1. Fourier Integral Theorem

If $f(x)$ satisfies Dirichlet's Conditions in every finite interval of length 2L, and is absolutely integrable in $(-\infty, \infty)$, then the Fourier integral representation of $f(x)$ is

$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos[\lambda(t-x)]dtd\lambda.$

Alternative form of Fourier Integral : Denoting $A(\lambda) = \int_{-\infty}^\infty f(t)\cos(\lambda t)\,dt$ and

$B(\lambda) = \int_{-\infty}^\infty f(t)\sin(\lambda t)\,dt$, the Fourier Integral representation of $f(x)$ can be expressed as $f(x) = \frac{1}{\pi}\int_0^\infty [A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)]d\lambda.$

2. Fourier Cosine and Sine Integrals

- If $f(x)$ is even, then the Fourier integral is called the **Fourier Cosine integral**, and is

  $f(x) = \frac{1}{\pi}\int_0^\infty A(\lambda)\cos(\lambda x)\,d\lambda$, where $A(\lambda) = 2\int_0^\infty f(t)\cos(\lambda t)\,dt.$

- If $f(x)$ is odd, then the Fourier integral is called the **Fourier Sine integral**, and is

  $f(x) = \frac{1}{\pi}\int_0^\infty B(\lambda)\sin(\lambda x)\,d\lambda$, where $B(\lambda) = 2\int_0^\infty f(t)\sin(\lambda t)\,dt.$

3. Complex Form of the Fourier integral

If $f(x)$ satisfies Dirichlet's Conditions in every finite interval of length 2L, and is absolutely integrable in $(-\infty, \infty)$, then the complex form of the Fourier integral of $f(x)$ is

$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{i\lambda(t-x)}dtd\lambda$. Alternatively, the complex form of the Fourier integral of $f(x)$ can also be expressed as $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)e^{-i\lambda x}d\lambda$, where $F(\lambda) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{\mathrm{i}\lambda \mathrm{t}}dt$.

Fourier Convergence Theorem : The complex form of the Fourier integral $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)e^{-i\lambda x}d\lambda$ converges to $f(x)$ at points of continuity and converges to $\frac{1}{2}[f(x-) + f(x+)]$ at points of discontinuity, where $f(x-)$ and $f(x-)$ are the left and right hand limits of $f(x)$.

4. Fourier Transform pair

The function $F(\lambda) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}\lambda \mathrm{x}}dx$ is called the Fourier transform of $f(x)$, and the complex form of the Fourier integral of the function $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)e^{-i\lambda x}d\lambda$ is called the inverse Fourier transform of $F(s)$. The pair $(f(x,), F(\lambda))$ is called a Fourier Transform pair, where λ is called the transform variable.

5. PROPERTIES OF FOURIER TRANSFORMS

Fourier Transform of $f(x)$ is defined by $F(\lambda) = \int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx$.

S.No.	Property Name	Function, $f(x)$	Fourier Transform function, $F(\lambda)$
1	Linearity	$af(x) + bg(x)$	$aF(\lambda) + bG(\lambda)$
2	Scaling	$f(ax)$	$\frac{1}{a}F\left(\frac{\lambda}{a}\right)$
3	Shifting	$f(x-a)$	$e^{ia\lambda}F(\lambda)$
4	Modulation	$f(x)\cos(ax)$	$\frac{1}{2}[F(\lambda+a) + F(\lambda-a)]$
		$f(x)\sin(ax)$	$\frac{-i}{2}[F(\lambda+a) + F(\lambda-a)]$
5	Convolution	$f(x) * g(x)$	$F(\lambda)G(\lambda)$
6	Parseval's Identity	(i) $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\bar{G}(\lambda)d\lambda = \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx$ (ii) $\frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert F(\lambda)\rvert^2 d\lambda = \int_{-\infty}^{\infty} \lvert f(x)\rvert^2 dx$	
7	Differentiation in the time domain	$f^n(x)$	$(i\lambda)^n F(\lambda)$
8	Differentiation in the frequency domain	$x^n f(x)$	$i^n F^n(\lambda)$
9	Transform of the integral in time domain	$\int_{-\infty}^{x} f(t)dt$	$\frac{F(\lambda)}{i\lambda}$

6. PROPERTIES OF FOURIER COSINE AND SINE TRANSFORMS

Fourier Cosine Transform (FCT) of $f(x)$ is defined by $F_C(\lambda) = \int_0^\infty f(x)\cos(\lambda x)dx$.

Fourier Sine Transform (FST) of $f(x)$ is defined by $F_S(\lambda) = \int_0^\infty f(x)\sin(\lambda x)dx$.

Property Name	Function, $f(x)$	FCT function, $F_C(\lambda)$	FST function, $F_S(\lambda)$
Linearity	$af(x) + bg(x)$	$aF_C(\lambda) + bG_C(\lambda)$	$aF_S(\lambda) + bG_S(\lambda)$
Scaling	$f(ax)$	$\frac{1}{a}F_C\left(\frac{\lambda}{a}\right)$	$\frac{1}{a}F_S\left(\frac{\lambda}{a}\right)$
Shifting	$f(x-a)$	$e^{ia\lambda}F_C(\lambda)$	$e^{ia\lambda}F_S(\lambda)$
Modulation	$f(x)\cos(ax)$	$\frac{1}{2}[F_C(\lambda-a) + F_C(\lambda+a)]$	$\frac{1}{2}[F_S(\lambda-a) + F_S(\lambda+a)]$
	$f(x)\sin(ax)$	$\frac{-1}{2}[F_S(\lambda-a) - F_S(\lambda+a)]$	$\frac{1}{2}[F_C(\lambda-a) - F_C(\lambda+a)]$
Parseval's Identity	$\frac{2}{\pi}\int_0^\infty F_C(\lambda)G_C(\lambda)d\lambda = \int_0^\infty f(x)g(x)dx$ $\frac{2}{\pi}\int_0^\infty \lvert F_C(\lambda)\rvert^2 d\lambda = \int_0^\infty \lvert f(x)\rvert^2 dx$		$\frac{2}{\pi}\int_0^\infty F_S(\lambda)G_S(\lambda)d\lambda = \int_0^\infty f(x)g(x)dx$ $\frac{2}{\pi}\int_0^\infty \lvert F_S(\lambda)\rvert^2 d\lambda = \int_0^\infty \lvert f(x)\rvert^2 dx$

7. Images of Fourier Transform pairs