Differential Equations Module - 5 - Tutorial sheet -2
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(I)
Check whether the following functions are odd or even and find the Fourier series expansion of them.

1. $f(x) = \pi - |x| \quad (-\pi < x < \pi)$

2. $f(x) = 2x|x| \quad (-1 < x < 1)$

3. $f(x) = \begin{cases} x & \text{if} \quad -\pi/2 < x < \pi/2 \\ \pi - x & \text{if} \quad \pi/2 < x < 3\pi/2 \end{cases}$

4. $f(x) = \begin{cases} \pi e^{-x} & \text{if } -\pi < x < 0 \\ \pi e^{x} & \text{if } 0 < x < \pi \end{cases}$

5. $f(x) = \begin{cases} 2 & \text{if } -2 < x < 0 \\ 0 & \text{if } 0 < x < 2 \end{cases}$

(II)
Show that the Fourier series of

$$f(x) = \begin{cases} k\,; & -1 < x < 0 \\ x\,, & 0 < x < 1 \end{cases}$$

is:

$$f(x) = \left(\frac{k}{2} + \frac{1}{4}\right) - \frac{2}{\pi^2} \sum_{n=1,3,5,\ldots,\infty} \frac{1}{n^2}\cos(n\pi x) - \frac{1}{\pi}\sum_{n=1}^{\infty}[k(1-(-1)^n) + (-1)^n]\sin(n\pi x).$$

Hence find the sum of the series: $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ by using the above Fourier series.

(III)
Find the Half-range Sine series of $f(x) = x\sin(x)$ on $(0, 2)$.

(IV)
Find the half-range cosine series of $f(x) = x(\pi - x)$ in $(0, \pi)$. Hence find the sum of the series $\frac{1}{1^4} + \frac{1}{2^4} + \ldots\infty$.

(V)
The function $f(x) = x - x^2$ is defined only over the range $0 < x < 1$. Find the half-range cosine Fourier expansion of *f(x)*. Hence prove that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

(VI)
Use Parseval's identity to evaluate $\int_{-\pi}^{\pi}(2\sin^2 5x - \cos 3x)^2\,dx$.

(VII)

(a)

Suppose the Fourier coefficients of a $2L$ periodic function $f(x)$ are a_n (n = 0, 1, 2 …) and b_n (n = 1, 2 …). And if the Fourier coefficients of the shifted periodic function $f(x-k)$ for the same f and for any positive constant k are A_n (n = 0, 1, 2 …) and B_n (n = 1, 2, …), then show that $A_0 = a_0$, $A_n = a_n \cos\frac{n\pi k}{L} - b_n \sin\frac{n\pi k}{L}$ and $B_n = a_n \sin\frac{n\pi k}{L} + b_n \cos\frac{n\pi k}{L}$. Find the relation between RMS values of $f(x)$ and $f(x-k)$.

(b)

Write the functions of the following two graphs defined in the range [-2π, 2π] with their fundamental periods.

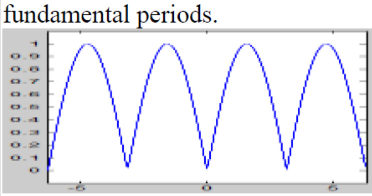

Fig 1

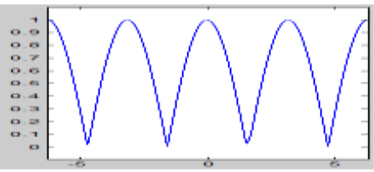

Fig 2

(c)

Check the Fourier coefficients of the first function in Fig-1 are $a_0 = \frac{2}{\pi}$, $a_n = \frac{-4}{\pi(4n^2-1)}$ and $b_n = 0$. Hence find the Fourier coefficients of the second function in Fig-2 (by using the result in (a)).

(VIII)

Find the Half-range Sine series of $f(x) = (x-1)^2$, $0 < x < 1$.

(IX)

Prove that in $0 < x < l$,

$$x = \frac{l}{2} - \frac{4\,l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \cdots\cdots\right).$$

(X)

(i) Let $f(x) = \begin{cases} -x^2, & x \in [-\pi, -0.5) \\ 3x, & x \in [-0.5, 0.5) \\ e^{2x}, & x \in [0.5, \pi] \end{cases}$. Does the Fourier series of $f(x)$ exist? If so, find a function $g(x)$ on $[-\pi, \pi]$ such that the Fourier series of $f(x)$ converges to $g(x)$.

(<u>Note:</u> The Fourier series of $f(x)$ is not needed to answer this.)

(ii) Let $h(x) = \begin{cases} 1, & x \in \{\textit{rational numbers in } [0,1]\} \\ 0, & x \in \{\textit{irrational numbers in } [0,1]\} \end{cases}$ be a periodic function with period 1. Do all Dirichlet's conditions are satisfied for $h(x)$? Justify your answer.

(<u>Note:</u> The Fourier series of $h(x)$ is not needed to answer this.)