DAY 2

1 Represent $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & 1 \le t < 3 \\ 2, & t \ge 3 \end{cases}$ using heaviside step function.

$$f(t) = 1.\left(h(t-1) - h(t-3)\right) + 2h(t-3) = h(t-1) + h(t-3)$$

2. Use L.T to solve the initial value problem $y' + 4y = g(t)$, $y(0) = 2$

where $g(t) = \begin{cases} 0, & 0 \le t < 1 \\ 12, & 1 \le t < 3 \\ 0, & t \ge 3 \end{cases} = 12\left(h(t-1) - h(t-3)\right)$

$$y' + 4y = g(t)$$

$$L(y'(t)) + 4L(y(t)) = L(g(t))$$

$$\left[sL(y(t)) - y(0)\right] + 4L(y(t)) = L\left(12\left(h(t-1) - h(t-3)\right)\right)$$

$$L(y(t))(s+4) - 2 = 12\left(\frac{e^{-s}}{s} - \frac{e^{-3s}}{s}\right)$$

$$L(y(t)) = \left[12\left(\frac{e^{-s} - e^{-3s}}{s}\right) + 2\right]\frac{1}{s+4}$$

$$y(t) = 12L^{-1}\left(\frac{e^{-s} - e^{-3s}}{s(s+4)}\right) + 2\underbrace{L^{-1}\left(\frac{1}{s+4}\right)}_{= 2e^{-4t}} \quad \text{---(1)}$$

$$\frac{1}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4}$$

$$1 = A(s+4) + Bs$$

$s = 0 \Rightarrow A = 1/4$ $\quad s = -4 \Rightarrow \Rightarrow B = -1/4$

Substituting in (1) $\quad y(t) = \frac{12}{4}L^{-1}\left(\frac{e^{-s} - e^{-3s}}{s}\right) - \frac{12}{4}L^{-1}\left(\frac{e^{-s} - e^{-3s}}{s+4}\right) + 2e^{-4t}$

(2)

$$= 3L^{-1}\left(\frac{e^{-s}}{s}\right) - 3L^{-1}\left(\frac{e^{-3s}}{s}\right) - 3L^{-1}\left(\frac{e^{-s}}{s+4}\right) + 3L^{-1}\left(\frac{e^{-3s}}{s+4}\right) + 2e^{-4t}$$

$$= 3u(t-1) - 3u(t-3) - 3u(t-1)e^{-4(t-1)} + 3e^{-4(t-3)}u(t-3) + 2e^{-4t}$$

$L(f(t-a)u(t-a))$
$= e^{-as}L(f(t))$

$L(f(t-a)u(t-a))$
$= e^{-as}L(f(t))$

$\Rightarrow f(t-a)u(t-a)$
$= L^{-1}\left(e^{-as}L(f(t))\right)$

2 Solve $y''-y'+5y = 4 + u_2(t)e^{4-2t}$, $y(0)=2$
$u_2(t) = u(t-2)$, $y'(0)=-1$

II shifting theory:

$$L(y''(t)) - L(y'(t)) + 5L(y(t)) = 4L(1) + L(u(t-2)e^{-2(t-2)})$$

$$\{s^2L(y(t)) - sy(0) - y'(0)\} - \{sL(y(t)) - y(0)\} + 5L(y(t)) = \frac{4}{s} + \frac{e^{-2s}}{s+2}$$

$y(0)=2$, $y'(0)=-1$

$$(s^2-s+5)L(y(t)) - 2s+1+2 = \frac{4}{s} + \frac{e^{-2s}}{s+2}$$

$$(s^2-s+5)L(y(t)) = \frac{4}{s} + \frac{e^{-2s}}{s+2} + 2s-3$$

$$L(y(t)) = \frac{2s^2-3s+4}{s(s^2-s+5)} + \frac{e^{-2s}}{(s+2)(s^2-s+5)}$$

$$L^{-1}\{\text{II term}\} = \frac{1}{5}\left\{\frac{4}{s} + \frac{6s-11}{s^2-s+5}\right\} + \frac{1}{11}\left\{\frac{1}{s+2} - \frac{s-3}{s^2-s+5}\right\}e^{-2s}$$

$$= \frac{1}{11}u(t-2)e^{-2(t-2)} - \frac{1}{11}\left\{e^{\frac{1}{2}t}\cos\frac{\sqrt{19}}{2}t - \frac{5}{2}\times\frac{2}{\sqrt{19}}\sin\frac{\sqrt{19}}{2}t\right\}u(t-2)$$

$t \to t-2$

$$= \frac{1}{11}u(t-2)e^{-2(t-2)} - \frac{1}{11}\left\{e^{(t-2)/2}\cos\frac{\sqrt{19}}{2}(t-2) - \frac{5}{\sqrt{19}}\sin\frac{\sqrt{19}}{2}(t-2)\right\}u(t-2)$$

$$\frac{s-3}{s^2-s+5} = \frac{s}{(s-\frac{1}{2})^2+\frac{19}{4}} - \frac{3}{(s-\frac{1}{2})^2+\left(\frac{\sqrt{19}}{2}\right)^2}$$

$$= \frac{s}{(s-\frac{1}{2})^2+\left(\frac{\sqrt{19}}{2}\right)^2} - \frac{3}{(s-\frac{1}{2})^2+\left(\frac{\sqrt{19}}{2}\right)^2}$$

$$= \frac{s-\frac{1}{2}+\frac{1}{2}-3}{(s-\frac{1}{2})^2+\left(\frac{\sqrt{19}}{2}\right)^2}$$

$$= \frac{s-\frac{1}{2}}{(s-\frac{1}{2})^2+\left(\frac{\sqrt{19}}{2}\right)^2} - \frac{5/2}{(s-\frac{1}{2})^2+\left(\frac{\sqrt{19}}{2}\right)^2}$$

$$L^{-1}(\text{I term}) = \frac{1}{5}\left(4 + 6e^{t/2}\cos\left(\frac{\sqrt{19}}{2}\right)t - \frac{16}{\sqrt{19}}e^{t/2}\sin\frac{\sqrt{19}}{2}t\right)$$

$$\text{Answer} = L^{-1}(\text{I term}) + L^{-1}(\text{II term})$$

$$\frac{6s-11}{s^2-s+5}$$

$$= \frac{6(s-\frac{1}{2}+\frac{1}{2})-11}{(s-\frac{1}{2})^2+\left(\frac{\sqrt{19}}{2}\right)^2}$$

$$= \frac{6(s-\frac{1}{2})}{(s-\frac{1}{2})^2+\left(\frac{\sqrt{19}}{2}\right)^2} - \frac{8}{(s-\frac{1}{2})^2+\left(\frac{\sqrt{19}}{2}\right)^2}$$

Questions:

(i) $y''(t) - y'(t) = \cos 2t + \cos(2t-12)u_6(t)$, $y(0) = -4$, $y'(0) = 0$.

Answer: $$y(t) = \frac{1}{5}\left(e^t - \cos 2t - \frac{1}{2}\sin 2t\right) + \frac{u(t-6)}{5}\left(e^{t-6} - \cos 2(t-6) - \frac{1}{2}\sin 2(t-6)\right) - 4$$

(ii) $y'' - 5y' - 14y = 9 + u(t-3) + 4(t-1)u(t-1)$, $y(0) = 0$, $y'(0) = 10$

$$y(t) = 9\left(-\frac{1}{14} + \frac{1}{63}e^{7t} + \frac{1}{18}e^{-2t}\right) + u(t-3)\left(-\frac{1}{14} + \frac{e^{7(t-3)}}{63} + \frac{1}{18}e^{-2(t-3)}\right)$$

$$+ 4u(t-1)\left(\frac{5}{196} - \frac{1}{14}(t-1) + \frac{1}{441}e^{7(t-1)} - \frac{1}{36}e^{-2(t-1)}\right)$$

$$+ \frac{10}{9}e^{7t} - \frac{10}{9}e^{-2t}$$

(iii) $y'' + 3y' + 2y = g(t)$, $y(0) = 0$, $y'(0) = -2$, $g(t) = \begin{cases} 2, & t < 6 \\ t, & 6 \le t < 10 \\ 4, & t \ge 10 \end{cases}$

Answer:

$$y(t) = 1 - 2e^{-t} + e^{-2t} - 2e^{-t} + 2e^{-2t}$$

$$+ u(t-6)\left(4f(t-6) + g(t-6)\right)$$

$$- u(t-10)\left(6f(t-10) + g(t-10)\right)$$

$$= 2 + (t-2)u(t-6) + (4-t)u(t-10)$$

$$= 2 + (t-6+4)u(t-6) - (t-10+6)u(t-10)$$

$$L(g(t)) = \frac{2}{s} + e^{-6s}\left(\frac{1}{s^2} + \frac{4}{s}\right) - e^{-10s}\left(\frac{1}{s^2} + \frac{6}{s}\right)$$

where $f(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$

$$g(t) = -\frac{3}{4} + \frac{1}{2}t + e^{-t} - \frac{1}{4}e^{-2t}$$