Evaluating triple integrals

Aim: Evaluating triple integrals (Cartesian, Cylindrical and Spherical coordinates) and visualizing regions using Matlab.

MATLAB Syntax used

int(f,v)	uses the symbolic object v as the variable of integration, rather than the variable determined by symvar
fill(X,Y,C)	fill(X,Y,C) creates filled polygons from the data in X and Y with vertex color specified by C.
fliplr(A)	If A is a row vector, then fliplr(A) returns a vector of the same length with the order of its elements reversed. If A is a column vector, then fliplr(A) simply returns A.
fsurf(f)	fsurf(f) creates a surface plot of the function z = f(x,y) over the default interval [-5 5] for x and y.
fsurf(f,xyinterval)	fsurf(f,xyinterval) plots over the specified interval. To use the same interval for both x and y, specify xyinterval as a two-element vector of the form [min max]. To use different intervals, specify a four-element vector of the form [xmin xmax ymin ymax].

Note: We invite your suggestions for the improvement of the topic triple integrals(Matlab codes and contents). mail id : kaliyappan.m@vit.ac.in

Example 1

Evaluate the iterated integral

$$\int_0^1 \int_0^z \int_0^{x+z} 6xz \, dy \, dx \, dz$$

Matlab code

```
syms x y z
sol = int(int(int(x+z,y,0,x+z),x,0,z),z,0,1)
```

Command window

```
sol = 7/12
```

Example 2

Evaluate the triple integral $\iiint_E 6xy\, dV$, where E lies under the plane z =1+x+y and above the region in the xy-pane bounded by the curves $y = \sqrt{x}$, y=0 and x=1.

Sol

Here $E=\{(x,y,z)|0\le x\le 1, 0\le y\le \sqrt{x}, 0\le z\le 1+x+y\}$

$$\iiint_E 6xy\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\,dz\,dy\,dx$$

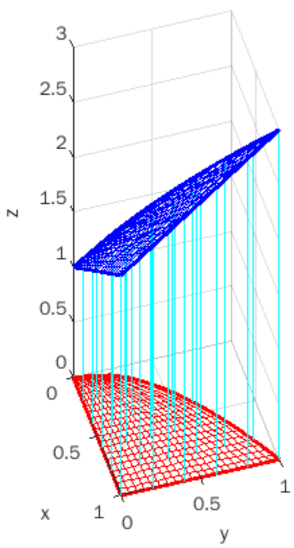
Matlab code

```
syms x y z
sol = int(int(int(6*x*y,z,0,1+x+y),y,0,sqrt(x)),x,0,1)
viewSolid(z,0+0*x*y,1+x+y,y,0+0*x,sqrt(x),x,0,1);
axis equal; grid on;
```

Command window

sol = 65/28

The region E is shown below (between two surfaces)



Example 3

Evaluate the triple integral $\iiint_E y\, dV$, where E is bounded by the planes x =0, y = 0, z=0, and 2x+2y+z = 4.

Sol:

$$\iiint_E y\,dV = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} y\,dz\,dy\,dx$$

Matlab code

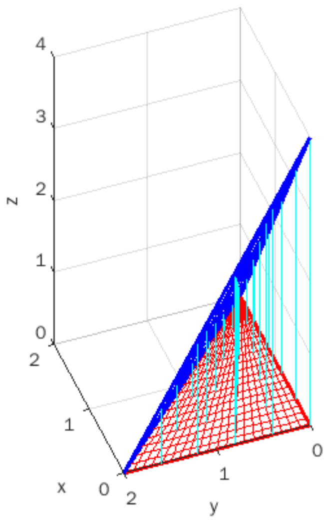
```
syms x y z
sol = int(int(int(y,z,0,4-2*x-2*y),y,0,2-x),x,0,2)
```

```
viewSolid(z,0+0*x*y,4-2*x-2*y,y,0+0*x,2-x,x,0,2);
axis equal; grid on;
```

Output in the command window

```
sol = 4/3
```

The region E is shown below



Example 4

A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane z =4, and above the paraboloid $z = 1 - x^2 + y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

Sol

In cylindrical coordinates the cylinder is r =1 and the paraboloid is z = 1- r^2, so we can write

$$E = \{(r, \theta, z) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant 1,\ 1 - r^2 \leqslant z \leqslant 4\}$$

Since the density at (x, y, z) is proportional to the distance from the z-axis, the density function is $f(x, y, z) = K\sqrt{x^2 + y^2} = Kr$ where is the proportionality constant. The mass of E is

$$m = \iiint_E K\sqrt{x^2 + y^2}\, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 (Kr)\, r\, dz\, dr\, d\theta$$

Matlab code

```
syms r z theta K
Ma= int(int(int((K*r)*r, z, 1-r^2,4), r ,0, 1),theta,0,2*pi) % integration

x = r*cos(theta), y = r*sin(theta), s = sym(4)
```

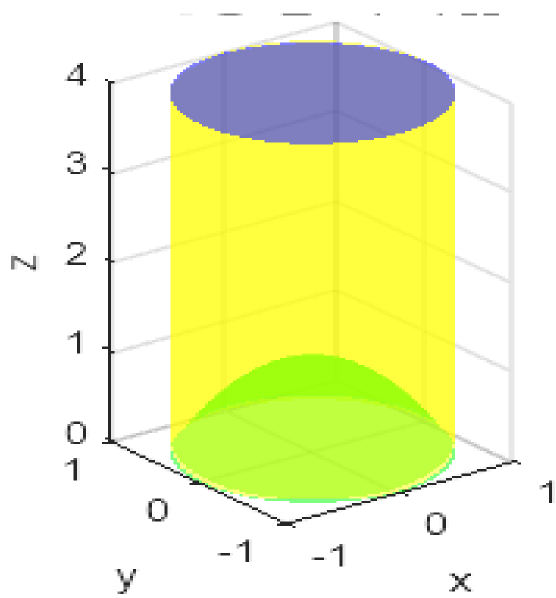
```
fsurf(x,y,1-r^2, [0 1 0 2*pi], 'g', 'EdgeColor', 'none');  % plotting paraboloid
hold on
fsurf(1*cos(theta), 1*sin(theta), r, 'y', [0  4  0 2*pi], 'EdgeColor', 'none')  % plotting
                                                            % cylinder of radius 1 with height z = 4
fsurf(x,y,s, [0 1 0 2*pi], 'b', 'EdgeColor', 'none'); % plotting circular  plane z=4.
hold on
axis equal; xlabel('x'); ylabel('y'); zlabel('z');
alpha 0.5
```

Output; In the command window

```
Ma =(12*pi*K)/5
```

In the figure window

The region E is shown below(above the paraboloid and below the surface z=4 inside the cylinder)



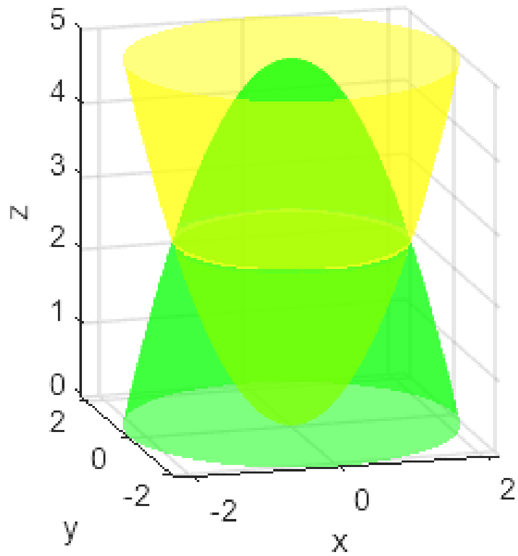
Example 5

Use Matlab to draw the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 5 - x^2 - y^2$

Matlab code

```
syms r z theta
x = r*cos(theta); y = r*sin(theta);
fsurf(x,y,5-r^2,[0 sqrt(5) 0 2*pi], 'g', 'EdgeColor', 'none');
hold on
fsurf(x,y,r^2, [0 sqrt(5) 0 2*pi], 'y', 'EdgeColor', 'none');
axis equal; xlabel('x'); ylabel('y'); zlabel('z');
alpha 0.5
```

Output: In the figure window



Example 6

Evaluate $\iiint_E e^z\, dV$, where E is enclosed by the paraboloid $z = 1 + x^2 + y^2$, the cylinder $x^2 + y^2 = 5$, and the xy-plane.

Sol

By Converting Cartesian to Cylindrical coordinates we get

$$\iiint_E e^z\, dV = \int_0^{2\pi} \int_0^{\sqrt{5}} \int_0^{1+r^2} e^z\, r dz dr d\theta$$

Matlab code

```
clc
clear all
syms x y r z theta
Sol= int(int(int(exp(z)*r,z,0,1+r^2),r,0,sqrt(5)),theta,0,2*pi) % integration
f=1+(x^2+y^2);
fsurf(f,[-sqrt(5) sqrt(5) -sqrt(5) sqrt(5)])
hold on
fsurf(sqrt(5)*cos(theta), sqrt(5)*sin(theta), r, 'y', [0  8  0 2*pi], 'EdgeColor', 'none')
alpha 0.5
```

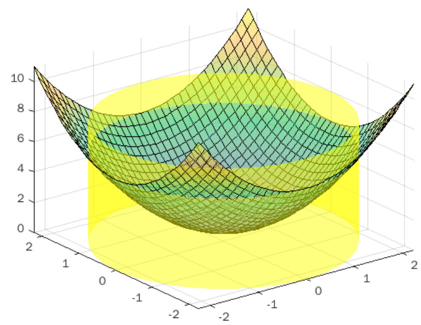
Output

In the command window

Sol =

-pi*(exp(1) - exp(6) + 5)

The region E is shown below

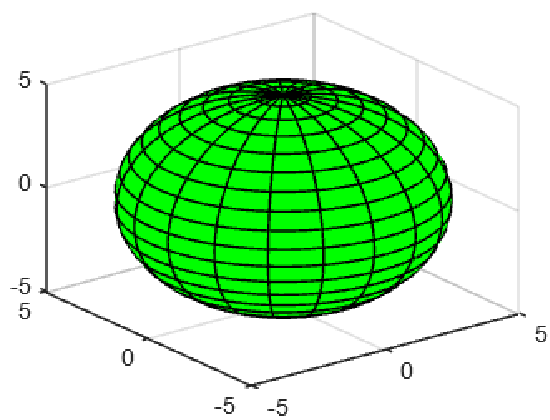


Example 7

Draw a sphere of radius 5 with centre at (0,0,0)

Matlab code

```
syms r z phi rho theta
rho=5
x= rho*sin(phi)*cos(theta), y = rho*sin(phi)*sin(theta), z= rho*cos(phi) ;
fsurf(x,y,z, [0 pi 0 2*pi], 'g', 'MeshDensity', 20);
```

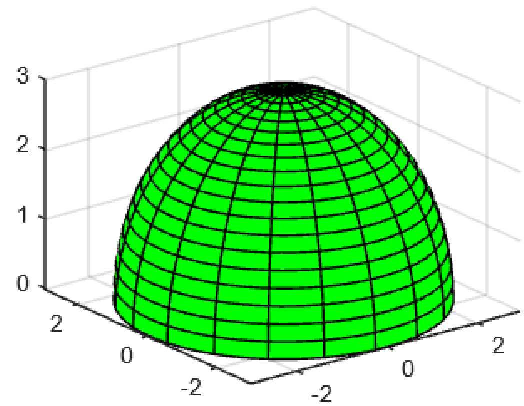


Example 8

Draw a hemisphere of radius 3 with centre at (0,0,0)

Matlab code

```
syms r z phi rho theta
rho=3
x= rho*sin(phi)*cos(theta), y = rho*sin(phi)*sin(theta), z= rho*cos(phi) ;
fsurf(x,y,z, [0 pi/2  0 2*pi], 'g', 'MeshDensity', 20);
```



Example 9

Evaluate $\iiint_E e^{\sqrt{x^2+y^2+z^2}}\, dV$, where E is enclosed by the sphere $x^2 + y^2 + z^2 = 9$ in the first octant.

Sol: By Changing Cartesian to Spherical coordinates we can get

$$\iiint_E e^{\sqrt{x^2+y^2+z^2}}\, dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{3} \rho^2 e^{\rho} \sin\phi \, d\rho \, d\phi \, d\theta$$

Matlab code

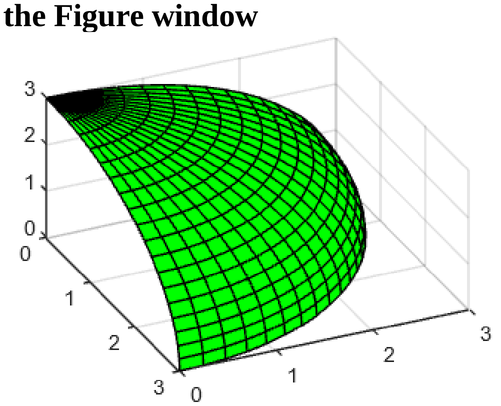
```
syms r  phi rho theta

Sol=int(int(int((exp(rho))*(rho)^2*sin(phi), rho,0,3), phi ,0,
pi/2),theta,0,pi/2)
rho=3
x = rho*sin(phi)*cos(theta), y = rho*sin(phi)*sin(theta), z =
rho*cos(phi) ;
fsurf(x,y,z, [0 pi/2 0 pi/2], 'g', 'MeshDensity', 20);
```

Output: In the command window

Sol =(pi*(5*exp(3) - 2))/2

In the Figure window



Example 10

Evaluate $\iiint_E z\, dV$, where E is enclosed by the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant.

Sol: By Changing Cartesian to Spherical coordinates we can get

$$\iiint_E z\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 (\rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

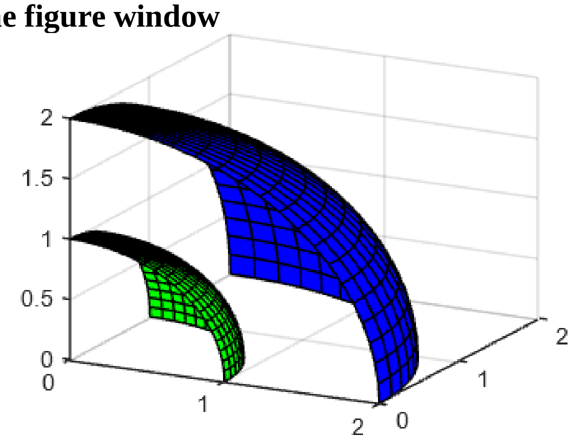
Matlab code

```
clc
clear all
syms r  phi rho theta
Sol=int(int(int((rho*cos(phi))*(rho)^2*sin(phi), rho,1,2), phi ,0, pi/2),theta,0,pi/2)
rho=1;
x = rho*sin(phi)*cos(theta), y = rho*sin(phi)*sin(theta), z = rho*cos(phi) ;
fsurf(x,y,z, [0 pi/2 0 pi/2], 'g', 'MeshDensity', 20);
hold on
rho=2;
x = rho*sin(phi)*cos(theta), y = rho*sin(phi)*sin(theta), z = rho*cos(phi) ;
fsurf(x,y,z, [0 pi/2 0 pi/2], 'b', 'MeshDensity', 20);
```

Output: In the command window

Sol = (15*pi)/16

In the figure window



Exercise

1. Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy –plane, and below the cone $z = \sqrt{x^2 + y^2}$.
2. Sketch the solid whose volume is given by the integral and evaluate the integral $\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_1^2 \rho^2 \sin(\varphi)\, d\rho d\varphi d\theta$

3. Evaluate $\iiint_E \sqrt{x^2 + y^2}\ dV$, where E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.