METHOD OF VARIATION OF PARAMETERS

This method is applied to the equation of the form $y'' + py' + qy = X$ where p, q and X are functions of x.

$$C.F = c_1 y_1 + c_2 y_2$$

$$P.I = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

where y_1 and y_2 are the solutions of

$y'' + py' + qy = 0$ which we get it from C.F

and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is called the

wronskian of y_1 and y_2

If $W \neq 0$ then y_1, y_2 are linearly independent

If $W = 0$ then y_1, y_2 are linearly dependent.

METHOD OF UNDETERMINED COEFFICIENTS

Let $f(D)y = X$ — (1) be the given eqn.

Then to find P.I

we assume the trial solution of P.I which contains unknown constant which are determined by substituting in the given equation. This is called the method of undetermined coefficients.

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STEPS

1° Assume the trial solution for P.I based on the RHS ie X of the given equation.

If $X = Ae^{ax}$

then trial solution is $y = a_1 e^{ax}$

If $X = A \sin ax$ (or) $A \cos ax$

then trial solution is

$$y = a_1 \cos ax + a_2 \sin ax$$

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If $X = Ax^m$ then the trial solution is

$$y = a_0 + a_1 x + a_2 x^2 + \dots + a_m x^m$$

2° check whether the trial solution has the same terms as the C.F if that is the case then multiply trial solution by the lowest +ve integral power of x, which is large enough so that none of the terms which are present in the C.F gets repeated.

3° Find D^2y, Dy from the trial soln and

substitute in (1)

4° compare the coefficient on both sides of (1),

we can determine the constants and

we will obtain the solution for the

~~trial~~ particular Integral.

NOTE

This method fails for $\tan x$/$\sec x$ because

the no. of terms obtained by differentiating $x = \tan x$

$\cos \sec x$ is infinite.

1) solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} + a^2y = \sec ax.$$

C.F $\quad m^2+a^2=0$

$$m^2 = -a^2$$

$$m = \pm ai$$

$$C.F = e^{0x}\left[\underbrace{c_1 \cos ax}_{y_1} + \underbrace{c_2 \sin ax}_{y_2}\right]$$

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$y_1 = \cos ax$ $\qquad$ $y_2 = \sin ax$

$y_1' = -a \sin ax$ $\qquad$ $y_2' = a \cos ax$

$$W[y_1, y_2] = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$$

$$= a\left[\cos^2 ax + \sin^2 ax\right]$$

$$\Rightarrow W = a$$

$$P.I = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$= -\cos ax \int \frac{\sin ax \cdot \sec ax}{a} dx$$

$$+ \sin ax \int \frac{\cos ax \cdot \sec ax}{a} dx$$

$$= -\frac{\cos ax}{a} \int \sin ax \cdot \frac{1}{\cos ax} dx$$

$$+ \frac{\sin ax}{a} \int \cos ax \cdot \frac{1}{\cos ax} dx$$

$$= -\frac{\cos ax}{a}\int \tan ax \, dx + \frac{\sin ax}{a}\int dx$$

$$= -\frac{\cos ax}{a}\left[-\frac{\log(\cos ax)}{a}\right] + \frac{\sin ax}{a}\cdot x$$

$$P.I. = \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \log(\cos ax)$$

$$\therefore y = c_1 \cos ax + c_2 \sin ax + \frac{x}{a}\sin ax$$

$$+ \frac{1}{a^2}\cos ax \log(\cos ax)$$

2) $y'' - 2y' + 2y = e^{x} \tan x$

C.F $y'' - 2y' + 2y = 0$.

$m^2 - 2m + 2 = 0$

$m = 1 \pm i$

$C.F = e^{x}[c_1 \cos x + c_2 \sin x]$

$y_1 = e^{x} \cos x \quad y_2 = e^{x} \sin x$

$y_1' = e^{x}[-\sin x + \cos x] \quad y_2' = e^{x}[\cos x + \sin x]$

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$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{x}\cos x & e^{x}\sin x \\ -e^{x}\sin x + e^{x}\cos x & e^{x}\cos x + e^{x}\sin x \end{vmatrix}$$

$$= e^{2x}\left[\cos^2 x + \cancel{\cos x \sin x} + \sin^2 x - \cancel{\sin x \cos x}\right]$$

$$W = e^{2x}$$

$$P.I = -y_1 \int \frac{y_2 X}{W}\,dx + y_2 \int \frac{y_1 X}{W}\,dx$$

$$= -e^{x}\cos x \int \frac{e^{x}\sin x \cdot e^{x}\tan x}{e^{2x}}\,dx$$

$$+ e^{x}\sin x \int \frac{e^{x}\cos x \cdot e^{x}\tan x}{e^{2x}}\,dx$$

$$= -e^{x}\cos x \int \frac{\cancel{e^{2x}}\left(\sin x \cdot \frac{\sin x}{\cos x}\right)}{\cancel{e^{2x}}}\,dx$$

$$+ e^{x}\sin x \int \frac{\cancel{e^{2x}}\left(\cancel{\cos x} \cdot \frac{\sin x}{\cancel{\cos x}}\right)}{\cancel{e^{2x}}}\,dx$$

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$$= -e^x \cos x \int \frac{\sin^2 x}{\cos x}\,dx$$

$$+ e^x \sin x \int \sin x\, dx$$

$$= -e^x \cos x \int \left(\frac{1-\cos^2 x}{\cos x}\right) dx$$

$$- e^x \sin x \cos x\, dx$$

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$$= -e^{x}\cos x\int\left(\frac{1}{\cos x}-\cos x\right)dx$$

$$-e^{x}\sin x\cos x$$

$$= -e^{x}\cos x\int(\sec x-\cos x)\,dx$$

$$-e^{x}\sin x\cos x$$

$$= -e^{x}\cos x\,\log(\sec x+\tan x)$$

$$+e^{x}\cancel{\cos x}\sin x-e^{x}\sin x\cancel{\cos x}$$

$$P.I = -e^{x}\cos x \log(\sec x + \tan x)$$

The complete solution is

$$y = C.F + P.I$$

$$y = c_1 e^{x}\cos x + c_2 e^{x}\sin x$$

$$- e^{x}\cos x \log(\sec x + \tan x)$$

3) Solve by the method of undetermined coefficients

$$(D^2 - 3D + 2)y = x^2 + e^x \quad \text{---} (1)$$

<u>C.F</u>

A.E $m^2 - 3m + 2 = 0$

$m = 2, 1$

$$C.F = c_1 e^{2x} + c_2 e^x$$

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P.I

Assume the trial solution as

$$P.I = a_0 + a_1 x + a_2 x^2 + a_3 x e^x$$

$\because$ C.F has this term so multiplied by x.

Trial soln

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x e^x$$

$$Dy = a_1 + 2x a_2 + a_3 \left[e^x + x e^x \right]$$

Trial soln

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x e^x$$

$$Dy = a_1 + 2x a_2 + a_3 [e^x + x e^x]$$

$$D^2 y = 2a_2 + a_3 e^x + a_3 [e^x + x e^x]$$

$$= 2a_2 + 2a_3 e^x + a_3 x e^{2x}$$

substitute in the gives eqn.

$$(D^2 - 3D + 2) y = x^2 + e^x$$

$$2a_2 + 2a_3 e^x + a_3 x e^{2x}$$

$$-3\left[a_1 + 2x a_2 + a_3 (e^x + x e^x)\right]$$

$$+ 2(a_0 + a_1 x + a_2 x^2 + a_3 x e^x) = x^2 + e^x$$

compare coefficient of x^2 on both sides

$$2a_2 = 1$$

$$\boxed{a_2 = \frac{1}{2}}$$

compare coefficient of e^x

$$2a_3 - 3a_3 = 1$$

$$\boxed{a_3 = -1}$$

compare coefficient of x

$$-6a_2 + 2a_1 = 0$$

$$-6\left(\frac{1}{2}\right) + 2a_1 = 0$$

$$-3 + 2a_1 = 0$$

$$\boxed{a_1 = \frac{3}{2}}$$

compare constants .

$$2a_2 - 3a_1 + 2a_0 = 0$$

$$2\left(\frac{1}{2}\right) - 3\left(\frac{3}{2}\right) + 2a_0 = 0$$

$$1 - \frac{9}{2} + 2a_0 = 0$$

$$-\frac{7}{2} + 2a_0 = 0$$

$$\boxed{a_0 = \frac{7}{4}}$$

$$\therefore \; P.I = \frac{7}{4} + \frac{3}{2}x + \frac{1}{2}x^2 - xe^x$$

The complete soln.

$$y = C.F + P.I$$

$$y = c_1 e^{2x} + c_2 e^x + \frac{7}{4} + \frac{3}{2}x + \frac{x^2}{2} - xe^x .$$

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