1 Given $f(t) = \begin{cases} 2 & 0 < t < 2 \\ -1 & 2 \le t < 3 \\ 1 & t \geq 3 \end{cases}$.

Find $L[f(t)]$...

Sol:

$$f(t) = (2)[u(t-0) - u(t-2)] + (-1)[u(t-2) - u(t-3)] + (1)[u(t-3)]$$

$$\Rightarrow \boxed{f(t) = 2u(t) - 3u(t-2) + 2u(t-3)}$$

$$\therefore L[f(t)] = 2L[u(t)] - 3L[u(t-2)] + 2L[u(t-3)]$$

$$= 2\left(\frac{1}{s}\right) - 3\left(\frac{e^{-2s}}{s}\right) + 2\left(\frac{e^{-3s}}{s}\right)$$

$$\therefore L[f(t)] = \frac{1}{s}\left[2 - 3e^{-2s} + 2e^{-3s}\right] //$$

② Find $L[t^2 u(t-1)]$.

Sol: consider $t^2 = (t-1+1)^2 = (t-1)^2 + 2(t-1) + 1$

$$t^2 u(t-1) = (t-1)^2 u(t-1) + 2(t-1)u(t-1) + u(t-1)$$

$$\therefore L[t^2 u(t-1)] = L[(t-1)^2 u(t-1)] + 2L[(t-1)u(t-1)] + L[u(t-1)] \quad \text{Ⓐ}$$

Now,

$$L[(t-1)^2 u(t-1)] = e^{-s} L[t^2]$$

$$= e^{-s}\left[\frac{2!}{s^3}\right]$$

$$L[(t-1)u(t-1)] = e^{-s} L[t] = e^{-s}/s^2$$

$$L[u(t-1)] = \frac{e^{-s}}{s}$$

Substituting in Ⓐ, we have

$$L[t^2 u(t-1)] = \frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s^2} + \frac{e^{-s}}{s}$$

$$= \frac{e^{-s}}{s}\left[1 + \frac{2}{s} + \frac{2}{s^2}\right] //$$

③ Find $L[\sin t \, u(t-\pi)]$

Sol: WKT, $\sin t = \sin(t-\pi+\pi)$

$= -\sin(t-\pi)$

$\therefore \quad L[\sin t \, u(t-\pi)] = -L[\sin(t-\pi)\, u(t-\pi)]$

$= -e^{-\pi s} L[\sin(t)]$

$= \dfrac{-e^{-\pi s}}{s^2+1}$ //

④ $$f(t)=\begin{cases}\sin\omega t, & 0<t<\pi/\omega\\ 0, & \pi/\omega<t<2\pi/\omega\end{cases}$$

Sol:

Now, $f(t) = (\sin\omega t)\left[u(t) - u\left(t-\frac{\pi}{\omega}\right)\right]$

(i) $L[\sin\omega t \, u(t)] = L[\sin\omega t]$

$= \dfrac{\omega}{s^2+\omega^2}$.

(ii) $L\left[\sin\omega t \, u\left(t-\frac{\pi}{\omega}\right)\right]$

$\sin\omega t = \sin\omega\left(\underline{t-\frac{\pi}{\omega}}+\underline{\frac{\pi}{\omega}}\right)$

$= -\sin\omega\left(t-\frac{\pi}{\omega}\right)$

$$L\left[\sin\omega t \, u\left(t-\frac{\pi}{\omega}\right)\right]$$

$$= -L\left[\sin\omega\left(t-\frac{\pi}{\omega}\right) u\left(t-\frac{\pi}{\omega}\right)\right]$$

$$= -e^{-\frac{\pi}{\omega}s} L\left[\sin\omega t\right]$$

$$= \frac{-\omega e^{-\frac{\pi}{\omega}s}}{s^2+\omega^2}.$$

$$\therefore \quad L\left[f(t)\right] = \frac{\omega}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2} e^{-\frac{\pi}{\omega}s}$$

$$= \frac{\omega}{s^2+\omega^2}\left[1+e^{-\frac{\pi}{\omega}s}\right] \quad //$$

Problems involving impulse function

(1) $L[t^3 \delta(t-5)] =$

$$\int_0^\infty \underline{e^{-st} t^3} \delta(t-5)\, dt$$

$$= 5^3 e^{-5s} = 125 e^{-5s}$$

[**Property of delta function**

$$\int \underline{f(x)\delta(x-a)}\, dx = f(a)]$$

(2) $L[t^3 \delta(t-5)] = (-1)^3 \frac{d^3}{ds^3}\left[L[\delta(t-5)]\right]$

$$= -\frac{d^3}{ds^3}\left[e^{-5s}\right]$$

$$= -\frac{d^2}{ds^2}\left[-5e^{-5s}\right] = 5\frac{d^2}{ds^2}\left[e^{-5s}\right]$$

$$= 5\frac{d}{ds}\left[-5e^{-5s}\right] = -25\frac{d}{ds}\left[e^{-5s}\right]$$

$$= 125 e^{-5s} \,//$$

② $$L\left[\frac{\delta(t-\pi)}{t}\right] = \int_s^\infty e^{-\pi s}\,ds$$

$$= \left(\frac{e^{-\pi s}}{-\pi}\right)_s^\infty = \frac{e^{-\pi s}}{\pi} \quad //$$

③ $$L\left[e^{-\pi t}\delta(t-a)\right] = L\left[\delta(t-a)\right]\Big|_{s\to s+\pi}$$

$$= e^{-as}\Big|_{s\to s+\pi}$$

$$= e^{-a(s+\pi)} \quad //$$

(ii) $L\left[\cos(t)\log(t)\delta(t-\pi)\right] = -\log(\pi)e^{-s\pi}$

Sol:

$$L[\cos t \log t \, \delta(t-\pi)]$$

$$= \int_0^\infty \underline{e^{-st}\cos t \log t}\, \delta(t-\pi)\, dt$$

$$= e^{-\pi s}\cos\pi \log\pi$$

$$= -\log\pi \, e^{-\pi s} \; //$$