① $L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right]$

Sol:

Let $F(s) = \frac{5s+3}{(s-1)(s^2+2s+5)}$

Sol: By PFM, $s^2+2s+5 = (s+1)^2+4$

$$F(s) = \frac{A}{s-1} + \frac{B(s+1)+C(2)}{(s+1)^2+4}$$

Equating numerators, we get

$$5s+3 = A\left((s+1)^2+4\right) + \left(B(s+1)+2C\right)(s-1)$$

$s=1$: $8 = 8A \Rightarrow \boxed{A=1}$

$s=-1$: $-2 = 4A - 4C \Rightarrow 4C = 4A+2 = 6$

$\Rightarrow \boxed{C=3/2}$

$s=0$: $3 = 5A - B - 2C = 2 - B$

$\Rightarrow \boxed{B=-1}$

$$\therefore F(s) = \frac{1}{s-1} - \frac{s+1}{(s+1)^2+4} + \frac{3}{2}\cdot\frac{2}{(s+1)^2+4}$$

Hence,

$$L^{-1}[F(s)] = e^{t} - e^{-t}.\cos 2t + \frac{3}{2}\cdot e^{-t}.\sin 2t$$ //

Solution by Laplace Transform method

1. Solve the following initial-value problem

$$y'' + 2y' + 2y = \delta(t-1) + \delta(t-5)$$

with $y(0) = y'(0) = 0$

Let $L[y(t)] = Y(s)$

Sol: Applying Laplace transform on the DE, we get

$$L[y'' + 2y' + 2y] = L[\delta(t-1) + \delta(t-5)]$$

$$\Rightarrow L[y''] + 2L[y'] + 2L[y] = L[\delta(t-1)] + L[\delta(t-5)]$$

$$\Rightarrow \left[s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0}\right] + 2\left[sY(s) - \cancel{y(0)}^{0}\right] + 2Y(s) = e^{-s} + e^{-5s}$$

$$\Rightarrow Y(s)\left[s^2 + 2s + 2\right] = e^{-s} + e^{-5s}$$

$$\therefore \boxed{Y(s) = \frac{e^{-s} + e^{-5s}}{s^2 + 2s + 2}}$$

Applying Inverse LT on $Y(s)$, we get

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{e^{-s} + e^{-5s}}{s^2 + 2s + 2}\right]$$

$$y(t) = L^{-1}\left[\frac{e^{-s} + e^{-5s}}{(s+1)^2+1}\right]$$

$$= L^{-1}\left[\frac{e^{-s}}{(s+1)^2+1}\right] + L^{-1}\left[\frac{e^{-5s}}{(s+1)^2+1}\right]$$

$$= u(t-1)\,f(t-1) + u(t-5)\,f(t-5)$$

When $f(t) = L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$

(using second shifting property)

But $f(t) = L^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t}.\sin t$

$$\therefore \quad y(t) = u(t-1)\,e^{-(t-1)}\sin(t-1) + u(t-5)\,e^{-(t-5)}\sin(t-5)$$