ORDINARY DIFFERENTIAL EQUATIONS

Differential equation

A differential equation is an equation which involves differential coefficients or differentials

Eg: $e^x\,dx + e^y\,dy = 0$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \frac{d^2y}{dx^2} = c$$

Differential equation

ODE	PDE
single independent variable	2 or more independent variables
diff. coefficient with reference to single independent variable	partial differential coefficient

Order

Order of a differential equation is the order of the highest derivative appearing in it.

Degree

Degree of the highest derivative ocurring in it, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.

$$y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + x$$

order = 1

degree = 2

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = c \Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = c\frac{d^2y}{dx^2}$$

$$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3} = c^2\left(\frac{d^2y}{dx^2}\right)^2$$

$\therefore$ order = 2

degree = 2

Types of ODE

① Linear differential equation

② Non-linear differential equation

Linear differential equation

These equations are those in which the dependent variable and its derivatives occur only in the 1st degree and are not multiplied together. Thus, the general linear differential equation of nth order is of the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = X \quad \text{— ①}$$

where $P_1, P_2, \cdots, P_n$ and X are functions of x only.

Types of Linear differential equation:

- L.D.E with constant coefficients
- L.D.E with variable coefficients

Linear differential equation with constant coefficients

Operator D will be used to denote the derivativ

i.e. $\frac{d}{dx}, \frac{d^2}{dx^2}, \ldots$ as $D, D^2, \ldots$

so that $\frac{dy}{dx} = Dy$, $\frac{d^2y}{dx^2} = D^2y$

General equation ① can be written as

$$(D^n + p_1 D^{n-1} + p_2 D^{n-2} + \cdots + p_n)\, y = X \quad \text{—} \quad (2)$$

i.e. $f(D)y = X$

The symbol D is called the operator of differentiation.

Solution of Linear Differential equation

The complete solution of (1) is given by

$$y = \underbrace{C.F}_{\text{Complementary function}} + \underbrace{P.I}_{\text{Particular Integral}}$$

To find C.F

Equate RHS to 0

$$\therefore (1) \Rightarrow \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0$$

In terms of D

$$\Rightarrow (D^n + p_1 D^{n-1} + \cdots + p_n)\, y = 0$$

Step: 1 write the Auxillary equation (A.E)

Put $D = m$

$$\Rightarrow m^n + p_1 m^{n-1} + \cdots + p_n = 0$$

Step: 2 Solve A.E and get the roots

say $m_1, m_2, \ldots, m_n$.

Case (i) Suppose say m_1 and m_2 are the roots.

Roots are real and distinct

then $C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

case (ii)

Roots are real and equal. $m_1 = m_2 = m$

$C.F = (c_1 + c_2 x) e^{mx}$

In general, $C.F = (c_1 + c_2 x + c_2 x^2 + \cdots + c_n x^n) e^{mx}$

case (iii)

Roots are complex say $\alpha \pm i\beta$

$$C.F = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x \right]$$

If the two pair of equal imaginary roots say $\alpha \pm i\beta$, $\alpha \pm i\beta$, then

$$C.F = e^{\alpha x}\left[(c_1 + c_2 x) \cos\beta x + (c_3 + c_4 x) \sin\beta x \right]$$

PROBLEMS

1. $$\frac{d^2x}{dt^2} + 3a\frac{dx}{dt} - 4a^2x = 0$$

$-4a^2$

$4a \quad -a$

$$(D^2 + 3aD - 4a^2)x = 0$$

A.E $\quad m^2 + 3a - 4a^2 = 0$

$$m = a, -4a$$

$$C.F = c_1 e^{at} + c_2 e^{-4at}$$

here $x \to$ dependent variable

$t \to$ independent variable

2. $y'' - 2y' + 10y = 0$ given that $y(0) = 4$,

$y'(0) = 1$

$$(D^2 - 2D + 10)y = 0$$

A.E $\quad m^2 - 2m + 10 = 0$

$$a = 1,\ b = -2,\ c = 10$$

$$m = \frac{-(-2) \pm \sqrt{4 - 4(1)(10)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

$y = C.F = e^{x}\left[c_1 \cos 3x + c_2 \sin 3x\right]$

$y(x) = e^{x}\left[c_1 \cos 3x + c_2 \sin 3x\right]$ — ①

diff. ① w.r.to x

$y'(x) = e^{x}\left[-c_1(3\sin 3x) + 3c_2\cos 3x\right]$

$+ e^{x}\left[c_1 \cos 3x + c_2 \sin 3x\right]$ — ②

Apply the conditions $y(0)=4$ and $y'(0)=1$ in ①

and ②, respectively to get the values

of c_1 and c_2.

$y(0)=4$

① $\Rightarrow$ $\boxed{4 = c_1}$

$y'(0)=1$

② $\Rightarrow$ $1 = 3c_2 + c_1$

$3c_2 = -3$

$\boxed{c_2 = -1}$

$\therefore\ y(x) = e^{x}\left[4\cos 3x - \sin 3x\right]$

3) $(D^2+1)^2 (D-1) y = 0$

A.E $(m^2+1)^2 (m-1) = 0$

$m = \pm i, \pm i, 1$

$\therefore$ C.F $= c_1 e^x + e^{0x}\left[(c_2 + c_3 x)\cos x + (c_4 + c_5 x)\sin x\right]$

C.F $= c_1 e^x + \left[(c_2 + c_3 x)\cos x + (c_4 + c_5 x)\sin x\right]$.

METHOD OF UNDETERMINED COEFFICIENTS

Let $f(D)y = X$ — ① be the given eqn.

Then to find P.I

We assume the trial solution of P.I which contains unknown constant which are determined by substituting in the given equation. This is called the method of undetermined coefficients.

STEPS

1° Assume the trial solution for P.I based on the RHS ie X of the given equation.

If $X = Ae^{ax}$

then trial solution is $y = a_1 e^{ax}$

If $X = A \sin ax$ or $A \cos ax$

then trial solution is

$$y = a_1 \cos ax + a_2 \sin ax$$

If $X = A x^m$ then the trial solution is

$$y = a_0 + a_1 x + a_2 x^2 + \dots + a_m x^m$$

2° Check whether the trial solution has the same terms as the C.F if that is the case then multiply trial solution by the lowest +ve integral power of x, which is large enough so that none of the terms which are present in the C.F gets repeated.

3° Find D^2y, Dy from the trial soln and substitute in ①

4° compare the coefficient on both sides of ①, we can determine the constants and we will obtain the solution for the ~~trial~~ particular Integral.

NOTE This method fails for $\tan x / \sec x$ because the no. of terms obtained by differentiating $x = \tan x$ (or $\sec x$) is infinite.

Solve by the method of undetermined coefficients

$$(D^2 - 3D + 2)y = x^2 + e^x \quad — ①$$

C.F

A.E $m^2 - 3m + 2 = 0$

$$m = 2, 1$$

$$C.F = c_1 e^{2x} + c_2 e^x$$

P.I

Assume the trial solution as

$$P.I = a_0 + a_1 x + a_2 x^2 + a_3 x e^x$$

$\because$ C.F has this term so multiplied by x.

Trial soln

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x e^x$$

$$Dy = a_1 + 2x a_2 + a_3 [e^x + x e^x]$$

plied by x

Trial soln

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x e^x$$

$$Dy = a_1 + 2x a_2 + a_3 [e^x + x e^x]$$

$$D^2 y = 2a_2 + a_3 e^x + a_3 [e^x + x e^x]$$

$$= 2a_2 + 2a_3 e^x + a_3 x e^{2x}$$

substitute in the given eqn.

$$(D^2 - 3D + 2)y = x^2 + e^x$$

$$2a_2 + 2a_3 e^x + a_3 x e^{2x}$$

$$-3\left[a_1 + 2xa_2 + a_3(e^x + xe^x)\right]$$

$$+2(a_0 + a_1 x + a_2 x^2 + a_3 x e^x) = x^2 + e^x$$

compare coefficient of x^2 on both sides

$$2a_2 = 1$$

$$\boxed{a_2 = \frac{1}{2}}$$

compare coefficient of e^x

$$2a_3 - 3a_3 = 1$$

$$\boxed{a_3 = -1}$$

compare coefficient of x

$$-6a_2 + 2a_1 = 0$$

$$-6\left(\frac{1}{2}\right) + 2a_1 = 0$$

$$-3 + 2a_1 = 0$$

$$\boxed{a_1 = \frac{3}{2}}$$

compare constants .

$$2a_2 - 3a_1 + 2a_0 = 0$$

$$2\left(\frac{1}{2}\right) - 3\left(\frac{3}{2}\right) + 2a_0 = 0$$

$$1 - \frac{9}{2} + 2a_0 = 0$$

$$-\frac{7}{2} + 2a_0 = 0$$

$$\boxed{a_0 = \frac{7}{4}}$$

$$\therefore\ P.I = \frac{7}{4} + \frac{3}{2}x + \frac{1}{2}x^2 - xe^x$$

The complete soln.

$$y = C.F + P.I$$

$$y = c_1 e^{2x} + c_2 e^x + \frac{7}{4} + \frac{3}{2}x + \frac{x^2}{2} - xe^x.$$