1) Solve the Simultaneous equation

$$\frac{dx}{dt} - y = e^t \quad \text{—} ①$$

$$\frac{dy}{dt} + x = \sin t \quad \text{—} ②$$

using the Laplace transform method which satisfies the condition $x(0) = 1$, $y(0) = 0$

consider ① $\quad \frac{dx}{dt} - y = e^t$

Take Laplace transform on both sides

$$L[x'(t)] - L[y(t)] = L[e^t]$$

$$\{s L[x(t)] - x(0)\} - L[y(t)] = \frac{1}{s-1}$$

$$s X(s) - x(0) - Y(s) = \frac{1}{s-1}$$

$$s X(s) - Y(s) = \frac{1}{s-1} + 1$$

$$s X(s) - Y(s) = \frac{s}{s-1} \quad \text{—} (1*)$$

consider (2) $y'(t) + x(t) = \sin t$

Take Laplace transform on both sides

$$L[y'(t)] + L[x(t)] = L[\sin t]$$

$$s\,Y(s) - \cancel{y(0)}^{0} + X(s) = \frac{1}{s^2+1}$$

$$X(s) + s\,Y(s) = \frac{1}{s^2+1} \quad \text{— } (2*)$$

$$(1*) \times s \Rightarrow s^2 X(s) - s\,Y(s) = \frac{s^2}{s-1}$$

$$(1+s^2)X(s) = \frac{1}{s^2+1} + \frac{s^2}{s-1}$$

$$= \frac{s-1+s^2(s^2+1)}{(s-1)(s^2+1)}$$

$$= \frac{s^4+s^2+s-1}{(s-1)(s^2+1)}$$

$$X(s) = \frac{s^4+s^2+s-1}{(s-1)(s^2+1)^2}$$

Consider $\dfrac{s^4+s^2+s-1}{(s-1)(s^2+1)^2} = \dfrac{A}{(s-1)} + \dfrac{Bs+C}{(s^2+1)}$

$+ \dfrac{Ds+E}{(s^2+1)^2}$

$s^4+s^2+s-1 = A(s^2+1)(s^2+1) + (Bs+C)(s-1)(s^2+1)$

$+ (Ds+E)(s-1)$

Put $s=1$

$2 = 4A \Rightarrow \boxed{A=\frac{1}{2}}$

compare the coefficient of s^4

$1 = A+B \Rightarrow B = 1-\frac{1}{2}$

$\boxed{B=\frac{1}{2}}$

compare the coefficient of s^3

$0 = -B+C \Rightarrow \boxed{C=\frac{1}{2}}$

compare the coefficient of s^2

$1 = 2A+B-C+D$

$1 = \cancel{2}\left(\frac{1}{\cancel{2}}\right)+\cancel{\frac{1}{2}}-\cancel{\frac{1}{2}}+D \Rightarrow \boxed{D=0}$

compare the coefficient of 's'

$$1 = -B + C - D + E$$

$$1 = -\tfrac{1}{2} + \tfrac{1}{2} - 0 + E$$

$$\Rightarrow \boxed{E = 1}$$

$A = \frac{1}{2}$

$B = \frac{1}{2}$

$C = \frac{1}{2}$

$D = 0$

$E = 1$

$$\frac{s^4 + s^2 + s - 1}{(s-1)(s^2+1)^2} = \frac{1}{2(s-1)} + \frac{s}{2(s^2+1)} + \frac{1}{2(s^2+1)} + \frac{1}{(s^2+1)^2}$$

$$x(t) = L^{-1}\left[\frac{s^4 + s^2 + s - 1}{(s-1)(s^2+1)^2}\right] = \frac{1}{2}L^{-1}\left[\frac{1}{(s-1)}\right] + \frac{1}{2}L^{-1}\left[\frac{s}{s^2+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s^2+1}\right] + L^{-1}\left[\frac{1}{(s^2+1)^2}\right]$$

$$= \frac{1}{2}e^t + \frac{1}{2}\cos t + \frac{1}{2}\sin t + L^{-1}\left[\frac{1}{(s^2+1)^2}\right]$$

$$L^{-1}\left[\frac{1}{(s^2+1)^2}\right] = \sin t - t\cos t$$

$$\Rightarrow x(t) = \frac{1}{2}\left[e^t + \cos t + \sin t + (\sin t - t\cos t)\right]$$

$$L^{-1}\left[\frac{1}{(s^2+1)^2}\right] = L^{-1}\left[\frac{1}{(s^2+1)}\cdot\frac{1}{(s^2+1)}\right]$$

$$= \sin t * \sin t$$

$f(u) = \sin u$

$g(t-u) = \sin(t-u)$

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \sin u \,\sin(t-u)\, du$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$= \frac{1}{2}\left[\cos(u-t+u) - \cos(u+t-u)\right]$$

$$= \frac{1}{2}\left(\cos\overset{(2u+}{t)} - \cos t\right)$$

$$\int_0^{\cancel{k}} \left(\frac{1}{2}\cos(2u \cancel{*} t) - \frac{1}{2}\cos t\right) du = \frac{1}{2}\left[\int_0^t \cos(2u+t)\, du - \int_0^t \cos t\, du\right]$$

$$= \frac{1}{2}\left[\frac{\sin(2u \cancel{*} t)}{2} - \cos t . u\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin \cancel{t}}{2} - t\cos t - \left(\frac{\sin t}{2}\right)\right] = \frac{1}{2}\left[\sin t - t\cos t\right]$$

$$① \Rightarrow \frac{dx}{dt} - y = e^t$$

$$\frac{d}{dt}\left(\frac{1}{2}\left(e^t + \cos t + \sin t + (\sin t - t\cos t)\right)\right)$$

$$- y = e^t$$

$$\Rightarrow y(t) = \frac{1}{2}\left(e^t - \cancel{\cos t} + \cancel{\cos t} + \cos t - (-t\sin t + \cancel{\cos t})\right)$$

$$- e^t$$

$$= \frac{1}{2}\left(e^t + t\sin t\right) - e^t$$

$$\frac{1}{2}\frac{d}{dt}\left(e^t + \cos t + \sin t + \sin t - t\cos t\right) - e^t = y$$

$$\frac{1}{2}\left(e^t - \sin t + \cos t + \cos t - (t(-\sin t) + \cos t)\right) - e^t = y$$

$$\frac{1}{2}\left(e^t - \sin t + \cos t + \cancel{\cos t} + t\sin t - \cancel{\cos t}\right) - e^t = y$$

$$y = \frac{1}{2}\left(t\sin t - e^t + \cos t - \sin t\right)$$

Laplace transformation for PDE

Given a function $u(x,t)$ defined for all $t>0$ and assumed to be bounded then we can apply Laplace transform in t considering x as a parameter

$$L[u(x,t)] = \int_0^\infty e^{-st} u(x,t)\,dt$$

$$= U(x,s)$$

$$L[u_t(x,t)] = \int_0^\infty e^{-st} u_t(x,t)\,dt$$

By Leibnitz's rule.

$$\left[u(x,t)\,e^{-st}\right]_0^\infty - \int_0^\infty (-s)\,e^{-st} u(x,t)\,dt$$

$$= \left[e^{-st} u(x,t)\right]_0^\infty + s\int_0^\infty e^{-st} u(x,t)\,dt$$

$$= -u(x,0) + s\,U(x,s)$$

$$L[u_t(x,t)] = s\,U(x,s) - u(x,0)$$

|||ly

$$L[u_{tt}(x,t)] = s^2 U(x,s) - s\,u(x,0) - u_t(x,0)$$

Transforms of x derivatives

$$L[u_x(x,t)] = \int_0^{\infty} e^{-st} u_x(x,t)\,dt$$

$$= U_x(x,s)$$

$$L[u_{xx}(x,t)] = \int_0^{\infty} e^{-st} u_{xx}(x,t)\,dt$$

$$= U_{xx}(x,s)$$

PROBLEMS

Solve $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = x, \quad x>0, t>0$

with initial and boundary condition

$u(0,t) = 0, \ t>0 \rightarrow$

and $u(x,0) = 0, \ x>0 \rightarrow$ I.C

Taking L.T

$$L\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right] = L[x]$$

$$L\left[\frac{\partial u}{\partial x}\right] + L\left[\frac{\partial u}{\partial t}\right] = x\,L[1]$$

$$\frac{dU(x,s)}{dx} + sU(x,s) - \underset{\downarrow \atop 0}{u(x,0)} = \frac{x}{s}$$

$$\Rightarrow \quad \frac{dU}{dx} + sU = \frac{x}{s}$$

It is of the form $\frac{dy}{dx} + P(x)y = Q(x)$.

soln is $\quad y e^{\int P dx} = \int Q e^{\int P dx} dx$

here $e^{\int P dx}$ = integrating factor

here $P = s$, $\quad Q = \frac{x}{s}$

$$U e^{\int s dx} = \int \frac{x}{s} e^{\int s dx} + c$$

$$U e^{sx} = \frac{1}{s}\left[\frac{x e^{sx}}{s} - \frac{e^{sx}}{s^2}\right] + c$$

$$U = \frac{e^{-sx}}{s}\left(\frac{x e^{sx}}{s} - \frac{e^{sx}}{s^2}\right) + c e^{-sx}$$

where c is an arbitrary constant.

Apply the condition $u(0,t) = 0$

$$L[u(0,t)] = U(0,s) = 0.$$

$$0 = \frac{e^{0x}}{s}\left(0 - \frac{1}{s^2}\right) + c$$

$$\Rightarrow \boxed{c = \frac{1}{s^3}}$$

$$\therefore \quad U(x,s) = \frac{x}{s^2} - \frac{1}{s^3} + \frac{e^{-sx}}{s^3}$$

Take inverse Laplace transform

$$L^{-1}[U(x,s)] = x L^{-1}[1/s^2] - L^{-1}[1/s^3] + L^{-1}\left[\frac{e^{-xs}}{s^3}\right]$$

$$= xt - \frac{t^2}{2} + \frac{(t-x)^2}{2} U(t-x)$$

By second shifting property.

$$\therefore \quad u(x,t) = xt - \frac{t^2}{2} + \frac{(t-x)^2}{2} \underset{\text{unit step fn.}}{\underbrace{U(t-x)}}$$

2) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + u = 0, \quad x > 0, \quad t > 0$

with boundary and initial condition

$u(0,t) = 0, \quad t > 0$

$u(x,0) = \sin x, \quad x > 0.$

$$L\left[\frac{\partial u}{\partial x}\right] + L\left[\frac{\partial u}{\partial t}\right] + L\left[u(x,t)\right] = 0$$

$$\frac{dU(x,s)}{dx} + s\,U(x,s) - u(x,0) + U(x,s) = 0$$

$$\frac{dU(x,s)}{dx} + (s+1)\,U(x,s) = \sin x.$$

$I.F = (s+1)$

$P(x)$

$$U e^{\int(s+1)dx} = \int \sin x \, e^{\int(s+1)dx} dx + c$$

$$U e^{(s+1)x} = \int e^{(s+1)x} \sin x \, dx + c$$

$$U e^{(s+1)x} = \frac{e^{(s+1)x}}{(s+1)^2+1^2}\left((s+1)\sin x - \cos x\right) + c$$

$$U = \frac{(s+1)(\sin x - \cos x)}{s^2+2s+2} + c e^{-(s+1)x}$$

$0 = U(0,s)$

$\Rightarrow \quad U(0,s) = 0 = \dfrac{-1}{s^2+2s+2} + c$

$$\Rightarrow \quad \boxed{c = \frac{1}{s^2+2s+2}}$$

$$U(x,s) = \frac{(s+1)\sin(x)}{(s+1)^2+1} - \frac{\cos x}{(s+1)^2+1} + \frac{e^{-(s+1)x}}{(s+1)^2+1}$$

Taking the inverse Laplace transform,

$$u(x,t) = e^{-t}\sin x \cdot \cos t - e^{-t}\cos x \sin t + e^{-t}U(t-x)\sin(t-x)$$

$$= e^{-t}\left[(\sin x\cos t - \cos x\sin t) + U(t-x)\sin(t-x)\right]$$

$$= e^{-t}\left[\sin(x-t) + U(t-x)\sin(t-x)\right] \quad //$$