INVERSE Z- TRANSFORMS BY CONVOLUTION THEOREM

Convolution Theorem (Statement)

If $z^{-1}[U(z)] = u_n$ and $z^{-1}[V(z)] = v_n$, then

$$z^{-1}[U(z) \cdot V(z)] = \sum_{m=0}^{n} u_m v_{n-m}$$

$$= u_n * v_n$$

where the symbol $*$ denotes the convolution.

Problems

1. Using convolution theorem evaluate

$$z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$$

Solution

$$z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = z^{-1}\left[\frac{z}{z-1}\cdot\frac{z}{z-3}\right]$$

$$= z^{-1}\left[\frac{z}{z-1}\right] * z^{-1}\left[\frac{z}{z-3}\right]$$

$$= 1^n * 3^n = 3^n * 1^n$$

$$= \sum_{m=0}^{n} 3^m \, 1^{n-m}$$

$$= \sum_{m=0}^{n} 3^m$$

$$= 1+3+3^2+\cdots+3^n$$

$$= \frac{3^{n+1}-1}{3-1} = \frac{1}{2}\left(3^{n+1}-1\right)$$

2) Find $z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$ using convolution theorem

<u>**Solution**</u> $z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-b}\right]$

$$= z^{-1}\left[\frac{z}{z-a}\right] * z^{-1}\left[\frac{z}{z-b}\right]$$

$$= a^n * b^n$$

Scanned with CamScanner

$$= a^n * b^n$$

$$= \sum_{m=0}^{n} a^m b^{n-m}$$

$$= \sum_{m=0}^{n} a^m b^n b^{-m}$$

$$= b^n \sum_{m=0}^{n} a^m b^{-m}$$

$$= b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m$$

$$= b^n \left[1 + \frac{a}{b} + \left(\frac{a}{b}\right)^2 + \cdots + \left(\frac{a}{b}\right)^n\right]$$

Scanned with CamScanner

$$= b^n \left[\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\left(\frac{a}{b}\right) - 1} \right]$$

$$= b^n \frac{a^{n+1} - b^{n+1}}{b^{n+1}} \times \frac{b}{a-b}$$

$$= \frac{\cancel{b^{n+1}}}{\cancel{b^{n+1}}} \frac{a^{n+1} - b^{n+1}}{a-b}$$

$$= \frac{a^{n+1} - b^{n+1}}{a-b}, \quad n = 0, 1, 2, \ldots$$