Solve

$$y'' + 4y = \sin 5x + x - 1 \quad — (1)$$

This is a second order non homogeneous LDE with constant coefficients

The complete solution of (1) is $y = C.F + P.I$

C.F $\quad y'' + 4y = 0 \rightarrow$ homogeneous LDE

$$(D^2+4)y = 0$$

$\alpha \pm i\beta$

$e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$

A.E $\quad m^2 + 4 = 0$

$$m^2 = -4$$

$$m = \pm 2i$$

$$C.F = e^{0x}[c_1 \cos 2x + c_2 \sin 2x]$$

$$\Rightarrow C.F = c_1 \cos 2x + c_2 \sin 2x$$

P.I Method of undetermined coefficients

RHS : $X = \sin 5x + x - 1$

The trial solution $y = a_0 \cos 5x + a_1 \sin 5x + a_2' + a_3 x + a_4$

$$\boxed{y = a_0 \cos 5x + a_1 \sin 5x + a_2 + a_3 x} \quad — (2)$$

where $a_2 = a_2' + a_4$

differentiate ② w.r.to x

$$y' = -5a_0 \sin 5x + 5a_1 \cos 5x + a_3$$

differentiate y' w.r.to x again

$$y'' = -25a_0 \cos 5x - 25a_1 \sin 5x$$

Substitute the trial solution and its derivatives in ①

$$① \Rightarrow \quad y'' + 4y = \sin 5x + x - 1$$

$$\underline{-25a_0} \cos 5x - 25a_1 \sin 5x + 4(a_0 \cos 5x + a_1 \sin 5x + a_2 + a_3 x) = \sin 5x + x - 1$$

$$\cos 5x(-25a_0 + 4a_0) + \sin 5x(-25a_1 + 4a_1) + 4a_2 + 4a_3 x = \sin 5x + x - 1$$

Compare the constants on both sides

$$4a_2 = -1 \Rightarrow a_2 = -\frac{1}{4}$$

compare coefficient of x

$$4a_3 = 1 \Rightarrow a_3 = \frac{1}{4}$$

compare coefficient of $\sin 5x$ $\quad -21a_1 = 1 \Rightarrow a_1 = -\frac{1}{21}$

compare coefficient of $\cos 5x$ $\quad -21a_0 = 0 \Rightarrow a_0 = 0$

$a_0 = 0$

$a_1 = \frac{-1}{21}$

$a_2 = \frac{-1}{4}$

$a_3 = \frac{1}{4}$

The trial soln is $\quad y = a_0 \cos 5x + a_1 \sin 5x + a_2 + a_3 x$

$$P.I = \frac{-1}{21} \sin 5x - \frac{1}{4} + \frac{1}{4} \cdot x$$

$\therefore$ The complete soln $y = C.F + P.I$

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{21} \cdot \sin 5x - \frac{1}{4} + \frac{x}{4} \quad //$$

Solve $\quad y'' + 2y' + 2y = e^x \cos 2x \quad — (1)$

This is a second order non-homogeneous LDE with constant coefficients

The complete solution of (1) is $y = C.F + P.I$

C.F $y'' + 2y' + 2y = 0 \Rightarrow$ homogeneous LDE

$(D^2 + 2D + 2)y = 0$

A.E $m^2 + 2m + 2 = 0$ $\quad a = 1,\ b = 2,\ c = 2$

$$m = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2}$$

$$m = -1 \pm i$$

$$C.F = e^{-x}\left[c_1 \cos x + c_2 \sin x\right]$$

P.I RHS: $X = e^{x} \cos 2x$

The trial solution $y = a_0 e^{x} (a_1 \cos 2x + a_2 \sin 2x)$

$$y = \underbrace{a_0 a_1}_{c_3} e^{x} \cos 2x + \underbrace{a_0 a_2}_{c_4} e^{x} \sin 2x$$

$$y = c_3 e^{x} \cos 2x + c_4 e^{x} \sin 2x \quad — (2)$$

differentiate (2) w.r.to x

$$y' = c_3\left[e^{x}(-2\sin 2x) + e^{x}\cos 2x\right] + c_4\left[e^{x}(2\cos 2x) + e^{x}\sin 2x\right]$$

$$y'' = c_3\left[-2\left(e^{x}(2\cos 2x) + \sin 2x \cdot e^{x}\right) + \left(e^{x}(-2\sin 2x) + e^{x}\cos 2x\right)\right]$$
$$+ c_4\left[2\left(e^{x}(-2\sin 2x) + e^{x}\cos 2x\right) + e^{x}(2\cos 2x) + e^{x}\sin 2x\right]$$

substitute the trial soln. and its derivatives in ①

$$① \Rightarrow y'' + 2y' + 2y = e^{x} \cos 2x$$

$$c_3\left[-4e^{x}\cos 2x - 2e^{x}\sin 2x - 2e^{x}\sin 2x + e^{x}\cos 2x\right]$$
$$+ c_4\left[-4e^{x}\sin 2x + 2e^{x}\cos 2x + 2e^{x}\cos 2x + e^{x}\sin 2x\right]$$
$$+ 2\left[c_3\left(-2e^{x}\sin 2x + e^{x}\cos 2x\right) + c_4\left(2e^{x}\cos 2x + e^{x}\sin 2x\right)\right]$$
$$+ 2\left[c_3 e^{x}\cos 2x + c_4 e^{x}\sin 2x\right] = e^{x}\cos 2x$$

$$e^{x}\cos 2x\left(-4c_3 + c_3 + 2c_4 + 2c_4 + 2c_3 + 4c_4 + 2c_4\right)$$
$$+ e^{x}\sin 2x\left(-2c_3 - 2c_3 - 4c_4 + c_4 - 4c_3 + 2c_4 + 2c_4\right) = e^{x}\cos 2x$$

$$e^{x}\cos 2x\left(-c_3 + 10c_4\right) + e^{x}\sin 2x\left(-8c_3 + c_4\right) = e^{x}\cos 2x$$

compare coefficient of $e^{x}\cos 2x$ on both sides

$$-c_3 + 10c_4 = 1 \quad — ②$$

compare coefficient of $e^{x}\sin 2x$ on both sides

$$-8c_3 + c_4 = 0 \quad — (4)$$

$$-8c_3 + 80c_4 = 8$$

$$\overset{(+)}{\cancel{-8c_3}} \overset{(-)}{+} c_4 \overset{(-)}{=} 0$$

$$79c_4 = 8 \Rightarrow \boxed{c_4 = \frac{8}{79}}$$

substitute c_4 in (4) $\Rightarrow$ $-8c_3 + \frac{8}{79} = 0$

$$+\cancel{8}c_3 = +\frac{\cancel{8}}{79} \Rightarrow \boxed{c_3 = \frac{1}{79}}$$

The trial soln is $c_3 e^{x}\cos 2x + c_4 e^{x}\sin 2x$

$$\therefore P.I = \frac{e^{x}}{79}\cos 2x + \frac{8}{79}e^{x}\sin 2x$$

The complete solution is $y = C.F + P.I$

$$y = e^{-x}\left[c_1\cos x + c_2\sin x\right] + \frac{e^{x}}{79}\cos 2x + \frac{8}{79}e^{x}\sin 2x \quad //$$