1) we obtain by integration

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s} \qquad (s > 0).$$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[1] = \int_0^\infty e^{-st}(1)\,dt$$

$$= \int_0^\infty e^{-st}\,dt$$

$$= \left(\frac{e^{-st}}{-s}\right)_0^\infty$$

$$= \frac{-1}{s}\left(e^{-st}\right)_0^\infty$$

$$= -\frac{1}{s}(0 - 1) = 1/s, \quad s > 0$$

Laplace Transform $\mathcal{L}(e^{at})$ of the Exponential Function e^{at}

$f(t) = e^{at}$

$L[f(t)] = \int_0^\infty e^{-st} e^{at} \, dt$

Let $f(t) = e^{at}$ when $t \geqq 0$, where a is a constant. Find $\mathcal{L}(f)$.

Solution. Again by (1),

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at} \, dt = \frac{1}{a-s} e^{-(s-a)t} \Big|_0^\infty ;$$

$= \int_0^\infty e^{-(s-a)t} \, dt$

hence, when $s - a > 0$,

$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a} .$$

$= \frac{-1}{(s-a)} \left[0 - 1 \right]$

$= \frac{1}{s-a}$

lly $L[e^{-at}] = \frac{1}{s+a}$

$$1)\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)$$

$$2)\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)$$

$$3)\,e^{\infty} = \infty \ \ \&\ \ e^{-\infty} = 0.$$

1) Prove that $L(\sin at) = \dfrac{a}{s^2 + a^2}$ if $s \rangle 0$.

Proof :

$$Given \quad f(t) = \sin at$$

$$W.K.T \quad L\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\, dt$$

$$\therefore \; L(\sin at) = \int_0^\infty e^{-st} \sin at \, dt$$

$$= \left[\frac{e^{-st}}{s^2 + a^2}\left(-s \sin at - a \cos at\right)\right]_0^\infty$$

$$= \left[(0) - \left(\frac{1}{s^2 + a^2}(-a)\right)\right] = \frac{a}{s^2 + a^2} \text{ if } s \rangle 0.$$

1) Prove that $L(\cos at) = \dfrac{s}{s^2 + a^2}$ if $s \rangle 0$.

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Another method

$$L\left(e^{iat}\right)=\frac{1}{s-ia}=\frac{s+ia}{s^2+a^2} \qquad (1)$$

But $e^{iat}=\cos at+i\sin at$

Hence $L\left(e^{iat}\right)=\int_0^\infty e^{-st}\left(\cos at+i\sin at\right)dt$

$$=\int_0^\infty e^{-st}\cos at\,dt+i\int_0^\infty e^{-st}\sin at\,dt=L\left(\cos at\right)+iL\left(\sin at\right) \qquad (2)$$

From (1) & (2) *we have on equating real and imaginary parts,*

$$L\left(\cos at\right)=\frac{s}{s^2+a^2},\quad L\left(\sin at\right)=\frac{a}{s^2+a^2}$$

$$Show\ that\ (i)\ L(\cosh at) = \frac{s}{s^2 - a^2},\ if\ \ s \rangle |a|.$$

$$(ii)\ L(\sinh at) = \frac{a}{s^2 - a^2},\ if\ \ s \rangle |a|.$$

$$(iii)\ L[F(t)] = \frac{5(1 - e^{-3s})}{s},\ if\ \ F(t) = \begin{cases} 5 & 0 \langle t \langle 3 \\ 0 & t \rangle 3 \end{cases}$$

$$H\,\text{int}: \cosh at = \frac{e^{at} + e^{-at}}{2}\ \ \&\ \ \sinh at = \frac{e^{at} - e^{-at}}{2}.$$

Sol:-

$$\mathcal{L}(\cosh at) = \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\mathcal{L}(e^{at}) + \frac{1}{2}\mathcal{L}(e^{-at})$$

$$= \frac{1}{2}\left\{\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\}$$

$$= \frac{1}{2}\left\{\frac{s+\cancel{a}+s-\cancel{a}}{s^2-a^2}\right\} = \frac{s}{s^2-a^2}$$

for $s > |a|$

2) $$\mathcal{L}(\sinh at) = \mathcal{L}\left(\frac{e^{at} - e^{-at}}{2}\right)$$

$$= \frac{1}{2}\mathcal{L}(e^{at}) - \frac{1}{2}\mathcal{L}(e^{-at})$$

$$= \frac{1}{2}\left\{\mathcal{L}(e^{at}) - \mathcal{L}(e^{-at})\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\}$$

$$= \frac{1}{2}\left\{\frac{\cancel{s} + a - \cancel{s} + a}{s^2 - a^2}\right\} = \frac{a}{s^2 - a^2}$$

for $s > |a|$

3) WKT

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\{F(t)\} = \int_0^3 e^{-st}(5)\,dt + \underbrace{\int_3^\infty e^{-st}(0)\,dt}_{=0}$$

$$= 5\int_0^3 e^{-st}\,dt = 5\left(\frac{e^{-st}}{-s}\right)_0^3$$

$$= \frac{5}{-s}\left[e^{-3s} - 1\right] = \frac{5(1-e^{-3s})}{s}$$