Differential Equations and Transforms BMAT102L

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Module:1 Ordinary Differential Equations (ODE)

Second order non- homogenous differential equations with constant coefficients- Differential equations with variable coefficients- method of undetermined coefficients-method of Variation of parameters- Solving Damped forced oscillations and LCR circuit theory problems.

15|2|22.

Intro :

diff eqn — derivatives.
one or more. dependent Variable.

eg $$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

$$x\frac{dy}{dx} = y-1$$

D. E

Types
1) Ordinary. DE

order
1^{st}, 2^{nd}.

linearity
linear
nonlinear..

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \quad ✓$$

2) Partial DE

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

Order → highest derivative → 2.

degree → 1

linear.

$$y\frac{dy}{dx} + \sin x = 0.$$

→ non linear

$y'' + y' = \sqrt{x}$ → linear.

order → 2.

$y^2 + y' = \sqrt{x}$ → non-linear

$a_0(x)y^n + a_{n-1}(x)y^{n-1} + \dots$ → linear.

Homogeneous & Non Homogeneous eqn.

$$a_0 y''' + a_1 y'' + a_2 y' + a_3 y = g(x)$$

→ non homog.

$g(x) = 0$ → homogeneous.

$\frac{dy}{} \; P \dots$

$\frac{}{dx} + \cdot y = Q$.

P, Q - fun of x. I.F

Second order linear DE

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)\,y = G(x) \quad (1)$$

$P, Q, R, G \to$ continu. f.

non-homogeneous.

If

$G(x) = 0$ in eqn (1) $\to$ homogeneous eqn

Thm:-

If $y_1(x)$ and $y_2(x) \to$ solns of a linear homogeneous eqn.

$$p(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)\,y = 0 \quad \text{—— (2)}$$

then,

y'' $\quad$ y'

$y(x) = \underline{c_1 y_1(x) + c_2 y_2(x)}$ is a soln. of eqn (2)

$c_1, c_2 \rightarrow$ constant

$y' = c_1 y_1' + c_2 y_2'$

$y'' = c_1 y_1'' + c_2 y_2''$

Pf :- If y_1 & y_2 are soln. of eqn (2)

$$\left.\begin{aligned} p(x)\,y_1'' + Q(x)\,y_1' + R(x)\,y_1 &= 0 \\ p(x)\,y_2'' + Q(x)\,y_2' + R(x)\,y_2 &= 0 \end{aligned}\right\} \text{—— } x$$

$P(x)\,y'' + Q(x)\,y' + R(x)\,y$

$= P(x)\left[c_1 y_1'' + c_2 y_2''\right] + Q(x)\left[c_1 y_1' + c_2 y_2'\right] + R(x)\left[c_1 y_1 + c_2 y_2\right]$

$= c_1\left[\underline{P(x)\,y_1'' + Q(x)\,y_1' + R(x)\,y_1}\right] +$

$c_2\left[P(x)\,y_2'' + Q(x)\,y_2' + R(x)\,y_2\right]$

$= c_1\,(0) + c_2\,(0)$

$\ldots = 0$

Solution 1- :

soln

$y(x) = c_1 y_1(x) + c_2 y_2(x)$

$P(x)$, $Q(x)$, $R(x) \to$ constant

$\downarrow$ a, $\downarrow$ b, $\downarrow$ c

eqn reduces to.

$$a y'' + b y' + c y = 0 \qquad \text{--- (3).}$$

Auxiliary eqn. or characteristic eqn.

$$a m^2 + b m + c = 0$$

quadratic.

let $y = e^{mx}$.

$y' = m e^{mx}$

$y'' = m^2 e^{mx}$

Case i) $b^2 - 4ac > 0$

roots are real & distinct.

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Case (ii) $b^2 - 4ac = 0$

roots are equal.

$m_1 = m_2 = m$.

$$y = c_1 e^{mx} + x c_2 e^{mx}.$$

case (iii) $b^2 - 4ac < 0$

complex roots $\alpha \pm i\beta$ (real) (imag.)

$$y = e^{\alpha x} (c_1 \cos\beta x + c_2 \sin\beta x)$$

Summary :-

(i) real & distinct

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

(ii) equal.

$$y = e^{mx} (c_1 + x c_2)$$

$y = e \quad (c_1 + x c_2)$

(iii) complex.

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

Problems :-

1) Solve $y'' + y' - 6y = 0$

Second order linear homogeneous DE

Auxillary eqn

$$m^2 + m - 6 = 0 \qquad -6 \begin{array}{l} 3 \\ -2 \\ \hline 1 \end{array}$$

$$(m+3)(m-2) = 0$$

$$m = -3, 2$$

roots are real & distincts.

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

$$\boxed{y = c_1 e^{-3x} + c_2 e^{2x}.}$$

2) $3\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$

A.E $\quad 3m^2 + m - 1 = 0$

$$m = \frac{-1 \pm \sqrt{1+12}}{6} = \frac{-1 \pm \sqrt{13}}{6}$$

real & distinct

$$m_1 = \frac{-1+\sqrt{13}}{6} \qquad m_2 = \frac{-1-\sqrt{13}}{6}$$

$$y = c_1 e^{\frac{-1+\sqrt{13}}{6}x} + c_2 e^{\frac{-1-\sqrt{13}}{6}x}.$$

3) $4y'' + 12y' + 9y = 0$

A E $\quad 4m^2 + 12m + 9 = 0$

$$(2m+3)^2 = 0$$

$$m = -3/2, -3/2$$

real & equal - $m_1 = m_2 = m = -3/2$

$$y = e^{mx}(c_1 + xc_2)$$

$$= e^{-3/2 x}(c_1 + x c_2)$$

4) $y'' - 6y' + 13y = 0.$

A E $m^2 - 6m + 13 = 0.$

$$m = \frac{6 \pm \sqrt{36-52}}{2}.$$

$$= 3 \pm 2i$$

complex roots -

$\alpha = 3 \qquad \beta = 2.$

$y = e^{\alpha x}($

$\quad c_1 \cos \beta x + c_2 \sin \beta x)$

$= e^{3x} (c_1 \cos 2x + c_2 \sin 2x)$

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Non- homogeneous linear eqns.

$$ay'' + by' + cy = G(x)$$

$a, b, c \rightarrow$ constants

$G \rightarrow$ continuous fn.

Consider homogen. eq. $\rightarrow$ $ay'' + by' + cy = 0$

y_c 1 part. soln complementary soln.

non. homoge. $y_p \rightarrow$ particular soln.

General soln. $\boxed{y = y_c + y_p}$ $\quad \boxed{y = y_c}$

Methods for findin, y_p

1. Method of undetermined coeff.
2. " " variation of parameter.

Method of Undetermined Coefficient.

UDC.

$$ay'' + by' + cy = G(x)$$

Basic rules :-

$f(x)$	Initial guess for y_p
$P(x)$	$Q(x)$
$c e^{ax}$	$d e^{ax}$
$\alpha \cos bx$ (or) $\alpha \sin bx$	$c \cos bx + d \sin bx$.
[illegible] e^{ax}	$Q(x) e^{ax}$

$p(x)\cos bx$ (or) $a(x)\cos bx + h(x)\sin bx.$

$p(x)\sin bx$

$p(x)e^{ax}\cos bx$ $\quad Q(x)e^{ax}\cos bx + R(x)e^{ax}\sin bx.$

$p(x)$

Problems

1) solve $y'' + 4y' - 2y = 2x^2 - 3x + 6$ — (1)

Homogeneous part

$$y'' + 4y' - 2y = 0$$

A.E $\quad m^2 + 4m - 2 = 0$

$$m = -2 \pm \sqrt{6}$$

$$y_c = c_1 e^{(-2+\sqrt{6})x} + c_2 e^{(-2-\sqrt{6})x}. \quad — \; y_c$$

RHS → Choice of y_p

$2x^2 \rightarrow Ax^2 + Bx + C$

$-3x \rightarrow Dx + E$

$6 \rightarrow F$

$\dfrac{(B+D)x}{B}$

$2x^2 - 3x + 6 \qquad Ax^2 + Bx + C$

generalize

$$\left.\begin{aligned} \text{Choice of } y_p &= Ax^2 + Bx + C \\ y_p' &= 2Ax + B \\ y_p'' &= 2A \end{aligned}\right\} \text{— } 2$$

$$y_p'' + 4y_p' - 2y_p = 2x^2 - 3x + 6$$

$$2A + 4(2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 - 3x + 6$$

$$2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6$$

equating like ~~term~~

$$-2A = 2. \Rightarrow A = -1 \checkmark$$

$8A - 2B = -3$

$-8 - 2B = -3 \Rightarrow B = -5/2$ ✓

$2A + 4B - 2C = 6 \Rightarrow C = -9$ ✓

$$y_p = -x^2 - \frac{5}{2}x - 9$$

$$y = y_c + y_p$$
$$= c_1 e^{(-2+\sqrt{6})x} + c_2 e^{(-2-\sqrt{6})x} - x^2 - \frac{5}{2}x - 9.$$

---

2) $(D^2+1)y = \sin x$.

Consider Homogeneous part

$$(D^2+1)y = 0$$

$$\frac{d^2y}{dx^2} + y = 0.$$

$m^2 + 1 = 0 \Rightarrow m = \pm i$

A·E [illegible]

$$\alpha = 0 \qquad \beta = 1.$$

$$y_c = e^{0x}(c_1 \cos x + c_2 \cancel{\sin} \, x)$$

$$= c_1 \cos x + c_2 \sin x.$$

<u>RHS</u> $\rightarrow$ $\sin x$.

$$y_p = A\cos x + B\sin x.$$

[Note :- If choice of y_p term appears in y_c multiply of x] again x^2 . . . . .]

$$y_p = x(A\cos x + B\sin x)$$

$$y_p' = x(-A\sin x + B\cos x) + (A\cos x + B\sin x)(1)$$

$$y_p'' = x(-A\cos x - B\sin x) + (-A\sin x + B\cos x)(1) + (-A\sin x + B\cos x)$$

$$= -Ax\cos x - Bx\sin x - A\sin x + B\cos x - A\sin x + B\cos x.$$

$= -\sin x(2A + x) + \cos x(2B - Ax)$

$y'' + 1 = \sin x$

$-\sin x(2A + Bx) + \cos x(2B - Ax) + x(A\cos x + B\sin x) = \sin x.$

equation, $A = -1/2$ $B = 0$

$y_p = -x/2 \cos x.$

$y = y_c + y_p$

$=$ ______ $+$ ______ //..

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1) use the method of undetermined coeff. to solve the second order linear non-homogeneous ODE

$$\frac{d^2y}{dx^2} + \frac{2dy}{dx} + 4y = 2x^2 + 3e^{-x}.$$

Homogeneous eqn. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 0$

A.E $m^2 + 2m + 4 = 0$

$$m = \frac{-2 \pm \sqrt{4 - 16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$\sqrt{12}$

$\sqrt{4 \times 3}$

$$= -1 \pm \sqrt{3}\, i$$

The roots are complex. $\alpha = -1$

$$\boxed{y_c = e^{-x}(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x)}$$ $\beta = \sqrt{3}$

Choose y_p depending on RHS $2x^2 + 3e^{-x}$.

$$y_p = Ax^2 + Bx + C + De^{-x}$$

$$y_p' = 2Ax + B - De^{-x}$$

$$y_p'' = 2A + De^{-x}.$$

Sub y_p, y_p', y_p'' in the soln of given eqn.

$$y_p'' + 2y_p' + 4y_p = 2x^2 + 3e^{-x}.$$

$$2A + De^{-x} + 2(2Ax + B - De^{-x}) + 4(Ax^2 + Bx + C + De^{-x})$$
$$= 2x^2 + 3e^{-x}.$$

$$2A + De^{-x} + 4Ax + 2B - 2De^{-x} + 4Ax^2 + 4Bx + 4C + 4De^{-x}$$
$$= 2x^2 + 3e^{-x}.$$

equating coeffs

$$A = \tfrac{1}{2}$$
$$B = -\tfrac{1}{2}$$
$$C = 0$$
$$D = 1$$

$$y_p = \frac{1}{2}x^2 - \frac{1}{2}x + e^{-x}.$$

$$y = y_c + y_p$$

$$= \_\_\_\_ + \_\_\_\_$$

Try this

2) $y'' - 9y' + 14y = 3x^2 - 5\sin 2x + 7xe^{6x}.$

$m_1 = 2 \quad m_2 = 7$

find. $y_p = Ax^2 + Bx + C + \underline{D\cos 2x + E\sin 2x} + \underline{(Fx + G)e^{6x}}$.

Ans $A = 3/14 \quad B = \frac{27}{98} \quad C = \frac{201}{1372} \quad D = \frac{-4}{212} \quad E = \frac{-25}{212} \quad F = \frac{-7}{4} \quad G = \frac{-21}{16}$

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.$$

Method of Variation of parameters.

1. homogeneous eqn. $y = c_1 y_1 + c_2 y_2$

$y_1 :$

$y_2 :$

2. Assume $y = Ay_1 + By_2$ as soln.

wronskian :-

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$A = -\int \frac{Ry_2}{W} dx + c_1$$

$$B = \int \frac{R y_1}{W} dx + c_2 \qquad W \neq 0$$

$\therefore$ General soln. $y_p = A y_1 + B y_1$

Problem :

1) $y'' - 4y' + 4y = \underline{(x+1)e^{2x}}$. $\rightarrow R.$

Note :- (Coeff of y'' = 1.)

Homogeneous eqn.

$$y'' - 4y' + 4y = 0$$

A E $\quad m^2 - 4m + 4 = 0$

$$m = 2, 2.$$

$$y_c = (c_1 + c_2 x) e^{2x}.$$

$$y_c = c_1 e^{2x} + c_2 x e^{2x}.$$

To find P.I $(y_p) = ?$

$y_1 = e^{2x}$, $y_2 = x e^{2x}$.

$y_1' = 2e^{2x}$ $\qquad y_2' = 2x e^{2x} + e^{2x}$

$y_2 \begin{vmatrix} e^{2x} & x e^{2x} \end{vmatrix}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} & \\ 2e^{2x} & 2xe^{2x}+e^{2x} \end{vmatrix}$$

$$= e^{2x}(2xe^{2x}+e^{2x}) - xe^{2x} \cdot 2e^{2x}$$

$$= e^{4x}.$$

$$A = -\int \frac{R\,y_2}{W}\,dx + c_1$$

$$= -\int \frac{(x+1)e^{2x} \cdot xe^{2x}}{e^{4x}}\,dx + c_1$$

$$= -\int (x^2+x)\,dx + c_1$$

$$= -\left[\frac{x^3}{3} + \frac{x^2}{2}\right] + c_1.$$

$$B = \int \frac{R\,y_1}{W}\,dx + c_2$$

$$= \int \frac{(x+1)e^{2x} \cdot e^{2x}}{e^{4x}}\,dx + c_2$$

$$= \int (x+1)\, dx = \frac{x^2}{2} + x + c_2$$

soln.

$$y_p = Ay_1 + By_2$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + c_1\right) e^{2x} + \left[\frac{x^2}{2} + x + c_2\right) x e^{2x}.$$

$$= e^{2x}\left[-\frac{x^3}{3} - \frac{x^2}{2} + \frac{x^3}{2} + x^2\right] + c$$

$$y_p = e^{2x}\left[\frac{x^3}{6} + \frac{x^2}{2}\right] + c$$

$$y = y_c + y_p$$

$$= (c_1 + c_2 x) e^{2x} + e^{2x}\left(\frac{x^3}{6} + \frac{x^2}{2}\right) + c$$

21/2/22.

d^2y

2) $\frac{d^2y}{dx^2} + 4y = \tan 2x$.

A.E $\quad m^2 + 4 = 0$

$$m = \pm 2i \qquad \alpha = 0 \qquad \beta = 2$$

$$y_c = e^{0x}\left(c_1 \cos 2x + c_2 \sin 2x\right)$$

$$= c_1 \cos 2x + c_2 \sin 2x.$$

To find y_p :-

$$y_1 = \cos 2x \qquad y_2 = \sin 2x.$$

$$y_1' = -2\sin 2x \qquad y_2' = 2\cos 2x.$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x.$$

$$W = 2.$$

$$A = -\int \frac{R\, y_2}{W}\, dx$$

$$= -\int \frac{\tan 2x \cdot \sin 2x}{2}\, dx$$

$$= -\frac{1}{2}\int \frac{\sin 2x}{\cos 2x} \cdot \sin 2x\, dx$$

$$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x}\, dx$$

$$= -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\, dx$$

$$= -\frac{1}{2}\left[\int \frac{1}{\cos 2x} - \int \cos 2x\, dx\right]$$

$$= -\frac{1}{2}\int \sec 2x + \frac{1}{2}\int \cos 2x\, dx$$

$$= -\frac{1}{2}\cdot\frac{1}{2}\log(\sec 2x + \tan 2x) + \frac{1}{2}\left(\frac{\sin 2x}{2}\right)$$

$$= \frac{-1}{4} \log(\sec 2x + \tan 2x) + \frac{1}{4} \sin 2x$$

$$B = \int \frac{\tan 2x \quad \cos 2x}{2} \, dx = \frac{1}{2} \int \sin 2x \, dx.$$

$$= -\frac{1}{4} \cos 2x.$$

$$y_p = Ay_1 + By_2$$

$$= \frac{1}{4} \left[- \log(\sec 2x + \tan 2x) + \sin 2x \right] \cos 2x$$

$$-\frac{1}{4} \cos 2x \cdot \sin 2x.$$

$$y = y_c + y_p$$

3) $(D^2 + a^2)y = \sec ax.$

$$\frac{d^2y}{dx^2} + a^2 y = \sec ax.$$

$D = \frac{d}{dx}.$

$Dy = \frac{dy}{dx}.$

A.E $m^2 + a^2 = 0$

$$m = \pm ai$$

$$\alpha = 0 \qquad \beta = a.$$

$$y_c = e^{0x}(c_1 \cos ax + c_2 \sin ax)$$

$$= c_1 \cos ax + c_2 \sin ax.$$

To find y_p by the method of variation of parameters,

$$y_1 = \cos ax \qquad y_2 = \sin ax.$$

$$W = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a.$$

$$A = -\int \frac{R y_2}{W}\,dx = \frac{-1}{a^2} \log(\sec ax)$$

$$B = \int \frac{R y_1}{W}\,dx = \frac{x}{a}$$

$$y_p = \frac{-1}{a^2} \log(\sec ax)(\cos ax) + \frac{x}{a} \sin ax.$$

$y = y_c + y_p$

solve :-

$y'' + 2y' + y = 0$; Gn $y(0) = 1$

$y(1) = 3.$

A. E

$m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$m = -1, -1.$

Roots are real & equal.

$y_c = c_1 e^{-x} + c_2 \cdot x e^{-x}.$

$y = e^{-x}(c_1 + c_2 x).$

Gn $y(0) = 1$, $y(1) = 3.$

$y(x) = c_1 e \qquad + c_2 \cdot x$

$y(0) = 1.$ $\quad y(0) = c_1 + 0 = 1$

$\Rightarrow \boxed{c_1 = 1}$

Ilrly. $y(1) = 3$

$y(1) = c_1 e^{-1} + c_2 (1) e^{-1} = 3.$

wkg e. $\quad c_1 + c_2 = 3e$

$c_2 = 3e - 1$

soln. of the boundary value problm.

$y = e^{-x} + (3e-1)\, x e^{-x}.$

5) $(D^2 - 6D + 13) y = 2^x.$ Variat.

$= e^{\log 2^x}.$

$= e^{x \log 2}$

$= e^{(\log 2) x}.$ - R.

A.E $\quad m^2 - 6m + 13 = 0$

$$m = 3 \pm 2i$$

$\alpha = 3 \quad \beta = 2$

$$y_c = e^{3x}\left(c_1 \cos 2x + c_2 \sin 2x\right)$$

To find y_p?

23/2/22.

Cauchy & Legendre's eqn.

linear diff. eqns with variable coefficients:

General form of linear 2nd order DE

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{———} \; (1)$$

Euler - Cauchy Eqn:

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots a_n y = f(x)$$

— (2)

reduce → constant coeff.s

let $x = e^z$

$z = \log x$.

$$\frac{dz}{dx} = \frac{1}{x}.$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$$

$$x \frac{dy}{dx} = \frac{dy}{dz}.$$

$x \, Dy = D' y.$ ✓

$D = \frac{d}{dx}$

$Dy = \frac{dy}{dx}$

$D^2 y = \frac{d^2 y}{dx^2}$

$D' =$

$D' y = \frac{dy}{dz}$

$D'^2 y = \frac{d^2 y}{dz^2}$

Illy $x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz}$ ✓

$$x^2 D^2 y = D' y - D \quad = D'(D'-1) y$$

$$\text{IIrly} \quad x^3 D^3 y = D'(D'-1)(D'-2) y \ldots$$

Note :-

cauchy eqn.

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

$$(x^2 D^2 + x D + 1) y = 0$$

Let $x = e^z$

$\log x = z$

$x D = D'$

$x^2 D^2 = D'(D'-1)$

problems

1) $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = 0$

linear second order ODE with Variable coeff

This is of the form of Cauchys eqn.

Let $x = e^z$

$\log x = z$

$xD = D'$

$x^2 D^2 = D'(D'-1)$

$(x^2 D^2 - xD + 1) y = 0$

$(D'(D'-1) - D' + 1) y = 0$

$(D'^2 - D' - D' + 1) y = 0$

$(D'^2 - 2D' + 1) y = 0.$

$Dy = \frac{dy}{dx}$

$D'y = \frac{dy}{dz}$

A.E $m^2 - 2m + 1 = 0$

$(m-1) \quad = 0$

$m = 1, 1$ real & equal.

$y = c_1 e^{z} + z c_2 e^{z}$

$= c_1 e^{\log x} + c_2 \log x \, e^{\log x}.$

$y = x c_1 + x \log x \cdot c_2$

2) $(x^2 D^2 - 7x D + 12) y = x^2$

This is cauchy eqn.

$x = e^z$

$\log x = z$

$x D = D'$

$x^2 D^2 = D'(D'-1)$

$(D'(D'-1) - 7D' + 12) y = (e^z)^2$

$(D'^2 - D' - 7D' + 12) y = e^{2z}$

$2z$

$$(D^2 - 8D + 12)y = e$$

A E $\quad m^2 - 8m + 12 = 0$

$m = 2, 6$ real & unequal.

$$y = c_1 e^{2z} + c_2 e^{6z}.$$

using variation of parameter,

$y_1 = e^{2z}$ $\quad y_2 = e^{6z}$.

$$W = \begin{vmatrix} e^{2z} & e^{6z} \\ 2e^{2z} & 6e^{6z} \end{vmatrix} = 4e^{8z}$$

$$A = -z/4$$

$$B = \frac{e^{-4z}}{-16}$$

$$y_c = Ay_1 + By_2 = -\frac{z}{4} e^{2z} + \frac{e^{-4z}}{-16} e^{6z}$$

$$y = c_1 e^{2z} + c_2 e^{6z} - \frac{z}{4} e^{2z} - \frac{e^{2z}}{16}$$

$$= c_1 e^{2\log x} + c_2 e^{6\log x} - \frac{\log x}{4} e^{2\log x} - \frac{1}{16} e^{2\log x}.$$

$$= c_1 x^2 + c_2 x^6 - \frac{\log x}{4} \cdot x^2 - \frac{1}{16} x^2 .$$

3) $(x^2 D^2 + 4xD + 2) y = \log x .$

Cauchy ~~Eqn~~

$xDy = D'y$

$x = e^z$

$\log x = z$

$xD = D'$

$x^2 D^2 = D'(D'-1)$

$(D'^2 - D' + 4D' + 2) y = \log(e^z)$

$(D'^2 + 3D' + 2) y = z .$

AE $(m^2 + 3m + 2) = 0$

$m = -1, -2$

$y = c_1 e^{-z} + c_2 e^{-2z} .$

$y_p =$

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Legendre's linear diff. eqn.

$$(ax+b)^n \frac{d^n y}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1}}{dx^{n-1}} + \dots k_n y = Q.$$

$k_1, k_2 \dots k_n \to$ const.

$Q \to$ fn. of x.

$$(ax+b)^2 \frac{d^2 y}{dx^2} + k_1 (ax+b) \frac{dy}{dx} + y = Q.$$

$$\left((ax+b)^2 D^2 + \cancel{k}(ax+b) D + 1 \right) y = Q.$$

Assumption: $ax+b = e^z$

$z = \log(ax+b)$

Recall.

$(ax+b) D = a D'$

$D = \frac{d}{dx}$

$(ax+b)^2 D^2 = a^2 D' (D'-1)$

$D' = \frac{d}{dz}$

$(a x+b)$

$D^2 = \frac{d^2}{dx^2}$

$D'^2 = \frac{d^2}{dz^2}$.

Problem

1) $(x+2)^2 \frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + y = 3x+4$.

This is legendre's eqn.
2nd order ODE with variable coeff.

$x+2 = e^z$

$x = e^z - \cancel{2} - 2$.

$z = \log(x+2)$

$(x+2)D = (1)D'$ | $x+2$
$(x+2)^2 D^2 = (1)^2 D'(D'-1)$ | $ax+b$, $a=1$, $b=2$

Substituting.

$((x+2)^2 D^2 - (x+2)D + 1)y = 3x+4$

$(D'(D'-1) - D' + 1)y = 3x+4$

$(D'^2 - 2D' + 1)y = 3x+4 = 3(e^z-2)+4$
$= 3e^z - 2$.

(, 2nd order ODE with constant coeff.

A.E

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$y_c = c_1 e^z + c_2 z e^z$

$Ae^z \cdot z^2$

RHS $\Rightarrow 3e^z - 2$.

By the method of undetermined coeff.

choice of $y_p = z^2 A e^z + B$

$$y_p = A z^2 e^z + B$$

$$y_p' = A [z^2 e^z + 2ze^z]$$

$$y_p'' = A [z^2 e^z + 2ze^z + 2ze^z + 2e^z]$$

$$= A [z^2 e^z + 4ze^z + 2e^z]$$

$(D'^2 - 2D' + 1) y = 3e^z - 2$.

$A (z^2 e^z + 4ze^z + 2e^z) - 2A(\quad) + (\quad) =$

-2

$A = 3/2$

$B = 2$.

$A = 3/2 \qquad B = -2.$

$$y_p = \frac{3}{2} z^2 e^z - 2.$$

$$y = y_c + y_p$$

$$= c_1 e^z + c_2 z e^z + \frac{3}{2} z^2 e^z - 2.$$

$$= c_1 (x+2) + c_2 \log(x+2)\cdot(x+2) + \frac{3}{2}(\log(x+2))^2 (x+2) - 2.$$

$x = e^z - 2.$

$e^z = x+2$

$z = \log(x+2)$

2) $$(2x+3)^2 \frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x.$$

$$2x+3 = e^z$$

$$z = \log(2x+3)$$

$$2x = e^z - 3$$

$$x = \frac{e^z - 3}{2}.$$

$a = 2$

$b = 3$

$(2x+3)D = aD' = 2D'$

$(2x+3)^2 D^2 = a^2 D'(D'-1)$

$= (2)^2 D'(D'-1)$

$= 4D'(D'-1)$.

Sub.

$\left(D'^2 - \frac{3}{2}D' - 3\right)y = \frac{3}{4}(e^z - 3)$

$\frac{1}{4}(3e^z - 9)$

$m =$

y_c

y_p

$y = y_c + y_p$

Application.

LCR - circuit

26/2/2022

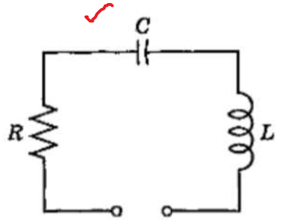
New Microsoft

APPLICATIONS OF SECOND ORDER ODE

LCR CIRCUITS

Kirchhoff's Voltage Law (KVL).[7] *The voltage (the electromotive force) impressed on a closed loop is equal to the sum of the voltage drops across the other elements of the loop.*

In Fig. 60 the circuit is a closed loop, and the impressed voltage $E(t)$ equals the sum of the voltage drops across the three elements R, L, C of the loop.



$E(t) = E_0 \sin \omega t$

Voltage Drops. Experiments show that a current I flowing through a resistor, inductor or capacitor causes a voltage drop (voltage difference, measured in volts) at the two ends; these drops are

RI	**(Ohm's law)** Voltage drop for a resistor of resistance R ohms (Ω),
$LI' = L\frac{dI}{dt}$	Voltage drop for an inductor of inductance L henrys (H),
$\frac{Q}{C}$	Voltage drop for a capacitor of capacitance C farads (F).

C

Here Q coulombs is the charge on the capacitor, related to the current by

$$I(t) = \frac{dQ}{dt}, \qquad \text{equivalently,} \qquad Q(t) = \int I(t)\, dt.$$

This is summarized in Fig. 61.

According to KVL we thus have in Fig. 60 for an RLC-circuit with electromotive force $E(t) = E_0 \sin \omega t$ (E_0 constant) as a model the **"integro-differential equation"**

$$(1') \qquad LI' + RI + \frac{1}{C}\int I\, dt = E(t) = E_0 \sin \omega t.$$

To get rid of the integral, we differentiate (1′) with respect to t, obtaining

$$(1) \qquad LI'' + RI' + \frac{1}{C} I = E'(t) = E_0\omega \cos \omega t.$$

This shows that the current in an RLC-circuit is obtained as the solution of this nonhomogeneous second-order ODE (1) with constant coefficients.

From (1′), using $I = Q'$, hence $I' = Q''$, we also have directly

$$(1'') \qquad LQ'' + RQ' + \frac{1}{C} Q = E_0 \sin \omega t.$$

But in most practical problems the current $I(t)$ is more important than the charge $Q(t)$, and for this reason we shall concentrate on (1) rather than on (1″).

Gn PROBLEM

R

L

C

E

Find the current $I(t)$ in an RLC-circuit with $R = 11\ \Omega$ (ohms), $L = 0.1$ H (henry), $C = 10^{-2}$F (farad), which is connected to a source of voltage $E(t) = 100 \sin 400t$ (hence $63\frac{2}{3}$ Hz $= 63\frac{2}{3}$ cycles/sec, because $400 = 63\frac{2}{3} \cdot 2\pi$). Assume that current and charge are zero when $t = 0$.

Solution. ***Step 1. General solution of the homogeneous ODE.*** Substituting R, L, C, and the derivative $E'(t)$ into (1), we obtain

$$0.1I'' + 11I' + 100I = 100 \cdot 400 \cos 400t.$$

Hence the homogeneous ODE is $0.1I'' + 11I' + 100I = 0$. Its characteristic equation is

$$0.1\lambda^2 + 11\lambda + 100 = 0.$$

The roots are $\lambda_1 = -10$ and $\lambda_2 = -100$. The corresponding general solution of the homogeneous ODE is

$$L\frac{di}{dt} + Ri + \frac{q}{C} = E \quad \text{— (1)}$$

dependent → i, q

$$i = \frac{dq}{dt}$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E \quad \text{— (2)}$$

2nd order DE with constant coeff.

$$L\frac{di}{dt} + Ri + \frac{q}{C} = E \quad \text{— (1)}$$

diff. w.r. to t

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C} i = \frac{dE}{dt} \quad (3)$$

2nd OD E

$$i = \frac{dq}{dt}$$

$I_h(t) = c_1 e^{\;---} + c_2 e^{\;----}$.

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = \frac{dE}{dt}$$

2nd ODE

AE

$$0.1 \frac{d^2 i}{dt^2} + 11 \frac{di}{dt} + \frac{1}{10^{-2}} i = 100 \cdot 400 \cos 400t$$

$0.1 m^2 + 11 m + 100 = 0 \qquad m = -10, -100$

$y_c \qquad i_c = c_1 e^{-10t} + c_2 e^{-100t}$ real, unequal.

PS $\qquad i_p$

$i = i_c + i_p$

$i =$

Step 2. Particular solution I_p of (1). We calculate the reactance $S = 40 - 1/4 = 39.75$ and the steady-state current

$$I_p(t) = a \cos 400t + b \sin 400t$$

with coefficients obtained from (4)

$$a = \frac{-100 \cdot 39.75}{11^2 + 39.75^2} = -2.3368, \qquad b = \frac{100 \cdot 11}{11^2 + 39.75^2} = 0.6467.$$

Hence in our present case, a general solution of the nonhomogeneous ODE (1) is

(6) $$I(t) = c_1 e^{-10t} + c_2 e^{-100t} - 2.3368 \cos 400t + 0.6467 \sin 400t.$$

Step 3. Particular solution satisfying the initial conditions. How to use $Q(0) = 0$? We finally determine c_1 and c_2 from the initial conditions $I(0) = 0$ and $Q(0) = 0$. From the first condition and (6) we have

(7) $$I(0) = c_1 + c_2 - 2.3368 = 0, \qquad \text{hence} \qquad c_2 = 2.3368 - c_1.$$

$i(0) = 0$

$q(0) = 0$

Furthermore, using (1$'$) with $t = 0$ and noting that the integral equals $Q(t)$ (see the formula before (1$'$)), we obtain

$$LI'(0) + R \cdot 0 + \frac{1}{C} \cdot 0 = 0, \qquad \text{hence} \qquad I'(0) = 0.$$

Differentiating (6) and setting $t = 0$, we thus obtain

$$I'(0) = -10c_1 - 100c_2 + 0 + 0.6467 \cdot 400 = 0, \qquad \text{hence} \qquad -10c_1 = 100(2.3368 - c_1) - 258.68.$$

c_1 ?

The solution of this and (7) is $c_1 = -0.2776$, $c_2 = 2.6144$. Hence the answer is

$$I(t) = -0.2776 e^{-10t} + 2.6144 e^{-100t} - 2.3368 \cos 400t + 0.6467 \sin 400t.$$

Figure 62 on p. 96 shows $I(t)$ as well as $I_p(t)$, which practically coincide, except for a very short time near $t = 0$ because the exponential terms go to zero very rapidly. Thus after a very short time the current will practically execute harmonic oscillations of the input frequency $63\frac{2}{3}$ Hz $= 63\frac{2}{3}$ cycles/sec. Its maximum amplitude and phase lag can be seen from (5), which here takes the form

$$I_p(t) = 2.4246 \sin(400t - 1.3008). \quad \blacksquare$$

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DAMPED FORCED VIBRATIONS.

Forces acting upon the mass.

Forces. ↓ the

1) $F_1 \to$ force of gravity with magnitude mg.

$F = mg$

$F_1 = mg$

2) $F_2 \rightarrow$ restoring force of spring.

$x + l \rightarrow$ elongation of string

Hooks law, $\rightarrow$ spring constant

$F = k\,s \rightarrow$ amount of elongation

magnitude of force.

$F_2 = -k(x+l) = -kx - kl$ force.

$F = -mg$

$-kx - kl = -mg$

If $x = 0 \Rightarrow mg = kl$.

$F_2 = -kx - kl$

$F_2 = -kx - mg$

$F_3 \rightarrow$ resisting force called damping

$F_3 = -a\,\dfrac{dx}{dt}$ $\quad a > 0$

↳ damping
coeff.

$F_4 \rightarrow$ external impressed force. $\underline{\underline{F(t)}}$

$F = ma$

$F = F_1 + F_2 + F_3 + F_4$

$$\cancel{m}g - kx - \cancel{m}g - a\frac{dx}{dt} + \cancel{F} + F(t) = ma$$

$$= m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = F(t)$$

2nd order ODE with constant coeff.

<u>Note :-</u>

If $a = 0 \Rightarrow$ undamped.

$\therefore$ eqn $\underline{m\frac{d^2x}{dt^2} + kx = 0}$.

If $a^2 - 4mk = 0 \Rightarrow$ critically damped

$< 0 \Rightarrow$ under damped

$> 0 \Rightarrow$ over damped.

Forced oscillation.

$$F(t) = F_1 \cos \omega t \quad \forall\ t \geq 0.$$

$\therefore$ eqn

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = F_1 \cos \omega t.$$

1 inch = 1 ft
6 inch = 1/2

Problem:

1. **A 32 lb weight is attached to the lower end of a coil spring suspended from the ceiling. The weight comes to rest in its equilibrium position, there by stretching the spring 2 ft. The weight is then pulled down 6 inch below its equilibrium position and released at t=0. No external forces are present, but the resistence of the medium in pounds is numerically equal to 4(dx/dt) where dx/dt is the instantaneous velocity in feet per second. Determine the resulting motion of the weight of the spring.**

$F = 32$

$s = 2.$

act → force due to gravity.

By Hooks law

$$F = ks$$

$$32 = k(2) \Rightarrow k = 16$$

32 = 1

$$m = \frac{w}{g} = \frac{\text{--}}{32} = 1$$

damping constant $a = 4$

eqn. $$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0$$

(1) $$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 16x = 0$$

A E $\quad m^2 + 4m + 16 = 0.$

$$m = \frac{-4 \pm \sqrt{16 - 64}}{2}$$

$$= \frac{-4 \pm \sqrt{-48}}{2}$$

$$= \frac{-4 \pm 4\sqrt{3}i}{2}$$

$$= -2 \pm 2\sqrt{3}i$$

$\alpha = -2 \qquad \beta = 2\sqrt{3}.$

$$x = e^{-2t}\left(A\cos 2\sqrt{3}\,t + B\sin 2\sqrt{3}\,t\right).$$

Initial cond.

$$x(0) = 1/2 \quad =$$

$$x'(0) = 0$$

when $t = 0$, $x = 1/2$

$$1/2 = 1\left(A(1) + 0\right)$$

$$\Rightarrow \boxed{A = 1/2}$$

$$x = e^{-2t}\left(A\cos 2\sqrt{3}t + B\sin 2\sqrt{3}t\right).$$

$$\frac{dx}{dt} = e^{-2t}\left(-2\sqrt{3}A\sin 2\sqrt{3}t + 2\sqrt{3}B\cos 2\sqrt{3}t\right)$$

$$+ \left(A\cos 2\sqrt{3}t + B\sin 2\sqrt{3}t\right)\left(-2e^{-2t}\right).$$

$$x'(0) = 0.$$

$$\Rightarrow \quad B = \frac{1}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{6}$$

$$x = e^{-2t}\left(\frac{1}{2}\cos 2\sqrt{3}t + \frac{\sqrt{3}}{6}\sin 2\sqrt{3}t\right).$$

2) 16 lb wt is attached - lower end

Spring constant $\to$ 10 lb / ft

wt comes to rest in equilibrium position.

at $t = 0$ an external ~~position~~. force

is gn. $f(t) = 5 \cos 2t$

damped force $\to$ $2\left(\frac{dx}{dt}\right)$

Sln

$k = 10 \qquad a = 2,$

$$m = \frac{w}{g} = \frac{16}{32} = \frac{1}{2}$$

$F(t) = 5 \cos 2t$

Initial cond.

$x(0) = 0$

$x'(0)$

$x(0) = 0.$

eqn.

$$m \cancel{\frac{d^2x}{dt^2}} + a\frac{dx}{dt} + kx = F(t)$$

$$\frac{1}{2}\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 5\cos 2t$$

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 20x = 10\cos 2t$$

AE $\quad m^2 + 4m + 20 = 0$

$$m = \frac{-4 \pm \sqrt{16 - 80}}{2} = -2 \pm 4i$$

$$x_c = e^{-2t}(A\cos 4t + B\sin 4t)$$

$$x_p = \frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t.$$

$$x = e^{-2t}(A\cos 2t + B\sin 2t) + \frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t$$

Initial cond. $\left.\begin{array}{l} x(0) = 0 \\ x'(0) = 0. \end{array}\right\}$

$A = -1/2$

$B = -3/8$.

$x =$

End module I.