(i) $L\left[t\sin at\right]$

<u>Sol:</u> Let $f(t) = \sin(at)$

Then $F(s) = L[f(t)] = \dfrac{a}{s^2+a^2}$

By derivative of transform property,

$$L[tf(t)] = -\frac{d}{ds}[F(s)]$$

$$\therefore \quad L[t\sin(at)] = -\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right]$$

$$= -\left[\frac{(s^2+a^2)(0) - a(2s)}{(s^2+a^2)^2}\right]$$

$$\boxed{L[t\sin(at)] = \frac{2as}{(s^2+a^2)^2}}$$

(ii) $L\left[t^2 \sin 2t\right]$

Sol: Let $f(t) = \sin 2t$

Then $F(s) = \dfrac{2}{s^2+4}$

Now, $L\left[t^2 f(t)\right] = (-1)^2 \cdot \dfrac{d^2}{ds^2}\left[F(s)\right]$

$$\therefore \quad L\left[t^2 \sin 2t\right] = \frac{d^2}{ds^2}\left[\frac{2}{s^2+4}\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+4)(0) - 2(2s)}{(s^2+4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-4s}{(s^2+4)^2}\right]$$

$$= \frac{(s^2+4)^2\left[-4\right] - (-4s)\left[2(s^2+4)(2s)\right]}{(s^2+4)^4}$$

$$= \frac{1}{(s^2+4)^3}\left[-4(s^2+4) + 16s^2\right]$$

$$= \frac{12s^2-16}{(s^2+4)^3} = \frac{4(3s^2-4)}{(s^2+4)^3} \;//$$

(iv) $L\left[t^n\right]$:

Sol:

$$L\left[t^n\right] = L\left[t^n \cdot 1\right]$$

$$= (-1)^n \cdot \frac{d^n}{ds^n}\left[\frac{1}{s}\right]$$

$$= (-1)^n \left[\frac{(-1)^n \cdot n!}{s^{n+1}}\right]$$

$$= \frac{n!}{s^{n+1}} \quad //$$

$$\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{-1}{s^2}$$

$$\frac{d^2}{ds^2}\left(\frac{1}{s}\right) = \frac{(-1)\cdot(-2)}{s^3}$$

$$= \frac{(-1)^2 \cdot 2!}{s^3}$$

(iii) $L\left[te^{-t}\cos t\right]$

Sol:

$$L[t\cos t] = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]$$

$$= -\left[\frac{(s^2+1)(1) - s(2s)}{(s^2+1)^2}\right]$$

$$L[t\cos t] = \frac{s^2-1}{(s^2+1)^2} = F(s)$$

$$\therefore \quad L[te^{-t}\cos t] = L[e^{-t}.\, t\cos t]$$

$$= F(s+1)$$

$$= \frac{(s+1)^2-1}{[(s+1)^2+1]^2}$$

$$= \frac{s^2+2s}{(s^2+2s+2)^2}$$

$$\therefore \quad L[te^{-t}\cos t] = \frac{s(s+2)}{(s^2+2s+2)^2} \quad //$$

(i) $L\left[\frac{1-e^{-t}}{t}\right]=\log\left(\frac{s+1}{s}\right)$

Sol:

Let $f(t)=1-e^{-t}$

Then $F(s)=\frac{1}{s}-\frac{1}{s+1}$

By integral of transform property,

$$L\left[\frac{f(t)}{t}\right]=\int_s^\infty F(s)\,ds$$

$$\therefore \quad L\left[\frac{1-e^{-t}}{t}\right]=\int_s^\infty\left(\frac{1}{s}-\frac{1}{s+1}\right)ds$$

$$=\left[\log s-\log(s+1)\right]_s^\infty$$

$$=\left[\log\left(\frac{s}{s+1}\right)\right]_s^\infty$$

$$=\{\log(1)\}-\log\left(\frac{s}{s+1}\right)$$

$$=0-\log\left(\frac{s}{s+1}\right)$$

$$=\log\left(\frac{s+1}{s}\right) \quad ||$$

(i) $L\left[\int_0^t \sin(t)dt\right]$

Sol: Let $f(t) = \sin t$

Then $F(s) = \frac{1}{s^2+1}$

By transform of integral property,

$$L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$$

$$\therefore \quad L\left[\int_0^t \sin t \, dt\right] = \frac{1}{s}\left[\frac{1}{s^2+1}\right]$$

(iv) $L\left[\int_0^t u e^{-u} \sin(4u)\,du\right]$

Sol: Let $f(u) = u e^{-u} \sin 4u$

$$F(s) = L[f(u)]$$

$$= L[e^{-u} . u \sin 4u] \quad \text{--- } (1)$$

Now,

$$L[u \sin 4u] = -\frac{d}{ds}\left[\frac{4}{s^2+16}\right]$$

$$= -\left[\frac{-4(2s)}{(s^2+16)^2}\right]$$

$$= \frac{8s}{(s^2+16)^2}$$

$\therefore$ From (1), we have

$$F(s) = L[u\sin 4u]\Big|_{s\to s+1}$$

$$= \frac{8s}{(s^2+16)^2}\Bigg|_{s\to s+1}$$

$$= \frac{8(s+1)}{[(s+1)^2+16]^2}$$

$$\Rightarrow F(s) = \frac{8(s+1)}{(s^2+2s+17)^2}$$

$$\therefore L\left[\int_0^t u e^{-u}\sin 4u\,du\right]$$

$$= L\left[\int_0^t f(u)\,du\right] = \frac{F(s)}{s}$$

$$= \frac{8(s+1)}{s(s^2+2s+17)^2} \,//$$