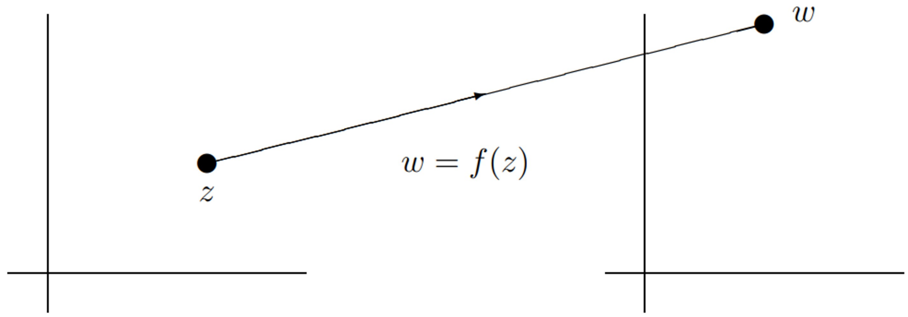
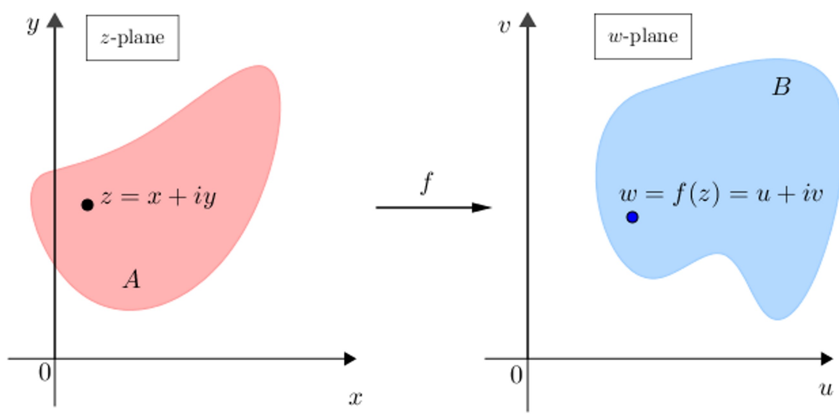
Module 2 - Conformal and Bilinear transformations

25 August 2022 09:35

For every point (x, y) in the z -plane, the relation $w = f(z)$ defines a corresponding point (u, v) in the w -plane. This is called as "**transformation** or **mapping** of z -plane into w -plane". If a point z_0 maps into the point w_0, w_0 is also known as the image of z_0.



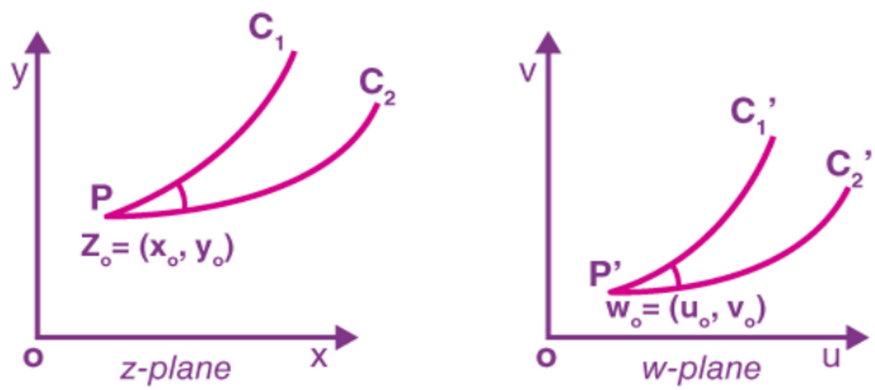


Example:

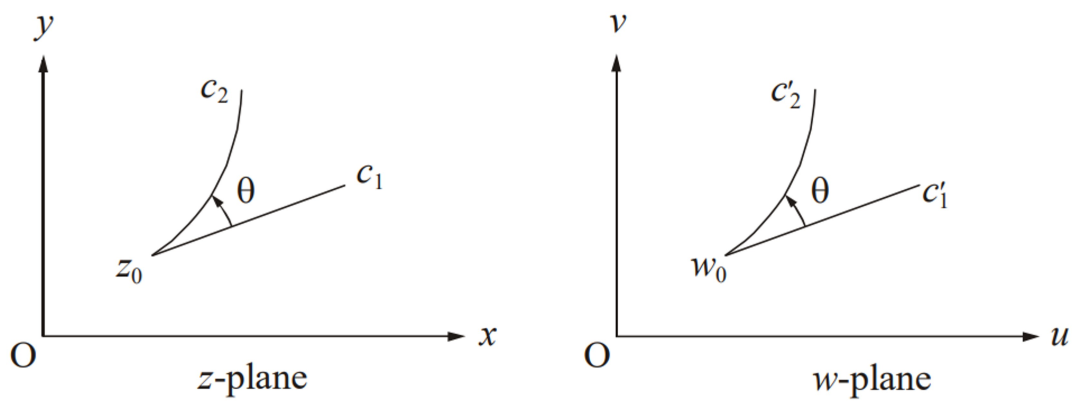
$$F(z) = z + 1, bounded\ by\ x = 0, y = 0, x = 1\ and\ y = 1$$

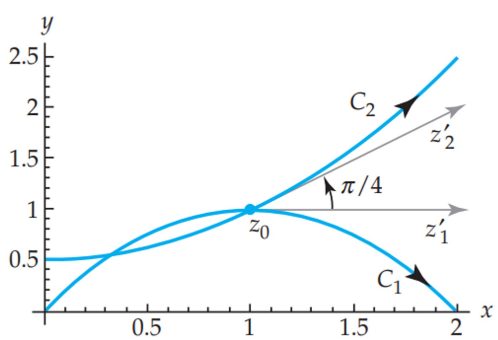
A transformation is said to be **isogonal** if two curves in the z-plane intersecting at the point z_0 at an angle θ are transformed into two corresponding curves in the w-plane intersecting at the point w_0 which corresponds to the point z_0 at the same angle θ.

We say equal in magnitude if both the angles are same



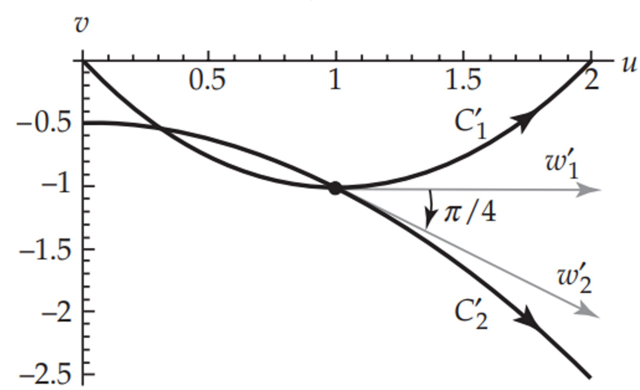
If the **sense** (direction of the rotation) of the rotation as well as the magnitude of the angle is preserved, then the transformation is called **conformal.**





(a) Curves C_1 and C_2 in the z-plane

$w = \bar{z}$



(b) Images of the curves in (a) under $w = \bar{z}$

In the above examples the mapping is not conformal since the angles are not equal in sense

Theorem: If f is an analytic function in a domain D containing z_0, and if $f'(z_0) \neq 0$, then $w = f(z)$ is a conformal mapping at z_0

Examples:

1. $f(z) = e^z$ is conformal mapping for all z in C

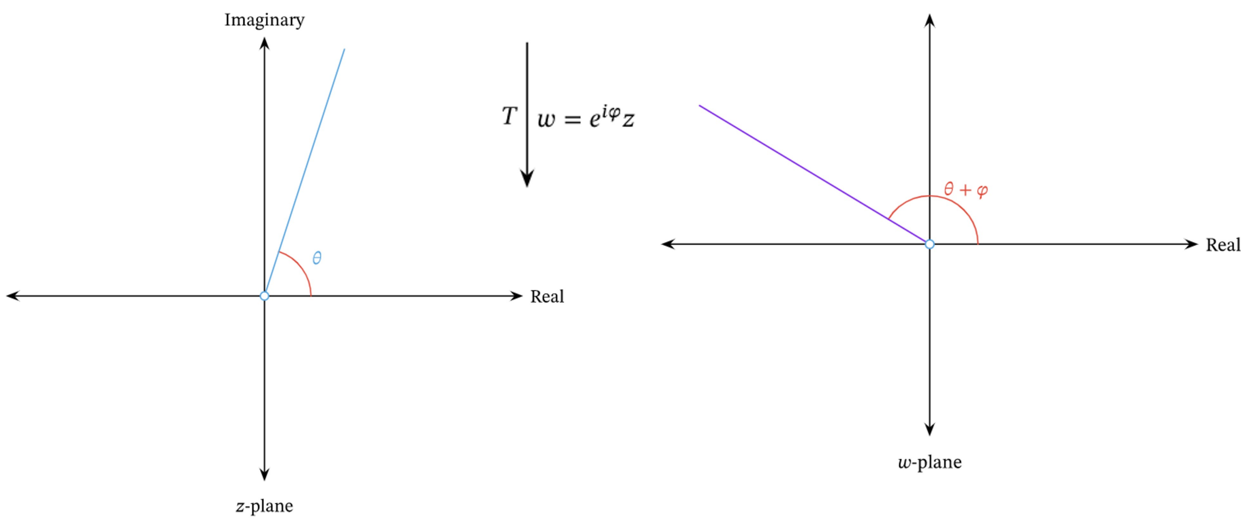
2. $f(z) = z^2$ is conformal for all z in C mapping except at $z = 0$.
   Since $f'(0) = 0$

3. $f(z) = az + b$ for $a \neq 0$

4. For what values of z, the function $f(z) = \sin(z)$ is conformal.

Under conformal mapping the tangent line of a curve is rotated by ω =arg(f′(z)) at z_0, in W plane.

ω is called as **angle of rotation** and $|f'(z)|$ is called as coefficient of magnification or scale factor at z_0



Find the angle of rotation of $f(z) = z^2$ at the point $z_0 = 1 + i$

Sol: angle of rotation is $\frac{\pi}{4}$ scale factor is $2\sqrt{2}$

Examples:

- Determine the angle of rotation at the point $z = \frac{1+i}{2}$ under the mapping $w = z^2$. find its scale factor also.

Solutions:

$\frac{\pi}{4}$ and $\sqrt{2}$

Example

For the conformal transformation $w = z^2$, show that

a. The coefficient of magnification at $z = 2 + i$ is $2\sqrt{5}$.
b. The angle of rotation at $z = 2 + i$ is $\tan^{-1}(0.5)$.
c. The coefficient of magnification at $z = 1 + i$ is $2\sqrt{2}$.
d. The angle of rotation at $z = 1 + i$ is $\dfrac{\pi}{4}$.

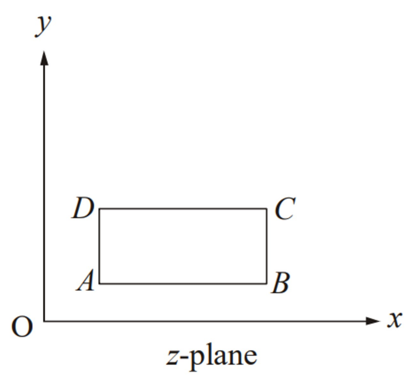
Standard Transformations:

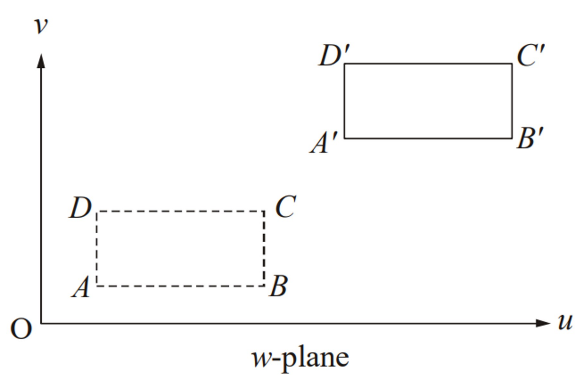
- Translation
- Rotation
- Mafnification
- Inversion

Translation:

The transformation $w = z + c$ where $c = a + ib$ is called as translation.

$w = z + c$ $u + iv = x + iy + a + ib$
$u = x + a$ and $v = y + b$
$x = u - a$ and $y = v - b$





Examples:

- Find the image of the region bounded by $x = 0, x = 1, y = 0, y = 1$ under the mapping $w = z + 1 - i$