MODULE 3 Laplace Transform

16 March 2022 14:50

$L(T)$

16/3/22

Definition- Properties of Laplace transform-Laplace transform of standard functions – Laplace transform of periodic functions-Unit step function-Impulse function. Inverse Laplace transform- Partial fractions method and by Convolution theorem

L. Transformation.

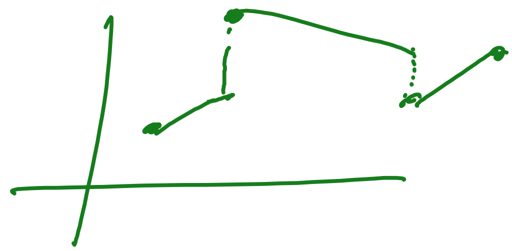
IVP $\Rightarrow$ alg. pr. $\to$ soln of alg.

$\downarrow$

soln. of IVP

piece wise continuous fn:

$f(t)$ ( ) ( )



Defn LT

Let $f(t) \to$ fn. of the variable t defined for all +ve values of t.

Let $s \to$ real, complex parameter.

If $\int_0^\infty e^{-st} f(t)\,dt$ exists $= F(s)$

$F(s)$ is the LT of $f(t)$.

denoted by $L[f(t)]$

$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt.$

Sufficient Condition for L.T to exist:

1) $f(t) \rightarrow$ continuous (or)

piecewise continuous in the closed interval $[a, b]$, $a > 0$

2) $f(t)$ sd. be of exponential order.

(i.e., $\lim_{t \to \infty} e^{-st} f(t) = 0$.

Result..

① $L[1] = \frac{1}{s}.$

WKT $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

(Gn $f(t) = 1$)

$= \int_0^\infty e^{-st} (1)\,dt$

$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$

$\frac{-1}{s}\left[e^{-\infty} - e^0\right]$

$= \frac{-1}{s}(0-1) = \frac{1}{s}$

1) $L[1] = \frac{1}{s}$

2) $L[t^n] = \frac{n!}{s^{n+1}}$

3) $= \frac{\Gamma_{n+1}}{s^{n+1}}$

4) $L[e^{at}] = \frac{1}{s-a}$

5) $L[e^{-at}] = \frac{1}{s+a}$

6) $L[\sin at] = \frac{a}{s^2+a^2}$

7) $L[\cos at] = \frac{s}{s^2+a^2}$

8) $L[\sinh at] = \frac{a}{s^2-a^2}$

(9) $L[\cosh at] = \frac{s}{s^2-a^2}$.

Properties

1. Linearity.

$L[af(t) \pm bg(t)]$

$$= a\, L[f(t)] + b\, L[g(t)]$$

Problems

(1) $L[t] = ?$

$$L[t^n] = \frac{n!}{s^{n+1}} =$$

WKT

$$L[t^1] = \frac{1!}{s^{1+1}} = \frac{1}{s^2}$$

2) $$L[\sqrt{t}] = \frac{\Gamma(\frac{1}{2}+1)}{s^{\frac{1}{2}+1}} = \frac{\Gamma(\frac{3}{2})}{s^{3/2}} = \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{s^{3/2}} = .$$

3) $$L[e^{3t+5}] = L[e^{3t} \cdot e^5]$$

$$= e^5 \cdot \frac{1}{s-3}$$

4) $$L[e^{-\sqrt{7}t}] = \frac{1}{s+\sqrt{7}}$$

5) $$L[2^t] = L[e^{\log 2^t}]$$

$$= L[e^{t(\log 2)}]$$

$$= L[e^{(\log 2)t}]$$

$$= \frac{1}{s - \log 2}$$

6) $L[\sin \underset{a}{2} t] = \dfrac{2}{s^2 + 4}$

7) $L[\sin t \cos t] = L\left[\dfrac{\sin 2t}{2}\right]$

$$= \frac{1}{2} \cdot \frac{2}{s^2+4} = \frac{1}{s^2+4}.$$

8) $L[\sin^2 2t] = L\left[\dfrac{1 - \cos 4t}{2}\right]$

$$= \frac{1}{2}\left[L[1] - L[\cos 4t]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$$

9) $L[\sin 2t \cos 3t]$

$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$

$\Rightarrow \quad \dfrac{1}{2}\left[L[\sin 5t] - L[\sin t]\right]$

$$= \frac{1}{\,}\cdot\left[\frac{5}{\,} - \frac{1}{\,}\right]$$

$$\frac{\quad}{2}\left[\frac{\quad}{s^2+25} \quad \frac{\quad}{s^2+1}\right]$$

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I) First shifting theorem.

(i) If $L[f(t)] = \phi(s)$ then
$L[e^{at} f(t)] = \phi(s-a)$

(ii) If $L[f(t)] = \phi(s)$ then
$L[e^{-at} f(t)] = \phi(s+a)$

Pf :- $L[f(t)] = \phi(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$L[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-t(s-a)} f(t)\,dt$$

$$= \phi(s-a)$$

1) $L[t^n e^{-at}] = L[t^n]_{s \to s+a}$ (by 1st shif)

$$= \left.\frac{n!}{s^{n+1}}\right]_{s \to s+a}$$

$$= \frac{n!}{\quad^{n+1}}$$

$(s+a)$ //.

2) $L\left[e^{t} t^{-1/2}\right] = L\left[t^{-1/2}\right]_{s \to s-1}$

(by first shift)

$= \left.\dfrac{\Gamma(-1/2+1)}{s^{-\frac{1}{2}+1}}\right]_{s \to s-1.}$

$= \left.\dfrac{\Gamma(1/2)}{s^{1/2}}\right]_{s \to s-1}$

$= \dfrac{\sqrt{\pi}}{(s-1)^{1/2}} = \sqrt{\dfrac{\pi}{s-1}}.$

3) $L\left[e^{-at}\cos bt\right] = L\left[\cos bt\right]_{s \to s+a}$

$= \left.\dfrac{s}{s^2+b^2}\right]_{s \to s+a}$

$= \dfrac{s+a}{(s+a)^2+b^2}$

4) $L\left[e^{t}\left(\cosh 2t + \frac{1}{2}\sinh 2t\right)\right]$

(linearity property)

$= L\left[e^{t}\cosh 2t\right] + L\left[\frac{1}{2} e^{t} \sinh 2t\right]$

(1st shifting property)

$= L[\cosh 2t] \quad + \frac{1}{2} L[\sinh 2t]$

$= L[\cosh 2t]_{s \to s-1} + \frac{1}{2} L[\sinh 2t]_{s \to s-1}$

$= \frac{s-1}{(s-1)^2 - 4} + \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{(s-1)^2 - 4}$

[iii] Change of scale property

If $L[f(t)] = F(s)$ then

$L[f(at)] = \frac{1}{a} F(s/a)$, $a > 0$.

5) $f(t) = \cos 4t$

direct $\rightarrow$ $L[f(t)] = \frac{s}{s^2+16}$

$L[\cos 4t] = \frac{1}{4} \frac{s/4}{(s/4)^2 + 1}$

$= \frac{1}{4} \cdot \frac{s/4}{\frac{s^2}{16} + 1}$

$= \frac{s}{16} \cdot \frac{1}{\frac{s^2+16}{16}} = \frac{s}{s^2+16}$

[iv] Derivatives of transforms

If $L[f(t)] = F(s)$ then

$L[t f(t)] = -\frac{d}{ds} F(s)$

In general ... n ...

In general

$$L\left[t^n f(t)\right] = (-1)^n \frac{d}{ds} F(s)$$

b) $L\left[t \sin t\right] = -\frac{d}{ds} F(s)$

$f(t) = \sin t$

$F(s) = L[f(t)] = \frac{1}{s^2+1}$.

$$L[t \cdot \sin t] = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= -\left[\frac{0 - 1(2s)}{(s^2+1)^2}\right]$$

$$= \frac{2s}{(s^2+1)^2}$$

2) $L\left[t^2 e^{-3t}\right] = (-1)^2 \frac{d^2}{ds^2} L\left[e^{-3t}\right]$

$$= \frac{d^2}{ds^2}\left[\frac{1}{s+3}\right]$$

$$= \frac{d}{ds}\left[\frac{-1}{(s+3)^2}\right] = \frac{2}{(s+3)^3}$$

3) $L\left[t e^{-t} \sin t\right]$

$$= -\frac{d}{ds} L\left[e^{-t} \sin t\right]$$

(derivative of num.)

$$= \frac{-d}{ds} L[\sin t]_{s \to s+1}$$

(1st shift)

$$= -\frac{d}{ds}\left[\frac{1}{\ldots}\right]$$

$$\frac{\quad}{ds}\left[(s+1)^2+1\right]$$

$$= \frac{-d}{ds}\left[\frac{1}{s^2+2s+2}\right]$$

$$= \frac{2s+2}{(s^2+2s+2)^2} \quad //$$

22/3/22.

Second Shifting Theorem.

If $L[f(t)] = F(s)$ and

$$G(t) = \begin{cases} f(t-a), & t > a \\ 0, & 0 \le t \le a \end{cases}$$

then $L[G(t)] = e^{-as} \cdot F(s)$

1) find the L.T of $G(t)$ where

$$G(t) = \begin{cases} \cos(t - 2\pi/3), & t > 2\pi/3 \\ 0, & t < 2\pi/3 \end{cases}$$

$a = 2\pi/3$

$f(t-a) = \cos(t - 2\pi/3)$

$f(t) = \cos t$

using 2nd shifting.

$L[G(t)] = e^{-as} F(s)$

$f(t) = \cos t$

$F(s) = L[f(t)]$ $L[\cos t] = \frac{s}{\quad}$

$F(s) = L[f(t)] = L[\cos t] = \frac{s}{s^2+1}$

$$L[G(t)] = e^{-2\pi/3\, s} \cdot \frac{s}{s^2+1}$$

2) $f(t) = \begin{cases} 0 & , 0 < t < 2 \\ 3 & , t > 2 \end{cases}$

$a = 2$ $\quad f(t-a) = 3$

$f(t) = 3$

$$L[f(t)] = L[3] = 3L[1]$$

$$= 3 \cdot \frac{1}{s}.$$

$$L[G(t)] = e^{-2s} \cdot \frac{3}{s}$$

3) find $L[f(t)]$ where

$$f(t) = \begin{cases} \cos t & , 0 < t < \pi \\ \sin t & , t > \pi \end{cases}$$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\pi e^{-st} \cos t\, dt + \int_\pi^\infty e^{-st} \sin t\, dt$$

$e^{-st}\left[-s\cos t + \sin t\right]_0^\pi + e^{-st}\left[-s\sin\right.$

ans. $= \frac{e}{s^2+1}\left[-s\ldots\right]_0 \quad \frac{}{s^2+1}$

$$\left[\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$\left[\int e^{ax}\sin bx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right)\right]$$

$$\underline{\text{Ans}} :- \frac{1}{s^2+1}\left(s + e^{-\pi s}(s-1)\right)$$

Integrals of transform :-

If $L[f(t)] = F(s)$ ~~then~~ and if $\frac{1}{t}f(t)$ has a lim. as $t\to 0$ then.

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$$

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1) $$L\left[\frac{\sin at}{t}\right] = \int_s^\infty L[\sin at]\,ds$$

$$= \int_s^\infty \frac{a}{s^2+a^2}\,ds = a\left(\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\right)_s^\infty$$

$$\left(\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a) \right)$$

$$= \tan^{-1}(s/a)\Big]_s^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/a)$$

$$= \pi/2 - \tan^{-1}(s/a)$$

2) $L\left[\frac{e^{-3t} - e^{-4t}}{t}\right]$

$$= L\left[\frac{e^{-3t}}{t}\right] - L\left[\frac{e^{-4t}}{t}\right] \quad \text{(using linearity.)}$$

(using Integ. of fun.

$$= \int_s^{\infty} L[e^{-3t}]\,dt - \int_s^{\infty} L[e^{-4t}]\,dt$$

$$= \int_s^{\infty} \frac{1}{s+3}\,dt - \int_s^{\infty} \frac{1}{s+4}\,dt$$

$$= \log(s+3) - \log(s+4)\Big]_s^{\infty}$$

$$= -\log(s+3) + \log(s+4)$$

$$\left[\log m - \log n = \log\frac{m}{n} \right]$$

$= \log\left(\frac{s+4}{s+3}\right)$ //.

CAT I PORTIONS

Module - I → fully

II → fully.

III → problems based on Result linearity, 1st shifting, cha

Mark distribution

I → 15 marks.

II → 10 "

III → 5 ".

All the best.

4/4/22

Periodic fns

$f(t+T) = f(t)$

$T \rightarrow$ +ve constant

least value of $T > 0$ $\rightarrow$ period of $f(t)$

$$L(f(t)) = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\,dt$$

1) find the L.T of the rectangular wave give by

$$f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$$

with $f(t+2b) = f(t)$ $\longrightarrow$ periodic $2b$

$$L[f(t)] = \frac{1}{1-e^{-2bs}} \left[\int_0^b e^{-st} f(t)\,dt + \int_b e^{-st} f(t)\,dt \right]$$

$f(t) \to 1$, $f(t) \to -1$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^b e^{-st}\,dt - \int_b^{2b} e^{-st}\,dt \right]$$

$$= \frac{1}{1-e^{-2bs}} \left\{ \left[\frac{e^{-st}}{-s} \right]_0^b - \left[\frac{e^{-st}}{-s} \right]_b^{2b} \right\}$$

$$= \frac{1}{1-e^{-2bs}} \left\{ \frac{e^{-sb}}{-s} + \frac{1}{s} + \frac{e^{-2bs}}{s} - \frac{e^{-bs}}{s} \right\}$$

$$= \frac{1}{1-e^{-2bs}} \left\{ \frac{1}{s} - \frac{2e^{-sb}}{s} + \frac{e^{-2bs}}{s} \right\}$$

$$= \frac{1}{s(1-e^{-2bs})} \left[1 - 2e^{-sb} + e^{-2sb} \right]$$

$a^2 - 2ab + b^2$

$(1 - e^{-sb})^2$

$$= \frac{1}{s(1-e^{-2bs})}$$

$a^2 - b^2 = (a+b)(a-b)$

$-bs)^2$

$$= \frac{1}{s(1+e^{-bs})(1-\cancel{e^{-bs}})} \cdot (1-e \quad)$$

$$= \frac{(1-e^{-bs})}{s(1+e^{-bs})}$$

$$\left[\tanh x = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right]$$

$$= \frac{1}{s}\left[\frac{(1-e^{-bs/2}\cdot e^{-bs/2})}{1+e^{-bs/2}\cdot e^{-bs/2})}\right]$$

$$\frac{-bs}{2} - \frac{bs}{2}$$

$$\frac{-2bs}{2}$$

$$-bs$$

$$= \frac{1}{s}\left[\frac{\cancel{e^{-bs/2}}(e^{bs/2}-e^{-bs/2})}{\cancel{e^{-bs/2}}(e^{bs/2}+e^{-bs/2})}\right]$$

$$= \frac{1}{s}\frac{(e^{bs/2}-e^{-bs/2})}{(e^{bs/2}+e^{-bs/2})}$$

$$= \frac{1}{s}\tanh\left(\frac{bs}{2}\right) //.$$

3) find the LT of half wave rectifier fn.

$$f(t) = \begin{cases} \sin \omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$$

This fn. is periodic with period $2\pi/\omega$ in the interval $(0, 2\pi/\omega)$

$$L[f(t)] = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\int_0^{\pi/\omega} e^{-st} \sin\omega t \, dt + \int_{\pi/\omega} 0 \right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-st}}{s^2+\omega^2} \left[-s \sin\omega t - \omega\cos\omega t\right] \right]_0^{\pi/\omega}$$

$$\left[\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b\cos bx) \right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-\frac{s\pi}{\omega}}}{s^2+\omega^2} (\omega) - \frac{1}{s^2+\omega^2} (-\omega) \right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\frac{\omega}{s^2+\omega^2} \left(e^{-s\pi/\omega} + 1 \right) \right]$$

$a^2 - b^2$

$$= \frac{\omega}{s^2+\omega^2} \cdot \frac{\cancel{(1+e^{-s\pi/\omega})}}{\cancel{(1+e^{-\pi s/\omega})}(1-e^{-\pi s/\omega})}$$

$$= \frac{\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$$ //

3) HW find the L.T of $|\sin t|$

periodic fn

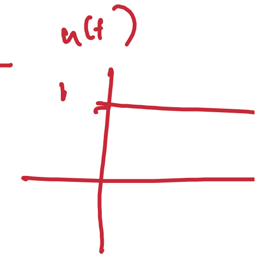
period : $\to (0, \pi)$

$$L[f(t)] = \frac{1}{1-e^{-\pi s}} \int_0^{\pi} e^{-st} \sin t \, dt$$

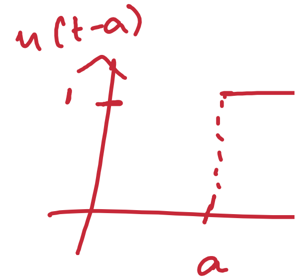
Ans $\frac{1}{s^2+1} \coth\left(\frac{\pi s}{2}\right)$

unit step function. 5/4/22

→ heaviside side fun



$$u(t-a) = \begin{cases} 0, & \text{if } t < a \\ 1, & \text{if } t > a. \end{cases}$$



for any arbitrary +ve 'a'.

L.T of unit step fn.

$$L[u(t-a)] = \frac{e^{-as}}{s}, \quad s>0, \quad a \text{ +ve}.$$

Pf

$$L[u(t-a)] = \int_0^\infty e^{-st} u(t-a)\, dt$$

$$= \int_a^\infty e^{-st} (1)\, dt.$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{-1}{s}\left[0 - e^{-as}\right]$$

$$= \frac{e^{-as}}{s}, \quad s>0$$

Spl. case. if $a = 0$

$$L[u(t-a)] = \frac{e^{-as}}{s}$$

$\Downarrow$

$$L[u(t)] = \frac{1}{S} //.$$

Ans

$$L[f(t)\, u(t-a)] = e^{-as}\, L[f(t+a)]$$

$$L[f(t-a)\, u(t-a)] = e^{-as}\, L[f(t)]$$

i) find $L[t^2\, u(t-2)]$ $\quad a = 2$

$f(t) = t^2$

$f(t-(2)) = (t+2)^2$

$$= e^{-2s}\, L[(t+2)^2]$$

$$= e^{-2s}\, [L(t^2 + 4t + 4)]$$

$$= e^{-2s}\left[\frac{2}{S^3} + \frac{4}{S^2} + \frac{4}{S}\right]$$

(ii) $L[u_{\pi/2}(t) \quad \sin t]$

$u_a(t)$

$$= e^{-\pi/2\, s}\, L[\sin(t + \pi/2)]$$

$$= e^{-\pi/2\, s}\, L[\cos t]$$

$$= e^{-\pi s/2} \cdot \frac{s}{s^2+1}. //.$$

(iii) $L[u_{\pi/2}(t) \cdot \sin(t - \pi/2)]$

$$= e^{-\frac{\pi}{2}s}\, L[\sin(t - \pi/2)]$$

$$= e^{-\pi s/2} \quad L[\sin t]$$

$$= e^{-\pi/2} \cdot \frac{1}{s^2+1}$$

(iv) $L[u_2(t)\, e^{7t}]$

$$= e^{-2s} \quad L\left[e^{7(t+2)}\right]$$

$f(t) = e^{7t}$

$f(t+2) = e^{7(t+2)}$

$$= e^{-2s} \quad L\left[e^{7t} \cdot e^{14}\right]$$

$$= e^{-2s} \cdot e^{14} \cdot \frac{1}{s-7}$$

$$= \frac{e^{14-2s}}{s-7}$$

$u_a(t)$

Impulse fn (or) Dirac Delta fn.

$\delta(t)$

(i) $\delta(t) = 0$ if $t \neq 0$

$= \infty$ if $t = 0$

(ii) $\int_{-\infty}^{\infty} \delta(t)\, dt = 1.$

Translation of $\delta(t)$

at $t = c$. instead of $t = 0$

(i) $\delta(t-c) = 0$; if $t \neq c$

(ii) $\int_{-\infty}^{\infty} \delta(t-c)\, dt = 1.$

$$L[\delta(t)] = 1$$

$$L[\delta(t-c)] = e^{-cs} \quad , c \geq 0$$

$$L[\delta(t-c) f(t)] = f(c) e^{-cs}.$$

1) $L[\delta(t) + \delta(t-2)]$

$= L[\delta(t)] + L[\delta(t-2)]$

$= 1 + e^{-2s}$

2) $L[e^{-\pi t} \delta(t-a)]$

$L[\delta(t-a)]_{s \to s+\pi}$

$= e^{-as}]_{s \to s+\pi}$

$= e^{-a(s+\pi)}$

Representation of fns. with Jump..

$u(t)$

$u(t-a)$

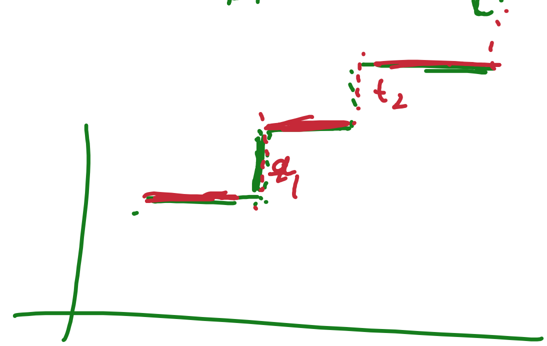
$g(t) = \begin{cases} g_1(t) & ; \; 0 < t < t_1 \\ \vdots \end{cases}$

$$\vdots$$

$$g_k(t), \quad t_{k-1} < t < t_k.$$

$$g(t) = g_1(t) + [g_2(t) - g_1(t)]\, u(t - t_1)$$

$$+ [g_3(t) - g_2(t)]\, u(t - t_2) + \dots [g_k(t) - g_{k-1}(t)]\, u(t - t_{k-1})$$



Inverse L.T

$$L[f(t)] = F(s)$$

$$f(t) = L^{-1}[F(s)]$$

$$L^{-1}\left[\frac{1}{s}\right] = 1 \qquad L[1] = \frac{1}{s}$$

$$L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!} = \frac{t^{n-1}}{\Gamma(n)}$$

$$L^{-1}\left[\frac{1}{s^2}\right] = t$$

$$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$$

$$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

$$L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$$

<u>6/4/22</u>.

$$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$$

$$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$$

$$L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at$$

Linearity prope.

$$L^{-1}[aF(s)+bG(s)] = a\,L^{-1}F(s)) + b\,L^{-1}[G(s)]$$

first shifting property

$$L^{-1}[F(s+a)] = e^{-at}\,L^{-1}[F(s)]$$

$$L^{-1}[F(s-a)] = e^{at}\,L^{-1}[F(s)]$$

CONVOLUTION THEOREM.

Defn :- convolution.

$f(t)$, $g(t)$

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du.$$

Convolution thm. :-

$f(t), g(t), t \geq 0$

$$L[f(t) * g(t)] = L[f(t)] \cdot L[g(t)]$$
$$= F(s) \cdot G(s)$$

$$\times L^{-1}[F(s)\, G(s)] = f(t) * g(t)$$
$$= L^{-1}[F(s)] * L^{-1}[G(s)]$$

Problem :-

1) using convolution thm. find

$$L^{-1}\left(\frac{1}{s(s^2+1)}\right) = L^{-1}\left[\frac{1}{s}\right] * L^{-1}\left[\frac{1}{s^2+1}\right]$$
$$= 1 * \sin t$$
$$= \underset{f(t)}{\sin t} * \underset{g(t)}{1}$$

By defn $f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du.$

$f(t) = \sin t$
$f(u) = \sin u$
$g(t) = 1$
$g(t-u) = 1$

$$= \int_0^t \sin u \,(1)\, du$$
$$= -\cos u\Big]_0^t = -[\cos t - \cos 0]$$
$$= 1 - \cos t$$

2) $L^{-1}\left(\frac{1}{s^2(s+5)}\right)$ using convolution.

$$= L^{-1}\left[\frac{1}{s^2}\right] * L^{-1}\left[\frac{1}{s+5}\right]$$

$$= t * e^{-5t}$$

$$= \int_0^t u \cdot e^{-5t} \cdot e^{5u}\, du.$$

$$= e^{-5t}\int_0^t u e^{5u}\, du.$$

$$= e^{-5t}\left[\left[\frac{u e^{5u}}{5}\right]_0^t - \int_0^t \frac{e^{5u}}{5}\, du\right]$$

$$= e^{-5t}\left\{\left(\frac{t e^{5t}}{5}\right) - \left[\frac{1}{5}\,\frac{e^{5u}}{5}\right]_0^t\right\}$$

$$= e^{-5t}\left\{\frac{t e^{5t}}{5} - \frac{1}{25}\left(e^{5t} - 1\right)\right\}$$

$$= e^{-5t}\left\{\frac{t e^{5t}}{5} - \frac{e^{5t}}{25} + \frac{1}{25}\right\}$$

$f(t) = t$

$f(u) = u$

$g(t) = e^{-5t}$

$g(t-u) = e^{-5(t-u)}$

$= e^{-5t} \cdot e$

$\int u\,dv = uv - \int v$

$dv = e^{5u}$

$v = \frac{e^{5u}}{5}$

$\int \frac{e^{5u}}{5}\, du$

---

3) $L^{-1}\left(\frac{s}{(s^2+2s+5)^2}\right) = L^{-1}\left[\frac{s}{s^2+2s+5} \cdot \frac{1}{s^2+2s+5}\right]$

$$= L^{-1}\left[\frac{s}{s^2+2s+5}\right] * L^{-1}\left[\frac{1}{s^2+2s+5}\right]$$

$$s^2+2s+5 = s^2+2s+5-1+1.$$
$$= \underline{s^2+2s+1} + 4$$
$$= (s+1)^2 + 2^2$$
$$s^2 + a^2 -$$

take coeff of s
÷ by 2
+, − sqrs. 1^2-

$$L^{-1}\left[\frac{s+1-1}{(s+1)^2+2^2}\right] * L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$L^{-1}\left[\frac{s+1}{(s+1)^2+2^2} - \frac{1}{(s+1)^2+2^2}\right] * L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$\left[L^{-1}\left(\frac{s+1}{(s+1)^2+2^2}\right) - L^{-1}\left(\frac{1}{(s+1)^2+2^2}\right)\right] * L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$\left[e^{-t}\cos 2t - e^{-t}\cdot\frac{\sin 2t}{2}\right] * \left[\frac{e^{-t}\sin 2t}{2}\right]$$

$$\underbrace{\left[e^{-t}\left(\cos 2t - \frac{\sin 2t}{2}\right)\right]}_{f(t)} * \underbrace{\frac{e^{-t}\sin 2t}{2}}_{g(t)}$$

$$\int_0^t e^{-u}\left(\cos 2u - \frac{\sin 2u}{2}\right)\cdot \frac{e^{-(t-u)}}{2}\sin 2(t-u)\,du$$

$$\int_0^t e^{-u}\left(\cos 2u - \frac{1}{2}\sin 2u\right)\frac{e^{-t}}{2}\cdot e^{u}\sin(2t-2u)$$

$$\frac{e^{-t}}{2}\int_0 \left(\cos 2u - \frac{1}{2}\sin 2u\right) \sin(2t-2u)\, du.$$

$$\frac{e^{-t}}{2}\int_0^t \cos 2u \sin(2t-2u) - \frac{1}{2}\sin 2u \sin(2t-2$$

$$\frac{e^{-t}}{2}\int_0^t$$

Ans $\frac{e^{-t}}{2}(2t\sin 2t + t\cos 2t - \sin 2t)$

ILT using 8/4/22

Partial fractions.

$$\frac{ax+b}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$\frac{ax+b}{(x+2)^2(x+1)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$\frac{ax+b}{(x+5)(x^2+7)} = \frac{A}{x+5} + \frac{Bx+C}{x^2+7}$$

i) find the L^{-1} of $\dfrac{s^2-2s+3}{(s+1)(s-2)(2s-1)}$

$$s^2-2s+3 = \frac{A}{} + \frac{B}{} + \frac{C}{} \quad \text{— (1)}$$

$(s+1)\cancel{(s-2)}(2s-1)$ $\quad s+1 \quad s-2 \quad 2s-1$

$$= \frac{A(s-2)(2s-1) + B(s+1)(2s-1) + C(s+1)(s-2)}{(s+1)\cancel{(s-2)}(2s-1)}$$

$$s^2-2s+3 = A(2s^2-s-4s+2) + B(2s^2-s+2s-1) + C(s^2-2s+s-2)$$ — equating ...

$$s^2-2s+3 = A(s-2)(2s-1) + B(s+1)(2s-1) + C(s+1)(s-2)$$

put $s=-1 \Rightarrow A(-3)(-3) = 1+2+3$

$9A = 6$

$\boxed{A = 2/3}$

put $s=2 \Rightarrow B(3)(3) = 4-4+3$

$9B = 3$

$B = 1/3$.

put $s = 1/2 \Rightarrow \boxed{C = -1}$

$A = 2/3 \qquad B = 1/3, \qquad C = -1.$

$$L^{-1}(\quad) = L^{-1}\left(\frac{2}{3}\cdot\frac{1}{s+1}\right) + L^{-1}\left(\frac{1}{3}\,\frac{1}{s-2}\right) + L^{-1}\left(\frac{1}{2s-1}\right)$$

$$= \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t} + \frac{1}{2}e^{t/2}$$

$L^{-1}\left(\frac{1}{2(s-1/2)}\right)$

2) $L^{-1}\left(\frac{3s+1}{(s-1)(s^2+1)}\right)$

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}.$$

$$= \frac{A(s^2+1) + Bs(s-1) + c(s-1)}{(s-1)(s^2+1)}$$

$$3s+1 = A(s^2+1) + Bs(s-1) + c(s-1)$$

Let $s = 1 \Rightarrow 3+1 = A(1+1) \Rightarrow \underline{\underline{A = 2}}.$

Let $s = 0 \Rightarrow 1 = A(1) + C(-1)$

$2 - C = 1 \Rightarrow \underline{\underline{C = 3}}.$

equate the coef. of s^2

$0 = A + B \Rightarrow \underline{\underline{B = -2}}.$

$$L^{-1}(\quad) = L^{-1}\left(\frac{2}{s-1}\right) + L^{-1}\left(\frac{-2s+1}{s^2+1}\right).$$

$$= 2L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left(\frac{-2s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= 2e^{t} - 2\cos t + \sin t$$

3) $L^{-1}\left(\frac{4s+5}{(s-1)^2(s+2)}\right)$

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s+2} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$= \frac{A(s-1)^2 + B(s+2)(s-1) + c(s+2)}{(s+2)(s-1)^2}$$

$$4s+5 = A(s-1)^2 + B(s+2)(s-1) + C(s+2)$$

Let $s = 1 \Rightarrow A = -1/3$

$B = 1/3$

$C = 3$

$$L^{-1}(\;) = L^{-1}\left(-\frac{1}{3}\cdot\frac{1}{s+2}\right) + L^{-1}\left(\frac{1}{3}\cdot\frac{1}{s-1}\right) + L^{-1}\left(3\cdot\right.$$

$$= -\frac{1}{3}e^{2t} + \frac{1}{3}e^{t} + 3te^{t}$$

L

first shift

$L^{-1}\left(\frac{}{s}\right.$

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