MAT2002 – APPLICATIONS OF DIFFERENTIAL AND DIFFERENCE EQUATIONS
Module 3 – Lecture Notes

PARTICULAR SOLUTION – METHOD OF VARIATION OF PARAMETERS

Theorem1 (VARIATION OF PARAMETERS-I)

Consider the standard form of the non-homogeneous differential equation(DE)

$$y'' + py' + qy = r(x) \qquad [1]$$

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the associated homogeneous DE, and let $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ be the Wronskian function of $y_1(x)$ and $y_2(x)$.

The particular solution of the non-homogeneous DE [1] has the form

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x),$$

where u and v are given by

$$u(x) = -\int \frac{y_2(x)r(x)}{W}dx \text{ and } v(x) = \int \frac{y_1(x)r(x)}{W}dx.$$

Theorem2 (VARIATION OF PARAMETERS-II)

Consider the general form of the non-homogeneous differential equation(DE)

$$a_1 y'' + a_2 y' + a_3 y = f(x) \qquad [1]$$

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the associated homogeneous DE, and let $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$ denote the Wronskian function of $y_1(x)$ and $y_2(x)$.

The particular solution of the non-homogeneous DE [1] has the form

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x),$$

where u and v are given by

$$u(x) = -\int \frac{y_2(x)f(x)}{a_1 W}dx \text{ and } v(x) = \int \frac{y_1(x)f(x)}{a_1 W}dx.$$

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Worked Example1 :

Solve the differential equation $y'' - 2y' - 3y = 2e^x - 10sinx$.

Solution.

Step1: To find the complementary function

The auxiliary equation of the associated homogeneous equation is $m^2 - 2m - 3 = 0 \Rightarrow m = -1,3$

Hence the complementary function is

$y_c(x) = c_1e^{-x} + c_2e^{3x}$, where c_1 and c_2 are arbitrary constants.

Step2: To find the particular solution by the method of variation of parameters.

Let $y_1(x) = e^{-x}$ and $y_2(x) = e^{2x}$. The Wronskian function is given by $W = \begin{vmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3e^{3x} \end{vmatrix} = 4e^{2x}$

From the given DE, we have $r(x) = 2e^x - 10sinx$.

The particular solution is given by $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$,

where u and v are given by $u(x) = -\int \frac{y_2(x)r(x)}{W}dx$ and $v(x) = \int \frac{y_1(x)r(x)}{W}dx$.

Now, $\frac{y_2(x)r(x)}{W} = \frac{e^{3x}[2e^x - 10sinx]}{4e^{2x}} = \frac{1}{2}[e^{2x} - 5e^x sinx]$.

Hence, $u(x) = -\int \frac{y_2(x)r(x)}{W}dx = -\int \frac{1}{2}[e^{2x} - 5e^x sinx]dx = -\frac{1}{2}\left[\frac{e^{2x}}{2} - 5\int e^x sinxdx\right]$

$$= -\frac{1}{2}\left[\frac{e^{2x}}{2} - 5\left(\frac{e^x}{2}\{sinx - cosx\}\right)\right] = -\frac{1}{4}e^{2x} + \frac{5}{4}e^x\{sinx - cosx\}$$

Also, $\frac{y_1(x)r(x)}{W} = \frac{e^{-x}[2e^x - 10sinx]}{4e^{2x}} = \frac{1}{4}[2e^{-2x} - 10e^{-3x}sinx]$.

Hence, $v(x) = \int \frac{y_1(x)r(x)}{W}dx = \frac{1}{4}[-e^{-2x} - 10\int e^{-3x}sinxdx] = \frac{1}{4}\left[-e^{-2x} - 10\left(\frac{e^{-3x}}{10}\{-3sinx - cosx\}\right)\right]$

$= -\frac{1}{4}e^{-2x} + \frac{1}{4}e^{-3x}\{3sinx + cosx\}$. Thus, we have

$u(x)y_1(x) == -\frac{1}{4}e^x + \frac{5}{4}\{sinx - cosx\}$ and $v(x)y_2(x) = -\frac{1}{4}e^x + \frac{1}{4}\{3sinx + cosx\}$

Hence the particular solution is $y_p(x) = u(x)y_1(x) + v(x)y_2(x) = -\frac{1}{2}e^x + 2sinx - cosx$

Thus the general solution of the given DE is

$$y(x) = y_c(x) + y_p(x) = c_1e^{-x} + c_2e^{2x} - \frac{1}{2}e^x + 2sinx - cosx$$

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