2. Apply convolution theorem to evaluate

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)} \cdot \frac{s}{(s^2+b^2)}\right]$$

$$\downarrow \qquad\qquad \downarrow$$

$$F(s) \qquad\qquad G(s)$$

$$f(t) = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$$

$$g(t) = L^{-1}\left[\frac{s}{s^2+b^2}\right] = \sin bt$$

By convolution theorem, we get

$$L^{-1}\left[\frac{s}{(s^2+a^2)}\cdot\frac{s}{(s^2+b^2)}\right] = \int_0^t \cos au \cos b(t-u)\,du$$

$$\because (f*g)(t) = \int_0^t f(u)\,g(t-u)\,du$$

$$\cos A \cos B = \frac{\cos(A-B) + \cos(A+B)}{2}$$

$$\cos au \cos b(t-u) = \frac{\cos(au-bt+bu) + \cos(au+bt-bu)}{2}$$

$$= \frac{\cos((a+b)u - bt) + \cos((a-b)u + bt)}{2}$$

$$\Rightarrow \frac{1}{2}\int_0^t \left\{\cos((a+b)u - bt) + \cos((a-b)u + bt)\right\} du$$

$$= \frac{1}{2}\left[\frac{\sin((a+b)u - bt)}{(a+b)} + \frac{\sin((a-b)u+bt)}{(a-b)}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin at + \sin bt}{(a+b)} + \frac{\sin at - \sin bt}{(a-b)}\right]$$

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$$= \frac{1}{2}\left[\frac{(a-b)(\sin at+\sin bt)+(a+b)(\sin at-\sin bt)}{a^2-b^2}\right]$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2} \quad //$$

PROBLEMS BASED ON UNIT STEP AND UNIT IMPULSE FUNCTION

1. Express the function $f(t) = \begin{cases} 0, & 0<t<1 \\ t-1, & 1<t<2 \\ 1, & t>2 \end{cases}$

in terms of unit step function and find its Laplace transforms.

$$f(t) = (t-1)\left\{u(t-1) - u(t-2)\right\} + 1.\ u(t-2)$$

Unit step fn or Heavisides fn is represented by $u(t-c)$ (or $H(t-c)$)

$$f(t) = (t-1)\,u(t-1) - (t-1)\,u(t-2) + 1 \cdot u(t-2)$$

$$= (t-1)\,u(t-1) - (t-2)\,u(t-2)$$

By second shifting property,

$$L[f(t-a)\,u(t-a)] = e^{-as}\,L[f(t)]$$

$$L[f(t)] = L[(t-1)u(t-1)] - L[(t-2)u(t-2)]$$

$$= e^{-s} L[t] - e^{-2s} L[t]$$

$$= e^{-s} \cdot \frac{1}{s^2} - e^{-2s} \cdot \frac{1}{s^2}$$

$$= \frac{e^{-s} - e^{-2s}}{s^2}$$

2. Find inverse Laplace transform of

$$\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$$

$$\Rightarrow \frac{se^{-s/2}}{s^2+\pi^2}+\frac{\pi e^{-s}}{s^2+\pi^2}$$

$$L^{-1}\left[\frac{se^{-s/2}}{s^2+\pi^2}+\frac{\pi e^{-s}}{s^2+\pi^2}\right]$$

$$= L^{-1}\left[e^{-s/2}\cdot\frac{s}{s^2+\pi^2}\right]+L^{-1}\left[e^{-s}\cdot\frac{\pi}{s^2+\pi^2}\right]$$

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$$= \cos\pi\left(t-\frac{1}{2}\right)u\left(t-\frac{1}{2}\right) + \sin\pi(t-1)\,u(t-1).$$

$$= \sin\pi t \cdot u\left(t-\frac{1}{2}\right) - \sin\pi t \cdot u(t-1)$$

$$= \left[u(t-1/2) - u(t-1)\right]\sin\pi t.$$