Special cases [of first-order nonlinear PDEs]

① $f(p,q)=0$

$$z = ax+by+c$$

$$p = \frac{\partial z}{\partial x} = a$$

$$q = \frac{\partial z}{\partial y} = b$$

$$f(p,q)=0 \Rightarrow \boxed{f(a,b)=0}$$

$$\Downarrow$$

$$a = \phi(b) \text{ or}$$

$$b = \psi(a)$$

Solution of $f(p,q)=0$ is

$$z = ax+\psi(a)y+c$$

a, c - constants (or)

$$z = \phi(b)x+by+c$$

b, c - constants.

(1) Solve $p - q = 1$

<u>Sol:</u> Let $z = ax + by + c$ be the reqd. solution.

Then $p = a$ and $q = b$

From pde, we have $a - b = 1$

$\Rightarrow a = 1 + b$

$\therefore$ The complete solution of $p - q = 1$ is

$$z = (1+b)x + by + c,$$

where b and c are arbitrary constants.

(2) $f(z, p, q) = 0$

Let $z = z(u)$, where

$u = x + ay$. be the solution.

Now, $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial z}{\partial u} = \frac{dz}{du}$

Also, $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = a\frac{\partial z}{\partial u} = a\frac{dz}{du}$

$f(z, p, q) = 0 \Rightarrow f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0$

[I order ODE in z and u]

$$\frac{dz}{du} = \phi(z, a)$$

$$\int \frac{dz}{\phi} = \int du$$

$$\Rightarrow F(z, a) = u + b$$

$$\boxed{F(z, a) = x + ay + b}$$

(Complete soln.)

(1) Solve $p(1+q) = qz$

Sol: Let $z = z(u)$, where $u = x + ay$.

Then $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$

Substituting p and q in pde, we get

$$\cancel{\frac{dz}{du}}\left[1 + a\frac{dz}{du}\right] = \left(a\cancel{\frac{dz}{du}}\right)z$$

$$a\frac{dz}{du} = az - 1$$

$$\frac{a\,dz}{az-1} = du$$

Put $t = az - 1$. Then $dt = a\,dz$

$$\int \frac{a\,dz}{az-1} = \int \frac{dt}{t} = \log(t) = \log(az-1)$$

upon integration, we get

$$\log(az-1) = u + b$$

i.e., $\log(az-1) = x + ay + b$

(2) solve $p(1+q) = qz$

Sol: Let $z = z(u)$, where

$u = y + ax$.

Then $p = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = a\frac{dz}{du}$

and $q = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = \frac{dz}{du}$

substituting p and q in the pde, we get

$$a\cancel{\frac{dz}{du}}\left(1+\frac{dz}{du}\right) = \cancel{\frac{dz}{du}}\,z$$

$$1+\frac{dz}{du} = \frac{z}{a}$$

$$\frac{dz}{du} = \frac{z}{a} - 1 = \frac{z-a}{a}$$

$$\frac{a\,dz}{z-a} = du$$

Integrating, we get

$$a\log(z-a) = u + b$$

$$\boxed{a\log(z-a) = y + ax + b}$$

(Complete soln. of PDE)

③ $f(x,p) = F(y,q)$

Let $f(x,p) = a = F(y,q)$

$f(x,p) = a \Rightarrow p = \phi(x,a)$

$F(y,q) = a \Rightarrow q = \psi(y,a)$

$dz = p\,dx + q\,dy$

$$\boxed{z = g_1(x,a) + g_2(y,a) + b}$$

① Solve $p^2 + q^2 = x + y$.

Sol: The given pde can be written as

$$p^2 - x = y - q^2$$

Let $p^2 - x = a = y - q^2$

$p^2 - x = a \Rightarrow p^2 = x + a \Rightarrow p = \sqrt{x+a}$

$y - q^2 = a \Rightarrow q^2 = y - a \Rightarrow q = \sqrt{y-a}$

$dz = p\,dx + q\,dy$

$z = \int p\,dx + \int q\,dy + b$

$$z = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(y-a)^{3/2} + b$$

(4) $z = px + qy + f(p, q)$

The complete soln. of this pde is

$$z = ax + by + f(a, b).$$