Solution of PDE

PDE can be of the form

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0$$

Soln. of a PDE will be of the form $f(x, y, z, a, b) = 0$

Types of Solution

1. Complete solution
2. Particular solution
3. Singular solution
4. General solution

Complete soln

PDE: $F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0$

then soln has two arbitrary fn and it is of the form $f(x, y, z, a, b) = 0$ is a COMPLETE SOLUTION

Particular soln.

PDE: $F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0$

In complete soln. the particular values of arbitrary constants are obtained then it becomes a particular solution.

Singular solution PDE: $F(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) = 0$

complete soln is $f(x, y, z, a, b) = 0$ — ①

diff ① w.r.to 'a' and equate to 'zero'

$$\frac{\partial f}{\partial a} = 0 \quad — ②$$

diff ① w.r.to 'b' and equate to 'zero'

$$\frac{\partial f}{\partial b} = 0 \quad — ③$$

Now, using ①, ②, ③ eliminate the arbitrary constants 'a' and 'b' then we will get a singular solution.

General solution PDE: $F(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) = 0$

Soln. g a PDE : $f(x, y, z, a, b) = 0$ — ①

COMPLETE SOLN.

In ① consider $a = \phi(b)$.

then ① $\Rightarrow$ $f(x, y, z, \phi(b), b) = 0$

(or) consider $b = \phi(a)$

then ① $\Rightarrow$ $f(x, y, z, a, \phi(a)) = 0$ — ②

diff ② partially w.r.to 'a' and equate it to 'zero'

$$\frac{\partial}{\partial a}\{ f(x, y, z, a, \phi(a)) \} = 0 \quad — ③$$

Eliminate 'a' using ② & ③, we will give a general solution.

NON-LINEAR PDE

Form I: $f(p,q)=0$ —① eqn. consists of p and q only

here $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

The soln of ① is $z = ax + by + c$

diff ① w.r.to 'x' $\frac{\partial z}{\partial x} = a \Rightarrow p = a$

diff ① w.r.to 'y' $\frac{\partial z}{\partial y} = b \Rightarrow q = b$

① becomes $f(a,b)=0$

a in terms of b

or b in terms of a

then substitute in the trial soln which will be the complete soln.

Solve

① $p+q=1$ —①

Replace p by a & q by b in ①

① $\Rightarrow a+b=1$

$$\boxed{b = 1-a}$$

The complete soln. of ① is $z = ax + by + c$

$\Rightarrow z = ax + (1-a)y + c$ here a and c are arbitrary constants.

Form I: $f(p,q) = 0$. — ①

Replace p by a, q by b in the given eqn ①

① $\Rightarrow f(a,b) = 0$.

a in terms of b or b in terms of a.

$a = \phi(b)$

$\therefore$ The complete soln is $z = \phi(b)x + by + c$

Eg: $\sqrt{p} + \sqrt{q} = 1$ — ① It comes under form I: $f(p,q) = 0$.

Replace p by a, q by b in ①

① $\Rightarrow \sqrt{a} + \sqrt{b} = 1 \Rightarrow \sqrt{b} = 1 - \sqrt{a}$

$b = (1-\sqrt{a})^2$

The complete soln is $z = ax + by + c$

$$\boxed{z = ax + (1-\sqrt{a})^2 y + c}$$

here a, c are arbitrary constants

Solve $p + q = pq$ — ① It comes under form I $f(p,q) = 0$.

Replace p by a, q by b in ①

$$a + b = ab$$

$$b = ab - a$$

$$b = a(b-1) \Rightarrow \boxed{a = \frac{b}{b-1}}$$

Now, The complete soln is $z = ax + by + c$

$$\Rightarrow z = \frac{b}{b-1}x + by + c \Rightarrow \text{complete soln.}$$ — ②

<u>Singular soln.</u>

diff ② w.r.to b and equate to 0.

diff ② w.r.to c and equate to 0.

$$\frac{\partial z}{\partial c} = 1 = 0 \text{ is impossible}$$

$1 \neq 0$ ∴ Singular soln doesn't exist

<u>General soln</u> In ② consider $c = \phi(b)$

$$② \Rightarrow z = \frac{b}{b-1}x + by + \phi(b) \quad \text{— } (2^*)$$

diff (2^*) w.r.to b & equate to 0

$$0 = \frac{\partial}{\partial b}\left\{ \frac{b}{b-1}x + by + \phi(b) \right\} \quad \text{— (3)}$$

eliminating 'b' using (2*) & (3) will give the general soln.

Solve.

$p+q=1$ — (1)

complete soln will be of the form $z = ax+by+c$

Replace p by a & q by b (1)

$(1) \Rightarrow \quad a+b=1 \Rightarrow b=(1-a)$

$\therefore$ complete soln is $z = ax+(1-a)y+c$ //

solve $p - q = 1$ — ① This is of the form $f(p,q)=0$

Replace p by a and q by b in ①

$\Rightarrow a - b = 1$

$\Rightarrow \boxed{a = 1+b}$

Trial soln is $z = ax + by + c$

$\therefore$ complete soln is $z = (1+b)x + by + c$

Singular soln

Consider the complete soln. $z = (1+b)x + by + c$

b & c are arbitrary constants

i.e $\frac{\partial z}{\partial b} = 0$

$\frac{\partial z}{\partial c} = 0$

$\frac{\partial z}{\partial b} = 0 = x + y$

$\frac{\partial z}{\partial c} = 0 = 1 \Rightarrow 0 = 1$ is impossible

$\therefore$ Singular soln doesn't exist

General soln

Consider complete soln.

$z = (1+b)x + by + c$

$c = f(b)$

$\Rightarrow z = (1+b)x + by + f(b)$ — ①

$\frac{\partial z}{\partial b} = 0$

$\frac{\partial z}{\partial b} = 0 = x + y + f'(b)$ — ②

Using ① & ②, eliminate 'b' to get the general soln.

Form I : $F(p,q)=0$ Eqn. containing p and q only

Complete soln.

There is no singular soln.

General soln will exist

Form II : $F(z,p,q)=0$ Eqn is not containing x and y

$z \downarrow u$ $\quad u = x+ay \Rightarrow \frac{\partial u}{\partial x} = 1$

$$\frac{\partial u}{\partial y} = a$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x}$$

$$p = \frac{dz}{du} \cdot 1 = \frac{dz}{du} \Rightarrow \boxed{p = \frac{dz}{du}}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du} a$$

$$\Rightarrow \boxed{q = a \cdot \frac{dz}{du}}$$

Given PDE $F(z,p,q)=0$

becomes $F\left(z, \frac{dz}{du}, a\frac{dz}{du}\right)=0$ which is an ordinary differential eqn

Solve $z^2(p^2+q^2+1)=1$

This is of the form $F(z,p,q)=0$ (form: II)

Assume $z =$ fn. of u

where $u = x+ay$

$$p = \frac{dz}{du}$$

$$q = a\cdot\frac{dz}{du}$$

Substitute p, q in the given eqn.

$$z^2(p^2+q^2+1)=1$$

$$\Rightarrow z^2\left(\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2+1\right)=1$$

$$z^2\left(\underbrace{\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2}+1\right)=1$$

$$z^2\left(\left(\frac{dz}{du}\right)^2(1+a^2)\right)+z^2=1$$

$$z^2\left(\frac{dz}{du}\right)^2(1+a^2)=1-z^2$$

Taking square root on both sides,

$$\left(z^2\left(\frac{dz}{du}\right)^2(1+a^2)\right)^{1/2}=(1-z^2)^{1/2}$$

$$z\left(\frac{dz}{du}\right)\sqrt{1+a^2}=\sqrt{1-z^2}$$

$$\sqrt{1+a^2}\,\frac{z\,dz}{\sqrt{1-z^2}}=du$$

Integrating, $$\int\sqrt{1+a^2}\,\frac{z\,dz}{\sqrt{1-z^2}}=\int du$$

$$\sqrt{1+a^2}\int\frac{-2}{-2}\frac{z\,dz}{\sqrt{1-z^2}}=\int du$$

$$-\frac{\sqrt{1+a^2}}{2}\int\frac{(-2z)\,dz}{\sqrt{1-z^2}}=u+c$$

$$-\frac{\sqrt{1+a^2}}{2}\int\frac{(-2z)\,dz}{\sqrt{1-z^2}} = u+c$$

$$-\frac{\sqrt{1+a^2}}{2}\int\underbrace{(-2z)(1-z^2)}^{-1/2}dz = u+c$$

$$-\frac{\sqrt{1+a^2}}{2}\;\frac{(1-z^2)^{-1/2+1}}{-1/2+1} = u+c$$

$$-\frac{\sqrt{1+a^2}}{2}\;\frac{(1-z^2)^{1/2}}{1/2} = u+c$$

$$-\frac{\sqrt{1+a^2}}{\cancel{2}}\;\cancel{2}(1-z^2)^{1/2} = u+c$$

$$-\sqrt{1+a^2}\,\sqrt{1-z^2} = x+ay+c$$

The complete soln is $\sqrt{1-z^2} = \frac{-1}{\sqrt{1+a^2}}(x+ay+c)$

here a, c are arbitrary consts.

Singular Solution

consider the complete soln.

$$\sqrt{1-z^2} = \frac{-1}{\sqrt{1+a^2}}(x+ay+c) \quad \text{—} \; ①$$

diff ① partially w.r.to a and c respectively and equate it to 0.

$$\frac{\partial z}{\partial a} = 0 = \frac{\partial}{\partial a}\left(\frac{-1}{\sqrt{1+a^2}}(x+ay+c)\right)$$

$$\frac{\partial z}{\partial c} = 0 = \frac{-1}{\sqrt{1+a^2}}(1) \quad \text{which is not possible} \quad \Rightarrow 0 = 1$$

$\therefore$ There is no singular solution

General solution

consider the complete solution

$$\sqrt{1-z^2} = \frac{-1}{\sqrt{1-a^2}}(x+ay+c)$$

. here a & c are arbitrary constants.

put $c = \phi(a)$

$$\sqrt{1-z^2} = \frac{-1}{\sqrt{1-a^2}}(x+ay+\phi(a)) \quad \text{—} \; ②$$

diff ② partially w.r.to a and equate to '0'

$$\frac{\partial z}{\partial a} = 0 = \frac{\partial}{\partial a}\left(\frac{-1}{\sqrt{1-a^2}}(x+ay+\phi(a))\right) \quad \text{—} \; ③$$

Using ② and ③, eliminate 'a' we will get the general solution.

2) Solve $z = p^2 + q^2$

This is of the form $F(z, p, q) = 0$ (Form: $\underline{\text{II}}$)

Assume $z \downarrow u$ where $u = x + ay$

$p = \frac{dz}{du}$, $q = a \cdot \frac{dz}{du}$

substitute p and q in the given eqn.

$$z = \left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2$$

$$z = \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2$$

$$z = \left(\frac{dz}{du}\right)^2 (1+a^2)$$

Taking square root on both sides

$$\sqrt{z} = \left(\left(\frac{dz}{du}\right)^2 (1+a^2)\right)^{1/2}$$

$$\sqrt{z} = \left(\frac{dz}{du}\right)\sqrt{1+a^2}$$

$$du = \frac{dz}{\sqrt{z}} \sqrt{1+a^2}$$

Integrating, $\int du = \int \frac{dz}{\sqrt{z}} \sqrt{1+a^2}$

$$u + c = \sqrt{1+a^2} \int \frac{dz}{\sqrt{z}}$$

$$u + c = \sqrt{1+a^2} \int z^{-1/2} dz$$

$$u + c = \sqrt{1+a^2} \; \frac{z^{-1/2+1}}{-1/2+1}$$

$$u + c = \sqrt{1+a^2} \; \frac{z^{1/2}}{1/2}$$

$$u + c = 2\sqrt{1+a^2} \sqrt{z}$$

$$x + ay + c = 2\sqrt{1+a^2} \sqrt{z}$$

$$\Rightarrow \sqrt{z} = \frac{x+ay+c}{2\sqrt{1+a^2}}$$

$$z = \frac{(x+ay+c)^2}{4(1+a^2)}$$ is the complete soln.

here a & c are arbitrary constants

Singular soln.

There is no singular soln.

General soln consider the complete soln.

$$z = \frac{(x+ay+c)^2}{4(1+a^2)}$$

put $C = \phi(a)$

$$\Rightarrow z = \frac{(x+ay+\phi(a))^2}{4(1+a^2)} \quad \text{--- } ①$$

diff. ① partially w.r.to a and equate it to zero

$$\frac{\partial z}{\partial a} = 0 = \frac{\partial}{\partial a}\left(\frac{(x+ay+\phi(a))^2}{4(1+a^2)}\right) \quad \text{--- } ②$$

Using ① & ② eliminate 'a' to get the general solution.

3) Find the complete integral of $p(1+q^2) = q(z-a)$

this comes under Form: II $F(z,p,q)=0$.

Assume $z \downarrow u$ where $u = x+by$

$$p = \frac{dz}{du}, \quad q = b \cdot \frac{dz}{du}$$

substitute p. q is the given eqn

$$p(1+q^2) = q(z-a)$$

$$\cancel{\frac{dz}{du}}\left(1+\left(b\frac{dz}{du}\right)^2\right) = b\cancel{\frac{dz}{du}}(z-a)$$

$$1 + b^2\left(\frac{dz}{du}\right)^2 = b(z-a)$$

$$b^2\left(\frac{dz}{du}\right)^2 = bz-ab-1$$

Taking square root on both sides

$$b\frac{dz}{du} = \sqrt{bz-ab-1}$$

$$\frac{b\,dz}{\sqrt{bz-(ab+1)}} = du$$

Integrating. $$\int \frac{b\,dz}{\sqrt{\underline{\underline{bz}}-(ab+1)}} = \int du$$

$$\int b\left(bz-(ab+1)\right)^{-1/2} dz = u + c$$

$$\Rightarrow \quad 2b\sqrt{bz-(ab+1)} = x+by+c$$

$$\sqrt{bz-(ab+1)} = \frac{x+by+c}{2b}$$ is the complete soln.

here b, c are arbitrary consts.

Form III $\quad F(x,y,p,q)=0 \quad z$ is absent.

Rewrite the given PDE like

$$f(x,p) = g(y,q)$$

Assume the trial soln as

$$f(x,p) = g(y,q) = a \text{ (say)}$$

$$f(x,p) = a \quad , \quad g(y,q) = a$$

$$p = \phi(x) \quad , \quad q = \psi(y)$$

Total differential $dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$

$$dz = p \cdot dx + q \cdot dy$$

$$dz = \phi(x) dx + \psi(y) dy$$

Integrating, $\int dz = \int \phi(x) dx + \int \psi(y) dy + c$

which is a complete soln.

$$\Rightarrow z = \int \phi(x) dx + \int \psi(y) dy + c$$

$F(x, p, q) = 0$ (or $F(y, p, q) = 0$

$\Rightarrow$

$f(x, p) = g(q) = a$ $f(p) = g(y, q) = a$

SINGULAR SOLUTION There is no singular soln.

$F(x, p, q) = 0$
$f(x, p) = g(q) = a$
$F(y, p, q) = 0$
$f(p) = g(y, q) = a$

1) Solve $p^2 + q^2 = x + y$

This is of the form $F(x, y, p, q) = 0$

$$p^2 + q^2 = x + y$$

$$\Rightarrow p^2 - x = y - q^2$$

The trial soln is $p^2 - x = y - q^2 = a$

$p^2 - x = a, \qquad y - q^2 = a$

$p^2 = a + x \qquad q^2 = y - a$

$p = \sqrt{a+x} \qquad q = \sqrt{y-a}$

substitute p and q in the total differential

$$dz = p\,dx + q\,dy$$

$$dz = \sqrt{a+x}\,dx + \sqrt{y-a}\,dy$$

Integrating, $\int dz = \int \sqrt{a+x}\,dx + \int \sqrt{y-a}\,dy + c$

$$z = \int (a+x)^{1/2} dx + \int (y-a)^{1/2} dy + c$$

$$= \frac{(a+x)^{1/2+1}}{1/2+1} + \frac{(y-a)^{1/2+1}}{1/2+1} + c$$

$$= \frac{(a+x)^{3/2}}{3/2} + \frac{(y-a)^{3/2}}{3/2} + c$$

$\Rightarrow \quad z = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(y-a)^{3/2} + c$ is a complete solution

where a, c are arbitrary constants.

Singular solution consider the complete soln.

$$z = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(y-a)^{3/2} + c \quad \text{—} \quad (1)$$

diff. (1) partially w.r.to a and c and then equate it to 0.

$$\frac{\partial z}{\partial a} = 0 = \frac{\partial}{\partial a}\left(\frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + c\right)$$

$$\frac{\partial z}{\partial c} = 0 = \frac{\partial}{\partial c}\left(\frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + c\right)$$

$0 = 1$ which is impossible $\therefore$ There is no singular soln.

soln.

General solution consider the complete soln.

$$z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + c$$

Put $c = \phi(a)$

$$\Rightarrow \quad z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + \phi(a) \quad \text{—} \quad (2)$$

diff (2) w.r.to a and equate to zero

$$\frac{\partial z}{\partial a} = 0 = \frac{\partial}{\partial a}\left(\frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + \phi(a)\right) - (3)$$

Using (2) and (3), 'a' can be eliminated to get the general solution.

2) Solve $\sqrt{p} + \sqrt{q} = 2x$

This is of the form $F(x, p, q) = 0$

Rewrite the given eqn of the form

$$f(x, p) = g(q)$$

consider $\sqrt{p} + \sqrt{q} = 2x$

$$\Rightarrow \sqrt{p} - 2x = -\sqrt{q}$$

Assume the trial soln. as $\sqrt{p} - 2x = -\sqrt{q} = a$

$$\sqrt{p} - 2x = a, \quad -\sqrt{q} = a$$

$$\sqrt{p} = a + 2x, \quad \sqrt{q} = -a$$

$$p = (a+2x)^2 \qquad q = (-a)^2$$

substitute p and q in the total differentice

$$dz = p\,dx + q\,dy$$

$$dz = (a+2x)^2\,dx + (-a)^2\,dy$$

Integrating,

$$\int dz = \int (a+2x)^2\,dx + \int (-a)^2\,dy + c$$

$$z = \frac{(a+2x)^3}{2 \cdot 3} + (-a)^2 y + c$$

$$z = \frac{(a+2x)^3}{6} + (-a)^2 y + c$$ which is a complete soln. where a, c are arbitrary constants.

Singular Solution

There is no singular soln. $\because \frac{\partial z}{\partial c} = 0 \Rightarrow 0 = 1$ which is not possible

General solution

Consider the complete soln.

$$z = \frac{(a+2x)^3}{6} + (-a)^2 y + c \quad -(1)$$

Put $c = \phi(a)$

$$\Rightarrow z = \frac{(a+2x)^3}{6} + (-a)^2 y + \phi(a) \quad -(2)$$

diff (2) w.r.to a and equate to 0

$$\frac{\partial z}{\partial a} = 0 = \frac{\partial}{\partial a}\left(\frac{(a+2x)^3}{6} + (-a)^2 y + \phi(a)\right) \quad -(3)$$

Using (2), (3) 'a' can be eliminated to get the general soln.

3) Find the complete integral of $p^2+q^2 = x^2+y^2$

This is of the form $F(x, y, p, q) = 0$

$$p^2+q^2 = x^2+y^2$$

$$p^2-x^2 = y^2-q^2$$

Trial soln is $p^2 - x^2 = y^2 - q^2 = a$

$p^2 - x^2 = a$, $y^2 - q^2 = a$

$p^2 = a + x^2$ $\quad$ $q^2 = y^2 - a$

$p = \sqrt{a + x^2}$ $\quad$ $q = \sqrt{y^2 - a}$

consider the total differential $dz = p\,dx + q\,dy$

$$dz = \sqrt{x^2+a}\,dx + \sqrt{y^2-a}\,dy$$

Integrating.
$$\int dz = \int \sqrt{x^2+a}\,dx + \int \sqrt{y^2-a}\,dy + c$$

$$z = \frac{x}{2}\sqrt{x^2+a} + \frac{a}{2}\log\left(x + \sqrt{x^2+a}\right) + \frac{y}{2}\sqrt{y^2-a} - \frac{a}{2}\log\left(y - \sqrt{y^2-a}\right) + c$$

which is a complete soln. where a, c are arbitrary constants.

Form: IV Eqn of the form $z = px + qy + f(p,q)$

CLAIRAUT'S FORM

The complete soln is $z = ax + by + f(a,b)$

Replace p by a
q by b

where a, c are arbitrary constants.

* Singular soln exists for this type

Find the complete integral of $z = px + qy - p^2 - q^2$ — ①

$$z = px + qy - (p^2 + q^2)$$

complete soln: Replace p by a, q by b in (1)

$\Rightarrow$ $z = ax + by - (a^2 + b^2)$ which is a complete soln.