Method of Separation of Variables

PDE: $f(x, y, z, p, q) = 0$

Assume $z = X(x)\,Y(y)$ to be the solution of Pde.

① Solve $2x\frac{\partial z}{\partial x} - 3y\frac{\partial z}{\partial y} = 0$ by the method of separation of variables.

Sol: Let $z = X(x)\,Y(y)$ be a solution of the given Pde.

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(XY) = Y\frac{\partial}{\partial x}(X)$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{dX}{dx}Y = X'Y$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(XY) = XY'$$

Substituting in pde, we get:

$$2xX'Y - 3yXY' = 0$$

$$2xX'Y = 3yXY'$$

$$\frac{2xX'}{X} = \frac{3yY'}{Y} = k$$

(k – constant)

i.e., $\boxed{\frac{2xX'}{X} = k}$ ① and $\boxed{\frac{3yY'}{Y} = k}$ ②

① $\Rightarrow$

$$2\frac{X'}{X} = \frac{k}{x}$$

Integrating, we get

$$2\log X = k\log x + \log A$$

$$\log X^2 = \log x^k + \log A$$
$$= \log(Ax^k)$$
$$\therefore \quad X^2 = Ax^k$$

Hence $\boxed{X(x) = \alpha x^{k/2}}$

② $\Rightarrow$ $$\frac{3Y'}{Y} = \frac{k}{y}$$

Integrating, we get

$$3\log Y = k\log y + \log B$$
$$\log Y^3 = \log y^k + \log B$$
$$= \log(By^k)$$
$$\therefore \quad Y^3 = By^k$$

Hence $\boxed{Y(y) = \beta y^{k/3}}$

Required solution of

$$2x\frac{\partial z}{\partial x} - 3y\frac{\partial z}{\partial y} = 0$$ is

$$z = XY$$

$$= \alpha x^{k/2}\ \beta y^{k/3}$$

$$= \alpha\beta\, x^{k/2} y^{k/3}$$

$$z = \gamma x^{k/2} y^{k/3}$$, where γ ~~and k are the~~ and k are the arbitrary constants.

(2) Solve $p = 2q + z$, by the method of separation of variables.

<u>Sol:</u> Let $z = X(x)\,Y(y)$ be a solution of the given pde.

$$p = X'Y, \qquad X' = \frac{dX}{dx}$$

$$q = XY', \qquad Y' = \frac{dY}{dy}$$

Substituting in pde, we get

$$X'Y = 2XY' + XY$$

Dividing throughout by XY, we get

$$\frac{X'}{X} = \frac{2Y'}{Y} + 1 = k \quad \leftarrow (1)$$

(k - const.)

From 1, we have

$$\boxed{\frac{X'}{X} = k} \text{ and } \boxed{\frac{2Y'}{Y} + 1 = k}$$

$\quad$ 2 $\quad\quad\quad$ 3

General soln. of 2 is

$$X(x) = \alpha e^{kx}$$

3 $\Rightarrow$

$$\frac{2Y'}{Y} + 1 = k$$

$$\frac{Y'}{Y} = \frac{k-1}{2}$$

$$\frac{Y'}{Y} = \beta, \quad \text{where } \beta = \frac{k-1}{2}$$

General soln. of 3 is

$$Y(y) = c \cdot e^{\beta y}$$

$$\boxed{Y(y) = c\, e^{\left(\frac{k-1}{2}\right)y}}$$

Reqd. soln. of $p = 2q + z$ is

$$z = \alpha e^{kx} \cdot c \cdot e^{\left(\frac{k-1}{2}\right)y}$$

$$\boxed{z = \beta e^{kx} e^{\left(\frac{k-1}{2}\right)y}},$$

when β and k are the arbitrary constants.

③ Solve $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, given $u(x,0) = 4e^{-x}$, using the method of separation of variables.

Sol: By the method of separation of variables,

let $u(x,y) = X(x)Y(y)$

be the reqd. solution.

Substituting in the pde, we get

$$3X'Y + 2XY' = 0$$

$$3X'Y = -2XY'$$

$$\boxed{\frac{3X'}{X} = \frac{-2Y'}{Y} = k}$$

(i)

$$\frac{3X'}{X} = k \Rightarrow \frac{X'}{X} = \frac{k}{3} \Rightarrow X' - \frac{k}{3}X = 0$$

$$\boxed{X(x) = \alpha e^{\frac{k}{3}x}}$$

④ Solve the following Pde's using method of separation of variables.

<u>Ex</u>

(i) $$\frac{\partial z}{\partial x} + 4z = \frac{\partial z}{\partial t},$$

given $z(x,0) = 4e^{-3x}$

(ii) $$x^2 q + y^3 p = 0$$

(ii) $\frac{-2Y'}{Y} = k$

$$\Rightarrow \frac{Y'}{Y} = \frac{-k}{2}$$

$$Y' + \frac{k}{2}Y = 0$$

$$\boxed{Y(y) = \beta e^{-\frac{k}{2}y}}$$

Reqd. soln. is

$$u(x,y) = \alpha e^{\frac{k}{3}x} \cdot \beta e^{-\frac{k}{2}y}$$

$$\boxed{u(x,y) = c e^{\frac{k}{3}x} \cdot e^{-\frac{k}{2}y}}$$

Now $u(x,0) = c e^{\frac{k}{3}x}$

Using the given condition,

$$4e^{-x} = ce^{\frac{k}{3}x}$$

$$\therefore \quad c = 4 \quad \text{and} \quad \frac{k}{3} = -1$$

$$\Downarrow$$

$$k = -3$$

Reqd. solution is

$$\boxed{u(x,y) = 4e^{-x}\, e^{\frac{3}{2}y}}$$