15.10 EQUATIONS OF THE FORM dx/P = dy/Q = dz/R

(1) Method of grouping

See if it is possible to take two fractions $dx/P = dz/R$ from which y can be cancelled or is absent, leaving equations in x and z only.

If so, integrate it by giving $\phi(x, z) = c$. ...(1)

Again see if one variable say : x is absent or can be removed may be with the help of (1), from the equation $dy/Q = dz/R$.

Then integrate it by giving $\psi(y, z) = c'$ ...(2)

These two independent solutions (1) and (2) taken together constitute the complete solution required.

$\frac{a}{b} = \frac{c}{d}$

$= \frac{\alpha a + \beta c}{\alpha b + \beta d}$

(2) Method of multipliers

By a proper choice of the multipliers l, m, n which are not necessarily constants, we write

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l dx + m dy + n dz}{lP + mQ + nR} \text{ such that } lP + mQ + nR = 0.$$

Then $l dx + m dy + n dz = 0$ can be solved giving the integral $\phi(x, y, z) = c$ ...(1)

Again search for another set of multipliers λ, μ, γ

so that $\lambda P + \mu Q + \gamma R = 0$

giving $\lambda dx + \mu dy + \gamma dz = 0$,

which on integration gives the solution $\psi(x, y, z) = c'$ ...(2)

These two solutions (1) and (2) taken together constitute the required solution.

Example 15.14. *Solve* $\frac{dx}{z^2 y} = \frac{dy}{z^2 x} = \frac{dz}{y^2 x}$.

Solution. Taking the first two fractions and cancelling z^2, we get

$$\frac{dx}{y} = \frac{dy}{x} \quad \text{or} \quad x dx - y dy = 0$$

$\frac{x^2}{2} - \frac{y^2}{2} = C$

which on integration gives $x^2 - y^2 = c$. ...(i)

Again taking the second and third fractions and cancelling x, we have

$x^2 - y^2 = 2C$

$$\frac{dy}{z^2} = \frac{dz}{y^2}, \text{ i.e., } y^2 dy - z^2 dz = 0.$$

Its integral is $y^3 - z^3 = c'$. ...(ii)

Thus (i) and (ii) taken together constitute the required solution of the given equations.

$y^3 - z^3 = 3c$ $\quad \frac{y^3}{3} - \frac{z^3}{3} = c$ $\quad x^2 - y^2 = a$

$y^3 - z^3 = b$

$\phi(x^2 - y^2, y^3 - z^3) = 0$ is the general solution.

Theorem: The general solution of a quasilinear partial differential equation

$$P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z)$$

is

$$\phi(u,v) = 0,$$

where ϕ is an arbitrary function. The arguments u and v are obtained from the subsidiary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

for which the two independent solutions are $u(x,y,z) = a$ and $v(x,y,z) = b$

(1) Solve $(mz-ny)p+(nx-lz)q = ly-mx.$

Sol: Here the subsidiary equations are

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} \quad \text{--- (1)}$$

(i) Using the multipliers x, y and z,

each fraction $= \dfrac{x\,dx+y\,dy+z\,dz}{0}$

$\therefore \quad x\,dx+y\,dy+z\,dz = 0$

Integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = c_1$$

$$\boxed{x^2+y^2+z^2 = a}$$

$\therefore \quad u = x^2+y^2+z^2$

(ii) Using the multipliers l, m and n,

$$\text{each fraction} = \frac{l\,dx + m\,dy + n\,dz}{0}$$

$$\therefore\ l\,dx + m\,dy + n\,dz = 0$$

Integrating, we get

$$\boxed{lx + my + nz = b}$$

$$\therefore\ V = lx + my + nz$$

General solution of the given Pde is

$$\phi(u, v) = 0$$

$$\text{i.e., } \phi(x^2 + y^2 + z^2,\ lx + my + nz) = 0$$

where ϕ is an arbitrary function.

(2) Solve $(x^2-y^2-z^2)p + 2xyq = 2xz$.

Sol: The subsidiary equations are

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \quad \text{(1)}$$

(i) Considering second and third fractions, we get

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, we get

$$\log y = \log z + \log a$$

$$\log\left(\frac{y}{z}\right) = \log a$$

$$\Rightarrow \boxed{\frac{y}{z} = a}$$

$\therefore u = y/z$

(iii)

Choosing multipliers as x, y and z,

each fraction $= \dfrac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$

i.e.,

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$$

Now, considering third and fourth fractions, we get

$$\frac{dz}{2xz} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$$

$$\Rightarrow \quad \frac{dz}{z} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$$

Taking $\phi = x^2+y^2+z^2$, we get

$$\frac{dz}{z} = \frac{d\phi}{\phi}$$

Integrating, we get

$$\log z - \log \phi = \log k$$

$$\log(z/\phi) = \log k$$

$$\Rightarrow \quad \frac{z}{\phi} = k$$

$$\Rightarrow \quad \boxed{\frac{z}{x^2+y^2+z^2} = k}$$

$$\therefore \quad V = \frac{z}{x^2+y^2+z^2}$$

The general solution of the given pde is

$$\checkmark\ F(u, v) = 0 ,$$

where $u = y/z$,

$v = \dfrac{z}{x^2+y^2+z^2}$ and F is an arbitrary function.

Another Soln:

The general Soln. is

$$v = G(u),$$ where

G is arbitrary function.

(3) solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Ans: $xyz = \phi\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

(4) solve $(x^2-yz)p + (y^2-zx)q = z^2-xy$

Ans: $\frac{x-y}{y-z} = \phi(xy+yz+zx)$

i.e.,

$$\frac{x^2+y^2+z^2}{z} = G\left(\frac{y}{z}\right)$$

(or)

$$x^2+y^2+z^2 = z\,G\left(\frac{y}{z}\right)$$