MODULE 4 Solution to ODE and PDE by Laplace Transform

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Solution of ODE's – Non-homogeneous terms involving Heaviside function, Impulse function - Solving Non-homogeneous system using Laplace transform - solution to First order PDE by Laplace transform.

11/4/2022

Differentiation of f(t)

$$L[f'(t)] = s\,L[f(t)] - f(0)$$

$$L[f''(t)] = s^2\,L[f(t)] - s f(0) - f'(0)$$

Results

1) $L^{-1}\left[\dfrac{s^2-a^2}{(s^2+a^2)^2}\right] = t\cos at$

2) $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right] = \dfrac{1}{2a}\sin at$

3) $L^{-1}\left[\dfrac{1}{(s^2+a^2)^2}\right] = \dfrac{1}{2a^3}(\sin at -$

Applications of solns. of DE

(1) Solve using LT

$$\frac{dy}{dt} - y = 2 \quad \text{given } y(0) = 2.$$

$$y' - y = 2. \quad ; \quad y(0) = 2.$$

taking LT on both sides

$$L[y'] - L[y] = L[2] \qquad f'(t) = y'$$

$$sL[y] - y(0) - L[y] = 2L[1] \qquad f(t)=y$$

$$sL[y] - 2 - L[y] = \frac{2}{s}$$

$$L[y]\{s-1\} = \frac{2}{s} + 2$$

$$= \frac{2(1+s)}{s(s-1)}$$

$$L[y] = \frac{2(s+1)}{s(s-1)}$$

using partial fractions.

$$\frac{2(s+1)}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1} = \frac{A(s-1)+Bs}{s(s-1)}$$

$$2(s+1) = A(s-1) + Bs$$

put $s = 1 \Rightarrow B = 4$

$s = 0 \Rightarrow A = -2$

$$L[y] = \frac{-2}{s} + \frac{4}{s-1}$$

$$y = L^{-1}\left[-\frac{2}{s} + \frac{4}{s-1}\right]$$

$$= L^{-1}\left[\frac{-2}{s}\right] + L^{-1}\left[\frac{4}{s-1}\right]$$

$$= -2(1) + 4e^{t}$$

$$= 4e^{t} - 2$$

2) solve using L T

$$y'' + y = \sin t \quad ; y(0) = 1$$
$$y'(0) = 1/2$$

taking LT

$$L[y''] + L[y] = L[\sin t]$$

$$s^2 L[y] - sy(0) - y'(0) + L[y] = \frac{1}{s^2+1}$$

$$s^2 L[y] - s(1) - \frac{1}{2} + L[y] = \frac{1}{s^2+1}$$

$$L[y]\{s^2+1\} = \frac{1}{s^2+1} + \frac{1}{2} + s$$

$$L[y] = \frac{1}{(s^2+1)^2} + \frac{1}{2(s^2+1)} + \frac{s}{s^2+1}$$

$$\Rightarrow y = L^{-1}\left[\frac{1}{(s^2+1)^2}\right] + L^{-1}\left[\frac{1}{2(s^2+1)}\right] + L^{-1}\left[\frac{s}{s^2+1}\right]$$

formula.

$$= \frac{1}{2}(\sin t - t\cos t) + \frac{1}{2}\sin t + \cos t$$

$$= \frac{1}{2}\sin t - \frac{t}{2}\cos t + \frac{1}{2}\sin t + \cos t$$

$$= \sin t + \cos t - \frac{t}{2}\cos t$$

3) solve: $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x}$; $y(0) = y'(0) = 0$

$$y'' - 3y' + 2y = e^{3x}.$$

taking LT on both side.

3) ?

$$L[y''] - 3L[y'] + 2L(y) = L[e^{3t}]$$

$$s^2 L[y] - s\,\underset{0}{y(0)} - \underset{0}{y'(0)} - 3[s\,L[y] - \underset{0}{y(0)}] + 2L[y] = \frac{}{s}$$

$$L[y][s^2 - 3s + 2] = \frac{1}{s-3}$$

$$L[y] = \frac{1}{(s-3)(s^2-3s+2)}$$

$$= \frac{1}{(s-3)(s-1)(s-2)}$$

$$\frac{1}{(s-3)(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$= \frac{A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)}{(s-1)(s-2)(s-3)}$$

put $s = 1 \Rightarrow A = 1/2$

$s = 2 \Rightarrow B = -1$

$s = 3 \Rightarrow C = 1/2$

$$L[y] = \frac{1}{2(s-1)} + \frac{-1}{s-2} + \frac{1/2}{s-3}$$

$$y = L^{-1}[\quad]$$

$$= \frac{1}{2}e^{t} - e^{2t} + \frac{1}{2}e^{3t}$$

2 2

13/4/22

LT → non homogeneous part N.P → unit step, dirac delta fn.

$u(t-a)$

$u_a(t)$

Solve the DE with initial cond.

$y(0)=3$, $y'(0)=2$.

$$3y'' - 2y' + 2y = 2 - 2u_2(t)$$

Taking LT on both sides.

$L[\quad] = L[\quad]$

using linearity property.

$$3L[y''] - 2L[y'] + 2L[y] = 2L[1] - 2L[$$

$$3[s^2 L[y] - s\,y(0) - y'(0)] - 2[sL[y] - y(0)]$$

$$+ 2L[y] = \frac{2}{s} - \frac{2e^{-2s}}{s}$$

$$3s^2 L[y] - 3s(3) - 3(2) - 2sL[y] + 2(3)$$
$$+ 2L[y] = \frac{2}{s} - \frac{2e^{-2s}}{s}$$

$$L[y][3s^2 - 2s + 2] - 9s - \cancel{6} + 6 = \frac{2}{s} - \frac{2e^{-2s}}{s}$$

$$\mathcal{L}(y)\,(3s^2-2s+2) = 9s + \frac{2}{s} - \frac{2e^{-2s}}{s}$$

$$= \frac{9s^2+2}{s} - \frac{2e^{-2s}}{s}$$

$$\mathcal{L}[y] = \frac{9s^2+2}{s(3s^2-2s+2)} - \frac{2e^{-2s}}{s(3s^2-2s+2)}$$

I II

$y = ?$ $\quad y_{\mathrm{I}} + y_{\mathrm{II}}$

$\mathrm{I} \Rightarrow$

$$\frac{9s^2+2}{s(3s^2-2s+2)} = \frac{A}{s} + \frac{Bs+C}{3s^2-2s+2}$$

$$= \frac{A(3s^2-2s+2)+Bs^2+Cs}{s(3s^2-2s+2)}$$

$A = 1 \quad ; \quad B = 6 \quad ; \quad C = 2$

$$\mathcal{L}[\mathrm{I}] = \frac{1}{s} + \frac{6s+2}{3s^2-2s+2}$$

$$y_{\mathrm{I}} = \mathcal{L}^{-1}\left[\frac{1}{s}\right] + \mathcal{L}^{-1}\left[\frac{6s+2}{3s^2-2s+2}\right]$$

$$= 1 + \mathcal{L}^{-1}\left[\frac{6s}{3s^2-2s+2} + \frac{2}{3s^2-2s+2}\right]$$

$$y_{\mathrm{I}} = 1 + \frac{6}{3}\mathcal{L}^{-1}\left[\frac{s}{s^2-\frac{2}{3}s+\frac{2}{3}}\right] + 2\,\mathcal{L}^{-1}\left[\frac{1}{3s^2-2s+2}\right]$$

$$= 1 + 2L^{-1}\left[\frac{s-\frac{1}{3}+\frac{1}{3}}{\left(s-\frac{1}{3}\right)^2+\frac{5}{9}}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{\left(s-\frac{1}{3}\right)^2+\frac{5}{9}}\right]$$

$$y_I = 1 + 2L^{-1}\left[\frac{s-1/3}{\left(s-1/3\right)^2+\frac{5}{9}}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{\left(s-1/3\right)^2+5/9}\right] +$$

$$= 1 + 2L^{-1}\left[\frac{s-1/3}{\left(s-1/3\right)^2+5/9}\right] + \frac{4}{3}L^{-1}\left[\frac{1}{\left(s-1/3\right)^2+5/9}\right]$$

$$y_I = 1 + 2\cdot\cos\sqrt{\frac{5}{9}}t \cdot e^{1/3\, t} + \frac{4}{\cancel{3}}\,\frac{1}{\frac{\sqrt{5}}{\cancel{3}}}\sin\sqrt{\frac{5}{9}}$$

$$y_I = 1 + e^{1/3\, t}\, 2\cos\frac{\sqrt{5}}{3}t + e^{1/3\, t}\frac{4}{\sqrt{5}}\sin\frac{\sqrt{5}}{9}$$

(1)

$$L^{-1}\left[e^{-2s}\frac{2}{s(3s^2-2s+2)}\right]$$

$$\boxed{\begin{aligned}A&=1\\B&=-3\\C&=2\end{aligned}}$$

$$= 2L^{-1}\left[\frac{e^{-2s}}{s}\right] + 2L^{-1}\left[e^{-2s}\left(\frac{-3s+2}{3s^2-2s+2}\right)\right]$$

$$= 2u(t-2)\cdot(1) + 2\cdot\left(\quad\right)\left(\quad\right)$$

$$=$$

$y = y_I + y_{II}$

$=$ ______ ______ .

<u>direcdelta fn</u> - $y'(0) = 0 = y(0)$

$t = 0$

(eg) $y'' - y = \delta(t)$,

$L(y'') - L[y] = L[\delta(t)]$

RHS

$s^2 L(y) - s y(0) - y'(0) - L(y) = 0$

$L(y)$ ( )

NAP → e^x, fns, alg, const, unit, dirc

SOLUTION OF Simultaneous ODE

18/4/22

1) Solve using LT

$$\frac{dx}{dt} - y = t$$

$$x + \frac{dy}{dt} = t^2$$

given $x(0) = 0$; $y(0) = 0$.

$x'(t) - y(t) = t$

$x(t) + y'(t) = t^2$ } gn. $x(0) = 0 = y(0)$

$x'(t) - y(t) = t$

Taking LT on both sides.

$L[x'(t)] - L[y(t)] = L[t]$

$\Rightarrow sL[x(t)] - x(0) - L[y(t)] = \frac{1}{s^2}$

$\Rightarrow s\bar{x} - 0 - \bar{y} = \frac{1}{s^2}$

$\Rightarrow s\bar{x} - \bar{y} = \frac{1}{s^2}$ —— (1)

$\Rightarrow L[x(t)] + L[y'(t)] = L[t^2]$

$\bar{x} + s\bar{y} - 0 = \frac{2}{s^3}$

$\bar{x} + s\bar{y} = \frac{2}{s^3}$ —— (2)

Solving (1) & (2)

(1) $\times s \Rightarrow s^2\bar{x} - s\bar{y} = \frac{1}{s}$

(2) $\Rightarrow \bar{x} + s\bar{y} = \frac{2}{s^3}$

$(s^2+1)\bar{x} = \frac{1}{s} + \frac{2}{s^3} = \frac{s^2+2}{s^3}$

$\Rightarrow \bar{x} = \frac{s^2+2}{s^3(s^2+1)}$

$L[x(t)] = \frac{s^2+2}{s^3(s^2+1)}$

$\Rightarrow x(t) = L^{-1}\left[\frac{s^2+2}{s^3(s^2+1)}\right]$

using partial fraction,

$\frac{s^2+2}{\ldots} = \frac{A}{\ldots} + \frac{B}{\ldots} + \frac{C}{\ldots} + \frac{Ds+E}{\ldots}$

$$\frac{s^2+2}{s^3(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2+1}$$

$$s^2+2 = As^2(s^2+1) + Bs(s^2+1) + C(s^2+1) + Ds(s^3) + \dots$$

put $s=0 \Rightarrow 2 = C(0+1)$

$\Rightarrow \boxed{C=2}$

equate

coeff of $s^4 \Rightarrow 0 = A + D$

coeff of $s^3 \Rightarrow 0 = B + E$

coeff of $s^2 \Rightarrow 1 = A + C \Rightarrow \boxed{A=-1}$

coeff of $s \Rightarrow \boxed{0 = B}$

$D = 1$

$E = 0$

$A=-1$; $B=0$, $C=2$, $D=1$, $E=0$.

$$x(t) = L^{-1}\left[\frac{-1}{s} + \frac{0}{s^2} + \frac{2}{s^3} + \frac{1.s+0}{s^2+1}\right]$$

$$= L^{-1}\left[\frac{-1}{s}\right] + 2L^{-1}\left[\frac{1}{s^3}\right] + L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$\boxed{x(t) = -1 + \cancel{2}\frac{t^2}{\cancel{2}} + \cos t}$$

$x'(t) - y(t) = t$ — (1)

$2t - \sin t - y(t) = t$

$y(t) = 2t - \sin t - t$

$$\boxed{y(t) = t - \sin t}$$

$\therefore$ solns are $x(t) = -1 + t^2 + \cos t$

$y(t) = t - \sin t$

2) Solve :

$$\frac{dx}{dt} + 2y = 5e^{t}$$

$$\frac{dy}{dt} - 2x = 5e^{t}$$

given $x = -1, \quad y = 3$ when $t = 0$

$\Rightarrow x(0) = -1, \quad y(0) = 3$

$x'(t) + 2y(t) = 5e^{t}$

$y'(t) - 2x(t) = 5e^{t}$

$s\bar{x} - x(0) + 2\bar{y} = 5L[e^{t}]$

$\Rightarrow s\bar{x} + 1 + 2\bar{y} = 5\frac{1}{s-1}$

$s\bar{x} + 2\bar{y} = \frac{5}{s-1} - 1 = \frac{5-s+1}{s-1}$

$s\bar{x} + 2\bar{y} = \frac{-s+6}{s-1}$ —— 1

$-2\bar{x} + s\bar{y} = \frac{3s+2}{s-1}$ —— 2

Solving

$$\boxed{\begin{aligned} x(t) &= -e^{t} \\ y(t) &= 3e^{t} \end{aligned}}$$

20/4/22.

1) $\dfrac{d^2y}{dt^2} + 9y = \cos 4t$, $y(0) = 1$

$y(\pi/2) = 1.$

$\boxed{y'(0) = k.}$

$y'' + 9y = \cos 4t$

$L[y''] + 9L[y] = L[\cos 4t]$

$s^2 L[y] - s\,y(0) - y'(0) + 9L[y] = \dfrac{s}{s^2+16}$

$s^2 L[y] - s(1) - k + 9L[y] = \dfrac{s}{s^2+16}$

$L[y][s^2+9] = \dfrac{s}{s^2+16} + s + k.$

$L[y] = \dfrac{s}{(s^2+16)(s^2+9)} + \dfrac{s+k}{s^2+9}$

$y = L^{-1}\left[\dfrac{s}{(s^2+9)(s^2+16)}\right] + L^{-1}\left[\dfrac{s+k}{s^2+9}\right]$

$\dfrac{As+B}{s^2+9} + \dfrac{Cs+D}{s^2+16}$ partial fn.

$L^{-1}\left[\dfrac{s}{s^2+9}\right] + k\,L^{-1}\left[\dfrac{1}{s^2+}\right.$

$A = \dfrac{1}{7}$ $B = 0$ $C = -\dfrac{1}{7}$ $D = 0$

$y = \dfrac{1}{7}L^{-1}\left[\dfrac{s}{s^2+9}\right] + L^{-1}\left[\dfrac{-1/7 \cdot s}{s^2+16}\right] + L^{-1}\left[\dfrac{s}{s^2+9}\right] + L$

$y = \dfrac{1}{7}\cos 3t - \dfrac{1}{7}\cos 4t + \cos 3t + k\sin 3t$

7 7 3

$\frac{1}{7}+1$

$y(t) = \frac{8}{7}\cos 3t - \frac{1}{7}\cos 4t + \frac{k}{3}\sin 3t$

When $t = \pi/2$, $y(\pi/2) = -1$ (Gn).

$y(\pi/2) = \frac{8}{7}\cos 3\pi/2 - \frac{1}{7}\cos 4\pi/2 + \frac{k}{3}\sin 3\pi/2$

$-1 = \frac{8}{7}(0) - \frac{1}{7}(1) + \frac{k}{3}(-1)$

$k = \frac{18}{7}$

$\therefore y(t) = \frac{8}{7}\sin 3t - \frac{1}{7}\cos 4t + \frac{\cancel{18}^{6}}{7}\frac{\sin 3t}{\cancel{3}}$

2) $\frac{dx}{dt} - 2y = \sin 2t$

$\frac{dy}{dt} + 2x = \cos 2t$ Gn $x(0)=0=y(0)$

$L[x'] - 2L[y] = L[\sin 2t$

$s\bar{x} - 2\bar{y} = \frac{2}{s^2+1}$ —— 1

$L[y'] + 2L[x) = L[\cos 2t]$

$s\bar{y} + 2\bar{x} = \frac{s}{s^2+4}$ —— 2

Ans

$$x = t\sin 2t$$

$$y = t\cos 2t$$