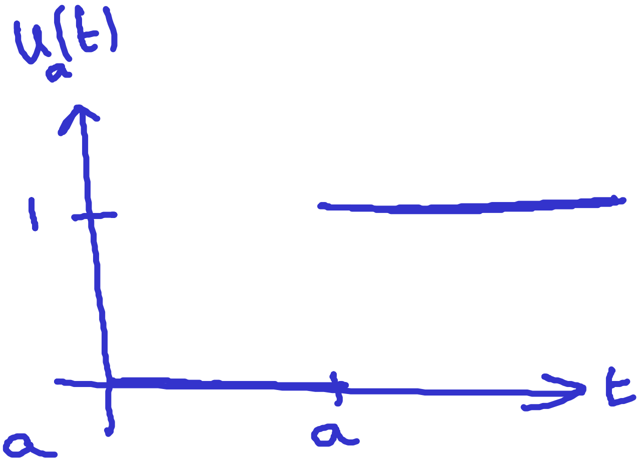
$u_a(t)$

$= u(t-a)$

$= \begin{cases} 1 & \text{for } t \geq a \\ 0 & \text{else} \end{cases}$



Unit step function / Heaviside's unit function:

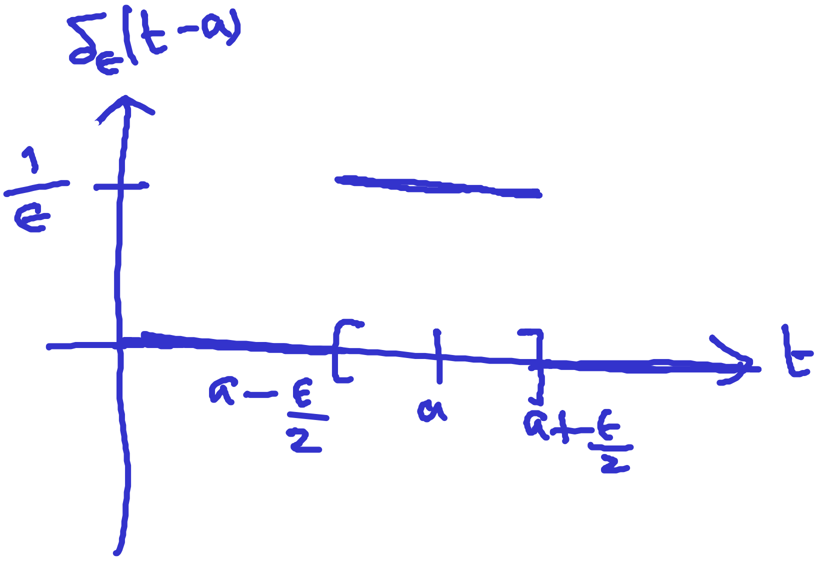
① $$L[u(t-a)] = \int_0^\infty e^{-st} \cdot u(t-a)\,dt$$

$$= \int_0^a e^{-st} \cancel{(0)\,dt} + \int_a^\infty e^{-st} (1)\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty = (0) - \left(\frac{e^{-as}}{-s}\right)$$

$$\boxed{\therefore\ L[u(t-a)] = \frac{e^{-as}}{s}}$$

Dirac-delta function



$$\delta_\epsilon(t-a) = \begin{cases} \frac{1}{\epsilon} & \text{for } t \in \left(a-\frac{\epsilon}{2}, a+\frac{\epsilon}{2}\right) \\ 0 & \text{else} \end{cases}$$

$$\boxed{\delta(t-a) = \lim_{\epsilon \to 0} \left[\delta_\epsilon(t-a)\right]}$$

① Find $L[\delta(t-a)]$

Sol:

$$\delta(t-a) = \lim_{\epsilon \to 0} [\delta_\epsilon(t-a)]$$

$$L[\delta(t-a)] = \lim_{\epsilon \to 0} \left[L[\delta_\epsilon(t-a)] \right]$$

Now,

$$L[\delta_\epsilon(t-a)] = \int_0^\infty e^{-st} \delta_\epsilon(t-a)\,dt$$

$$= \int_{a-\frac{\epsilon}{2}}^{a+\frac{\epsilon}{2}} e^{-st} \cdot \frac{1}{\epsilon} \cdot dt$$

$$= \frac{1}{\epsilon} \left[\frac{e^{-st}}{-s} \right]_{a-\frac{\epsilon}{2}}^{a+\epsilon/2}$$

$$= \frac{-1}{s\epsilon} \left[e^{-s(a+\frac{\epsilon}{2})} - e^{-s(a-\frac{\epsilon}{2})} \right]$$

$$= \frac{e^{-as}}{s\epsilon}\left[e^{s\epsilon/2} - e^{-s\epsilon/2} \right]$$

$$L[\delta_\epsilon(t-a)] = \frac{e^{-as}}{s\epsilon}\left[2\sinh\left(\frac{s\epsilon}{2}\right) \right]$$

$$\therefore \quad L[\delta(t-a)]$$

$$= \lim_{\epsilon \to 0}\left[\frac{e^{-as}}{s\epsilon} 2\sinh\left(\frac{s\epsilon}{2}\right) \right]$$

$$= \lim_{\epsilon \to 0} e^{-as} \cdot \left[\frac{\sinh\left(\frac{s\epsilon}{2}\right)}{\left(\frac{s\epsilon}{2}\right)} \right]$$

$$= e^{-as} \cdot \lim_{\epsilon \to 0}\left[\frac{\sinh \alpha\epsilon}{\alpha\epsilon} \right],$$

when $\alpha = s/2$

$$= e^{-as} \cdot \lim_{\epsilon \to 0}\left[\frac{\cancel{\alpha} \cosh \alpha\epsilon}{\cancel{\alpha}} \right]$$

$$\therefore L[\delta(t-a)] = e^{-as}$$

Second Shifting Property

$$L[u(t-a)\, f(t-a)] = e^{-as} F(s)$$

Proof:

Let $g(t) = u(t-a)\, f(t-a)$

$$\Rightarrow \quad g(t) = \begin{cases} f(t-a), & t \geq a \\ 0 & t < a \end{cases}$$

$$L[g(t)] = \int_0^\infty e^{-st} g(t)\, dt$$

$$= \int_a^\infty e^{-st} f(t-a)\, dt$$

$a<t$, $\quad b<t$, $\quad t<c$

1

$$f(t) = \begin{cases} \phi_1(t) & a \leq t < b \\ \phi_2(t) & b < t \leq c \end{cases}$$

$$f(t) = \phi_1(t)\left[u(t-a) - u(t-b)\right] + \phi_2(t)\left[u(t-b) - u(t-c)\right]$$

(i) consider $t \in [a, b]$.

Now, $f(t) = \phi_1(t)[1-0] + \phi_2(t)[0-0]$

$= \phi_1(t)$ //

(ii) consider $t \in [b, c]$.

Now, $f(t) = \phi_1(t)[1-1]$

$+ \phi_2(t)[1-0]$

$= \phi_2(t)$ //

1.
$$f(t) = \begin{cases} 2 & 0 < t < 2 \\ -1 & 2 \leq t < 3 \\ 1 & t \geq 3 \end{cases}$$

Sol:
$$f(t) = (2)[u(t-0) - u(t-2)] + (-1)[u(t-2) - u(t-3)] + (1)[u(t-3)]$$

$$f(t) = 2u(t) - 3u(t-2) + 2u(t-3)$$

Check: (i) For $0 < t < 2$,

$$f(t) = 2 - 3(0) + 2(0) = 2$$

(ii) For $2 \leq t < 3$,

$$f(t) = 2(1) - 3(1) + 2(0) = -1$$

(iii) For $t \geq 3$,

$$f(t) = 2(1) - 3(1) + 2(1) = 1$$