FOURIER SERIES

INTRODUCTION

Periodic phenomena occur in many applications such as sound waves, flow of heat, mechanical vibrations, motion of earth and so on. These periodic phenomena are represented by periodic functions which may be complicated.

A representation of the periodic function in the form of a trigonometric series of sines and cosines was given by the French Mathematical physicist Jean Baptiste Joseph Fourier (1768-1830) in 1822 in his work on the mathematical theory of heat and is known as the Fourier series.

Such a representation had a profound influence on the general development of mathematical analysis over the past 150 years.

1. PERIODIC FUNCTION

A function $f(x)$ is said to be periodic with period T

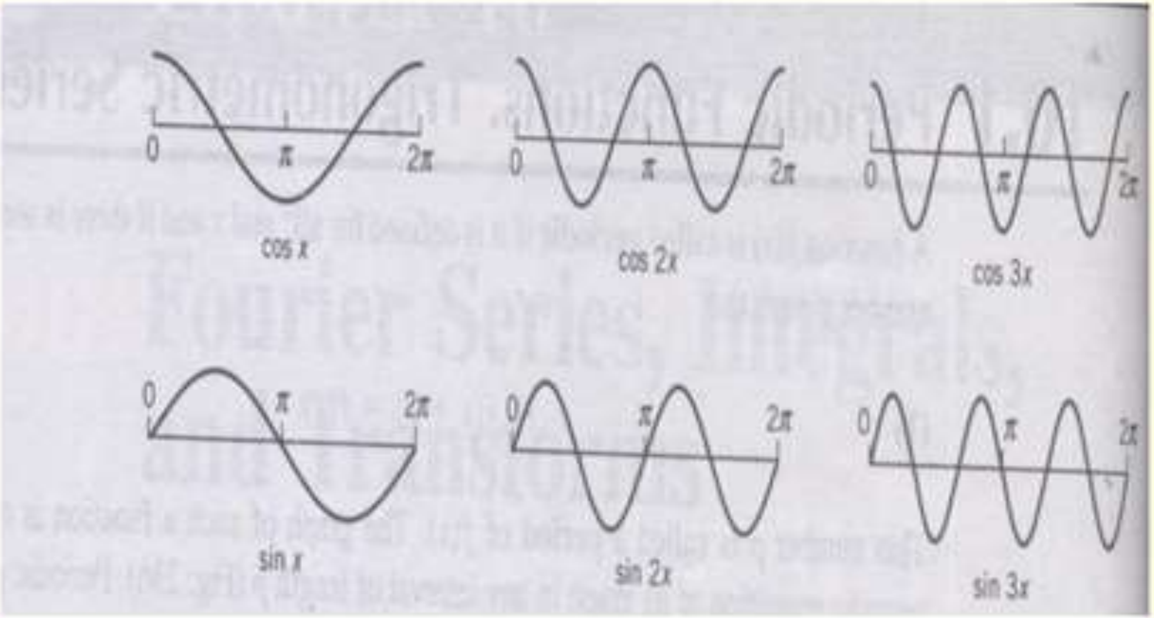
if $f(x+T) = f(x)\ for\ all\ x$, where T is a positive constant. The smallest value of T is called the fundamental period of $f(x)$ or simply period of $f(x)$.

For each positive integer n, it can be shown that $f(x+nT) = f(x)$ for all x and hence nT will also be a period of $f(x)$ when T is a period. i.e., Each member of the set $\{T, 2T, 3T, \dots, nT, \dots\}$ is a period of $f(x)$ and fundamental period is T.

Example: The function *f(x)=sin x* has periods 2π, 4π, 6π, …. and 2π is the fundamental period of *f(x).*

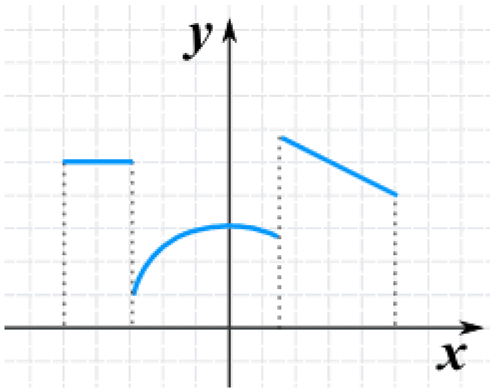
Example: The function *f(x)=tanx is a periodic function with period* π*.*

Example: The period of *sin nx* and *cos nx* is $2\pi/n$*.*


0
π
2π
cos x
cos 2x
cos 3x
sin x
sin 2x
sin 3x

2. Piecewise continuous function

A function is called piecewise continuous on an interval [a,b] if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval (i.e., subinterval without its endpoints) and has a finite limit at the endpoints of each subinterval.



3. Dirichlet's condition for a function $f(x)$

The Dirichlet's conditions for a function $f(x)$ defined on a finite interval $[a,b]$ are summarized by the following points :

- $f(x)$ is periodic, single-valued and finite in $[a,b]$
- $f(x)$ is either continuous or piecewise continuous with finite number of finite discontinuities in $[a,b]$
- $f(x)$ has at the most a finite number of maxima and minima in $[a,b]$

Note : Dirichlet's conditions are the sufficient conditions for the existence of the Fourier series of a function.

4. **Fourier series of** $f(x)$ in $[0, 2\pi]$

Let $f(x)$ be a function defined in an interval $[0, 2\pi]$. If $f(x)$ satisfies the Dirichlet's conditions, then the function $f(x)$ can be expanded as a Fourier series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx].$$

The Fourier coefficients are computed from the following **Euler's formulae**

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx \quad ;$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx \qquad ; \qquad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx$$

5. Fourier series of $f(x)$ in $[c, c+2L]$ by change of interval

Let $f(x)$ be a function defined in an interval $[c, c+2L]$. If $f(x)$ satisfies the Dirichlet's conditions, then the function $f(x)$ can be expanded as a Fourier series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right].$$

The Fourier coefficients are computed from the following **Euler's formulae**

$$a_0 = \frac{1}{L} \int_{c}^{c+2L} f(x)\,dx \qquad ;$$

$$a_n = \frac{1}{L} \int_{c}^{c+2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad ; \qquad b_n = \frac{1}{L} \int_{c}^{c+2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Case (i) : $c = 0$. $[c, c+2L] = [0, 2L]$.

Fourier series of $f(x)$ in $[0, 2L]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right],$$

where

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)dx \ ;$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \ ;$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Case (ii) : $c=-L$. $[c,c+2L]=[-L,L]$.

Fourier series of $f(x)$ in $[-L,L]$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right],$$

where

$$a_0=\frac{1}{L}\int_{-L}^{L}f(x)dx\ ;$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi x}{L}\right)dx\ ;$$

$$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

When $c = -\pi,\ L = \pi$. $[c, c+2L] = [-\pi, \pi]$.

Fourier series of $f(x)$ in $[-\pi, \pi]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right],$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \ ;$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \ ;$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

When $c=0,\ L=\pi$. $[c, c+2L]=[0,2\pi]$.

Fourier series of $f(x)$ in $[0,2\pi]$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n \cos nx+b_n \sin nx\right],$$

where

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)dx \ ;$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx \ ;$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$$

Bernoulli's formulae for integration by parts

(i) $\int u dv = uv - \int v du$

(ii) $\int uv dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

where

$u' = \frac{du}{dx}$ $\quad u'' = \frac{du'}{dx}$ $\quad u''' = \frac{du''}{dx}$ $\quad \ldots$

$v_1 = \int v dx$ $\quad v_2 = \int v_1 dx$ $\quad v_3 = \int v_2 dx$ $\quad \ldots$

Note : In the LIATE rule for deciding u, the order of preference is

L stands for logarithmic functions
I stands for inverse trigonometric functions
A stands for algebraic functions
T stands for trigonometric functions
E stands for exponential functions

The function lower on LIATE have easier antiderivatives than the functions above them and hence *dv* should be chosen from the functions lower on the LIATE.

Example1: Find the Fourier series for

$$f(x) = \begin{cases} 1 & -1 \le x < 0 \\ \frac{1}{2} & x = 0 \\ x & 0 < x \le 1 \end{cases}$$

Solution :

The general Fourier series on $[-L, L]$ is:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(\frac{k\pi x}{L}) + b_k \sin(\frac{k\pi x}{L})$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx, \;\; n = 0, 1, 2, ...$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx, \;\; n = 1, 2, 3, ...$$

For the present problem, we have

$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx \; = \int_{-1}^{0} dx + \int_{0}^{1} x\,dx = \frac{3}{2}$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{0} f(x)\cos(n\pi x)\,dx + \int_{0}^{1} f(x)\cos(n\pi x)\,dx$$

$$= \int_{-1}^{0} \cos(n\pi x)\,dx + \int_{0}^{1} x\cos(n\pi x)\,dx \;\; = \left[\frac{\sin(n\pi x)}{n\pi}\right]_{-1}^{0} + \left[\frac{x\sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{n^2\pi^2}\right]_{0}^{1}$$

$$= \frac{\cos(n\pi) - 1}{n^2\pi^2} = \frac{(-1)^n - 1}{n^2\pi^2} \;\; n = 1, 2, 3, ...$$

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_{-1}^{0} f(x)\sin(n\pi x)\,dx + \int_{0}^{1} f(x)\sin(n\pi x)\,dx$$

$$= \int_{-1}^{0} \sin(n\pi x)\,dx + \int_{0}^{1} x\;\sin(n\pi x)\,dx \;\; = \left[-\;\frac{\cos(n\pi x)}{n\pi}\right]_{-1}^{0} + \left[-\;\frac{x\;\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{n^2\pi^2}\right]_{0}^{1}$$

$$= -\;\frac{1}{n\pi}$$

So the Fourier series is:

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2} \cos(n\pi x) - \frac{1}{n\pi} \sin(n\pi x)$$

Setting $x = 0$ gives $\dfrac{\pi^2}{8} = 1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \ldots$

Example2: Find the Fourier series for $f(x) = x$ on $[0,1]$.

Solution:

Fourier series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2n\pi x + b_n \sin 2n\pi x$$

$$a_0 = 2\int_0^1 x\,dx = 1$$

$$a_n = 2\int_0^1 x \cos 2n\pi x \;\; dx \;\; = \;\; 0$$

$$b_n = 2\int_0^1 x \sin(2n\pi x)\,dx \quad = 2\left(\left[\frac{-x\cos 2n\pi x}{2n\pi}\right]_0^1 + \int_0^1 \frac{\cos 2n\pi x}{2n\pi}\,dx\right)$$

$$= \frac{-1}{n\pi} + \left[\frac{\sin 2n\pi x}{(2n\pi)^2}\right]_0^1 = \frac{-1}{n\pi}$$

So the Fourier series for $f(x)$ is

$$f(x) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n\pi}$$

Definitions of an Even function

- A function $f(x) = \phi(x)$ in $(-L, L)$ is said to be even if $\phi(-x) = \phi(x)$.

- A function $f(x) = \begin{cases} \phi_1(x) & x \in (-L, 0) \\ \phi_2(x) & x \in (0, L) \end{cases}$ in $(-L, L)$ is said to be even if

  $\phi_2(x) = \phi_1(-x)$ or $\phi_1(x) = \phi_2(-x)$.

Examples : x^4, $x^6 + x^2 + 2, cos2x, \sin^2(x)$, $|x|$

Definitions of an Odd function

- A function $f(x) = \phi(x)$ in $(-L, L)$ is said to be odd if $\phi(-x) = -\phi(x)$.

- A function $f(x) = \begin{cases} \phi_1(x) & x \in (-L, 0) \\ \phi_2(x) & x \in (0, L) \end{cases}$ in $(-L, L)$ is said to be odd if

  $\phi_2(x) = -\phi_1(-x)$ or $\phi_1(x) = -\phi_2(-x)$.

Examples : $x^3, sinx, tanx, x^5 + 2x$

Fourier series of an even function [Fourier cosine series]

The Fourier series of an even function $f(x)$ defined in $[-L, L]$ is given by

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$, where the Fourier coefficients are given by

$$a_0 = \frac{2}{L}\int_0^L f(x)dx \quad ; \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

6. Fourier series of an odd function [Fourier sine series]

The Fourier series of an odd function $f(x)$ defined in $[-L, L]$ is given by

$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$, where the Fourier coefficients are given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Example3

Find the Fourier series expansion of $f(x) = x$ in $-\pi < x < \pi$

SOLUTION:

$$f(x) = x \qquad f(-x) = -x = -f(x)$$

$\therefore$ $f(x)$ is odd function

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{-2(-1)^n}{n}$$

$$f(x) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx.$$

7. Fourier Convergence Theorem

Let the function $f(x)$ defined in $[c, c+2L]$ satisfies the Dirichlet's conditions. Then the Fourier series of $f(x)$ represented by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right],$$

converges to

- $f(x)$ if x is a point of continuity
- $\frac{1}{2}[f(x-) + f(x+)]$ if x is a point of discontinuity

Note :

(i) Dirichlet's conditions ensures the convergence of the Fourier series of $f(x)$ to the function $f(x)$. They are the sufficient conditions for the existence of the Fourier series.

(ii) At a point of discontinuity, say $x = x_d$, the Fourier series of the function will overshoot its value. Although as more terms are included, the overshoot moves to a position arbitrarily close to the discontinuity, it never disappears even in the limit of an infinite number of terms. This phenomenon is known as Gibb's phenomenon.

Thus, the Gibb's phenomenon represents the difficulty of Fourier series in approximating functions near their jump discontinuities.

The Gibbs phenomenon is shown below.

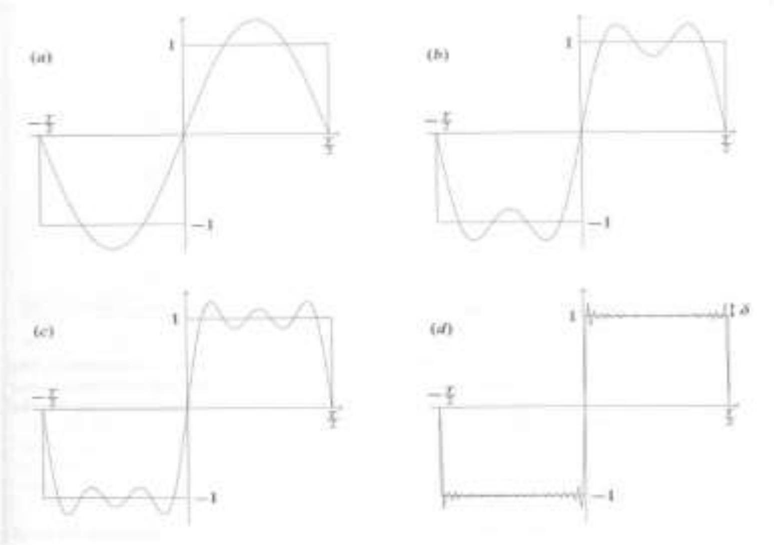


FIG. 3: The convergence of a Fourier series expansion of a square-wavefunction, including (a) one term, (b) two terms, (c) three terms and (d) 20 terms. The overshoot δ is shown in (d).

Example4

Find the Fourier series for $f(x) = x^2$ in $-\pi \leq x \leq \pi$ and deduce that

(i) $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots = \dfrac{\pi^2}{6}$ (ii) $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \cdots = \dfrac{\pi^2}{12}$

(iii) $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots = \dfrac{\pi^2}{8}$

Solution

$$f(x) = x^2$$

$$f(-x) = (-x)^2 = x^2 = f(x)$$

$\therefore$ $f(x)$ is even

$$\therefore \; f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 \, dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{3\pi}\left[\pi^3 - 0\right] = \frac{2}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(0 + 2\pi\frac{(-1)^n}{n^2} - 0\right) - (0 + 0 + 0)\right]$$

$$= \frac{2}{\pi}\left(2\pi\frac{(-1)^n}{n^2}\right) \quad = \frac{4(-1)^n}{n^2}$$

$$\therefore\ f(x) = \frac{\frac{2}{3}\pi^2}{2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx + \sum_{n=1}^{\infty} (0) \sin nx$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \qquad (1)$$

(i) By putting the point of continuity $x = \pi$ in Fourier convergence Theorem, we get

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \qquad (2)$$

(ii) **By putting the point of continuity $x = 0$ in Fourier convergence Theorem, we get**

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$$

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots = \frac{-\pi^2}{12}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12} \qquad (3)$$

(iii) Add the results of (2) and (3).

$$(2)+(3) \Rightarrow \quad \frac{2}{1^2}+\frac{2}{3^2}+\cdots=\frac{\pi^2}{6}+\frac{\pi^2}{12}$$

$$2\left(\frac{1}{1^2}+\frac{1}{3^2}+\cdots\right)=\frac{2\pi^2+\pi^2}{12}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\cdots=\frac{3\pi^2}{12\,(2)}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{8}.$$

Definition: *Given a function* $f(t)$ *defined on the interval* $0 \leq t < L$*, then we define the* **even periodic extension** *of* f *as the function defined for* $-\infty < t < \infty$*, which satisfies* $f(-t) = f(t)$ *and* $f(t + 2L) = f(t)$ *for all* t.

Definition: *Given a function* $f(t)$ *defined on the interval* $0 \leq t < L$*, then we define the* **odd periodic extension** *of* f *as the function defined for* $-\infty < t < \infty$*, which satisfies* $f(-t) = -f(t)$ *and* $f(t + 2L) = f(t)$ *for all* t.

Let $f(x)$ be a function defined and integrable on $[0, \pi]$. Set

$$f_1(x) = \begin{cases} -f(-x), & -\pi \le x < 0 \\ f(x), & 0 \le x \le \pi \end{cases} \quad \text{and}$$

$$f_2(x) = \begin{cases} f(-x), & -\pi \le x < 0 \\ f(x), & 0 \le x \le \pi \end{cases}$$

Then f_1 is odd and f_2 is even.

The function f_1 is called the **odd extension** of $f(x)$, while f_2 is called its **even extension**.

$f_1(x)$ and $f_2(x)$ are equal to $f(x)$ on $[0, \pi]$

8. Half range Fourier series

Suppose $f(x)$ is defined in the half range interval $(0, L)$ instead of the full range interval $(-L, L)$. The Fourier series of $f(x)$ in $(0, L)$ is called Half range Fourier series and can be represented by either a cosine series or a sine series, due to the two choices: (i) even extension of $f(x)$ and (ii) odd extension of $f(x)$.

8.1 Fourier Half range Cosine Series

If $f(x)$ is defined in $(0, L)$, then the Fourier half range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \text{ where}$$

$$a_0 = \frac{2}{L}\int_0^L f(x)dx \ ; \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

8.2 Fourier Half range Sine Series

If $f(x)$ is defined in $(0, L)$, then the Fourier half range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \text{ where}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Problem

Find the half range (i) cosine series and (ii) sine series for $f(x) = \pi - x$ in $(0, \pi)$.

9. Root Mean Square [R.M.S.] value of a function

The Mean Square value of a function $y(x)$ in [a,b], denoted by $\overline{y^2}$, is defined as

$$\overline{y^2} = \frac{1}{(b-a)}\int_a^b y^2 dx.$$

The Root Mean Square value of this function $y(x)$ is the square root of its mean square value, and is denoted by $\overline{y}$.

Note: If $y(x)$ is regarded as a signal, then the total power of the signal is its mean square value.

10. Parseval's Identity

If a function $y(x)$ defined in $[c, c+2L]$ has a Fourier series of the form

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right],$$

then its mean square value $\overline{y^2}$ is given by $\overline{y^2}=\frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}[a_n^2+b_n^2]$.

Proof : $$\overline{y^2}=\frac{1}{2L}\int_c^{c+2L} y^2 dx=\frac{1}{2L}\int_c^{c+2L} y\left\{\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right]\right\}dx$$

$$\overline{y^2}=\frac{a_0}{4L}\int_c^{c+2L} y dx+\frac{1}{2}\sum_{n=1}^{\infty}\left\{a_n\frac{1}{L}\int_c^{c+2L} y\cos\left(\frac{n\pi x}{L}\right)dx+b_n\frac{1}{L}\int_c^{c+2L} y\sin\left(\frac{n\pi x}{L}\right)dx\right\}$$

$$\overline{y^2}=\frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}[a_n^2+b_n^2]$$

Corollary 1 :

If the Fourier half-range cosine series of a function $y(x)$ defined in $[0, L]$ is $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$, then its mean square value $\overline{y^2}$ is given by

$$\overline{y^2}=\frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty} a_n^2 \text{, where } \overline{y^2}=\frac{1}{L}\int_0^L y^2 dx.$$

Corollary 2 :

If the Fourier half-range sine series of a function $y(x)$ defined in $[0, L]$ is $\sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right)$, then its mean square value $\bar{y}^2$ is given by

$$\overline{y^2}=\frac{1}{2}\sum_{n=1}^{\infty} b_n^2 \text{, where } \overline{y^2}=\frac{1}{L}\int_0^L y^2 dx.$$

Problems [R.M.S. & Parseval's Identity]

1. Find the root mean square value of the following functions

   (i) x^2 defined in $(-\pi, \pi)$

   (ii) x defined in $(1, 2)$

   (iii) $f(x) = x - x^2$ in (-1,1)

2. Find the mean square value of the function $f(x) = x - x^2$ in (-1,1) using Parseval's Identity

   Ans : $a_0 = -2/3$; $a_n = \dfrac{4}{n^2\pi^2}(-1)^{n+1}$; $b_n = \dfrac{2}{n\pi}(-1)^{n+1}$; $\sqrt{8/15}$

3. Prove that in $0 < x < L$,

$$x = \frac{L}{2} - \frac{4L}{\pi^2}\left[\cos\left(\frac{\pi x}{L}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{L}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{L}\right) + \ldots\right].$$

Hence, deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}$.

4. Prove that in $0 < x < \pi$,

$$\pi x - x^2 = \frac{\pi^2}{6} - \left[\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \ldots\right].$$

Hence, deduce that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

5. By using the half range sine series for $f(x) = 1$ in $0 < x < \pi$, show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^4}{8}.$$

11. COMPUTATION OF HARMONICS

Consider a function $f(x)$ represented in a tabular form shown below.

x	$c = x_0$	x_1	x_2	$\cdots$	x_{n-1}	$x_n = c + 2L$
$f(x)$	f_0	f_1	f_2	$\cdots$	f_{n-1}	$f_n = f_0$

We can have the Fourier series for this function as

$$f(x) \approx \frac{a_0}{2} + \sum_{m=1}^{\infty}\left[a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right].$$

The term $a_1 \cos\left(\frac{\pi x}{L}\right) + b_1 \sin\left(\frac{\pi x}{L}\right)$ is called the **<u>fundamental harmonic or first harmonic</u>**. The term $a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)$ is called the m^{th} harmonic.

The Fourier coefficients are computed numerically from Euler's formulae.

$$a_0 = \frac{1}{L}\int_c^{c+2L} f(x)dx = 2.\frac{1}{2L}\int_c^{c+2L} f(x)dx \text{ = 2 x mean value of } f(x) = \frac{2}{n}\sum_{i=0}^{n-1} f_i$$

$$\boxed{a_0 = \frac{2}{n}\sum_{i=0}^{n-1} f_i} \quad \text{or} \quad \boxed{a_0 = \frac{2}{n}\sum_{x} f(x)}$$

$$a_m = \frac{1}{L}\int_c^{c+2L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx = 2.\frac{1}{2L}\int_c^{c+2L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx$$

$$\text{= 2 x meanvalue of } f(x)\cos\left(\frac{m\pi x}{L}\right) = \frac{2}{n}\sum_{i=0}^{n-1} f_i \cos\left(\frac{m\pi x_i}{L}\right)$$

$$\boxed{a_m = \frac{2}{n}\sum_{i=0}^{n-1} f_i \cos\left(\frac{m\pi x_i}{L}\right)} \quad \text{or} \quad \boxed{a_m = \frac{2}{n}\sum_{x} f(x)\cos\left(\frac{m\pi x}{L}\right)}$$

$$b_m = \frac{1}{L}\int_{c}^{c+2L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = 2.\frac{1}{2L}\int_{c}^{c+2L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

= 2 x meanvalue of $f(x)\sin\left(\frac{m\pi x}{L}\right) = \frac{2}{n}\sum_{i=0}^{n-1} f_i \sin\left(\frac{m\pi x_i}{L}\right)$

$$\boxed{b_m = \frac{2}{n}\sum_{i=0}^{n-1} f_i \sin\left(\frac{m\pi x_i}{L}\right)} \quad \text{or} \quad \boxed{b_m = \frac{2}{n}\sum_{x} f(x)\sin\left(\frac{m\pi x}{L}\right)}$$

Worked Example : Find the first two harmonics of Fourier series from the following table.

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$
y	.8	0.6	0.4	0.7	0.9	1.1

Solution .

The Fourier series up to 2^{nd} harmonic is given by

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x$$

x	y	$\cos x$	$\cos 2x$	$\sin x$	$\sin 2x$	$y \cos x$	$y \cos 2x$	$y \sin x$	$y \sin 2x$
0	0.8	1	1	0	0	0.8	0.8	0	0
$\frac{\pi}{3}$	0.6	0.5	- 0.5	0.866	0.866	0.3	-0.3	0.52	0.52
$\frac{2\pi}{3}$	0.4	-0.5	-0.5	0.866	-0.866	-0.2	-0.2	0.346	-0.346
π	0.7	-1	1	0	0	-0.7	0.7	0	0
$\frac{4\pi}{3}$	0.9	-0.5	-0.5	-0.866	0.866	-0.45	-0.45	-0.78	0.78
$\frac{5\pi}{3}$	1.1	0.5	-0.5	-0.866	-0.866	0.55	-0.55	-0.953	-.953
	4.5					0.3	0	-0.867	.001

$$a_0 = 2 \times \frac{[\sum y]}{6} = 2 \times \frac{[4.5]}{6} = 1.5$$

$$a_1 = 2 \times \frac{1}{6}\left[\sum y \operatorname{coc} x\right] = 2 \times \frac{[0.3]}{6} = 0.1$$

$$a_2 = 2 \times \frac{1}{6}\left[\sum y \cos 2x\right] = 2 \times \frac{[0]}{6} = 0$$

$$b_1 = 2 \times \frac{1}{6}\left[\sum y \sin x\right] = 2 \times \frac{[-0.867]}{6} = -0.289$$

$$b_2 = 2 \times \frac{1}{6}\left[\sum y \sin 2x\right] = 2 \times \frac{[.001]}{6} = 0.00033$$

Substituting the values of $a_0, a_1, a_2, b_1,$ and b_2, we have

$$f(x) = \frac{1.5}{2} + 0.1 \cos x - 0 \times \cos 2x - 0.289 \sin x + 0.00033 \sin 2x$$

Exercise :

Find the Fourier cosine series, upto 3rd harmonics for $y(x)$ tabulated below.

X:	0	1	2	3	4	5
Y:	4	8	15	7	6	2

Hint : $2L = 6 \Rightarrow L = 3; \quad n = 6$

Ans : $a_0 = 14;\ \ a_1 = -2.8;\ \ a_2 = -1.5,\ \ a_3 = 2.7$