TRANSFORMS OF DERIVATIVES

If $f'(t)$ be continuous and $L[f(t)] = F(s)$, then

$$L[f'(t)] = sF(s) - f(0)$$

$$L[f''(t)] = s^2 F(s) - s f(0) - f'(0)$$

$$L[f'''(t)] = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$$

$$\vdots$$

$$L[f^n(t)] = s^n F(s) - s^{n-1} f(0) - \cdots - f^{n-1}(0)$$

TRANSFORMS OF INTEGRALS

If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(u)\,du\right] = \frac{1}{s} . F(s)$

Multiplication by t^n

If $L[f(t)] = F(s)$ then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$

where $n = 1, 2, 3, \ldots$

$n = 1$ $\quad L[t\, f(t)] = (-1) \frac{d}{ds} F(s)$

$L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} F(s)$

Division by t

If $L[f(t)] = F(s)$, then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\, ds$

provided the integral exists..

Problems

1) Find $L[t^2 \cos at]$

$$L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} F(s)$$

$$F(s) = L[\cos at] = \frac{s}{s^2+a^2}$$

$$\frac{d^2}{ds^2}F(s) = \frac{d}{ds}\left\{\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)\right\}$$

$$= \frac{d}{ds}\left\{\frac{(s^2+a^2)(1) - s(2s)}{(s^2+a^2)^2}\right\}$$

$$= \frac{d}{ds}\left\{\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right\}$$

$$= \frac{d}{ds}\left\{\frac{a^2-s^2}{(s^2+a^2)^2}\right\}$$

$$= \frac{(s^2+a^2)^2(-2s) - (a^2-s^2)\,2(s^2+a^2)(2s)}{(s^2+a^2)^4}$$

$$= \frac{2s^3-6a^2s}{(s^2+a^2)^4}$$

2) $L[t e^{-t} \sin 2t]$

$= F(s+1)$

$$\{L[t \sin 2t]\}_{s \to s+1} = \left\{-\frac{d}{ds} L[\sin 2t]\right\}_{s \to s+1}$$

$$= \left\{-\frac{d}{ds}\left(\frac{2}{s^2+4}\right)\right\}_{s \to s+1}$$

$$= -\left[\frac{(s^2+4)(0) - 2(2s)}{(s^2+4)^2}\right]_{s \to s+1}$$

$$= \left[\frac{4s}{(s^2+4)^2}\right]_{s \to s+1}$$

$$= \frac{4(s+1)}{((s+1)^2+4)^2}$$

3. Find $L\left[\frac{e^{-at}-e^{-bt}}{t}\right]$

Division by t

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$$

$$L\left[e^{-at}-e^{-bt}\right] = F(s)$$

$$\Rightarrow \frac{1}{s+a} - \frac{1}{s+b}$$

$$\Rightarrow L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty = \left[\log \frac{\cancel{s}\left(1+\frac{a}{s}\right)}{\cancel{s}(1+b/s)}\right]_s^\infty$$

$$= \log\left(\frac{s+b}{s+a}\right)$$

4. $L\left[\int_0^t e^t \frac{\sin t}{t}\, dt\right]$

Transforms of Integrals

$$L\left[\int_0^t f(u)\, du\right] = \frac{1}{s} F(s)$$

$$L\left[\int_0^t e^t \frac{\sin t}{t}\, dt\right] = \frac{1}{s} L\left[e^t \frac{\sin t}{t}\right] \quad \text{--- (1)}$$

$$L\left[e^t \frac{\sin t}{t}\right] = \left\{L\left[\frac{\sin t}{t}\right]\right\}_{s \to s-1}$$

$$= \left\{\int_s^\infty L[\sin t]\, ds\right\} s \to s-1$$

$$= \left\{ \int_s^{\infty} L[\sin t]\, ds \right\}_{s \to s-1}$$

$$= \left\{ \int_s^{\infty} \frac{1}{s^2+1}\, ds \right\}_{s \to s-1}$$

$$= \int_s^{\infty} \frac{1}{(s-1)^2+1}\, ds$$

$$= \left[\tan^{-1}(s-1)\right]_s^{\infty} = \frac{\pi}{2} - \tan^{-1}(s-1)$$

$$= \cot^{-1}(s-1)$$

$$= \frac{1}{s}\cot^{-1}(s-1)$$

Scanned with CamScanner

Evaluate

5. $$\int_0^\infty \frac{e^{-t}\sin^2 t}{t}\,dt$$

$$\int_0^\infty e^{-t}\,\frac{\sin^2 t}{t}\,dt = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \Big[F(s)\Big]_{s=1}$$

$$F(s) = L\left[\frac{\sin^2 t}{t}\right]$$

$$= \int_s^\infty L[\sin^2 t]\,ds$$

$$= \int_s^\infty L\left[\frac{1-\cos 2t}{2}\right] ds$$

$$= \frac{1}{2} \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4} \right) ds$$

$$= \frac{1}{2} \left[\log s - \frac{1}{2} \log (s^2+4) \right]_s^\infty$$

$$= \frac{1}{2} \log \frac{\sqrt{s^2+4}}{s}$$

$\underline{\underline{s=1}}$ $\Rightarrow$ $\int_0^\infty e^{-t} \frac{\sin^2 t}{t} dt = \frac{1}{4} \log 5.$