$\therefore \quad p - \sin x = \sin y - q = a$, say

$\therefore \quad p = a + \sin x, \quad \text{and} \quad q = \sin y - a$

We know that $\quad dz = pdx + qdy = (a + \sin x)dx + (\sin y - a)dy$

Integrating, $\quad z = \int (a + \sin x)dx + \int (\sin y - a)dy$

$\Rightarrow \quad z = ax - \cos x - \cos y - ay + c$

This is the complete integral.

EXAMPLE 11

Solve $p^2 + q^2 = x^2 + y^2$.

Solution.

Given $\quad p^2 + q^2 = x^2 + y^2$. $\quad$ It is separable type.

$\therefore \quad p^2 - x^2 = y^2 - q^2 = a$

$\therefore \quad p^2 = x^2 + a \Rightarrow p = \sqrt{x^2 + a} \quad \text{and} \quad q^2 = y^2 - a \Rightarrow q = \sqrt{y^2 - a}$

We know that $\quad dz = pdx + qdy = \sqrt{x^2 + a}\, dx + \sqrt{y^2 - a}\, dy$

Integrating, $\quad z = \int \sqrt{x^2 + a}\, dx + \int \sqrt{y^2 - a}\, dy$

$$\Rightarrow \quad z = \frac{x}{2}\sqrt{x^2 + a} + \frac{a}{2}\log\left(x + \sqrt{x^2 + a}\right) + \frac{y}{2}\sqrt{y^2 - a} - \frac{a}{2}\log\left(y - \sqrt{y^2 - a}\right) + c \qquad (1)$$

This is the complete integral.
There is no singular integral.
The general integral is found by putting $c = \phi(a)$ in (1), differentiating w.r.to a and eliminating a.

14.4.3 Equations Reducible to Standard Forms

(A) Equation of the form $\quad F(x^m p, y^n q) = 0 \qquad (1)$

and $\quad F(z, x^m p, y^n q) = 0 \qquad (2)$

Case 1: If $m \neq 1, \quad n \neq 1$, put $x^{1-m} = X, \quad y^{1-n} = Y$

$$\therefore \quad \frac{dX}{dx} = (1-m)x^{-m}, \quad \frac{dY}{dy} = (1-n)y^{-n}$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{dX}{dx} = \frac{\partial z}{\partial X}(1-m)x^{-m}$$

$$\Rightarrow \quad p = (1-m)x^{-m}P, \quad \text{where } P = \frac{\partial z}{\partial X}$$

$$\Rightarrow \quad x^m p = (1-m)P.$$

Similarly, $$y^n q = (1-n)Q, \quad \text{where } Q = \frac{\partial z}{\partial Y}$$

Then (1) becomes $f(P, Q) = 0$, which is standard type (1)

and (2) becomes $f(z, P, Q) = 0$, which is standard type 3(a).

Case 2: If $m = 1$, $n = 1$, then the equations are $F(xp, yp) = 0$ and $F(z, xp, yp) = 0$

Put $X = \log x$, $Y = \log y$, then $\dfrac{dX}{dx} = \dfrac{1}{x}$ and $\dfrac{dY}{dy} = \dfrac{1}{y}$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{dX}{dx} = P\frac{1}{x} \quad \Rightarrow \quad px = P$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\cdot\frac{dY}{dy} = Q\frac{1}{y} \quad \Rightarrow \quad qy = Q$$

where $$P = \frac{\partial z}{\partial X}, \quad Q = \frac{\partial z}{\partial Y}$$

Then (1) becomes $F(P, Q) = 0$, which is standard type (1)

and (2) becomes $F(z, P, Q) = 0$, which is standard type 3(a).

(B) Equation of the form $F(x^m z^k p, y^n z^k q) = 0$

Case 1: If $m \neq 1$, $n \neq 1$, $k \neq -1$. Put $X = x^{1-m}$, $Y = y^{1-n}$ and $Z = z^{k+1}$.

$$\therefore \frac{dX}{dx} = (1-m)x^{1-m-1} \Rightarrow \frac{dx}{dX} = \frac{x^m}{1-m} \quad \text{and} \quad \frac{\partial Z}{\partial z} = (k+1)z^k.$$

$$\therefore \quad P = \frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial x}\cdot\frac{dx}{dX} = (k+1)z^k p\cdot\frac{x^m}{1-m} \quad \Rightarrow \quad x^m \cdot z^k \cdot p = \frac{(1-m)}{k+1}P$$

Similarly, $$y^n z^k q = \frac{(1-n)}{k+1}Q$$

$\therefore$ the equation reduces to $F(P, Q) = 0$, which is standard type (1).

Case 2: If $m = 1$, $n = 1$, $k = -1$, then the equation is $F\left(\dfrac{px}{z}, \dfrac{qy}{z}\right) = 0$.

Put $X = \log_e x$, $Y = \log_e y$ and $Z = \log_e z$. $\therefore$ $\dfrac{dX}{dx} = \dfrac{1}{x}, \dfrac{dY}{dy} = \dfrac{1}{y}$ and $\dfrac{\partial Z}{\partial z} = \dfrac{1}{z}$

$$\therefore \quad P = \frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial x}\cdot\frac{dx}{dX} = \frac{px}{z}$$

Similarly, $Q = \dfrac{qy}{z}$

$\therefore$ the equation becomes $F(P, Q) = 0$, which is standard type (1).

(C) Equation of the form $F(z^k p, z^k q) = 0$

Case 1: If $k \neq -1$, put $Z = z^{k+1}$, then $\dfrac{\partial Z}{\partial z} = (k+1)z^k$.

$\therefore$ $$P = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} = (k+1)z^k \cdot p \quad \Rightarrow \quad \frac{P}{k+1} = z^k p$$

Similarly, $$\frac{Q}{k+1} = z^k q$$

$\therefore$ the equation is $F(P, Q) = 0$, which is standard type 1.

Case 2: If $k = -1$, the equation is $F\left(\dfrac{p}{z}, \dfrac{q}{z}\right) = 0$

Put $Z = \log_e z$, then $P = \dfrac{p}{z}$, $Q = \dfrac{q}{z}$

$\therefore$ the equation is $F(P, Q) = 0$, which is standard type 1.

WORKED EXAMPLES

EXAMPLE 12

Solve $x^2 p^2 + y^2 q^2 = z^2$.

Solution.

Given $$x^2 p^2 + y^2 q^2 = z^2$$

$\Rightarrow$ $$\left(\frac{xp}{z}\right)^2 + \left(\frac{yq}{z}\right)^2 = 1 \qquad \text{(Dividing by } z^2\text{)}$$

It is the form $$F\left(\frac{px}{z}, \frac{qy}{z}\right) = 0 \qquad [m = 1, n = 1, k = -1, \text{(B) Case 2}]$$

Put $X = \log_e x$, $Y = \log_e y$, $Z = \log_e z$

$\therefore$ $$P = \frac{\partial Z}{\partial X} = \frac{px}{z}, \quad \text{and} \quad Q = \frac{\partial Z}{\partial Y} = \frac{qy}{z}$$

$\therefore$ the equation is $P^2 + Q^2 = 1$, which is standard type (1).

The complete integral is $Z = aX + bY + c$, where $a^2 + b^2 = 1 \Rightarrow b = \sqrt{1 - a^2}$

$\therefore$ $$\log_e z = a \log_e x + \sqrt{1 - a^2} \log_e y + C \qquad (1)$$

which is the complete integral.

There is no singular integral.

The general integral is found by putting $c = \phi(a)$ in (1), differentiating w.r.to a and eliminating a.

EXAMPLE 13

Solve $x^4 p^2 - yzq = z^2$.

Solution.

Given $$x^4 p^2 - yzq = z^2 \quad \Rightarrow \quad \frac{x^4 p^2}{z^2} - \frac{yz}{z^2} q = 1$$

$$\Rightarrow \qquad \frac{x^4 p^2}{z^2} - \frac{yq}{z} = 1 \quad \Rightarrow \quad \left(\frac{x^2 p}{z}\right)^2 - \left(\frac{yq}{z}\right) = 1 \qquad (1)$$

It is of the form $F(x^m z^k p, y^n z^k q) = 0$ with $m = 2, \quad k = -1, \quad n = 1$.

$\therefore$ put $\quad X = x^{1-m} = x^{-1}, \quad Y = \log_e y$ and $Z = \log_e z$

$$\therefore \qquad \frac{dX}{dx} = (-1)x^{-2} \Rightarrow \frac{dX}{dx} = -x^2, \frac{dY}{dy} = \frac{1}{y} \Rightarrow \frac{dy}{dY} = y \text{ and } \frac{\partial Z}{\partial z} = \frac{1}{z}.$$

$$\therefore \qquad P = \frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} \cdot \frac{dx}{dX} = \frac{1}{z} \cdot p(-x^2) = -\frac{x^2 p}{z}$$

$$\text{and} \qquad Q = \frac{\partial Z}{\partial Y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{dy}{dY} = \frac{1}{z} qy$$

Substituting for P and Q in (1) the equation is $P^2 - Q = 1$

So, the complete integral is $Z = aX + bY + c, \quad$ where $\quad a^2 - b = 1 \quad \Rightarrow \quad b = a^2 - 1$

$$\therefore \qquad \log_e z = ax^{-1} + b \log_e y + c$$

$$\Rightarrow \qquad \log_e z = \frac{a}{x} + (a^2 - 1)\log_e y + c, \text{ which is the complete integral.}$$

There is no singular integral.
The general integral is found by putting $c = \phi(a)$, differentiating w.r.to a and eliminating a.

EXAMPLE 14

Solve $z^2(p^2x^2 + q^2) = 1$.

Solution.
Given $\qquad z^2(p^2x^2 + q^2) = 1 \quad \Rightarrow \quad (xzp)^2 + (qz)^2 = 1 \qquad (1)$

This is of the form $\qquad F(x^m z^k p, y^n z^k q) = 0$
Here $m = 1, \quad k = 1, \quad n = 0 \neq 1$

$\therefore$ Put $\quad X = \log_e x, \qquad Y = y^{1-n} = y \quad$ and $\quad Z = z^{k+1} = z^2$

$$\therefore \qquad \frac{dX}{dx} = \frac{1}{x} \Rightarrow \frac{dx}{dx} = x, \frac{dY}{dy} = 1 \Rightarrow \frac{dy}{dY} = 1 \text{ and } \frac{\partial Z}{\partial z} = 2z.$$

$$\therefore \qquad P = \frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} \cdot \frac{dx}{dX} = 2z \cdot p \cdot x \quad \Rightarrow \quad \frac{P}{2} = xzp$$

$$\text{and} \qquad Q = \frac{\partial Z}{\partial Y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{dy}{dY} = 2z \cdot q \cdot 1 \quad \Rightarrow \quad \frac{Q}{2} = zq$$

Substituting in (1), we get

$$\frac{P^2}{4}+\frac{Q^2}{4}=1 \quad \Rightarrow \quad P^2+Q^2=4$$

This is of standard type (1),

$\therefore$ the complete integral is $\qquad Z = aX + bY + c$

where $\qquad a^2+b^2=4 \quad \Rightarrow \quad b=\sqrt{4-a^2}$

$$\therefore \qquad z^2 = a\log_e x + \sqrt{4-a^2}\,y + c \tag{2}$$

This is the complete integral.

There is no singular integral.

The general integral is found by putting $c = \phi(a)$ in (2), differentiating w.r.to a and eliminating a.

EXAMPLE 15

Solve $(x + pz)^2 + (y + qz)^2 = 1$.

Solution.

Given $\qquad (x+pz)^2+(y+qz)^2=1$

Put $\qquad Z = z^{1+k} = z^2 \qquad \therefore \qquad \dfrac{\partial Z}{\partial z} = 2z \qquad [\because k = 1]$

$$\therefore \qquad P=\frac{\partial Z}{\partial x}=\frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial x}=2z\cdot p \quad \Rightarrow \quad zp=\frac{P}{2}$$

$$Q=\frac{\partial Z}{\partial y}=\frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial y}=2z\cdot q \quad \Rightarrow \quad zq=\frac{Q}{2}$$

$\therefore$ the equation becomes $\left(x+\dfrac{P}{2}\right)^2+\left(y+\dfrac{Q}{2}\right)^2=1$

$$\Rightarrow \qquad \left(x+\frac{P}{2}\right)^2=1-\left(y+\frac{Q}{2}\right)^2=a$$

$$\therefore \qquad x+\frac{P}{2}=\sqrt{a}, \quad y+\frac{Q}{2}=\sqrt{1-a}$$

$$\Rightarrow \qquad P=2\left(\sqrt{a}-x\right), \quad Q=2\left(\sqrt{1-a}-y\right)$$

We know that $\qquad dZ = Pdx + Qdy = 2\left(\sqrt{a}-x\right)dx+2\left(\sqrt{1-a}-y\right)dy$

Integrating, $\qquad Z = 2\int\left(\sqrt{a}-x\right)dx+2\int\left(\sqrt{1-a}-y\right)dy$

$$\Rightarrow \qquad z^2=2\left(\sqrt{a}x-\frac{x^2}{2}\right)+2\left(\sqrt{1-a}y-\frac{y^2}{2}\right)+c$$

$$z^2 = 2\sqrt{a}x-x^2+2\sqrt{1-a}y-y^2+c$$

There is no singular integral.

The general integral will be found by putting $c = \phi(a)$ in (2), differentiating w.r.to a and eliminating a.

EXERCISE 14.4

Solve the following partial differential equations.

1. $p(1+q) = qz$
2. $ap + bq + cz = 0$
3. $z^2 = 1 + p^2 + q^2$
4. $z^2(p^2z^2 + q^2) = 1$
5. $pz = 1 + q^2$
6. $p^3 + q^3 = 8z$
7. $p^2 - q^2 = z$
8. $p(1+q^2) = q(z-a)$
9. $p^2z^2 + q^2 = p^2q$
10. $p = 2qx$
11. $\sqrt{p} + \sqrt{q} = 2x$
12. $pq = y$
13. $q = py + p^2$
14. $\sqrt{p} + \sqrt{q} = x + y$
15. $p^2 + q^2 = x^2 + y^2$
16. $p^2 + q^2 = x + y$
17. $pq = xy$
18. $\dfrac{x}{p} + \dfrac{y}{q} + 1 = 0$
19. $px + qy = 1$
20. $p^2x^4 + y^2zq = 2z^2$

    [**Hint:** $x^2z^{-1}p + y^2z^{-1}q = 2;\quad m = 2,\quad n = 2,\quad k = -1.$ Put $X = x^{-1},\quad Y = y^{-1},\quad Z = \log_e z$]
21. $p^2 + q^2 = z^2(x^2 + y^2)$

    $\left[\textbf{Hint:} \left(\dfrac{p}{z}\right)^2 + \left(\dfrac{q}{z}\right)^2 = x^2 + y^2.\right.$ Here $k = -1$, Put $Z = \log_e z$ then $\left. p = \dfrac{\partial z}{\partial x} = \dfrac{p}{z},\quad Q = \dfrac{y}{z}\right]$
22. $2x^4p^2 - yzq - 3z^2 = 0$

    $\left[\textbf{Hint:} 2(x^2z^{-1}p)^2 - (yz^{-1}q) = 3.\right.$ Here $m = 2,\quad n = 1,\quad k = -1.$

    Put $X = x^{-1},\quad Y = \log y,\quad Z = \log z \quad \therefore x^2z^1p = -p,\quad Q = yz^{-1}q,$

    where $P = \dfrac{\partial z}{\partial X},\quad Q = \dfrac{\partial Z}{\partial Y} \quad \therefore \quad \left. 2P^2 - Q^2 = 3\right]$
23. $pz^2\sin^2 x + qz^2\cos^2 y = 1$

    $\left[\textbf{Hint:} \text{ Put } Z = z^3 \qquad P = \dfrac{\partial Z}{\partial z} = \dfrac{\partial Z}{\partial z}\cdot\dfrac{\partial z}{\partial x} = 3z^2p;\quad Q = \dfrac{\partial Z}{\partial y} = 3z^2q^2\right.$

    $\left. P\sin^2 x + Q\cos^2 y = 3 \;\therefore\; P\sin^2 x = 3 - \cos^2 y = a \right]$

ANSWERS TO EXERCISE 14.4

Complete integral is given below

1. $\log_e(az - 1) = x + ay + c$
2. $\log_e z = -\dfrac{c}{a + bk}(x + ky) + c'$
3. $\log_e\left(z + \sqrt{z^2 - 1}\right) = \dfrac{1}{\sqrt{1 + a^2}}(x + ay) + c$
4. $(z^2 + a^2)^{3/2} = 3(x + ay) + c$
5. $z^2 - z\sqrt{z^2 - 4a^2} + 4a^2\log_e\left(z + \sqrt{z^2 - 4a^2}\right) = 4(x + ay) + c$
6. $3(1 + a^3)^{1/3}\cdot z^{2/3} = 4(x + ay) + c$
7. $2\sqrt{1 - a^2}\sqrt{z} = x + ay + c$

8. $2\sqrt{bz-ab-1} = x+ay+c$

9. $z = a\tan(x+ay+c)$

10. $z = ax^2+ay+c$

11. $z = \frac{1}{6}\left(2x-\sqrt{a}\right)^3+ay+c$

12. $z = ax+\frac{y^2}{2a}+c$

13. $z = ax+\frac{a^2}{2}y^2+a^2y+c$

14. $3z = (x+a)^3+(y-a)^3+c$

15. $2z = a\sin^{-1}\frac{x}{\sqrt{a}}+x\sqrt{x^2+a}+y\sqrt{y^2-a}-a\log_e\left(y+\sqrt{y^2-a}\right)+c$

16. $z = \frac{2}{3}\{(x+a)^{3/2}+(y-a)^{3/2}\}+c$

17. $z = \frac{ax^2}{2}+\frac{y^2}{2a}+c$

18. $z = -\frac{a}{a+1}\cdot\frac{x^2}{2}+\frac{a}{2}y^2+c$

19. $z(1+a) = \log_e x+a\log_e y+c$

20. $\log_e z = \frac{a}{x}+\frac{(a^2-2)}{y}+c$

21. $2\log_e z = x\sqrt{x^2+a}+a\log_e\left(x+\sqrt{x^2+a}\right)+y\sqrt{y^2-a}-a\log_e\left(y+\sqrt{y^2-a}\right)+c$

22. $\log_e z = \frac{a}{x}+(2a^2-3)\log_e y+c$

23. $z^3 = -a\cot x+(3-a)\tan y+c$

14.5 LAGRANGE'S LINEAR EQUATION

A partial differential equation of the form $Pp+Qq=R$, where P, Q, R are functions of x, y, z and $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$, is called **Lagrange's linear equation**.

We have seen already that elimination of ϕ from $\phi(u, v)=0$, where u and v are functions x, y, z leads to Lagrange's equation.

$\therefore$ $\phi(u, v)=0$ is the general solution of $Pp+Qq=R$, where ϕ is an arbitrary function.

The method to find the solution of $Pp+Qq=R$ is known as **Lagrange's method**.

Working Rule: To solve $Pp+Qq=R$, where P, Q, R are functions of x, y, z.

(i) Form the auxiliary equations or subsidiary equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

(ii) Solving the subsidiary equations, find two independent solutions $u(x, y, z)=a$ and $v(x, y, z)=b$, where a and b are arbitrary constants.

(iii) Then the required general solution is $\phi(u, v)=0$ [or $u=f(v)$ or $v=g(u)$] where ϕ (or f or g) is an arbitrary function.