Method of variation of parameters

Let $y_c(x) = c_1\, y_1(x) + c_2\, y_2(x)$ be the complementary function of *the linear ODE*

$$y'' + p(x)y' + q(x)y = r(x) \qquad (1)$$

A particular solution y_p *to the linear ODE is given by*

$$y_p(x) = -y_1(x)\int \frac{y_2(x)r(x)}{W(y_1, y_2)}dx + y_2(x)\int \frac{y_1(x)r(x)}{W(y_1, y_2)}dx, \qquad (2)$$

where $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is the Wronskian of two functions y_1 *and* y_2.

The required general solution of (1) *is*

$$y(x) = y_c(x) + y_p(x) \qquad (3)$$

This method has much more applicability than the method of undetermined coefficients.

First, the ODE need not be with constant coefficients.

Second, the nonhomogeneos part $r(x)$ can be a much more general function.

Proof: *Consider* $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$

$$y_p'(x) = u'(x)y_1(x) + v'(x)y_2(x) + u(x)y_1'(x) + v(x)y_2'(x).$$

Now to make calculations easier , we take

$$u'(x)y_1(x) + v'(x)y_2(x) = 0. \tag{4}$$

$$y_p''(x) = u'(x)y_1'(x) + v'(x)y_2'(x) + u(x)y_1''(x) + v(x)y_2''(x).$$

Subtituting $y_p(x), y_p'(x)$ and $y_p''(x)$ into (1)

(and the fact that y_1 and y_2 are solutions of the homogeneos part),

we get $u'(x)y_1'(x) + v'(x)y_2'(x) = r(x).$ (5)

solve u', v' from (4) and (5) (Cramer's rule):

$$u' = -\frac{r(x)y_2(x)}{W(y_1, y_2)}, \quad v' = \frac{r(x)y_1(x)}{W(y_1, y_2)} \qquad W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Integrating $u(x) = -\int \frac{y_2(x)r(x)}{W(y_1, y_2)}dx, \quad v(x) = \int \frac{y_1(x)r(x)}{W(y_1, y_2)}dx.$

Substituting u and v in $y_p(x) = y_1(x)u(x) + y_2(x)v(x)$,

we find the required form of y_p given in (2).

Example 1. *Consider* $y'' - 2y' - 3y = xe^{-x}$.

The LI solutions of the homogenous part are $y_1(x) = e^{-x}$ and $y_2(x) = e^{3x}$.

$$y_c(x) = C_1 e^{-x} + C_2 e^{3x}$$

$y_p(x) = y_1(x)u(x) + y_2(x)v(x)$ where

$$u(x) = -\int \frac{y_2(x)r(x)}{W(y_1, y_2)}dx, \quad v(x) = \int \frac{y_1(x)r(x)}{W(y_1, y_2)}dx.$$

Now $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 4e^{2x}$.

$$u(x) = -\int \frac{x}{4}dx = -\frac{x^2}{8} \qquad v(x) = \int \frac{xe^{-4x}}{4}dx = -\frac{x}{16}e^{-4x} - \frac{1}{64}e^{-4x}$$

Thus, $y_p(x) = -\frac{x^2}{8}e^{-x} + e^{3x}\left(-\frac{x}{16}e^{-4x} - \frac{1}{64}e^{-4x}\right)$

Hence, the general solution is
$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= C_1 e^{-x} + C_2 e^{3x} - \frac{x^2}{8}e^{-x} + e^{3x}\left(-\frac{x}{16}e^{-4x} - \frac{1}{64}e^{-4x}\right) \end{aligned}$$

Example 2. *Consider* $y'' + y = \tan x$.

Solution: (This cannot be solved by the method of undetermined coefficients.)

LI solutions of the homogenous part are $y_1(x) = \cos x$ and $y_2(x) = \sin x$.

$y_C(x) = C_1 \cos x + C_2 \sin x$

$y_p(x) = y_1(x)u(x) + y_2(x)v(x)$ where $u(x) = -\int \frac{y_2(x)r(x)}{W(y_1, y_2)}dx, \quad v(x) = \int \frac{y_1(x)r(x)}{W(y_1, y_2)}dx.$

$W(y_1, y_2) = 1.$

$u(x) = -\int \sin x \tan x \, dx = -\ln|\sec x + \tan x| + \sin x$

$v(x) = \int \sin x \, dx = -\cos x$

$y_p(x) = -\cos x \ln|\sec x + \tan x|$

Hence, the general solution is $y(x) = y_C(x) + y_p(x)$

$$= C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x + \tan x|$$

Problems for practice (Try variation of parameters method)

Differential equation	Solution by UC method
(a) $y'' - 2y' - 3y = 2e^{x} - 10\sin x$	$y(x) = c_1 e^{3x} + c_2 e^{-x} - \frac{1}{2}e^{x} + 2\sin x - \cos x$
(b) $y'' - 3y' + 2y = 2x^2 + (1+2x)e^{x} + 4e^{3x}$	$y(x) = c_1 e^{x} + c_2 e^{2x} + x^2 + 3x + \frac{7}{2} + 2e^{3x} - x^2 e^{x} - 3xe^{x}$