INVERSE Z-TRANSFORMS

Convergence of Z-Transforms

Z-transform operation is performed on a sequence u_n, which may exist in the range of integers $-\infty < n < \infty$

$$U(z) = \sum_{n=-\infty}^{\infty} u_n \cdot z^{-n} \quad \text{--- (1)}$$

where u_n represents a number in the sequence for $n =$ an integer.

The region of the z-plane in which (1) converges absolutely is known as the region of convergence (ROC) of $U(z)$

We have discussed for $n \geq 0$ here the sequence is always right-sided and the region of convergence is always outside a prescribed circle say $|z| > |a|$.

Inverse Z-transforms for some standard functions

1. $Z^{-1}\left[\dfrac{z}{z-a}\right] = a^n$

2. $Z^{-1}\left[\dfrac{1}{z-a}\right] = a^{n-1}, \quad n \geq 1$

3. $Z^{-1}\left[\dfrac{z}{(z-1)^2}\right] = n.$

4. $z^{-1}\left[\frac{z}{(z-a)^2}\right] = na^{n-1}$

5. $z^{-1}\left[\frac{az}{(z-a)^2}\right] = na^{n}$

6. $z^{-1}\left[\frac{2z}{(z-1)^3}\right] = n(n-1)$

7. $z^{-1}\left[\frac{1}{(z-a)^2}\right] = (n-1)a^{n-2}$

1) METHOD OF PARTIAL FRACTIONS

This method is similar to the method of inverse Laplace transforms by using partial fractions. Here, generally we express $\frac{U(z)}{z}$ into partial fractions and multiply each partial term by z and then apply inverse transform z^{-1} to each term.

Problems

1. Find the inverse z-transform of $\frac{5z}{(2-z)(3z-1)}$

Given $U(z) = \frac{5z}{(2-z)(3z-1)}$

$$\frac{U(z)}{z} = \frac{-5}{(z-2)(3z-1)}$$

Let $\frac{-5}{(z-2)(3z-1)} = \frac{A}{(z-2)} + \frac{B}{(3z-1)}$

$$-5 = A(3z-1) + B(z-2)$$

Put $z=2$

$$-5 = A(5)$$

$$\Rightarrow \boxed{A=-1}$$

Put $z=\frac{1}{3}$

$$-5 = B\left(\frac{1}{3}-2\right)$$

$$-5 = -\frac{5}{3}B$$

$$\Rightarrow \boxed{B=3}$$

$$\therefore \frac{U(z)}{z} = \frac{-5}{(z-2)(3z-1)} = \frac{-1}{(z-2)} + \frac{3}{(3z-1)}$$

$$U(z) = \frac{-z}{z-2} + \frac{3z}{3\left(z-\frac{1}{3}\right)}$$

$$U(z) = -\frac{z}{z-2} + \frac{z}{z-\frac{1}{3}}$$

Taking inverse on both sides
(z^{-1})

$$\Rightarrow \quad u_n = z^{-1}\left[\frac{-z}{z-2}\right] + z^{-1}\left[\frac{z}{z-\frac{1}{3}}\right]$$

$$u_n = -2^n + \left(\frac{1}{3}\right)^n, \quad n = 0, 1, 2, \ldots$$

2) Find the inverse z-transform of

$$\frac{2z^2 - 10z + 13}{(z-2)(z-3)^2}, \quad 2<|z|<3.$$

Given $U(z) = \dfrac{2z^2 - 10z + 13}{(z-2)(z-3)^2}$

$$\frac{2z^2 - 10z + 13}{(z-2)(z-3)^2} = \frac{A}{(z-2)} + \frac{B}{(z-3)} + \frac{C}{(z-3)^2}$$

$$2z^2 - 10z + 13 = A(z-3)^2 + B(z-2)(z-3) + C(z-2)$$

Put $z=2$,

$$1 = (-1)^2 A$$

$$\Rightarrow \boxed{A=1}$$

Scanned with CamScanner

Put $z=3$

$1=C$

$\Rightarrow \boxed{C=1}$

Equating coefficients of z^2,

$2 = A+B$

$2-1=B$

$\boxed{B=1}$

$\therefore \quad U(z) = \dfrac{1}{z-2} + \dfrac{1}{z-3} + \dfrac{1}{(z-3)^2}$

Scanned with CamScanner

Given ROC $2 < |z| < 3$

$$\Rightarrow \left|\frac{2}{z}\right| < 1 \quad \text{and} \quad \left|\frac{z}{3}\right| < 1$$

$$\Rightarrow U(z) = \frac{1}{z\left(1-\frac{2}{z}\right)} + \frac{1}{-(3-z)} + \frac{1}{9\left(1-\frac{z}{3}\right)^2}$$

$$U(z) = \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{3\left(1-\frac{z}{3}\right)} + \frac{1}{9\left(1-\frac{z}{3}\right)^2}$$

$$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{3}\left(1-\frac{z}{3}\right)^{-1} + \frac{1}{9}\left(1-\frac{z}{3}\right)^{-2}$$

$$= \frac{1}{z}\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\cdots\right]$$

$$= \frac{1}{z}\left[1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots\right]$$

$$-\frac{1}{3}\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \cdots\right]$$

$$+\frac{1}{9}\left[1 + 2\cdot\frac{z}{3} + 3\cdot\frac{z^2}{3^2} + 4\cdot\frac{z^3}{3^3} + \cdots\right]$$

$$= \frac{1}{z} + 2\cdot\frac{1}{z^2} + 2^2\cdot\frac{1}{z^3} + \cdots$$

$$-\left[\frac{1}{3} + \frac{1}{3^2} z + \frac{1}{3^3} z^2 + \cdots\right]$$

$$+\left[\frac{1}{3^2} + \frac{2}{3^3} z + \frac{3}{3^4} z^2 + \cdots\right]$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n+1} z^{n}$$

$$+ \sum_{n=0}^{\infty} (n+1) \left(\frac{1}{3}\right)^{n+2} z^{n}$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \left[\frac{1}{3^{n+1}} - (n+1)\frac{1}{3^{n+2}}\right] z^{n}$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \frac{1}{3^{n+2}} (3-n-1) z^{n}$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \frac{1}{3^{n+2}} (2-n) z^{n}$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} 3^{-n-2} (2-n) z^{n}$$

replace n by $-m$ in the second term

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{-m=0}^{\infty} 3^{m-2} (2+m) z^{-m}$$

$$\Rightarrow \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{m \leq 0} 3^{m-2} (2+m) z^{-m}$$

In first term

$u_n = 2^{n-1}$ if $n \geq 1$

In second term

$u_n = -3^{n-2} (2+n)$ if $n \leq 0$.

Scanned with CamScanner