Z TRANSFORMS

1. Introduction

Z transform is a very powerful mathematical tool for solving difference equations, just as Laplace transforms and Fourier transforms are useful tools for solving differential equations. For example, a continuous-time system is described by a system of differential equations, whereas, a discrete-time system is governed by a set of difference equations.

It is used extensively today in the areas of applied mathematics, digital signal processing, control theory, population science, economics.

2. Definition of the Z transform

(a) One-sided or Unilateral Z transform of a sequence :

If $\{f(n)\}$ is a sequence defined for $n = 0,1,2,...,$ then the series $\sum_{n=0}^{\infty} f(n)z^{-n}$ is called the unilateral Z transform of the sequence $\{f(n)\}$ and is denoted by $Z\{f(n)\}$ or $\bar{f}(z)$, where z is a complex variable in general.

The series $\sum_{n=-\infty}^{\infty} f(n)z^{-n}$ will be convergent only for those values of z in a certain region of the Z-plane. This region in the Z-Plane is called the Region Of Convergence (R.O.C.) of the Z transform and it depends on the sequence $\{f(n)\}$.

(b) One-sided or Unilateral Z transform of a function :

If a continuous function $f(t)$ is defined by means of a sequence of its sampled values at $t = 0, T, 2T, ...,$ then the Z transform of the function $f(t)$ is given by

$\bar{f}(z) = Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)z^{-n}$, where T is the sampling period.

3. RELATION BETWEEN Z AND LAPLACE TRANSFORMS

The transform variables z and s in the Z transform and Laplace transform are related by

$z = e^{sT}$, where T is the sampling period of the discrete function.

4. Definition of the Inverse Z transform

If $\bar{f}(z) = Z\{f(n)\}$, then the Inverse Z transform of $\bar{f}(z)$, denoted by $Z^{-1}\{\bar{f}(z)\}$, is defined as

$Z^{-1}\left[\bar{f}(z)\right] = \{f(n)\} = \frac{1}{2\pi i}\oint_C z^{n-1}\bar{f}(z)dz$, where C is the circle whose centre is origin and radius is sufficiently large to include all isolated singularities of transform function $\bar{f}(z)$.

5. Definition of Convolution of two sequences

The convolution of two sequences $\{f(n)\}$ and $\{g(n)\}$, denoted by $\{f(n)*g(n)\}$, is defined as

$\{f(n)*g(n)\} = \sum_{r=0}^{n} f(r)g(n-r)$.

6. **Properties of Z transforms**

(P1) Linearity Property : Z transform operator is linear. That is,

$$Z\{af(n)+bg(n)\}=aZ\{f(n)\}+bZ\{g(n)\}$$

(P2) Scaling Property : If $Z\{f(n)\}=\bar{f}(z)$, then $Z\{a^n f(n)\}=\bar{f}\left(\frac{z}{a}\right)$

(P3) Shifting Property : If $Z\{f(n)\}=\bar{f}(z)$, then

(i) $Z\{f(n-k)\}=z^{-k}\bar{f}(z)$ (shifting to the right)

(ii) $Z\{f(n+k)\}=z^{k}\left[\bar{f}(z)-f(0)-\frac{f(1)}{z}-\frac{f(2)}{z^2}-\ldots-\frac{f(k-1)}{z^{k-1}}\right]$ (shifting to the left)

(P4) Differentiation in the Z domain : If $Z\{f(n)\}=\bar{f}(z)$, then $Z\{nf(n)\}=-z\frac{d}{dz}\left[\bar{f}(z)\right]$

(P5) <u>Initial Value Theorem</u> : If $Z\{f(n)\} = \bar{f}(z)$, then $f(0) = \lim_{z\to\infty} \bar{f}(z)$.

Corollary : If $f(0)$ is known, then the values of $f(1), f(2), \ldots,$ can be found as follows :

$f(1) = \lim_{z\to\infty}[z(\bar{f}(z) - f(0))]$, $f(2) = \lim_{z\to\infty}[z^2(\bar{f}(z) - f(0) - \frac{f(1)}{z})]$, and so on.

(P6) <u>Final Value Theorem</u> : If $Z\{f(n)\} = \bar{f}(z)$, then $f(\infty) = \lim_{z\to 1}\left[(z-1)\bar{f}(z)\right]$.

(P7) <u>Convolution Theorem</u> :

If $Z\{f(n)\} = \bar{f}(z)$ and $Z\{g(n)\} = \bar{g}(z)$, then $Z\{f(n) * g(n)\} = \bar{f}(z)\bar{g}(z)$, where the symbol $*$ denotes the convolution operation.

<u>Final Value Theorem</u> : If $Z\{f(n)\} = \bar{f}(z)$, then $f(\infty) = \lim_{z \to 1}\left[(z-1)\bar{f}(z)\right]$

Proof : By the definition of Z transform, we have

$$Z\{f(n+1) - f(n)\} = \sum_{n=0}^{\infty}[f(n+1) - f(n)]z^{-n}$$

$$= \lim_{k\to\infty}\sum_{n=0}^{k}[f(n+1) - f(n)]z^{-n}$$

$$= \lim_{k\to\infty}\left[-f(0) + f(1)\left\{1 - \frac{1}{z}\right\} + f(2)\left\{\frac{1}{z} - \frac{1}{z^2}\right\} + f(3)\left\{\frac{1}{z^2} - \frac{1}{z^3}\right\} + \cdots + f(k+1)z^{-k}\right]$$

Taking the limit of both sides as $z \to 1$, we get

$$\lim_{z\to 1} Z\{f(n+1) - f(n)\} = -f(0) + \lim_{k\to\infty} f(k+1) \qquad [1]$$

By shifting property, we have $Z\{f(n+1)\} = z\left(\bar{f}(z) - f(0)\right)$

$\therefore \quad Z\{f(n+1) - f(n)\} = (z-1)\bar{f}(z) - zf(0)$ [2]

Using [2] in [1], we get

$$\lim_{z\to 1}\left[(z-1)\bar{f}(z) - zf(0)\right] = -f(0) + \lim_{k\to\infty} f(k+1)$$

i.e, $\lim_{z\to 1}\left[(z-1)\bar{f}(z)\right] = \lim_{k\to\infty} f(k+1)$ or $\lim_{k\to\infty} f(k+1) = \lim_{z\to 1}\left[(z-1)\bar{f}(z)\right]$

$\Rightarrow f(\infty) = \lim_{z\to 1}\left[(z-1)\bar{f}(z)\right].$

PROPERTIES OF Z TRANSFORMS [TABULAR FORM]

S. No.	Name of the Property	$f(n)$	$\bar{f}(z)=Z\{f(n)\}$
1	Linearity	$af(n)+bg(n)$	$a\bar{f}(z)+b\bar{g}(z)$
2	Change of Scale	$a^n f(n)$	$\bar{f}(z/a)$
3	Shifting	$f(n-k)$	$z^{-k}\bar{f}(z)$
		$f(n+k)$	$z^k\left[\bar{f}(z)-f(0)-\frac{f(1)}{z}-\frac{f(2)}{z^2}-\dots-\frac{f(k-1)}{z^{k-1}}\right]$
4	Derivative of the transform	$nf(n)$	$-z\frac{d}{dz}[\bar{f}(z)]$
5	Convolution	$f(n)*g(n)$	$\bar{f}(z)\bar{g}(z)$
6	Initial Value Theorem	$f(0)=\underset{z\to\infty}{Lt}\left[\bar{f}(z)\right]$	
7	Final Value Theorem	$f(\infty)=\underset{z\to 1}{Lt}[(z-1)\bar{f}(z)]$	

7. **Methods for obtaining Inverse Z Transforms**

(i) Partial Fractions method : When $\bar{f}(z)$ is a rational function of the form $\bar{f}(z)=\dfrac{g(z)}{h(z)}$ in which the denominator can be factorized, $\bar{f}(z)$ is resolved into partial fractions and then $Z^{-1}\{\bar{f}(z)\}$ is derived as the sum of the inverse Z transforms of the partial fractions.

(ii) Convolution theorem : $Z^{-1}\left[\bar{f}(z)\bar{g}(z)\right]=Z^{-1}\left[\bar{f}(z)\right]*Z^{-1}\left[\bar{g}(z)\right]$

8. Z TRANSFORMS TO SOLVE DIFFERENCE EQUATIONS

If $Z\{y(n)\} = \overline{y}(z)$, then it can be shown that

(i) $Z\{y(n+1)\} = z(\overline{y}(z) - y(0))$,

(ii) $Z\{y(n+2)\} = z^2(\overline{y}(z) - y(0)) - zy(1)$,

(iii) $Z\{y(n+3)\} = z^3(\overline{y}(z) - y(0)) - z^2y(1) - zy(2)$, and so on.

We can use Z transforms to solve finite difference equation with constant coefficients.
Consider a second order difference equation with constant coefficients, of the form $ay(n+2) + by(n+1) + cy(n) = R(n)$, given $y(0)$ and $y(1)$. Applying Z transform, we get $\overline{y}(z)$. Then $\{y(n)\} = Z^{-1}(\overline{y}(z))$.

PROBLEMS UNDER Z TRANSFORMS

1. Find the Z transform of the following sequences :

i) Unit impulse sequence, $\delta(n-k)=\begin{cases}1 & \text{for} \quad n=k\\ 0 & \text{for} \quad n\neq k\end{cases}$ Ans : $\dfrac{1}{z^k}$

ii) Unit step sequence, $U(n-k)=\begin{cases}1 & \text{for} \quad n=k,k+1,k+2,\ldots\\ 0 & \text{for} \quad n<k\end{cases}$

Ans : $\dfrac{1}{z^{k-1}(z-1)}$; R.O.C. : $|z|>1$

iii) Constant sequence, k Ans : $\dfrac{kz}{z-1}$; R.O.C. : $|z|>1$

iv) a^n Ans : $\dfrac{z}{z-a}$; R.O.C. : $|z|>a$

v) $\dfrac{1}{n}$ Ans : $\log\left(\dfrac{z}{z-1}\right)$; R.O.C. : $|z|>1$

vi) $\dfrac{a^n}{n!}$ Ans : $e^{a/z}$

2. Find the Z transform of the following functions :

i) Unit impulse function, $\delta(t)=\begin{cases}1 & \text{for} \quad t=0\\ 0 & \text{for} \quad t=T,\ 2T,\ 3T,\ldots\end{cases}$ Ans : 1

ii) Unit step sequence, $U(t)=\begin{cases}1 & \text{for} \quad t=0, T, 2T,\ldots\\ 0 & \text{for} \quad t<0\end{cases}$ Ans : $\dfrac{z}{z-1}$; R.O.C. : $|z|>1$

3. Using the properties, find the Z transform of the following functions / sequences :

i) n, n^2, n^p, na^n, $\frac{1}{n+1}$, $\frac{1}{n-1}$ ii) a^{n-1}

iii) $n(n-1)$, $\frac{1}{2}(n+1)(n+2)$, an^2+bn+c, where a, b and c are constants

iv) $\frac{1}{n(n-1)}$, $\frac{2n+3}{(n+1)(n+2)}$

v) e^{an}, $r^n e^{in\theta}$, $r^n\cos(n\theta)$, $r^n\sin(n\theta)$, $\cos\left(\frac{n\pi}{2}\right)$, $\sin\left(\frac{n\pi}{2}\right)$, $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$, $\sin^2\left(\frac{n\pi}{4}\right)$

vi) $a^n\cosh(bn)$, $a^n\sinh(bn)$ vii) $n\cos(n\theta)$, $a^n\sin(a\theta)$

viii) Find $f(0)$ and $f(\infty)$, if $\bar{f}(z)=\frac{0.4z^2}{(z-1)(z^2-0.736z+0.136)}$ Ans : 0, 1

ix) Find $f(2)$ and $f(3)$, if $\bar{f}(z)=\frac{2z^2+3z+12}{(z-1)^4}$ Ans : 2, 11

4. Find the Z transform of $f(n)*g(n)$, where

i) $f(n)=U(n)$ and $g(n)=\delta(n)+\left(\frac{1}{2}\right)^n U(n)$

ii) $f(n)=\left(\frac{1}{2}\right)^n$ and $g(n)=\cos(n\pi)$

5. Given that $\bar{f}(z)=\log\left(1+\frac{a}{z}\right)$ for $|z|>|a|$, find $f(n)$ and $Z\{nf(n)\}$.

6. Find $Z^{-1}\left[\bar{f}(z)\right]$ by partial fractions method, when $\bar{f}(z)$ is given as

i) $\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}$ ii) $\frac{z^2+2z}{z^2+2z+4}$

iii) $\frac{4z^3}{(2z-1)^2(z-1)}$ iv) $\frac{z^2}{(z+2)(z^2+4)}$

v) $\frac{4-8z^{-1}+6z^{-2}}{(1+z^{-1})(1-2z^{-1})^2}$

7. Find $Z^{-1}\left[\bar{f}(z)\right]$ by convolution theorem, when $\bar{f}(z)$ is given as

i) $\frac{z^2}{(z-a)(z-b)}$ iv) $\frac{8z^2}{(4z+1)(2z-1)}$

ii) $\frac{az}{(z-a)^2}$ v) $\frac{1}{z^2+a^2}$

iii) $\frac{z}{(z-a)^2}$ vi) $\frac{z^2}{z^2+a^2}$

8. Solve the following difference equations, using Z transforms :

i) $y(n+3)-3y(n+1)+2y(n)=0$ given that $y(0)=y(1)=0$ and $y(2)=8$

ii) $f(n)+3f(n-1)-4f(n-2)=0$ given that $f(0)=3$ and $f(1)=-2$

iii) $y_{n+2}-7y_{n+1}+12y_n=2^n$ given that $y_0=y_1=0$

iv) $x_{n+2}-5x_{n+1}+6x_n=36$ given that $x_0=x_1=0$

v) $y(x+2)+4y(x+1)=4y(x)=x$ given that $y(0)=0$ and $y(1)=1$

vi) $y_{n+2}+y_n=n2^n$

vii) $4u_n-u_{n+2}=0$ given that $u_0=0$ and $u_1=2$

LIST OF IMPORTANT FORMULAS

(i) $(1-x)^{-1}=1+x+x^2+...$ if $|x|<1$

(ii) $(1+x)^{-1}=1-x+x^2-x^3+...$ if $|x|<1$

(iii) $(1-x)^{-2}=1+2x+3x^2+4x^3+...$ if $|x|<1$

(iv) $(1+x)^{-2}=1-2x+3x^2-4x^3+...$ if $|x|<1$

(v) $1+x+x^2+...+x^n=\dfrac{1-x^{n+1}}{1-x}$ if $|x|<1$

(vi) $x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+....=\log(1+x)$ if $|x|<1$

(vii) $x+\dfrac{x^2}{2}+\dfrac{x^3}{3}+\dfrac{x^4}{4}+....=-\log(1-x)$ if $|x|<1$

(viii) $1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+....=e^x$

Summation	Definite sum
$\sum_{k=0}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=0}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=0}^{n} k^3$	$\left[\frac{n(n+1)}{2}\right]^2$
$\sum_{k=0}^{n} k^4$	$\frac{n(6n^4+15n^3+10n^2-1)}{30}$
$\sum_{k=0}^{n} a^k$	$\begin{cases} (a^{n+1}-1)/a-1 & \text{if } a \neq 1 \\ n+1 & \text{if } a = 1 \end{cases}$
$\sum_{k=0}^{n} ka^k, \quad a \neq 1$	$\frac{(a-1)(n+1)a^{n+1}-a^{n+2}+a}{(a-1)^2}$

TABLE OF Z TRANSFORM PAIRS

S. No.	$f(n)$	$\bar{f}(z) = Z\{f(n)\}$	Region of convergence
1	$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$	1	
2	$\delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases}$	$\frac{1}{z^k}$	
3	$u(n) = \begin{cases} 1 & \text{for } n = 0,1,2,\ldots \\ 0 & \text{for } n < 1 \end{cases}$	$\frac{z}{z-1}$	$\|z\| > 1$
4	$u(n-k) = \begin{cases} 1 & \text{for } n = k, k+1,\ldots \\ 0 & \text{for } n < k \end{cases}$	$\frac{1}{z^{k-1}(z-1)}$	
5	k	$\frac{kz}{z-1}$	$\|z\| > 1$
6	1	$\frac{z}{z-1}$	
7	a^n	$\frac{z}{z-a}$	$\|z\| > \|a\|$
8	$\frac{a^n}{n!}$	$e^{a/z}$	
9	$\frac{1}{n}$	$\log\left[\frac{z}{z-1}\right]$	$\|z\| > 1$
10	a^{n-1}	$\frac{1}{z-a}$	
11	$(n-1)a^{n-2}$	$\frac{1}{(z-a)^2}$	
12	$a^n \sin\left(\frac{n\pi}{2}\right)$	$\frac{az}{z^2+a^2}$	
13	$a^n \cos\left(\frac{n\pi}{2}\right)$	$\frac{z^2}{z^2+a^2}$	
14	$-a^{n-2} \cos\left(\frac{n\pi}{2}\right)$	$\frac{1}{z^2+a^2}$	
15	na^n	$\frac{az}{(z-a)^2}$	

RELATIONSHIP BETWEEN LAPLACE TRANSFORM AND Z TRANSFORM

Let $x_d(t)$ denote the discrete-time signal obtained from a continuous-time signal $x_c(t)$ with sampling period T.

In terms of the Dirac-delta function, we can write the discrete-time signal as

$$x_d(t) = \sum_{n=0}^{\infty} x_c(nT)\delta(t-nT)$$

Applying Z transform to the discrete signal $x_d(t)$, we get

$$Z[x_d(t)] = \sum_{n=0}^{\infty} x_d(nT)z^{-n} \ , \qquad [1]$$

where z is the transform variable in the Z transform domain.

Applying Laplace Transform to the discrete signal $x_d(t)$, we get

$$L[x_d(t)] = \int_0^{\infty} e^{-st}[\sum_{n=0}^{\infty} x_c(nT)\delta(t-nT)]dt = \sum_{n=0}^{\infty} x_c(nT)\int_0^{\infty} e^{-st}\delta(t-nT)dt = \sum_{n=0}^{\infty} x_d(nT)e^{-snT} \ , \qquad [2]$$

where s is the transform variable in the Laplace transform domain.

By substituting $z = e^{sT}$ and comparing with [1], we get

$$L[x_d(t)] = Z\{x_d(t)\} \qquad [3]$$

Applying Laplace Transform to the continuous signal $x_c(t)$ and using [2], we get

$$L[x_c(t)] = \int_0^{\infty} e^{-st}x_c(t)dt \approx \sum_{n=0}^{\infty} e^{-st_n}x_c(t_n)T$$

$$\approx \sum_{n=0}^{\infty} e^{-snT}x_d(nT)T = TL[x_d(t)]$$

By using [3], we get $L[x_c(t)] \approx TZ[x_d(t)]$ [4]

Note(s) :

- **Z transform is the discrete analogue of the Laplace transform. The transform variables in both transform domains are related by** $z = e^{sT}$.

- **The Laplace transform of a discrete-time signal can be obtained from the corresponding Z transform of the signal, by substituting the transform variable** z **in the Z transform domain with** e^{sT}**.**

- **As** $T \to 0$**,** $x_d(t) \to x_c(t)$ **and** $T.Z\{x_d(t)\} \to L[x_c(t)]$**.**