$$\{f(n)\} = \{f(0), f(1), f(2), \dots\}$$

$$\{g(n)\} = \{g(0), g(1), g(2), \dots\}$$

$$\{f(n) * g(n)\} = \{f(0)g(0),\ f(0)g(1)+f(1)g(0),$$
$$f(0)g(2)+f(1)g(1)+f(2)g(0),$$
$$f(0)g(3)+f(1)g(2)+f(2)g(1)+$$
$$f(3)g(0), \dots\}$$

$$\{f(n)*g(n)\}_{n=0}^{\infty} = \{f(0)*g(0),\ f(1)*g(1),\ f(2)*g(2), \dots,$$
$$f(n)*g(n), \dots\}$$

$$f(0)*g(0) = f(0)g(0)$$

$$f(1)*g(1) = f(0)g(1) + f(1)g(0)$$

$$f(2)*g(2) = f(0)g(2) + f(1)g(1) + f(2)g(0)$$

$$= \sum_{r=0}^{2} f(r)g(2-r)$$

$$\boxed{f(n)*g(n) = \sum_{r=0}^{n} f(r)g(n-r)}$$

Convolution Theorem

$$Z\{f(n)*g(n)\} = Z\{f(n)\}\,Z\{g(n)\}$$

$$= \bar{f}(z)\,\bar{g}(z)$$

$$Z^{-1}[\bar{f}\bar{g}] = f(n)*g(n)$$

✓ $$\boxed{Z^{-1}[\bar{f}\bar{g}] = Z^{-1}(\bar{f}) * Z^{-1}(\bar{g})}$$

Convolution Theorem for Inverse Z transforme.

Inverse Z transforms

(A) By Partial fractions

(1) $Z^{-1}\left[\dfrac{3z^2-18z+26}{(z-2)(z-3)(z-4)}\right]$.

Let $F(z) = \dfrac{3z^2-18z+26}{(z-2)(z-3)(z-4)}$

By Partial fraction method, let

$$F(z) = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{z-4}$$

$$A = \lim_{z\to 2}\left[(z-2)F(z)\right] = \lim_{z\to 2}\left[\frac{3z^2-18z+26}{(z-3)(z-4)}\right]$$

$$= \frac{12-36+26}{(-1)(-2)} = \frac{2}{2} = 1$$

$$A = 1$$

$$B = \lim_{z\to 3}\left[(z-3)F(z)\right]$$

$$= \lim_{z\to 3}\left[\frac{3z^2-18z+26}{(z-2)(z-4)}\right] = \frac{27-54+26}{(-1)}$$

$$\boxed{B = 1}$$

$$c = \lim_{z\to 4}\left[(z-4)\,F(z)\right]$$

$$= \lim_{z\to 4}\left[\frac{3z^2-18z+26}{(z-2)(z-3)}\right] = \frac{48-72+26}{(2)(1)}$$

$$\boxed{c = 1}$$

$$\therefore \quad F(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{z-4}$$

$$Z^{-1}\left[F(z)\right] = Z^{-1}\left[\frac{1}{z-2}\right] + Z^{-1}\left[\frac{1}{z-3}\right] + Z^{-1}\left[\frac{1}{z-4}\right]$$

$$\therefore Z^{-1}\left[\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}\right] = 2^{n-1} + 3^{n-1} + 4^{n-1} \quad //$$

② Find $z^{-1}\left[\dfrac{4z^3}{(2z-1)^2(z-1)}\right]$ using PFM.

Sol: Here, $F(z) = \dfrac{4z^3}{(2z-1)^2(z-1)}$

$$= 1 + \frac{A}{z-1} + \frac{B_1 z + B_2}{(2z-1)^2}$$

Equating numerators, we get

$$4z^3 = (2z-1)^2(z-1) + A(2z-1)^2 + (B_1 z + B_2)(z-1)$$

$z=1 \Rightarrow$ $\boxed{4 = A}$

$z=\frac{1}{2} \Rightarrow$ $\frac{1}{2} = \left(\frac{B_1}{2} + B_2\right)\left(\frac{-1}{2}\right)$

$\frac{B_1}{2} + B_2 = -1 \Rightarrow$ $B_1 = 2(-1-B_2)$

$B_1 = -8$

$z=0 \Rightarrow$ $0 = -1 + A - B_2 \Rightarrow$ $B_2 = A-1$

$\therefore$ $B_2 = 3$

$$\therefore F(z) = 1 + \frac{4}{z-1} - \frac{8z}{(2z-1)^2} + \frac{3}{(2z-1)^2}$$

$$F(z) = 1 + \frac{4}{z-1} - \frac{2z}{\left(z-\frac{1}{2}\right)^2} + \frac{3}{4}\cdot\frac{1}{\left(z-\frac{1}{2}\right)^2}$$

Hence

$$Z^{-1}\left[F(z)\right] = Z^{-1}[1] + 4Z^{-1}\left[\frac{1}{z-1}\right]$$

$$+ 4Z^{-1}\left[\frac{\frac{1}{2}z}{\left(z-\frac{1}{2}\right)^2}\right]$$

$$+ \frac{3}{4}Z^{-1}\left[\frac{1}{\left(z-\frac{1}{2}\right)^2}\right]$$

$$= \delta(n) + 4(1)^{n-1} + 4n\left(\frac{1}{2}\right)^n$$

$$+ \frac{3}{4}(n-1)\left(\frac{1}{2}\right)^{n-2} \quad //$$

(B) <u>Inverse Z transform using Convolution Theorem</u>

(1) Find $Z^{-1}\left[\frac{1}{(z-a)^2}\right]$.

<u>Sol:</u> By Convolution Theorem,

$$Z^{-1}\left[\frac{1}{(z-a)^2}\right] =$$

$$Z^{-1}\left[\frac{1}{z-a}\right] * Z^{-1}\left[\frac{1}{z-a}\right]$$

$$= a^{n-1} * a^{n-1}$$

$$= \sum_{r=0}^{n} a^{r-1} a^{n-r-1}$$

$$= \sum_{r=1}^{n-1} a^{r-1} a^{n-r-1}$$

$$= \sum_{r=1}^{n-1} a^{n-2}$$

$$= a^{n-2} \sum_{r=1}^{n-1} (1)$$

$$\therefore Z^{-1}\left[\frac{1}{(z-a)^2}\right] = (n-1)\, a^{n-2} \quad //$$

(2) $Z^{-1}\left[\dfrac{z^2}{(z-a)(z-b)}\right]$

Sol: By Convolution Theorem,

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-b}\right]$$

$$= a^n * b^n$$

$$= \sum_{r=0}^{n} a^r b^{n-r}$$

$$= b^n \sum_{r=0}^{n} \left(\frac{a}{b}\right)^r$$

$$= b^n\left[1 + \frac{a}{b} + \left(\frac{a}{b}\right)^2 + \dots + \left(\frac{a}{b}\right)^n\right]$$

$$= b^n\left[\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\frac{a}{b} - 1}\right]$$

$$= \cancel{b^n}\left[\frac{a^{n+1} - b^{n+1}}{\cancel{b^n}(a-b)}\right]$$

$$\therefore Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{a^{n+1} - b^{n+1}}{a-b} \quad //$$

③ Find $Z^{-1}\left[\frac{1}{(z-a)(z-b)}\right]$

By Convolution Theorem,

Sol: $Z^{-1}\left[\frac{1}{(z-a)(z-b)}\right] = a^{n-1} * b^{n-1}$

$$= \sum_{r=1}^{n-1} a^{r-1}\, b^{n-r-1}$$

$$= a^{-1} b^{n-1} \sum_{r=1}^{n-1} \left(a^{r}\, b^{-r}\right)$$

$$= \frac{b^{n-1}}{a} \sum_{r=1}^{n-1} \left(\frac{a}{b}\right)^{r}$$

$$= \frac{b^{n-1}}{a} \cdot \left[\frac{a}{b} + \left(\frac{a}{b}\right)^{2} + \dots + \left(\frac{a}{b}\right)^{n-1}\right]$$

$$= \frac{b^{n-1}}{b}\left[1 + \frac{a}{b} + \left(\frac{a}{b}\right)^{2} + \dots + \left(\frac{a}{b}\right)^{n-2}\right]$$

$$= \frac{b^{n-1}}{b}\left[\frac{\left(\frac{a}{b}\right)^{n-2+1} - 1}{\frac{a}{b} - 1}\right]$$

$$= \frac{b^{n-1}}{b}\left[\frac{\left(\frac{a}{b}\right)^{n-1} - 1}{\frac{a}{b} - 1}\right]$$

$$Z^{-1}\left[\frac{1}{(z-a)(z-b)}\right] = \frac{\cancel{b^{n-1}}}{\cancel{b}}\left[\frac{a^{n-1}-b^{n-1}}{\cancel{b^{n-1}}\left(\frac{a-b}{\cancel{b}}\right)}\right]$$

$$= \frac{a^{n-1}-b^{n-1}}{a-b} \quad //$$

4 $Z^{-1}\left[\dfrac{z}{(z-a)^2}\right]$

$$\frac{z}{(z-a)^2} = \frac{z}{z-a} \cdot \frac{1}{z-a}$$

By convolution Theorem,

$$Z^{-1}\left[\frac{z}{(z-a)^2}\right] = a^n * a^{n-1}$$

$$= \sum_{r=0}^{n-1} a^r a^{n-r-1}$$

$$= a^{n-1} \sum_{r=0}^{n-1} (1)$$

$$\boxed{Z^{-1}\left[\frac{z}{(z-a)^2}\right] = n a^{n-1}}$$