① Using Laplace transform, solve

$y'' + 4y' + 3y = 10 \sin t$, given

$y(0) = 0$, $y'(0) = 0$.

Sol: Let $Y(s) = L[y(t)]$ be the Laplace transform function.

Applying Laplace transform on DE, we get

$$\left[s^2 Y(s) - s\,y(0) - y'(0)\right] + 4\left[sY(s) - y(0)\right] + 3Y(s) = \frac{10}{s^2+1}$$

$$\Rightarrow \left[s^2 + 4s + 3\right] Y(s) = \frac{10}{s^2+1}$$

$$\Rightarrow \boxed{Y(s) = \frac{10}{(s+3)(s+1)(s^2+1)}}$$

Applying Inverse LT, we get

$$y(t) = 10\, L^{-1}\left[\frac{1}{(s+3)(s+1)(s^2+1)}\right]$$

Now, by Partial fraction expansion

$$\frac{1}{(s+3)(s+1)(s^2+1)} = \frac{A_1}{s+3} + \frac{A_2}{s+1} + \frac{B_1 s + B_2}{s^2+1} \quad \text{--- (1)}$$

$$A_1 = \lim_{s \to -3} \left[\frac{1}{(s+1)(s^2+1)}\right] = \frac{-1}{20}$$

$$A_2 = \lim_{s \to -1} \left[\frac{1}{(s+3)(s^2+1)}\right] = \frac{1}{4}$$

Equating numerator terms of (1), we get

$$1 = A_1(s+1)(s^2+1) + A_2(s+3)(s^2+1) + (B_1 s + B_2)(s+3)(s+1)$$

$s=0 \Rightarrow$
$$1 = A_1 + 3A_2 + 3B_2$$
$$= \frac{-1}{20} + \frac{3}{4} + 3B_2$$
$$3B_2 = 1 + \frac{1}{20} - \frac{3}{4} = \frac{20+1-15}{20}$$
$$\Rightarrow \boxed{B_2 = \frac{1}{10}}$$

$s^3: \quad 0 = A_1 + A_2 + B_1$

$$\Rightarrow B_1 = -A_1 - A_2 = \frac{1}{20} - \frac{1}{4}$$

$$\Rightarrow \boxed{B_1 = \frac{-1}{5}}$$

$\therefore$ (1) becomes

$$\frac{1}{(s+3)(s+1)(s^2+1)} = \frac{-1}{20} \cdot \frac{1}{s+3} + \frac{1}{4} \cdot \frac{1}{s+1}$$

$$- \frac{1}{5} \cdot \frac{s}{s^2+1} + \frac{1}{10} \cdot \frac{1}{s^2+1}$$

Reqd. soln. is

$$y(t) = 10 L^{-1}\left[\frac{1}{(s+3)(s+1)(s^2+1)}\right]$$

$$= 10 \left[\frac{-1}{20} e^{-3t} + \frac{1}{4} e^{-t} - \frac{1}{5} \cos t + \frac{1}{10} \sin t\right]$$

$$= \frac{-1}{2} e^{-3t} + \frac{5}{2} e^{-t} - 2 \cos t + \sin t$$ //

(2) Using Laplace transform, solve

$x'' + 9x = 18t, \quad x(0) = 0,$

$x\left(\frac{\pi}{2}\right) = 0$

Sol:

Let $L[x(t)] = X(s)$.

$$L[x''] + 9L[x] = L[18t]$$

$$\left[s^2 X - s x(0) - x'(0)\right] + 9X = \frac{18}{s^2}$$

Since $x(0)=0$ and $x'(0) = 0$, we get

$$(s^2+9)X = \frac{18}{s^2} + x'(0)$$

$$\Rightarrow X(s) = \frac{18}{s^2(s^2+9)} + \frac{x'(0)}{s^2+9}$$

Applying inverse laplace transform, we get

$$x(t) = L^{-1}[X(s)]$$

$$= 18\, L^{-1}\left[\frac{1}{s^2(s^2+9)}\right] + \frac{x'(0)}{3}\sin 3t$$

$\hookrightarrow$ (1)

Now,

$$L^{-1}\left[\frac{1}{s^2(s^2+9)}\right] = t * \frac{1}{3}\sin 3t$$

$$= \frac{1}{3}\sin 3t * t$$

$$= \int_0^t \left[\frac{1}{3}\sin 3u\right]\left[t-u\right] du$$

$$= \frac{1}{3}t\left[\frac{-\cos 3u}{3}\right]_0^t - \frac{1}{3}\int_0^t u\sin 3u\,du$$

$$= \frac{-1}{9}t\left[\cos 3t - 1\right]$$

$$- \frac{1}{3}\left[(u)\left\{\frac{-\cos 3u}{3}\right\} - (1)\left\{\frac{-\sin 3u}{9}\right\} + (0)\right]_{u=0}^{t}$$

$$= \frac{-1}{9}t\cancel{\cos 3t} + \frac{1}{9}t - \frac{1}{3}\left[\frac{-t\cancel{\cos 3t}}{3} + \frac{\sin 3t}{9}\right]$$

$$= \frac{t}{9} - \frac{1}{27}\sin 3t$$

$\therefore$ ① gives

$$\boxed{x(t) = 2t - \frac{2}{3}\sin 3t + \frac{x'(0)}{3}\sin 3t}$$

Using $x(\pi/2) = 0$, we get

$$0 = \pi - \frac{2}{3}\sin\frac{3\pi}{2} + \frac{x'(0)}{3}\sin\frac{3\pi}{2}$$

$$= \pi + \frac{2}{3} - \frac{1}{3}x'(0)$$

$$x'(0) = 3\left[\pi + \frac{2}{3}\right] = 3\pi + 2$$

Reqd soln. is

$$x(t) = 2t - \frac{2}{3}\sin 3t$$

$$+\left(\frac{1}{3}\sin 3t\right)(2 + 3\pi)$$

$$= 2t + \pi\sin 3t \quad //$$

Check: $x(0) = 0$; $x\left(\frac{\pi}{2}\right) = 0$.