Fourier Series

① Find the Fourier series of $f(x) = e^{-x}$ in $(-\pi, \pi)$.

Sol: The Fourier series of $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right].$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-x}\, dx$$

$$= \frac{1}{\pi} \left[-e^{-\pi} + e^{\pi} \right] = \frac{2}{\pi} \sinh \pi$$

$$\boxed{a_0 = \frac{2}{\pi} \sinh \pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-x} \cos nx\, dx$$

WKT,
$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2} \left[a \cos bx + b \sin bx \right]$$

$$\therefore \quad a_n = \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(-\cos nx + n\sin nx\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[\left\{e^{-\pi}\left(-\cos n\pi + \cancel{n\sin n\pi}\right)\right\} - \left\{e^{\pi}\left(-\cos n\pi - \cancel{n\sin n\pi}\right)\right\}\right]$$

$$= \frac{\cos n\pi}{\pi(1+n^2)}\left[-e^{-\pi} + e^{\pi}\right]$$

$$\boxed{a_n = \frac{2[\sinh\pi](-1)^n}{\pi(1+n^2)}} \qquad \left[\because e^{x} - e^{-x} = 2\sinh x\right]$$

$n = 1, 2, \ldots$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\sin nx\, dx$$

$$\left[\text{WKT, } \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)\right]$$

$$\therefore \quad b_n = \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(-\sin nx - n\cos nx\right)\right]_{-\pi}^{\pi}$$

$$b_n = \frac{1}{\pi(1+n^2)}\left[\left\{e^{-\pi}(-\cancel{\sin n\pi} - n\cos n\pi)\right\} - \left\{e^{\pi}(\cancel{\sin n\pi} - n\cos n\pi)\right\}\right]$$

$$= \frac{n\cos n\pi}{\pi(1+n^2)}\left\{-e^{-\pi} + e^{\pi}\right\}$$

$$\boxed{b_n = \frac{2n(-1)^n \sinh\pi}{\pi(1+n^2)}}, \quad n = 1, 2, \ldots$$

$\therefore$ Fourier Series of $f(x) = e^{-x}$ in $(-\pi, \pi)$ is

$$e^{-x} = \frac{1}{\pi}\sinh\pi + \sum_{n=1}^{\infty} \frac{2(-1)^n \sinh\pi}{\pi(1+n^2)}\cos nx + \sum_{n=1}^{\infty} \frac{2n(-1)^n \sinh\pi}{\pi(1+n^2)}\sin nx$$

$$\Rightarrow e^{-x} = \frac{1}{\pi}\sinh\pi + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}(-1)^n\left[\frac{\cos nx + n\sin nx}{1+n^2}\right]$$

② Find the Fourier Series of $f(x) = \sqrt{1-\cos x}$ in $(-\pi, \pi)$. $\boxed{f(-x) = f(x)}$

Sol: WKT, $\cos 2\theta = 1 - 2\sin^2\theta$.

$$\therefore \quad f(x) = \sqrt{1-\cos x} = \sqrt{1-\left(1-2\sin^2\left(\frac{x}{2}\right)\right)}$$

$$= \pm\sqrt{2}\cdot\sin\left(\frac{x}{2}\right) \quad \text{in } (-\pi, \pi)$$

$$\Rightarrow \quad f(x) = \begin{cases} -\sqrt{2}\sin\left(\frac{x}{2}\right) & -\pi < x < 0 \\ \sqrt{2}\sin\left(\frac{x}{2}\right) & 0 \le x < \pi \end{cases}$$

$\therefore$ f is even in $(-\pi, \pi)$ since $f(-x) = f(x)$.

The Fourier Series of $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\sqrt{2}\sin\left(\frac{x}{2}\right)dx$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{-\cos\left(\frac{x}{2}\right)}{\left(\frac{1}{2}\right)}\right]_0^{\pi}$$

$$\Rightarrow \quad a_0 = \frac{-4\sqrt{2}}{\pi}\left[\cos\left(\frac{x}{2}\right)\right]_0^{\pi}$$

$$= \frac{-4\sqrt{2}}{\pi}\left[(0) - (1)\right] = \frac{4\sqrt{2}}{\pi}$$

$$\therefore \boxed{a_0 = \frac{4\sqrt{2}}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx$$ (since integrand is even)

$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{2}\,\sin\left(\frac{x}{2}\right)\cos nx \, dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi} \left[2\cos nx \sin\left(\frac{x}{2}\right)\right] dx$$

$$\left[2\cos C \sin D = \sin(C+D) - \sin(C-D)\right]$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi}\left[\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x\right]dx$$

$$a_n = \frac{\sqrt{2}}{\pi}\left[\frac{-\cos\left(n+\frac{1}{2}\right)x}{n+\frac{1}{2}} + \frac{\cos\left(n-\frac{1}{2}\right)x}{n-\frac{1}{2}}\right]_0^{\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\left\{\frac{-\cos\left(n+\frac{1}{2}\right)\pi}{n+\frac{1}{2}} + \frac{\cos\left(n-\frac{1}{2}\right)\pi}{n-\frac{1}{2}}\right\} - \left\{\frac{-1}{n+\frac{1}{2}} + \frac{1}{n-\frac{1}{2}}\right\}\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\{0\} - \left\{\frac{-1}{n+\frac{1}{2}} + \frac{1}{n-\frac{1}{2}}\right\}\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1}{n+\frac{1}{2}} - \frac{1}{n-\frac{1}{2}}\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-1}{n^2-\frac{1}{4}}\right] = \frac{-4\sqrt{2}}{\pi(4n^2-1)}$$

$$\therefore \boxed{a_n = \frac{-4\sqrt{2}}{\pi(4n^2-1)}}, \quad n = 1, 2, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx = 0$$ since the integrand is odd.

$\therefore$ The reqd. Fourier Series of $f(x)$ in $(-\pi, \pi)$ is

$$\boxed{f(x) = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos nx}$$

(3) Find the Fourier Series of

Exercise

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 2 & 1 < x < 3 \end{cases}$$

Sol: Here, $2L = 3 \Rightarrow \boxed{L = \frac{3}{2}}$

Fourier Series of $f(x)$ in $(0,3)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi x}{3}\right) + b_n \sin\left(\frac{2n\pi x}{3}\right)\right],$$

where

$$a_0 = \frac{2}{3}\int_0^3 f(x)\,dx$$

$$a_n = \frac{2}{3}\int_0^3 f(x)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$b_n = \frac{2}{3}\int_0^3 f(x)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$a_0 = \frac{2}{3}\left[\int_0^1 dx + \int_1^3 2\,dx\right]$$

$$= \frac{2}{3}\left[(1) + (4)\right] = \frac{10}{3}$$

$$a_n = \frac{2}{3}\int_0^1 \cos\left(\frac{2n\pi x}{3}\right)dx + \frac{4}{3}\int_1^3 \cos\left(\frac{2n\pi x}{3}\right)dx$$