<u>Second order linear differential equations with variable coefficients reducible to constant coefficients</u>

- (A) **<u>Cauchy-Euler DE</u> :**

A second order linear differential equation of the form

$x^2y'' + k_1xy' + k_2y = X(x)$ (or) $[x^2D^2 + k_1xD + k_2]y = X(x)$ (DE1)

where $D \equiv \frac{d}{dx}$ and k_1, k_2 are constants, is called Cauchy-Euler DE.

Consider the substitution

$x = e^z$ or $z = \log_e x = \ln(x)$ (1)

Now,

$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$ (by chain rule) $\Rightarrow$ $\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz}$ $\Rightarrow$ $x\frac{dy}{dx} = \frac{dy}{dz}$

Denoting $\theta \equiv \frac{d}{dz}$, we have $xy' = \theta y$ or $xD = \theta$ (2a)

Also,

$\frac{d^2y}{dx^2} = \frac{dy\prime}{dx} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$ (by chain rule)

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} \quad \Rightarrow \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

Hence, we have $x^2y'' = \theta^2y - \theta y = \theta(\theta - 1)y$ or $x^2D^2 = \theta(\theta - 1)$ (2b)

Using (1), (2a) and (2b) in (DE1), we get a differential equation with constant coefficients,

$[\theta(\theta - 1) + k_1\theta + k_2]y = X(e^z)$

which can be solved using the methods already considered.

- (B) **<u>Cauchy-Legendre DE</u> :**

A second order linear differential equation of the form

$(ax+b)^2y'' + k_1(ax+b)y' + k_2y = X(x)$ (or)

$$[(ax+b)^2D^2 + k_1(ax+b)D + k_2]y = X(x) \tag{DE1}$$

where $D \equiv \frac{d}{dx}$ and k_1, k_2 are constants, is called Cauchy-Legendre DE.

Consider the substitution

$$ax + b = e^z \quad \text{or} \quad z = \log_e(ax+b) = \ln(ax+b) \tag{1}$$

so that $x = \frac{e^z - b}{a}$.

Now,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \quad \text{(by chain rule)} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{a}{ax+b}\frac{dy}{dz} \quad \Rightarrow \quad (ax+b)\frac{dy}{dx} = a\frac{dy}{dz}$$

Denoting $\theta \equiv \frac{d}{dz}$, we have $(ax+b)y' = a\theta y$ or $(ax+b)D = a\theta$ (2a)

Also,

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{a}{ax+b}\frac{dy}{dz}\right) = -\frac{a^2}{(ax+b)^2}\frac{dy}{dz} + \frac{a}{ax+b}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{a^2}{(ax+b)^2}\frac{dy}{dz} + \frac{a}{ax+b}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} \quad \text{(using chain rule)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{a^2}{(ax+b)^2}\frac{dy}{dz} + \frac{a^2}{(ax+b)^2}\frac{d^2y}{dz^2} \quad \Rightarrow \quad (ax+b)^2\frac{d^2y}{dx^2} = a^2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

Hence, we have

$$(ax+b)^2\, y'' = a^2(\theta^2 y - \theta y) = a^2\theta(\theta-1)y \quad \text{or} \quad (ax+b)^2D^2 = a^2\theta(\theta-1) \tag{2b}$$

Using (1), (2a) and (2b) in (DE1), we get a differential equation with constant coefficients,

$$[a^2\theta(\theta-1) + k_1a\theta + k_2]y = X\left(\frac{e^z-b}{a}\right)$$

which can be solved using the methods already considered.

<u>Note :</u> By setting $a = 1$ and $b = 0$ in the Cauchy-Legendre equation, we get Cauchy-Euler equation.

Worked Problem 1 : Solve $x^2y'' - xy' + y = \log(x)$

Solution : The given differential equation is a Cauchy-Euler DE.

Let $x = e^z$ so that $z = \log(x)$.

Denoting $\theta \equiv \frac{d}{dz}$, it can be shown that

$xy' = \theta y$ and $x^2y'' = \theta(\theta - 1)y$

Substituting in the given differential equation, we get

$[\theta(\theta - 1) - \theta + 1]y = z$

$\Rightarrow \frac{d^2y}{dz^2} - 2\frac{dy}{dz} + y = z$ ---- (1)

Complementary function of (1) is

$y_c = c_1e^z + c_2ze^z$

By the method of undetermined coefficients, the particular solution of (1) is of the form

$y_p = A_1 + A_2z$

$\therefore$ $y_p' = A_2$ and $y_p'' = 0$

Substituting y_p, y_p' and y_p'' in (1), we get

$-2A_2 + A_1 + A_2z = z$ ---- (2)

Equate the coefficients of the independent functions of (2) as follows.

Coefficient of z : $A_2 = 1$
Constant term : $-2A_2 + A_1 = 0 \Rightarrow A_1 = 2$

Hence, $y_p = 2 + z$. The general solution of (1) is

$$y = y_c + y_p = c_1e^z + c_2ze^z + z + 2$$

Required solution of the given differential equation is

$$y = c_1x + c_2xlogx + logx + 2$$

Worked Problem 2 : Solve $(2x-1)^2y'' + (2x-1)y' - 2y = 8x^2 - 2x + 3$

Solution : The given differential equation is a Cauchy-Legendre DE.

Let $2x - 1 = e^z$ so that $z = \log(2x-1)$ and $x = \frac{e^z+1}{2}$.

Denoting $\theta \equiv \frac{d}{dz}$, it can be shown that

$(2x-1)y' = 2\theta y$ and $(2x-1)^2y'' = 2^2\theta(\theta-1)y = 4\theta(\theta-1)y$

Substituting in the given differential equation, we get

$$[4\theta(\theta-1) + 2\theta - 2]y = 8\left(\frac{e^z+1}{2}\right)^2 - 2\left(\frac{e^z+1}{2}\right) + 3$$

$\Rightarrow 2\frac{d^2y}{dz^2} - \frac{dy}{dz} - y = e^{2z} + \frac{3}{2}e^z + 2$ ---- (1)

Complementary function of (1) is

$y_c = c_1e^z + c_2e^{-z/2}$

By the method of undetermined coefficients, the particular solution of (1) is of the form

$y_p = A_1e^{2z} + B_1ze^z + C_1$
$\therefore\ y_p' = 2A_1e^{2z} + B_1e^z + B_1ze^z$ and $y_p'' = 4A_1e^{2z} + 2B_1e^z + B_1ze^z$

Substituting y_p, y_p' and y_p'' in (1), we get

$2[4A_1e^{2z} + 2B_1e^z + B_1ze^z] - [2A_1e^{2z} + B_1e^z + B_1ze^z] - [A_1e^{2z} + B_1ze^z + C_1] = e^{2z} + \frac{3}{2}e^z + 2$
---- (2)

Equate the coefficients of the independent functions of (2) as follows.
Coefficient of e^{2z} : $8A_1 - 2A_1 - A_1 = 1 \Rightarrow A_1 = \frac{1}{5}$
Coefficient of e^z : $4B_1 - B_1 = 3/2 \Rightarrow B_1 = \frac{1}{2}$
Constant term : $-C_1 = 2 \Rightarrow C_1 = -2$

Hence $y_p = \frac{1}{5}e^{2z} + \frac{1}{2}ze^z - 2$.

The general solution of (1) is

$$y = y_c + y_p = c_1e^z + c_2e^{-z/2} + \frac{1}{5}e^{2z} + \frac{1}{2}ze^z - 2$$

Required solution of the given differential equation is

$$y = c_1(2x-1) + c_2(2x-1)^{-1/2} + \frac{1}{5}(2x-1)^2 + \frac{1}{2}(2x-1)\log(2x-1) - 2$$

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1. Solve the following Cauchy's linear differential equations.

(i)	$x^2y'' + 3xy' + y = \dfrac{1}{(1-x)^2}$
	Ans : $y = \frac{c_1}{x} + \frac{c_2\log(x)}{x} + \frac{1}{x}\log\left(\frac{x}{x-1}\right)$
(ii)	$x^2y'' + xy' + y = logxsin(logx)$
	Ans : $y = c_1\cos(\log x) + c_2\sin(logx) - \frac{1}{4}(logx)^2\cos(logx) + \frac{1}{4}\log(logx)\sin(logx)$
(iii)	$x^2y'' - 3xy' + y = logx\dfrac{\sin(logx)+1}{x}$
	Ans : $y = x^2(c_1x^{\sqrt{3}} + c_2x^{-\sqrt{3}} + \dfrac{1}{x}\left[\dfrac{1}{6}\log(x+1) + \dfrac{logx}{61}\{5\sin(logx) + 6\cos(logx)\} + \dfrac{2}{3721}\{27\sin(logx) + 191\cos(logx)\}\right]$
(iv)	$x^2y'' + 4xy' + 2y = e^x$
	Ans : $y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{e^x}{x^2}$

2. Solve the following Legendre's linear differential equations.

(i)	$(1+x)^2y'' + (1+x)y' + y = 2\sin(\log(1+x))$
	Ans : $y = c_1\cos(\log(1+x)) + c_2\sin(\log(1+x)) - \log(1+x)\ \cos(\log(1+x))$
(ii)	$(1+x)^2y'' + (1+x)y' + y = \sin(2\log(1+x))$
	Ans : $y = c_1\cos(\log(1+\mathrm{x})) + c_2\sin(\log(1+x)) - \frac{1}{3}\sin(2\log(1+x))$
(iii)	$(3x+2)^2y'' + 5(3x+2)y' - 3y = x^2 + x + 1$
	Ans : $y = c_1(3x+2)^{\frac{1}{3}} + c_2(3x+2)^{-1} + \frac{1}{27}\left[\frac{1}{15}(3x+2)^2 + \frac{1}{4}(3x+2) - 7\right]$
(iv)	$(2x+3)^2y'' - (2x+3)y' - 12y = 6x$
	Ans : $y = c_1(2x+3)^a + c_2(2x+3)^b - \frac{3}{14}(2x+3) + \frac{3}{4}$