DAY3

Solving pde using L.T

This arises from physical situations like heat equⁿ, wave equⁿ etc.

Here u is a function of 2 variables x and t

x - location t - time

Let $\mathcal{L}[u(x,t)] = U(x,s)$ or U

$$\mathcal{L}(u_t(x,t)) = \mathcal{L}\left(\frac{\partial u}{\partial t}\right) = sU(x,s) - u(x,0)$$

$$\mathcal{L}(u_{tt}(x,t)) = \mathcal{L}\left(\frac{\partial^2 u}{\partial t^2}\right) = s^2 U(x,s) - su(x,0) - u_t(x,0)$$

$$\mathcal{L}\left(\frac{\partial u}{\partial x}\right) = \frac{dU}{dx}$$

$$\left(\mathcal{L}(u_x(x,t)) = \int_0^\infty e^{-st}\frac{\partial u}{\partial x}\,dt = \frac{d}{dx}\int_0^\infty u(x,t)e^{-st}\,dt\right.$$

$$\left.\frac{d\,\mathcal{L}(u(x,t))}{dx} = \frac{dU(s,x)}{dx} = \frac{dU}{dx}\right)$$

$$\mathcal{L}\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{d^2 U}{dx^2}$$

Solve: $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = x$, $x>0$, $t>0$, $u(x=0,t)=0$, $u(x,t=0)=0$

Taking L.T

$$\frac{dU}{dx} + sU - \underbrace{u(x,0)}_{0} = \mathcal{L}(x)$$

$$= x\,\mathcal{L}(1)$$

$$= \frac{x}{s}$$

$$\frac{dU}{dx} + sU = \frac{x}{s}$$

This is a linear pde as it is of the form $\frac{dy}{dx} + Py = Q$

Solution is $y e^{\int p\,dx} = \int Q e^{\int p\,dx}\,dx + c$

Here $y \leftrightarrow U$

$P \leftrightarrow s$

$Q \leftrightarrow x/s$

$\therefore$ The solution is $U.e^{\int s\,dx} = \int \frac{x}{s} e^{\int s\,dx}\,dx + c$

$$\Rightarrow Ue^{sx} = \int \frac{x}{s} e^{sx}\,dx + c$$

$$\Rightarrow Ue^{8x} = \frac{1}{8}\left\{ \frac{xe^{8x}}{8} - 1\left(\frac{e^{8x}}{8^2}\right)\right\} + c$$

$$\Rightarrow Ue^{8x} = (x-1)e^{8x}\left(\frac{8-1}{8^3}\right) + c \qquad \div e^{8x}$$

$$\Rightarrow U = \frac{(x-1)(8-1)}{8^3} + ce^{-8x}$$

How to solve linear ~~ode~~ ODE?

A linear ~~ode~~ ODE will be of the form $\frac{dy}{dx} + Py = Q$ whose solution is given by $y.e^{\int p\,dx} = \int Q e^{\int p\,dx} dx + c$, where $e^{\int p\,dx}$ is the integrating factor

Example 1: $y' - y = xe^{x}$ $\quad (y' + Py = Q)$

$P = -1$

$Q = xe^{x}$.

Integrating factor $= e^{\int p\,dx}$

$= e^{\int -1\,dx} = e^{-x}$.

Solution is $y.e^{-x} = \int xe^{x}.e^{-x}dx + c$.

$\Rightarrow ye^{-x} = \int x\,dx + c$.

$\Rightarrow ye^{-x} = \frac{x^2}{2} + c$

$\Rightarrow y = e^{x}\left(\frac{x^2}{2} + 2\right)$

Example 2: $xy' = y + 2x^3 \Rightarrow xy' - y = 2x^3$ Here we need to divide the ODE by x to bring it to standard form

$y' - \frac{y}{x} = 2x^2$ $\qquad P = -1/x \quad Q = 2x^2$

$I.F = e^{\int p\,dx} = e^{-\int 1/x\,dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = 1/x$

The Solution is $y\, e^{\int P dx} = \int Q e^{\int P dx} dx + c$

$$y \frac{1}{x} = \int 2x^2 \, \frac{1}{x} \, dx + c$$

$$\frac{y}{x} = x^2 + c$$

$$\underline{y = x^3 + cx}$$

Solving pde using LT

2)

$$y_t = -\alpha y_x \quad , x>0 \, , t>0 \ \& \ y(0,t) = k \, , y(x,0) = 0$$

$$\cancel{\mathcal{L}(y'(t)) = -\alpha \mathcal{L}(y')}$$

Here $\mathcal{L}(y(x,t)) = Y(x,s) = Y$

$$L\left(\frac{\partial y}{\partial t}\right) = -\alpha \, L\left(\frac{\partial y}{\partial x}\right)$$

$$sY(x,s) - \underbrace{y(x,0)}_{0} = -\alpha \frac{dY}{dx}$$

$$\cancel{Y(x,s)(s+\alpha)} \Rightarrow \alpha \frac{dY}{dx} + sY = 0 \quad —(1)$$

The initial condition $y(0,t) = k$ is for the pde. Now we have converted it into ODE. So the initial condition sho also be converted

Given $y(0,t) = k \quad Y(0) = \mathcal{L}(y(0,t)) = \mathcal{L}(k) = \frac{k}{s}$

Now the pde has been converted into ODE as

$$\alpha \frac{dY}{dx} + sY = 0 \text{ with } Y(0) = \frac{k}{s} \Rightarrow \frac{dY}{dx} + \frac{s}{\alpha} Y = 0 \, , Y(0) = \frac{k}{s}$$

This is a linear ODE. $I.F = e^{\int \frac{s}{\alpha} dx} = e^{s/\alpha \, x}$

Solution is $Y e^{s/\alpha \, x} = \int \underbrace{0 \, e^{s/\alpha \, x} dx}_{0} + C \quad \therefore Y e^{s/\alpha \, x} = C.$

$Y(0) = \frac{k}{s} \Rightarrow \frac{k}{s} e^{s/\alpha(0)} = c \Rightarrow c = \frac{k}{s}$

$\therefore \; Y e^{s/\alpha \, x} = \frac{k}{s}$

(or) $Y = \frac{k}{s} e^{-s/\alpha \, x}$

But $Y = L(y(x,t))$

$\therefore \; L(y(x,t)) = \frac{k}{s} e^{-s/\alpha \, x}$

$y(x,t) = k \, L^{-1}\left(\frac{e^{-s/\alpha \, x}}{s}\right)$

$= k \, u\left(t - \frac{x}{\alpha}\right)(1)$ (II shifting)

$= k u\left(t - \frac{x}{\alpha}\right)$

(or) $y(x,t) = \begin{cases} 0, & t < x/\alpha \\ k, & \text{otherwise} \end{cases}$

3. $y_t + y_x + y = 0$, $x > 0, t > 0$, $y(0,t) = \sin t$, $y(x,0) = 0$

Let $L(y(x,t)) = Y(x,s)$

$sY(x,s) + Y'(x) + Y(x,s) = 0$, $Y(0) = L(y(0,t)) = L(\sin t) = \frac{1}{s^2+1}$

$(s+1)Y + Y'(x) = 0$

(or) $Y' + Y(s+1) = 0$ — linear ODE with $Y(0) = \frac{1}{s^2+1}$.

$IF = e^{\int (s+1)dx} = e^{(s+1)x}$.

Solution $Y e^{\int p dx} = \cancel{\int Q e^{\int p dx} dx} + c$ $\rightarrow$ 0 as $Q = 0$

$\therefore \; Y e^{(s+1)x} = C$.

Given $Y(0) = \frac{1}{s^2+1}$.

$\therefore \quad Y(0)e^{(s+1)(0)} = C$

$\Rightarrow \frac{1}{s^2+1} = C$.

$\therefore \quad Ye^{(s+1)x} = \frac{1}{s^2+1}$.

$$Y = \frac{e^{-(s+1)x}}{s^2+1}.$$

(i.e) $L(y(x,t)) = \frac{e^{-(s+1)x}}{s^2+1}$.

$$y(x,t) = L^{-1}\left\{\frac{e^{-sx} \cdot e^{-x}}{s^2+1}\right\}$$

$f(t-a)u(t-a) = L^{-1}\{f(t)e^{-as}\}$.

Here $a = x$

$$= e^{-x}\, u(t-x)\sin(t-x).$$