Module:5 Fourier Series

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Fourier series - Euler's formulae- Dirichlet's conditions - Change of interval - Half range series – RMS value – Parseval's identity.

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Periodic fn

$f(x+T) = f(x)$ T → the

period. eg, sinx.

$f(x) = \sin x = \sin(x + 2\pi)$

single valued fn::

$x = 4$

$f(x) = x^2 \Rightarrow f(4) = 16$ → single

$f(x) = \sqrt{x} \Rightarrow f(4) = \pm 2$ → not s

$\sqrt{4}$

limits, conti -

piecewise continuous fn:

DIRICHLETS CONDITION

$f(x)$ $c \le x \le c+2l$. → infinite trigonometric ser.

$$\frac{a_0}{2} + \sum \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right)$$

(i) $f(x)$ sd be defined & single valued fn. $(c, c+2l)$

(ii) $f(x)$ cont. (or) piecewise cont. $(c, c+2l)$

(iii) $f(x)$ has no (or) finite no. of max. or minima. in $(c, c+2l)$

Fourier Series

$f(x)$ → periodic & satisfies Dirichlet's cond.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

a_0, a_n, b_n .. → Fourier

coeff.

Euler's formula:

F.S of a fn $f(x)$ in $(c, c+2l)$ is infinite trigonometric Series.

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}}$$

where

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx.$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx, \quad n \geq 0 \text{ (+ve)}$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx. \quad n \geq 1$$

For Interval $(0, 2\pi)$

$(c, c+2l)$: $c \to 0$, $l = \pi$, $(0, 2\pi)$.

$c = 0 \quad l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$\frac{n\cancel{\pi}x}{\cancel{\pi}}$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$(0, 2\pi)$
$(0, 2l)$
$(-\pi, \pi)$
$(-l, l)$

$b_n = \frac{1}{\pi}\int^{2\pi}$

$$\frac{\ \ }{\pi}\int_0 f(x)\sin nx\,dx.$$

In the interval $(0, 2l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx.$$

$(0, 2\pi)$

$(0, 2l)$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right) dx.$$

Expand $f(x) = x$ as a fourier series in the interval $(0, 2\pi)$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left.\frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}(4\pi^2 - 0) = 2\pi$$

2π

$$a_n = \frac{1}{\pi}\int_0 f(x)\ \cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\ \cos nx\, dx.$$

[Bernoulli's formula.

$\int u\, dv = uv_1 - u'v_2 + u''v_3 - \ldots$]

Consider

$$\int_0^{2\pi} x\ \cos nx\ dx.$$

$x \uparrow u$, $\cos nx\, dx \downarrow dv$

$u = x$ $\qquad dv = \cos nx\, dx.$

$u' = 1$ $\qquad v_1 = \frac{\sin nx}{n}$

$v_2 = -\frac{\cos nx}{n^2}$

$$\Rightarrow (x)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\Big]_0^{2\pi}$$

$$2\pi\frac{\sin 2n\pi}{n} + \frac{\cos 2n\pi}{n^2} - \left(0 + \frac{\cos 0}{n^2}\right)$$

$$\boxed{\begin{aligned}\sin 2n\pi &= 0\\ \cos 2n\pi &= 1.\end{aligned}}$$

$$\Rightarrow 2\pi(0) + \frac{1}{n^2} - \frac{1}{n^2} = 0$$

$$a_n = \frac{1}{\pi}(0) = 0 \qquad \boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin\frac{n\pi x}{l}\,dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin nx\,dx. \qquad (uv_1 - u'v_2)$$

$u = x.$ $\qquad dv = \sin nx\,dx$

$u' = 1$ $\qquad v_1 = -\frac{\cos nx}{n}$

$v_2 = -\frac{\sin nx}{n^2}.$

$$\int x\sin nx = \left[(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= 2\pi\left(-\frac{\cos 2n\pi}{n}\right) + \frac{\sin 2n\pi}{n^2} - \left(0 + \frac{\sin}{n}\right.$$

$$= 2\pi\left(-\frac{1}{n}\right) + 0 = -\frac{2\pi}{n}.$$

$$b_n = \frac{1}{\pi}\left[\frac{-2\pi}{n}\right] = \underline{-\frac{2}{n}.}$$

$$a_0 = 2\pi \;;\; a_n = 0 \;;\; b_n = -\frac{2}{n}.$$

$$f(x) = \frac{2\pi}{2} + \sum_{n=1}^{\infty} 0 + \sum_{n=1}^{\infty} \frac{-2}{n}\sin nx.$$

$$f(x) = \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

2) find the F.S of $f(x) = x^2$ in $(0, 4)$

$(0, 4)$

$(0, 2l) \Rightarrow 2l = 4$

$\boxed{l = 2}$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx.$$

$$= \frac{1}{2}\int_0^4 x^2\,dx = \frac{1}{2}\left.\frac{x^3}{3}\right]_0^4 = \frac{64}{6} = \frac{32}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx.$$

$$= \frac{1}{2}\int_0^4 x^2 \cos\frac{n\pi x}{2}\,dx.$$

$u = x^2$ $\quad dv = \cos\frac{n\pi x}{2}\,dx.$

$u' = 2x$ $\quad v_1 = \dfrac{+\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right.$

$u'' = 2$

$$v_2 = \left(\frac{2}{n\pi}\right)^2\left(-\cos\frac{n\pi x}{2}\right)$$

$$v_3 \quad \left(\frac{2}{n\pi}\right)^3 \left(-\sin\frac{n\pi x}{2}\right) = \frac{-8}{n^3\pi^3}$$

$$\int u\,dv = uv_1 - u'v_2 + u''v_3$$

$$= (x^2)\frac{2}{n\pi}\sin\frac{n\pi x}{2} - (2x)\left(\frac{2}{n\pi}\right)^2\left(-\cos\right.$$

$$+ (2)\left(\frac{8}{n^3\pi^3}\sin\left(\frac{n\pi x}{2}\right)\right]$$

$$=$$

$$= 0 + \frac{2(4)(4)}{n^2\pi^2} + 0 \ldots$$

$$= \frac{32}{n^2\pi^2}$$

$$a_n = \frac{1}{2}\left(\frac{32}{n^2\pi^2}\right) = \frac{16}{n^2\pi^2}.$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

$$= \frac{-16}{\quad}$$ check

$n\pi$.

$$a_0 = \frac{32}{3} \qquad a_n = \frac{16}{n^2\pi^2} \qquad b_n = \frac{-16}{n\pi}.$$

$$f(x) = \frac{16}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\cos\left(\frac{n\pi x}{2}\right) - \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{2}\right).$$

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In interval $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

for the interval $(-l, l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx.$$

l

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx.$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx.$$

$(-l, l)$ $(-\pi, \pi)$

even fn, odd fn.

$f(-x) = f(x) \Rightarrow$ even fn.

eg. $\cos x$, x^2.

$f(-x) = -f(x) \Rightarrow$ odd fn.

eg. $\sin x$, x^3.

For $(-\pi, \pi)$

$f(x) \Rightarrow$ even.

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \, dx.$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx.$$

$b_n = 0$ ($\because$ f(x) is an even fn)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

If $f(x) \rightarrow$ odd,

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx.$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx. \quad //.$$

For inter. $(-l, l)$

$f(x)$ is even, $a_0 = \frac{2}{l} \int_0^{l} f(x) dx.$

$$a_n = \frac{2}{l} \int_0^{l} f(x) \cos \frac{n\pi x}{l} \, dx.$$

$$b_n = 0$$

$$f(x) =$$

If $f(x)$ is odd.

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{l} \int_0^{l} f(x) \sin \frac{n\pi x}{l} \, dx.$$

$f(x) =$

Problem.

i) find F. S $f(x) = x^2$ in $-\pi \le x \le \pi$
and hence deduce.

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$.

Gn. fn. $f(x) = x^2$.
$f(-x) = (-x)^2 = x^2 = f(x)$

$\Rightarrow$ even function.
$\Rightarrow b_n = 0$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx.$$

$$= \frac{2}{\pi} \int_0^{\pi} x^2\, dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2\pi^2}{3}.$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \cancel{f(x)}\, \cos nx\, dx.$$

$\pi \quad 0$

$$= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx.$$

$u = x^2 \qquad dv = \cos nx$

$u' = 2x \qquad v_1 = \frac{\sin nx}{n}$

$u'' = 2 \qquad v_2 = -\frac{\cos nx}{n^2}$

$v_3 = -\frac{\sin nx}{n^3}.$

$$a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right.$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi^2}{n}\underbrace{\sin n\pi}_{(0)} + \frac{2\pi}{n^2}\cos n\pi + 0\right) - \left(0+0+0\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{n^2}\cos n\pi\right] = \frac{4}{n^2}\cos n\pi.$$

$\cos n\pi = -1, 1, -1, 1$ ($n=1$, $n=2$, $n=3$)

$\cos n\pi = (-1)^n, \; n = 1, 2, \ldots$

$$a_n = \frac{4}{n^2}(-1)^n$$

$$b_n = 0$$

$$f(x) = \frac{\frac{2\pi^2}{3}}{2} + 4\sum_{n=1}^{\infty} \underbrace{\frac{1}{n^2}(-1)^n}_{a_n} \cos nx.$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n \cos nx \quad \text{—} (1)$$

interval $[-\pi, \pi]$

(i) If $x = 0 \Rightarrow f(x) = f(0) = 0$

sub in (1)

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n \cdot (1)$$

$$4\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n = -\frac{\pi^2}{3}$$

$$\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\cdots = -\frac{\pi^2}{3} \times \frac{1}{4}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\cdots = \frac{\pi^2}{12} \quad //$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

(ii) If $x = \pi$ (cont)

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n$$

$$(-1)^n (-1)^n = (-1)^{2n} = 1$$

$$\frac{2\pi^2}{3} \times \frac{1}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \checkmark$$

(iii) Add (i) & (ii)

$$2\left(\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right) = \frac{\pi^2}{6} + \frac{\pi^2}{12}$$

$$= \frac{3\pi^2}{12}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \cdots = \frac{3\pi^2}{12} \times \frac{1}{2} = \frac{\pi^2}{8}$$

2) $f(x) = \begin{cases} 1+x, & -2 \le x \le 0 \\ 1-x, & 0 \le x \le 2. \end{cases}$

hence deduce $\sum_{1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx.$$

$(-l, l)$
$(-2, 2)$

$$= \frac{1}{2}\left[\int_{-2}^{0} (1+x)\,dx + \int_{0}^{2} (1-x)\,dx\right]$$

$$= \frac{1}{2}\left[\left[x + \frac{x^2}{2}\right]_{-2}^{0} + \left[x - \frac{x^2}{2}\right]_{0}^{2}\right]$$

$= 0$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx.$$

$$= \frac{1}{2}\left[\int_{-2}^{0}(1+x)\cos\left(\frac{n\pi x}{l}\right)dx + \int_{0}^{2}(1-x)\cos\left(\frac{n\pi x}{l}\right.\right.$$

I   II

$$\mathrm{I} \Rightarrow \int_{-2}^{0}(1+x)\cos\left(\frac{n\pi x}{l}\right)dx.$$

$$u = 1+x \qquad dv = \cos\left(\frac{n\pi x}{2}\right).$$

$$u' = 1 \qquad v_1 = \frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} = \frac{2}{n\pi}\sin\left(n\right.$$

$$v_2 = -\left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{2}\right)$$

$$\mathrm{I} = \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$\mathrm{II} = \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$a_n = \frac{1}{2}\left[2\left(\frac{4}{n^2\pi^2}\left(1-(-1)^n\right)\right)\right]$$

$$= \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)$$

$$= \begin{cases} 0, & n \text{ is even.} \\ \frac{8}{n^2\pi^2}. & n \text{ is odd.} \end{cases}$$

Illly. $b_n = 0$. (Calculate)

$$f(x) = \sum_{odd} \frac{8}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right)$$

In deduction. $\sum_{1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

If $x = 0$ → continuity.

$f(0) = 1$

$$\Rightarrow 1 = \sum_{odd} \frac{8}{n^2\pi^2} \cos 0$$

$$\sum_{odd} \frac{1}{n^2} = \frac{\pi^2}{8}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

$$\sum_{odd} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{3^2} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2}$$