Power - Series

$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots \dots \Rightarrow$ sum $= \frac{a}{1-r} \rightarrow \boxed{\frac{1}{1-z}}$

for $\frac{1}{1-z}$ → Region of convergence is $|z| < 1$

points at which f^n fails to be analytic are called singular point of $f(z)$

If $f(z)$ → not analytic at z_0 then $f(z)$ can't be expressed as taylor series about z_0. If radius of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is R_1 and if ROC of $\sum_{n=0}^{\infty} b_n(z-z_0)^n$ is R_2 → then sum of series is $\min(R_1, R_2)$

Q) find taylor series exp of $f(z)$ about $z = z_0$ and region of convergence.

① $f(z) = \sin z$ about $z_0 = \pi$ →

ans) power series → $\sum a_n(z-z_0)^n$, $a_n = \frac{f^n(z_0)}{n!}$

$a_0 = f(z_0) = \sin \pi = 0$, $a_1 = \frac{f'(z_0)}{1!} = \frac{\cos \pi}{1!} = \frac{-1}{1!}$

$a_2 = \frac{f''(z_0)}{2!} = \frac{0}{2!} = 0$, $a_3 = \frac{f'''(z_0)}{3!} = \frac{1}{6}$

$\sin(z)|_{\pi} = 0 + \left(\frac{-1}{1!}(z-\pi)\right) + 0 + \frac{1}{6}(z-\pi)^3 + \dots \dots$

Q) $f(z) = \frac{z}{(z+1)(z-2)}$

ans) $\frac{z}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2} \rightarrow A = \frac{+1}{3}, B = \frac{2}{3}$

$f(z) = \frac{2}{z-2} \frac{-1}{z-1}$

Complex Integration

Radius of convergence and circle of convergence let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series and R → any +ve real constant and the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is convergent for all z lying within circle $|z-z_0| = R$ then R → radius of convergence and $|z-z_0| = R$ is called circle of convergence and $|z-z_0| < R$ → region of convergence. every power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ can be expressed as analytic f^n within its region of convergence $|z-z_0| < R$

every analytic f^n within its Region of convergence can be expressed as a complex power series or taylor series

let $f(z) \to$ analytic within circle $|z-z_0| = R$.
then $f(z)$ can be expressed as infinite series of form $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, $a_n = \frac{f^n(z_0)}{n!}$

Note $\to$ $|z|<1$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 \dots (-1)^n x^n + \dots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots x^n$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 \dots + (-1)^n n x^n$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots n x^{n-1}$

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \frac{x^n}{n!}$

$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \dots \frac{(-1)^n x^n}{n!}$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^n x^{2n-1}}{(2n-1)!}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots (-1)^n \frac{x^{2n}}{(2n)!}$ $|x|<\infty$

$|z|<\infty$ for all these

Q) (i) $\sin z$ in taylor about $z_0 = \pi$

(ii) obtain exp of $\frac{z}{(z+1)(z-2)}$ about $z=1$

ans) (i) $\sin z = \sin(z - \pi + \pi)$

$= \sin((z-\pi)+\pi)$

$= \sin(\pi + (z-\pi))$

$= -\sin(z-\pi)$

so $f(z) = -\left[(z-\pi) - \frac{(z-\pi)^3}{3!} + \frac{(z-\pi)^5}{5!} \ldots\right]$ (ans)

(ii) ans) $f(z) = \frac{1}{3(z+1)} + \frac{2}{3(z-2)}$ {partial fract}

as $z = 1 \rightarrow \boxed{z-1=0}$

let $u = z-1 \rightarrow z = u+1$

so $f(z) = \frac{1}{3}\left\{\frac{1}{u+1+1}\right\} + \frac{2}{3}\left\{\frac{1}{u+1-2}\right\}$

$= \frac{1}{3(u+2)} + \frac{2}{3(u-1)}$

$= \frac{1}{3}\left\{\frac{1}{2\left(1+\frac{u}{2}\right)}\right\} + \frac{2}{3}\left(\frac{1}{(-1)(1-u)}\right)$

$= \frac{1}{6}\left(1+\frac{u}{2}\right)^{-1} - \frac{2}{3}(1-u)^{-1}$

$= \frac{1}{6}\left\{1 - \frac{u}{2} + \left(\frac{u}{2}\right)^2 - \left(\frac{u}{2}\right)^3 + \ldots\right] - \frac{2}{3}\left[1+u+u^2+u^3 \ldots\right]$

$= \frac{1}{6}\left\{1 - \left(\frac{z-1}{2}\right) + \left(\frac{z-1}{2}\right)^2 - \frac{(z-1)^3}{2} + \ldots\right\} - \frac{2}{3}\left[1 + (z-1) + (z-1)^2 + (z-1)^3 + \ldots\right\}$

RO convergence of 1st is $\rightarrow |z-1| < 2$ so $\boxed{R_1 = 2}$

RO convergence of 2nd is $|z-1| < 1$ so $\boxed{R_2 = 1}$

RO of entire is $\min(R_1, R_2)$ so $\boxed{R = 1}$ (ans).

(iii) $f(z) = \frac{z-1}{z+1} \rightarrow z = 0$

$\rightarrow$ ans) $(z-1)(1+z)^{-1}$

$= (z-1)\left[1 - z + z^2 - z^3 + z^4\right]$

$= (z - z^2 + z^3 - z^4 + z^5 \ldots) + (-1 + z - z^2 + z^3 \ldots)$

$= -1 + 2z - 2z^2 + 2z^3 - 2z^4 \ldots$

so $|z|<1$ as no possible pattern found as closest thing matching is $-2+2z-8z^3+16z^4 \cdots$ for which

or $-2+2z-(2z)^3+(2z)^4 \cdots$

$|2z|<1$ and this is different.

<u>diff method (same Q) →</u>

$z=1 \rightarrow$ so $z-1=0$ so let it be u

$\boxed{u=z-1}$

so $f(z)=\dfrac{u}{u+2} = \dfrac{u}{2}\left(1+\dfrac{u}{2}\right)^{-1}$

$$\rightarrow \frac{u}{2}\left[1-\frac{u}{2}+\left(\frac{u}{2}\right)^2-\left(\frac{u}{2}\right)^3+\cdots\right)$$

$$\rightarrow \frac{u}{2}-\left(\frac{u}{2}\right)^2+\left(\frac{u}{2}\right)^3-\left(\frac{u}{2}\right)^4 \cdots$$

$$=\frac{z-1}{2}-\left(\frac{z-1}{2}\right)^2+\left(\frac{z-1}{2}\right)^3 \cdots$$

so ROC $\rightarrow \left|\dfrac{z-1}{2}\right|<1 \rightarrow \boxed{|z-1|<2}$

so ROC is 2

<u>Note</u> → (1) 1, 2, 3, 4, ...

(2) 1, 2, 3, 4 ......

(3) 1, 2, 3 ...

(4) 1, 2, 3 ....

out of this (4) only is correct as ... is the number of correct dots. it means <u>and so on</u>

Laurents Series

Let $C_1 : |z-z_0| < r_1$ and $C_2 : |z-z_0| < r_2$, $r_2 < r_1$
Let $f(z)$ be analytic fn in the annulus region $r_2 < |z-z_0| < r_1$
then power series expansion of $f(z)$ about the point $z = z_0$ given by $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ (1)

where $a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{(z-z_0)^{n+1}} dz$

$b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{(z-z_0)^{-n+1}} dz$

the power series eq(1) is laurents series of given fn

Q) expand $\frac{-1}{(z-1)(z-2)}$ as power series in $|z| < 1$, $1 < |z| < 2$

$|z| > 2$

ans) $f(z) = \frac{1}{z-1} - \frac{1}{z-2}$

→ ① if → $|z| < 1$ → $\frac{1}{(-1)\{1-z\}} - \frac{1}{-2(1-\frac{z}{2})}$

$\Rightarrow -(1-z)^{-1} + \frac{1}{2}(1-\frac{z}{2})^{-1}$

$\Rightarrow -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ (ans)

② if $1 < |z|$ or $|z| < 2$ → $|z| > 1$ or $2 > |z|$

→ $f(z) = \frac{1}{z-1} - \frac{1}{z-2}$

$= \frac{1}{z(1-\frac{1}{z})} - \frac{1}{-2(1-\frac{z}{2})}$

$= \frac{1}{z}\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 \cdots\right] + \frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 \cdots\right]$

$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ (ans)

③ $|z| > 2$ → $f(z) = \frac{1}{z-1} - \frac{1}{z-2}$

$\Rightarrow \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1}$

$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$

$f(z) = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$ (ans)

Q) $\frac{z}{(z-1)(z-2)}$ in laurent for $|z|<1$, $1<|z|<2$, $|z|>2$, $|z-1|>0$, $0<|z-1|<1$

ans $f(z) = \frac{1}{z-1} - \frac{2}{z-2}$

(4) for $|z-1|>0 \rightarrow z-1=u \rightarrow z=u+1$ now $|u|>0$.

$f(z) = \frac{1}{u} - \frac{2}{u-1}$

$= \frac{1}{u-0} - \frac{2}{u-1}$

$= \frac{1}{u} - \frac{2}{u(1-\frac{1}{u})}$

$= \frac{1}{u} - \frac{2}{u}\sum_{n=0}^{\infty}\left(\frac{1}{u}\right)^n$

$= \frac{1}{z-1} - 2\sum_{n=0}^{\infty}\frac{1}{(z-1)^{n+1}}$ (ans)

(2) $0<|z-1|<1 \rightarrow$

let $z=1+u$

$f(z) = \frac{1}{z-1} - \frac{2}{z-2}$

$= \frac{1}{u} - \frac{2}{u-1}$

$= \frac{1}{u} + 2(1-u)^{-1}$

$= \frac{1}{u} + 2\left\{1 + u + (u)^2 + (u)^3 \dots\right\}$

$= \frac{1}{u} + 2\sum_{n=0}^{\infty} u^n$

$= \frac{1}{z-1} + 2\sum_{n=0}^{\infty}(z-1)^n$

if we have $|z-1|=1$ we only take number common

Q) $f(z) = \frac{2z-5}{(z^2-1)(2-z)}$ in $|z|>1$, $1<|z|<2$, $|z|>2$

ans) $A = \frac{7}{6}$, $B = -\frac{3}{2}$, $C = \frac{1}{3}$

$\rightarrow f(z) = \frac{7}{6}\frac{1}{z+1} - \frac{3}{2}\frac{1}{(z-1)} + \frac{1}{3}\left(\frac{1}{z-2}\right)$

for $|z|>1 \rightarrow$

$f(z) = \frac{7}{6}\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{3}{2}\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{3}\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$

$= \frac{7}{6}\frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}\cdots\cdots(-1)^n\frac{1}{z^n}\right)$

$-\frac{3}{2}\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}\cdots\cdots\right]$

$+\frac{1}{3}\frac{1}{z}\left[1+\frac{2}{z}+\frac{2^2}{3}+\cdots\right]$

$= \frac{7}{6}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{n+1}} - \frac{3}{2}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \frac{1}{z}\sum_{n=0}^{\infty}\frac{2}{z^{n+1}}$

for $1<|z|<2 \rightarrow$ $|z|>1$, $|z|<2$

$\frac{7}{6}\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{3}{2}\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1-\frac{z}{2}\right)^{-1}$

$= \frac{7}{6}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{n+1}} - \frac{3}{2}\sum_{n=\infty}^{2}\frac{1}{z^{n+1}} - \frac{1}{3}\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$

NOTE

when $|z|>1 \rightarrow$ we take z common everywhere

when $|z|<1 \rightarrow$ number common everywhere

$|z|<2 \rightarrow$ wherever number is 2 we take number common everywhere else $\rightarrow z$ common

when $|z|>2 \rightarrow z$ common everywhere

Singularities

Singular point types → Removable, → Pole, → essential

if no negative power of z → removable
if infinite no of negative power of z → essential
if finite → pole

Singular point → point $z = z_0$ is said to be singular point of $f(z)$ when $f(z)$ fails to be analytic at the point $z = z_0$

note → let $z = z_0$ be a singular point of $f(z)$ and the laurents series expansion of $f(z)$ about $z = z_0$ is

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{analytic part}} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{principle part}}$$

classification → there are 3 types of S.P →
(i) Removable S.P (ii) Pole (iii) Essential S.P

(i) if principle part of laurent series exp. of $f(z)$ about z_0 has no terms
} no -ve power of z }

(ii) if principle part of laurent series exp of $f(z)$ about z_0 has m terms where m is +ve integer and finite then $z = z_0$ is pole of order m
pole of order 1 is simple pole

(iii) if infinite terms in principal terms. then it is essential S.P

Q) $f(z) = \frac{e^z}{z^2}$ and $\frac{1-\sin z}{z^5}$ of $z=0$

ans) (i) $\frac{1}{z^2}\left[1 + \frac{z}{1!} + \frac{z^2}{2!} \cdots\right]$

$= \frac{1}{z^2} + \frac{1}{z \cdot 1!} + \frac{1}{2!} + \frac{z}{3!} + \cdots$

→ $\boxed{z = 0}$ is ~~simple~~ double pole of order 2
for order 2

(ii) $\frac{1}{z^5}\{1 - \sin z\} \to \frac{1}{z^5}\left\{1 - \left\{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \cdots\right\}\right\}$

$= \frac{1}{z^5} - \frac{1}{z^4} + \frac{1}{3!}\frac{1}{z^2} - \frac{1}{5!} + \frac{z^2}{7!} \cdots$

$z=0$ has pole order 5 (ans)

Q) find singularity of $f(z)=\dfrac{\tan z}{z^4-\frac{1}{z^3}}$

ans) $\tan z = z+\frac{z^3}{3}+\frac{z^5}{5}\dots$

$$f(z)=\frac{\tan z}{\frac{z^2-1}{z^2}}$$

$$=\frac{-z^3}{(1-z^7)}\tan z$$

$$=\frac{-z^3}{(1-z^7)}\left[z+\frac{z^3}{3}+\frac{z^5}{5}+\dots\right]$$

$$f(z)=-z^3(1-z^7)^{-1}\left(z+\frac{z^3}{3}+\frac{z^5}{5}\dots\right)$$

$$f(z)=-z^3\left(1+z^7+(z^7)^2+(z^7)^3\dots\right)\left(z+\frac{z^3}{3}+\frac{z^5}{5}\dots\right)$$

$\to$ no $-$ve power of $z \to$ Removable.

Q) $\dfrac{1}{z^3(z-5)^2}$ $\to$ find singularity

ans) $z^3(z-5)^2=0$

$z=0$ and $z=5$ are singularpt

$z=0 \to$ order 3, $z=5 \to$ order 2

Q) $f(z)=\dfrac{1}{z^2(z^2+1)(z-2)^2}=0$

ans) $z^2(z^2+1)(z-2)^2=0$

$z^2 \to 0 \to$ order 2

$(z^2+1) \to +1,-1 \to$ order 1

$(z-2)^2 \to 2 \to$ order 2

Residues $\to$ coeff of $\frac{1}{z-z_0}$ in laurent expansion of $f(z)$ about singular point is residue

Residue at simple pole $\to$ if $z=z_0$ is a sample pole of $f(z)$ then $\underset{z=z_0}{\text{Res}}\, f(z)=\lim_{z\to z_0}(z-z_0)f(z)$

→ Residue at pole of order m:

if z_0 has pole of order m then $\underset{z=z_0}{\text{Res}} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left((z-h)^m f(z) \right)$

Residue at ∞

$\underset{z=\infty}{\text{Res}} f(z) = -\underset{w=0}{\text{Res}}\, g(w)$ where $g(w) = \frac{1}{w^2} f\left(\frac{1}{w}\right)$

Note → if $f(z) = \frac{\phi(z)}{\psi(z)}$ such that $z = z_0$ is simple solⁿ

$\underset{z=z_0}{\text{Res}} f(z) = \frac{\phi(z_0)}{\psi'(z_0)}$ provides $\phi(z)$ is analytic and $\psi'(z) \neq 0$

Q) find residue at singular point → $\frac{z}{(z+1)(z-2)}$, $\cot z$, $\frac{1}{z^3+z^5}$, $\frac{z+3}{z(z-1)(z+2)}$

ans (a) $z = -1$ → simple pole
$z = 2$ → simple pole

residue at $z=-1$ ∴ $z_0 = -1$

$\underset{z=-1}{\text{Res}} f(z) = \lim_{z \to -1} \cancel{z-(-1)} \left(\frac{z}{\cancel{(z+1)}(z-2)} \right)$

$= \lim_{z \to -1} \frac{z}{z-2}$

$= \frac{-1}{-3} = \frac{1}{3}$ (ans) now we do for $z_0 = 2$

$\underset{z \to 2}{\text{Res}} f(z) = \lim_{z \to 2} \cancel{(z-2)} \frac{z}{(z+1)\cancel{(z-2)}}$

$= \frac{2}{3}$ (ans)

(b) $\cot z = \frac{\cos z}{\sin z}$

singular points → $\sin z = 0 \to z = n\pi \to 0, \pm\pi, \pm 2\pi \ldots$

Residue is $\left. \frac{\phi(z)}{\psi'(z)} \right|_{z=z_0}$

$\psi'(z) = \cos z$

so Residue $= \frac{\phi(n\pi)}{\psi'(n\pi)} = \frac{\cancel{\cos n\pi}}{\cancel{\cos n\pi}} = 1$ (ans)

(c) $f(z) = \frac{1}{z^3+z^5}$

$\rightarrow z^3(1+z^2)=0 \rightarrow \pm i \rightarrow$ order 1 pole

$\downarrow$

0, pole of order 3

residue at $z=0 \rightarrow m=3$

$$\operatorname{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$$

$$\operatorname{Res}_{z=0} f(z) = \frac{1}{2!} \lim_{z\to 0}\left(\frac{z^3}{z^3(z+i)(z-i)}\right)^{**}$$ (double derivative)

$= -1$ (ans)

residue at $z=i$

$\lim_{z=i} \frac{1}{z^3(z+i)}$

$= \frac{1}{i^3(2i)}$

$= \frac{1}{2}$ (ans)

at $z=-i \rightarrow \lim_{z\to -i} \frac{1}{z^3(z-i)} \Rightarrow \frac{+1}{-i^3(-2i)} \Rightarrow \frac{1}{2}$ (ans)

Complex integration

smooth curve $\rightarrow z = z(t)$ $a \le t \le b$ if $z(t)$ is smooth curve $\rightarrow$ it is [redacted] continuously differentiable

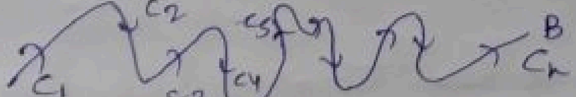
C_1, C_2, C_3, C_4

contour $\rightarrow$ smooth curve / piecewise smooth curve

Complex integral over a contour $\rightarrow$

let $f(z)$ be conti f^n defined in domain D and let C be a contour taken in D, then complex integral of $f(z)$ over contour C is given by $\int_C f(z)\,dz$

Note $\rightarrow$ if 'c' consists of 'n' piecewise smooth curves $C_1, C_2, \ldots\ldots C_n$ then $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz \ldots\ldots \int_{C_n} f(z)\,dz$



if piecewise smooth curve is in parametric form as

$C: z(t) = x(t) + iy(t)$ where $a \le t \le b$

then $\int f(z)dz = \int_{t=a}^{t=b} f(z(t))\, z'(t)\, dt$.

Q) evaluate $\int_C z^2 dz$ where $|z|=2$ and θ from 0 to $\frac{\pi}{3}$ $\rightarrow x^2+y^2=2$

ans) $|z|=2$, $\theta: 0$ to $\frac{\pi}{3}$

let $z = 2e^{i\theta}$

$dz = 2ie^{i\theta} d\theta$

$\int_C z^2 dz = \int_{\theta=0}^{\pi/3} (2e^{i\theta})^2 (2ie^{i\theta}) d\theta$

$= \int_{\theta=0}^{\pi/3} 4e^{2i\theta} \times 2ie^{i\theta} d\theta$

$= 8i\left(\frac{e^{3i\theta}}{3i}\right)\Big|_0^{\pi/3}$

$= \frac{-16}{3}$ (ans)

Q) $\int_C z^2 dz$ where $C \rightarrow$ straight line from $z=0$ to $z=2+i$

given that curve $C \rightarrow$ st. line

ans) $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = t$

$\frac{x-0}{2-0} = \frac{y-0}{1-0} = t \rightarrow \frac{x}{2} = y = t$

so $x = 2t$ | $y = t$

$dx = 2dt$ | $dy = dt$

at $O(0,0) \rightarrow x=0,\ 2t=0,\ t=0$

$y=0 \rightarrow t=0$

at $A(2,1) \rightarrow t=1,\ 1=t$

Parametric form of z is

$z(t) = 2t + it$

$z'(t) = (2+i)\, dt$

$\int_C z^2 dz = \int_{t=0}^{t=1} (2t+it)^2 (2dt + idt)$

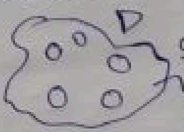
$$= \int_{t=0}^{1} (2+i)^2 t^2 (2+i)\, dt$$

$$= (2+i)^3 \int_{t=0}^{1} t^2 dt = (2+i)^3 \left(\frac{t^3}{3}\right)\Big|_{t=0}^{1}$$

$$= \frac{(2+i)^3}{3} \text{ (ans)}$$

Simple connected domain

a domain D is simple connected domain if every simple closed curve C is taken in D lies totally in D



simple connected domain

→ not simple connected

multiple connected domain

a domain not simple connected is called multiple connected

ex →

Cauchy's - Goursat theorem | cauchy's theorem

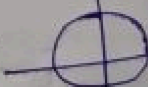
if $f(z)$ analytic in simple connected domain D then $\oint_C f(z)dz = 0$ for every simple closed curve C in D

Cauchy theorem for multiple connected domain →

if $f(z)$ is analytic in MCD 'D' bounding simple closed curves $C1, C2, C3 \ldots Cn$ then $\oint_C f(z)dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)dz + \ldots \int_{C_n} f(z)\, dz$

Q) evaluate $\oint_C (z^2+1)\,dz$ where $C \to |z|=1$

ans) $|z|=1 \to x^2+y^2=1$



simple connected

so by C-G theorem $\oint_{C:|z|=1} (z^2+1)\,dz = 0$

verification → $z = e^{i\theta} \to dz = ie^{i\theta} d\theta$

$\theta \to$ b/w $0, 2\pi$

$$\oint_{C:|z|=1} f(z)\,dz = \int_{C:|z|=1} (z^2+1)\,dz$$

$$= \int_0^{2\pi} (e^{i\theta})^2 + 1 \times ie^{i\theta} d\theta$$

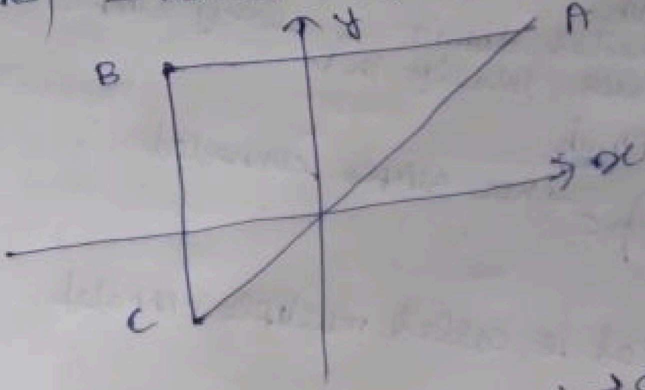
$$= i\int_0^{2\pi} (e^{2i\theta}+1) e^{i\theta} d\theta$$

$$= i\left[\frac{e^{3i\theta}}{3i} + \frac{e^{i\theta}}{i}\right]\Big|_0^{2\pi}$$

$= 0$ (ans)

Q2) verify C-G-T and integration along boundary of Δ $(1+i), (-1+i), (-1,-i)$ for e^{iz}

ans) Δ vertices $-(1,1), (-1,1), (-1,-1)$



along AB → line is $y = 1 \to dy = 0$

$z = x + iy$ so $z = x + i \to dz = dx$

$x \to 1$ to -1

$$\int_{C_1} f(z)\,dz = \int^{-1} e^{i(x+i)}\,dz$$

$$= \frac{1}{i}\left[e^{-i-1} - e^{i-1}\right]$$

along BC →

$x = -1 \to dx = 0$

$y = 1$ to $-1 \to dz = i\,dy$

$z = -1 + iy$

$$\int_{C_2} f(z)\,dz = \int_1^{-1} i e^{i(-1+iy)}\,dy$$

$$= -ie^{-i+1} + ie^{-i-1}$$

along CA →

$y = x \to dx = dy$

$z = x + ix \to dz = (1+i)\,dx$

$$\int_C f(z)\,dz = \int_{-1}^{1} e^{i(1+i)x} \times (1+i)\,dx$$

$$= -(1+i)\left[e^{i-1} - e^{1+i}\right]$$