Linearity

If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$, then
$L\{a\,f_1(t) + b\,f_2(t)\} = a\,F_1(s) + b\,F_2(s)$
where a and b are constants.

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

$$L\{af_1(t) + bf_2(t)\} = \int_0^\infty e^{-st}\{af_1(t) + bf_2(t)\}\,\mathrm{d}t$$

$$= a\int_0^\infty e^{-st} f_1(t)\,\mathrm{d}t + b\int_0^\infty e^{-st} f_2(t)\,\mathrm{d}t$$

$$= a\,F_1(s) + b\,F_2(s)$$

Change of Scale

If $L\{f(t)\} = F(s)$, then $L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\mathrm{d}t$$

Putting $at = x$, $\mathrm{d}t = \frac{\mathrm{d}x}{a}$

$$L\{f(at)\} = \int_0^\infty e^{-s\left(\frac{x}{a}\right)} f(x)\frac{\mathrm{d}x}{a} = \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)x} f(x)\mathrm{d}x$$

$$= \frac{1}{a}F\left(\frac{s}{a}\right)$$

First Shifting Theorem

If $L\left\{f(t)\right\} = F(s)$, then $L\left\{e^{-at} f(t)\right\} = F(s+a)$

Proof:

$$L\left\{f(t)\right\} = \int_0^{\infty} e^{-st} f(t)\mathrm{d}t$$

$$L\left\{e^{-at} f(t)\right\} = \int_0^{\infty} e^{-st} e^{-at} f(t)\mathrm{d}t = \int_0^{\infty} e^{-(s+a)t} f(t)\mathrm{d}t = F(s+a)$$

Second Shifting Theorem

If $L\{f(t)\} = F(s)$

and $g(t) = f(t-a) \quad t > a$

$\quad\quad = 0 \quad\quad t < a$

then $L\{g(t)\} = e^{-as} F(s)$

Proof: $L\{g(t)\} = \int_0^\infty e^{-st} g(t)\mathrm{d}t = \int_0^a e^{-st} \cdot 0 \, \mathrm{d}t + \int_a^\infty e^{-st} f(t-a)\mathrm{d}t$

Putting $t-a=x \quad \mathrm{d}t = \mathrm{d}x \quad$ When $t=a,\ x=0 \quad t\to\infty,\ x\to\infty$

$$L\{g(t)\} = \int_0^\infty e^{-s(a+x)} f(x)\mathrm{d}x = e^{-as}\int_0^\infty e^{-sx} f(x)\mathrm{d}x = e^{-as}\int_0^\infty e^{-st} f(t)\mathrm{d}t = e^{-as}F(s)$$

Multiplication by *t*

If $L\{f(t)\} = F(s)$, then $L\{t^n f(t)\} = (-1)^n \dfrac{\mathrm{d}^n}{\mathrm{d}s^n} F(s)$

Proof: $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t$

$$\frac{\mathrm{d}}{\mathrm{d}s}F(s) = \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty e^{-st} f(t)\mathrm{d}t = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\mathrm{d}t$$

$$= \int_0^\infty (-t\, e^{-st}) f(t)\mathrm{d}t = \int_0^\infty e^{-st}\{-t\, f(t)\}\mathrm{d}t = -L\{t\, f(t)\}$$

$$L\{t\, f(t)\} = (-1)\frac{\mathrm{d}}{\mathrm{d}s}F(s) \quad \text{Similarly,} \quad L\{t^2 f(t)\} = (-1)^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2}F(s)$$

In general, $L\{t^n\, f(t)\} = (-1)^n \dfrac{\mathrm{d}^n}{\mathrm{d}s^n}F(s)$

Division by t

If $\quad L\{f(t)\} = F(s)$, then $L\left\{\dfrac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

Proof: $\quad L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

Integrating both the sides w.r.t s from s to ∞,

$$\int_s^\infty F(s)\,ds = \int_s^\infty \int_0^\infty e^{-st} f(t)\,dt\,ds$$

Since s and t are independent variables, interchanging the order of integration,

$$\int_s^\infty F(s)\,ds = \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\,ds\right] dt = \int_0^\infty \left[\frac{e^{-st}}{-t} f(t)\right]_s^\infty dt$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t}\,dt = L\left\{\frac{f(t)}{t}\right\}$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

Laplace Transforms of Derivatives

If $L\{f(t)\} = F(s)$, then

$L\{f'(t)\} = s\,F(s) - f(0)$

$L\{f''(t)\} = s^2\,F(s) - s\,f(0) - f'(0)$

In general

$L\{f^n(t)\} = s^n\,F(s) - s^{n-1}\,f(0) - s^{n-2}\,f'(0) - s^{n-3}\,f''(0) \ldots - f^{(n-1)}(0)$

Proof: $L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$

$L\{f'(t)\} = \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-se^{-st}) f(t)\,dt = -f(0) + s\int_0^\infty e^{-st} f(t)\,dt = -f(0) + s\,L\{f(t)\}$

Similarly, $\quad L\{f''(t)\} = -f'(0) + s\,L\{f'(t)\} = -f'(0) + s\left[-f(0) + sL\{f(t)\}\right] = -f'(0) - s\,f(0) + s^2 L\{f(t)\}$

In general, $\quad L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \ldots - f^{n-1}(0)$

Laplace Transforms of Integrals

If $L\{f(t)\} = F(s)$, then $L\left\{\int_0^t f(t)\mathrm{d}t\right\} = \dfrac{F(s)}{s}$

Proof: $L\left\{\int_0^t f(t)\mathrm{d}t\right\} = \int_0^\infty e^{-st}\left\{\int_0^t f(t)\mathrm{d}t\right\}\mathrm{d}t$

Integrating by parts

$$L\left\{\int_0^t f(t)\mathrm{d}t\right\} = \left[\int_0^t f(t)\mathrm{d}t\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty - \int_0^\infty\left[\left(\frac{e^{-st}}{-s}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(t)\mathrm{d}t\right)\right]\mathrm{d}t$$

$$= \int_0^\infty \frac{1}{s}e^{-st} f(t)\mathrm{d}t = \frac{1}{s}L\{f(t)\} = \frac{F(s)}{s}$$

Initial Value Theorem

If $L\{f(t)\} = F(s)$, then $\lim\limits_{t\to 0} f(t) = \lim\limits_{s\to\infty} s\,F(s)$

Proof: We know that,

$$L\{f'(t)\} = s\,F(s) - f(0)$$

$$s\,F(s) = L\{f'(t)\} + f(0) = \int_0^\infty e^{-st} f'(t)\mathrm{d}t + f(0)$$

$$\lim_{s\to\infty} s\,F(s) = \lim_{s\to\infty}\int_0^\infty e^{-st} f'(t)\mathrm{d}t + f(0)$$

$$= \int_0^\infty \lim_{s\to\infty}\left[e^{-st} f'(t)\right]\mathrm{d}t + f(0)$$

$$= 0 + f(0) = f(0) = \lim_{t\to 0} f(t)$$

Final Value Theorem

If $L\left\{f(t)\right\} = F(s)$, then $\lim\limits_{t\to\infty} f(t) = \lim\limits_{s\to 0} s\,F(s)$

Proof: $L\left\{f'(t)\right\} = s\,F(s) - f(0)$

$$s\,F(s) = L\left\{f'(t)\right\} + f(0) \;= \int_0^\infty e^{-st} f'(t)\mathrm{d}t + f(0)$$

$$\lim_{s\to 0} s\,F(s) = \lim_{s\to 0}\int_0^\infty e^{-st} f'(t)\mathrm{d}t + f(0)$$

$$= \int_0^\infty \lim_{s\to 0}\left[e^{-st} f'(t)\right]\mathrm{d}t + f(0) \;= \int_0^\infty f'(t)\,\mathrm{d}t + f(0)$$

$$= \left|f(t)\right|_0^\infty + f(0) \;\; = \lim_{t\to\infty} f(t) - f(0) + f(0) \;= \lim_{t\to\infty} f(t)$$