EQUATIONS REDUCIBLE TO STANDARD FORMS

1. Find the complete integral of $x^2p^2 + y^2q^2 = z^2$

$$\left(x \cdot \frac{\partial z}{\partial x}\right)^2 + \left(y \cdot \frac{\partial z}{\partial y}\right)^2 = z^2 \quad \text{— ①}$$

Put $X = \log x$ $\qquad Y = \log y$

$$\frac{\partial X}{\partial x} = \frac{1}{x} \qquad \frac{\partial Y}{\partial y} = \frac{1}{y}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y}$$

$$\frac{\partial z}{\partial x} = \frac{1}{x} \cdot \frac{\partial z}{\partial X} \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{1}{y}$$

$$\Rightarrow x \cdot \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \qquad \Rightarrow y \cdot \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}$$

substitute the transformed form in the given eqn.

$$① \Rightarrow \left(\frac{\partial z}{\partial X}\right)^2 + \left(\frac{\partial z}{\partial Y}\right)^2 = z^2$$

$$P^2 + Q^2 = z^2 \quad \text{where } P = \frac{\partial z}{\partial X}, \; Q = \frac{\partial z}{\partial Y}$$

It is of the form $f(z, P, Q) = 0$.

Assume $z \to u$ where $u = X + aY$

$$\Rightarrow p = \frac{dz}{du}, \quad Q = \frac{a\,dz}{du}$$

then our eqn. becomes

$$\left(\frac{dz}{du}\right)^2 + \left(\frac{a\,dz}{du}\right)^2 = z^2$$

$$\left(\frac{dz}{du}\right)^2 (1+a^2) = z^2$$

$$\frac{dz}{du}\sqrt{1+a^2} = z$$

$$\frac{dz}{z}\sqrt{1+a^2} = du$$

Integrating $\sqrt{1+a^2}\int \frac{dz}{z} = \int du + c$

$$\Rightarrow \sqrt{1+a^2}\log z = u + c$$

$$\Rightarrow \sqrt{1+a^2}\log z = X + aY + c$$

$$\Rightarrow \sqrt{1+a^2}\log z = \log x + a\log y + c$$

which is the complete integral.

classmate

Date

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2) Solve $z^2(p^2x^2+q^2)=1$

Given eqn. can be reduced to any one of the standard form

$$z^2\left[\left(x\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=1$$

consider $X=\log x$

$$\frac{\partial X}{\partial x}=\frac{1}{x}$$

$$z \to X \to x$$

$$\Rightarrow \frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x}$$

$$\frac{\partial z}{\partial x}=\frac{1}{x}\cdot\frac{\partial z}{\partial X}$$

$$\Rightarrow x\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}$$

$\therefore$ The given equation becomes

$$z^2\left[\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=1$$

$$z^2\left[P^2+q^2\right]=1 \quad \text{where} \quad P=\frac{\partial z}{\partial X}$$

It is of the form $F(z,P,q)=0$

Assume $z \to u$ where $u=X+ay$

$$p = \frac{dz}{du}, \quad q = \frac{a\,dz}{du}$$

$$z^2\left[\left(\frac{dz}{du}\right)^2 + \left(\frac{a\,dz}{du}\right)^2\right] = 1$$

$$z^2\left(\frac{dz}{du}\right)^2(1+a^2) = 1$$

$$\sqrt{1+a^2}\, z\, dz = du$$

$$\int \sqrt{1+a^2}\, z\, dz = \int du + c$$

$$\sqrt{1+a^2}\, \frac{z^2}{2} \pm u + c$$

$$\sqrt{1+a^2}\, \frac{z^2}{2} = x + ay + c$$

$\Rightarrow \sqrt{1+a^2}\, \frac{z^2}{2} = \log x + ay + c$ which is the complete solution.

LINEAR EQUATIONS OF THE FIRST ORDER

A linear partial differential equation of the form $Pp + Qq = R$, where P, Q, R are functions of x, y, z and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$ is called a Lagrange's linear PDE.

P, Q, R are functions x, y, z then it is a quasi-linear equation.
When P, Q, R are independent of z it is known as linear equation.

The solution of this PDE will be of the form $\phi(u,v) = 0$

PROCEDURE FOLLOWED TO SOLVE $Pp + Qq = R$

1) Form the auxiliary equation or Lagrange's subsidiary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

2) Solving the subsidiary equations and find two independent solutions

$$u(x,y,z) = a \quad \text{and} \quad v(x,y,z) = b$$

where a and b are arbitrary constants.

The subsidiary equations are solved by

(I) METHOD OF GROUPING

(II) METHOD OF MULTIPIERS.

3) write the complete solution as $\phi(u,v)=0$ or $u=\phi(v)$ where ϕ is an arbitrary function.

This becomes the general solution.

If $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, then by properties of ratio and proportion,

$$\text{each ratio} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

where l, m, n are constants or suitable functions of x, y, z and they are called Lagrange multipliers

If l, m, n are found such that the denominator $lP + mQ + nR = 0$ then we get

$$l\,dx + m\,dy + n\,dz = 0.$$

Integrating we get $lx + my + nz = a$, where a is an arbitrary constant. This is one solution $u = a$.

Similarly, we can find another independent set of multipliers l_1, m_1, n_1 such that $l_1 P + m_1 Q + n_1 R = 0$, then $l_1 dx + m_1 dy + n_1 dz = 0$

Integrating, we get $l_1 x + m_1 y + n_1 z = b$ where b is an arbitrary constant and it is the second solution $v = b$.

(or) one solution may be obtained by method of grouping.

1) Solve $pyz + qzx = xy$

$$\Rightarrow (yz)p + (zx)q = xy$$

This is of the form $Pp + Qq = R$

where $P = yz$

$Q = zx$

$R = xy$

Subsidiary equations are $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$

Consider $\frac{dx}{\cancel{y}z} = \frac{dz}{x\cancel{y}}$

$$x\,dx = z\,dz$$

$x\,dx - z\,dz = 0$ Integrating, we get

$$\frac{x^2}{2} - \frac{z^2}{2} = a$$

$$\Rightarrow x^2 - z^2 = \underbrace{2a}_{c_1} \Rightarrow x^2 - z^2 = c_1$$

first solution

consider $\frac{d\cancel{z}x}{y\cancel{z}} = \frac{dy}{\cancel{z}x}$

$x\,dx = y\,dy$

$x\,dx - y\,dy = 0$, Integrating, we get

$\frac{x^2}{2} - \frac{y^2}{2} = b$

$\Rightarrow \quad x^2 - y^2 = \underbrace{2b}_{c_2}$

$\Rightarrow x^2 - y^2 = c_2$ is the second Independent solution.

$\therefore$ The general solution is

$$\phi(x^2-y^2, y^2-z^2) = 0$$

(or) $x^2 - y^2 = \phi(y^2 - z^2)$

2) Solve $x(y-z)p + y(z-x)q = z(x-y)$

This is Lagrange's eqn $Pp + Qq = R$

where $P = x(y-z)$

$Q = y(z-x)$

$R = z(x-y)$

The subsidiary equations are

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

First solution

Adding all the denominators

$x(y-z)+y(z-x)+z(x-y)$

$\cancel{xy}-\cancel{zx}+\cancel{yz}-\cancel{xy}+\cancel{zx}-\cancel{zy}=0.$

$$\frac{dx+dy+dz}{x(y-z)+y(z-x)+z(x-y)}=\frac{dx+dy+dz}{0}=c.$$

Integrating, $\boxed{x+y+z=c_1}$

Second solution

$l=\frac{1}{x}, m=\frac{1}{y}, n=\frac{1}{z}$

$$\frac{l\,dx+m\,dy+n\,dz}{l.x(y-z)+m.y(z-x)+n.z(x-y)}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\frac{1}{x}.x(y-z)+\frac{1}{y}y(z-x)+\frac{1}{z}z(x-y)}$$

$$=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y-z+z-x+x-y}=\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}=c$$

$\Rightarrow \log x+\log y+\log z=\log c_2'$

$\Rightarrow \boxed{xyz=c_2}$

$\therefore$ The general solution is

$$\phi(x+y+z,\ xyz) = 0$$