Module 4 – SOLUTION TO ODE & PDE BY L.T

LAPLACE TRANSFORM OF DERIVATIVES:

THEOREM: If f is continuous in $t \geq 0$, $f'(t)$ is piecewise continuous in every finite interval in the range $t \geq 0$ & $f(t)$ & $f'(t)$ are of exponential order, then

$$L(f'(t)) = sL(f(t)) - f(0)$$

Proof: $L(f'(t)) = \int_0^\infty e^{-st} f'(t)\,dt$

$dv = f'(t)\,dt$

$v = f(t)$

$u = e^{-st}$

$du = (-s)e^{-st}\,dt$

$$= \left(f(t)e^{-st}\right)_0^\infty - \int f(t)(-s)e^{-st}\,dt$$

$$= 0 - f(0)e^0 + s\underbrace{\int f(t)e^{-st}\,dt}_{L(f(t))}$$

$$L(f'(t)) = sL(f(t)) - f(0)$$

Similarly $L(f''(t)) = s^2L(f(t)) - sf(0) - f'(0)$

1. Using L.T solve $L\frac{di}{dt} + Ri = Ee^{-at}$, $i(0) = 0$, L, R, E & a are constants.

$$Li' + Ri = E(e^{-at})$$

Taking Laplace transform

$$\mathcal{L}(Li') + R\mathcal{L}(i) = E\,\mathcal{L}(e^{-at})$$

$$L(s\mathcal{L}(i(t)) - \underbrace{\cancel{i(0)}}_{0}) + R\mathcal{L}(i) = \frac{E}{s+a}$$

$$\mathcal{L}(i(t))(Ls + R) = \frac{E}{s+a}$$

$$\mathcal{L}(i(t)) = \frac{E}{(s+a)(Ls+R)} \Rightarrow i(t) = \mathcal{L}^{-1}\left(\frac{E}{(s+a)(Ls+R)}\right)$$

Consider $\dfrac{E}{(s+a)(LS+R)} = \dfrac{A}{(s+a)} + \dfrac{B}{Ls+R}$ (say)

$$\Rightarrow \quad E = A(Ls+R) + B(s+a)$$

$$s = -a \Rightarrow E = A(-aL+R)$$

$$\Rightarrow A = \frac{E}{R-aL}$$

$$s = -R/L \Rightarrow E = B\left(-\frac{R}{L}+a\right)$$

$$\Rightarrow B = \frac{LE}{aL-R}$$

$$\mathcal{L}^{-1}\left\{\frac{E}{(s+a)(Ls+R)}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{E}{R-aL}}{s+a}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{LE}{aL-R}}{Ls+R}\right\}$$

$$Ls+R$$

$$L\left(s+\frac{R}{L}\right)$$

$$= \frac{E}{R-aL} e^{-at} + \frac{LE}{aL-R} \times \frac{1}{L} e^{-R/L\, t}$$

$$= E\left\{\frac{1}{R-aL} e^{-at} + \frac{1}{aL-R} e^{-R/L\, t}\right\}$$

$$= \frac{E}{aL-R}\left\{e^{-R/L\, t} - e^{-at}\right\}$$

2. Solve $y''(t) - 4y'(t) + 8y = e^{2t}$, $y(0) = 2$, $y'(0) = -2$

$$L(y''(t)) - 4L(y'(t)) + 8L(y(t)) = L(e^{2t})$$

$$\left\{s^2 L(y(t)) - sy(0) - y'(0)\right\} - 4\left\{sL(y(t)) - y(0)\right\} + 8L(y(t)) = \frac{1}{s-2}$$

$$L(y(t))(s^2 - 4s + 8) - 2s + 2 + 8 = \frac{1}{s-2}$$

$$L(y(t))(s^2 - 4s + 8) = \frac{1}{s-2} + 2s - 10$$

$$L(y(t)) = \frac{1}{(s-2)(s^2-4s+8)} + \frac{2s-10}{s^2-4s+8} \quad \text{——— (1)}$$

$$\frac{1}{(s-2)(s^2-4s+8)} = \frac{A}{s-2} + \frac{Bs+C}{s^2-4s+8}$$

$$\Rightarrow 1 = A(s^2-4s+8) + (Bs+C)(s-2)$$

Comparing coeffts of s^2: $\cancel{A=0}$ $A+B=0$

Coeffts of s: $-4A-2B+C=0$

Const. terms: $8A-2C=1$

$\cancel{A=0 \Rightarrow -2C=1 \Rightarrow C=-1/2}$

$A+B=0 \Rightarrow B=-A$

Sub. $A=1/4$, $B=-1/4$, $C=1/2$

$$\frac{1}{(s-2)(s^2-4s+8)} = \frac{1}{4} \cdot \frac{1}{s-2} + \frac{\left(-\frac{1}{4}s+\frac{1}{2}\right)}{s^2-4s+8}$$

$$(1) \rightarrow L(y(t)) = \frac{1}{4(s-2)} - \frac{1}{4}\frac{s}{(s-2)^2+4} + \frac{1}{2}\frac{1}{(s-2)^2+4}$$

$$+ \frac{2s}{(s-2)^2+4} - \frac{10}{(s-2)^2+4}$$

adding 2 & subtracting 2

$$= \frac{1}{4(s-2)} + \frac{7}{4}\left(\frac{s-2+2}{(s-2)^2+4}\right) - \frac{19}{2}\frac{1}{(s-2)^2+2^2}$$

$$y(t) = \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}\cos 2t + \left(-\frac{19}{2} + \frac{7}{2}\right) e^{2t}\sin 2t$$

$$= \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}\cos 2t + 3e^{2t}\sin 2t$$

3. Solve $y''+y'-2y = 3\cos 3t - 11\sin 3t$, $y(0)=0$, $y'(0)=6$

$$L(y'') + L(y') + 2L(y(t)) = \frac{3s}{s^2+9} - \frac{11\times 3}{s^2+9}$$

$$\{s^2 L(y(t)) - s\cancel{y(0)} - \underset{\overset{\shortparallel}{6}}{y'(0)}\} + \{sL(y(t)) - \cancel{y(0)}\} + 2L(y(t)) = \frac{3s}{s^2+9} - \frac{33}{s^2+9}$$

$y(0) \to 0$, $y(0) \to 0$

$$(s^2+s+2)\,L(y(t)) = \frac{3s-33}{s^2+9} + 6$$

$$= \frac{6s^2+3s+21}{s^2+9}$$

$$L(y(t)) = \frac{6s^2+3s+21}{(s^2+9)(s+2)(s-1)}$$

$$= \frac{As+B}{s^2+9} + \frac{C}{s+2} + \frac{D}{s-1}$$

$A=0$, $B=3$, $C=-1$ & $D=1$

$$L(y(t)) = \frac{3}{s^2+9} - \frac{1}{s+2} + \frac{1}{s-1}$$

$$y(t) = \sin 3t - e^{-2t} + e^{t}$$

4. Solve $(D^2+4D+13)\,y = e^{-t}\sin t$, $y(0)=0$, $D(y(0))=0$ $\quad D = \frac{d}{dt}$.

$$(s^2 L(y(t)) - s\cancel{y(0)} - \cancel{y'(0)}) + 4(sL(y(t)) - \cancel{y(0)}) + 13L(y(t)) = \frac{1}{(s+1)^2+1}$$

$y(0) \to 0$, $y'(0) \to 0$, $y(0) \to 0$

$$(s^2+4s+13)\,L(y(t)) = \frac{1}{(s+1)^2+1} \Rightarrow L(y(t)) = \frac{1}{(s^2+4s+13)(s^2+2s+2)}$$

$$L(y(t)) = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+4s+13}$$

$$A = -\frac{2}{85}, \; B = \frac{7}{85}, \; C = \frac{2}{85} \; \& \; D = -3/85$$

$$L(y(t)) = \frac{1}{85}\left\{\frac{-2s+7}{(s^2+2s+2)} + \frac{2s-3}{s^2+4s+13}\right\}$$

$$L(y(t)) = \frac{1}{85}\left(\frac{-2(s+1)+9}{(s+1)^2+1^2} + \frac{2(s+2)-7}{(s+2)^2+3^2}\right)$$

$$= -\frac{2}{85}\left(\frac{s+1}{(s+1)^2+1^2}\right) + \frac{9}{85}\left(\frac{1}{(s+1)^2+1^2}\right)$$

$$+ \frac{2}{85}\left(\frac{s+2}{(s+2)^2+3^2}\right) - \frac{7}{85}\left(\frac{1}{(s+2)^2+3^2}\right)$$

$$= -\frac{2}{85} e^{-t}\cos t + \frac{9}{85} e^{-t}\sin t + \frac{2}{85} e^{-2t}\cos 3t - \frac{7}{85 \times 3} e^{-2t}\sin 3t$$