MAT2002 – APPLICATIONS OF DIFFERENTIAL AND DIFFERENCE EQUATIONS
Module 3 – Lecture Notes

METHOD OF UNDETERMINED COEFFICIENTS

The method of undetermined coefficients can be used to find the particular integral of a linear differential equation with constant coefficients. But this method is applicable only to a limited class of functions, called UC functions.

UC function : A function is a UC function if it is either

(i) a function defined by one of the following four types

- (a) $a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$, where n is a positive integer or zero ;
- (b) e^{ax}, where $a \neq 0$ is a constant ;
- (c) $\sin(bx+c)$, where $b \neq 0$ and c are constants ; and
- (d) $\cos(bx+c)$, where $b \neq 0$ and c are constants, or

(ii) a function defined as a finite product of two or more functions of these four types.

Example1 : The functions $3x^2 + 2x - 1$, $x^2 \sin x$, $xe^{2x} \cos x$, $xe^{2x} \sin^2 x \cos 2x$ are UC functions.

UC set : Given a UC function f, each successive derivative of f is either a constant multiple of itself or a linear combination of linearly independent UC functions. The set of functions consisting of f itself and all linearly independent UC functions is called the UC set of f .

Example2 : Let $f(x) = x^2 \sin x$. The successive derivatives of f are given by $f^{'}(x) = x^2 \cos x + 2x \sin x$, $f^{''}(x) = -x^2 \sin x + 4x \cos x + 2 \sin x$, $f^{'''}(x) = -x^2 \cos x - 2x \sin x - 4x \sin x + 6 \cos x$ and so on. The UC set of f is $S = \left\{ x^2 \sin x, x^2 \cos x, x \sin x, x \cos x, \sin x, \cos x \right\}$.

MAT2002 – APPLICATIONS OF DIFFERENTIAL AND DIFFERENCE EQUATIONS
Module 3 – Lecture Notes

Table of UC functions and their UC sets

S.No.	UC function	UC set
1	$a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$	$\{x^n, x^{n-1}, ..., x, 1\}$
2	e^{ax}	$\{e^{ax}\}$
3	$\sin(ax+b)$ **or** $\cos(ax+b)$	$\{\sin(ax+b), \cos(ax+b)\}$
4	$f\ g$	**UC set of** f $\times$ **UC set of** g
5	$x^2 \sin x$	$\{x^2, x, 1\} \times \{\sin x, \cos x\}$ $= \{x^2 \sin x, x^2 \cos x, x \sin x, x \cos x, \sin x, \cos x\}$

METHOD OF UNDETERMINED COEFFICIENTS

Problem : Find the particular solution of an nth order linear differential equation with constant coefficients

$$[a_0D^n + a_1D^{n-1} + ... + a_{n-1}D + a_n]\ y = X(x), \qquad [1]$$

where $X(x) = A_1 f_1(x) + A_2 f_2(x) + ... A_k f_k(x)$ is a linear combination of UC functions $f_1, f_2, ..., f_k$

Step1 : Obtain the complementary function $y_c(x)$ of the given differential equation.

Step2 : To seek a particular (or trial) solution, $y_p(x)$, proceed as follows.

- For each of the UC functions in $X(x)$, form the UC sets, say, $S_1, S_2, ..., S_k$.
  From these UC sets, obtain those UC sets that are mutually disjoint. [i.e., If $S_i \subseteq S_j$ then remove smaller set S_i and retain larger set S_j.] Suppose there are 'm' UC sets that are mutually disjoint.
- Check whether each of the 'm' UC set contain UC members of the complementary function or not. [If the UC set S_ℓ includes the members of the complementary function, obtain the revised UC set $S_\ell^{'}$, that doesn't contain any UC member of the complimentary function, by multiplying the set S_ℓ with the lowest positive integral power of x.].
- Now we have 'm' UC sets (including the revised UC set) that are mutually disjoint. The particular integral $y_p(x)$ is chosen to be the linear combination of the members of the 'm' UC sets.

Step3: Determine the unknown coefficients by substituting $y_p(x)$ in the differential equation.

Step4: General solution of the given differential equation is $y(x) = y_c(x) + y_p(x)$.

Worked Example1 :

Solve the differential equation

$y'' - 2y' - 3y = 2e^x - 10sinx.$

Solution.

Step1: To find the complementary function

The auxiliary equation of the associated homogeneous equation is $m^2 - 2m - 3 = 0 \Rightarrow m = -1,3$

Hence the complementary function is

$y_c(x) = c_1e^{-x} + c_2e^{3x}$, where c_1 and c_2 are arbitrary constants.

Step2: To find the particular solution by the method of undetermined coefficients.

Let the particular solution be

$y_p(x) = Ae^x + B_1sinx + B_2cosx$, where A, B_1, B_2 are the coefficients to be determined.

$y_p'(x) = Ae^x + B_1cosx - B_2sinx$ and $y_p''(x) = Ae^x - B_1sinx - B_2cosx$.

As $y_p(x)$ satisfies the given DE, we have

$[Ae^x - B_1sinx - B_2cosx] - 2[Ae^x + B_1cosx - B_2sinx] - 3[Ae^x + B_1sinx + B_2cosx] = 2e^x - 10sinx$

To determine the coefficients, equate the coefficients of UC functions as follows:

e^x: $A - 2A - 3A = 2 \quad \Rightarrow \quad A = -\frac{1}{2}$

$sinx$: $-B_1 + 2B_2 - 3B_1 = -10 \quad \Rightarrow \quad -2B_1 + B_2 = -5$

$cosx$: $-B_2 - 2B_1 - 3B_2 = 0 \quad \Rightarrow \quad -2B_1 - 4B_2 = 0$

On solving, we get $B_2 = -1$ and $B_1 = 2$.

Hence the particular solution is $y_p(x) = -\frac{1}{2}e^x + 2sinx - cosx$

Thus the general solution of the given DE is

$y(x) = y_c(x) + y_p(x) = c_1e^{-x} + c_2e^{3x} - \frac{1}{2}e^x + 2sinx - cosx$

Worked Example2 :

Find the complete solution of the differential equation

$y'' - 3y' + 2y = 2x^2 + (1+2x)e^x + 4e^{3x}$.

Solution.

Step1: To find the complementary function

The auxiliary equation of the associated homogeneous equation is $m^2 - 3m + 2 = 0 \Rightarrow m = 1,2$

Hence the complementary function is

$y_c(x) = c_1e^x + c_2e^{2x}$, where c_1 and c_2 are arbitrary constants.

Step2: To find the particular solution by the method of undetermined coefficients.

Let the particular solution be

$y_p(x) = [Ax^2 + Bx + C] + [De^{3x}] + [Ex^2e^x + Fxe^x]$, where A, B, C, D, E, F are the coefficients to be determined.

$y_p'(x) = 2Ax + B + 3De^{3x} + E[2xe^x + x^2e^x] + F[e^x + xe^x]$

$y_p''(x) = 2A + 9De^{3x} + E[2e^x + 4xe^x + x^2e^x] + F[2e^x + xe^x]$

As $y_p(x)$ satisfies the given DE, we have

$$\{2A + 9De^{3x} + E[2e^x + 4xe^x + x^2e^x] + F[2e^x + xe^x]\}$$
$$-3\{2Ax + B + 3De^{3x} + E[2xe^x + x^2e^x] + F[e^x + xe^x]\}$$
$$+2\{[Ax^2 + Bx + C] + [De^{3x}] + [Ex^2e^x + Fxe^x]\} = 2x^2 + (1+2x)e^x + 4e^{3x}$$

To determine the coefficients, equate the coefficients of UC functions as follows:

$x^2\colon\ 2A = 2 \quad\Rightarrow\quad A = 1$

$x\colon\ -6A + 2B = 0 \quad\Rightarrow\quad B = 3$

$1\colon 2A - 3B + 2C = 0 \quad\Rightarrow\quad C = \dfrac{7}{2}$

$e^{3x}\colon 9D - 9D + 2D = 4 \quad\Rightarrow\quad D = 2$

$xe^x\colon 4E + F - 6E - 3F + 2F = 2 \quad\Rightarrow\quad E = -1$

$e^x\colon 2E + 2F - 3F = 1 \quad\Rightarrow\quad F = -3$

Hence, the particular solution is $y_p(x) = x^2 + 3x + \frac{7}{2} + 2e^{3x} - x^2e^x - 3xe^x$

Thus the general solution of the given DE is

$$y(x) = y_c(x) + y_p(x) = c_1 e^x + c_2 e^{2x} + x^2 + 3x + \frac{7}{2} + 2e^{3x} - x^2 e^x - 3xe^x$$

MAT2002 – APPLICATIONS OF DIFFERENTIAL AND DIFFERENCE EQUATIONS
Module 3 – Lecture Notes

Problems for practice

1. Set up the particular solution by the method of undetermined coefficients and hence determine the particular solution for the following differential equations.
   - $y'' + 6y' + 13y = xe^{-3x}sin2x$
   - $y'' + 4y' + 5y = e^{-2x}(1 + cosx)$
   - $y'' + 9y = 2sinh3x + e^{3x}sin3x$
   - $y'' - 6y' + 8y = x^3 + x + e^{-2x}$

2. Set the following differential equations using the method of undetermined coefficients.
   - $y'' - 2y' - 3y = 2e^x - 10sinx; \quad y(0) = 0; \quad y'(0) = 4$
   - $y'' - 3y' + 2y = 2x^2 + e^x + 2xe^x$