20.3.23

Basis and Dimension :

Basis : A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is a basis of V, if the vectors are linearly independent and set spans vector space V.

Eg 1. $\{(1,0,0), (0,1,0), (0,0,1)\}$ $\mathbb{R}^3$

$\rightarrow$ $c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$

$$c_1 = c_2 = c_3 = 0$$

$\therefore$ The vectors are Linearly Independent.

$\rightarrow$ Let $(x, y, z) \in \mathbb{R}^3$

$$(x, y, z) = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$$

$$c_1 = x, \quad c_2 = y, \quad c_3 = z.$$

Ex. $\left(\frac{1}{2}, 5, 7, 5\right)$

$$c_1 = 1/2, \quad c_2 = 5, \quad c_3 = 7.5$$

Eg. 2. $(1,1,0)\ (0,-1,1)\ (1,0,1)$, verify if S is a basis of $\mathbb{R}^3$.

Eg. 3. $\{(1,1,1), (0,1,1), (0,0,1)\}$

Eg. 2.

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1(-1) - 1(0-1) = -1+1 = 0$$

$$c_1(1,1,0) + c_2(0,-1,1) + c_3(1,0,1) = (0,0,0)$$

$$(c_1 + c_3,\ c_1 - c_2,\ c_2 + c_3) = (0,0,0)$$

$c_1 = -c_3$

$c_1 = c_2$

$c_2 = -c_3$

$c_1 = 1 \quad c_2 = 1 \quad c_3 = -1$

Since, vectors are L.D. it cannot form bases.

Eg. 3.

$$c_1(1,1,1) + c_2(0,1,1) + c_3(0,0,1) = (0,0,0)$$

$$c_1 = c_2 = c_3 \equiv 0$$

Span:

Let $(x, y, z) \in \mathbb{R}^3$

$$(x,y,z) = c_1(1,1,1) + c_2(0,1,1) + c_3(0,0,1)$$

$$c_1 = x, \quad c_1 + c_2 = y \Rightarrow c_2 = y - c_1 = y - x$$

$$c_1 + c_2 + c_3 = z$$

$$\Rightarrow c_3 = z - c_1 - c_2 = z - x - (y - x)$$

$$= z - x - y + x$$

$$= z - y$$

(4) Verify

If $\{6x-3,\ 3x^2,\ 1-2x-x^2\}$ forms bases of P_2

$$\begin{vmatrix} 6x-3 & 3x^2 & 1-2x-x^2 \\ 6 & 6x & -2-2x \\ 0 & 6 & -2 \end{vmatrix} = 0 \Rightarrow L \cdot D$$

It cannot form a bases.

Note :

1. If one vector is removed from bases they can't form bases. Since, it cannot span V.

2. If a vector is added to the basis, the set becomes linearly dependent, hence, can't form basis.

3. Basis of a vector space need not be unique.

4. No. of Elements in any basis for a given vector space will be same.

Dimension :

Dimension of a vector space B is the number of elements in its basis and is denoted by dim V.

$\dim \mathbb{R}^3 = 3$

$\dim \mathbb{P}_2 = 3$ $\quad \{1, x, x^2\}$: $5x^2 + 9x + 11$

$= 5(x^2) + 9(x) + 11(1)$

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Basis and dimension of

(i) $y'' - y = 0$

(ii) $y'' + y = 0$

~~Def~~ :

Finite dimensional vector space.

Solution :

(i) Aux. Eqn : $m^2 - 1 = 0$

$m = \pm 1$ (Roots are real and distinct)

$y_c = c_1 e^{x} + c_2 e^{-x}$

Basis = $\{e^{x}, e^{-x}\}$; Dimension = 2

~~Complementary Functions / General solutions~~ ⇒

(ii) Aux. Eqn : $m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

$y_c = \cancel{c_1 e^{x} +} e^{0x} (c_1 \cos x + c_2 \sin x)$

Basis : $\{\cos x, \sin x\}$

Dimension : 2

(1) Real and distinct :

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

(2) Real and equal :

$y_c = c_1 e^{mx} + c_2 x e^{mx}$

(3) complex Conjugate :

$y_c = e^{ax} (c_1 \cos bx + c_2 \sin bx)$

Finite Dimension Vector Space:

A vector space whose dimension is finite is called finite dimensional vector space otherwise the vector space is infinite dimension.

Eg:

Finite:

Dimension of $\mathbb{R}_n$ is n

" " $\mathbb{P}_n$ is $n+1$

" " $M_{m \times n}$ is $m \cdot n$

$$M_{2\times 2} = \left\{ \underset{M_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \underset{M_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 3 \\ 5 & 9 \end{bmatrix} = \left\{ M_1 + 3M_2 + 5M_3 + 9M_4 \right\}$$

Infinite:

Set of all polynomials, set of all matrices.

Qs: Find basis and dimension of $x + 2z = 0$ in $\mathbb{R}_3$.

$x = -2z$, y can be anything.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2z \\ y \\ z \end{bmatrix} = z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Basis $= \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ Dimension is 2.

2) Find basis and dimension of $x + 2z = 0$ & $x + y = 0$

$$\cancel{x} + 2z = 0$$
$$\cancel{x} + y = 0$$
$$\underline{\quad - \quad\quad - \quad}$$
$$2z - y = 0$$

$2z = y \Rightarrow \cancel{z = y/2}$

$x = -2z$ $\cancel{x = 2z}$ $\Rightarrow$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

Basis $= \{(-2, 2, 1)\}$ (Basis is not supposed to be unique)

Dimension $= 1$

3) $\{(a, b, c, d) \mid a = 2b, c = d\}$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2b \\ b \\ c \\ c \end{bmatrix} = b \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Basis $= \{(+2, 1, 0, 0), (0, 0, 1, 1)\}$

Dimension $= 2$.

4)

$$2x + y + 2z = 0 \Rightarrow y = -2x - 2z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -2x-2z \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Basis $= \{(1,-2,0)\ (0,-2,1)$

Dimension $= 2$.

Row space & Column space of a matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 5 & 7 & 9 \end{bmatrix}$$

Row Space $R(A) = \{c_1(1,2,3) + c_2(2,4,6) + c_3(5,7,9)\}$

Column space $C(A)$

$= \{c_1(1,2,5) + c_2(2,4,7) + c_3(3,6,9)\}$

Row Space of Matrix A:

A ~~space~~ subspace $\underset{\text{linear combinations}}{\text{spanned}}$ by Rows of matrix A.

$$R(A) = \{c_1 r_1 + c_2 r_2 + \dots + c_m r_m\}$$

Column Space of Matrix A:

A subspace spanned by columns of matrix A

$$C(A) = \cancel{\{ac_1 + bc_2} \ \{a_1 c_1 + a_2 c_2 + \dots + a_n c_n\}$$

$c_1 - c_n$: column vectors.

Null Space of Matrix A:

Set of all solution of $AX = 0$

Eg: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2 \times \boxed{3 \quad 3} \times 1$ $\qquad$ 2×1

$N(A) = \{(0,0,0)\}$

$1 \cdot x_1 + 2x_2 + 3x_3 = 0$

$4x_1 + 5x_2 + 6x_3 = 0.$

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Row Space, Column Space and Null Space:

For matrix A, $R(A) = \{c_1 r_1 + c_2 r_2 + \dots + c_m r_m \mid c_1, c_2, \dots, c_m \in \mathbb{R}\}$

r_i is i^{th} row.

$\boxed{\text{FAT } X}$

$C(A) = \{a_1 c_1 + a_2 c_2 + \dots + a_n c_n \mid a_1, a_2, \dots, c_m \in \mathbb{R}\}$

$N(A)$ is solution set of $AX = 0$.

Row Space: A subspace spanned by rows of matrix.

Col Space: A subspace spanned by column of matrix.

We need to find basis and dimension of $R(A)$, $C(A)$ and $N(A)$.

$R(A) = C(A^T)$, $R(A^T) = C(A)$

We reduce the matrix to row reduced ~~to row~~ echelon form.

Ex:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & -1 & 2 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to \frac{R_2}{-2}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_1 \to R_1 - R_2]{R_2 \to R_2/2} \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis of $R(A)$:

The non-zero rows in row reduced echelon form of A.

Basis of $R(A) = \{ (1, 0, 3/2), (0, 1, -1/2) \}$

dimension of $R(A) = 2$.

(ii) Basis of $C(A)$ = The columns corresponding to the columns with leading entries

Basis $C(A) = \{ (1, 2, 1)\ (1, 2, -1) \}$ dimension = 2

(iii) $N(A) \Rightarrow$

$$\begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

~~x_1~~ $\longrightarrow$ $\boxed{\text{Incomplete}}$

(2)

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$$

Find basis of $R(A)$, $C(A)$ and $N(A)$.

Row Reduced echelon form:

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R(A) = \{ (1,2,0,2,5), (0,1,-1,-3,-2), (0,0,0,1,1) \}$

dim $= 3$,

$C(A) = \{ (1,-2,0,3) (2,-5,-3,6) (2,-1,4,-7) \}$

dim $= 3$.

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

~~4x5~~

FAT :

Module	Weightage
1	20
2	15
3	20
4	20
5	15
6	10
7	20
	120

Tutorial on Module - 4 :

① Check if $\left\{ \begin{bmatrix} 0 & a \\ -a & b \end{bmatrix} \middle/ a, b \in \mathbb{R} \right\}$

② Find basis and dimension of $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 5 & -2 \end{bmatrix}$

verify Rank nullify Theorem.

Solution:

① ⓘ Consider $\begin{bmatrix} 0 & a \\ -a & b \end{bmatrix}, \begin{bmatrix} 0 & c \\ -c & d \end{bmatrix}$

Sum of $\begin{bmatrix} 0 & a \\ -a & b \end{bmatrix} + \begin{bmatrix} 0 & c \\ -c & d \end{bmatrix} = \begin{bmatrix} 0 & a+c \\ -a-c & b+d \end{bmatrix} \in W$

(ii) Scalar Multiplipcation :

$k \begin{bmatrix} 0 & a \\ -a & b \end{bmatrix} = \begin{bmatrix} 0 & ka \\ -ka & kb \end{bmatrix} \in W$

Hence, its is a subspace of $M_{2\times 2}$.

② ~~State~~ Statement :

$\dim(R(A)) + \dim(N(A)) = n$ (no. of column)

$\dim(N(A^T))$ = m (no. of rows)

(or) $\dim(C(A)) +$

Solutions

(2)

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & +1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is echelon form.

Basics of $R(A) = \{(1,2,1,2), (0,1,-1,2)\}$

dimension $R(A) = 2$.

Basics of $C(A) = \{(1,1,2), (2,1,1)\}$

dimension $C(A) = 2$.

for $N(A)$:

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + x_3 + 2x_4 = 0$$

$$x_2 - x_3 + 2x_4 = 0$$

$$x_3 = x_2 + 2x_4.$$

$$x_1 = -2x_2 - x_3 - 2x_4$$

$$= +2x_2 - x_2 - 2x_4 - 2x_4 = -3x_2 - 4x_4$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 - 4x_4 \\ x_2 \\ x_2 + 2x_4 \\ x_4 \end{bmatrix} = x_2\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -4 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\dim N(A^T) = m - \dim(\mathcal{C}(A))$$

Find basis and dimension of

$$x_1 + x_2 - x_3 + 2x_4 = 0$$

$$x_2 + x_3 - x_4 = 0$$

$$3x_1 + 4x_2 - 2x_3 + 5x_4 = 0$$

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 3 & 4 & -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x = -2y$$

$$z = y$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y \\ y \\ y \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

5-6 $$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 + x_3 - x_4 = 0$$

$$\boxed{x_4 = x_2 + x_3}$$

$$x_1 + x_2 - x_3 + 2x_4 = 0$$

$$x_1 = x_3 - x_2 - 2x_4$$

$$= x_3 - x_2 - 2x_2 - 2x_3$$

$$\boxed{x_1 = -3x_2 - x_3}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 - x_3 \\ x_2 \\ x_3 \\ x_2 + x_3 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

④ check if following are L.I or not :

$\{(1,2,-1,4),(1,0,1,0),(1,-1,0,1)\}$

⑤ Express t^2-2t+1 as Linear combination of $\{t^2-3t+2,\ t^2-4,\ 2t-1\}$.

⑥ Express $(1,2,3)$ as linear combination, of $\{(1,2,-4),(3,-1,2),(2,-3,0)\}$

⑦ check if following are subspaces of R^3.

ⓐ $\{(x,y,z) \mid xyz=0\}$

ⓑ $\{(x,y,z) \mid x^2+y^2-z^2=0\}$

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Module 5 : Linear Transformation

Let V and W be two vector spaces and T be a transformation

$T: V \to W$ T is said to be linear,

if :

i) $T(x+y) = T(x)+T(y)$; $x, y \in V$

ii) $T(kx) = kT(x)$ $\forall$ scalar k.

Ex :

$T: \mathbb{R}^3 \to R^2$, $T((x,y,z)) = (x+y,\ y+z) \to T((1,3,2))$
$= (4,5)$

$T: \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbb{R}^2$, $T((x,y)) = (x+2y,\ 2x+y)$, $T((3,5))$
$= (13,11)$

1) Prove that : $T((x,y)) : (x+2y, 2x+y)$ is linear $T: \mathbb{R}^2 \to \mathbb{R}^2$

Soln,

Let $(x_1, y_1)\ (x_2, y_2) \in \mathbb{R}^2$

(i) $T(\underbrace{(x_1, y_1)}_{x} + \underbrace{(x_2, y_2)}_{y}) = T(\cancel{x+} (x_1 + x_2, y_1 + y_2))$

$= [(x_1 + x_2) + 2(y_1 + y_2),\ 2(x_1 + x_2) + (y_1 + y$ $\to$ ①

Let us consider now,

$\Rightarrow T(x_1, y_1) + T(x_2, y_2)$

$\Rightarrow (x_1 + 2y_1, 2x_1 + y_1) + (x_2 + 2y_2, 2x_2 + y_2)$

$= (x_1 + 2y_1 + x_2 + 2y_2,\ 2x_1 + y_1 + 2x_2 + y_2)$

$= (x_1 + x_2) + 2(y_1 + y_2),\ 2(x_1 + x_2) + y_1 + y_2)$

$\to$ ②

since, eqn ① = ②

$$T(kx) \Rightarrow T(k(x_1, y_1))$$

$$= T(kx_1, ky_1)$$

$$= (kx_1 + 2ky_1, 2kx_1 + ky_1)$$

$$= k(x_1 + 2y_1, 2x_1 + ky_1)$$

$$= k\,T(x_1, y_1) = k(T(x))$$

$\therefore$ Given Transformation is linear.

<u>Ex:</u>

$T: \mathbb{R}^2 \to \mathbb{R}^2$, $T((x, y)) = (x^2, x + y)$

(i) $T((x_1, y_1) + (x_2, y_2))$

$$= T((x_1 + x_2), (y_1 + y_2))$$

$$= T((x_1 + x_2)^2, x_1 + x_2 + y_1 + y_2)$$

$$= (\underbrace{x_1^2 + x_2^2 + 2x_1x_2}_{\text{Extra}}, x_1 + x_2 + y_1 + y_2)$$

$$T(x_1, y_1) + T(x_2, y_2) = (x_1^2, x_1 + y_1) + (x_2^2, x_2 + y_2)$$

$$= (x_1^2 + x_2^2, x_1 + y_1 + x_2 + y_2)$$

③ $T : \mathbb{R}^2 \to \mathbb{R}^2 \qquad T((x, y)) = (x+y, 1)$

$$T(x_1 + x_2, \; y_1 + y_2) = T(x_1, y_1) + T(\cancel{x_2}, y_2)$$

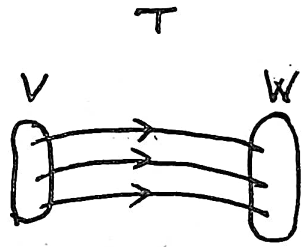
$$T((x_1, y_1)) + T((x_2, y_2))$$

$$\neq (-x_2^2, \; x_1 + y_1) + (x_2^2; \; x_2 + y_2)$$

Kernal of a Transformation:

$$\ker(T) = \{x \in V \mid T(x) = 0 \in W\}$$

The set of ~~ever~~ vector in V that are mapped to zero vector in W.



Ex:

$T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T(x, y) = (x+y, x-y)$

$$\ker T = \{(x, y) \in \mathbb{R}^2 \mid T(x, y) = (0,0)\}$$

~~T =~~

$$T(x, y) = (x+y, x-y) = (0,0)$$

$$x + y = 0$$

$$x - y = 0$$

$$x = y = 0$$

$$\ker T = \{(0, 0) \mid (0,0) \in \mathbb{R}^2\}$$

Ex :

$T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(x, y, z) = (x + y, y - z)$

Find ker (T)

$T(x, y, z) = (x + y, y - z) = (0, 0)$

$$\cancel{T(x,y,z)} \Rightarrow \left.\begin{array}{l} x + y = 0 \\ \cancel{x} \\ y - z = 0 \end{array}\right\} \quad \begin{array}{l} x = -y \\ y = z \end{array}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y \\ y \\ y \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$\ker T = \left\{ k \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \middle| \, k \in \mathbb{R} \right\}$ dim ker T = 1

Ex :

Find kernal of $T : \mathbb{R}^3 \to \mathbb{R}^2$

$T(x, y, z) = (x + 2y + z, -x + 3y + z)$

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Image of a transformation $T: V \to W$ is ~~going~~ given by,

$$IM(T) = \{T(x) \in W \mid x \in V\}$$

$\rightarrow$ $\ker(T) \subseteq V$, $Im(T) \subseteq W$.

Properties of Linear Transformation :

i) $\underset{\in V}{T(0)} = 0 \in W$

ii) $T(c_1 x_1 + c_2 x_2 + \cdots + c_n x_n) = c_1 T(x_1) + c_2 T(x_2) + \cdots + c_n T(x_n)$

x_i - vector

c_i - scalar

① Let $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(e_1) = (1,0)$, $T(e_2) = (2,-1)$, $T(e_3) = (4,3)$

$\{e_1, e_2, e_3\}$ is a std. basis

$\{(1,0,0), (0,1,0), (0,0,1)\}$

if,

$T(1,0,0) = (1,0)$

$T(0,1,0) = (2,-1)$

$T(0,0,1) = (4,3)$

find,

$T((x,y,z)) = ?$

Soln,

Let,

$(x, y, z) = c_1 (1,0,0) + c_2 (0,1,0) + c_3 (0,0,1)$

$(x, y, z) = x (1,0,0) + y (0,1,0) + z (0,0,1)$

$T((x,y,z)) = x T(1,0,0) + y (0,1,0) + z T(0,0,1)$

$T(x,y,z) = x (1,0) + y (2,-1) + z (4,+3)$

$T(x,y,z) = (x + 2y + 4z, -y + 3z)$

② $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(1,1,1) = (1,0)$ Find Transformation.

$T(1,1,0) = (2,-1)$

$T(0,0,0) = (4,3)$

Solution,

$T((1,1,1)) = (1,0)$

$T((1,1,0)) = (2,-1)$

$T((0,0,0)) = (4,+3)$

$(x,y,z) = c_1 (1,1,1) + c_2 (1,1,0) + c_3 (1,0,0)$

$(x,y,z) = z T(1,1,1) + (y-z)(T(1,1,0)) + (x-y) T(1,0,0)$

$= (z,0) + (2y - 2z, -y+z) + (4x - 4y, 3x - 3y)$

$(x,y,z) = (4x - 2y - z, 3x - 4y + z)$

Find basis and dimension of $Im(T)$, $ker(T)$

$T(x,y,z) = (x+z, x+y+2z, 2x+y+3z)$

Solution,

$\mathbb{R}^3 \to \mathbb{R}^3 : T(x,y,z) = (x+z, x+y+2z, 2x+y+3z)$

$= (0,0,0)$

$$\begin{aligned} x+0+z &= 0 \\ x+y+2z &= 0 \\ 2x+y+3z &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$ker(T) = \{k(-1,-1,1) \mid k \in \mathbb{R}\}$ $\dim(ker(T)) = 1$

Image space:

$Im(T) = \{T(x,y,z) \mid (x,y,z) \in \mathbb{R}^3\}$

= Column Space of Mat A.

Bases $Im(T) = \{(1,1,2), (0,1,1)\}$, $\dim Im(T) = 2$.

31.3.23

Matrix of Linear Transformation :

Let $T : V \to W$, the matrix representation of T is given by $\begin{bmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{bmatrix}$, where $\{e_1, e_2, \dots, e_m\}$ is a standard basis of V, $\dim(V) = m$, $\dim(W) = n$.

Example :

$T : \mathbb{R}^3 \to \mathbb{R}^3$; $T((x, y, z)) = (x+y, y+z, x+y+z)$

$\dim(\mathbb{R}^3) = 3$

bases $= \{(1,0,0), (0,1,0), (0,0,1)\}$

$$\text{Matrix} = \begin{bmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{bmatrix}$$

$$M_T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3\times 3}$$, $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(\mathbb{R}^3 \to \mathbb{R}^3)$

$$T((x, y, z)) = M_T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$T((x, y, z)) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= (x+y, y+z, x+y+z)$$

<u>Theorem :</u>

Rank Nullify Theorem for transformation $T : V \to W$

$\dim(\operatorname{Im}(T)) + \dim(\ker(T)) = \dim(V)$

Verify Rank Nullify Theorem for,

$T(x, y, z) = (x + y, y + z, x + y + z)$, $T : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{array}{l} x + y = 0 \\ y + z = 0 \quad \Rightarrow y = 0 \\ z = 0 \end{array}$$

$\operatorname{cer}(T) = \{(0, 0, 0)\}$

$\dim \ker T = 0$

Basis ξ $\operatorname{Im}(T) = C(A) = \{(1, 0, 1), (1, 1, 1), (0, 1, 1)\}$

$\dim \operatorname{Im}(T) = 3$

Theorem : $\dim \ker(T) + \dim(\operatorname{Im} T) = 0 + 3 = 3$

$= \dim(V)$

<u>Square of a transformation :</u>

Let $T : V \to W$, Square of the transformation T is denoted by $T^2(x)$

$$T^2(x,y) = T(T(x,y)) = T(x+3y\ ,\ 3x+y)$$

$$= (x+3y + 3(3x+y\ ,$$

$$3(x+3y) + 3x+y)$$

$$= (10x + 6y\ ,\ 6x + 10y)$$

Transformation of Mat.

$$\{(0,1),(1,0)\} \Rightarrow \begin{bmatrix} T(1,0) & T(0,1) \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

Inverse of Transformation : $T : V \to W$

Inverse $T^{-1} : W \to V,\ T^{-1}(v) = u\ ,\quad v \in W\ ,\ u \in V$

$$T(u) = v\ ,\quad u \in V\ ,\ v \in W$$

Find inverse of $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T(x,y) = (x+3y,\ x+5y)$$

Matrix $A = \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}$ $\quad$ $\det A = 5-3 = 2$

Hence T^{-1}.

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 5 & -3 \\ -1 & 1 \end{bmatrix}$$

$$T^{-1}(x,y) = \begin{bmatrix} 5/2 & -3/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \left(\frac{5}{2}x - \frac{3}{2}y\ ,\ -\frac{x}{2} + \frac{y}{2}\right)$$

Let $M = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

(i) $\cancel{M = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}}$ write an expression for

(ii) Find $T(1,0)$, $T(0,1)$ using Matrix

(iii) Find (x, y) such that $T(x,y) = (1,0)$

(iv) Find T^2

(v) Find T^{-1}.

5.3.23

Matrix representation with a different bases:

$T: R^2 \to R^2$, $T(x,y) = (4x - 6y, x - y)$

(i)

$M = \begin{bmatrix} 4 & -6 \\ 1 & -1 \end{bmatrix}$ w.r.t standard bases $\{(1,0), (0,1)\}$

$T(1,0) = (4,-1) \Rightarrow 4(1,0) + \_(0,1)$

$T(0,1) = (-6,-1) \Rightarrow -6(1,0) + -1(0,1)$

$= (4x - 6y, x + y)$

(ii) $\{(3,1), (2,1)\}$

$\cancel{T(1,0)}$

M

$$T(3,1) = (6,2) = c_1(3,1) + c_2(2,1)$$

$$T(2,1) = (2,1) = c_3(3,1) + c_4(2,1)$$

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = (3c_1 + 2c_2, c_1 + c_2)$$

$\cancel{c_1 =}$ $\cancel{c_3 =}$

$\cancel{c_2 =}$ $\cancel{c_4 =}$

$$\Rightarrow (6,2) = (3c_1 + 2c_2, c_1 + c_2)$$

$$\Rightarrow c_1 = 2, \; c_2 = 0$$

$$c_3 = 0, \; c_4 = 0$$

Let $T: V \to W$, $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be basis of V,

$\{\beta_1, \beta_2, \ldots, \beta_n\}$ of W.

$[T]_\alpha^\beta \Rightarrow$ if $\alpha = \beta$, $[T]_\alpha^\alpha = [T]_\alpha = [T]_\beta$.

$T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(x, y) = (1x + 2y, \ 0, \ 2x + 3y)$

$\quad\quad 1 \quad 2 \quad 0 \quad 2$

$\alpha = \{(1,0)(0,1)\}$

$\beta = \{(0,0,1)(0,1,0), (1,0,0), (1,0,0)\}$

find $[T]_\alpha^\beta$

$T(1,0) = (1,0,2) = c_1(0,0,1) + c_2(0,1,0) + c_3(1,0,0)$

$T(0,1) = (2,0,3) = c_4(0,0,1) + c_5(0,1,0) + c_6(1,0,0)$

$$\boxed{\begin{array}{lll} c_1 = 2 & c_2 = 0 & c_3 = 1 \\ c_4 = 3 & c_5 = 0 & c_6 = 2 \end{array}}$$

$$\begin{bmatrix} c_1 & c_4 \\ c_2 & c_5 \\ c_3 & c_6 \end{bmatrix}$$

~~I.V~~

(2) $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(x,y,z) = (x+y,\ y-z)$

$\alpha = \{(1,0,1), (0,1,1), (1,1,1)\}$

$\beta = \{(1,2), (-1,1)\}$

$[T]_\alpha^\beta = ?$

$$\begin{bmatrix} c_1 & c_3 & c_5 \\ c_2 & c_4 & c_6 \end{bmatrix}_{2\times 3}$$

~~$\alpha =$~~

$c_1(1,2) + c_2(-1,1)$

~~$T(1,0) = (+$~~

$(1,-1) = (\ , \ )$

$T(1,0,1) = \cancel{(1,-1)}\ \cancel{(2,1,2)}$

$T(0,1,1) = (1,0) = c_3(1,2) + c_4(-1,1)$

$T(1,1,1) = (2,0) = c_5(1,2) + c_6(-1,1)$

$$\begin{bmatrix} c_1 & c_3 & c_5 \\ c_2 & c_4 & c_6 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 2/3 \\ -1 & -2/3 & -4/3 \end{bmatrix}$$

Similarity :

Identity transformation denoted by $Id : V \to V$

or $T(4,5) = (4,5)$, $T : \mathbb{R}^2 \to \mathbb{R}^2$

~~or T(~~

Similarity Theorem :

Let V be a vector space, α and β be its bases consider $T: V \to V$, let $Q = [Id]^{\alpha}_{\beta}$ and $Q^{-1} = [Id]^{\beta}_{\alpha}$. For the matrices $[T]_\alpha$ and $[T]_\beta$ if there is Q such that $[T]_\beta = Q^{-1}[T]_\alpha Q$

Then, $[T]_\alpha$ and $[T]_\beta$ are similar.

6.4.23

Ex:

On similarity theorem,

$T: \mathbb{R}^3 \to \mathbb{R}^3$, $T((x,y,z)) = (x+2y+z, -y, x+4z)$

① Find $[T]_\alpha$

② Find $[T]_\beta$ using similarity theorem where,

$\alpha = \{(1,0,0), (0,1,0), (0,0,1)\}$; $\beta = \{(1,0,0), (1,1,0), (1,1,1)\}$

Solution:

(i) $[T]_\alpha$

$T((1,0,0)) = (1,0,1) = 1(1,0,0) + 0(0,1,0) + 1(0,0,1)$

$T((0,1,0)) = (2,-1,0) = 2(1,0,0) + (-1)(0,1,0)$
$+ 0.(0,0,1)$

$T((0,0,1)) = (1,0,4) = 1(1,0,0) + 0(0,1,0) +$
$4(0,0,1)$

$$[T]_\alpha = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

$[T]_\beta = Q^{-1}[T]_\alpha Q, \quad Q = [Id]_\beta^\alpha \quad Q^{-1} = [Id]_\alpha^\beta$

$Q = [Id]_\beta^\alpha$

$T(1,0,0) = (1,0,0) = 1(1,0,0) + 0(0,1,0) +$
$0(0,0,1)$

$T(1,1,1) = (1,1,1) = 1(1,0,0) + 1(0,1,0) +$
$1(0,0,1)$

$$[Id]_\beta^\alpha = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$Q^{-1} = [Id]_\alpha^\beta$

$Id(1,0,0) = (1,0,0) = \underline{1}(1,0,0) + \underline{0}(1,1,0)$
$+ \underline{0}(1,1,1)$

$Id(0,1,0) = (0,1,0) = \underline{-1}(1,0,0) + \underline{1}(1,1,0)$
$+ \underline{0}(1,1,1)$

$Id(0,0,1) = (0,0,1) = \underline{0}(1,0,0) + \underline{-1}(1,1,0)$
$+ \underline{1}(1,1,1)$

$$Q^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[I_\beta] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 1 \\ -1 & -1 & -4 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 5 \\ -1 & -2 & -6 \\ 1 & 1 & 5 \end{bmatrix}$$

Module-6 : Inner Product Space

Dot product of two vectors (x_1, x_2, x_3) and (y_1, y_2, y_3) is $xy = x_1 y_1 + x_2 y_2 + x_3 y_3$.

Length of a vectors $x = \sqrt{x_1^2 + \cancel{3}\, x_2^2 + x_3^2}$

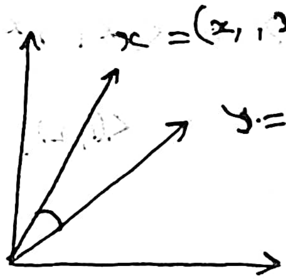
Distance between two vectors:

$x = (x_1, x_2, x_3)$

$y = (y_1, y_2, y_3)$

$$\|x-y\| = d(x,y) = \sqrt{(y_1-x_1)^2 + (y_2-x_2)^2 + (y_3-x_3)^2}$$

Angle between two vectors:



$$\cos\theta = \frac{x \cdot y}{\|x\| \cdot \|y\|} \quad \text{at origin}$$

$$\theta = \cos^{-1}\left[\frac{x \cdot y}{\|x\| \cdot \|y\|}\right]$$

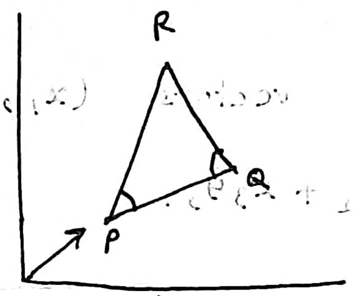
$$\cos\theta = \frac{(Q-P)\cdot(R-P)}{\|Q-P\| \, \|R-P\|}$$

Angle at Q, $\cos\theta = \dfrac{(P-Q)(R-Q)}{\|P-Q\| \cdot \|R-Q\|}$

$P = (1, 2, 3)$

$Q = (2, -1, 4)$

$R = (3, 5, -2)$



Find angles at P, Q, R if P, Q, R are forming a Δle.

14.4.23

Inner Product Space :

Eg : $\mathbb{R}^2 \to \mathbb{R}$

$\langle x, y \rangle = \langle x_1, x_2 \rangle = 2x_1 y_1 - x_1 y_2 - x_2 y_1 + 5 x_2 y_2$

$\langle (1,2), (3,1) \rangle = 2(1)(3) - (1)(1) - 2(3) + 5(2)(1)$

$= 9.$

Inner Product Space : is an vector space V is a function that assigns a real no. $\langle x, y \rangle$ to each point of vector (x, y) such that, (necessary circle)

Properties:

(i) $\langle x, y \rangle = \langle y, x \rangle$, symmetry

(ii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y + z \rangle$, additivity

(iii) $\langle kx, y \rangle = k\langle x, y \rangle$, homogeneity

(iv) $\langle x, x \rangle > 0$ and $\langle x, x \rangle = 0$ if $x = 0$, +ve definiteness

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Ex.

Find orthogonal basis of $\mathbb{R}^3$ for $\{(1,1,1), (-1,1,0), (1,2,1)\}$ using gram-schmidt orthogonalization.

$a = (1,1,1)$

$b = (-1,1,0)$

$c = (1,2,1)$

$A = a = (1,1,1)$

$$B = b - \frac{A \cdot b}{A \cdot A} A = (-1,1,0) - \frac{(1,1,1)(-1,1,0)}{(1,1,1)(1,1,1)} (1,1,1)$$

$B = (-1,1,0) - 0 = (-1,1,0)$

$$C = c - \frac{A \cdot c}{A \cdot A} A - \frac{B \cdot c}{B \cdot B} B = (1,2,1) - \frac{(1,1,1)\cancel{(-1,1,0)}(1,2,1)}{(1,1,1)(1,1,1)} (1,1,1) - \frac{(-1,1,0)(1,2,1)}{(-1,1,0)(-1,1,0)} (-1,1,0)$$

$$= (1,2,1) - \frac{4}{3}(1,1,1) - \frac{1}{2}(-1,1,0)$$

$$C = (1,2,1) - \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) - \left(\frac{-1}{2}, \frac{1}{2}, 0\right)$$

Till this step these are orthogonal

This is ortho normal

to normalize, → unit vector

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{\frac{1}{6}}6}, \frac{1}{\sqrt{\frac{1}{6}}6}, \frac{1}{\sqrt{\frac{1}{6}}3}\right)$$

② For $\{(1,2,1,0)\ (1,1,0,1)\}$ in $\mathbb{R}^4$, find an orthonormal basis using Gram Schmidt process.

Solution

A and B vector dot product should be zero for verification

$a = (1,2,1,0)$

$b = (1,1,0,1)$

$\underline{A} = (1,2,1,0)$

$$\underline{B} = b - \frac{A \cdot b}{A \cdot A}(A)$$

$$= (1,1,0,1) - \frac{(1,2,1,0)(1,1,0,1)}{(1,2,1,0)(1,2,1,0)}(1,2,1,0)$$

$$B = (1,1,0,1) - \frac{\cancel{3}^{1}}{\cancel{6}_{2}}(1,2,1,0)$$

$(a+b)^2 + ab =$

$a = 04H$

$b = 424$

$$B = (1,1,0,1) - \left(\frac{1}{2}, 1, \frac{1}{2}, 0\right)$$

$$B = \left(\frac{1}{2}, 0, \frac{-1}{2}, 1\right)$$

To normalize,

$$\left\{\left(\frac{1}{\sqrt6}, \frac{2}{\sqrt6}, \frac{1}{\sqrt6}, 0\right) \left(\frac{1}{\sqrt{\frac{3}{2}}\cdot 2}, \frac{1}{\sqrt{\frac{3}{2}}\ 2}, \frac{1}{2\sqrt{\frac{3}{2}}}, 0\right)\right\}$$

$$\Rightarrow \left\{\left(\frac{1}{\sqrt6}, \frac{2}{\sqrt6}, \frac{1}{\sqrt6}, 0\right), \left(\frac{1}{\sqrt6}, 0, \frac{-1}{\sqrt6}, \frac{\sqrt2}{\sqrt3}\right)\right\}$$

<u>Module 4:</u>

Verify if $7x^2+4x-3$ is span of $\{4x^2+x, x^2-2x+3\}$

$$7x^2+4x-3 = c_1(4x^2+x) + c_2(x^2-2x+3)$$

$$7x^2+4x-3 = (4c_1+c_2)x^2 + (c_1-2c_2)x + 3c_2$$

$$\boxed{c_1 = 2, \quad c_2 = -1}$$

↳ If we can't get constants it isn't spanned

Since, $7x^2+4x-3$ is linear combination of $4x^2+x$, x^2-2x+3. It is spanned by given set.

(2) $A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

check if A & B are in span of

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} 1 &= c_1 \\ c_2 &= 2 \end{aligned}}$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$0 = c_1 + 0 \cdot c_2$$

$$0 \cdot c_1 + 0 \cdot c_2 = 1$$

↳ Here, we cannot get the values of constant, so it is not spanned

③ Use Gram Schmidt on $\{(1,1,1,1), (1,2,3,4), (3,4,2,1)\}$

$a = (1,1,1,1)$

$b = (1,2,3,4)$

$c = (3,4,2,1)$

$A = (1,1,1,1)$

$$B = (1,2,3,4) - \frac{(1,1,1,1)\cdot(1,2,3,4)}{(1,1,1,1)\cdot(1,1,1,1)}(1,1,1,1)$$

$$= (1,2,3,4) - \frac{\overset{5}{\cancel{(10)}}}{\underset{2}{\cancel{4}}}(1,1,1,1)$$

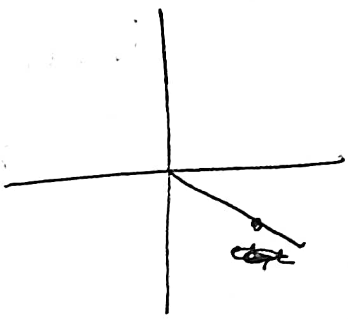
$$= (1,2,3,4) - \left(\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}\right)$$

incomplete

Rotation Matrix:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{\frac{\pi}{2}} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix}$$



$$R_{\frac{\pi}{2}}(1,-2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Projection Matrix,

on x - axis : $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$

on y - axis : $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$

Revision :

M7:

→ Eigen values & Eigen vectors — 10 M

→ System of Equation — 10 M

→ Inverse / Power of Matrix.

M6: (Inner Product Space)

→ Matrix Representation

→ Gram Schmidt Process — 10 M

M5 : Transformations → T_α, T_α^β, T_β^α

→ Proving Linearity

→ Square and Inverse either by definition (or) Matrix

→ Finding dimension of Im(T) / range and Ker (T) Null space.

$\boxed{\text{Solve Notes Problem first}}$

Module 5:

① Ex: Let $T, \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, y, z) = (2x+z, y+3z)$

$\alpha = \{(1,1,0), (1,0,1), (1,1,1)\}$

$\beta = \{(2,3), (3,2)\}$

Find T_{α}^{β}.

② Find inverse and square of $T : \mathbb{R}^3 \to \mathbb{R}^3$

$T(x,y,z) = (y+z, x+z, x+y)$.

③ Find dim of Im(T) / Range (T), Ker(T) / null Space (T).

$T : \mathbb{R}^4 \to \mathbb{R}^3$, $T(x, y, s, t)$

$= (x-y+s+t, x+2s-t, x+y+3s-3t)$

Solution

Take matrix
$$\begin{bmatrix} T(e_1) & T(e_2) & T(e_3) & T(e_4) \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}_{3\times 4}$$

Ker(T) = Null space of matrix

Im(T) = col space of matrix

$T^{-1}(u, v, w) = (x, y, z)$

$$\boxed{\begin{array}{l} x = \\ y = \\ z = \end{array} \Bigg\} \; u, v, w.}$$

$T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (x + 3y, x + 5y)$.

Matrix :

$$M_T = \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}$$

$$M_T^{-1} = \frac{1}{2}\begin{bmatrix} 5 & -3 \\ -1 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 5/2 & -3/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \left(\frac{5}{2}x - \frac{3}{2}y, \; \frac{-x}{2} + \frac{y}{2} \right).$$

$T(x, y) = (u, v) = T^{-1}(u, v) = (x, y)$

(or) $x + 3y = u$

$x + 5y = v$

$x = \frac{5}{2}u - \frac{3}{2}v$

$y = \frac{-u}{2} + \frac{v}{2}.$

Module 4:

→ Proving Subspace

~~Take 2 vectors and 1~~

→ Linearly Independent (constant is non-zero linearly dep.)

→ Span

→ Row Space . C (A), Null Space

Rank + Nullify