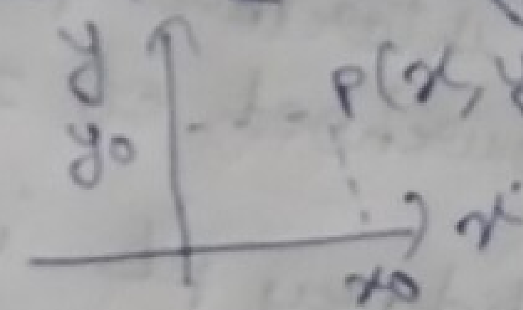
Maths

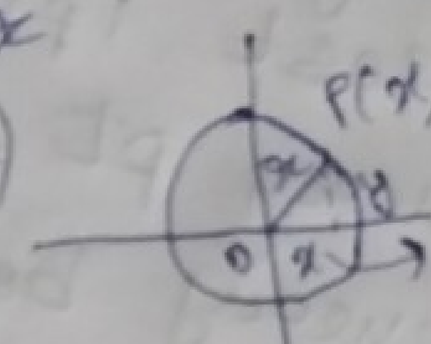
Module 1

Analytic functions

Complex variables → Cartesian → $z = x+iy = Re(z) + Im(z) = (x, y)$



Polar coord → $Z = e^{i\theta} r$
↓
(euler form)



$x = r\cos\theta,\ y = r\sin\theta$

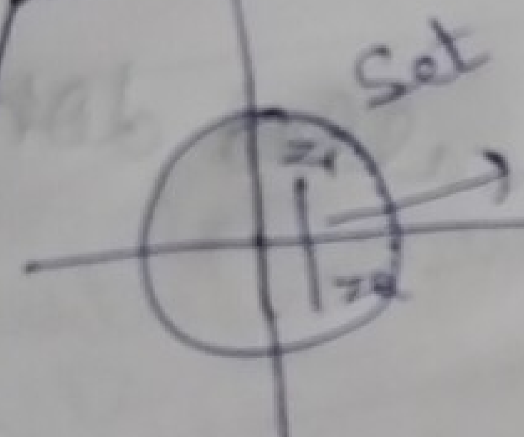
$P(x,y) = (r\cos\theta, r\sin\theta)$

$= r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$= e^{i\theta} r$

$|z| = |x+iy| = \sqrt{x^2+y^2} = r$



→ equal to $|z_1 - z_2|$

if $z_1 \in$ Set $z_2 \in$ Set and distance $|z_1 - z_2|$ → connected set

$|z| > r$ or $|z| < r$ → open set

$|z| \le r$ → closed set

↓ explained better later on

if z_1, z_2 → complex no → $|z_1 - z_2|$ is dist b/w them

① set of pts satisfying $|z - z_0| = \delta$ → define circle of centre z_0, radius δ

any pt z on this → $z = z_0 + \delta e^{i\theta}$

$|z - z_0| < \delta$ defines open disk of

② set of pts satisfying $r = \delta$, centre z_0

if $|z - z_0| \le \delta$ → closed disk " "

if $\delta_1 < |z - z_0| < \delta_2$ → open annulus | open circular ring
↓
set of pts lying b/w 2 concentric circles

if $\delta_1 \le |z - z_0| \le \delta_2$ → closed annulus, closed circular ring

an open circular disc of radius δ with centre z_0 can be called δ neighborhood of z_0.

if we exclude z_0 from open disk $|z-z_0|<\delta$
its called deleted neighbourhood of z_0 and denoted by
$0<|z-z_0|<\delta$

(iii) ~~~

a pt $z \in S$ is interior pt of S, if all pts are
some δ neighbourhood of z are in S

a pt $z \in S$ boundary pt if every δ neighbourhood
of z contains atleast 1 pt outside and inside S

Set S → open if every pt of S is interior pt
→ closed if every boundary pt of $S \in S$

→ bounded if there exist +ve real no m
such that $|z| \le m$ for all $z \in S$
→ connected if 2 pts $z_1, z_2 \in S$ can be
joined by polygonal line which is
totally contained in S

an open + connected set is called domain

(iv) D
domain → always a region
region → may or not be domain
ex open disk → domain + region but
closed disk is region not
domain

Analytic $f^n \rightarrow f(z) = \underset{\downarrow}{u} + i\underset{\swarrow}{v}$

f^n of x, y

analytic f^n if $\rightarrow$ (i) differentiable everywhere in Domain
(ii) if all first order partial derivates are continuous –

also must follow $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ {CR eqn}

Q) $f(z) = z^2$

ans) $f(z) = (x+iy)^2 = x^2 + y^2 i^2 + 2ixy$

$u = x^2 - y^2$, $v = 2xy$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = -2y$, $\frac{\partial v}{\partial x} = 2y$, $\frac{\partial v}{\partial y} = 2x$

so satisfies CR eqn

a f^n $F(z) \rightarrow$ analytic about some pt z_0 if its differentiable everywhere in some neighbourhood of z_0 in domain D

necessary and sufficient condn $\rightarrow$ {of complex variable $f(z)$} f^n in domain D

$\rightarrow$ necessary $\rightarrow$ for $f(z) = u + iv$ provided

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x}$

they all exist. called cauchy Remann eqn / CR eqn.

$\rightarrow$ Sufficient $\rightarrow$ for $f(z) = u + iv$ in Domain D $\rightarrow$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x}$ are continuous in domain D then u, v

if f^n $f(z)$ is analytic satisfy CR eqn

CR eqns are necessary for f^n $f(z)$ to be diff able at a pt

CR eqn is necessary but not sufficient for analytic

the CR eqn are sufficient if first order PD are conti in domain D then

if $f(z) = u + iv$ is analytic

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

properties

conjugate of analytic $f^n \rightarrow$ never analytic for $f(z) = z$

$\rightarrow$ {$\bar{z} \rightarrow$ not analytic}

Q) find value of C_1, C_2 such that fn $f(z) =$
$(x^2 + c_1y^2 + 2xy) + i(c_2x^2 + y^2 \pm 2xy)$ is
analytic, find $f'(z)$

ans) $\frac{\partial u}{\partial x} = 2x + 2y$, $\frac{\partial u}{\partial y} = 2c_1y + 2x$

$\frac{\partial v}{\partial x} = 2c_2x + 2y$, $\frac{\partial v}{\partial y} = +2x + 2y$

$\boxed{c_2 = -1}$ $\boxed{c_1 = -1}$

so $u = x^2 - y^2 + 2xy$, $v = -x^2 + y^2 + 2xy$
and both are conti.

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$
$= 2[x(1-i) + y(1+i)]$

Q) $f(z) = \frac{1}{z}$ satisfies CR or no?

ans) $f(z) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$

$u = \frac{x}{x^2+y^2}$, $v = \frac{-y}{x^2+y^2}$

$\frac{\partial u}{\partial x} = \frac{1 \times (x^2+y^2) - (2x) \times x}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y} = \frac{0 - (2x) \times x}{(x^2+y^2)^2}$, $\frac{\partial v}{\partial x} = \frac{0 - (2x)(-y)}{(x^2+y^2)^2}$

$\frac{\partial v}{\partial y} = \frac{-1(x^2+y^2) - (2y \times -y)}{(x^2+y^2)^2}$

$\left\{ d\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2} \right\}$

CR satisfied, so analytic at all
except at 0.

Q) $f(z) = u + iv$
$u \to$ constant, prove $f(z) \to$ const

ans) $\frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}$

$v = c_2 + g(x)$
$v = c_2 + h(y)$
so $h(y) = g(x) = 0$
so $\boxed{v = c_2}$
so constant fn.

Q) $f(z) = e^x(\cos y + i\sin y)$ analytic or not?
ans) $\Rightarrow e^x \times e^{iy} \to e^z$
$u = e^x \cos y$
$v = e^x \sin y$
$u_x = e^x \cos y$, $u_y = e^x \times -\sin y$
$v_x = e^x \sin y$, $v_y = e^x \cos y$
so yes, CR satisfied

Q) $f(z) = \sin(z)$
ans) $f(z) = \sin(x + iy)$
$= \sin x \cos(iy) + \cos x \sin(iy)$
$\Rightarrow$ now $\cos i\theta = \frac{e^{-\theta} + e^{\theta}}{2} = \cosh\theta$
similarly $\sin(i\theta) = i\sinh\theta$
$\left\{ \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \right\}$

so $\rightarrow$

$f(z) = \underbrace{\sin x \cosh y}_{u} + i\underbrace{\cos x \sinh\theta}_{v}$

and than do CR.

Q) $f(z) = \bar{z}$

ans) $\bar{z} = x - iy$

doing we get not analytic

Q) $f(z) = z\bar{z}$

ans) $f(z) = (x+iy)(x-iy)$

$= x^2 + y^2 + 0i$

$u = x^2 + y^2, \quad v = 0$

so not analytic

CR polar form

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = \frac{-\partial u}{r\,\partial \theta}$

Q) $f(z) = \log z$

ans) $= \log z = \log r e^{i\theta} = \log r + i\theta$

$u = \log r, \quad v = \theta$

$u_r = \frac{1}{r}, \quad u_\theta = 0, \quad v_r = 0, \quad v_\theta = 1$

So analytic everywhere except $(0,0)$

Q) $f(z) = r^2 \cos 2\theta + i r^2 \sin P\theta$ find P if $f(z)$ analytic

ans) $u = r^2 \cos 2\theta, \quad v = r^2 \sin P\theta$

$u_r = 2r \cos 2\theta, \quad v_r = 2r \sin P\theta$

$u_\theta = r^2 \times 2 \times -\sin 2\theta, \quad v_\theta = r^2 P \cos P\theta$

for analytic $\to$ comparing we get $\to$

$2r \cos 2\theta = \frac{1}{\cancel{r}} \times \cancel{r^2} P \cos P\theta$

$2r \cos 2\theta = r P \cos P\theta$

$\boxed{P = 2}$ (ans)

and

$2r \cos 2\theta = -\frac{1}{r} \times -2r^2 \sin 2\theta$ {no need to solve}

Q) PT $f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

satisfy CR but $f'(0)$ DNE

ans) Next Page $\to$

$$f(z) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} + i\,\dfrac{(x^3+y^3)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$u = \frac{x^3-y^3}{x^2+y^2}, \quad v = \frac{x^3+y^3}{x^2+y^2}$$

$$u = \frac{x^3-y^3}{x^2+y^2}, \quad v = \frac{x^3+y^3}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h}$$

$$\left.\frac{\partial u}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{u(0+h,0) - u(0,0)}{h} \to ①$$

$$\left.\frac{\partial u}{\partial y}\right|_{(0,0)} = \lim_{k\to 0} \frac{u(0,k) - u(0,0)}{k} \to ②$$

$$\left.\frac{\partial v}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{v(h,0) - v(0,0)}{h} \to ③$$

$$\left.\frac{\partial v}{\partial y}\right|_{(0,0)} = \lim_{k\to 0} \frac{v(0,k) - v(0,0)}{k} \to ④$$

$$\left.\frac{\partial u}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{\frac{h^3-0}{h^2} - 0}{h} = 1$$

$$\left.\frac{\partial u}{\partial y}\right|_{(0,0)} = -1$$

$$\left.\frac{\partial v}{\partial x}\right|_{(0,0)} = 1$$

$$\left.\frac{\partial v}{\partial y}\right|_{(0,0)} = 1$$

So CR satisfied for fn at (0,0)

hence, first order PD d.n.e at (0,0) so

not analytic at (0,0)

how to check $f'(z)$ at origin.

let $f(z) = u + iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

at $z_0 = 0 \rightarrow$

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z \to 0} \frac{f(z) - 0}{z - 0}$$

$$= \lim_{(x,y) \to 0} \frac{\frac{x^3 - y^3}{x^2 + y^2} + i\left(\frac{x^3 + y^3}{x^2 + y^2}\right)}{x + iy}$$

if $y = mx$ path

$$\Rightarrow \lim_{(x, mx) \to (0,0)} = \frac{\frac{x^3 - m^3x^3}{x^2 + m^2x^2} + i\left(\frac{x^3 + m^3x^3}{x^2 + m^2x^2}\right)}{x + imx}$$

$$\rightarrow \lim_{x \to 0} \frac{x\left\{\frac{1-m^3}{1+m^2} + i\left(\frac{1+m^3}{1+m^2}\right)\right\}}{x(1+im)}$$

$$\rightarrow \frac{\left\{\frac{1-m^3}{1+m^2} + i\left(\frac{1+m^3}{1+m^2}\right)\right\}}{1+im} = f'(0)$$

now this gives different values of limit for different m, hence limit values diff along $y = mx$.

so $f'(0)$ dne.

Constructing analytic fn

$f(z) = u(x,y) + iv(x,y)$ be analytic

if (i) Real part given
(ii) Imaginary part given

$\boxed{M-1}$

(i) Here u is known.

$v = v(x,y) \rightarrow dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \rightarrow (1)$

now CR $\rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow (2)$

so $dv = -\frac{\partial u}{\partial y} dx + i\left(\frac{\partial u}{\partial x}\right) dy = Mdx + Ndy$.

$\boxed{M-1}$ $dv = Mdx + Ndy \rightarrow (3)$

(3) is exact diff eqn.

$\int dv = \int Mdx + Ndy \rightarrow (4)$

$v = \int_{y \text{ as const}} Mdx + \int$ Independent terms of x in N dy

→ milnes thompson eqn

$\boxed{M-2}$ → u known then we want to find v

Partially diff $\rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

{wrt x}

$f(z) = u + iv$ $\rightarrow f'(z) = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} i$ {by CR}

replace $x \rightarrow z$, $y \rightarrow 0$ and integrate

$\int f'(z) dz = \int\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) dz + c$

$f(z) = \int\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) dz + c$

↓ (3) is reqd analytic fn

put $z = x + iy$ in (3) and we get v

Q) $u = x^2 - y^2 - x$ find v is real part of analytic

ans) Method (1) → $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$

$u = x^2 - y^2 - x \rightarrow \frac{\partial u}{\partial x} = 2x - 1$

$\frac{\partial u}{\partial y} = -2y$.

$\int dv = \int 2y dx + \int 2x - 1 dy$

$v = \int_{y \text{ as const}} 2y dx + \int 0 - 1 dy + c$ {independant of x}

$\boxed{v = 2xy - y + c}$

Method ② → milnes thompson

$u = x^2 - y^2 - x$

$v = ?$

$f'(z) = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} i$

$= 2x - 1 + i(2y) \rightarrow$ ①

$x \rightarrow z,\ y \rightarrow 0$

$f'(z) = 2z - 1 + i(0)$

$f'(z) = 2z - 1$

→ integrating wrt z

$\int f'(z)\,dz = \int 2z - 1\,dz + c$

$f(z) = \cancel{2}\frac{z^2}{\cancel{2}} - z + c$

$f(z) = z^2 - z + c$

$u + iv = (x+iy)^2 - x - iy + c$

$= x^2 - y^2 + i\,2xy - x - iy + c$

$= (x^2 - y^2 - x) + (2xy - y + c)i$

so $v = 2xy - y + c$

{ c can be made ic }

Q) $u = 3x - 2xy$, find v, express $f(z)$ in z $\boxed{m-2}$

ans) $\frac{\partial u}{\partial x} = 3 - 2y,\ \frac{\partial u}{\partial y} = -2x$

$f'(z) = 3 - 2y + i(2x)$

→ $x \rightarrow z,\ y \rightarrow 0$

$f'(z) = 3 + i(2z)$

integrating →

$f(z) = 3z + iz^2 + c$

$\boxed{f(z) = 3z + iz^2 + c}$

$u + iv = 3x + 3iy + i(x^2 - y^2 + i2xy) + c$

u + i

so

$\boxed{M-1}$

(ii) Con

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$u + iv = 3x - 2xy + i(3y + x^2 - y^2 + c)$

so $\boxed{v = 3y + x^2 - y^2 + c}$

$\boxed{M-1}$ →

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$\int dv = \int \underset{M}{2x}\,dx + \int \underset{N}{3 - 2y}\,dy$$

$$v = x^2 + 3y - y^2 + c$$

$$\boxed{v = x^2 - y^2 + 3y + c}$$

(ii) Construct analytic fn if u isnt given but v given →

Method 1 → $f(z) = u + iv$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \rightarrow (1)$$

CR eqn → $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(1) becomes →

$$du = \underset{M}{\left(\frac{\partial v}{\partial y}\right)}dx + \underset{N}{\left(-\frac{\partial v}{\partial x}\right)}dy$$

$$du = M\,dx + N\,dy \rightarrow (4)$$

$$\int du = \int M\,dx + N\,dy$$

$$u = \underset{y \text{ as const}}{\int M\,dx} + \int \text{independent terms of } x \text{ in } N\,dy + c$$

Method 2 → milnes Thompson method.

let $f(z) = u + iv$

v → known, u → find

$f(z) = u + iv \rightarrow (1)$

differentiate partially wrt x

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial v}{\partial y} + \frac{i\partial v}{\partial x} \quad \{CR\ eqn\} \to (2)$$

replace x by z, y by 0 and integrate

$$\int f'(z) = \int\left(\frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}\right) dz + c$$

$$f(z) = \int\left(\frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}\right) dz + c$$

analytic fn if v known.

Problems →

Q) analytic fn with $v = \sinh x \cos y$ find.

ans) Method (1)

$\rightarrow du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$

$\rightarrow du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dx.$

$\frac{\partial v}{\partial x} = \cosh x \cos y$, $\frac{\partial v}{\partial y} = -\sinh x \sin y$

so $u = \int(-\sinh x \sin y)\, dx + 0 + c$

y const {x independent}

$$\boxed{u = -\sin y \cosh x + c}$$

so $\boxed{f(z) = (-\sin y \cosh x + c) + i \sinh x \cos y}$

$\left\{ \begin{array}{l} \int \sin x = -\cos x \\ \int \cos x = \sin x \end{array} \right.$ $\quad$ $\left. \begin{array}{l} \int \cosh x = \sinh x \\ \int \sinh x = \cosh x \end{array} \right\}$

$\Rightarrow -\sin y \cosh x + i \sinh x \cos y + c$

$\Rightarrow -\sin y \cosh x + i \sinh x \cos y + c$

$\Rightarrow (-\sin y \cos ix + \sin ix \cos y) + c$

$\quad A \qquad B$

$\rightarrow \sin(ix - y) + c$

$\rightarrow \sin(ix + i \times iy) + c$

$\rightarrow \sin(i(x + iy)) + c$

$\rightarrow \sin iz + c$ (ans)

Method 2

$f'(z) = (-\sinh x \sin y) + i \cosh x \cos y$

$f'(z) \Rightarrow 0 + i \cosh z \quad \{x \to z, y \to 0\}$

$f(z) = \int i \cosh z \, dz + c$

$\boxed{f(z) = i \sinh z + c}$

$u + iv = i \sinh(x + iy) + c$

$= i[\sinh x \cosh iy + \cosh x \sinh iy] + c$

$= i[\sinh x \cos y + \cosh x \, i \sin y] + c$

$= i[\sinh x \cos y$

$\boxed{\{\cosh(iy)\} = \cos y}$

$\boxed{\{\sinh(iy)\} = i \sin y}$

so comparing $\rightarrow \boxed{u = -\cosh x \sin y + c}$

Q) $v = x^2 \cos 2\theta + x \cos\theta + 2$ find $f(z)$

ans) $f(z) = u + iv$

$\quad x \cos\theta + 2$

$v = x^2 \cos 2\theta + \cos\theta$

$v_x = 2x \cos 2\theta + \cos\theta$

$v_\theta = -2x^2 \sin 2\theta - x \sin\theta$

Partially diff →

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f'(z) = \frac{1}{r}\frac{\partial v}{\partial \theta} + i\frac{\partial v}{\partial x}$

$= \frac{1}{r}(-2r^2 \sin 2\theta - r\sin\theta) + i(2r\cos 2\theta + \cos\theta)$

$f'(z) = -2r\sin 2\theta - \sin\theta + i(2r\cos 2\theta + \cos\theta)$

$r \to z$ } milnes thompson
$\theta \to 0$

$f'(z) = (2z+1)i$

$f(z) = (z^2+z)i + c$ (ans)

$z = re^{i\theta}$ or $r(\cos\theta + i\sin\theta)$

$u+iv = \{(re^{i\theta})^2 + re^{i\theta}\}i + c$

$u+iv = i\{r\cos 2\theta + ir\sin 2\theta + r\sin\theta i + r\cos\theta\} + c$

→ comparing we get $u = -r^2\sin 2\theta - r\sin\theta$
and $c = 2$ as given in Q.

Harmonic f^n of an analytic f^n

let $f(z) = u+iv$ be analytic then

(i) Real part u of $f(z)$ is harmonic if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(ii) v harmonic if $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Harmonic conjugate → Real and imaginary part of analytic are always harmonic

① if $f(z) = u+iv$ be analytic then u, v are always harmonic but converse need not be true.

② if $u+iv$ be analytic f^n then v is harmonic conjugate of u

Q) find harmonic conjugate of $u = x^2 - y^2$ and check if it is
u is harmonic

ans) u harmonic conj v.

$u = x^2 - y^2$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial^2 u}{\partial x^2} = 2$

$\frac{\partial u}{\partial y} = -2y$, $\frac{\partial^2 u}{\partial y^2} = -2$

as $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + (-2) = 0$

$u \to$ harmonic proved.

$v \to$

$f(z) = u + iv$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$f'(z) = 2x + 2yi$

$x \to z$, $y \to 0$

$f'(z) = 2z$

$\boxed{f(z) = z^2 + c}$

$u + iv = x^2 + i^2y^2 + 2xyi + ic$

$u + iv = x^2 - y^2 + i(2xy + c)$

so $\boxed{v = 2xy + c}$ (harmonic conj of u).

Q) 10 marker HOT question $\to$

find $f(z) = u + iv$ where $u - v = e^x(\cos y - \sin y)$

? $f(z)$ has to be analytic ?

ans) $f(z) = u + iv$

$i f(z) = iu - v$

adding $\to$

$f(z)\{1 + i\} = u - v + i(u + v)$

$F(z) = U + iV$ where $F(z) = f(z)\{1 + i\}$

$U = u - v$, $V = u + v$

$U = e^x \cos y - e^x \sin y$ {given}

so $\frac{\partial V}{\partial x} = e^x \cos y - e^x \sin y$

$\frac{\partial U}{\partial y} = e^x\{-\sin y - \cos y\}$

$F'(z) = \frac{\partial U}{\partial x} + \frac{i\partial V}{\partial x} = \frac{\partial u}{\partial x} - \frac{i\partial u}{\partial y}$

$F'(z) = e^x(\cos y - \sin y) + e^x(\sin y + \cos y)i$

$x \to z, y \to 0$

$F'(z) = e^z + e^z \times i$

$F'(z) = e^z(1+i)$

$F(z) = e^z(1+i) + C = (1+i)f(z)$

$\boxed{f(z) = e^z + C_1 \{C_1 = \frac{C}{1+i}\}}$

Q) find $f(z)$, $z = x+iy$ if $u - v = (x-y)(x^2+4xy+y^2)$

ans) $F(z) = U + iV$

$F(z) \to (1+i)f(z)$, $U \to u-v$, $V \to u+v$

$U = (x-y)(x^2+4xy+y^2)$

$\frac{\partial U}{\partial x} = (x-y)[2x+4y] + 1[x^2+4xy+y^2]$

$\frac{\partial u}{\partial y} = x-y[4x+2y] - 1(x^2+4xy+y^2)$

$F'(z) = \frac{\partial U}{\partial x} - \frac{i\partial U}{\partial y}$

$= (x-y)(2x+4y) + (x^2+4xy+y^2)$

$- i\{x-y(4x+2y) - 1(x^2+4xy+y^2)\}$

$x \to z, y \to 0$

$F'(z) = 2z^2 + z^2 - i\{z^2 \times 4 - z^2\}$

$$F'(z) = 3z^2 - i3z^2$$

$$F'(z) = 3z^2(1-i)$$

$$F(z) = z^3(1-i) + c$$

$$\boxed{f(z) = \frac{1-i}{1+i} z^3 + c}$$

$$f(z) = -iz^3 + c$$