① Using Laplace transform, solve $y'' + 2y' + y = te^{-t}$ if $y(0) = 1$, $y'(0) = -2$.

Sol: $L[y''] + 2L[y'] + L[y] = L[te^{-t}]$

$$\left[s^2 Y(s) - s + 2\right] + 2\left[sY(s) - 1\right] + Y(s) = \frac{1}{(s+1)^2}$$

$$(s^2 + 2s + 1)\,Y(s) = \frac{1}{(s+1)^2} + s$$

$$\boxed{Y(s) = \frac{1}{(s+1)^4} + \frac{s}{(s+1)^2}}$$

$$\therefore\ y(t) = L^{-1}\left[\frac{1}{(s+1)^4}\right] + L^{-1}\left[\frac{s}{(s+1)^2}\right]$$

$$= e^{-t}\cdot\frac{t^3}{6} + L^{-1}\left[\frac{s+1-1}{(s+1)^2}\right]$$

$$= \frac{t^3}{6}e^{-t} + L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$= \frac{t^3}{6}e^{-t} + e^{-t}(1) - e^{-t}(t)$$

$$= e^{-t}\left[\frac{t^3}{6} + 1 - t\right] //$$

# Solution of Partial differential equation using Laplace transform method

$u(x,t)$    $t$ – Primary variable

$x$ – Secondary variable

$$U(x,s) = L[u(x,t);s]$$

$L\{f(t)\} = F(s)$

$L[y(t)]$

$t \to s$

$$L[y'(t)] = sY(s) - y(0)$$

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**Theorem:** Let $u(x,t)$ be a function of two variables $x$ and $t$, and let $U(x,s) = L[u(x,t)]$. Then we have

✓ (i) $L\left[\dfrac{\partial u}{\partial t}\right] = sU(x,s) - u(x,0)$

(ii) $L\left[\dfrac{\partial^2 u}{\partial t^2}\right] = s^2 U(x,s) - s\,u(x,0) - u_t(x,0)$

✓ (iii) $L\left[\dfrac{\partial u}{\partial x}\right] = \dfrac{d}{dx}\left[U(x,s)\right]$

(iv) $L\left[\dfrac{\partial^2 u}{\partial x^2}\right] = \dfrac{d^2}{dx^2}\left[U(x,s)\right]$

(v) $L\left[\dfrac{\partial^2 u}{\partial x \partial t}\right] = s\dfrac{d}{dx}\left[U(x,s)\right] - \dfrac{d}{dx}\left[u(x,0)\right]$

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(c) $L\left[\dfrac{\partial^2 u}{\partial t^2}\right] = L\left[\dfrac{\partial v}{\partial t}\right]$, when $v = \dfrac{\partial u}{\partial t}$

$= sV(x,s) - v(x,0)$, $V(x,s) = L[v(x,t)]$

$= s\,L\left[\dfrac{\partial u}{\partial t}\right] - u_t(x,0)$

$= s\left[sU(x,s) - u(x,0)\right] - u_t(x,0)$

$$\boxed{L\left[\dfrac{\partial^2 u}{\partial t^2}\right] = s^2 U(x,s) - s\,u(x,0) - u_t(x,0)}$$

(A) $$L\left[\frac{\partial u}{\partial t}\right] = \int_0^\infty e^{-st} \cdot \frac{\partial u}{\partial t}\, dt$$

$$= \lim_{p\to\infty}\left[\int_0^p e^{-st}\frac{\partial u}{\partial t}\, dt\right]$$

$$= \lim_{p\to\infty}\left[\left(u e^{-st}\right)_0^p + s\int_0^p u e^{-st}\, dt\right]$$

$$= \lim_{p\to\infty}\left[u(x,p)e^{-sp} - u(x,0) + s\int_0^p u e^{-st}\, dt\right]$$

$$= 0 - u(x,0) + s\int_0^\infty u e^{-st}\, dt$$

$$= sL[u(x,t)] - u(x,0)$$

$$\boxed{L\left[\frac{\partial u}{\partial t}\right] = sU(x,s) - u(x,0)}$$

③ $$L\left[\frac{\partial u}{\partial x}\right] = \int_0^\infty e^{-st}\,\frac{\partial u}{\partial x}\,dt$$

$$= \frac{\partial}{\partial x}\int_0^\infty e^{-st}\cdot u(x,t)\,dt$$

$$= \frac{d}{dx}\int_0^\infty e^{-st}\,u(x,t)\,dt$$

$$= \frac{d}{dx}\left[U(x,s)\right]$$

$$\boxed{L\left[\frac{\partial u}{\partial x}\right] = \frac{dU}{dx}}$$