6. $L\left\{ \int_0^t \int_0^t \int_0^t (t\sin t)\,dt\,dt\,dt \right\}$

$$L[\sin t] = \frac{1}{s^2+1}$$

$$L[t\sin t] = (-1)\frac{d}{ds}\left\{\frac{1}{s^2+1}\right\}$$

$$= -\frac{d}{ds}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{2s}{(s^2+1)^2}$$

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$$L\left\{\int_0^t\int_0^t\int_0^t t\sin t\,dt\right\} = \frac{1}{s^3} \underbrace{L[t\sin t]}_{F(s)}$$

By the property
Transforms of Integrals

$$= \frac{1}{s^3} \cdot \frac{2s}{(s^2+1)^2}$$

$$= \frac{2}{s^2(s^2+1)^2}$$

Evaluate

(17) $\int_0^\infty t e^{-3t} \sin t \, dt$

compare with the definition of Laplace transform

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

here $f(t) = t \sin t$

and $s = 3$

$$L[t \sin t] = \frac{2s}{(s^2+1)^2}$$

Put $s=3$ $\Rightarrow$ $\dfrac{2\times 3}{(3^2+1)^2} = \dfrac{3}{50}$

$$\therefore \int_0^\infty t e^{-3t} \sin t \, dt = \frac{3}{50}.$$

INVERSE LAPLACE TRANSFORMS

Inverse Laplace transform of some standard functions are as follows:

1. $L^{-1}\left[\frac{1}{s}\right] = 1.$

2. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

3. $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}, \quad n = 1, 2, 3, \ldots$

4. $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

5. $L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at$

6. $L^{-1}\left[\dfrac{1}{s^2-a^2}\right] = \dfrac{1}{a}\sinh at$

7. $L^{-1}\left[\dfrac{s}{s^2-a^2}\right] = \cosh at$

8. $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right] = \dfrac{1}{2a} t \sin at$

9. $L^{-1}\left[\frac{1}{(s-a)^n}\right] = e^{at}\frac{t^{n-1}}{(n-1)!}$

10. $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at}\cos bt$

11. $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b}e^{at}\sin bt$

12. $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos at)$

In general,
we can apply partial fraction to get the inverse Laplace transform of a function.

Problems

1. $L^{-1}\left[\dfrac{1}{s^2-5s+6}\right]$

consider $\dfrac{1}{s^2-5s+6} = \dfrac{1}{(s-3)(s-2)}$

Apply partial fraction,

$$\frac{1}{(s-3)(s-2)} = \frac{A}{(s-3)} + \frac{B}{(s-2)}$$

$$1 = A(s-2) + B(s-3)$$

Put $s=2$

$$\Rightarrow \quad 1 = A(0) + B(-1)$$

$$\boxed{B = -1}$$

Put $s=3$

$$1 = A(3-2) + B(0)$$

$$\Rightarrow \boxed{A = 1}$$

$$\Rightarrow \frac{1}{(s-3)(s-2)} = \frac{1}{(s-3)} - \frac{1}{(s-2)}$$

$$L^{-1}\left[\frac{1}{(s-3)(s-2)}\right] = L^{-1}\left[\frac{1}{s-3}\right] - L^{-1}\left[\frac{1}{s-2}\right]$$

$$= e^{3t} - e^{2t} \;//$$

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I. Shifting property for inverse Laplace transforms.

If $L^{-1}[F(s)] = f(t)$ then

$$L^{-1}[F(s-a)] = e^{at} L^{-1}[F(s)]$$

II

If $L^{-1}[F(s)] = f(t)$ and $f(0)=0$, then

$$L^{-1}[sF(s)] = \frac{d}{dt}\{f(t)\}$$

In general,

$$L^{-1}[s^n F(s)] = \frac{d^n}{dt^n}\{f(t)\}$$

provided $f(0) = f'(0) = f''(0) = \cdots = f^{n-1}(0) = 0$

III If $L^{-1}[F(s)] = f(t)$, then

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt$$

IV If $L^{-1}[F(s)] = f(t)$, then

$$L^{-1}\left[-\frac{d}{ds}F(s)\right] = t\,f(t)$$

V If $L^{-1}[F(s)] = f(t)$, then $L^{-1}\left[\int_s^\infty F(s)\,ds\right] = \frac{f(t)}{t}$

can be conveniently calculated