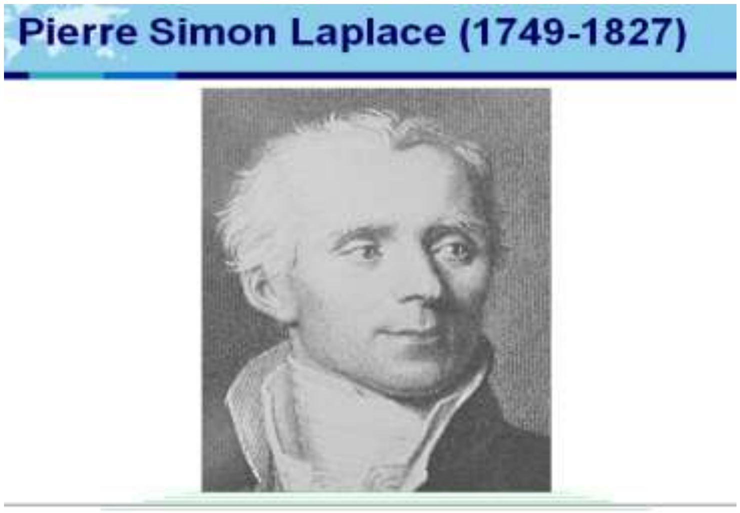
LAPLACE TRANSFORMS



- Laplace Transform (LT) is a powerful tool for solving linear differential equations in which the differential equation is converted to an algebraic equation. It is best suited for solving initial value problems as the initial conditions are taken care in the algebraic equation and the solution for the differential equation is obtained without resorting to find the general solution and the arbitrary constants.

- The name is due to the French Mathematician Pierre Simon de Laplace who used this transforms while developing the theory of probability.

DEFINITION OF THE LAPLACE TRANSFORM

Let $F(t)$ be a function of t specified for $t > 0$. Then the *Laplace transform* of $F(t)$, denoted by $\mathcal{L}\{F(t)\}$, is defined by

$$\mathcal{L}\{F(t)\} \quad = \quad f(s) \quad = \quad \int_0^\infty e^{-st}\, F(t)\, dt \tag{1}$$

where we assume at present that the parameter s is real. Later it will be found useful to consider s complex.

The Laplace transform of $F(t)$ is said to *exist* if the integral (*1*) *converges* for some value of s; otherwise it does not exist.

SECTIONAL OR PIECEWISE CONTINUITY

A function is called *sectionally continuous* or *piecewise continuous* in an interval $\alpha \leqq t \leqq \beta$ if the interval can be subdivided into a finite number of intervals in each of which the function is continuous and has finite right and left hand limits.

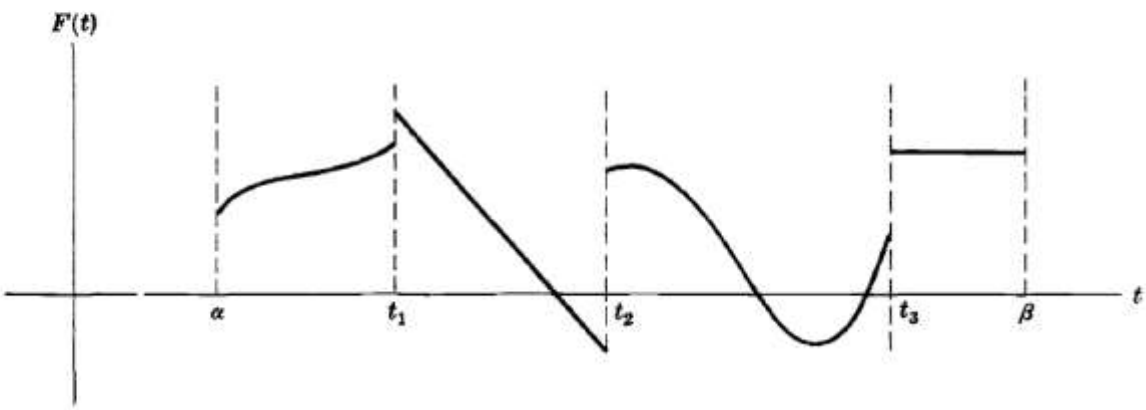


Fig. 1-1

An example of a function which is sectionally continuous is shown graphically in Fig. 1-1 above. This function has discontinuities at t_1, t_2 and t_3. Note that the right and left hand limits at t_2, for example, are represented by $\lim_{\epsilon \to 0} F(t_2 + \epsilon) = F(t_2 + 0) = F(t_2+)$ and $\lim_{\epsilon \to 0} F(t_2 - \epsilon) = F(t_2 - 0) = F(t_2-)$ respectively, where ϵ is positive.

FUNCTIONS OF EXPONENTIAL ORDER

If real constants $M > 0$ and γ exist such that for all $t > N$

$$|e^{-\gamma t}F(t)| < M \qquad \text{or} \qquad |F(t)| < Me^{\gamma t}$$

we say that $F(t)$ is a *function of exponential order* γ *as* $t \to \infty$ or, briefly, is of *exponential order*.

Example 1. $F(t) = t^2$ is of exponential order 3 (for example), since $|t^2| = t^2 < e^{3t}$ for all $t > 0$.

Example 2. $F(t) = e^{t^3}$ is not of exponential order since $|e^{-\gamma t}e^{t^3}| = e^{t^3 - \gamma t}$ can be made larger than any given constant by increasing t.

Intuitively, functions of exponential order cannot "grow" in absolute value more rapidly than $Me^{\gamma t}$ as t increases. In practice, however, this is no restriction since M and γ can be as large as desired.

Bounded functions, such as $\sin at$ or $\cos at$, are of exponential order.

DEFINITION: **exponential order**

A function f is said to be of **exponential order** c if there exist constants c, $M > 0$, $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$.

Basically this is saying that in order for $f(t)$ to have a Laplace Transform then in a race between $|f(t)|$ and e^{ct} as $t \to \infty$ then e^{ct} must approach its limit first, i.e. $\lim\limits_{t\to\infty} \dfrac{f(t)}{e^{ct}} = 0$.

SUFFICIENT CONDITIONS FOR EXISTENCE OF LAPLACE TRANSFORMS

Theorem 1-1. If $F(t)$ is sectionally continuous in every finite interval $0 \leq t \leq N$ and of exponential order γ for $t > N$, then its Laplace transform $f(s)$ exists for all $s > \gamma$.

If the conditions are not satisfied, however, the Laplace transform may or may not exist.

Problems under Laplace Transforms

1. Using the definition of the Laplace Transform, prove the following results :

(i) $L[k]=k/s$, provided $s>0$

(ii) $L[e^{t}]=\dfrac{1}{s-1}$, provided $s>1$

(iii) $L[e^{-t}]=\dfrac{1}{s+1}$, provided $s>-1$

(iv) $L[\sin t]=\dfrac{1}{s^{2}+1}$, provided $s>0$

(v) $L[\cos t]=\dfrac{s}{s^{2}+1}$, provided $s>0$

(vi) $L[t^{n}]=\dfrac{\Gamma(n+1)}{s^{n+1}}$, if $n>-1\ ; s>0$

(vii) $L[u(t-a)]=\dfrac{e^{-as}}{s}$

(viii) $L[\delta(t-a)]=e^{-as}$

2. Find $L[\sqrt{t}]$ and $L[1/\sqrt{t}]$. Ans : $\dfrac{\sqrt{\pi}}{2s^{3/2}}$; $\sqrt{\dfrac{\pi}{s}}$

3. Express $f(t)=\begin{cases} 2; & 0<t<2 \\ -1; & 2\le t<3 \\ 1; & t\ge 3 \end{cases}$ as a single expression in terms of unit step function.

Ans : $f(t)=2-3u(t-2)+2u(t-3)$

4. Prove the following results :

(i) $L\left[t^{3}\delta(t-5)\right]=5^{3}e^{-5s}$

(ii) $L\left[\cos(t)\log(t)\delta(t-\pi)\right]=-\log(\pi)e^{-s\pi}$

(iii) $L\left[\dfrac{\delta(t-\pi)}{t}\right]=\dfrac{e^{-s\pi}}{\pi}$

(iv) $L\left[e^{-\pi t}\delta(t-a)\right]=e^{-a(s+\pi)}$

5. Using the linearity property, prove the following results :

(i) $L[\sinh t]=\dfrac{1}{s^{2}-1}$, provided $s^{2}>1$

(ii) $L[\cosh t]=\dfrac{s}{s^{2}-1}$, provided $s^{2}>1$

6. Using the change of scale property, prove the following results :

(i) $L[e^{at}] = \frac{1}{s-a}$, provided $s > a$ (ii) $L[e^{-at}] = \frac{1}{s+a}$, provided $s > -a$

(iii) $L[\sin at] = \frac{a}{s^2+a^2}$, provided $s > 0$ (iv) $L[\cos at] = \frac{s}{s^2+a^2}$, provided $s > 0$

(v) $L[\sinh at] = \frac{a}{s^2-a^2}$, if $|s| > |a|$ (vi) $L[\cosh at] = \frac{s}{s^2-a^2}$, if $|s| > |a|$

(vii) $L[sin2tsin3t] = \frac{12s}{(s^2+1)(s^2+25)}$ (viii) $L[\sin^3 2t] = \frac{48}{(s^2+4)(s^2+36)}$

7. Using the First Shifting property, prove the following results :

(i) $L[e^{at}t^n] = \frac{n!}{(s-a)^{n+1}}$ for integer n.

(ii) $L\left[e^{-3t}\left(2\cos 5t - 3\sin 5t\right)\right] = \frac{2s-9}{s^2+6s+34}$

(iii) $L\left[\cosh(at)\cos(at)\right] = \frac{s^3}{s^4+4a^4}$

(iv) $L\left[e^{-t}\left(3\sinh 2t - 5\cosh 2t\right)\right] = \frac{1-5s}{s^2+2s-3}$

(v) $L[e^{2t}\cos^2 t] = \frac{1}{2}\left(\frac{1}{s-2} + \frac{s-2}{(s-2)^2+4}\right)$

8. Using the Second Shifting property for LT, prove the following results :

(i) $L\left[f(t)\right] = \frac{\omega}{s^2+\omega^2}\left[1+e^{-\frac{\pi s}{\omega}}\right]$, where $f(t) = \begin{cases} \sin\omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$

(ii) $L\left[t^2u(t-1)+\delta(t-1)\right] = \left(1+\frac{1}{s}+\frac{2}{s^2}+\frac{2}{s^3}\right)e^{-s}$

9. Using the derivative of the transform property, prove the following :

(i) $L\left[t\sin at\right] = \frac{2as}{(s^2+a^2)^2}$ (ii) $L\left[t^2\sin 2t\right] = \frac{4(3s^2-4)}{(s^2+4)^3}$

(iii) $L\left[te^{-t}\cos t\right] = \frac{s^2+2s}{(s^2+2s+2)^2}$ (iv) $L\left[t^n\right] = \frac{n!}{s^{n+1}}$, where n is +ve integer.

10. (a) Using the transform of the derivative property, find (i) $L[\sin(at)]$ (ii) $L[\cosh(at)]$

(b) If $L\left[\sin\sqrt{t}\right] = \dfrac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$, prove that $L\left[\dfrac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\dfrac{\pi}{s}} e^{-\frac{1}{4s}}$ (Hint : Use the transform of the derivative property).

11. Using the integral of the transform property, prove the following :

(i) $L\left[\dfrac{1-e^{-t}}{t}\right] = \log\left(\dfrac{s+1}{s}\right)$

(ii) $L\left[\dfrac{\cos(at)-\cos(bt)}{t}\right] = \dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$

(iii) $L\left[\dfrac{\cos(4t)\sin(2t)}{t}\right] = \dfrac{1}{2}\left[\tan^{-1}(s/2) - \tan^{-1}(s/6)\right]$

(iv) $L\left[\dfrac{\sin t}{t}\right] = \dfrac{\pi}{2} - \tan^{-1}(s) = \cot^{-1}(s)$

12. Using the transform of the integral property, find

(i) $L\left[\int_0^t \sin(t)dt\right]$

(ii) $L\left[\int_1^t \sin(t)dt\right]$ Hint : $L\left[\int_a^t f(t)dt\right] == \dfrac{1}{s}\left(F(s) + \int_a^0 f(t)dt\right)$

(iii) $L\left[\int_0^t \dfrac{\sin(u)}{u}du\right]$

(iv) $L\left[\int_0^t ue^{-u}\sin(4u)du\right]$ Ans : $\dfrac{s(s+1)}{s(s^2+2s+17)^2}$

13. Obtain the convolution function $f*g$, where $f(t) = e^{at}$ and $g(t) = e^{bt}$. Hence find $L\left[(f*g)(t)\right]$ using Convolution Theorem.

PERIODIC FUNCTION

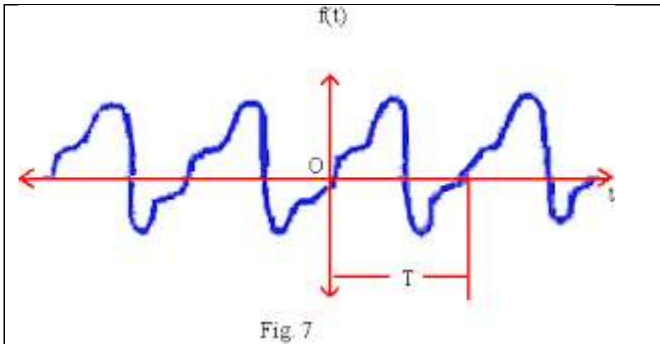
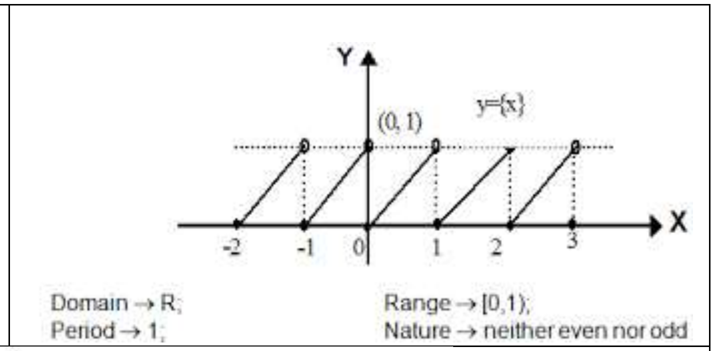
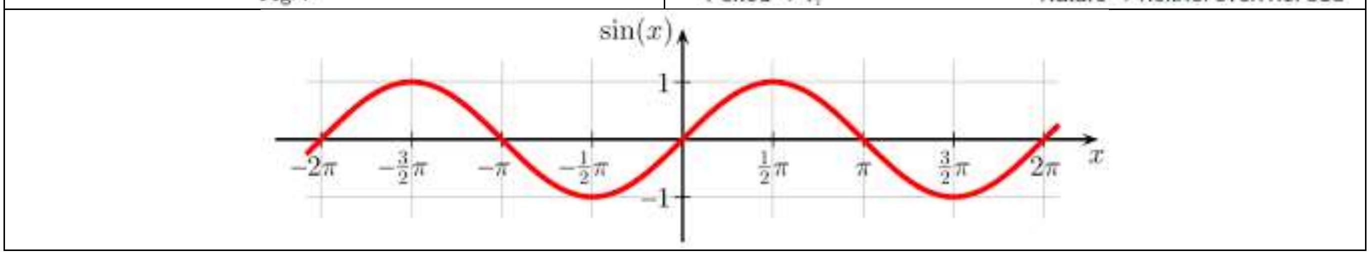


Fig. 7





14. (i) Find $L[f(t)]$, where $f(t)=\begin{cases} \sin\omega t, & 0<t<\pi/\omega \\ 0, & \pi/\omega<t<2\pi/\omega \end{cases}$ is a periodic function with period $2\pi/\omega$.

(ii) Find $L[f(t)]$, where $f(t)=\begin{cases} t, & 0<t<a \\ 2a-t, & a<t<2a \end{cases}$ is the triangular wave of period $2a$.

15. (A) If $L[f(t)]=\dfrac{3s^2+5s+2}{s^3+4s^2+2s}$, find $f(0)$ and $f(\infty)$.

(B) If $L\left[e^{-t}\sin t\right]=F(s)$, find $\lim\limits_{s\to\infty}[sF(s)]$ and $\lim\limits_{s\to 0}[sF(s)]$.

(C) If $L[f(t)]=\dfrac{1}{s}+\dfrac{1}{(s+1)^2+1}+\dfrac{s}{(s+1)^2+1}$, find $\lim\limits_{t\to 0}[f(t)]$ and $\lim\limits_{t\to\infty}[f(t)]$.

(D) Verify the Initial and Final Value Theorem’s for the following function f(t) :

(i) $e^{-t}(t+2)^2$

(ii) $1+e^{-t}(\sin t+\cos t)$

(iii) ae^{-bt}

(iv) $(2t-3)^2$

16. Apply Convolution Theorem to evaluate $L^{-1}\left[F(s)\right]$, where $F(s)$ is given by

(i) $\dfrac{s}{(s^2-a^2)}$ Ans : $\cosh at$

(ii) $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$ Ans : $\dfrac{1}{(a^2-b^2)}\left(a\sin at-b\sin bt\right)$

(iii) $\dfrac{s}{(s^2+a^2)^2}$ Ans : $\dfrac{t\sin at}{2a}$

(iv) $\dfrac{5s(s+1)}{(s^2+1)(s^2+2s+2)}$ Ans : $3\cos t+\sin t-3e^{-t}\cos t+e^{-t}\sin t$

(v) $\dfrac{1}{(s-2)(s+2)^2}$ Ans : $\dfrac{1}{16}\left(e^{2t}-e^{-2t}-4te^{-2t}\right)$

17. Find $L^{-1}\left[F(s)\right]$ using partial fractions method, where $F(s)$ is given by

(i) $\dfrac{1}{(s-a)(s+a)}$ Ans : $\dfrac{1}{a}\sinh at$

(ii) $\dfrac{s^3-4s+1}{s(s-1)^3}$ Ans : $-1+2e^t+te^t-t^2e^t$

(iii) $\dfrac{s-1}{s^2-6s+25}$ Ans : $\dfrac{e^{3t}}{2}\left(2\cos 4t+\sin 4t\right)$

(iv) $\dfrac{s+1}{s^2+4}$ Ans : $\cos 2t+\dfrac{\sin 2t}{2}$

(v) $\dfrac{2s^2-5}{(s+1)(s+2)}$ Ans : $2\delta(t)+\left(7e^{-t}-13e^{-2t}\right)u(t)$

(vi) $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$ Ans : $e^t-\dfrac{e^{-t}}{2}\left(2\cos 2t-3\sin 2t\right)$

18. Find $L^{-1}[F(s)]$, where $F(s)$ is given by

(i) $\log\left(\frac{s+b}{s+a}\right)$ Ans : $\frac{1}{t}\left(e^{-at}-e^{-bt}\right)$

(ii) $\tan^{-1}\left(\frac{1}{s}\right)$ Ans : $\frac{\sin t}{t}$

(iii) $s\log\left(\frac{s-1}{s+1}\right)$ Ans : $\frac{2}{t^2}\left(\sinh t - t\cosh t\right)$

(iv) $\cot^{-1}\left(\frac{a}{s}\right)$ Ans : $\frac{\sin at}{t}$

(v) $\tan^{-1}\left(\frac{s+2}{3}\right)$ Ans : $\frac{-1}{t}e^{-2t}\sin 3t$

19. Find $L^{-1}[F(s)]$, where $F(s) = \frac{s}{(s+6)^3}$

20. Find $L^{-1}[F(s)]$, where $F(s) = \frac{s}{(s^2+a^2)^2}$

21. Find $L^{-1}[F(s)]$, where $F(s) = \frac{1}{s(s+2)^3}$