3. **change of scale property**

If $L\{f(t)\} = F(s)$, then $L[f(at)] = \frac{1}{a}F(s/a)$

$$L[f(at)] = \frac{1}{a} F(s/a)$$

Problems

1. Find the laplace transform of the following functions

(i) $t^3 + 6t - 7 + 2e^{-6t} + 10 \sin 2t + 7 \cosh 3t$

$$L[t^3 + 6t - 7 + 2e^{-6t} + 10 \sin 2t + 7 \cosh 3t]$$

$$= L[t^3] + 6L[t] - 2L[e^{-6t}] + 10L[\sin 2t] + 7L[\cosh 3t)$$

(By linearity property)

$$= \frac{3!}{s^4} + 6 \cdot \frac{1}{s^2} - 2 \cdot \frac{1}{s+6} + 10 \cdot \frac{2}{s^2+4} + 7 \cdot \frac{s}{s^2-9}$$

$$= \frac{6}{s^4} + \frac{6}{s^2} - \frac{2}{s+6} + \frac{20}{s^2+4} + \frac{7s}{s^2-9}$$

(ii) $L[e^{-t} \sin^2 t]$

$$= L[\sin^2 t]_{s \to s+1}$$ By First shifting property

$$= L\left[\frac{1-\cos 2t}{2}\right]_{s \to s+1}$$

$$= \left\{ L\left[\frac{1}{2}\right] - L\left[\frac{\cos 2t}{2}\right] \right\}_{s \to s+1}$$

$$= \left\{ \frac{1}{2} L[1] - \frac{1}{2} L[\cos 2t] \right\}_{s \to s+1}$$

$$= \left[\frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \frac{s}{s^2+4} \right]_{s \to s+1}$$

$$= \frac{1}{2} \left[\frac{1}{(s+1)} - \frac{(s+1)}{(s+1)^2+4} \right]$$

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(iii) If $L\left[\frac{\sin t}{t}\right] = \tan^{-1}(1/s)$, then prove that

$$L\left[\frac{\sin at}{t}\right] = \tan^{-1}(a/s)$$

WKT the change of scale property is

If $L[f(t)] = F(s)$ then $L[f(at)] = \frac{1}{a} F(s/a)$

here $f(t) = \frac{\sin t}{t}$

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$$f(at) = \frac{\sin at}{at}$$

$$L\left[\frac{\sin at}{at}\right] = \frac{1}{a}\tan^{-1}\left(1/(s/a)\right)$$

$$= \frac{1}{a}\tan^{-1}(a/s)$$

$$\Rightarrow \quad L\left[\frac{\sin at}{t}\right] = \tan^{-1}(a/s)$$

(iii) $L[\cos^3 2t] = L\left\{\frac{1}{4}\left[3\cos 2t + \cos 6t\right]\right\}$

$$= \frac{1}{4}\left[L[3\cos 2t] + L[\cos 6t]\right]$$

$$= \frac{1}{4}\left[3 \cdot \frac{s}{s^2+4} + \frac{s}{s^2+36}\right]$$

$$= \frac{s}{4}\left[\frac{3}{s^2+4} + \frac{1}{s^2+36}\right]$$

(iv) $f(t) = \begin{cases} \cos(t - 2\pi/3) & , t > 2\pi/3 \\ 0 & , t < 2\pi/3 \end{cases}$

$$L[f(t)] = \int_0^{2\pi/3} e^{-st}(0)\,dt + \int_{2\pi/3}^{\infty} e^{-st}\cos(t - 2\pi/3)\,dt$$

$$= \int_{2\pi/3}^{\infty} e^{-st}\cos\left(t - \frac{2\pi}{3}\right)dt$$

Put $t - \frac{2\pi}{3} = y \Rightarrow t = y + \frac{2\pi}{3}$

$dt = dy$

limits $t = 2\pi/3 \Rightarrow y = 0$

$t = \infty \Rightarrow y = \infty$

$$= \int_0^\infty e^{-s\left(y+\frac{2\pi}{3}\right)} \cos y \, dy$$

$$= e^{-\frac{2\pi}{3}s} \int_0^\infty e^{-sy} \cos y \, dy$$

$$= e^{-\frac{2\pi}{3}s} \; L[\cos y]$$

$$= e^{-\frac{2\pi}{3}s} \cdot \frac{s}{s^2+1}$$

$$L[f(t)] = \frac{s e^{-\frac{2\pi}{3}s}}{s^2+1}.$$

TRANSFORMS OF PERIODIC FUNCTIONS

If $f(t)$ is a periodic function with period T

i.e. $f(t+T) = f(t)$, then

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Example

Find the Laplace transform of the periodic function

$$f(t) = \begin{cases} t, & 0<t<1 \\ 2-t, & 1<t<2 \end{cases}$$

with period 2.

WKT

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

here T is a period

$$\Rightarrow L[f(t)] = \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st}\, t\,dt + \int^2 e^{-st}(2-t)\,dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left\{ \left[\left(\frac{t\, e^{-st}}{-s} \right) - \left(\frac{e^{-st}}{s^2} \right) \right]_0^1 + \left[\left(\frac{(2-t)\, e^{-st}}{-s} \right) - \left(\frac{(-1)\, e^{-st}}{s^2} \right) \right]_1^2 \right\}$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{e^{-2s} - 2e^{-s} + 1}{s^2} \right]$$

$$= \frac{1}{s^2} \frac{(1-e^{-s})^2}{(1-e^{-s})(1+e^{-s})}$$

$$= \frac{1}{s^2} \frac{(1-e^{-s})}{(1+e^{-s})}$$

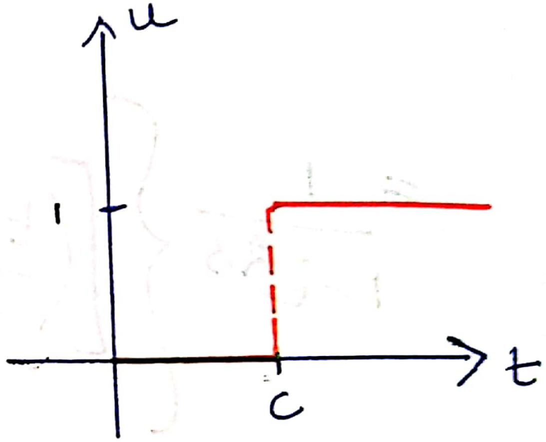
$$= \frac{1}{s^2} \frac{(e^{s/2}-e^{-s/2})}{(e^{s/2}+e^{-s/2})}$$

$$= \frac{1}{s^2} \tanh(s/2)$$

UNIT STEP FUNCTION OR HEAVISIDE FUNCTION

The step function or Heaviside function is defined as

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$$



It is also represented like $u(t-c)$ or $H(t-c)$

$$L[u(t-c)] = \frac{e^{-cs}}{s}$$

|||ly $f(t)\,u(t-c) = \begin{cases} 0 & \text{for } t < c \\ f(t) & \text{for } t \geq c \end{cases}$

If the function $f(t)$ multiplied with Heaviside fn.

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Transform of unit function

$$L\{u(t-c)\} = \int_0^\infty e^{-st}\, u(t-c)\, dt$$

$$= \int_0^c e^{-st}\, 0\, dt + \int_c^\infty e^{-st} \cdot 1\, dt$$

$$= \left(\frac{e^{-st}}{-s}\right)_c^\infty$$

$$\boxed{L\{u(t-c)\} = \frac{e^{-cs}}{s}}$$

The product $f(t)\, u(t-c) = \begin{cases} 0 & \text{for } t < c \\ f(t) & \text{for } t \geq c \end{cases}$

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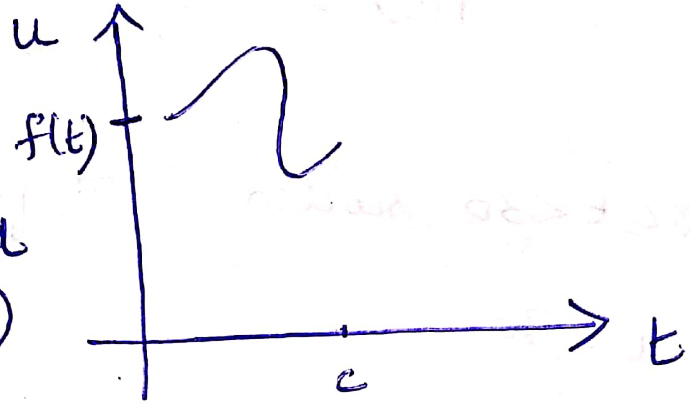
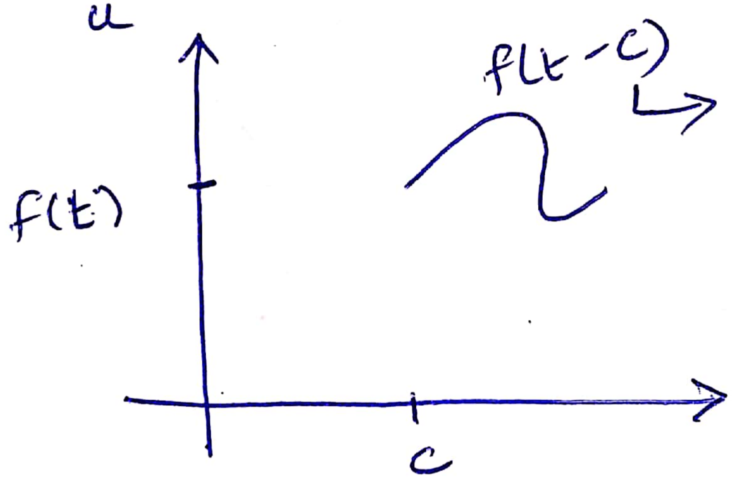
(instead of 1)



$$f(t-c)\,u(t-c) = \begin{cases} 0 & \text{for } t < c \\ f(t-c) & \text{for } t \geq c \end{cases}$$



graph of f(t) shifted through a distance 'c' to the right

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Second shifting property

$$\boxed{L\{f(t-c)\,u(t-c)\} = e^{-cs}F(s)}$$

(i) $L\{f(t-c)\,u(t-c)\} = \int_0^c e^{-st}(0)\,dt + \int_c^\infty e^{-st} f(t-c)\,dt$

$$= \int_c^\infty e^{-st} f(t-c)\,dt$$

Put $t-c=a$

$dt=da$

$t=a+c$

$$= e^{-cs}F(s).$$

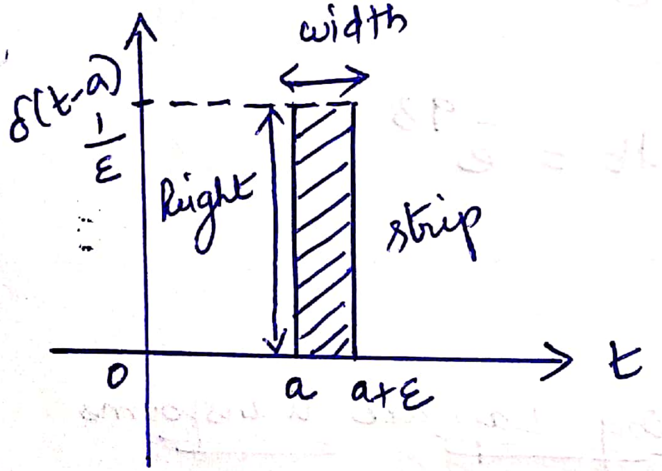
UNIT IMPULSE FUNCTION OR DIRAC DELTA FUNCTION

This kind of function occurs for the case where very large force acting for a very short time.

Thus unit impulse function is considered as the limiting form of the function

$$\delta_{\varepsilon}(t-a) = \begin{cases} 1/\varepsilon, & a \le t \le a+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$\varepsilon \rightarrow$ small negligible quantity



As $\varepsilon \to 0$ the height of the strip increases indefinitely and the width decreases in such away that its area is always unity

Thus the unit impulse fn. $\delta(t-a)$ is defined as follows

$$\delta(t-a) = \begin{cases} \infty & \text{for } t = a \\ 0 & \text{for } t \neq a \end{cases}$$

such that $\displaystyle\int_0^\infty \underbrace{\delta(t-a)\,dt}_{\text{Area}} = 1 \quad (a \geq 0)$

Transform of unit impulse fn.

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$

$$\boxed{L[\delta(t-a)] = e^{-as}}$$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[\delta(t-a)] = \int_0^\infty \underbrace{e^{-st}}_{f(t)} \delta(t-a)\,dt$$

$$= e^{-as}$$

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