Module 2 - Partial Differential Eqns. (PDE)

$z = f(x,y)$ $z \rightarrow$ dependent variable

$x, y \rightarrow$ independent variables

Partial derivatives : $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

PDE: $a_0 \frac{\partial z}{\partial x} + a_1 \frac{\partial z}{\partial y} = a_2 \rightarrow$ partial differential equation because partial derivatives are involved in it.

Order of a PDE: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 1 \rightarrow$ order $= 2$

$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = z \rightarrow$ order $= 1$

$\frac{\partial^3 z}{\partial x^3} + \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \rightarrow$ order $= 3$

Types of PDE

1. Quasi-linear PDE
2. Semi linear (or almost PDE
3. Linear PDE
4. Non-linear PDE

QUASI-LINEAR PDE

An equation of the form

$$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z)$$

where P, Q, R are fns. of independent as well as dependent variable then the eqn is said to quasi-linear PDE.

Eg: $P(z)\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z$ $\qquad P \cdot \frac{\partial z}{\partial x} + Q \cdot \frac{\partial z}{\partial y} = R$

SEMI-LINEAR (OR) ALMOST LINEAR PDE

An equation of the form

$$P(x,y)\frac{\partial z}{\partial x} + Q(x,y)\frac{\partial z}{\partial y} = R(x,y,z)$$

where P, Q are fns. of independent variables only but R can be a fn. of both independent and dependent variables then it is said to semi-linear or almost linear PDE.

Eg: $x.\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xz \rightarrow$ Almost linear

LINEAR PDE

An eqn. of the form

$$P(x,y)\frac{\partial z}{\partial x} + Q(x,y)\frac{\partial z}{\partial y} + R(x,y)z = S(x,y)$$

where P, Q, R, S are fns. of independent variables only then it is said to be a linear PDE

Eg: $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z \rightarrow$ linear PDE

Non-linear PDE

An eqn. which does not fit into the above categories is non-linear

Eg: $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1$

$\Rightarrow \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{\partial z}{\partial y} = 1$

$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = nz \rightarrow$ linear

$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = z^2 \rightarrow$ almost (or semi-linear PDE

1. FORMATION OF PDE

2. SOLUTION OF PDE

FORMATION OF PDE $\quad z = ax + by + c$

1. Eliminating arbitrary constants
2. Eliminating arbitrary fns. $\quad z = f(x) + g(y)$

If the No. of arbitrary constants $\leq$ No. of independent variables

then the resulting PDE will be of order one

If the No. of arbitrary constants $>$ No. of independent variables

then resulting PDE will be of order 2 or more

$z = ax + by$

No. of arbitrary constants = 2

No. of independent variable = 2

The resulting PDE will be of order

No. of arbitrary fns. = order of a PDE

① Form the PDE by eliminating arbitrary constants in $z = ax + by$ — ①

order of a resulting PDE is 1

No. of Arbitrary const. = 2 = No. of independent variables

diff. (1) partially w.r.to x and y

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(ax+by)$$

$$\boxed{\frac{\partial z}{\partial x} = a} \quad — (2)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(ax+by)$$

$$\boxed{\frac{\partial z}{\partial y} = b} \quad — (3)$$

substitute (2) and (3) in (1)

$$\Rightarrow \quad z = ax+by$$

$$\Rightarrow \quad z = \left(\frac{\partial z}{\partial x}\right)\cdot x + \left(\frac{\partial z}{\partial y}\right)\cdot y$$

$$\boxed{z = x\cdot\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y}}$$

<u>Usual notation</u>

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r,$$

$$\frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t$$

(2) Find the PDE by eliminating the arbitrary constants in $z = a(x+y)+b$ — (1)

The resulting PDE will of order 1

diff. (1) partially w.r.to. x

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(a(x+y)+b\right) = \frac{\partial}{\partial x}(ax+ay+b)$$

$$\frac{\partial z}{\partial x} = a \quad \text{— (2)}$$

diff (1) partially w.r.to y

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(a(x+y)+b\right) = \frac{\partial}{\partial y}(ax+ay+b)$$

$$\frac{\partial z}{\partial y} = a \quad \text{— (3)}$$

Equate (2) and (3) $\Rightarrow$ $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$

$\Rightarrow$ $p = q$ $\Rightarrow$ $p - q = 0$ is a PDE of order 1.

3) Find the PDE by eliminating the arbitrary constants a and b from the eqn. $(x-a)^2+(y-b)^2 = z^2 \underbrace{\cot^2\alpha}_{\text{fixed values}}$ — ①

No. of arbitrary constants = 2 = No. of independent variables

∴ The resulting PDE will be of order 1.

diff. ① partially w.r.to 'x'

$$\cancel{2}(x-a) = \cot^2\alpha \cdot \cancel{2}z \cdot \frac{\partial z}{\partial x}$$

$$(x-a) = \cot^2\alpha \cdot z \cdot p \quad \text{— ②}$$

diff. ① partially w.r.to 'y'

$$\cancel{2}(y-b) = \cot^2\alpha \cdot \cancel{2}z \cdot \frac{\partial z}{\partial y}$$

$$(y-b) = \cot^2\alpha \cdot zq \rightarrow ③$$

substitute ② and ③ in ①

$$① \Rightarrow (x-a)^2 + (y-b)^2 = z^2\cot^2\alpha$$

$$(\cot^2\alpha\, zp)^2 + (\cot^2\alpha\, zq)^2 = z^2\cot^2\alpha$$

$$\cot^4\alpha\, z^2p^2 + \cot^4\alpha\, z^2q^2 = z^2\cot^2\alpha$$

$$\cancel{z^2}\cot^{\cancel{4}\,2}\alpha\,(p^2+q^2) = \cancel{z^2}\cot^{\cancel{2}}\alpha$$

$$\cancel{z^2}\cot^4\alpha\,(p^2+q^2) = \cancel{z^2}\cot^2\alpha$$

$$p^2+q^2 = \frac{\cot^2\alpha}{\cot^4\alpha} = \frac{1}{\cot^2\alpha} = \tan^2\alpha$$

$\Rightarrow$ $p^2+q^2 = \tan^2\alpha$ is the required PDE

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \tan^2\alpha.$$

4) Find the PDE of the family of spheres having centres lie on the z-axis.

origin $x^2+y^2+z^2 = r^2$

Eqn of family of spheres is

$$x^2+y^2+(z-c)^2 = r^2 \quad \text{— (1)}$$

where c, r as arbitrary constants.

here centre $(0,0,c)$

The resulting PDE will be of order 1.

diff (1) partially w.r.to 'x'

$$2x + 2(z-c)\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \cancel{2}(z-c)\frac{\partial z}{\partial x} = -\cancel{2}x$$

$$\Rightarrow (z-c) = \frac{-x}{p} \quad \text{— (2)}$$

diff (1) partially w.r.to 'y'

$$2y + 2(z-c)\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \cancel{2}(z-c)\frac{\partial z}{\partial y} = -\cancel{2}y \Rightarrow (z-c) = \frac{-y}{q} \quad \text{— (3)}$$

Equate (2) & (3) $\Rightarrow$ $\cancel{-}\frac{x}{p} = \cancel{-}\frac{y}{q} \Rightarrow qx = py$

$$\Rightarrow \boxed{py - qx = 0}$$

is of order 1.

5) Eliminate the fn. from $z = f(x^2+y^2)$ and form a PDE — ①

Arbitrary fn = 1

The resulting PDE will be of order 1.

diff ① partially w.r.to x

$$\frac{\partial z}{\partial x} = f'(x^2+y^2)\,(2x)$$

$$\frac{p}{2x} = f'(x^2+y^2) \quad — ②$$

diff ① partially w.r.to 'y'

$$\frac{\partial z}{\partial y} = f'(x^2+y^2)\,(2y)$$

$$\Rightarrow \frac{q}{2y} = f'(x^2+y^2) \quad — ③$$

Equate ② & ③ $\Rightarrow$ $\frac{p}{\cancel{2}x} = \frac{q}{\cancel{2}y}$ $\Rightarrow$ $py = qx$

$\boxed{py - qx = 0}$

is of order 1.

6) Form a PDE by eliminating arbitrary fn. from

$$z^2 - xy = f\left(\frac{x}{z}\right) \quad — ①$$

The resulting PDE will be of order 1.

diff ① partially w.r.to 'x'

$$2z\cdot\frac{\partial z}{\partial x} - y = f'\left(\frac{x}{z}\right)\left[\frac{z\cdot 1 - x\cdot\frac{\partial z}{\partial x}}{z^2}\right]$$

$$\Rightarrow 2zp - y = f'\left(\frac{x}{z}\right)\left[\frac{z - xp}{z^2}\right] \quad — ②$$

diff ① partially w.r.to 'y' $\quad$ ① $\Rightarrow z^2 - xy = f\left(\frac{x}{z}\right)$

$$2z \cdot \frac{\partial z}{\partial y} - x = f'\left(\frac{x}{z}\right)\left(-\frac{x}{z^2} \cdot \frac{\partial z}{\partial y}\right)$$

$$2zq - x = f'\left(\frac{x}{z}\right)\left(-\frac{x}{z^2} \cdot q\right) \quad - ③$$

$$② \Rightarrow (2zp - y)\left(\frac{z^2}{z - xp}\right) = f'(x/z)$$

$$③ \Rightarrow (2zq - x)\left(\frac{-z^2}{xq}\right) = f'(x/z)$$

Equate ② & ③ $\Rightarrow$ $(2zp - y)\left(\frac{z^2}{z - xp}\right) = (2zq - x)\left(\frac{-z^2}{xq}\right)$

$$\frac{2zp - y}{z - xp} = \frac{x - 2zq}{xq}$$

$$\Rightarrow (2zp - y)xq = (z - xp)(x - 2zq)$$

$$\Rightarrow 2z\cancel{p}qx - xyq = zx + 2zx\cancel{p}q - x^2p - 2z^2q$$

$$-x^2p + (xy - 2z^2)q = -zx \;//$$

$$\Rightarrow x^2p - (xy - 2z^2)q = xz \qquad Pp + Qq = R$$

↳ Quasi-linear PDE

Resulting PDE will be of the form

$\phi(u,v)=0$

$$Pp + Qq = R$$

$u, v \rightarrow$ dependent variables

$\downarrow$

$x, y, z \rightarrow$ independent variables

where $P = \dfrac{\partial(u,v)}{\partial(y,z)}$, $Q = \dfrac{\partial(u,v)}{\partial(x,z)}$

$u, v \downarrow x, y, z \downarrow x, y.$

$R = \dfrac{\partial(u,v)}{\partial(x,y)}$

$$P = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \frac{\partial(u,v)}{\partial(y,z)}$$

$$Q = \frac{\partial(u,v)}{\partial(x,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial z} \end{vmatrix}$$

$$R = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

If the eqn is given as $\phi(u,v)=0$

$u, v \rightarrow$ dependent variables

$x, y, z \rightarrow$ independent variables

then the resulting PDE will be of the form

$$P \cdot p + Q \cdot q = R \qquad p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}$$

where $P = \dfrac{\partial(u,v)}{\partial(y,z)}$ $Q = \dfrac{\partial(u,v)}{\partial(x,z)}$, $R = \dfrac{\partial(u,v)}{\partial(x,y)}$

Form the PDE by eliminating arbitrary fn ϕ from

$$\phi\left(z^2 - xy, \frac{x}{z}\right) = 0$$

consider $u = z^2 - xy$, $v = \frac{x}{z}$

The resulting PDE will be of the form

$$Pp + Qq = R$$

$$P = \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix}$$

$u = z^2 - xy$

$\frac{\partial u}{\partial y} = -x$, $\frac{\partial u}{\partial z} = 2z$

$$= \begin{vmatrix} -x & 2z \\ 0 & -\frac{x}{z^2} \end{vmatrix}$$

$v = \frac{x}{z}$

$\frac{\partial v}{\partial y} = 0$, $\frac{\partial v}{\partial z} = \frac{-x}{z^2}$

$$P = \frac{x^2}{z^2}$$

$u = z^2 - xy$

$\frac{\partial u}{\partial x} = -y$

$v = \frac{x}{z} \Rightarrow \frac{\partial v}{\partial x} = \frac{1}{z}$

$$Q = \frac{\partial(u,v)}{\partial(x,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -y & 2z \\ 1/z & -x/z^2 \end{vmatrix}$$

$$= \frac{xy}{z^2} - \frac{2z}{z} = \frac{xy}{z^2} - 2$$

$$R = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -y & -x \\ \frac{1}{z} & 0 \end{vmatrix}$$

$$= \frac{x}{z}$$

The resulting PDE is $Pp + Qq = R$

$$\frac{x^2}{z^2} p + \left(\frac{xy}{z^2} - 2\right) q = \frac{x}{z}$$

$$\frac{x^2}{z^2} \cdot \frac{\partial z}{\partial x} + \left(\frac{xy}{z^2} - 2\right) \frac{\partial z}{\partial y} = \frac{x}{z}$$

$$\boxed{x^2 \frac{\partial z}{\partial x} + (xy - 2z^2) \frac{\partial z}{\partial y} = xz} \rightarrow \text{Quasi linear PDE}$$

$\phi(u,v) = 0$

$u = \phi(v)$

$\phi(u, v) = 0$

Assume $u = f(v)$

$$\phi\left(z^2 - xy, \frac{x}{z}\right) \Rightarrow z^2 - xy = \phi\left(\frac{x}{z}\right)$$

2) Form the PDE by eliminating arbitrary fn. from

$$F(\underbrace{x+y+z}_{u}, \underbrace{x^2+y^2+z^2}_{v}) = 0$$

$\phi(u,v) = 0$ can also be written as $u = \phi(v)$

Assume $x + y + z = f(x^2+y^2+z^2)$ — (1)

differentiate (1) partially w.r.to x

$z \rightarrow x, y$

$$1 + 0 + \frac{\partial z}{\partial x} = f'(x^2+y^2+z^2)\left(2x + 2z \cdot \frac{\partial z}{\partial x}\right)$$

$$1 + p = f'(x^2+y^2+z^2)\,(2x + 2zp)$$

$$\Rightarrow \frac{1+p}{2(x+zp)} = f'(x^2+y^2+z^2) \quad \text{— (2)}$$

differentiate (1) partially w.r.to y

$$0 + 1 + \frac{\partial z}{\partial y} = f'(x^2+y^2+z^2)\left(2y + 2z\cdot\frac{\partial z}{\partial y}\right)$$

$$1 + q = f'(x^2+y^2+z^2)\,(2y + 2zq)$$

$$\Rightarrow \frac{1+q}{2(y+zq)} = f'(x^2+y^2+z^2) \quad \text{— (3)}$$

Equating (2) & (3) $\Rightarrow \dfrac{1+p}{\cancel{2}(x+zp)} = \dfrac{1+q}{\cancel{2}(y+zq)}$

$$(1+p)(y+zq) = (1+q)(x+zp)$$

$$y + zq + py + p\cancel{q}z = x + zp + xq + p\cancel{q}z$$

$$py + zq + y - x - zp - xq = 0$$

$$(y-z)p + (z-x)q = x - y \rightarrow$$ First order quasi-linear