MODULE: 6

Z- TRANSFORMS

Z-transform plays the same role in discrete analysis as Laplace transforms in continuous systems. Z-transform has many properties similar to those of the Laplace transform.

The main difference is that the Z-transform operates not on functions of continuous arguements but on sequence of the discrete integer-valued arguements, $n = 0, \pm 1, \pm 2, \ldots$

The Laplace transform converts differential equation into algebraic equations. whereas the z-transform converts difference equation (discrete form of the differential equations) into algebraic equations.

Definition

If the function u_n is defined for discrete values $(n=0,1,2,\dots)$ and $u_n = 0$ for $n<0$, then its z-transform is defined to be

$$Z(u_n) = U(z) = \sum_{n=0}^{\infty} u_n z^{-n} \quad \text{whenever} \qquad (1)$$

the infinite series converges.

The inverse z-transform is written as

$$Z^{-1}[U(z)] = u_n.$$

If we insert a particular complex number z into power series (1), the resulting value of $Z(u_n)$ will be a complex number. Thus, the z-transform $U(z)$ is a complex valued function of a complex variable z.

SOME STANDARD Z-TRANSFORMS

1. $$Z(a^n) = \frac{z}{z-a}$$

2. $$Z(n^p) = -z\frac{d}{dz}Z(n^{p-1}),$$ p being a +ve integer

$$Z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots$$

$$= \frac{1}{1-(a/z)} = \frac{z}{z-a}$$

3. $$Z(1) = \frac{z}{z-1}$$

4. $Z(n) = \frac{z}{(z-1)^2}$

5. $Z(n^2) = \frac{z^2+z}{(z-1)^3}$

6. $Z(n^3) = \frac{z^3+4z^2+z}{(z-1)^4}$

7. $Z(n^4) = \frac{z^4+11z^3+11z^2+z}{(z-1)^5}$

Linearity Property

If a, b, c be any constants and u_n, v_n, w_n be any discrete functions, then

$$Z(au_n + bv_n - cw_n) = aZ(u_n) + bZ(v_n) - cZ(w_n)$$

Damping rule

If $Z(u_n) = U(z)$, then $Z(a^{-n} u_n) = U(az)$

$$Z(a^n u_n) = U(z/a)$$

SOME STANDARD RESULTS

1. $Z(na^n) = \dfrac{az}{(z-a)^2}$ (Apply damping rule in $Z(n)$)

2. $Z(n^2 a^n) = \dfrac{az^2 + a^2 z}{(z-a)^3}$

3. $Z(\cos n\theta) = \dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$

4. $$Z(\sin n\theta) = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

5. $$Z(a^n \cos n\theta) = \frac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$$

6. $$Z(a^n \sin n\theta) = \frac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$$

$$Z(e^{in\theta}) = Z(\cos n\theta + i\sin n\theta)$$

$$Z((e^{i\theta})^n) = \frac{z}{z-e^{i\theta}}$$

$$= \frac{z}{z-(\cos\theta + i\sin\theta)} = \frac{z}{(z-\cos\theta)+i\sin\theta}$$

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$$= \frac{z\left[(z-\cos\theta) + i\sin\theta\right]}{(z-\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{z(z-\cos\theta) + i\, z\sin\theta}{z^2 - 2z\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{z(z-\cos\theta) + i\, z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Equating real and imagenary parts seperately, we get

$$z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$$

$$z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

7. $$Z\left(\frac{1}{n}\right) = \log_e\left(\frac{z}{z-1}\right), \quad |z|>1, \quad n>0$$

$$u_n = \frac{1}{n}$$

$$Z\left(\frac{1}{n}\right) = \sum_{n=0}^{\infty} \frac{1}{n} z^{-n}$$

$$= z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{3} + \cdots$$

$$= -\log_e(1-z^{-1}) \quad \text{if } |z^{-1}|<1$$

$$\left(\because x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = -\log_e(1-x)\right)$$

$$= -\log\left(1-\frac{1}{z}\right) = -\log\left(\frac{z-1}{z}\right)$$

$$\Rightarrow \quad Z\left(\frac{1}{n}\right) = \log\left(\frac{z}{z-1}\right), \quad |z|>1.$$

8. $$Z\left(\frac{1}{n!}\right) = e^{1/z}$$

$$Z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots$$

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$$= 1 + \left(\frac{1}{z}\right)\frac{1}{1!} + \frac{1}{z^2}\cdot\frac{1}{2!} + \frac{1}{z^3}\cdot\frac{1}{3!} + \cdots$$

$$= e^{1/z} \quad \left(\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$z\left(\frac{1}{n!}\right) = e^{1/z}$$

9. $$z\left(\frac{1}{(n+1)!}\right) = z\left(e^{1/z} - 1\right)$$

10. $$z\left(\frac{a^n}{n!}\right) = e^{a/z}$$

11. $$z\left(e^{an}\right) = \frac{z}{z - e^a}$$

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Shifting Property

Shifting property to the Right.

If $Z(u_n) = U(z)$ then

$$Z(u_{n-k}) = z^{-k}\, U(z) \quad (k>0)$$

This rule will be very useful in applications to difference equation.

Shifting property to the left

If $Z(u_n) = U(z)$ then

$$Z(u_{n+k}) = z^{k}\left[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \dots - u_{k-1}\, z^{-(k-1)}\right]$$

In particular, we have the following standard results

$$Z(u_{n+1}) = z[U(z) - u_0]$$

$$Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$$

$$Z(u_{n+3}) = z^3[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$$

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Multiplication by n

If $Z(u_n) = U(z)$, then $Z(n u_n) = -z \dfrac{d\,U(z)}{dz}$

$$Z(n^2 u_n) = \left(-z\frac{d}{dz}\right)^2 U(z) = (-z)^2 \frac{d^2}{dz^2} U(z)$$

$$Z(n^m u_n) = \left(-z\frac{d}{dz}\right)^m U(z) = (-z)^m \frac{d^m}{dz^m} U(z).$$

Problems

1. Find $Z\left[2n + 5\sin\frac{n\pi}{4} - 3a^4\right]$

$= 2Z[n] + 5Z\left[\sin\frac{n\pi}{4}\right] - 3Z(a^4)$

$= 2Z[n] + 5Z\left[\sin\frac{n\pi}{4}\right] - 3a^4 Z[1]$ By linearity property

$= 2\frac{z}{(z-1)^2} + 5\frac{z\sin\pi/4}{z^2 - 2z\cos\frac{\pi}{4} + 1} - 3a^4 \cdot \frac{z}{z-1}$

$$= \frac{2z}{(z-1)^2} + \frac{5z}{\sqrt{2}\,(z^2 - 2z\cancel{\cos} + 1)} - 3a^4 \frac{z}{z-1}.$$

2. $Z((n+1)(n+2))$

$$= Z(n^2 + 3n + 2)$$

$$= Z(n^2) + 3\,Z(n) + 2\,Z(1)$$ By linearity property

$$= \frac{z(z+1)}{(z-1)^3} + 3 \cdot \frac{z}{(z-1)^2} + \frac{2z}{z-1}.$$

3. $Z(n\cos n\theta)$.

Function multiplied by n.

Multiplication by n. property.

$$Z(n\,u_n) = \frac{zd}{dz}\,U(z)$$

$$Z(n\cos n\theta) = -z\,\frac{d}{dz}\,Z(\cos n\theta)$$

$$= -z\,\frac{d}{dz}\left\{\frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}\right\}$$

$$= -z\left[\frac{(z^2-2z\cos\theta+1)(2z-\cos\theta) - (z^2-z\cos\theta)(2z-2\cos\theta)}{(z^2-2z\cos\theta+1)^2}\right]$$

$$= \frac{-z\,[2z - z^2\cos\theta - \cos\theta]}{(z^2-2z\cos\theta+1)^2} = \frac{(z^3+z)\cos\theta - 2z^2}{(z^2-2z\cos\theta+1)^2}$$

TWO BASIC THEOREMS

Initial Value Theorem

If $Z(u_n) = U(z)$, then $u_0 = \underset{z\to\infty}{Lt}\, U(z)$.

$u_1 = \underset{z\to\infty}{Lt} \{z(U(z) - u_0)\}$; $u_2 = \underset{z\to\infty}{Lt} \{z^2(U(z) - u_0 - u_1 z^{-1})\}$ and so on.

Final Value Theorem

If $Z(u_n) = U(z)$, then $\underset{n\to\infty}{Lt}\,(u_n) = \underset{z\to 1}{Lt}\,(z-1)U(z)$

Example

1. Given that $Z[u_n] = \dfrac{2z^2+3z+4}{(z-3)^3}$, $|z|>3$, show that

$u_1 = 2$, $u_2 = 21$, $u_3 = 139$

By Initial value theorem, we have

$$u_0 = \lim_{z\to\infty} U(z)$$

$$= \lim_{z\to\infty} \frac{2z^2+3z+4}{(z-3)^3}$$

$$= \lim_{z\to\infty} \frac{z^2\left(2+\frac{3z}{z^2}+\frac{4}{z^2}\right)}{z^3\left(1-\frac{3}{z}\right)^3}$$

$$= \lim_{z\to\infty} \frac{\left(2+\frac{3}{z}+\frac{4}{z^2}\right)}{z\left(1-\frac{3}{z}\right)^3} = 0$$

$$u_1 = \lim_{z \to \infty} z\left(U(z) - u_0\right)$$

$$= \lim_{z \to \infty} z \left\{ \frac{2z^2 + 3z + 4}{(z-3)^3} - 0 \right\}$$

$$= \lim_{z \to \infty} \frac{\cancel{z^3}\left(2 + \frac{3}{z} + \frac{4}{z^2}\right)}{\cancel{z^3}\left(1 - \frac{3}{z}\right)^3}$$

$$\boxed{u_1 = 2}$$

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$$u_2 = \lim_{z \to \infty} z^2 \left(U(z) - u_0 - u_1 z^{-1} \right)$$

$$= \lim_{z \to \infty} z^2 \left\{ \frac{2z^2+3z+4}{(z-3)^3} - 0 - \frac{2}{z} \right\}$$

$$= \lim_{z \to \infty} z^2 \left\{ \frac{z(2z^2+3z+4) - 2(z-3)^3}{z(z-3)^3} \right\}$$

$$= \lim_{z \to \infty} z \left[\frac{2z^3 + 3z^2 + 4z - 2(z^3 - 9z^2 + 27z - 27)}{(z-3)^3} \right]$$

$$= \lim_{z \to \infty} z \frac{(21z^2 - 50z + 54)}{z^3 \left(1 - \frac{3}{z}\right)^3}$$

$$= \lim_{z \to \infty} \frac{\cancel{z^3}}{\cancel{z^3}} \frac{\left(21 - \frac{50}{z} + \frac{54}{z^2}\right)}{\left(1 - \frac{3}{z}\right)^3} = 21$$

$$\boxed{u_2 = 21}$$

$$u_3 = \lim_{z \to \infty} z^3 \left(U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} \right)$$

$$= \lim_{z \to \infty} z^3 \left[\frac{2z^2+3z+4}{(z-3)^3} - 0 - \frac{2}{z} - \frac{21}{z^2} \right]$$

$$= \lim_{z \to \infty} z^3 \left[\frac{z^2(2z^2+3z+4) - 2z(z-3)^3 - 21(z-3)^3}{z^2(z-3)^3} \right]$$

$$= \lim_{z\to\infty} \frac{z\left[2z^4+3z^3+4z^2-2z(z^3-9z^2+27z-27) - 21(z^3-9z^2+27-27)\right]}{(z-3)^3}$$

$$= \lim_{z\to\infty} \frac{\cancel{z^3}\left[139-\frac{513}{z}+\frac{567}{z^2}\right]}{\cancel{z^3}\left(1-\frac{3}{z}\right)^3}$$

$$\boxed{u_3 = 139}$$

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Unit Impulse function

unit Impulse sequence $\delta(n) = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n \neq 0 \end{cases}$

$$Z[\delta(n)] = \sum_{n=0}^{\infty} \delta(n) z^{-n}$$

$$= 1+0+0+\cdots$$

$$= 1$$

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Unit step function

$$u(n) = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

$$Z[u(n)] = \sum_{n=0}^{\infty} u_n z^{-n}$$

$$= \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$= \left(1 - \frac{1}{z}\right)^{-1}$$

$$= \frac{z}{z-1} \text{ if } |z| > 1.$$