VECTOR CALCULUS

- Scalar and vector valued functions.
- Gradient–physical interpretation.
- Total derivative–directional derivative.
- Divergence and Curl –Physical interpretations-Statement of vector identities.
- Scalar and Vector potentials.
- Line, Surface and Volume integrals.
- Statement of Green's , Stoke's and Gauss divergence theorems.
- Verification and evaluation of vector integrals using them.

6

7

Vector Functions

When a particle moves through space during a time interval I, we think of the particle's coordinates as functions defined on I:

$$x = f(t), \qquad y = g(t), \qquad z = h(t), \qquad t \in I. \tag{1}$$

The points $(x, y, z) = (f(t), g(t), h(t))$, $t \in I$, make up the **curve** in space that we call the particle's **path**. The equations and interval in Equation (1) **parametrize** the curve. A curve in space can also be represented in vector form. The vector

$$\mathbf{r}(t) = \overrightarrow{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \tag{2}$$

from the origin to the particle's **position** $P(f(t), g(t), h(t))$ at time t is the particle's **position vector** (Figure 1). The functions f, g, and h are the **component functions (components)** of the position vector. We think of the particle's path as the **curve traced by r** during the time interval I.

Equation (2) defines $\mathbf{r}$ as a vector function of the real variable t on the interval I. More generally, a **vector function** or **vector-valued function** on a domain set D is a rule that assigns a vector in space to each element in D. For now, the domains will be intervals of real numbers resulting in a space curve.

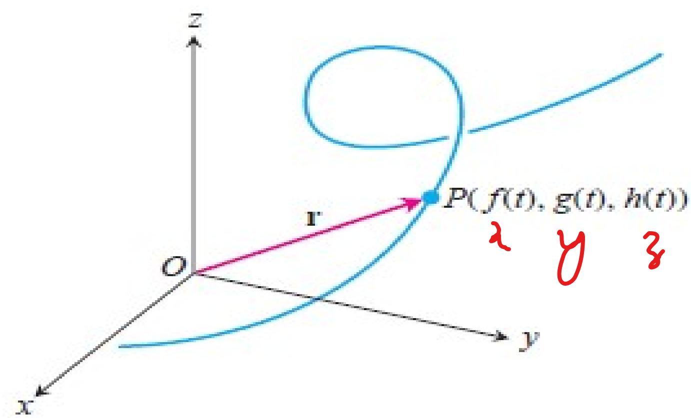


FIGURE 1 The position vector $\mathbf{r} = \overrightarrow{OP}$ of a particle moving through space is a function of time.

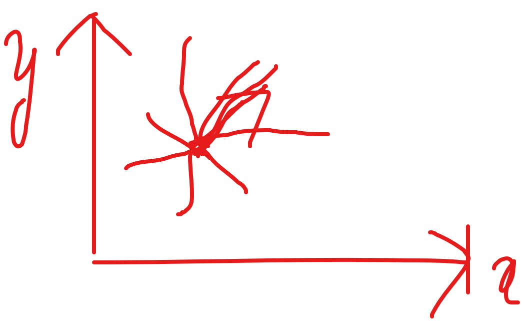
NOTE :- Vector functions will then represent surfaces in space. Vector functions on a domain in the plane or space also give rise to "vector fields," which are important to the study of the flow of a fluid, gravitational fields, and electromagnetic phenomena.

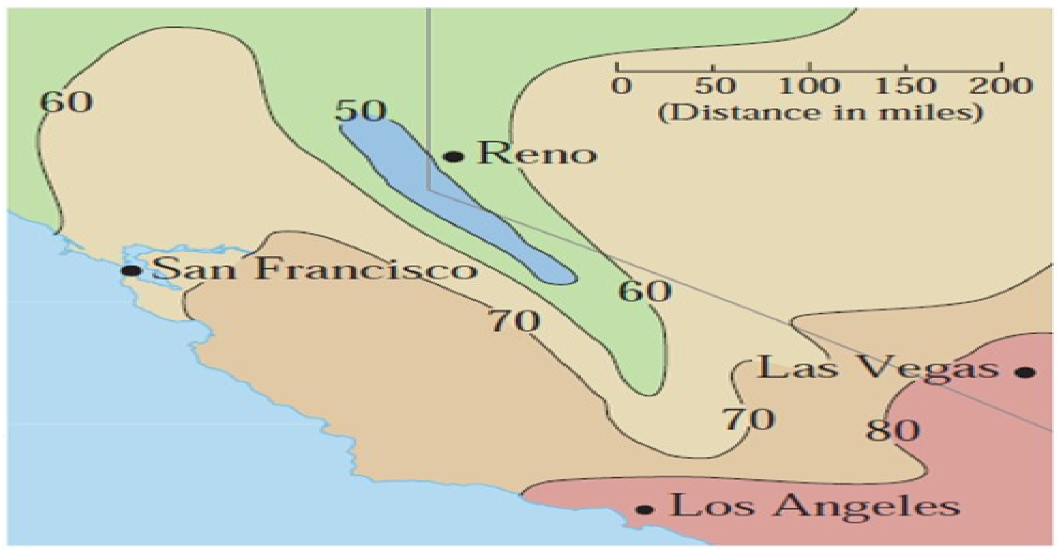
NOTE :-

We refer to real-valued functions as **scalar functions** to distinguish them from vector functions. The components of $\mathbf{r}$ are scalar functions of t. When we define a vector-valued function by giving its component functions, we assume the vector function's domain to be the common domain of the components.

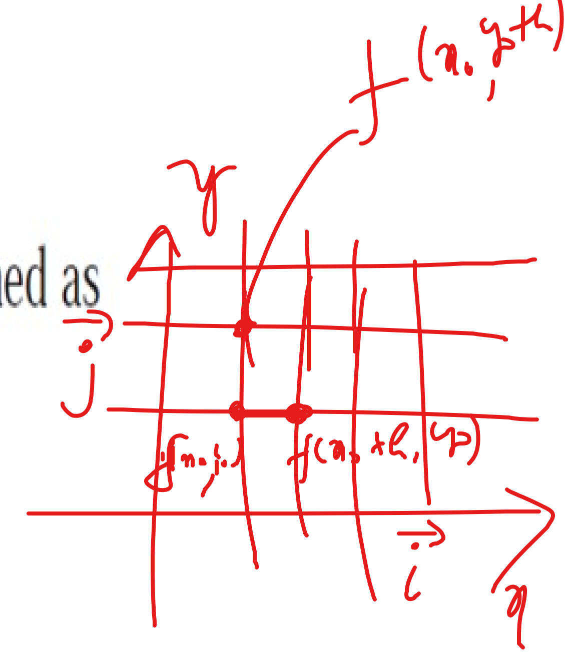
DIRECTIONAL DERIVATIVES AND THE GRADIENT VECTOR

The weather map in Figure shows a contour map of the temperature function $T(x, y)$ for the states of California and Nevada at 3:00 PM on a day in October. The level curves, or isothermals, join locations with the same temperature. The partial derivative T_x at a location such as Reno is the rate of change of temperature with respect to distance if we travel east from Reno; T_y is the rate of change of temperature if we travel north. But what if we want to know the rate of change of temperature when we travel southeast (toward Las Vegas), or in some other direction? In this section we introduce a type of derivative, called a *directional derivative*, that enables us to find the rate of change of a function of two or more variables in any direction.





DIRECTIONAL DERIVATIVES



Recall that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

and represent the rates of change of z in the x- and y-directions, that is, in the directions of the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$. (See Figure 1) To do this we consider the surface S with equation $z = f(x, y)$ (the graph of f) and we let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P in the direction of $\mathbf{u}$ intersects S in a curve C. (See Figure 2) The slope of the tangent line T to C at the point P is the rate of change of z in the direction of $\mathbf{u}$.

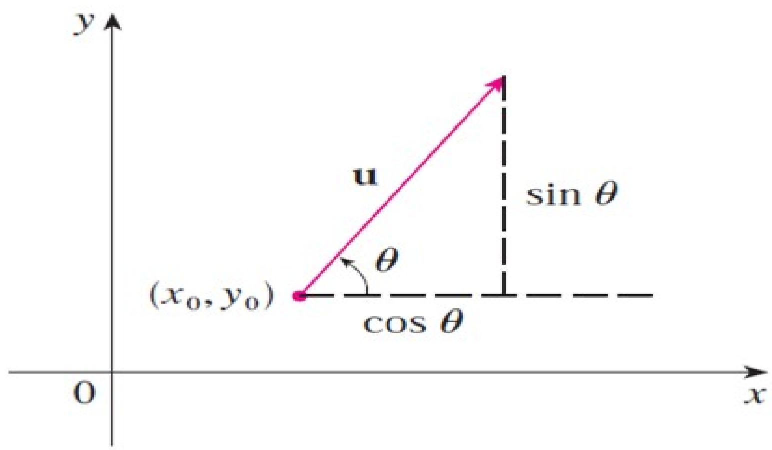


FIGURE 1
A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos\theta, \sin\theta \rangle$

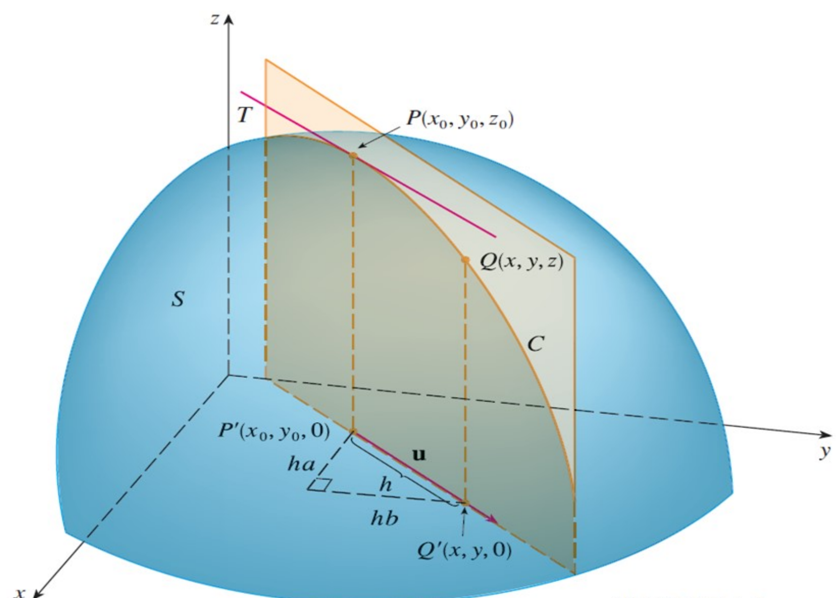


FIGURE 2

$\vec{2i} + \vec{3j}$

2 **DEFINITION** The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\vec{\mathbf{u}} = \langle a, b \rangle$ is $\langle 2, 3 \rangle$

1, 0

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \qquad \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}$$

if this limit exists.

THEOREM If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$\boxed{D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b}$$

Note :

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis (as in Figure 1), then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and the formula in Theorem 3 becomes

$$D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos\theta + f_y(x, y) \sin\theta$$

$D_u f(x,y)$ $\quad f_y = -3x + 8y$

Problem : Find the directional derivative $D_{\mathbf{u}} f(x, y)$ if

$f_x = 3x^2 - 3y$

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\mathbf{u}$ is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}} f(1, 2)$?

SOLUTION

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= f_x(x, y) \cos \frac{\pi}{6} + f_y(x, y) \sin \frac{\pi}{6} \\ &= (3x^2 - 3y) \frac{\sqrt{3}}{2} + (-3x + 8y) \tfrac{1}{2} \\ &= \tfrac{1}{2}\left[3\sqrt{3}\,x^2 - 3x + \left(8 - 3\sqrt{3}\right)y\right] \end{aligned}$$

Therefore

$$D_{\mathbf{u}} f(1, 2) = \tfrac{1}{2}\left[3\sqrt{3}\,(1)^2 - 3(1) + \left(8 - 3\sqrt{3}\right)(2)\right] = \frac{13 - 3\sqrt{3}}{2}$$

$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}$

THE GRADIENT VECTOR

$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$

DEFINITION If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Problem : If $f(x, y) = \sin x + e^{xy}$, then $f_x = \cos x + e^{xy} \cdot y$

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle \cos x + ye^{xy}, xe^{xy} \rangle$$

and $\nabla f(0, 1) = \langle 2, 0 \rangle$ $f_y = e^{xy} \cdot x = xe^{xy}$

$\nabla f = (\cos x + \underline{ye^{xy}})\vec{i} + xe^{xy}\vec{j}$

$$\left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}\right)\cdot\left(a\vec{i} + b\vec{j}\right)$$

$$f_x a + f_y b$$

NOTE :-

$$\boxed{D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}}$$

Problem : Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

$$u = \frac{v}{|v|}$$

SOLUTION We first compute the gradient vector at $(2, -1)$:

$$f_x = 2xy^3$$

$$f_y = 3x^2y^2 - 4$$

$$\nabla f(x, y) = 2xy^3\mathbf{i} + (3x^2y^2 - 4)\mathbf{j}$$

$$\nabla f(2, -1) = -4\mathbf{i} + 8\mathbf{j}$$

$$\nabla f = f_x\vec{i} + f_y\vec{j}$$

Note that $\mathbf{v}$ is not a unit vector, but since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of $\mathbf{v}$ is

$$= 2xy^3\vec{i} + (3x^2y^2 - 4)\vec{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}$$

$$\vec{u} = \frac{2\vec{i} + 5\vec{j}}{\sqrt{4 + 25}} = \frac{2\vec{i} + 5\vec{j}}{\sqrt{29}}$$

$$\nabla f_{(2,-1)} = 2(2)(-1)^3\vec{i} + ((3)(2^2)(-1)^2 - 4)\vec{j}$$

$$= -4\vec{i} + 8\vec{j}$$

$$D_u f(2,-1) = (-4\vec{i}+8\vec{j}) \cdot \left(\frac{2}{\sqrt{29}}\vec{i} + \frac{5}{\sqrt{29}}\vec{j}\right)$$

Therefore,

$$D_{\mathbf{u}} f(2, -1) = \nabla f(2, -1) \cdot \mathbf{u} = (-4\mathbf{i} + 8\mathbf{j}) \cdot \left(\frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}\right)$$

$$= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$

FUNCTIONS OF THREE VARIABLES

DEFINITION The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

For a function f of three variables, the **gradient vector**, denoted by ∇f or **grad** f, is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

for short,

$$\boxed{\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}}$$

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = \langle f_x, f_y, f_z \rangle$$

NOTE :-

$$\boxed{D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}}$$

$\hat{u}$

$\vec{v}$

Problem : If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

SOLUTION

(a) The gradient of f is $\frac{\partial f}{\partial y} = f_y = x \cos yz \cdot z = xz \cos yz$

$\frac{\partial f}{\partial x} = f_x = \sin yz (1) = \sin yz$

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

$\frac{\partial f}{\partial z} = f_z = x \cos yz \cdot y = xy \cos yz$

$$= \langle \sin yz, xz \cos yz, xy \cos yz \rangle$$

(b) At $(1, 3, 0)$ we have $\nabla f(1, 3, 0) = \langle 0, 0, 3 \rangle$. The unit vector in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is

$\nabla f(1,3,0) = 3\vec{k}$

$$\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

$$\hat{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{i} + 2\vec{j} - \vec{k}}{\sqrt{1+4+1}} = \frac{1}{\sqrt{6}}\vec{i} + \frac{2}{\sqrt{6}}\vec{j} - \frac{\vec{k}}{\sqrt{6}}$$

Therefore

$$D_{\mathbf{u}} f(1, 3, 0) = \nabla f(1, 3, 0) \cdot \mathbf{u}$$
$$= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}} \mathbf{i} + \frac{2}{\sqrt{6}} \mathbf{j} - \frac{1}{\sqrt{6}} \mathbf{k} \right)$$
$$= 3\left(-\frac{1}{\sqrt{6}} \right) = -\sqrt{\frac{3}{2}}$$

MAXIMIZING THE DIRECTIONAL DERIVATIVE

THEOREM Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \langle \frac{1}{2} - 2, 2 - 0 \rangle$
$= \langle -3/2, 2 \rangle$

Problem :-

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q\left(\frac{1}{2}, 2\right)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$

$f_x = e^y$

$f_y = xe^y(1)$
$= xe^y$

SOLUTION

(a) We first compute the gradient vector:

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle$$

$$\nabla f(2, 0) = \langle 1, 2 \rangle$$

$\nabla f = e^y\vec{i} + xe^y\vec{j}$

$\nabla f(2,0) = \vec{i} + 2\vec{j} = \langle 1, 2 \rangle$

The unit vector in the direction of $\overrightarrow{PQ} = \langle -1.5, 2 \rangle$ is $\mathbf{u} = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$, so the rate of change of f in the direction from P to Q is

$$D_{\mathbf{u}} f(2, 0) = \nabla f(2, 0) \cdot \mathbf{u} = \langle 1, 2 \rangle \cdot \left\langle -\tfrac{3}{5}, \tfrac{4}{5} \right\rangle$$
$$= 1\left(-\tfrac{3}{5}\right) + 2\left(\tfrac{4}{5}\right) = 1$$

$\hat{u} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{-3/2\,\vec{i} + 2\vec{j}}{\sqrt{(-3/2)^2 + 2^2}} = \frac{-3/2\,\vec{i} + 2\vec{j}}{\sqrt{25/4}} = \frac{-3/\cancel{2}}{5/\cancel{2}}\vec{i} + \frac{2}{5/2}\vec{j}$

$= -3/5\,\vec{i} + 4/5\,\vec{j}$
$= \langle -3/5, 4/5 \rangle$

(b) According to Theorem , f increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2 \rangle$. The maximum rate of change is

$|\nabla f(2,0)| = |\langle 1, 2\rangle|$
$= \sqrt{1^2 + 2^2}$
$= \sqrt{5}$

$$|\nabla f(2, 0)| = |\langle 1, 2 \rangle| = \sqrt{5}$$

Problem: Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

SOLUTION The gradient of T is

$\nabla T = \frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j} + \frac{\partial T}{\partial z}\vec{k}$

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}$$

$$= -\frac{160x}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{i} - \frac{320y}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{j} - \frac{480z}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{k}$$

$$= \frac{160}{(1 + x^2 + 2y^2 + 3z^2)^2}(-x\mathbf{i} - 2y\mathbf{j} - 3z\mathbf{k})$$

At the point $(1, 1, -2)$ the gradient vector is

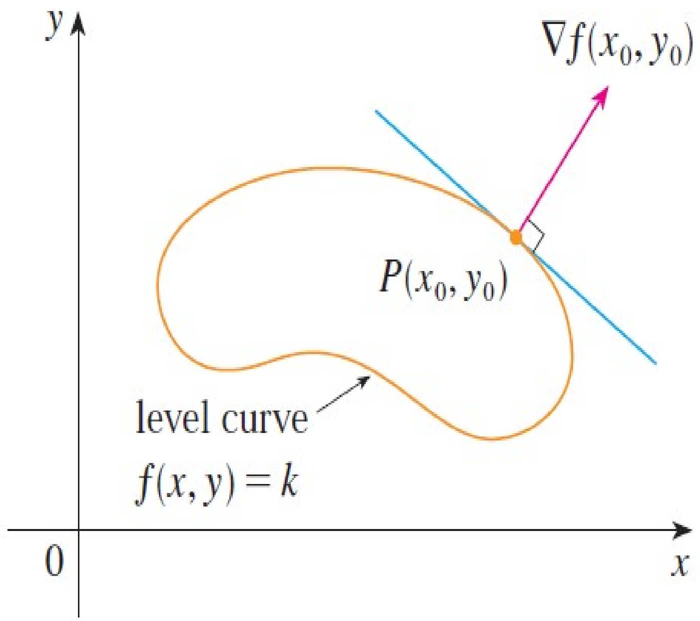
$$\nabla T(1, 1, -2) = \tfrac{160}{256}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) = \tfrac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$$

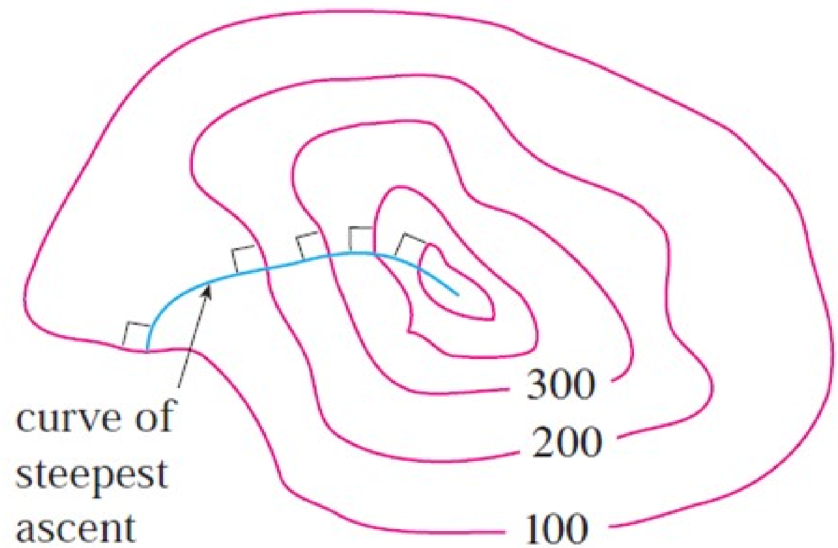
By Theorem the temperature increases fastest in the direction of the gradient vector $\nabla T(1, 1, -2) = \tfrac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ or, equivalently, in the direction of $-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ or the unit vector $(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})/\sqrt{41}$. The maximum rate of increase is the length of the gradient vector:

$$|\nabla T(1, 1, -2)| = \tfrac{5}{8}|-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}| = \tfrac{5}{8}\sqrt{41}$$

Therefore the maximum rate of increase of temperature is $\tfrac{5}{8}\sqrt{41} \approx 4^\circ\mathrm{C/m}$.

SIGNIFICANCE OF THE GRADIENT VECTOR

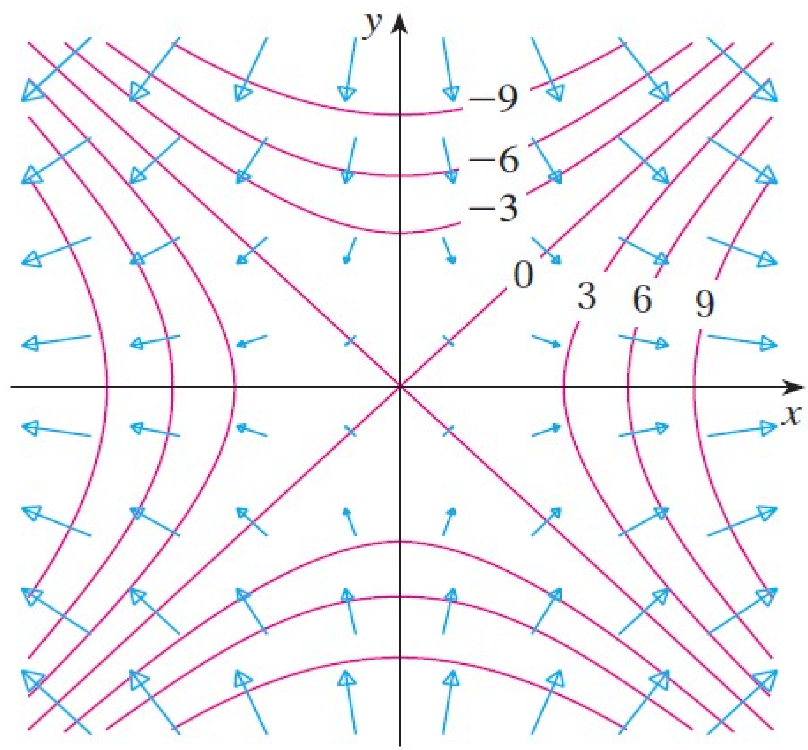




FIGURE

If we consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y), then a curve of steepest ascent can be drawn as in Figure by making it perpendicular to all of the contour lines.

Computer algebra systems have commands that plot sample gradient vectors. Each gradient vector $\nabla f(a, b)$ is plotted starting at the point (a, b). Figure shows such a plot (called a *gradient vector field*) for the function $f(x, y) = x^2 - y^2$ superimposed on a contour map of f. As expected, the gradient vectors point "uphill" and are perpendicular to the level curves.



$$\left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \partial/_{\partial z}\vec{k}\right)\cdot\left(V_1\vec{i} + V_2\vec{j} + V_3\vec{k}\right)$$

THE DIVERGENCE. Let $\mathbf{V}(x,y,z) = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$ be defined and differentiable at each point (x,y,z) in a certain region of space (i.e. $\mathbf{V}$ defines a differentiable vector field). Then the *divergence* of $\mathbf{V}$, written $\nabla\cdot\mathbf{V}$ or div $\mathbf{V}$, is defined by

$$\nabla\cdot\vec{q} = 0$$

$$\nabla\cdot\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}\right)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = 0$$

$$\vec{q} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Note the analogy with $\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$. Also note that $\nabla\cdot\mathbf{V} \neq \mathbf{V}\cdot\nabla$.

THE CURL. If $\mathbf{V}(x,y,z)$ is a differentiable vector field then the *curl* or *rotation* of $\mathbf{V}$, written $\nabla\times\mathbf{V}$, curl $\mathbf{V}$ or rot $\mathbf{V}$, is defined by

$$\nabla\times\vec{V}$$

$$\nabla\times\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\times\left(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_2 & V_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ V_1 & V_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_1 & V_2 \end{vmatrix} \mathbf{k}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\mathbf{k}$$

Find $\nabla\phi$ if (a) $\phi = \ln|\mathbf{r}|$, (b) $\phi = \frac{1}{r}$.

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

(a) $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then $|\mathbf{r}| = \sqrt{x^2+y^2+z^2}$ and $\phi = \ln|\mathbf{r}| = \frac{1}{2}\ln(x^2+y^2+z^2)$.

$\ln(x^2+y^2+z^2)^{1/2}$

$\frac{1}{2}\ln(x^2+y^2+z^2)$

$$\nabla\phi = \tfrac{1}{2}\nabla\ln(x^2+y^2+z^2)$$

$$= \tfrac{1}{2}\left\{\mathbf{i}\frac{\partial}{\partial x}\ln(x^2+y^2+z^2) + \mathbf{j}\frac{\partial}{\partial y}\ln(x^2+y^2+z^2) + \mathbf{k}\frac{\partial}{\partial z}\ln(x^2+y^2+z^2)\right\}$$

$$= \tfrac{1}{2}\left\{\mathbf{i}\frac{2x}{x^2+y^2+z^2} + \mathbf{j}\frac{2y}{x^2+y^2+z^2} + \mathbf{k}\frac{2z}{x^2+y^2+z^2}\right\} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{x^2+y^2+z^2} = \frac{\mathbf{r}}{r^2}$$

//

$$\nabla \frac{1}{\sqrt{x^2+y^2+z^2}} = \nabla(x^2+y^2+z^2)^{-1/2}$$

(b) $\nabla\phi = \nabla(\frac{1}{r}) = \nabla(\frac{1}{\sqrt{x^2+y^2+z^2}}) = \nabla\{(x^2+y^2+z^2)^{-1/2}\}$

$$= \mathbf{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} + \mathbf{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2} + \mathbf{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}$$

$$= \mathbf{i}\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2x\} + \mathbf{j}\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2y\} + \mathbf{k}\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2z\}$$

$$= \frac{-x\mathbf{i}-y\mathbf{j}-z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mathbf{r}}{r^3}$$

$f(x,y,z) = x^2y + 2xz - 4$

$\frac{\partial}{\partial}$

Find a unit normal to the surface $\underbrace{x^2y + 2xz = 4}$ at the point (2,−2,3).

∇f $\hat{n}$

$$\nabla(x^2y + 2xz) = (2xy + 2z)\mathbf{i} + x^2\mathbf{j} + 2x\mathbf{k} = -2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} \text{ at the point } (2,-2,3).$$

$x^2y+2xz = 4$

Then a unit normal to the surface $= \dfrac{-2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}}{\sqrt{(-2)^2+(4)^2+(4)^2}} = -\dfrac{1}{3}\mathbf{i} + \dfrac{2}{3}\mathbf{j} + \dfrac{2}{3}\mathbf{k}.$

Another unit normal is $\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$ having direction opposite to that above.

Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

$\phi_1 = x^2+y^2+z^2-9$

The angle between the surfaces at the point is the angle between the normals to the surfaces at the point.

$\phi_2 = x^2+y^2-3$ $\quad \nabla\phi_1$

A normal to $x^2+y^2+z^2=9$ at $(2,-1,2)$ is

$$\nabla\phi_1 = \nabla(x^2+y^2+z^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(x^2+y^2+z^2-9)$

A normal to $z=x^2+y^2-3$ or $x^2+y^2-z=3$ at $(2,-1,2)$ is

$$\nabla\phi_2 = \nabla(x^2+y^2-z) = 2x\,\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k} = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$(\nabla\phi_1)\cdot(\nabla\phi_2) = |\nabla\phi_1|\,|\nabla\phi_2|\cos\theta$, where θ is the required angle. Then

$= \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2-9)$

$$(4\mathbf{i}-2\mathbf{j}+4\mathbf{k})\cdot(4\mathbf{i}-2\mathbf{j}-\mathbf{k}) = |4\mathbf{i}-2\mathbf{j}+4\mathbf{k}|\,|4\mathbf{i}-2\mathbf{j}-\mathbf{k}|\cos\theta$$

$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$$16 + \cancel{4} - \cancel{4} = \underbrace{\sqrt{(4)^2+(-2)^2+(4)^2}\ \sqrt{(4)^2+(-2)^2+(-1)^2}}_{6\sqrt{21}}\cos\theta$$

and $\cos\theta = \dfrac{16}{6\sqrt{21}} = \dfrac{8\sqrt{21}}{63} = 0.5819$; thus the acute angle is $\theta = \text{arc}\cos 0.5819$.

$+\vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2-9)$

$+\vec{k}\frac{\partial}{\partial z}(x^2+y^2+z^2-9)$

$\nabla\phi_1\ (2,-1,2)$

$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

$= 4\vec{i} - 2\vec{j} + 4\vec{k}$

Prove: (*a*) $\nabla \times (\nabla\phi) = \mathbf{0}$ (curl grad $\phi = \mathbf{0}$), (*b*) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ (div curl $\mathbf{A} = 0$).

(*a*) $$\nabla \times (\nabla\phi) = \nabla \times \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\right]\mathbf{k}$$

$$= \underbrace{\left(\frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial y}\right)}_{0}\mathbf{i} + \underbrace{\left(\frac{\partial^2\phi}{\partial z\,\partial x} - \frac{\partial^2\phi}{\partial x\,\partial z}\right)}_{0}\mathbf{j} + \underbrace{\left(\frac{\partial^2\phi}{\partial x\,\partial y} - \frac{\partial^2\phi}{\partial y\,\partial x}\right)}_{0}\mathbf{k} = \underbrace{\mathbf{0}}_{=\vec{0}}$$

provided we assume that ϕ has continuous second partial derivatives so that the order of differentiation is immaterial.

$\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$

(b) $$\nabla\cdot(\nabla\times\mathbf{A}) = \nabla\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \nabla\cdot\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 A_3}{\partial x\,\partial y}} - \cancel{\frac{\partial^2 A_2}{\partial x\,\partial z}} + \cancel{\frac{\partial^2 A_1}{\partial y\,\partial z}} - \cancel{\frac{\partial^2 A_3}{\partial y\,\partial x}} + \cancel{\frac{\partial^2 A_2}{\partial z\,\partial x}} - \cancel{\frac{\partial^2 A_1}{\partial z\,\partial y}} = 0$$

assuming that $\mathbf{A}$ has continuous second partial derivatives.

$\nabla\cdot\vec{q} = 0$

$\nabla\cdot\nabla\phi = \nabla^2\phi$

$\vec{q} = \nabla\phi$

$\nabla \times \vec{V} = 0$ $\nabla \times \vec{V} = 0$

NOTE:-

A vector V is called irrotational if curl V = 0 .

(*a*) Find constants a, b, c so that

$$\mathbf{V} = (x + 2y + az)\mathbf{i} + (bx - 3y - z)\mathbf{j} + (4x + cy + 2z)\mathbf{k}$$

is irrotational.

(*b*) Show that **V** can be expressed as the gradient of a scalar function. $\vec{V} = \nabla \phi$

Solution:-

(*a*) curl $\mathbf{V} = \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = (c+1)\mathbf{i} + (a-4)\mathbf{j} + (b-2)\mathbf{k}$ $= \vec{0}$

$= 0\vec{i} + 0\vec{j} + 0\vec{k}$

$c+1=0$ $\quad a-4=0$

$c=-1$ $\quad a=4$

$b-2=0$

$b=2$

This equals zero when $a = 4$, $b = 2$, $c = -1$ and

$$\mathbf{V} = (x + 2y + 4z)\mathbf{i} + (2x - 3y - z)\mathbf{j} + (4x - y + 2z)\mathbf{k}$$

(*b*) Assume $\mathbf{V} = \nabla\phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$

Then (*1*) $\frac{\partial \phi}{\partial x} = x + 2y + 4z$, (2) $\frac{\partial \phi}{\partial y} = 2x - 3y - z$, (*3*) $\frac{\partial \phi}{\partial z} = 4x - y + 2z$.

$\nabla\phi = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$

Integrating (*1*) partially with respect to x, keeping y and z constant,

(4) $$\phi = \frac{x^2}{2} + 2xy + 4xz + f(y, z)$$

where $f(y, z)$ is an arbitrary function of y and z. Similarly from (2) and (*3*),

$\vec{V} = \nabla\phi$ $\quad \int \partial\phi = \int (x + 2y + 4z)\, \partial x$

$$\phi = \frac{x^2}{2} + 2y(x) + 4z(x) + f(y, z)$$

(5) $$\phi = 2xy - \frac{3y^2}{2} - yz + g(x,z)$$

$$\int \partial\phi = \int (2x - 3y - z)\,\partial y$$

$$\phi = 2xy - \frac{3y^2}{2} - zy + g(x,z)$$

(6) $$\phi = 4xz - yz + z^2 + h(x,y).$$

$$\int \partial\phi = \int (4x - y + 2z)\,\partial z$$

Comparison of (4), (5) and (6) shows that there will be a common value of ϕ if we choose

$$\phi = 4xz - yz + 2z^2/2 + h(x,y)$$

$$f(y,z) = -\frac{3y^2}{2} + z^2, \quad g(x,z) = \frac{x^2}{2} + z^2, \quad h(x,y) = \frac{x^2}{2} - \frac{3y^2}{2}$$

so that

$$\boxed{\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz + K}$$

NOTE :

If $\mathbf{F}$ is conservative, then curl $\mathbf{F} = \mathbf{0}$.

Problem : Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ is not conservative.

SOLUTION In Example 1 we showed that

$$\text{curl}\,\mathbf{F} = -y(2 + x)\,\mathbf{i} + x\,\mathbf{j} + yz\,\mathbf{k}$$

This shows that curl $\mathbf{F} \neq \mathbf{0}$ and so, by Theorem , $\mathbf{F}$ is not conservative.

Problem :

(a) Show that

$$\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$$

is a conservative vector field.

$\text{curl}\,\vec{F} = \vec{0}$

(b) Find a function ϕ such that $\mathbf{F} = \nabla\phi$

SOLUTION

(a) We compute the curl of $\mathbf{F}$:

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= (6xyz^2 - 6xyz^2)\,\mathbf{i} - (3y^2z^2 - 3y^2z^2)\,\mathbf{j} + (2yz^3 - 2yz^3)\,\mathbf{k}$$

$$= \mathbf{0}$$

Since $\text{curl}\,\mathbf{F} = \mathbf{0}$ and the domain of $\mathbf{F}$ is $\mathbb{R}^3$, $\mathbf{F}$ is a conservative vector field

$$y^2z^3\vec{i} + 2xyz^3\vec{j} + 3xy^2z^2\vec{k} = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

Assume $F = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$

$$\frac{\partial\phi}{\partial x} = y^2z^3 \qquad \int\partial\phi = \int y^2z^3\,\partial x \implies \qquad (1) \qquad \phi = y^2z^3x + f(y,z)$$

$$\frac{\partial\phi}{\partial y} = 2xyz^3 \qquad \int\partial\phi = \int 2xyz^3\,\partial y \qquad (2) \qquad \phi = \cancel{2}xz^3\frac{y^2}{\cancel{2}} + g(z,x)$$

$$\frac{\partial\phi}{\partial z} = 3xy^2z^2 \qquad \int\partial\phi = \int 3xy^2z^2\,\partial z \qquad (3) \qquad \phi = \cancel{3}xy^2\frac{z^3}{\cancel{3}} + \ell(x,y)$$

Integrating, (1) partially with respect to x, keeping y and z cons tan t,

$$\phi = xy^2z^3 + f(y,z) \qquad (4)$$

where $f(y,z)$ is an arbitrary function of y and z.

Similarly from (2) and (3)

$$\phi = \frac{2xy^2z^3}{2} = xy^2z^3 + g(x,z) \qquad (5)$$

$$\phi = \frac{3xy^2z^3}{3} = xy^2z^3 + h(x,y) \qquad (6)$$

Comparing (4), (5) & (6)

$$\phi = xy^2z^3 + K$$

NOTE:-

$\nabla \cdot \overline{V} = 0$

A vector such as v whose divergence is zero is sometimes called *solenoidal.*

Determine the constant a so that the vector $\mathbf{V} = (x+3y)\mathbf{i} + (y-2z)\mathbf{j} + (x+az)\mathbf{k}$ is solenoidal.

A vector **V** is solenoidal if its divergence is zero $\nabla \cdot \vec{V} = 0$

$$\nabla \cdot \mathbf{V} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 1 + 1 + a$$

Then $\nabla \cdot \mathbf{V} = a + 2 = 0$ when $a = -2$.