① $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$

Sol: Let $F(s) = \dfrac{s}{s^2+a^2}$ and

$$G(s) = \frac{s}{s^2+b^2}$$

$$L^{-1}[F] = \cos at \quad ; \quad L^{-1}[G] = \cos bt$$

By convolution Theorem for ILT,

$$L^{-1}[FG] = \cos at * \cos bt$$

$$= \int_0^t \cos(au)\cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \left[2\cos au \cos b(t-u)\right]du$$

$$= \frac{1}{2}\int_0^t \left[\cos\left((a-b)u+bt\right) + \cos\left((a+b)u-bt\right)\right]du$$

$$= \frac{1}{2}\left[\frac{\sin\left((a-b)u+bt\right)}{(a-b)} + \frac{\sin\left((a+b)u-bt\right)}{a+b}\right]_{u=0}^{t}$$

$$= \frac{1}{2}\left[\left\{\frac{\sin(at)}{a-b} + \frac{\sin(at)}{a+b}\right\} - \left\{\frac{\sin bt}{a-b} - \frac{\sin bt}{a+b}\right\}\right]$$

$$= \frac{1}{2}\left[\sin at \left[\frac{1}{a-b} + \frac{1}{a+b} \right] - \sin bt \left[\frac{1}{a-b} - \frac{1}{a+b} \right] \right]$$

$$= \frac{1}{2}\left[\frac{2a \sin at}{a^2-b^2} - \frac{2b \sin bt}{a^2-b^2} \right]$$

$$\therefore L^{-1}[FG] = \frac{a \sin at - b \sin bt}{a^2-b^2}$$

Hence

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)} \right] = \frac{a \sin at - b \sin bt}{a^2-b^2}$$

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② Find $L^{-1}\left[\frac{1}{(s-2)(s+2)^2}\right]$

Sol: Let $F(s)=\frac{1}{(s+2)^2}$ and $G(s)=\frac{1}{s-2}$

Now,

$$L^{-1}[G]=e^{2t}$$

$$L^{-1}[F]=L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$=e^{-2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$=te^{-2t}$$

By Convolution Theorem for ILT,

$$L^{-1}[FG]=te^{-2t}*e^{2t}$$

$$=\int_0^t ue^{-2u}e^{2(t-u)}\,du$$

$$=\int_0^t ue^{-4u+2t}\,du$$

$$=e^{2t}\int_0^t ue^{-4u}\,du$$

$$=e^{2t}\left[(u)\left\{\frac{e^{-4u}}{-4}\right\}-(1)\left\{\frac{e^{-4u}}{16}\right\}+(0)\right]_{u=0}^{t}$$

$$= e^{2t}\left[\left\{\frac{-t}{4}e^{-4t} - \frac{1}{16}e^{-4t}\right\} - \left\{\frac{-1}{16}\right\}\right]$$

$$= \frac{-t}{4}e^{-2t} - \frac{1}{16}e^{-2t} + \frac{1}{16}e^{2t}$$

$$= \frac{-t}{4}e^{-2t} + \frac{1}{16}\left[e^{2t} - e^{-2t}\right]$$

$$= \frac{-t}{4}e^{-2t} + \frac{1}{16}(2\sinh 2t)$$

$$\therefore L^{-1}\left[\frac{1}{(s-2)(s+2)^2}\right] = \frac{1}{8}\sinh 2t - \frac{t}{4}e^{-2t}$$

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(3) Find $L^{-1}\left[\dfrac{5s(s+1)}{(s^2+1)(s^2+2s+2)}\right]$

Sol: Let $F(s) = \dfrac{s}{s^2+1}$ and

$$G(s) = \frac{s+1}{s^2+2s+2} = \frac{s+1}{[(s+1)^2-1]+2}$$

$$= \frac{s+1}{(s+1)^2+1}$$

Now, $L^{-1}[F] = \cos t$

$$L^{-1}[G] = L^{-1}\left[\frac{s+1}{(s+1)^2+1}\right] = e^{-t}\cdot\cos t$$

By Convolution Theorem for ILT,

$$L^{-1}[FG] = \cos t * e^{-t}\cos t$$

$$= e^{-t}\cos t * \cos t \qquad [\text{since } f*g = g*f]$$

$$= \int_0^t e^{-u}\cos u\cdot\cos(t-u)\,du$$

$$= \frac{1}{2}\int_0^t e^{-u}\left[2\cos u\cos(t-u)\right]du$$

$$= \frac{1}{2}\int_0^t e^{-u}\left[\cos(t)+\cos(2u-t)\right]du$$

$[\because 2\cos C\cos D = \cos(C+D)+\cos(C-D)]$

$$= \frac{1}{2}\cos t\int_0^t e^{-u}\,du + \frac{1}{2}\int_0^t e^{-u}\cos(2u-t)\,dt$$

$$= \frac{1}{2}\cos t\left[\frac{e^{-u}}{-1}\right]_{u=0}^{t} + \text{''}$$

$$= \frac{-1}{2}\left[e^{-t}-1\right]\cos t + \text{''}$$

$$= \left(\frac{1-e^{-t}}{2}\right)\cos t + \frac{1}{2}\int_0^t e^{-u}\cos(2u-t)\,dt$$

$$= \text{''} + \frac{1}{2}\left[\frac{e^{-u}}{4+1}\left\{-\cos(2u-t)+2\sin(2u-t)\right\}\right]_{u=0}$$

$$= \text{''} + \frac{1}{10}\left[\left\{e^{-t}(-\cos t+2\sin t)\right\} - \left\{-\cos t - 2\sin t\right\}\right]$$

$$= \text{''} + \frac{1}{10}\left[(2\sin t-\cos t)e^{-t} + \cos t + 2\sin t\right]$$

$$= \frac{1}{2}\cos t \checkmark - \frac{1}{2}e^{-t}\cos t \checkmark + \frac{1}{5}\sin t\, e^{-t} - \frac{1}{10}e^{-t}\cos t \checkmark + \frac{1}{10}\cos t \checkmark + \frac{1}{5}\sin t \checkmark$$

$$= \frac{3}{5}\cos t + \frac{1}{5}\sin t - \frac{3}{5}e^{-t}\cos t + \frac{1}{5}e^{-t}\cdot\sin t \checkmark$$

$$= \frac{3}{5}(\cos t)\left[1-e^{-t}\right] + \frac{1}{5}(\sin t)\left(1+e^{-t}\right)$$

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$$\therefore L^{-1}\left[\frac{5s(s+1)}{(s^2+1)(s^2+2s+2)}\right]$$

$$= 5L^{-1}[F(s)]$$

$$= 3\cos t\left[1-e^{-t}\right] + \sin t\left(1+e^{-t}\right)$$

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$$= 3\cos t + \sin t + e^{-t}\left[\sin t - 3\cos t\right]$$

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