Relation between Fourier and Laplace transforms

Let $f(t) = \begin{cases} e^{-xt} g(t), & t > 0 \\ 0, & t < 0 \end{cases}$

Then $F[f(t)] = L[g(t)]$.

Proof: $F[f(t)] = \int_{-\infty}^{\infty} f(t) e^{i\lambda t} dt$

$$= \int_{0}^{\infty} e^{-xt} g(t) e^{i\lambda t} dt$$

$$= \int_{0}^{\infty} e^{-t(x-i\lambda)} g(t)\, dt$$

$$= \int_{0}^{\infty} e^{-st} g(t) dt, \quad \text{where } s = x - i\lambda.$$

$\therefore F[f(t)] = L[g(t)]$

Z-Transforms

Discrete Sequence — $\{x_n\}_{n=0}^{\infty}$

$$\{x_0, x_1, x_2, \ldots x_n, \ldots\}$$

$$\{x_n\}_{n=0}^{\infty} = \{1, 3, 5, 7, 9, \ldots\}$$

$$x_n = 1 + 2n, \quad n = 0, 1, 2, \ldots$$

$$\{2, 2^2, 2^3, \ldots\} = \{2^n\}_{n=0}^{\infty}$$

✓ $\{x_n\}_{n=-\infty}^{\infty}$ two sided | ✓ $\{x_n\}_{n=0}^{\infty}$ ✓ one sided

$$\{1, 3, 5, 7, 9, \ldots\}$$

$$t_0 = 1, \quad t_1 = 3, \quad t_2 = 5, \ldots$$

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$f(n) \xrightarrow{Z} \sum_{n=0}^{\infty} f(n) z^{-n}$$

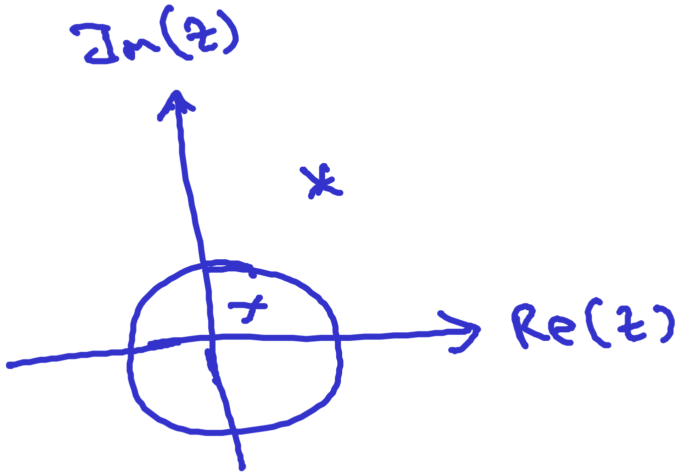
$$f(t) \longrightarrow \sum_{n=0}^{\infty} f(t_n) z^{-n}$$

$$t_n = t_0 + n\Delta t \qquad Z(f(t))$$

$$\{t_n\}_{n=0}^{\infty}$$

$$z = \alpha + i\beta$$

$$|z| = \sqrt{\alpha^2 + \beta^2}$$



$$\bar{f}(z) = Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$$

Problems

① Let $f(n) = k$

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} k z^{-n}$$

$$= k\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right]$$

$$= k\left[\frac{1}{1-\frac{1}{z}}\right] \text{ if } \left|\frac{1}{z}\right| < 1$$

$$\therefore \boxed{Z\{k\} = \frac{kz}{z-1} \text{ provided } |z| > 1}$$

Region of Convergence : $|z| > 1$

(2) $Z\{a^n\} = \sum_{n=0}^{\infty} a^n z^{-n}$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \cdots$$

$$= \frac{1}{1 - \frac{a}{z}} \quad \text{provided } \left|\frac{a}{z}\right| < 1$$

$$\therefore \boxed{Z\{a^n\} = \frac{z}{z-a}} \quad \text{provided } |z| > |a|$$

Note: $Z\{2^n\} = \frac{z}{z-2}$

$$Z\left\{\left(\frac{1}{2}\right)^n\right\} = \frac{z}{z - \frac{1}{2}} = \frac{2z}{2z-1}$$

$$\{ \ldots -2, -1, 0, 1, 2, \ldots \}$$

$$\{ \ldots x_{-2}\ x_{-1}\ x_0\ x_1\ x_2\ \ldots \}$$

$$\{x_n\}_{n=-\infty}^{\infty}$$

③ $Z\left\{\frac{1}{n}\right\} = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$

$= \frac{1}{z} + \frac{1}{2}\cdot\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots$

$= x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$, where $x = \frac{1}{z}$

$= -\log(1-x)$ Provided $|x|<1$

$= -\log\left(1-\frac{1}{z}\right)$ for $\left|\frac{1}{z}\right|<1$

$= -\log\left(\frac{z-1}{z}\right)$

$\therefore$ $\boxed{Z\left\{\frac{1}{n}\right\} = \log\left(\frac{z}{z-1}\right)}$ Provided $|z|>1$

4. $$Z\{\delta(n-k)\} = \sum_{n=0}^{\infty} \delta(n-k)\, z^{-n}$$

$$\therefore\ \boxed{Z\{\delta(n-k)\} = z^{-k}}$$

5. $$Z\{U(n-k)\} = \sum_{n=0}^{\infty} U(n-k)\, z^{-n}$$

$$= \sum_{n=k}^{\infty} z^{-n}$$

$$= z^{-k} + z^{-(k+1)} + z^{-(k+2)} + \cdots$$

$$= z^{-k}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right]$$

$$= z^{-k}\left[\frac{1}{1-\frac{1}{z}}\right]$$

$$= z^{-k}\left[\frac{z}{z-1}\right]$$

$$\therefore\ \boxed{Z\{U(n-k)\} = \frac{1}{z^{k-1}(z-1)}}$$

(6) $$Z\left\{\frac{a^n}{n!}\right\} = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{a}{z}\right)^n$$

$$= 1 + \frac{1}{1!}\left(\frac{a}{z}\right) + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \cdots$$

$$\therefore Z\left\{\frac{a^n}{n!}\right\} = e^{(a/z)}$$