Mechanical vibrations

To know how the second order differential equations can be applied to solve problems concerning the vibrations of springs and the analysis of electric circuits using MATLAB.

MATHEMATICAL FORM

$P(x)\dfrac{d^2y}{dx^2}+Q(x)\dfrac{dy}{dx}+R(x)y=G(x)$ Where P, Q, R and G are constants. If $G(x)=0, \forall x$ then its called as homogeneous linear equations or else non homogeneous.

VIBRATING STRINGS

The motion of an object with mass m at the end of a spring that is either vertical (Figure 1) or horizontal (Figure 2) on a level surface.

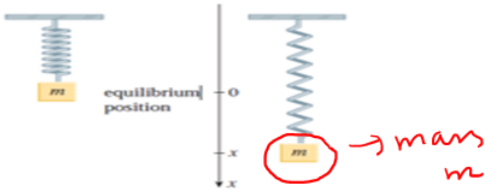

Figure – 1

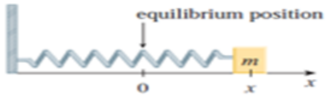

Figure – 2

If the string is stretched x units from its natural length then it exerts force (restoring force) that is proportional to x:

Restoring force $= -kx$. Where k is a positive constant (called the spring constant). If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's second law, we have $m\dfrac{d^2x}{dt^2}=-kx$ (or) $m\dfrac{d^2x}{dt^2}+kx=0$.

Its auxiliary equation is $mr^2+k=0$ with roots $r=\pm\omega i$, where $\omega=\sqrt{k/m}$.

Thus the solution is

$x(t)=c_1\cos\omega t+c_2\sin\omega t$ (or) $x(t)=A\cos(\omega t+\delta)$

where $\omega=\sqrt{k/m}$ (frequency) $A=\sqrt{c_1^2+c_2^2}$ (amplitude)

$\cos\delta=\dfrac{c_1}{A}$ $\sin\delta=-\dfrac{c_2}{A}$ (δ is the phase angle)

This type of motion is called **simple harmonic motion.**

DAMPED VIBRATIONS

The motion of a spring that is subject to a frictional force (in the case of horizontal spring of figure 2) or a damping force (in the case of vertical spring moves through a fluid as in Figure 3). An example is the damping force supplied by a shock absorber in a car or a bicycle.

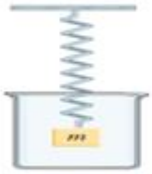

Figure – 3

The damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion.

$damping\ force=-c\dfrac{dx}{dt}$ where c is a positive constant, called the **damping constant.** In this case, Newton's second Law gives

$m\dfrac{d^2x}{dt^2}=restoring\ force+damping\ force=-kx-c\dfrac{dx}{dt}$ (or)

$m\dfrac{d^2x}{dt^2}+c\dfrac{dx}{dt}+kx=0$

Its auxiliary equation is $mr^2+cr+k=0$. The roots are

$r_1=\dfrac{-c+\sqrt{c^2-4mk}}{2m}$ $r_2=\dfrac{-c-\sqrt{c^2-4mk}}{2m}$

There are three cases shown below. The solution curves of the differential equations are going to be visualized by solving the problems.

CASE I $c^2 - 4mk > 0$ **(overdamping)**

CASE II $c^2 - 4mk = 0$ **(critical damping)**

CASE III $c^2 - 4mk < 0$ **(underdamping)**

Example 1:

Suppose that the spring with a mass of 2 kg has natural length 0.5 m. A force of 25.6 N is required to maintain it stretched to a length of 0.7 m. The spring is immersed in a fluid with damping constant $c = 40$. Find the position of the mass at any time t if it starts from the equilibrium position and is given a push to start with an initial velocity of 0.6m/s.

$m = 2\,\text{kg}$ damping constant $c = 40$

Solution:

$K = \frac{25.6}{2} \times 10 = 128$

The differential equation is $2x'' + 40x' + 128x = 0$ with $\underbrace{x(0) = 0; x'(0) = 0.6}$

$$\text{Solution } x \text{ for } 2x'' + 40x' + 128x = 0 \quad x(0) = 0; x'(0) = 0.6 \text{ in the interval } [0, 1.5].$$

$\boxed{\text{at } t=0 \quad x=0}$

Here $m = 2, c = 40, k = 128$ Therefore $c^2 - 4mk = 576 > 0$. $\underbrace{\text{Overdamping.}}$

Therefore no oscillations occur.

$\left(\frac{dx}{dt}\right)_{t=0} = 0.6$

$\boxed{x'(0) = 0.6}$

Suppose that the spring with a mass of 2 kg has natural length 0.5 m. A force of 25.6 N is required to maintain it stretched to a length of 0.7m. If the spring is stretched to a length of 0.7m and then released with initial velocity 0. The spring is immersed in a fluid with damping constant $c = 32$. Find the position of the mass at any time t if it starts from the equilibrium position and is given a push to start with an initial velocity of 0.6m/s.

Solution:

The differential equation is $2x'' + 32x' + 128x = 0$ with $x(0) = 0; x'(0) = 0.6$

Inference:

Solution x for $2x'' + 32x' + 128x = 0$ $x(0) = 0; x'(0) = 0.6$ in the interval $[0, 1.5]$.

Here $m = 2, c = 32, k = 128$ Therefore $c^2 - 4mk = 0$. **Critical damping.**