LAPLACE TRANSFORMS

* Used to get the solution of differential equations with boundary conditions

Definition

Let $f(t)$ be a function of t defined for all positive values of t. Then the Laplace transforms of $f(t)$, denoted by $L\{f(t)\}$ is defined by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= F(s)$$

provided the integral exists, s is a parameter which may be real or complex number.

$$L\{f(t)\} = F(s) = f(s)$$

Inverse Laplace transform is given by

$$L^{-1}\{F(s)\} = f(t)$$

Here L is called the Laplace Transformation operator.

Conditions for the existence (SUFFICIENT CONDITION)

$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt$ exists for $s > a$ if

(i) $f(t)$ is continuous

(ii) $\lim_{t\to\infty} \{e^{-at} f(t)\}$ is finite

Its a sufficient condition but not necessary.

Eg: $L\left[\frac{1}{\sqrt{t}}\right]$ exists though $\frac{1}{\sqrt{t}}$ is infinite at $t=0$.

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TRANSFORMS OF ELEMENTARY FUNCTIONS

1. $L[1] = \frac{1}{s}$

$$L[1] = \int_0^\infty e^{-st} \cdot 1 \, dt$$

$$= \left(\frac{e^{-st}}{-s} \right)_0^\infty = \frac{1}{s}$$

2. $L[e^{at}] = \frac{1}{s-a}$

$$L[e^{at}] = \int_0^\infty e^{-st} e^{at} \, dt = \int_0^\infty e^{-(s-a)t} \, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty = \frac{1}{s-a}$$

3. $L[\sin at] = \dfrac{a}{s^2+a^2}$

$$L[\sin at] = \int_0^\infty e^{-st} \sin at \, dt$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$$

$$\Rightarrow L[\sin at] = \left[\frac{e^{-st}}{s^2+a^2}(-s \sin at - a \cos at) \right]_0^\infty$$

$$= \frac{a}{s^2+a^2}$$

4. $L[\cos at] = \dfrac{s}{s^2+a^2}$

5. $L[t^n] = \dfrac{n!}{s^{n+1}}$ when $n = 0, 1, 2, 3, \ldots$

otherwise $\dfrac{\Gamma n+1}{s^{n+1}}$

6. $L[\sinh at] = \dfrac{a}{s^2-a^2}$

7. $L[\cosh at] = \dfrac{s}{s^2-a^2}$

Properties of Laplace Transforms

1. Linearity property

If a, b, c be any constants and f, g and h be functions of t, then

$$L[a f(t) + b g(t) - c h(t)] = a L[f(t)] + b L[g(t)] - c L[h(t)]$$

2. First Shifting property

If $L[f(t)] = F(s)$ then

$$L[e^{at} f(t)] = F(s-a)$$

SOME STANDARD RESULTS USING FIRST SHIFTING PROPERTY

1. $L[e^{at}] = \dfrac{1}{s-a}$

2. $L[e^{at}\sin bt] = \dfrac{b}{(s-a)^2+b^2}$

3. $L[e^{at}\cos bt] = \dfrac{s-a}{(s-a)^2+b^2}$

4. $L[e^{at}\cosh at] = \dfrac{s-a}{(s-a)^2-b^2}$

5. $L[e^{at}\sinh bt] = \dfrac{b}{(s-a)^2 - b^2}$

6. $L[e^{at} t^n] = \dfrac{n!}{(s-a)^{n+1}}$

where

in each case $s > a$