Inverse LT using Partial fractions

① Find $L^{-1}\left[\frac{1}{(s-a)(s+a)}\right]$

Sol:

$$F(s) = \frac{1}{(s-a)(s+a)}$$

By PFM, let

$$F(s) = \frac{A_1}{s-a} + \frac{A_2}{s+a}$$

$$A_1 = \lim_{s \to a}\left[(s-a)F(s)\right]$$

$$= \lim_{s \to a}\left[\frac{1}{s+a}\right] = \frac{1}{2a}$$

$$A_2 = \lim_{s \to -a}\left[(s+a)F(s)\right]$$

$$= \lim_{s \to -a}\left[\frac{1}{s-a}\right] = \frac{-1}{2a}$$

$$\therefore \quad F(s) = \frac{1}{2a}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$L^{-1}\left[F(s)\right] = \frac{1}{2a}\left[e^{at} - e^{-at}\right] = \frac{1}{a}\sinh(at)$$

$$\therefore \quad L^{-1}\left[\frac{1}{(s-a)(s+a)}\right] = \frac{1}{a}\sinh(at) \quad //$$

(2) Find $L^{-1}\left[\dfrac{s^3-4s+1}{s(s-1)^2}\right]$

Sol: The given function $\dfrac{s^3-4s+1}{s(s-1)^2}$ is not a proper rational function.

By PFM, let

$$F(s)=\frac{s^3-4s+1}{s(s-1)^2}=1+\left[\frac{A_1}{s}+\frac{B_1}{s-1}+\frac{B_2}{(s-1)^2}\right]$$

$$F(s)=1+\frac{A_1}{s}+\frac{B_1}{s-1}+\frac{B_2}{(s-1)^2}$$

Equating numerator terms, we get

$$s^3-4s+1=s(s-1)^2+A_1(s-1)^2+B_1 s(s-1)+B_2 s$$

$s=0$: $\boxed{1=A_1}$

$s=1$: $\boxed{-2=B_2}$

Coef. of s^2: $0=-2+A_1+B_1=-1+B_1$

$\boxed{B_1=1}$

$$\therefore F(s)=1+\frac{1}{s}+\frac{1}{s-1}-\frac{2}{(s-1)^2}$$

Hence $L^{-1}[F(s)] = \delta(t) + 1 + e^t - 2te^t.$ //

(3) Find $L^{-1}\left[\dfrac{s}{(s-a)^2}\right]$

Sol:

Let $F(s) = \dfrac{s}{(s-a)^2}$

By PFM,

$$F(s) = \frac{A_1}{s-a} + \frac{A_2}{(s-a)^2}$$

$$A_2 = \lim_{s\to a}\left[(s-a)^2 F(s)\right] = \lim_{s\to a}[s] = a$$

$$A_1 = \lim_{s\to a} \frac{d}{ds}\left[(s-a)^2 F(s)\right]$$

$$= \lim_{s\to a} \frac{d}{ds}[s] = \lim_{s\to a}[1] = 1$$

$$\therefore F(s) = \frac{1}{s-a} + \frac{a}{(s-a)^2}$$

$$L^{-1}[F(s)] = e^{at} + ae^{at}L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{at}[1 + at]$$

④ Find $L^{-1}\left[\frac{1}{(s-a)^2}\right]$

<u>Sol:</u> By convolution Theorem,

$$L^{-1}\left[\frac{1}{(s-a)^2}\right] = e^{at} * e^{at}$$

$$= \int_0^t e^{au} \cdot e^{a(t-u)}\, du$$

$$= e^{at} \cdot \int_0^t du = te^{at} \,//$$

⑤ Find $L^{-1}\left[\dfrac{1}{(s-a)^3}\right]$

Sol:

By Convolution Theorem,

$$L^{-1}\left[\frac{1}{(s-a)^3}\right] = L^{-1}\left[\frac{1}{(s-a)^2}\right] * L^{-1}\left[\frac{1}{s-a}\right]$$

$$= te^{at} * e^{at}$$

$$= \int_0^t ue^{au} \cdot e^{a(t-u)}\,du$$

$$= e^{at}\int_0^t u\,du = \frac{t^2}{2}e^{at}.$$

$$\therefore L^{-1}\left[\frac{1}{(s-a)^3}\right] = \frac{t^2}{2}e^{at} \,//$$

(6) Find $L^{-1}\left[\frac{s-1}{s^2-6s+25}\right]$

Sol: Let $F(s)= \frac{s-1}{s^2-6s+25}$

$$= \frac{s-1}{[(s-3)^2-9]+25}$$

$$= \frac{s-1}{(s-3)^2+16}$$

$$F(s)= \frac{s-1}{(s-3)^2+4^2}$$

By PFM,

$$F(s)= \frac{A(s-3)+B(4)}{(s-3)^2+4^2}$$

Equating numerator terms, we get

$$s-1 = A(s-3)+4B$$

coef of s : $1 = A$

const. term : $-1 = -3A+4B = -3+4B$ $\Rightarrow$ $4B=2$

$\Rightarrow$ $B = 1/2$

$$\therefore F(s) = (1)\left[\frac{s-3}{(s-3)^2+4^2}\right] + \left(\frac{1}{2}\right)\left[\frac{4}{(s-3)^2+4^2}\right]$$

$$L^{-1}[F(s)] = e^{3t}\cos 4t + \frac{1}{2}e^{3t}\sin 4t \quad //$$