Properties:

Transform of derivative

If $L[f(t)] = F(s)$, then

(i) $L[f'(t)] = sF(s) - f(0)$

(ii) $L[f''(t)] = s^2F(s) - sf(0) - f'(0)$

(iii) $L[f^{(n)}(t)] = s^n F(s) - s^{n-1} f(0)$
$- s^{n-2} f'(0) - s^{n-3} f''(0)$
$- \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

1. Find $L[\sin(at)]$ using transform of derivative property.

Sol: Let $f'(t) = \sin(at)$.

Then $f(t) = \int f'(t)\,dt$

$= \dfrac{-\cos(at)}{a}$

$\therefore F(s) = L[f(t)] = \dfrac{-1}{a} L[\cos(at)]$

$= \dfrac{-1}{a}\left[\dfrac{s}{s^2+a^2}\right]$

Laplace Transform of Periodic function

Let $f(t)$ be a Periodic function with Period T. Then

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

<u>Proof</u>:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \left[\int_0^T + \int_T^{2T} + \int_{2T}^{3T} + \cdots + \int_{(n-1)T}^{nT} + \cdots\right] (e^{-st} f(t)\,dt)$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \cdots$$

$$\Big\downarrow\ u = t - T$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(u+T)} f(u+T)\,du$$

$$+ \cdots$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-su} f(u)\,du + e^{-2sT}\int_0^T e^{-su} f(u)\,du + \dots$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-st} f(t)\,dt + e^{-2sT}\int_0^T e^{-st} f(t)\,dt + \dots$$

$$= \left(1 + e^{-sT} + e^{-2sT} + \dots\right)\int_0^T e^{-st} f(t)\,dt$$

$$\boxed{L[f(t)] = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}}$$

$\left[1 + x + x^2 + \dots = \frac{1}{1-x} \text{ for } |x|<1\right]$

(1) Find $L[f(t)]$, where $f(t)=\begin{cases} t, & 0<t<a \\ 2a-t, & a<t<2a \end{cases}$ is the triangular wave of period $2a$.

Sol: Here, $T = 2a$.

Consider $\int_0^{2a} e^{-st} f(t)\,dt$

$$= \int_0^a t e^{-st}\,dt + \int_a^{2a} (2a-t)e^{-st}\,dt \quad \to (1)$$

$$\int_0^a t e^{-st}\,dt = \left[(t)\left\{\frac{e^{-st}}{-s}\right\} - (1)\left\{\frac{e^{-st}}{s^2}\right\} + (0) \right]_{t=0}^{a}$$

$$= \left\{ -\frac{a}{s}e^{-as} - \frac{e^{-as}}{s^2} \right\} - \left\{ \frac{-1}{s^2} \right\}$$

$$\therefore \int_0^a t e^{-st}\,dt = \frac{1}{s^2} - \frac{1}{s^2}e^{-as} - \frac{a}{s}e^{-as}$$

$$\int_a^{2a} (2a-t)e^{-st}\,dt$$

$$= \left[(2a-t)\left\{\frac{e^{-st}}{-s}\right\} - (-1)\left\{\frac{e^{-st}}{s^2}\right\} + (0)\right]_{t=a}^{2a}$$

$$= \left\{\frac{e^{-2as}}{s^2}\right\} - \left\{\frac{-a}{s}e^{-as} + \frac{e^{-as}}{s^2}\right\}$$

$$= \frac{e^{-2as}}{s^2} + \frac{a}{s}e^{-as} - \frac{1}{s^2}e^{-as}$$

Hence ① becomes

$$= \frac{1}{s^2} - \frac{1}{s^2}e^{-as} - \cancel{\frac{a}{s}e^{-as}} + \frac{e^{-2as}}{s^2} + \cancel{\frac{a}{s}e^{-as}} - \frac{1}{s^2}e^{-as}$$

$$= \frac{1}{s^2}\left(e^{-2as} - 2e^{-as} + 1\right)$$

$$= \frac{(1-e^{-as})^2}{s^2}$$

$$\therefore \quad L[f(t)] = \frac{\int_0^{2a} e^{-st} f(t)\,dt}{1 - e^{-s(2a)}}$$

$$= \frac{(1 - e^{-as})^2}{s^2(1 - e^{-2as})}$$

$$= \frac{(1 - e^{-as})^2}{s^2(1 - e^{-as})(1 + e^{-as})}$$

$$= \frac{1 - e^{-as}}{s^2(1 + e^{-as})} \quad //$$

By transform of derivative property,

$$L[f'(t)] = sF(s) - f(0)$$

i.e.,

$$L[\sin(at)] = s\left(\frac{-s}{a(s^2+a^2)}\right) - \left(\frac{-1}{a}\right)$$

$$= \frac{1}{a} - \frac{s^2}{a(s^2+a^2)}$$

$$= \frac{1}{a}\left[\frac{\cancel{s^2}+a^2-\cancel{s^2}}{s^2+a^2}\right]$$

✓ $$\boxed{L[\sin(at)] = \frac{a}{s^2+a^2}}$$

If $L\left[\sin\sqrt{t}\right] = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$, prove that $L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}} e^{-\frac{1}{4s}}$ (Hint : Use the transform of the derivative property).

Sol: Let $f(t) = \sin(\sqrt{t})$

so that $F(s) = L[f(t)]$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$$

$$f'(t) = \left[\cos(\sqrt{t})\right] \cdot \left(\frac{1}{2} t^{-1/2}\right)$$

$$= \frac{1}{2}\left[\frac{\cos(\sqrt{t})}{\sqrt{t}}\right]$$

$$\therefore \frac{\cos(\sqrt{t})}{\sqrt{t}} = 2f'(t)$$

Applying Laplace transform on both sides of the equation, we get

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = 2L[f'(t)]$$

$$= 2\left[sF(s) - f(0)\right]$$

$$= 2\left[\frac{\sqrt{\pi}}{2\sqrt{s}} e^{-\frac{1}{4s}} - 0\right]$$

$$\therefore L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{\sqrt{s}} e^{-1/4s} \;//$$