$) + c$

$f(z) = -iz^3 + c$

application of analytic f^n → fluid flow
→ elec fields

$w = \phi + i\psi$ → stream function

complex f^n — velocity potential of a fluid flow

Consider a 2D flow of an irrotational and incompressible fluid ∥ xy plane. Let $\bar{V}$ be the velocity of fluid then it can be expressed $\bar{V} = \frac{\partial V}{\partial x} i + \frac{\partial V}{\partial y} j$

since motion is irrotation there exist a scalar f^n $\phi(x,y)$ such that $\bar{V} = \nabla\phi = \text{grad}\,\phi$

$\bar{V} = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y}$. from these 2, $\frac{\partial \phi}{\partial x} = \frac{\partial \bar{V}}{\partial x}$

and $\frac{\partial \phi}{\partial y} = \frac{\partial \bar{V}}{\partial y}$. Scalar f^n ϕ which gives velocity components $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ called velocity potential

if fluid is irrotational ~~[illegible]~~ $\bar{\nabla V} = 0$, $\nabla(\nabla\phi) = 0$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \rightarrow \phi \text{ is harmonic}$$

→ ϕ is always harmonic, can be real part of analytic f^n

$w = \phi + i\psi$, ψ → stream f^n

Q) if $w = \phi + iy$ represents complex pot of an elec field and $\psi = x^2 - y^2 + \frac{x^2}{x^2+y^2}$, then find ϕ

$L_1 = 207.14\,mm$ (ans)

ans) $w = \phi + i\psi$

ϕ → velocity pot, ψ → stream f^n

$$\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy.$$

CR of ① → $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$ and $\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$

$$\partial \phi = \frac{\partial \psi}{\partial y} dx - \frac{\partial \psi}{\partial x} dy$$

$$d\phi = \underbrace{-\left\{ 2y + \frac{2xy}{(x^2+y^2)^2} \right\}}_{M} dx \overbrace{- 2x + \frac{x^2-y^2}{(x^2+y^2)^2}}^{N} dy$$

$$\int d\phi = \int_{y \text{ as constant}} M\,dx + \int N\,dy$$

as x in N independent

$$\phi = -2y \int dx - y \int \frac{2x}{(x^2+y^2)^2} dx + c + 0$$

$$\phi = -2xy - y \int \frac{du}{u^2} \quad \left\{ \begin{array}{l} u = x^2+y^2 \\ du = 2x\,dx \end{array} \right\}$$

$$\phi = -2xy - \frac{y}{x^2+y^2} + c \text{ (ans)}$$

1Q) if velocity pot f^n is $\log\sqrt{x^2+y^2}$ find flux f^n ψ and complex potential f^n w

2Q) prove $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{4\partial^2}{\partial z \partial \bar{z}}$

3Q) if $u + iv$ → analytic then u, v are mutually orthogonal to each other

4Q) PT a analytic f^n with const real part is constant itself

Q1) ans) $\phi = \log\sqrt{x^2+y^2}$

$\rightarrow f'(z) = \frac{\partial\phi}{\partial x} + i\left(-\frac{\partial\phi}{\partial y}\right)$

$\rightarrow \phi = \log\sqrt{x^2+y^2}$

$\frac{\partial\phi}{\partial x} = \frac{x}{x^2+y^2}$, $\frac{\partial\phi}{\partial y} = \frac{y}{x^2+y^2}$

$x \rightarrow z$, $y \rightarrow 0$

$f'(z) = \frac{1}{z} \rightarrow \boxed{f(z) = \log z + c}$

$z = re^{i\theta}$

so $f(z) = \log(re^{i\theta}) + c$

$= \log r + i\theta$

$= \log r + i\tan^{-1}\frac{y}{x}$

$= \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$

$\downarrow$

ϕ

so $\boxed{\psi = \tan^{-1}\frac{y}{x}}$

Q2) $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\partial\bar{z}}$

ans) $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2} \times -i$

now $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z}$, $\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}}$

$x = \frac{1}{2}(z+\bar{z}) \Rightarrow \frac{\partial x}{\partial z} = \frac{1}{2}$, $\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$

$y = \frac{-i}{2}(z-\bar{z}) \Rightarrow \frac{\partial y}{\partial z} = \frac{-i}{2}$, $\frac{\partial y}{\partial \bar{z}} = \frac{i}{2}$

so $\frac{\partial f}{\partial z} = \frac{1}{2}\left\{\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right\}$

$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f$, $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \rightarrow (2)$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{i\partial}{\partial y}\right) f$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{i\partial}{\partial y}\right) \rightarrow (4)$$

$$(3)\times(4) \rightarrow \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{i\partial}{\partial y}\right)\frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{i\partial}{\partial y}\right)$$

$$\boxed{\frac{4\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}} \text{ proved.}$$

Q3) (ans)

let $u = c_1$, $v = c_2$

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \rightarrow \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \frac{\partial y}{\partial x} = m_1 \rightarrow (1)$$

similarly $\rightarrow \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0$

and $$\frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = \frac{dy}{dx} = m_2$$

by CR $\rightarrow$ $$\frac{\frac{\partial u}{\partial y}}{\frac{\partial y}{\partial x}} = m_2 \rightarrow (2)$$

(1)×(2) gives $\boxed{m_1 m_2 = -1}$

so orthogonal to each other.

$f(z) = u + iv$ is an analytic f^n

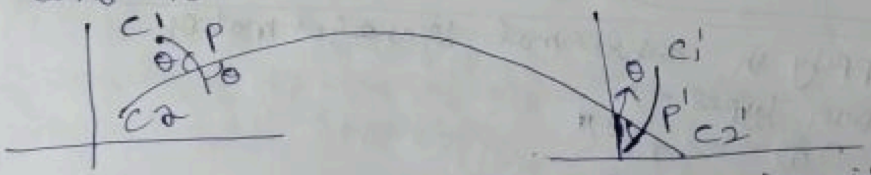
Module 2

Conformal and bilinear transformation

has both magnitude and sense

Conformal transformation

Conformal → let C_1, C_2 be two curves in z plane intersecting at P and their corresponding images C_1', C_2' at P' if angle of intersection of these curves at P in z plane is equal in magnitude and sense {direction} to the angle of intersection of the angles C_1', C_2' at P' then mapping is called to be conformal



the point at which $f(z)=0$ → critical points, where $\neq 0$ → ordinary points

at each point where $f(z)$ is analytic and $f'(z) \neq 0$ ~~the~~ mapping is conformal

The angle $\psi = \arg(f'(z))$ is called the angle of rotation and $|f'(z)|$ is called scale factor or coefficient of magnification magnification at z_0

A mapping that preserves an angle only in magnitude, but not in direction. then the mapping is isognal {isogonal}

The points at which fn is mapped onto itself is called fixed points. ie $f(z) = z$

Q) 1) determine angle of rotation at pt $z = \frac{1+i}{2}$ under mapping $w = z^2$ also find scale factor

2) find AOR produced by transformation $w = 2z - 1 + 2i$ at pt $z = 1 + 2i$

ans) 1) → GT → $w = z^2$, point $= \frac{1+i}{2}$

$f(z) = z^2$, $f'(z) = 2z$

$$\arg(f'(z_0)) = \arg\left(f'\left(\frac{1+i}{2}\right)\right)$$
$$= \arg\left(2\left(\frac{1+i}{2}\right)\right)$$
$$= \arg(1+i) = \frac{\pi}{4} \rightarrow \left\{\tan^{-1}\frac{1}{1}\right\}$$

so angle of rotation $= \frac{\pi}{4}$ (ans)

Scale factor $\rightarrow |f'(z_0)|$
$$= |1+i|$$

$\boxed{\text{Scale factor} = \sqrt{2} \text{ (ans)}}$

---

the conformal mapping in conformal transformation classified as four types $\rightarrow$

(i) translation $\{w = z + c\}$
(ii) Rotation $\{w = z e^{i\phi}\}$
(iii) magnification $\{w = cz\}$
(iv) inversion $\{w = \frac{1}{z}$ if $z \neq 0\}$

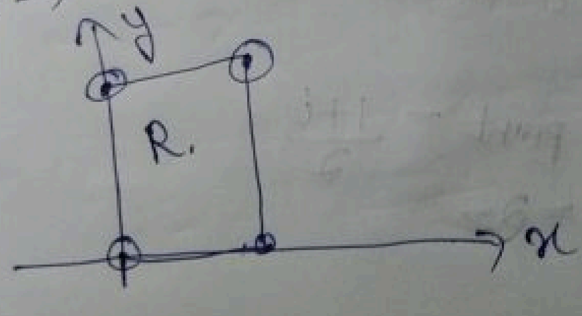
1) translation $\rightarrow$ $w = f(z) = z + c$

now $z = x + iy$
$$w = f(z) = x + iy + c_1 + ic_2 \quad \{c = c_1 + ic_2\}$$
$$= (x + c_1) + i(y + c_2)$$
$$= (x + c_1, y + c_2)$$

Problems

Q) let a rectangular region OABC with vertices $O(0,0)$, $A(1,0)$, $B(1,2)$, $C(0,2)$ in z plane find image of region in w plane under the mapping $w = z + 2 + i$.

ans) for z plane $\rightarrow$ x, y axes.
so $\rightarrow$



G.T → $w = f(z) = z + 2 + i$

$u + iv = (x+iy) + 2 + i$

$= (x+2) + (y+1)i$

so $(u, v) = (x+2, y+1)$

so points become → $(2,1)$, $(3,1)$, $(3,3)$, $(2,3)$

so rectangle (given) is mapped to $O'A'B'C'$ region in w plane.

Q) let rect domain R be bounded by $x=0$, $y=0$, $x=2$, $y=1$. determine region R' on w plane which R is mapped under transformation $w = z + (1-2i)$

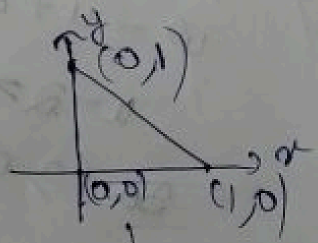
Rotation → Q) consider transformation $w = z.e^{i\frac{\pi}{4}}$ and determine the region R' in w-plane in the triangular region bounded by $x=0, y=0, x+y=1$ in z plane.

ans) let $z = x+iy$, $w = u+iv$

$u + iv = (x+iy)e^{i\frac{\pi}{4}}$

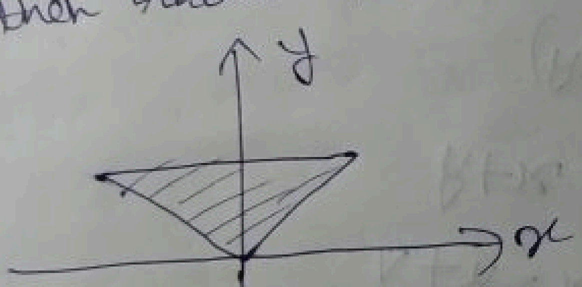
$u + iv = (x+iy)\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$u = \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}$, $v = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$



so → $(0,0)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

then redraw the triangle.



→ Next Page new Q

Q) a rect region ABCD with vertices $(2,1), (3,1), (3,3), (2,2)$ in z plane find image of region in w plane under mapping $w = ze^{i\frac{\pi}{4}}$

(iii) Magnification $\rightarrow w = cz$

let $z = re^{i\theta}$, $w = Re^{i\phi}$

$Re^{i\phi} = cre^{i\theta}$

$\phi = \theta$, $R = cr$

so $(r, \theta) \xrightarrow{w = cz} (cr, \theta)$

Q) consider $w = 2z$ and determine the region R' in w plane corresponding to the $\triangle$ region is mapped under the $x = 0, y = 0, x + y = 1$ map.

ans) $(0,0), (1,0), (0,1)$ are pts

$w = u + iv = 2z = 2(x + iy)$

$u = 2x$, $v = 2y$

so $(0,0)$, $(2,0)$, $(0,2)$

or

$u = 2x \rightarrow x = \frac{u}{2}$, $\frac{v}{2} = y$.

so $x = 0 \rightarrow u = 0$

$y = 0 \rightarrow v = 0$

$x + y = 1 \rightarrow u + v = 2$.

Q) $w = (1+i)z$

ans) $u + iv = (1+i)(x + iy)$

$\downarrow u = x - y$, $v = x + y$

$2x = u + v \rightarrow x = \frac{u+v}{2}$

$y = -\left(\frac{u-v}{2}\right)$

if $x=0 \rightarrow u+v=0$

$y=0 \rightarrow u-v=0$

$x+y=1 \longrightarrow \frac{u+v}{2}+\left(\frac{-(u-v)}{2}\right)=1 \rightarrow v=1$

(iv) inversion $\rightarrow w=\frac{1}{z}$

case 1 $\rightarrow w=u+iv, \quad z=x+iy$

$$u+iv=\frac{1}{x+iy}$$

$$=\frac{x}{x^2+y^2}+i\left(\frac{-y}{x^2+y^2}\right)$$

$$u=\frac{x}{x^2+y^2}, \quad v=\frac{-y}{x^2+y^2}$$

$$(x,y)\longrightarrow\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$

case 2 $\rightarrow$ polar form $\rightarrow$

$$z=re^{i\theta} \text{ and } w=Re^{i\phi}$$

$$w=\frac{1}{z} \rightarrow Re^{i\phi}=\frac{1}{re^{i\theta}}$$

$$\rightarrow Re^{i\phi}=\frac{1}{r}e^{-i\theta}$$

$$R=\frac{1}{r} \text{ and } \phi=-\theta$$

$$(r,\theta)\xrightarrow{w=\frac{1}{z}}\left(\frac{1}{r}, -\theta\right)$$

Q) considering the map $w=\frac{1}{z}$, $R' \Rightarrow \frac{1}{4}<\operatorname{Im}(z)<\frac{1}{2}$

infinite strip

ans) $\frac{1}{4}<y<\frac{1}{2}$

$$\frac{1}{4}<y \rightarrow \frac{1}{4}>\frac{-v}{u^2+v^2}$$

$$u^2+v^2+4v>0$$

centre $\rightarrow (0,-2)$

radius $\rightarrow 2$

$$y<\frac{1}{2} \longrightarrow \frac{-v}{u^2+v^2}<\frac{1}{2} \rightarrow u^2+v^2+2v>0$$

centre $\rightarrow (0,-1)$

$r=1$