Inverse Laplace Transform

$$f(t) \xrightarrow{L} F(s) = L[f(t)]$$

$$L^{-1}[F(s)] = f(t) \xleftarrow[L^{-1}]{} F(s)$$

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$f(t) = L^{-1}[F(s)] = \frac{1}{2\pi i}\int_C e^{st} F(s)\, ds$$

C - closed curve or contour.

The convolution function of two functions $f(t)$ and $g(t)$ over $(0, \infty)$ is defined by

$$(f * g)(t) = \int_0^t f(u) g(t-u)\, du$$

<u>Notations:</u> $(f * g)(t)$, $f(t) * g(t)$

① Let $f(t) = e^{at}$ and $g(t) = e^{bt}$.

Find the convolution function $f(t) * g(t)$

Sol: By definition,

$$f(t) * g(t) = \int_0^t e^{au} \, e^{b(t-u)} \, du$$

$$= e^{bt} \int_0^t e^{u(a-b)} \, du$$

$$= e^{bt} \left[\frac{e^{u(a-b)}}{a-b} \right]_{u=0}^{t}$$

$$= \frac{e^{bt}}{a-b} \left[\{ e^{(a-b)t} \} - \{ 1 \} \right]$$

$$\therefore \boxed{e^{at} * e^{bt} = \frac{e^{at} - e^{bt}}{a-b}}$$

② Obtain $L[e^{at} * e^{bt}]$.

Sol: By Convolution Theorem,

$$L[e^{at} * e^{bt}] = L[e^{at}] \, L[e^{bt}]$$

$$= \frac{1}{s-a} \cdot \frac{1}{s-b}$$

$$= \frac{1}{(s-a)(s-b)} \quad //$$

Convolution Theorem for Laplace Transform

Let $L[f(t)] = F(s)$ and

$L[g(t)] = G(s)$. Then

$$\boxed{L[f(t) * g(t)] = F(s)\,G(s)}$$

Note: $f * g = g * f$