Transformation of $w=z^2$, $w=e^z$

(i) $w = z^2 \rightarrow (1)$

let $z = x+iy$, $w = u+iv$

$u+iv = x^2 + 2xyi - y^2$

$u = x^2-y^2$, $v = 2xy$

Q) find image of region boundary $x=1$, $x=2$, $y=1$, $y=2$ under $w=z^2$

ans) given that transformation is $w=z^2$

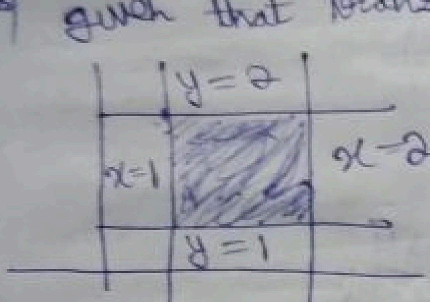


image of $x=1 \rightarrow$

$u = x^2 - y^2 = 1-y^2$.

$v = 2xy \rightarrow 2y$

so $u = 1-\left(\frac{v}{2}\right)^2$

$4u = 4 - v^2$

image of $x=2 \rightarrow$

$v^2 = -16(u-4)$

image of $y=1$

$u = x^2-1$

$v = 2x$

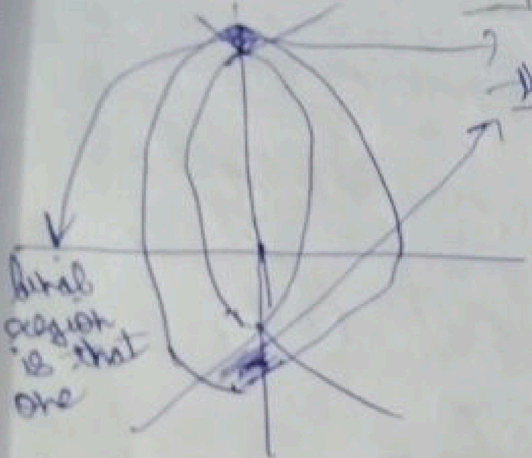
so $4u = v^2-4 \rightarrow v^2 = 4(u+1)$

image of $y=2$

if $y=2 \rightarrow$ $u = x^2-4$, $v = 4x$

$\rightarrow \boxed{16u = v^2-64}$

$v^2 = 16u+64$

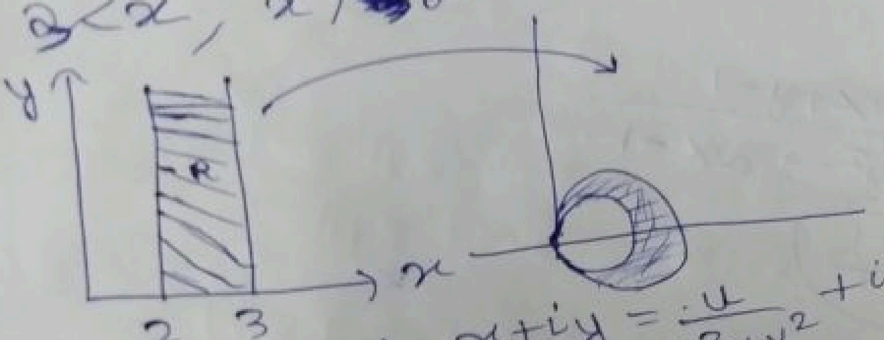


this or this

but as parabola are $\rightarrow$ +ve we pick the one on top.

Q) find image of $2 < x < 3$ under $w = \frac{1}{z}$

ans) $2 < x$, $x > 2$



as $w = \frac{1}{z} \rightarrow x + iy = \frac{u}{u^2+v^2} + i\left\{\frac{-v}{u^2+v^2}\right\}$

$2 < x < 3 \rightarrow$ we get $\frac{u}{u^2+v^2} > 2$, $\frac{u}{u^2+v^2} < 3$

$u^2+v^2 - 2 < 0$

$g = -\frac{1}{4}$, centre $\rightarrow \left(\frac{1}{4}, 0\right)$

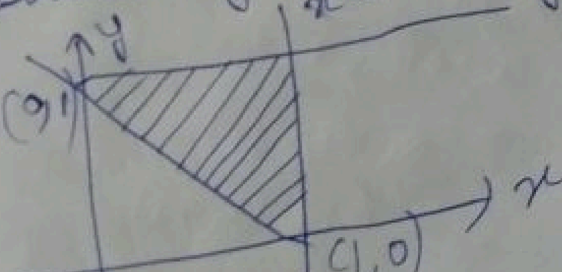
$r = \frac{1}{4}$

$u^2+v^2 - \frac{u}{3} > 0$

$g = -\frac{1}{6}$, $r = \frac{1}{6}$, centre $\rightarrow \left(\frac{1}{6}, 0\right)$

Q) find image of region bounded by $x=1, y=1, x+y=1$ under $w = z^2$ in z plane

ans) R is bounded by $x=1$, $y=1$, $x+y=1$



$w = z^2 \rightarrow u+iv = (x+iy)^2$

$= x^2 - y^2 + i(2xy)$

$u = x^2 - y^2, \; v = 2xy$

image of st line $x=1$

$u = 1 - y^2, \; v = 2y$

now line $\rightarrow v^2 = -4(u-1)$

image of st line $y=1$

$u = x^2 - 1, \; v = 2x$

$v^2 = 4u + 4$

image of $x+y=1$

$u = x^2 - y^2 \rightarrow 2x - 1$

$x = \frac{u+1}{2}$

$v = 2xy$

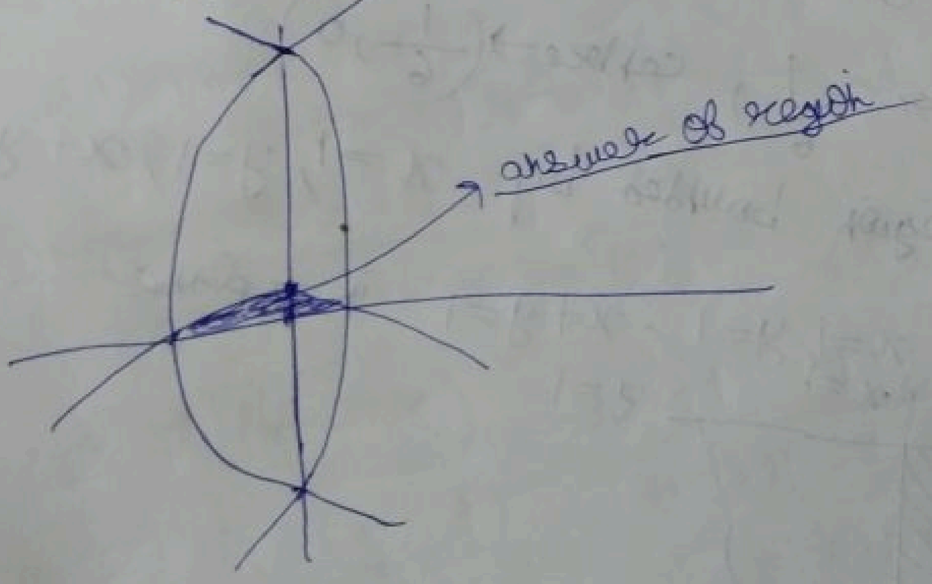
$= 2(x)(1-x)$

$= 2\left(\frac{u+1}{2}\right)\left(1 - \left(\frac{1+u}{2}\right)\right)$

$v = \frac{1}{2}(1-u^2)$

$2v = 1 - u^2$

$u^2 = 1 - 2v$



Transformation $w = e^z$ or exponential transformation

$z = x + iy$, $w = Re^{i\phi}$

Q) $Re^{i\phi} = e^{x} e^{iy}$ { for e^{x+iy} }

$R = e^{x}$, $\phi = y$ (ans)

Q) find image of $a \le x \le b$ and $c \le y \le d$ under the mapping $w = e^z$.

ans) $w = e^z$

let $z = x + iy$, $w = Re^{i\phi}$

$Re^{i\phi} = e^{x+iy}$

$Re^{i\phi} = e^{x} \times e^{iy} \to \phi$

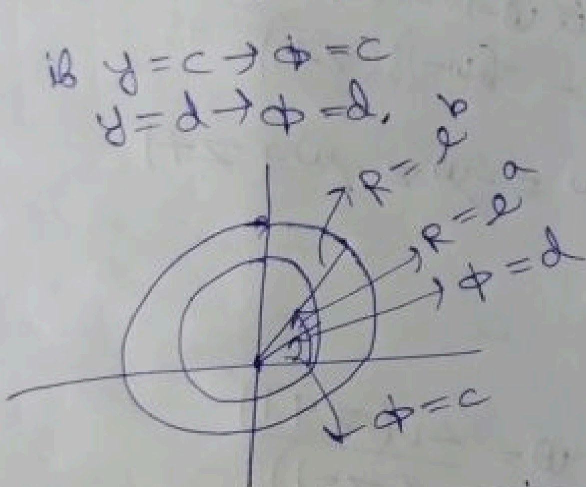
$\downarrow R$

$a \le x \le b$, $e^a \le e^x \le e^b$. if $x = a \to R = e^a$

if $x = b \to R = e^b$

if $y = c \to \phi = c$

$y = d \to \phi = d$,



Bilinear transformation or linear functional transformation or mobius transformation

$w = \frac{az+b}{cz+d}$ if $ad - bc \neq 0$ is bilinear

→ transformation

transformation, $a, b, c, d \to$ constants

if $w = \frac{az+b}{cz+d}$ is a BLT the inverse of a BLT is $z = \frac{b - dw}{cw - a}$

$w = \frac{az+b}{cz+d}$

fixed or invariant points

let $f(z) = w = \frac{az+b}{cz+d}$ is a BLT then fixed point of BLT is $f(z) = z$.

ie $\to z = \frac{az+b}{cz+d} \to cz^2 + (d-a)z - b = 0$

$z = \alpha, \beta \to$ fixed pts

Cross ratio

let z_1, z_2, z_3, z are pts in z plane transformed to points in w plane as w, w_1, w_2, w_3 then cross ratio is

$$\frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)} = \frac{(w-w_2)(w_1-w_3)}{(w-w_3)(w_1-w_2)}$$

Q) find BLT which maps $z_1=0, z_2=-i, z_3=-1$ into $w_1=i$, $w_2=1$, $w_3=0$ resp.

ans) $$\frac{(z+i)(0+1)}{(z+1)(0+i)} = \frac{(w-1)(i-0)}{(w-0)(i-1)}$$

$$\rightarrow \frac{z+i}{i(z+1)} = \frac{i(w-1)}{w(i-1)}$$

$$(z+i)\,w(i-1) = i(w-1)\,i(z+1)$$

$$=(wz+iw)(i-1) = -(w-1)(z+1)$$

$$\Rightarrow wiz - \cancel{wz} - \cancel{w} - iw = -\cancel{wz} - \cancel{w} + z + 1$$

$$\rightarrow wiz - iw = z+1$$

$$\rightarrow iw(z-1) = z+1$$

$$w = \frac{z+1}{i(z-1)}$$

$$w = \frac{(z+1)-i}{(-1+z)} \rightarrow w = \frac{-i(z+1)}{(z-1)}$$

verify with all points then to check.

Q) determine the BLT which maps $z_1=0, z_2=1, z_3=\infty$ into $w_1=i$, $w_2=-1$, $w_3=-i$ resp.

ans) $$\frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)} = \frac{(w-w_2)(w_1-w_3)}{(w-w_3)(w_1-w_2)}$$

$$\frac{(z-1)(0-\infty)}{(z-\infty)(0-1)} = \frac{(w+1)(i+i)}{(w+i)(i+1)}$$

$$\frac{(z-1)\, lt_{h\to\infty}(0-h)}{lt_{h\to\infty}(z-h)\,(-1)} = \frac{2i(w+1)}{(w+i)(i+1)}$$

by L Hopital rule →

$$\frac{(z-1)\lim_{h\to\infty}(-1)}{\lim_{h\to\infty}(w-1)(-1)} = \frac{2i(w+1)}{(w+i)(i+1)}$$

$$\to \frac{-(z-1)}{1} = \frac{2i(w+1)}{(w+i)(i+1)}$$

$$\to -(z-1) = \frac{2i(w+1)}{(w+i)(i+1)}$$

$$\to -(wiz+wz-z+iz-iw-w+1-i) = 2iw+2i$$

$$w(-i-iz-z+1) + (-i+z-iz-1) = 0$$

$$\to w(-i-iz-z+1) = -(-i+z-iz-1)$$

$$\to w(-i-iz-z+1)$$

$$\to w = \frac{-(-i+z-iz-1)}{(-i-iz-z+1)}$$

$$\to w = \frac{-((-i-1)+z(1-i))}{[(-i+1)+z(1+i)]}$$

$$w = \left[\frac{(1+i)-z(i-1)}{(1-i)-z(i+1)}\right] \text{ (ans)}$$

Q) find fixed point of $w = \frac{z-1}{z+1}$

ans) $f(z) = z \to w = z$

$$\to \frac{z-1}{z+1} = z$$

$$\to z^2+1 = 0$$

$$\to z = \pm i \to \text{fixed points}$$

Q) find BLT if fixed pts are i and $-i$

ans) if $z=\alpha$, $z=\beta$ are among fixed pts

$(z-\alpha)(z-\beta)=0$

$z^2-(\alpha+\beta)z+\alpha\beta=0$

$z^2+1=0$ comparing with $cz^2+(d-a)z-b=0$

so $c=1$, $b=-1$, $d-a=0 \rightarrow d=a$

now $\alpha+\beta=\frac{-b}{a} \rightarrow i-i=\frac{-1}{a} \rightarrow 0=\frac{1}{a}$

now $\alpha\beta=\frac{c}{a} \rightarrow i(-i)=\frac{1}{a} \rightarrow a=1$

so $d=1$

so $w=\frac{az+b}{cz+d}=\frac{1(z)-1}{z+1}=\frac{z-1}{z+1}$