Equations reducible to standard types

① Solve $x^2p^2 + y^2q^2 = z^2$

Sol: Dividing throughout by z^2, we get

$$\frac{x^2}{z^2}p^2 + \frac{y^2}{z^2}q^2 = 1$$

$$\left(\frac{x}{z}p\right)^2 + \left(\frac{y}{z}q\right)^2 = 1 \Rightarrow x^2\left(\frac{p}{z}\right)^2 + y^2\left(\frac{q}{z}\right)^2 = 1$$

put $Z = \log z$ so that

$$\frac{\partial Z}{\partial x} = \frac{1}{z}\frac{\partial z}{\partial x}$$

$$\boxed{P = \frac{1}{z}p}, \quad \text{where } P = \frac{\partial Z}{\partial x}$$

$$\frac{\partial Z}{\partial y} = \frac{1}{z}\frac{\partial z}{\partial y}$$

$$\boxed{Q = \frac{1}{z}q}, \quad \text{where } Q = \frac{\partial Z}{\partial y}$$

The given pde becomes

$$x^2P^2 + y^2Q^2 = 1 \qquad \text{(Type 3)}$$

$$\Rightarrow x^2P^2 = a^2 = 1 - y^2Q^2$$

Now, $x^2P^2 = a \Rightarrow P^2 = \frac{a^2}{x^2} \Rightarrow P = \frac{a}{x}$

Similarly $\quad 1 - y^2 q^2 = a^2$

$$\Rightarrow \quad y^2 q^2 = 1 - a^2$$

$$y q = \sqrt{1-a^2}$$

$$q = \frac{\sqrt{1-a^2}}{y}$$

But $\quad dz = P\,dx + q\,dy$

Integrating, we get

$$z = \int \frac{a}{x}\,dx + \int \frac{\sqrt{1-a^2}}{y}\,dy + b$$

$$z = a\log x + \sqrt{1-a^2}\,\log y + \log b$$

i.e., $\quad \log z = a\log x + \sqrt{1-a^2}\,\log y + \log b$

$$\log\left(\frac{z}{x^a y^{\sqrt{1-a^2}}}\right) = \log b$$

$$\Rightarrow \quad z = b\,x^a y^{\sqrt{1-a^2}}$$