System of first order linear DE's with constant coefficients – Solution by Laplace Transform method.

1. $\dfrac{dx}{dt} = -x + y$

$\dfrac{dy}{dt} = 2x$

$x(0) = 0, \quad y(0) = 1$

Sol: Let $L[x(t)] = X(s)$ and

$L[y(t)] = Y(s)$.

Applying Laplace transform on the system, we get

$$L[x'] = -L[x] + L[y]$$

$$L[y'] = 2L[x]$$

i.e., $$sX(s) - x(0) = -X(s) + Y(s)$$

$$\Rightarrow (s+1)X(s) = Y(s)$$

$$\boxed{(s+1)X(s) - Y(s) = 0} \quad \text{— (A)}$$

$$sY(s) - y(0) = 2X(s)$$

$$sY(s) - 2X(s) = y(0)$$

$$\boxed{-2X(s) + sY(s) = 1} \quad \text{— (B)}$$

Ⓐ & Ⓑ represent a linear system of two equations in two unknowns $X(s)$ and $Y(s)$.

By Cramer's rule, we have

$$X(s) = \frac{\begin{vmatrix} 0 & -1 \\ 1 & s \end{vmatrix}}{\begin{vmatrix} s+1 & -1 \\ -2 & s \end{vmatrix}} = \frac{1}{s(s+1)-2}$$

$$\boxed{X(s) = \frac{1}{s^2+s-2}}$$

$$Y(s) = \frac{\begin{vmatrix} s+1 & 0 \\ -2 & 1 \end{vmatrix}}{\begin{vmatrix} s+1 & -1 \\ -2 & s \end{vmatrix}} = \frac{s+1}{s^2+s-2}$$

$\therefore$ $X(s) = \dfrac{1}{s^2+s-2}$ and

$$Y(s) = \frac{s+1}{s^2+s-2}$$

Applying inverse LT, we get

$$x(t) = L^{-1}[X(s)] \text{ and}$$

$$y(t) = L^{-1}[Y(s)].$$

$$x(t) = L^{-1}\left[\frac{1}{s^2+s-2}\right]$$

$$= L^{-1}\left[\frac{1}{(s+\frac{1}{2})^2-\frac{1}{4}-2}\right]$$

$$= L^{-1}\left[\frac{1}{(s+\frac{1}{2})^2-(\frac{3}{2})^2}\right]$$

$$= \frac{2}{3} L^{-1}\left[\frac{(3/2)}{(s+\frac{1}{2})^2-(\frac{3}{2})^2}\right]$$

$$x(t) = \frac{2}{3} e^{-\frac{1}{2}t} \sinh\left(\frac{3}{2}t\right)$$

Now,

$$y(t) = L^{-1}[Y(s)]$$

$$= L^{-1}\left[\frac{s+1}{s^2+s-2}\right]$$

$$\Rightarrow \quad y(t) = L^{-1}\left[\frac{s+1}{\left(s+\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\right]$$

$$s+1 = s+\frac{1}{2}+\frac{1}{2}$$

$$= s+\frac{1}{2} + \frac{1}{3}\left(\frac{3}{2}\right)$$

$$= L^{-1}\left[\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\right] +$$

$$\frac{1}{3} L^{-1}\left[\frac{(3/2)}{\left(s+\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\right]$$

$$\therefore \quad y(t) = e^{-\frac{1}{2}t}\cosh\left(\frac{3}{2}t\right) + \frac{1}{3}e^{-\frac{1}{2}t}\sinh\left(\frac{3}{2}t\right)$$

Hence the required solution of the given system is

$$x(t) = \frac{2}{3}e^{-t/2}\sinh\left(\frac{3}{2}t\right)$$

$$y(t) = e^{-t/2}\left[\cosh\left(\frac{3}{2}t\right) + \frac{1}{3}\sinh\left(\frac{3}{2}t\right)\right]$$

$$=$$

Check: $x(0) = 0$, $y(0) = 1$