Problems (using Properties)

① $Z\left\{\frac{1}{n(n-1)}\right\}$

Sol: By Partial fraction expansion,

$$\frac{1}{n(n-1)} = \frac{A}{n} + \frac{B}{n-1}$$

$$A = \lim_{n\to 0}\left[\frac{1}{n-1}\right] = -1$$

$$B = \lim_{n\to 1}\left[\frac{1}{n}\right] = 1$$

$$\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$$

$$\therefore \quad Z\left\{\frac{1}{n(n-1)}\right\} = Z\left\{\frac{1}{n-1}\right\} - Z\left\{\frac{1}{n}\right\}$$

$$= \frac{1}{z}\log\left(\frac{z}{z-1}\right) - \log\left(\frac{z}{z-1}\right)$$

$$= \left(\frac{1}{z} - 1\right)\log\left(\frac{z}{z-1}\right) //$$

(2) $Z\{e^{an}\}$

Sol: $$e^{an} = (e^{a})^{n} = b^{n}, \text{ where } b = e^{a}$$

$$Z\{e^{an}\} = Z\{b^{n}\} = \frac{z}{z-b}$$

$$\therefore \boxed{Z\{e^{an}\} = \frac{z}{z-e^{a}}}$$

(3) $Z\{r^{n}e^{in\theta}\}$

Sol: $$r^{n}e^{in\theta} = r^{n}[\cos n\theta + i\sin n\theta]$$

$$= r^{n}\cos n\theta + i r^{n}\sin n\theta$$

$$Z\{r^{n}e^{in\theta}\} = Z\{(re^{i\theta})^{n}\}$$

$$= \frac{z}{z-re^{i\theta}} \quad \left(\because Z\{a^{n}\} = \frac{z}{z-a}\right)$$

$$= \frac{z}{z-r(\cos\theta + i\sin\theta)}$$

$$= \frac{z}{z-r\cos\theta - ir\sin\theta}$$

$$Z\{r^n e^{in\theta}\} = \frac{z(z - r\cos\theta + irsin\theta)}{(z - r\cos\theta)^2 + (r\sin\theta)^2}$$

$$\therefore Z\{r^n e^{in\theta}\} = \frac{z(z - r\cos\theta) + izr\sin\theta}{(z - r\cos\theta)^2 + r^2\sin^2\theta}$$

But $Z\{r^n e^{in\theta}\} = Z\{r^n\cos n\theta\} + iZ\{r^n \sin n\theta\}$

Equating real and imaginary parts, we get

$$Z\{r^n\cos n\theta\} = \frac{z(z - r\cos\theta)}{(z - r\cos\theta)^2 + r^2\sin^2\theta}$$

and

$$Z\{r^n\sin n\theta\} = \frac{zr\sin\theta}{(z - r\cos\theta)^2 + r^2\sin^2\theta}$$

<u>Sp. cases:</u>

(i) Putting $r = 1$, we get

$$Z\{\cos n\theta\} = \frac{z(z - \cos\theta)}{(z - \cos\theta)^2 + \sin^2\theta}$$

$$\therefore \boxed{Z\{\cos n\theta\} = \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}}$$

Similarly,

$$\boxed{Z\{\sin n\theta\} = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}}$$

④ $Z\left\{\cos\frac{n\pi}{2}\right\}$

<u>Sol:</u> WKT, $Z\{\cos n\theta\} = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$

Putting $\theta = \pi/2$, we get

$$Z\left\{\cos\frac{n\pi}{2}\right\} = \frac{z\left(z-\cos\frac{n\pi}{2}\right)}{z^2-2z\cos\frac{n\pi}{2}+1}$$

⑤ $Z\{a^n\cosh bn\}$

<u>Sol:</u> $a^n\cosh bn = a^n\left[\frac{e^{bn}+e^{-bn}}{2}\right]$

$$Z\{a^n\cosh bn\} = \frac{1}{2}Z\{a^n e^{bn}\} + \frac{1}{2}Z\{a^n e^{-bn}\}$$

$$= \frac{1}{2}\left[Z\{(ae^b)^n\} + Z\{(ae^{-b})^n\}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-ae^b} + \frac{z}{z-ae^{-b}}\right]$$

$$= \frac{1}{2}\left[\frac{z(z-ae^{-b})+z(z-ae^b)}{(z-ae^b)(z-ae^{-b})}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - az\left(e^{b}+e^{-b}\right)}{z^2 - az\left(e^{b}+e^{-b}\right) + a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - 2az\cosh b}{z^2 - 2az\cosh b + a^2}\right]$$

$$\therefore \boxed{Z\left\{a^n \cosh bn\right\} = \frac{z(z - a\cosh b)}{z^2 - 2az\cosh b + a^2}}$$

(6) $Z\left\{a^n \cos bn\right\}$

<u>Sol:</u> By change of scale property,

$$Z\left\{a^n f(n)\right\} = \bar{f}(z/a),$$

where $\bar{f}(z) = Z\left\{f(n)\right\}$.

Here $f(n) = \cosh n$.

$$\therefore \bar{f}(z) = \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}$$

Hence,

$$Z\{a^n \cos n\theta\} = \bar{f}(z/a)$$

$$= \frac{\left(\frac{z}{a}\right)\left(\frac{z}{a} - \cos\theta\right)}{\left(\frac{z}{a}\right)^2 - 2\left(\frac{z}{a}\right)\cos\theta + 1}$$

$$\therefore Z\{a^n \cos n\theta\} = \frac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$$

(7) Given $Z\{f(n)\} = \dfrac{0.4z^2}{(z-1)(z^2 - 0.736z + 0.136)}$

Find $f(0)$ and $f(\infty)$, $f(1)$, $f(2)$

Sol: (i) By initial Value Theorem,

$$f(0) = \lim_{z \to \infty} \left[Z\{f(n)\}\right]$$

$$= \lim_{z \to \infty} \left[\frac{0.4z^2}{(z-1)(z^2 - 0.736z + 0.136)}\right]$$

$$= \lim_{z \to \infty} \left[\frac{0.4\cancel{z^2}}{z\left(1 - \frac{1}{z}\right)\cancel{z}\left(1 - \frac{0.736}{z} + \frac{0.136}{z^2}\right)}\right]$$

$$= \lim_{z \to \infty} \left[\frac{(0.4/z)}{\left(1 - \frac{1}{z}\right)\left(1 - \frac{0.736}{z} + \frac{0.136}{z^2}\right)}\right] = 0$$

$\therefore \boxed{f(0)=0}$

(ii) By final value theorem,

$$f(\infty)=\lim_{z\to 1}\left[(z-1)\,Z\{f(n)\}\right]$$

$$=\lim_{z\to 1}\left[\cancel{(z-1)}\frac{0.4z^2}{\cancel{(z-1)}(z^2-0.736z+0.136)}\right]$$

$$=\frac{0.4}{1-0.736+0.136}=1$$

$$\therefore \boxed{f(\infty)=1}$$

(iii)

$$f(1)=\lim_{z\to\infty} z\left(\bar{f}(z)-f(0)\right)$$

$$=\lim_{z\to\infty}\left[z\bar{f}(z)-zf(0)\right]$$

$$=\lim_{z\to\infty}\left[\frac{0.4z^3}{(z-1)(z^2-0.736z+0.136)}-0\right]$$

$$\boxed{f(1)=0.4}$$

Note:

$$f(2)=\lim_{z\to\infty}\left[z^2\left(\bar{f}(z)-f(0)-\frac{f(1)}{z}\right)\right]$$

$$f(3)=\lim_{z\to\infty}\left[z^3\left(\bar{f}(z)-f(0)-\frac{f(1)}{z}-\frac{f(2)}{z^2}\right)\right]$$

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