02 March 2022 15:04

→ reducing.

2/3/2022.

Into PDE

If $z = f(x,y)$

$z \rightarrow$ dependent variable.

$x, y \rightarrow$ Independent variable.

partial derivatives of z w.r.to x & y

$$p = \frac{\partial z}{\partial x}. \qquad q = \frac{\partial z}{\partial y}$$

$$r = \frac{\partial^2 z}{\partial x^2} \qquad s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}. \qquad t = \frac{\partial^2 z}{\partial y^2}.$$

eg $p + qx = x + y \rightarrow$ PDE of order 1

$r + t = x^2 + y \rightarrow$ PDE of order 2.

Formation of PDE.

↙ eliminate arbitrary constants

↘ eliminate arbitrary functions.

FORMATION OF PDE BY ELIMINATING ARBITRARY CONSTANTS

$f(x, y, z, a, b) = 0$ —— (1)

$a, b \to$ const.

$z \to$ dependent var.

$x, y \to$ independent v.

diff (1) p. w. r. to x —— (2)

diff (1) " " " y —— (3)

fr (1), (2), (3)

Problems :-

arbitrary constants a.c

Indep. var. I v

1) form the PDE by eliminating the arbitrary constants from.

$$z = ax + by + a^2 + b^2 \quad \text{—— (1)}$$

a.c $\to a, b,$ -

I.v $\to x, y$ -

Hint a.c $\leq$ I v

p, q.

diff (1) p. w. r. to x

$$\frac{\partial z}{\partial x} = a$$

$$\Rightarrow \boxed{p = a} \quad \text{—— (2)}$$

diff (1) p. w. r. to y

$$\frac{\partial z}{\partial y} = b$$

$$\Rightarrow \boxed{q = b} \quad \text{—— (3)}$$

sub (2) & (3) in (1)

$$z = px + qy + p^2 + q^2$$ $\leftarrow$ any a.c

NO

2) $z = (x^2 + a)\ (y^2 + b)$ —— ①

d. p. w. r. to x,

$p = \frac{\partial z}{\partial x} = 2x(y^2 + b)$

$\Rightarrow y^2 + b = \frac{p}{2x}$ —— ②

a.c $\rightarrow$ 2

I.V $\Rightarrow$ 2.

diff p. w. r. to y.

$q = \frac{\partial z}{\partial y} = 2y(x^2 + a)$

$\Rightarrow x^2 + a = \frac{q}{2y}$ —— ③

sub ② & ③ in ①

$z = \left(\frac{q}{2y}\right)\left(\frac{p}{2x}\right) \Rightarrow$ $4xyz = pq$

3) find the PDE of all planes thro' the origin.

general eqn of plane.

$ax + by + cz = d$.

thro' origin $\Rightarrow (0,0,0) \Rightarrow d = 0$

$ax + by = -cz$

$ax + by + cz = 0$ —— ①

a.c $\rightarrow$

I.V $\Rightarrow$

p. d. w. r. to x.

$a + cp = 0$ —— ②

$\Rightarrow a = -cp$

p. d. w. r. to y

$$b + cq = 0 \quad \text{—} \ (3)$$

$$\Rightarrow b = -cq.$$

Sub (a) & (b) in eqn (1)

$$-cpx - cqy + cz = 0$$

$$\underline{px + qy = z}$$

4) find PDE of all spheres of radius having centers in XOY plane.

Center. $\rightarrow (a, b, 0)$

$c \rightarrow$ rad.

eqn of spher.

$$(x-a)^2 + (y-b)^2 + (z-0)^2 = c^2 \quad \text{—} \ (1)$$

a.c $\rightarrow$ 2

{ I V $\rightarrow x, y$

{ $z \Rightarrow$ dep

dif. p. w. r. to x

$$2(x-a) + 2zp = 0$$

$$\Rightarrow x - a = -zp \quad \text{—} \ (2)$$

diff p. w. r. to y.

$$2(y-b) + 2zq = 0$$

$$\Rightarrow y - b = -zq \quad \text{—} \ (3)$$

Sub (2) + (3) in (1) $\Rightarrow (-zp)^2 + (-zq)^2 + z^2 = c^2$

$$\underline{z^2(p^2 + q^2 + 1) = c^2}$$

5) $x^2 \quad y^2 + z^2 \quad 1$ ... a.c $\rightarrow$ 3

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{—— (1)} \qquad \text{I v} \rightarrow 2\text{-}$$

d.p. w.r. to x.

$$\frac{2x}{a^2} + \frac{2z}{c^2}\, p = 0.$$

$$\frac{x}{a^2} + \frac{z}{c^2}\, p = 0 \quad \text{—— (2)}$$

d.p.w.r. to y $\Rightarrow$ $\frac{2y}{b^2} + \frac{2z}{c^2} q = 0$

$$\frac{y}{b^2} + \frac{zq}{c^2} = 0 \quad \text{—— (}$$

diff eqn (2) w.r. to x. $\Rightarrow$ $\frac{1}{a^2} + \frac{1}{c^2}\left[z\frac{\partial^2 z}{\partial x^2} + \right.$

$$\Rightarrow \frac{1}{a^2} + \frac{1}{c^2}\left[zr + p \right.$$

d.p. eqn (3) w.r. to y.

$$\Rightarrow \frac{1}{b^2} + \frac{1}{c^2}\left[z\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} \right.$$

$$\frac{1}{b^2} + \frac{1}{c^2}\left[zt + q \right.$$

p.d eqn (3) w.r. to x.

$$\frac{y}{b^2} + \frac{z}{c^2}\, q = 0 \quad \text{—— (3)}$$

$$0 + \frac{1}{c^2}\left[z\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} \cdot \frac{\partial z}{\partial x} \right] = 0$$

$$\frac{1}{c^2}\left[zs + qp \right] = 0.$$

$$\boxed{zs + pq = 0}$$

4/3/22

Formation of PDE

- Eliminating arbitrary fn.

$\phi(u, v) = 0$

$u = u(x, y, z)$

$v = v(x, y, z)$

$\frac{\partial \phi}{\partial u}$ $\frac{\partial \phi}{\partial v}$ elimi.

1) eliminate f from

$$z = f(x^2 - y^2) \quad \text{—} (1)$$

arf → 1

1st Order..

p.d.w.r.to x

$$p = \frac{\partial z}{\partial x} = f'(x^2 - y^2) \times (2x) \quad \text{—} (2)$$

p.d.w.r.y

$$q = \frac{\partial z}{\partial y} = f'(x^2 - y^2) \times (-2y) \quad \text{—} (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{\cancel{f'(x^2 - y^2)} \, (2x)}{\cancel{f'(x^2 - y^2)} \, (-2y)}$$

$$\frac{p}{q} = \frac{\cancel{2}x}{-\cancel{2}y}$$

$$\boxed{py + qx = 0}$$

any fn? No

(2) eliminate f from,

$$z = f(x^2+y^2+z^2) \quad \text{— (1)}$$

dif p.w.r.to x.

$$p = \frac{\partial z}{\partial x} = f'(x^2+y^2+z^2) \times \left(2x + 2z\cdot\frac{\partial z}{\partial x}\right)$$

$$= f'(x^2+y^2+z^2) \times 2(x+zp) \quad \text{— (2)}$$

$$q = \frac{\partial z}{\partial y} = f'(x^2+y^2+z^2) \times 2(y+zq) \quad \text{— (3)}$$

$$\frac{(2)}{(3)} \quad \frac{p}{q} = \frac{x+zp}{y+zq}$$

$$\boxed{py - qx = 0}$$

3) $$xyz = \phi(x+y+z)$$

dif. p.w.r. to x.

$$y[x(p) + z(1)] = \phi'(x+y+z) \times \left(1+0+\frac{\partial z}{\partial x}\right)$$

$$y(xp+z) = \phi'(x+y+z)(1+p) \quad \text{— (2)}$$

dif. p.w.r. to y

$$x[yq+z] = \phi'(x+y+z)(1+q) \quad \text{— (3)}$$

$$\frac{(2)}{(3)} \quad \frac{y(xp+z)}{x(yq+z)} = \frac{1+p}{1+q}$$

$$p(xy - xz) + q(yz - xy) = xz - zy$$

4) $$z = f(\sin x + \cos y)$$

p. d. w.r. to x.

$$p = \frac{\partial z}{\partial x} = f'(\sin x + \cos y) \cdot (\cos x) \quad (2)$$

$$q = f'(\sin x + \cos y)\,(-\sin y) \quad (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{\cos x}{-\sin y}$$

$$p \sin y + q \cos x = 0$$

(5) $z = f(x + ct) + \phi(x - ct)$

2 fns $\to$ 2.
IV $\to x, t$

$$p = \frac{\partial z}{\partial x} = f'(x+ct) + \phi'(x-ct)$$

$$q = \frac{\partial z}{\partial t} = c f'(x+ct) + \phi'(x-ct)(-c)$$

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct)$$

$$t = \frac{\partial^2 z}{\partial t^2} = c^2 f''(x+ct) + c^2 \phi''(x-ct)$$

$$= c^2 [r]$$

$$\boxed{t = c^2 r}$$

Solutions of PDE

A soln (or) an integral of a PDE is a relation bet' dep. & Indep. variable, that satisfies the DE.

Complete Integral
Singular Integral.
General Integral.

Type I Eqns of the form $f(p, q) = 0$

ie., PDE containing p & q

Soln. of the form.

$$z = ax + by + c.$$

a.c → 3

I.V → 2

$$p = \frac{\partial z}{\partial x} = a \qquad q = \frac{\partial z}{\partial y} = b.$$

$$f(a, b) = 0$$

Let $b = \phi(a)$ (say)

$$z = ax + by + c$$

$$= ax + \phi(a)y + c. \quad \longrightarrow \text{Complete Integral.}$$

I.V → 2

a.c → 2

Singular Intg..

elim. 'a' & 'c'

p.d. w.r. to a & 'c'.

$$0 = x + \phi'(a)y$$

p.d. w.r. to c.

$$0 = 1 \text{ (absurd).}$$

⇒ No singular Integral.

General Solⁿ

put $c = f(a)$

$$z = ax + \phi(a)y + f(a) \quad \text{--- (2)}$$

diff. p. w.r.to a

$$0 = x + \phi'(a)y + f'(a) \quad \text{--- (3)}$$

eliminating 'a' between eqn (2) & (3) give General Integral.

7/3/22

1) Solve $pq + p + q = 0$. --- (1)

This contains p, q

This is of the form $f(p,q) = 0$ ---

Let $z = ax + by + c$ --- (2) be a solⁿ. of the eqn.

$p = \frac{\partial z}{\partial x} = a$, $q = \frac{\partial z}{\partial y} = b$. --- (3)

ac → 3
IV → 2

sub $p = a$, $q = b$ in (1)

$$ab + a + b = 0$$

$$ab + b = -a$$

$$b(a+1) = -a \Rightarrow b = \frac{-a}{a+1} \quad \text{--- (4)}$$

sub (4) in (1)

$$z = ax - \frac{a}{a+1} y + c \quad \text{---------} (5)$$

complete Integral. C.I $\left\| \begin{array}{l} a.c \to 2 \\ I\,v \to 2 \end{array} \right\|$

To find singular Integral :-

diff (5) p. w. r to 'a' & 'c'

p.d.w.r to a $\Rightarrow 0 = x - \left(\frac{a+1-a}{(a+1)^2} \right) y$

p.d.w.r. to c $0 = 1$ not possible. (absurd)

$\therefore$ No singular solution.

To find General soln. :-

Let $c = f(a)$

$$\Rightarrow z = ax - \frac{a}{a+1} y + f(a) \quad \text{---------} (6)$$

p d. w. r. to 'a'.

$$0 = x - \frac{1}{(a+1)^2} y + f'(a) \quad \text{---------} (7)$$

eliminating the constant 'a' from eqns (6) & (7) gives the general soln.

---

2) Solve $p^2 + q^2 = 4pq \quad \text{---------} (1)$

$p^2 + q^2 - 4pq = 0$

This is of the form $f(p,q)=0$

Let $z = ax+by+c$ be a soln of eqn (1) ——— (2)

$\Rightarrow p=a, \quad q=b.$

sub in (1)

$$a^2+b^2-4ab=0. \qquad b=?$$

$$b=(2\pm\sqrt{3})a$$

$$b^2-4ab+a^2=0$$

$$\Rightarrow b = \frac{4a\pm\sqrt{16a^2-4a^2}}{2}$$

sub (b) in (2)

$$z = ax+(2\pm\sqrt{3})a\,y+c \quad \text{——— (3)}$$

$\Rightarrow$ C I

a.c $\Rightarrow$

$2v \rightarrow 2$

To find Singular Integral.

p. d. w.r.to 'a' & 'c'

w.r.to c $\quad 0 = 1$ (absurd)

Not possible.

$\Rightarrow$ No singular integral -

To find general soln.

Let $c = f(a)$

$$z = ax+(2\pm\sqrt{3})\,ay+f(a) \quad \text{——— (4.)}$$

p.d. w.r. to a.

$$0 = x + (2+\sqrt{3})y + f'(a) \quad \text{—— (5)}$$

Eliminating 'a' between eqn (4) + (5) gives the general soln.

3) $\sqrt{p} + \sqrt{q} = 1$. —— (1)

This is of the form $f(p,q) = 0$

Let $z = ax + by + c$ is a soln. —— (2)

$\Rightarrow p = a, \quad q = b$

sub p, q in eqn (1)

$$\sqrt{a} + \sqrt{b} = 1$$

$$\sqrt{b} = 1 - \sqrt{a}$$

Squaring

$$b = (1-\sqrt{a})^2$$

Sub b in eqn (2)

$$z = ax + (1-\sqrt{a})^2 y + c \quad \text{—— (3)}$$

This is the C.I

To find Singular Integral.

p.d. eqn (3) w.r. to 'a' & 'c'

w.r. to a

$$0 = x + 2(1-\sqrt{a})\left(\frac{-1}{2\sqrt{a}}\right)y +$$

w.r to c

$$0 = 1 \quad \text{(absurd)}$$

$\therefore$ No singular Integral.

To find general soln.

Let $c = f(a)$

$$z = ax + (1-\sqrt{a})^2 y + f(a) \quad \text{——— (5)}$$

diff. (5) p. w. r to a

$$0 = x + 2(1-\sqrt{a})\left(\frac{-1}{2\sqrt{a}}\right) y + f'(a) \quad \text{——— (6)}$$

eliminating 'a' from eqns (5) & (6)

give the general soln.

Type II : Clairaut's form.

$$z = px + qy + f(p,q) \quad \text{——— (1)}$$

Let $z = ax + by + c$ ——— (2)

$$p = a, \quad q = b$$

$$z = ax + by + f(a,b) \quad \text{——— (3)}$$

from (2) & (3) $c = f(a,b)$

$$z = ax + by + f(a,b) \longrightarrow C.I.$$

To find S.I

p. d. w. r. to 'a' & 'b'.

Singular integral exist

G. I

7/3/22

1) solve $z = px + qy + pq$

This is of the form. $z = px + qy + f(p,q)$

$\Rightarrow$ clairaut's form.

Let $z = ax + by + c$ — 1

$p = a$; $q = b$

sub. in gn eqn. $\Rightarrow z = px + qy + pq$.

$z = ax + by + ab$ — 3

This is C.I

a, c → 2

I.v → 2

To find S.I

diff. p. w.r. to 'a' & 'b' eqn 3.

w.r. to a $\Rightarrow 0 = x + b \Rightarrow b = -x$.

w.r. to b $\Rightarrow 0 = y + a \Rightarrow -y = +a$

Sub. 'a' & 'b' in 3

$z = (-y)x + (-x)y + (-y)(-x)$

$= -yx - \cancel{xy} + \cancel{xy} = -yx$

$\Rightarrow \boxed{z + xy = 0}$ is the S.I

To find general soln.

Let $b = \phi(a)$.

$z = ax + \phi(a)y + a\,\phi(a)$

d. p. w.r. to a. ——— (4)

$0 = x + \phi'(a)y + a\phi'(a) + \phi(a)$ (1)

——— (5)

eliminating 'a' between eqns (4) & (5) gives the general eqn.

2) find S.S of.

$z = px + qy + \underline{p^2 + pq + q^2}$

This is of the form $z = px + qy + f(p,q)$

$\Rightarrow$ Clairaut's form.

$z = ax + by + c$.

$p = a$, $q = b$.

$z = ax + by + a^2 + ab + b^2$ is the C.S

To find S.S :-

diff. p w.r. to 'a' & 'b'

w.r. to a $\Rightarrow$ $0 = x + 2a + b$

w.r.to b $\Rightarrow$ $0 = y + a + 2b$

$a = \frac{1}{3}(y - 2x)$

$b = \frac{1}{3}(x - 2y)$

sub (a, b) in ...

$$z = \frac{x}{3}(y-2x) + \frac{y}{3}(x-2y) + \frac{1}{9}(y-2x)^2 + \ldots$$

$$z = \frac{1}{3}\left[xy - 2x^2 + xy - 2y^2\right] + \frac{1}{9}\left[y^2 - 4xy + 4x^2 + \ldots\right.$$

$$9z = 2xy - 3x^2 - 3y^2$$

$3z + x^2 + y^2 - 2y = 0$ is the S.I

Type III :- eqns are of the form.

$f(z, p, q)$ | $f(x, p, q)$, $f(y, p, q)$

Soln

$$z = \phi(x + ay)$$

$\phi \to$ arbi. fn.

$a \to$ const.

$u = x + ay.$

$z = \phi(u).$

$$\frac{\partial u}{\partial x} = 1 \quad ; \frac{\partial u}{\partial y} = a$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du}(1)$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du}(a)$$

— (3).

Sub p & q

$$f\left(z, \frac{\partial z}{\partial u}, a\frac{\partial z}{\partial u}\right) = 0 \longrightarrow ai$$

Thy S.I, G.S.

find the C.I of

3) $p(1+q) = qz$ —— (1)

$f(z, p, q) = 0$

Let $u = x + ay$

$$p = \frac{dz}{du}, \quad q = a.\frac{dz}{du}.$$

sub p & q in (1) $p(1+q) = qz$

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = \left(a\frac{dz}{du}\right)z$$

$$1 + a\frac{dz}{du} = az$$

$$a\frac{dz}{du} = az - 1$$

$$\frac{dz}{du} = \frac{az-1}{a}$$

Integrate both sides, $\int \frac{a}{az-1}\,dz = \int du.$ $a.\frac{1}{a}\log(az-1)$

$= u + C$

$\frac{1}{a}\log(az-1)$

$\log(az-1) = x + ay + c$

C.I

No S.I

4) find C.I of

$$9(p^2 z + q^2) = 4.$$

This is of the form $f(z, p, q) = 0$.

$$u = x + ay$$

$$9\left(\left(\frac{dz}{du}\right)^2 z + \left(a\frac{dz}{du}\right)^2\right) = 4.$$

$$\left(\frac{dz}{du}\right)^2 \left[9(z + a^2)\right] = 4$$

$$\left(\frac{dz}{du}\right)^2 = \frac{4}{9(z+a^2)}$$

$$\frac{dz}{du} = \sqrt{} = \frac{2}{3\sqrt{z+a^2}}$$

$$\int 3\sqrt{z+a^2}\,dz = \int 2\,du.$$

$$3 \cdot \frac{(z+a^2)^{3/2}}{3/2} = 2u + c$$

$$2(z+a^2)^{3/2} = 2u + c$$

$$4(z+a^2)^3 = (2u+c)^2$$

$$4(z+a^2)^3 = (2(x+ay) + c)^2$$

is the C.I

11/3/2022

Type IV

Seperable eqn.

$$f(x, p) = \phi(y, q)$$

put $f(x, p) = \phi(y, q) = a$ (say.

WKT $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$

$$dz = p\,dx + q\,dy.$$

Int. . . .

C.I

1) $p^2 y (1+x^2) = qx^2$ find C.I

$$\frac{p^2(1+x^2)}{x^2} = \frac{q}{y}$$

~~(1) $f(p,q) = 0$~~

~~(2) clairaut~~

$z = px + qy + f($

✓(3) $f(z, p, q)$

p, x . q, c

this eqn is seperable

$$f(p, x) = \phi(q, y).$$

$$\frac{p^2(1+x^2)}{x^2} = a \quad ; \quad \frac{q}{y} = a.$$

$$p^2(1+x^2) = ax^2$$

$$p^2 = \frac{ax^2}{1+x^2}$$

$$\Rightarrow p = \frac{x\sqrt{a}}{\sqrt{1+x^2}} \quad ; \quad q = ay.$$

WKT

$$dz = p\,dx + q\,dy$$

$$dz = \frac{x\sqrt{a}}{\sqrt{1+x^2}}dx + ay\,dy$$

Integrating both side.

$$\int dz$$

$$z = \sqrt{a}\,t + \frac{ay^2}{2} + b$$

$$z = \sqrt{a}\sqrt{1+x^2} + \frac{ay^2}{2} + b$$

$t^2 = 1+x^2$

$2t\,dt = 2x\,dx$

$x\,dx = t\,dt$

$$\sqrt{a}\int \frac{t\,dt}{\sqrt{t^2}} = \sqrt{a}\,t$$

is C.I

S.I d.f.s w.r.t. a, b

$0 = 1$ absurd. No singular sol.

2) find the C.I of

$$p - x^2 = q + y^2.$$

seperable. $f(p, x) = \phi(q, y)$

$p - x^2 = a \qquad q + y^2 = b.$

$\Rightarrow p = a + x^2 \qquad q = b - y^2$

$$dz = p\,dx + q\,dy$$

$$dz = (a+x^2)dx + (b-y^2)dy.$$

$$z = ax + \frac{x^3}{3} + by - \frac{y^3}{3} + c.$$

Soln $\to$ C.I.

NO. S.I.

G.S.

Method of Seperating of Variable:

$z \to$ dependent.

x, y - indep.

Soln. $z = f(x)\, g(y)$

$f \to$ fn. of x alone.

$g \to$ " " y alone.

1) Solve the eqn.

$$2x \frac{\partial z}{\partial x} - 3y \frac{\partial z}{\partial y} = 0$$ by the method of seperation of variables.

Let $Z = X(x)\, Y(y)$ be a soln. of the gn. eqn.

$X\ Y$

$$2x\, \partial z - 3y\, \partial z = 0$$

$$\frac{\partial x}{} \qquad \frac{}{\partial y}$$

$$2x \; z_x - 3y \; z_y = 0$$

$$\left\{ \begin{aligned} z_x &= X' Y \\ z_y &= X Y' \end{aligned} \right.$$

where,

$$X' = \frac{dX}{dx}$$

$$Y' = \frac{dY}{dy}.$$

$$2x \; z_x - 3y \; z_y = 0$$

$$2x \; X' Y - 3y \cdot X Y' = 0$$

$$\frac{2x \; X'}{X} = \frac{3y \; Y'}{Y}$$

LHS is a fn. of x alone.

RHS is a fn. of y alone.

$$\frac{2 \cancel{x} X'}{X} = \frac{3y \; Y'}{Y} = a \text{ (say)}$$

$$\frac{2X'}{X} = \frac{a}{x} \qquad \Big| \qquad \frac{3Y'}{Y} = \frac{a}{y}.$$

Integrating,

$$2\int \frac{X'}{X} dx = \int \frac{a}{x} dx \qquad \Big| \qquad 3\int \frac{Y'}{Y} dY = a\int \frac{dy}{y}$$

$2\log X = a\log x + \log A$ | $3\log Y = a\log y + \log B$

$\log X^2 = \log x^a + \log A$

$\Rightarrow X^2 = A x^a$ | $Y^3 = B y^a$

$\Rightarrow X = \sqrt{A x^a}$ | $Y = B_1 y^{a/3}$

$X = A_1 x^{a/2}$

$$Z = X Y$$

$$= A_1 x^{a/2} \cdot B_1 y^{a/3}$$

$$Z = C x^{a/2} y^{a/3}$$

Equations reducible to standard form.

Case (i) If a non-linear PDE is given by

$$f(x^m z^k p,\ y^m z^k q) = 0$$

It can be reduced to the standard type $F(P, Q) = 0$ using the substitution.

Hint.

$$X = \begin{cases} x^{1-m} & \text{if } m \neq 1 \\ \log x & \text{if } m = 1 \end{cases}$$

$$Y = \begin{cases} y^{1-n} & \text{if } n \neq 1 \\ \log y & \text{if } n = 1 \end{cases}$$

$$Z = \begin{cases} z^{k+1} & \text{if } k \neq -1 \\ \log z & \text{if } k = -1. \end{cases}$$

Hence $P = \dfrac{\partial Z}{\partial X} = \dfrac{\partial Z}{\partial z} \cdot \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial x}{\partial X}$

$Q = \dfrac{\partial Z}{\partial Y} = \dfrac{\partial Z}{\partial z} \cdot \dfrac{\partial z}{\partial y} \cdot \dfrac{\partial y}{\partial Y}$

Case(ii) $f(z, x^m, p, y^n, q) = 0$

reduces to $f(z, \underline{p, q})$

Case (iv) If $z^k p$, $z^k q$ occurs.

$F(z^k p, z^k q)$ (or)

$f_1(x, z^k, p) = f_2(y, z^k, q)$

$$Z = \begin{cases} z^{k+1}, & \text{if } k \neq -1 \\ \log z & \text{if } k = -1. \end{cases}$$

$P = \dfrac{\partial Z}{\partial z} \cdot \dfrac{\partial z}{\partial x}$; $Q = \dfrac{\partial Z}{\partial z} \cdot \dfrac{\partial z}{\partial y}$.

$\underline{4/3/22}$

C.I

1) Solve $x^2p^2 + y^2q^2 = z^2$ & find

$\dfrac{x^2p^2}{} + \dfrac{y^2q^2}{} = 1$

$$\left(\frac{xp}{z}\right)^2 + \left(\frac{yq}{z}\right)^2 = 1$$
$$(pxz^{-1})^2 + (qyz^{-1})^2 = 1 \quad \text{—— (1)}$$

eqns of the type $F(x^m z^k p, y^n z^k q) = 0$

$x^m z^k p = pxz^{-1} \Rightarrow m = 1 \quad k = -1$

$y^n z^k q = qyz^{-1} \Rightarrow n = 1 \quad k = -1$

Let $X = \log x \quad Y = \log y \quad Z = \log z$

$$\frac{\partial X}{\partial x} = \frac{1}{x} \qquad \frac{\partial Y}{\partial y} = \frac{1}{y} \qquad \frac{\partial Z}{\partial z} = \frac{1}{z}$$

$$\Rightarrow \frac{\partial x}{\partial X} = x \qquad \frac{\partial y}{\partial Y} = y.$$

$$P = \frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial X} = \frac{1}{z} p \cdot x = \frac{px}{z}.$$

$$Q = \frac{\partial Z}{\partial Y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial Y} = \frac{1}{z} q y = \frac{qy}{z}.$$

Sub in eqn (1)

$$P^2 + Q^2 = 1 \quad \text{—— (2)}$$

This is of the form $F(P, Q) = 0$.

Let $Z = aX + bY + c$

$P = a \quad ; \quad Q = b$

$\Rightarrow a^2 + b^2 = 1 \Rightarrow b^2 = 1 - a^2$

$b = \pm\sqrt{1-a^2}$

$\therefore Z = aX + \sqrt{1-a^2}\, Y + c.$

$\therefore \quad = a \log x + \sqrt{1-a^2} \log y + c$

$\log z = a \log x \ldots$
is the C.I

2) solve $p^2+q^2 = z^2(x+y)$

$$\frac{p^2}{z^2} + \frac{q^2}{z^2} = x+y$$

$$\left(\frac{p}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = x+y$$

$$(pz^{-1})^2 + (qz^{-1})^2 = x+y.$$ — (1)

This is of the form. $f_1(x, z^k, p) = f_2(y, z^k q)$

$$\frac{p}{z} = z^k p \Rightarrow k = -1.$$

$$qz^{-1} = z^k q \Rightarrow k = -1.$$

let $Z = \log z \Rightarrow \frac{\partial Z}{\partial z} = \frac{1}{z}$.

$$P = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} = \frac{1}{z} \cdot p = \frac{p}{z}.$$

$$Q = \frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} = \frac{1}{z} \cdot q = q/z.$$

$$P^2 + Q^2 = x+y.$$

This is of the form. ? IV

$$P^2 - x = y - Q^2 = a \text{ (say)}$$ — (2)

$$P^2 - x = a \Rightarrow P = \sqrt{a+x}.$$

$$y - Q^2 = a \Rightarrow Q = \sqrt{y-a}$$

C.I is $z = \int P\,dx + \int Q\,dy + b$

$= \int \sqrt{a+x}\,dx + \int \sqrt{y-a}\,dy + b$

$z = \dfrac{(x+a)^{3/2}}{3/2} + \dfrac{(y-a)^{3/2}}{3/2} + b$

$\log z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2}$. $\rightarrow$ C I

Lagrange's Linear eqn $Pp + Qq = R$.

$P, Q, R \rightarrow$ fn. of x, y, z

eliminates arbitrary fn. 'f'.

$f(u, v) = 0$.

---

form the PDE by eliminating 'f' from.

$f(x^2+y^2+z^2, x+y+z) = 0$.

rewrite the given eqn as

$x^2+y^2+z^2 = \phi(x+y+z)$ ——— (1)

d.p w.r.to x.

$2x + 2zp = \phi'(x+y+z)(1+p)$ ——— (2)

d.p w.r.to y

$2y + 2zq = \phi'(x+y+z)(1+q)$ ——— (3)

$\dfrac{(2)}{(3)}$ $\dfrac{\cancel{2}(x+zp)}{\cancel{2}(y+zq)} = \dfrac{1+p}{1+q}$

=

$x + q + zpq + xq = y + zq = yp + zpq$

$$Pp + Qq = R.$$

the general soln. $f(u, v) = 0$

$f \to$ function.

$u(x, y, z) = a$

$v(x, y, z) = b.$

A.E $\quad \dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}. \quad \Bigg/ \quad \dfrac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}.$

method of grouping

method of multiplier.

Method of grouping.

1) solve $px + qy = z$.

This is of the form.

$$P\underline{p} + Q\underline{q} = R.$$

$P = x, \quad Q = y \quad R = z$

Auxillar eqn or Subsidiary eqn.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

$$\frac{dx}{} = \frac{dy}{} = \frac{dz}{}$$

$x \quad y \quad z$

$\downarrow \quad \downarrow \quad \downarrow$

$\mathrm{I} \quad \mathrm{II} \quad \mathrm{III}$

consider.

I & II

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating both sides.

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\log x = \log y + \log c_1$$

$$x = y c_1$$

$$\Rightarrow c_1 = \frac{x}{y}$$

$$\boxed{u = \frac{x}{y}} \leftarrow u(x,y)$$

consider II & III

$$\frac{dy}{y} = \frac{dz}{z}$$

Integ.

$$\int \frac{dy}{y} = \int$$

$$\log y =$$

$$y =$$

c_2

$\Rightarrow v$

$\therefore$ soln is $\phi(u, v) = 0$

$$\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$$

2) solve $x^2 p + y^2 q + z^2 = 0$

form. $Pp + Qq = R$.

$P = x^2$; $Q = y^2$, $R = -z^2$

A.E $\quad dx = dy = \frac{dz}{}$

$$\overline{P} \qquad \overline{Q} \qquad R.$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2}$$

$$\text{I} \qquad \text{II} \qquad \underline{\overline{\text{III}}}$$

consider $\underline{\text{I} \ \& \ \text{II}}$

$$\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$

$$\frac{-1}{x} = -\frac{1}{y} + c_1$$

$$\frac{1}{x} - \frac{1}{y} = c_1$$

$$\therefore \quad u = \frac{1}{x} - \frac{1}{y^2}.$$

$$v = \frac{1}{y} + \frac{1}{z}$$

$$\phi(u, v) = 0$$

$$\phi\left(\frac{1}{x} - \frac{1}{y}, \quad \frac{1}{y} + \frac{1}{z}\right) = 0$$

3) find the general soln. of

$$p \tan x + q \tan y = \tan z.$$

$$P = \tan x$$

$$Q = \tan y \qquad R = \tan z.$$

15/3/22

Lagrange's → method of multipliers.

1) Solve $x(y-z)p + y(z-x)q = z(x-y)$

$$Pp + Qq = R.$$

$P = x(y-z) \quad Q = y(z-x) \quad R = z(x-y)$

Auxiliary eqn.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

lagrange's multipliers

$$\frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

find l, m, n ∋

$lP + mQ + nR = 0$ $\Rightarrow 1(xy - xz) + 1(yz - yx) + \ldots$

$\Rightarrow 1\,\cancel{xy} - \cancel{xz} + \cancel{yz} - \cancel{yx} + \ldots = 0$

Choose multipliers $1, 1, 1$

Nr $\Rightarrow$ $dx + dy + dz = 0$

Integ. $x + y + z = a$.

Consider $u = x + y + z$.

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

multiplier ∋ ~~$l\,dx + m\,dy$~~

$\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ $\quad lP + mQ + nR = 0$

$\frac{1}{\cancel{x}} \cdot \cancel{x}(y-z) + \frac{1}{\cancel{y}} \cdot \cancel{y}(z-x) + \frac{1}{\cancel{z}} \cdot \cancel{z}(x \ldots$

$\Rightarrow \quad y-z+z-x+x-y=0.$

$$l\,dx+m\,dy+n\,dz \Rightarrow \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=$$

integ.

$$\log x+\log y+\log z=\log b$$

$$\Rightarrow xyz=b$$

$$v=xyz$$

$\therefore$ soln. $\phi(u,v)=0$

$$\phi(x+y+z,\ xyz)=0.$$

2) $(x^2-yz)p+(y^2-zx)q=z^2-xy.$

$P=x^2-yz, \quad Q=y^2-zx, \quad R=z^2-$

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}$$

is grouping posible? $\times$

multiplier? $\exists\ lP+mQ+nR=0$

$1,-1,0$

$$\frac{dx-dy}{(x^2-yz)-(y^2-zx)}=\frac{dy-dz}{(y^2-zx)-(z^2-}$$

$$\frac{d(x-y)}{x^2 - y^2 + z(x-y)} = \frac{d(y-z)}{y^2 - z^2 + x(y-z)}$$

$(x+y)(x-y) + z(x-y)$ $\qquad$ $(y-z)[y+z+x]$

$(x-y)[x+y+z]$

$$\frac{d(x-y)}{(x-y)\cancel{(x+y+z)}} = \frac{d(y-z)}{(y-z)\cancel{(x+y+z)}}$$

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

Integ.

$$\int \frac{d(x-y)}{x-y} = \int \frac{d(y-z)}{y-z}$$

$$\log(x-y) = \log(y-z) + \log a.$$

$$a = \frac{x-y}{y-z}$$

$$\therefore \quad u = \frac{x-y}{y-z} \quad //$$