I. Shifting property for inverse Laplace transforms.

If $L^{-1}[F(s)] = f(t)$ then

$$L^{-1}[F(s-a)] = e^{at} L^{-1}[F(s)]$$

II If $L^{-1}[F(s)] = f(t)$ and $f(0)=0$, then

$$L^{-1}[sF(s)] = \frac{d}{dt}\{f(t)\}$$

In general,

$$L^{-1}[s^n F(s)] = \frac{d^n}{dt^n}\{f(t)\}$$

provided $f(0) = f'(0) = f''(0) = \cdots = f^{n-1}(0) = 0$

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III If $L^{-1}[F(s)] = f(t)$, then

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt$$

IV If $L^{-1}[F(s)] = f(t)$, then

$$L^{-1}\left[-\frac{d}{ds}F(s)\right] = t\,f(t)$$

This prop. is used to find $L^{-1}[\log(\,)]$, $L^{-1}[\tan^{-1}(\,)]$, $L^{-1}[\cot^{-1}(\,)]$

V If $L^{-1}[F(s)] = f(t)$, then $L^{-1}\left[\int_s^\infty F(s)\,ds\right] = \frac{f(t)}{t}$

can be conveniently calculated

1. Find $L^{-1}\left[\frac{s}{(s+2)^2}\right]$

we apply the property $L^{-1}[sF(s)] = \frac{d}{dt} f(t)$

provided $f(0)=0$.

$$L^{-1}[sF(s)] = L^{-1}\left[\frac{s \cdot 1}{(s+2)^2}\right]$$

$$= \frac{d}{dt} L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{d}{dt}\left(e^{-2t} \cdot t\right)$$

here $f(t) = te^{-2t}$

$f(0) = 0$.

$$= t(-2e^{-2t}) + e^{-2t}$$

$$= e^{-2t}(1-2t).$$

2. Find $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

$$= L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2+a^2}\right]$$

WKT $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt$

here $F(s) = \frac{1}{s^2+a^2}$

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= \frac{1}{a}\sin at = f(t)$$

$$L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2+a^2}\right] = \int_0^t \frac{1}{a}\sin at\, dt$$

$$= \frac{1}{a}\left[-\frac{\cos at}{a}\right]_0^t$$

$$= -\frac{\cos at}{a^2} + \frac{1}{a^2}$$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{(1-\cos at)}{a^2}$$

3. Find $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$

$L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$ For logarithmic function apply the property

$$L^{-1}\left[-\frac{d}{ds}F(s)\right] = t f(t)$$

$$\frac{d}{ds}F(s) = \frac{d}{ds}\left[\log\left(\frac{s+1}{s-1}\right)\right]$$

$$= \frac{d}{ds}\left[\log(s+1) - \log(s-1)\right]$$

$$= \frac{1}{s+1} - \frac{1}{s-1}$$

$$L^{-1}\left[\frac{d}{ds}F(s)\right] = -t\,f(t)$$

$$\Rightarrow L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right] = e^{-t} - e^{t} = -t\,f(t)$$

$$\therefore f(t) = \frac{-(e^{-t} - e^{t})}{t}$$

$$\Rightarrow \boxed{f(t) = \frac{e^{t} - e^{-t}}{t}}$$

4) Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

$$f(t) = L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$$

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$$

$$= \int_s^\infty \frac{s}{(s^2+a^2)^2}\,ds$$

$$= \frac{1}{2}\int_s^{\infty} \frac{2s}{(s^2+a^2)^2}\,ds$$

$$\left\{ \begin{aligned} &\text{Put} \quad t = s^2+a^2 \\ &\qquad\quad dt = 2s\,ds \end{aligned} \right. \qquad \left. \begin{aligned} \int \frac{dt}{t^2} &= \frac{t^{-2+1}}{\cancel{t}\,-2+1} \\ &= -\frac{1}{t} \end{aligned} \right\}$$

$$= \frac{1}{2}\int_s^{\infty} \frac{2s\,ds}{(s^2+a^2)^2} = \frac{-1}{2}\left[\frac{1}{s^2+a^2}\right]_s^{\infty}$$

$$= \frac{1}{2}\cdot\frac{1}{s^2+a^2}$$

$$L^{-1}\left[L\left[\frac{f(t)}{t}\right]\right] = L^{-1}\left[\frac{1}{2}\cdot\frac{1}{s^2+a^2}\right]$$

$$\frac{f(t)}{t} = \frac{1}{2a}\sin at$$

$$\boxed{f(t) = \frac{t}{2a}\sin at}$$

5. Find $L^{-1}\left[\frac{(s+2)^2}{(s^2+4s+8)^2}\right]$

$$= L^{-1}\left[\frac{(s+2)^2}{((s^2+4s+4)+4)^2}\right]$$

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$$= L^{-1}\left[\frac{(s+2)^2}{((s+2)^2+4)^2}\right]$$

$$= e^{-2t}\, L^{-1}\left[\frac{s^2}{(s^2+4)^2}\right]$$

$$= e^{-2t}\, L^{-1}\left[\frac{(s^2+4)-4}{(s^2+4)^2}\right]$$

$$= e^{-2t} L^{-1}\left[\frac{s^2+4}{(s^2+4)^2} - \frac{4}{(s^2+4)^2}\right]$$

$$= e^{-2t} L^{-1}\left[\frac{1}{s^2+4} - \frac{4}{(s^2+4)^2}\right]$$

$$= e^{-2t} \frac{\sin 2t}{2} - 4e^{-2t} L^{-1}\left[\frac{1}{(s^2+4)^2}\right]$$

$$= e^{-2t} \frac{\sin 2t}{2} - 4e^{-2t}\left[\frac{1}{4}\left(\frac{\sin 2t}{4} - \frac{t\cos 2t}{2}\right)\right]$$

$$= e^{-2t}\left[\frac{\sin 2t}{4} + \frac{t\cos 2t}{4}\right]$$

CONVOLUTION THEOREM

Statement

If $L^{-1}[F(s)] = f(t)$ and $L^{-1}[G(s)] = g(t)$,

then $L^{-1}[F(s)\,G(s)] = f(t) * g(t)$

$$= \int_0^t f(u)\, g(t-u)\, du$$

$f*g$ is called the convolution or falting of f & g

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1.

$$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$$

$$= L^{-1}\left[\frac{1}{(s+a)}\cdot\frac{1}{(s+b)}\right]$$

$$L^{-1}\left[\frac{1}{(s+a)}\right] = e^{-at}$$

$$L^{-1}\left[\frac{1}{(s+b)}\right] = e^{-bt}$$

$$f * g = \int_0^t f(u)\, g(t-u)\, du$$

$$f(t) = e^{-at} \Rightarrow f(u) = e^{-au}$$

$$g(t) = e^{-bt} \Rightarrow g(t-u) = e^{-b(t-u)}$$

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$$\Rightarrow \quad f*g = \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_0^t e^{-au}\, e^{-b(t-u)}\, du$$

$$= e^{-bt} \int_0^t e^{-au} \cdot e^{bu}\, du$$

$$= e^{-bt} \int_0^t e^{-(a-b)u}\, du$$

$$= e^{-bt}\left[\frac{e^{-(a-b)u}}{-(a-b)}\right]_0^t$$

$$= e^{-bt}\left[\frac{e^{-(a-b)t} - 1}{-(a-b)}\right]$$

$$= e^{-bt}\left[\frac{1 - e^{-(a-b)t}}{a-b}\right]$$

$$= \frac{e^{-bt} - e^{-at}}{a-b}.$$