1. $$L[t^n] = \int_0^\infty e^{-st} . t^n \, dt$$

Put $u = st$. Then $du = s\,dt$

and $t = u/s$.

$$\therefore L[t^n] = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \left[\frac{du}{s}\right]$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} . u^n \, du$$

$$= \frac{1}{s^{n+1}} \int_0^\infty u^{(n+1)-1} . e^{-u} . du$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \qquad \boxed{\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du}$$

Note: If n is an +ve integer, then

$$\Gamma(n+1) = n!$$

In this case, $L[t^n] = \dfrac{n!}{s^{n+1}}$ //

Ex: $$L[t^3] = \frac{3!}{s^4} = \frac{6}{s^4}$$ //

Linearity Property

1. $$L[\sinh t] = L\left[\frac{e^{t} - e^{-t}}{2}\right]$$

$$= L\left[\frac{1}{2}e^{t} - \frac{1}{2}e^{-t}\right]$$

$$= \frac{1}{2}L[e^{t}] - \frac{1}{2}L[e^{-t}]$$

$$= \frac{1}{2}\left[\frac{1}{s-1}\right] - \frac{1}{2}\left[\frac{1}{s+1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-1} - \frac{1}{s+1}\right]$$

$$\boxed{L[\sinh t] = \frac{1}{s^2-1}}$$

② $L[\cosh t] = L\left[\frac{e^{t}+e^{-t}}{2}\right]$

$= \frac{1}{2}\left[L(e^{t}) + L(e^{-t})\right]$

$= \frac{1}{2}\left[\frac{1}{s-1} + \frac{1}{s+1}\right]$

$$\boxed{L[\cosh t] = \frac{s}{s^2-1}}$$

Change of scale property

① Find $L[e^{at}]$.

Sol: Let $f(at) = e^{at}$

$\Big\downarrow\, t \to t/a$

Then $f(t) = e^{t}$

$F(s) = L[f(t)] = \frac{1}{s-1}$

By change of scale property,

$L[f(at)] = \frac{1}{a} F(s/a)$

i.e., $L[e^{at}] = \frac{1}{a} \frac{1}{\left(\frac{s}{a}-1\right)} = \frac{1}{s-a}$

② Find $L[\sin(at)]$

Sol: Let $f(at) = \sin(at)$

so that $f(t) = \sin t$.

$$F(s) = L[f(t)] = \frac{1}{s^2+1}$$

By change of scale property,

$$L[f(at)] = \frac{1}{a} F(s/a)$$

i.e., $$L\{\sin(at)\} = \frac{1}{a} \frac{1}{\left[\left(\frac{s}{a}\right)^2+1\right]}$$

$$\therefore \boxed{L[\sin(at)] = \frac{a}{s^2+a^2}}$$

③ Find $L[\sin 2t \sin 3t]$

Sol: [WKT,

$$2\sin C \sin D = \cos(C-D) - \cos(C+D)]$$

$$\therefore \sin 2t \sin 3t = \frac{1}{2}[\cos(t) - \cos 5t]$$

$$L[\sin 2t \sin 3t] = \frac{1}{2}\{L[\cos t] - L[\cos 5t]\}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right]$$

$$= \frac{s}{2}\left[\frac{\cancel{s^2}+25-\cancel{s^2}-1}{(s^2+1)(s^2+25)}\right]$$

$$\boxed{L[\sin 2t \sin 3t] = \frac{12s}{(s^2+1)(s^2+25)}}$$

<u>First shifting property</u>

① $L[e^{at}\sin bt]$

<u>Sol:</u> let $f(t) = \sin bt$

$$F(s) = L[f(t)] = \frac{b}{s^2+b^2}$$

By First Shifting property,

$$L[e^{at}f(t)] = F(s-a)$$

i.e., $$L[e^{at}\sin bt] = \frac{b}{(s-a)^2+b^2} \;//$$

② Find $L\left[e^{-3t}(2\cos 5t - 3\sin 5t)\right]$

Sol:

Let $f(t) = 2\cos 5t - 3\sin 5t$

$$F(s) = L\left[f(t)\right]$$

$$= 2\left[\frac{s}{s^2+25}\right] - 3\left[\frac{5}{s^2+25}\right]$$

$$= \frac{2s-15}{s^2+25}$$

By FSP,

$$L\left[e^{at} f(t)\right] = F(s-a)$$

i.e., $L\left[e^{-3t}(2\cos 5t - 3\sin 5t)\right]$

$$= F(s+3)$$

$$= \frac{2(s+3)-15}{(s+3)^2+25}$$

$$= \frac{2s-9}{s^2+6s+34}$$