Module 4

Vector Space →

Euclidean space →

→ $x = (x_1, x_2) \to R^2 \to$ plane

→ $y = (y_1, y_2) \to R^2 \to$ plane

→ $(x_1, x_2, x_3) \to R^3 \to$ space

over a real field → called euclidean space

over a complex field → called unitary space

set R^n is known as Euclidean n space.

vector space – non empty set of V vectors with 2 algebraic operations (vector +, vector ·) that satisfy →

1. for every vector x and $y \in V$, $x + y \in V$
2. for every vector $x, y, z \in V$ → $x + (y + z) = (x + y) + z = x + y + z$
3. there is a unique vector 0 in V such that $x + 0 = x = 0 + x$ for all $x \in V$ called zero vector
4. for any $x \in V$ there is a vector $-x \in V$ such that $x + (-x) = (-x) + x$ called negative of x
5. for every vector x and $y \in V$, $x + y = y + x$
6. for every vector $x \in V$ and k any scalar, a unique vector $kx \in V$
7. x and $y \in V$ and k any scalar $k(x + y) = kx + ky$
8. for every vector $x \in V$ and k, l any scalar $(k + l)x = kx + lx$
9. for every vector $x \in V$ and k, l any scalars $k(lx) = (kl)x$
10. for every vector $x \in V$ $1x = x$

Q) $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in R$, $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in R$

$C = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} \in R$

ans) ① $A + B = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \in R$ so ok

② $A + (B + C) = \begin{bmatrix} a_1 + a_2 + a_3 & b_1 + b_2 + b_3 \\ c_1 + c_2 + c_3 & d_1 + d_2 + d_3 \end{bmatrix}$

$= (A + B) + C$ so ok

③ $A + O = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0+a_1 & 0+b_1 \\ 0+c_1 & 0+d_1 \end{bmatrix} = O + A$

$A + O = O + A = A$

④ $A + (-A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\rightarrow A + (-A) = 0 = (-A) + A = 0$

so ok

and so on

$(f+g)(x) = f(x) + g(x)$

$(kf)(x) = kf(x)$

→ let $C(R)$ denote set of real valued conti fn

→ for 2 fns $f(x)$ and $g(x)$ and a real number k and the the sum $f+g$ and the scalar multiplication kf of them are defined by ~~this~~, then we can easily verify that set $C(R)$ is vector space under these operations, the 0 vector in this space is const fn whose value at each pt is 0.

→ Shortcut for Vector space → always verify

① $x \in V$, $y \in V$ ~~then~~ then $x + y \in V$

and

② $k \rightarrow$ scalar, $x \in V$ then $kx \in V$ ↓ scalar multiplication

in beginning.

[redacted]

Q) $(x, y) + (u, v) = (x + 2u, y + 2v)$

$k(x, y) = (kx, ky)$ is this vector space?

ans) if we check $(x+y)+z = x+(y+z)$ property, we get $\rightarrow$

$x+(y+z) = (a_1, b_1) + (a_2 + 2a_3, b_2 + 2b_3)$

$x+(y+z) = (a_1 + 2(a_2 + 2a_3)), (b_1 + 2(b_2 + 2b_3))$

$x+(y+z) = (a_1 + 2a_2 + 4a_3, b_1 + 2b_2 + 4b_3)$

$(x+y)+z = (a_1 + 2a_2, b_1 + 2b_2) + (a_3, b_3)$

$(x+y)+z = (a_1 + 2a_2 + 2a_3, + b_1 + 2b_2 + 2b_3)$

so fails

Q) let V be set of all 2×2 matrices with trace equal to 0. check if its a vector space or not

ans) let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in V$

① $a_1 + d_1 = 0$ $\qquad a_2 + d_2 = 0$

$A + B = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$

if $tr(A+B) = 0$ then $A+B \in V$

$tr(A+B) = 0 \rightarrow a_1 + a_2 + d_1 + d_2 = 0$

$A+B \in V$

② $A + (B+C) = (A+B) + C$

holds true

③ $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$

$tr(0) = 0 \rightarrow 0 \in V$

so $A + 0 = A = 0 + A$.

holds true

.... and so on we see, all 11 are satisfied. so true vector space

Subspace → condition to be subspace is →

$a + kb \in W$, $a, b \in W$

k → scalar.

(examples in printout)

Steps → ① take a, b from vector space V

② find $a + kb$

③ prove $a + kb$ satisfies condition of W while assuming condition of W applies / works on V as well {on $a + kb$}

if these restrictions given →

$x > 0$ $y < 0$

$x \geq 0$ $y \leq 0$ → wont be subspace. these restrictions arent allowed

$y > 0$ $x \neq 0$

$y \geq 0$ $y \neq 0$

$xy = 0$

→ if U, W be subspace of V, then $U \cap W$ and $U + W$ is also subspace of V

→ $U \cup W$ will be subspace if and only if U is contained in W or W is contained in U

$\cup$ → union

If rank of matrix < no of variables → non-trivial solⁿ, linearly dependant

rank = no of non zero row in matrix or min (m, n)

m → row, n → column

Q) $(6, 14, -8)$ in R^3 can be written as combination of $(1, 2, 3)$, $(2, 3, 7)$, $(3, 5, 6)$

ans $c_1(1, 2, 3) + c_2(2, 3, 7) + c_3(3, 5, 6)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 7 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ -8 \end{bmatrix}$$

represents →

$c_1 + 2c_2 + 3c_3 = 6$

$2c_1 + 3c_2 + 5c_3 = 14$

$3c_1 + 7c_2 + 6c_3 = -8$

$C_2 \to C_2 - C_1$ (1)
$R_3 \to R_3 - R_2$ (2)
$R_2 \to R_2 - R_1$ (3)
gives

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ -8 \end{bmatrix}$$

$c_2 + c_3 = 6$
$c_1 + c_3 = 14$
$c_1 + 5c_2 = -8$
solve now.

or just solve eqn we found earlier {those 3} for answer

Q) find basis for R^3 of $\{(0,0,0), (1,2,3), (-1,0,1)\}$
$(0,0,0) \to x_1$, $(1,2,3) \to x_2$, $(-1,0,1) \to x_3$

ans
$c_1 x_1 + c_2 x_2 + c_3 x_3 = 0$
$\to c_1(0,0,0) + c_2(1,2,3) + c_3(-1,0,1) = (0,0,0)$
$(c_2 - c_3,\ c_2,\ 3c_2 + c_3) = (0,0,0)$
$c_2 = c_3 = 0$
$c_1 = ? \to$ arbitrary
so L.D

for Basis $\to$
$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

putting $R_3 \to R_3 - 3R_2$
$$\to \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so Basis $\to \{(1,1,0),\ (-1,0,1)\}$ (ans)

dimension $\to 2$

Q) $B = \left\{ \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right\}$

$\downarrow A \quad \downarrow C \quad \downarrow D \quad \downarrow E$

→ find Basis

ans) $c_1A + c_2C + c_3D + c_4E = 0$

$= \begin{bmatrix} c_1 + 2c_2 + c_4 & -c_1 + c_2 + c_3 \\ 4 - c_2 + c_3 + 2c_4 & c_2 - c_3 + c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & 1 \end{bmatrix}$ now using transformation we reach →

$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

so $6c_4 = 0$, $-2c_3 + 4c_4 = 0$, $c_1 + 2c_2 + c_4 = 0$

$c_2 - c_3 + c_4 = 0$,

so $c_1 = c_2 = c_3 = c_4 = 0$

so LI

→ basis is $\{(1,0,0,0), (2,1,0,0), (0,-1,-2,0), (1,1,4,6)\}$

dimension is 4

Q) $B = \{1, 1+x^3, 1+x^2\}$ find Basis

ans) $c_1(1) + c_2(1+x^3) + c_3(1+x^2) = 0$

$c_1 + c_2 + c_2x^3 + c_3 + c_3x^2 = 0 \times 1 + 0 \times x_2 + 0 \times x + 0 \times x^3$

$(c_1 + c_2 + c_3) \times 1 + c_3x^2 + c_2x^3 = 0 \times 1 + 0 \times x + 0 \times x^2 + 0 \times x^3$

$c_1 = c_2 = c_3 = c_4$ so its a

so c_1

→ basis.

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

Basis → $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

→ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Basis $= \{(1,0,0), (0,1,0), (0,0,1)\}$

Q) find basis for $W = \{(x_1, x_2, x_3, x_4) \in R^4 [$
$x_1 + x_2 - 2x_3 + x_4 = 0,\ 2x_1 - x_2 + x_3 = 0,$
$4x_1 + x_2 - 3x_3 + 2x_4 = 0\}$

ans) $\boxed{M-1}$

$x_1 = 2x_3 - x_2 - x_4$
$x_2 = +2x_1 + x_3$
$x_3 = \frac{4x_1}{3} + \frac{x_2}{3} + \frac{2x_4}{3}$
$x_4 = x_4$

so $\left(2x_3 - x_2 - x_4,\ 2x_1 + x_3,\ \frac{4x_1}{3} + \frac{x_2}{3} + \frac{2x_4}{3},\ x_4\right)$

$\to \left(x_1(0, 2, \frac{4}{3}, 0) + x_2(-1, 0, \frac{1}{3}, 0) + x_3(2, 1, 0, 0)\right.$
$+ x_4(-1, 0, \frac{1}{3}, 1)$

$\to$ Basis $\to \{(0, 2, \frac{4}{3}, 0), (-1, 0, \frac{1}{3}, 0), (2, 1, 0, 0),$
$(-1, 0, \frac{1}{3}, 1)\}$

dimension $= 2$ as repeating

Q) $S = \{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 4 \\ 0 & 5 \end{pmatrix}\}$ LI or no

ans) $c_1 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + c_2 \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 5 & 4 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 4 - c_2 + 5c_3 & 2c_1 + 4c_3 \\ 3c_1 + 2c_2 & 4c_1 + 5c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

solve for all.

Q) $W = \{(a_1, a_2, a_3, a_4, a_5) \in R^5;$
$a_1 + a_3 + a_5 = 0,\ a_2 = a_4\}$
find Basis, dimension

ans using $\boxed{M-1}$

$\{(-a_3 - a_5, a_4, a_3, a_4, a_5)\}$

$= \{a_3(-1, 0, 1, 0, 0) + a_4(0, 1, 0, 1, 0)$
$+ a_5(-1, 0, 0, 0, 1)\}$

$= \{(-1,0,1,0,0), (0,1,0,1,0), (-1,0,0,0,1)\}$
so dimension 3

Q) does $W = \{ P(x) \in P_3(R) \mid P(1) = P(2) = 0 \}$ a subspace of $P_3(R)$ if so, find basis

ans) let $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ & $P(1)P(2)=0$
$q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$

$(P+kq)(x) = (P+kq)$
$(P+kq)(1) = P(1) + kq(1)$
$= 0$

$\boxed{P+kq(1) = 0 \in W}$

similarly for $x = 2$
$(P+kq)(x) = P(x) + kq(x)$
$(P+kq)(2) = 0 + 0$
$= 0$

so its a subspace

<u>Basis</u> $\to$ $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$
$P(1) = 0 = a_0 + a_1 + a_2 + a_3$
$P(2) = 0 = a_0 + 2a + 4a_2 + 8a_3 = 0$

$P(1) = P(2) = 0$
$a_1 + 3a_2 + 7a_3 = 0$

$\to$ so $(a_0, a_1, a_2, a_3) \in P_3(x) \mid a_1 + 3a_2 + 7a_3 = 0 \}$

$W = \{(a_0, a_1, a_2, a_3)\}$
using M-1
$\{(a_0, -3a_2 - 7a_3, a_2, a_3)$

$\{ a_0(1,0,0,0) + a_2(0,-3,1,0) + a_3(0,-7,0,1)\}$

$= \{(1,0,0,0), (0,-3,1,0), (0,-7,0,1)\}$
$\downarrow x_1 \quad \downarrow x_2 \quad \downarrow x_3$

$W = \{(1,0,0,0), \ldots$ $\downarrow$ Basis

$\boxed{\text{dimension} = 3}$

for it to be basis $\to c_1 x_1 + c_2 x_2 + c_3 x_3 = 0$
solving it we would get that it is

Row and column Spaces

if A is an m×n matrix then subspace of R^n spanned by row vector is row space and subspace of R^m spanned by the column vectors of A is called the column space of A the solution space of homogenous system of eqn $Ax=0$ which is a subspace of R^n is null space of A

note → if matrix A is row echelon form. then the row vectors with the leading 1's (the non 0 vectors) form a basis for the row space of A | and column vectors with leading 1's of the row vectors form basis for column space of A

Q) find basis for row space of matrix $A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$

ans) $A = \begin{pmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & -2 & -4 & -1 & 0 \\ 0 & -1 & -2 & -2 & -3 \end{pmatrix}$ after $R_4 \to R_4 - R_1$, $R_5 \to R_5 - 2R_1$

$= \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$ after $R_4 \to R_4 + 2R_2$, $R_5 \to R_5 + R_2$

$= \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ after $R_4 \to R_4 - R_3$, $R_5 \to R_5 + R_3$

row space → $R_1 = (1, 1, 4, 1, 2)$, $R_2 = (0, 1, 2, 1, 1)$, $R_3 \to (0, 0, 0, 1, 2)$

basis → $\{ (1, 1, 4, 1, 2), (0, 1, 2, 1, 1), (0, 0, 0, 1, 2) \}$

dimension = 3

we do row transforms till we get that triangle of 0's for row

Q) find basis of column space $\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$

ans) do till prev problem
then do →

$C_3 \to C_3 - 4C_2$
$C_4 \to C_4 - C_2$
$C_5 \to C_5 - 2C_2$
then $C_5 \to 2C_5 - C_3$
then $C_3 \to \frac{C_3}{2}$, $C_5 \to \frac{C_5}{4}$

so we get →

$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

aim is to make all this zero and simplify

column space →

$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $C_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$, $C_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

↓ basis

dimension = 3

→ dimension of column or row space of matrix A is rank of matrix A, denoted by R(A)

if A is a m×n matrix then the set of all soln of homogenous system of linear eqns $Ax = 0$ is subspace of $\mathbb{R}^n$ called null space of A and denoted by N(A)

$N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$

dimension of N(A) is nullity of A

Q) find basis for null space of matrix $A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$

ans) after row echelon we get →

$A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$Ax = 0$

so $\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 + x_2 + 4x_3 + x_4 + 2x_5 = 0 \to (1)$
$x_2 + 2x_3 + x_4 + x_5 = 0 \to (2)$
$x_4 + 2x_5 = 0$

let $x_5 = t \to x_4 = -2t$
$x_3 = s$
so $x_2 = -2s + t$
$x_1 = -2s - t$
so column space is $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s-t \\ -2s+t \\ s \\ -2t \\ t \end{bmatrix}$ (ans)

$= s\begin{pmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}$

so $\left\{\begin{pmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}\right\}$

so nullity = dimension of null space = 2

Rank nullity $\to R(A) + N(A) = n$
so in prev question $\to 3 + 2 = 5$
hence proved

Q) $A = \begin{bmatrix} 2 & 4 & -3 & -6 \\ 7 & 14 & -6 & -3 \\ -2 & -4 & 1 & -2 \\ 2 & 4 & -2 & -2 \end{bmatrix}$

find Basis of row space, column space, null space, rank, nullity and verify rank nullity

ans) $R_2 \to 2R_2 - 7R_1$, $R_3 \to R_3 + R_1$, $R_4 \to R_4 - R_1$ gives $\begin{bmatrix} 2 & 4 & -3 & -6 \\ 0 & 0 & 9 & 36 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$R_4 \to 9R_4 - R_2$, $R_3 \to 9R_3 + 2R_2 \to \begin{bmatrix} 2 & 4 & -3 & -6 \\ 0 & 0 & 9 & 36 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_1 \to \frac{R_1}{2}$, $R_2 \to \frac{R_2}{9} \to \begin{bmatrix} 1 & 2 & -\frac{3}{2} & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Row space $\to R_1 \to (1, 2, -\frac{3}{2}, -3)$, $R_2 = (0, 0, 1, 4)$
Basis $\to \{(1, 2, -\frac{3}{2}, -3), (0, 0, 1, 4)\}$

so Rank = dimension of Row space = 2 (iv) ans

(ii) row echelon form is →

$$A = \begin{pmatrix} 1 & 2 & -\frac{3}{2} & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow$$ C_1, C_3 have leading 1 according to echelon form so C_1, C_3 of question matrix is column space

column space is → $C_1 =$

$$C_1 = \begin{pmatrix} 2 \\ 7 \\ -2 \\ 2 \end{pmatrix} \quad C_3 = \begin{pmatrix} -3 \\ -6 \\ 2 \\ 2 \end{pmatrix}$$

(iii) $$A = \begin{pmatrix} 1 & 2 & -\frac{3}{2} & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$Ax = 0$ ~~null space giving eq~~

$$\begin{pmatrix} 1 & 2 & -\frac{3}{2} & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 + 2x_2 - \frac{3x_3}{2} - 3x_4 = 0$ — (1)

$x_3 + 4x_4 = 0$ — (2)

let $x_4 = t \rightarrow x_3 = -4t$

$x_1 + 2x_2 - \frac{3}{2}(-4t) - 3t = 0$ (1)

$x_1 + 2x_2 + 6t - 3t = 0$

→ $x_1 + 2x_2 + 3t = 0.$

let $x_2 = s \rightarrow \boxed{x_1 = -2s - 3t}$

soln set → $$x = \begin{pmatrix} -2s - 3t \\ s \\ -4t \\ t \end{pmatrix}$$

$$= s\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 0 \\ -4 \\ 1 \end{pmatrix}$$

so null space → $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -4 \\ 1 \end{pmatrix}$

Basis → $\left[\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -4 \\ 1 \end{pmatrix} \right]$

so nullity of A = 2

as $R(A)+N(A)=n$

$2+2=4$

so Rank nullity theorem verified

Linear transformation

Consider 2 vector spaces V, W over same field F a mapping $T: V \to W$ is said to be a linear mapping or linear {from V to W} transformation if it satisfies

(i) $T(\alpha+\beta) = T(\alpha) + T(\beta)$, $\forall\, \alpha, \beta \in V$

(ii) $T(c\alpha) = cT(\alpha)$, $c \in F$, $\alpha \in V$

~~$T(c\alpha) = T(cx_1, cx_2, cx_3) = (cx_1, cx_2, 0)$~~

~~$cT(\alpha) = c(x_1, x_2, 0)$~~

~~$V = R^3 = \{(x_1, x_2, x_3) \mid x_i \in R\}$ defining a mapping $T: R^3 \to R^3$ by $T(x_1, x_2, x_3) = (x_1, x_2, 0)$, $(x_1, x_2, x_3) \in R^3$ is a linear transformation for $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3) \in R^3$~~

~~$\alpha + \beta = (x_1+y_1, x_2+y_2, x_3+y_3)$~~

~~(i) $T(\alpha+\beta) = (x_1+y_1, x_2+y_2, 0) = (x_1, x_2, 0) + (y_1, y_2, 0) = T(\alpha) + T(\beta)$~~

~~→ (ii) zero mapping → $T: V \to W$ {0 = zero element in W}~~

~~$T(\alpha) = 0$, $\forall\, \alpha \in V$~~

$$\boxed{\begin{aligned}&\to \text{Shortcut for linear transformation} \to \\ &\text{prove } T(x+ky) = T(x) + kT(y)\end{aligned}}$$

let

Q) let $T: R^2 \to R^2$ defined by $T(x, y) = (x-y, 2x)$
is T linear transformation
ans) let $x = (x_1, x_2) \in R^2$ and $y = (y_1, y_2) \in R^2$

To prove is yes or no →

$T(x+ky) = T(x) + kT(y)$

LHS → $T((x_1, x_2) + k(y_1, y_2))$
$= T(x_1+ky_1, x_2+ky_2)$
$= ((x_1+ky_1) - (x_2+ky_2), 2(x_1+ky_1))$
$= (x_1 - x_2 + ky_1 - ky_2, 2x_1 + 2ky_1)$

RHS → $T(x) + kT(y)$
$= T(x_1, x_2) + kT(y)$
$= (x_1 - x_2, 2x_1) + k(y_1 - y_2, 2y_1)$
$= (x_1 - x_2 + ky_1 - ky_2, 2x_1 + 2ky_1)$

LHS = RHS

Q) $T \to R^2 : R^2 \to T(x, y) = (xy, x^2+1)$
is linear or no
ans) proving → $T(x+ky) = T(x) + T(y)k$

LHS → $T(x_1 + ky_1, x_2 + ky_2)$
→ $((x_1+ky_1)(x_2+ky_2),$
$(x_1+ky_1)^2 + 1)$

⇒ $(x_1x_2 + k(x_1y_2 + x_2y_1)$
$+ k^2 y_1y_2, x_1^2 + 2kx_1y_1 +$
$k^2y_1^2 + 1)$

RHS → $T((x_1, x_2)) + kT((y_1, y_2))$
$= ((x_1x_2, x^2+1) + k(y_1y_2, y_1^2+1))$

$= ((x_1 x_2, x_1^2+1) + (ky_1 y_2 + ky_1^2 + k))$

$= (x_1 x_2 + ky_1 y_2,\ x_1^2 + ky_1^2 + k + 1)$

RHS $\neq$ LHS

so not LT.

Q) $(x^2, y^2) = T(x, y)$

ans) $T(x+ky) = T(x_1+ky_1, x_2+ky_2)$ ← LHS

$= (x_1^2 + 2kx_1y_1 + k^2y^2,\ x_2^2 + 2kx_2y_2 + k^2y_2^2)$

RHS $\Rightarrow T(x) + kT(y)$

$= T((x_1, x_2)) + kT((y_1, y_2))$

$= (x_1^2 + ky_1^2,\ x_2^2 + ky_2^2)$

so not LT

Q) $T(x,y) = (\sin x, y)$

ans) $T(x+ky) = T((x_1, x_2) + k(y_1, y_2))$ ← LHS

$= (\sin(x_1+ky_1),\ x_2+ky_2)$

$= (\sin x \cos ky_1 + \cos x_1 \sin ky_1,\ x_2+ky_2)$

RHS → $T(x) + kT(y)$

$T(x_1, x_2) + kT(y_1, y_2)$

$= T(x_1, x_2) + k(\sin y_1, y_2)$

$= (\sin x_1, x_2)$

$= (\sin x_1 + k \sin y_1,\ x_2 + ky_2)$

LHS $\neq$ RHS

not LT

Q) let $\alpha = \{P_1, P_2, P_3\}$ be standard basis for R^3

ie $\rightarrow P_1(1,0,0),\ P_2(0,1,0),\ P_3(0,0,1)$

let $T: R^3 \rightarrow R^2$ be a LT defined by $T(P_1) = w_1$, $T(P_2) = w_2$, $T(P_3) = w_3$ find formula for $T(x_1, x_2, x_3)$ and then use it to compute $T(2,-3,5)$

$w_1 = (1,0)$, $w_2 = (2,-1)$, $w_3 = (4,3)$

ans) $T: V \rightarrow W$

ie $T: R^3 \rightarrow R^2$ let $w_1 = (1,0)$, $w_2 = (2,-1)$, $w_3 = (4,3)$

$S = \{P_1, P_2, P_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$

$T: R^3 \rightarrow R^2 \Rightarrow T(e_i) = w_i$

$T(e_1) = w_1 \rightarrow T(1,0,0) = (1,0)$

$T(e_2) = w_2 \rightarrow T(0,1,0) = (2,-1)$

$T(e_3) = w_3 = (4,3)$

let $x = (x_1, x_2, x_3) \in V$ ie $(x_1, x_2, x_3) \in R^3$

linear combination of vector space (x_1, x_2, x_3)

is $(x_1, x_2, x_3) = c_1 P_1 + c_2 e_2 + c_3 e_3$

$(x_1, x_2, x_3) = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$

$(x_1, x_2, x_3) = (c_1, c_2, c_3)$

$c_1 = x_1,\ c_2 = x_2,\ c_3 = x_3$

$(x_1, x_2, x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$

apply LT on both sides $\rightarrow$

$T(x_1, x_2, x_3) = x_1 T(1,0,0) + x_2 T(0,1,0) + x_3 T(0,0,1)$

$= x_1 T(P_1) + x_2 T(P_2) + x_3 T(P_3)$

$= x_1 w_1 + x_2 w_2 + x_3 w_3$

$= x_1(1,0) + x_2(2,-1) + x_3(4,3)$

$T(x_1, x_2, x_3) = (x_1 + 2x_2 + 4x_3,\ 3x_3 - x_2)$

so $T(2,-3,5) = (16, 18)$

if matrix is asked $\rightarrow$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 3 \end{bmatrix}_{2\times 3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 4x_3 \\ -x_2 + 3x_3 \end{bmatrix}$$

Q) let $\beta = \{v_1, v_2, v_3\}$, $v_1 = (1,1,1)$, $v_2 = (1,1,0)$, $v_3 = (1,0,0)$ be a basis for R^3 and let $T: R^3 \to R^2$ be a LT defined by $T(v_1) = w_1$, $T(v_2) = w_2$, $T(v_3) = w_3$ where $w_1 = (1,0)$, $w_2 = (2,-1)$ and $w_3 = (4,3)$ then find formula for $T(x_1, x_2, x_3)$ and to compute $T(2,-3,5)$

ans) $T: R^3 \to R^2$, $V = R^3$, $W = R^2$ given that basis of vector space is $\to \beta = \{v_1, v_2, v_3\} = \{(1,1,1), (1,1,0), (1,0,0)\}$

$w_1 = (1,0)$, $w_2 = (2,-1)$, $w_3 = (4,3)$

$T(v_1) = w_1 = (1,0)$, $T(v_2) = (2,-1)$, $T(v_3) = (4,3)$

linear combination of vector space (x_1, x_2, x_3) with basis $\beta = \{v_1, v_2, v_3\}$

$$(x_1, x_2, x_3) = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$(x_1, x_2, x_3) = (c_1 + c_2 + c_3,\ c_1 + c_2,\ c_1)$$

$$x_3 = c_1,\ x_2 = c_1 + c_2,\ x_1 = c_1 + c_2 + c_3$$

$$c_3 = x_1 - x_2,\ c_2 = x_2 - x_3$$

$$\to (x_1, x_2, x_3) = x_3(1,1,1) + (x_2 - x_3)(1,1,0) + (x_1 - x_2)(1,0,0)$$

applying LT $\longrightarrow$

$$T(x_1, x_2, x_3) = x_3 T(1,1,1) + (x_2 - x_3) T(1,1,0) + (x_1 - x_2) T(1,0,0)$$

$$T(x_1, x_2, x_3) = x_3(1,0) + (x_2 - x_3)(2,-1) + (x_1 - x_2)(4,3)$$

$$= (x_3 + 2x_2 - 2x_3 + 4x_1 - 4x_2,\ -x_2 + x_3 + 3x_1 - 3x_2)$$

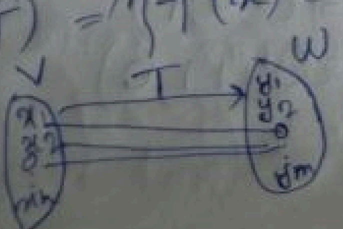
$$T(2,-3,5) = (9, 23) \text{ (ans)}$$

Definition $\to$ let V, W be 2 vector spaces & let $T: V \to W$ be a LT from V into W

(i) $\ker(T) = \{x \in V : T(x) = 0\} \subseteq V$

(ii) $\mathrm{Im}(T) = \{T(x) \in W : x \in V\} \subseteq W$

ex $\to$



$\ker(T) = \{0, x_1, x_{n-1}, x_n\}$

dim (ker T) = nullity dim (Im (T)) = rank

dim (ker T) + dim (Im (T)) = dim(V)

Q) find LT $T: R^3 \to R^3$ which maps basis {(0,1,1), (1,0,1), (1,1,0)} of R^3 to {(1,1,1)} verify rank nullity

given → $V = R^3$, $W = R^3$, $x = (x_1, x_2, x_3) \in R^3$

ans) $(x_1, x_2, x_3) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$

linear combination

$(x_1, x_2, x_3) = (\underset{\downarrow x_1}{c_2 + c_3}, \underset{\downarrow x_2}{c_1 + c_3}, \underset{\downarrow x_3}{c_1 + c_2})$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

→ as $x_1 \neq x_2 \neq x_3 \neq 0$

its not homogenous

so

$$\rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & x_1 \\ 1 & 0 & 1 & x_2 \\ 1 & 1 & 0 & x_3 \end{array}\right]$$

$R_1 \leftrightarrow R_2$, $R_3 \to R_3 - R_1$, $R_3 \to R_3 - R_2$

gives →

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & x_2 \\ 0 & 1 & 1 & x_1 \\ 0 & 0 & -2 & x_3 - x_2 - x_1 \end{array}\right]$$

echelon form

so → $c_1 + c_3 = x_2$

$c_2 + c_3 = x_1$

$-2c_3 = x_3 - x_2 - x_1$

$c_3 = \frac{1}{2}(x_1 + x_2 - x_3)$

$c_2 = x_1 - c_3 \Rightarrow$ $c_2 = \frac{x_1 - x_2 - x_3}{2}$

$\boxed{c_1 = \frac{1}{2}(-x_1 + x_2 + x_3)}$

$(x_1, x_2, x_3) = \frac{1}{2}(-x_1 + x_2 + x_3)(0,1,1)$

$+ \frac{1}{2}(x_1 - x_2 + x_3)(1,0,1)$

$+ \frac{1}{2}(x_1 + x_2 - x_3)(1,1,0)$

taking LT →

$$T(x_1, x_2, x_3) = \frac{1}{2}[-x_1 + x_2 + x_3]T(0,1,1)$$
$$+ \frac{1}{2}(x_1 - x_2 + x_3)T(1,0,1)$$
$$+ \frac{1}{2}(x_1 + x_2 - x_3)T(1,1,0)$$
$$= \frac{1}{2}(-x_1 + x_2 + x_3)(1,1,1) + \frac{1}{2}(x_1 - x_2 + x_3)(1,1,1)$$
$$+ \frac{1}{2}(x_1 + x_2 - x_3)(1,1,1)$$
$$T(x_1, x_2, x_3) = \frac{1}{2}(x_1 + x_2 + x_3)(1,1,1)$$

$\rightarrow$ ker(t) = by definition $\rightarrow T(x_1, x_2, x_3) = 0$
$$\frac{1}{2}[x_1 + x_2 + x_3)(1,1,1) = (0,0,0)$$
ie $\rightarrow$
$$\Rightarrow \frac{1}{2}(x_1 + x_2 + x_3) = 0, \quad x_1 + x_2 + x_3 = 0$$
$$\rightarrow \boxed{x_1 = -x_2 - x_3}$$
$$\ker(T) = \{(x_1, x_2, x_3) \mid x_1 = -x_2 - x_3\}$$
$$= \{(-x_2 - x_3, x_2, x_3)\}$$
x_2 repeated, x_3 repeated
$$= \{(-1,1,0), (-1,0,1)\}$$
$$\boxed{\dim(\ker(T)) = 2}$$

$$\rightarrow \operatorname{Im}(T) = \{T(x_1, x_2, x_3) \in W \mid (x_1, x_2, x_3) \in V$$
$$= \{T(0,1,1), T(1,0,1), T(1,1,0)\}$$
$$\operatorname{Im}(T) = \{(1,1,1)\}$$
$$\boxed{\dim(\operatorname{Im}(T)) = 1}$$

rank nullity $\rightarrow$ $2 + 1 = \dim(V) = 3$ (ans)

Q) Determine LT $T: R^3 \to R^3$ which transforms basis domain of R^3 is $\{(0,1,1), (1,0,1), (1,1,0)\}$ to basis of codomain of vectors of R^3 is $\{(2,0,0), (0,2,0), (0,0,2)\}$ resp. find ker T, Im T and prove rank nullity

ans) $\{(0,1,1), (1,1,0), (1,0,1)\}$ of $R^3 \to$ domain basis
codomain of basis $\to \{(2,0,0), (0,2,0), (0,0,2)\}$ of R^3

$$(x_1, x_2, x_3) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$$

$$(x_1, x_2, x_3) = (c_2 + c_3, c_1 + c_3, c_1 + c_2)$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} 0 & 1 & 1 & x_1 \\ 1 & 0 & 1 & x_2 \\ 1 & 1 & 0 & x_3 \end{array}\right]$$

$R_1 \leftrightarrow R_2$, $R_3 \to R_3 - R_1$, $R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & x_2 \\ 0 & 1 & 1 & x_1 \\ 0 & 0 & -2 & x_3 - x_2 \end{array}\right]$$

$\to$ RE form

$$c_1 + c_3 = x_2$$
$$c_2 + c_3 = x_1$$
$$-2c_3 = x_3 - x_2 - x_1$$

$$c_3 = \frac{1}{2}(x_1 + x_2 - x_3)$$

$$c_1 = \frac{2x_2 - x_1 - x_2 + x_3}{2}$$

$$(x_1, x_2, x_3) = \frac{1}{2}(-x_1 + x_2 + x_3)(0,1,1)$$
$$+ \frac{1}{2}(x_1 - x_2 + x_3)(1,0,1)$$
$$+ \frac{1}{2}(x_1 + x_2 - x_3)(1,1,0)$$

LT on both and using codomain $\to$ $\frac{1}{2}(-x_1 + x_2 + x_3)(2,0,0) + \frac{1}{2}(x_1 - x_2 + x_3)(0,2,0) + \frac{1}{2}(x_1 + x_2 - x_3)(0,0,2)$

$T(x_1, x_2, x_3) = (-x_1 + x_2 + x_3,\ x_1 - x_2 + x_3,\ x_1 + x_2 - x_3) \to$ read LT

$\ker(T) = \{(x_1, x_2, x_3) \in R^3 \mid T(x_1, x_2, x_3) = 0$

$T(x_1, x_2, x_3) = 0$

$\rightarrow -x_1 + x_2 + x_3 = 0$

$x_1 - x_2 + x_3$

$x_1 + x_2 - x_3 = 0$

$x_1 = x_2 = x_3 = 0$

so $\ker(T) = \{(0,0,0)\} = \{0\}$

so $\boxed{\dim(\ker(T)) = 0}$

$Im(T) = \{(2,0,0), (0,2,0), (0,0,2)\}$

$\dim(Im(T)) = 3$

so $0 + 3 = \dim(V) = 3$ proved.

Transformation of a matrix

Q) Consider LT is $T: R^2 \rightarrow R^3$ given by $T(x,y) = (2x, x+y, x-y)$ find LT matrix wrt domain basis of R^2 is $\{(1,1), (1,2)\}$ and codomain basis of R^3 is $\{(1,1,0), (1,2,0), (0,0,3)\}$

ans) $T(x,y) = (2x, x+y, x-y)$

$T: R^2 \rightarrow R^3$, $V = R^2$ and $W = R^3$

given domain basis of R^2 is $\{(1,1), (1,2)\}$

$B_{R^2} = \{\alpha_1, \alpha_2\}$, $\alpha_1 = (1,1)$, $\alpha_2 = (1,2)$

codomain basis of $R^3 \rightarrow B_{R^3} = \{\beta_1, \beta_2, \beta_3\} = \{(1,1,0), (1,2,0), (0,0,3)\}$

transformation matrix $\rightarrow$ wrt B_{R^2}, B_{R^3} $\rightarrow$

$$[T; B_{R^2}, B_{R^3}] = \begin{array}{c} \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{array} \begin{array}{cc} T(\alpha_1) & T(\alpha_2) \\ \multicolumn{2}{c}{\left[\begin{array}{cc} \quad & \quad \\ & \\ & \end{array}\right]} \end{array}$$

let $(x_1, x_2, x_3) \in R^3$

$(x_1, x_2, x_3) = (c_1(1,1,0) + c_2(1,2,0) + c_3(0,0,3))$

$c_1 + c_2 = x_1$, $c_1 + 2c_2 = x_2$

$3c_3 = x_3$

$(x_1, x_2, x_3) = (2x_1 - x_2)(1,1,0) + (-x_1 + x_2)(1,2,0)$
$+ \frac{1}{3}x^3 (0,0,3)$

$T(x,y) = (2x, x+y, x-y)$

$T(1,1) = (2, 2, 0)$

$\uparrow x_1 \quad \uparrow x_2 \quad x_3$

$= (2(2) - 2)(1,1,0) + (-2+2)(1,2,0)$
$+ 0(0,0,3)$

$T(1,1) = 2(1,1,0)$

$\alpha_2 = (1,2) = T(\alpha_2) = (3,3,-1)$

$T(1,2) = (1)(1,1,0) + (1)(1,2,0) + (-\frac{1}{3})(0,0,3)$

$[T: B_{R^2}, B_{R^3}] = [T]_{B_{R^2}}^{B_{R^3}}$

$$= \begin{array}{c} \beta_1 \\ \beta_2 \\ \beta_3 \end{array} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 0 & -\frac{1}{3} \end{bmatrix}$$

So Transformation of matrix wrt basis is →

$$T = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 0 & -\frac{1}{3} \end{bmatrix} \text{ (ans)}$$

Module 7

→ $A = [a_{ij}]$ is a sq matrix of order n,
the characteristic eqn of A is $|A - \lambda I| = 0$ the roots of characteristic eqn are <u>eigen values</u>

<u>eigen vector</u> → $A = [a_{ij}]$ is a square matrix of order n if there is non 0 vector x such that $Ax = \lambda x$ then X is eigen vector of A corresponding to eigen value

Q) find eigen value $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

ans) → $|A - \lambda I| = 0$ → ①

$$\rightarrow \left| \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\rightarrow \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

$$\rightarrow (1-\lambda)\left[(2-\lambda)(1-\lambda) - 1\right] + 1\left(-1(1-\lambda) - 0\right) + 0 = 0$$

$$\rightarrow (1-\lambda)\left[2 - 2\lambda - \lambda + \lambda^2 - 1\right] - 1 + \lambda = 0$$

$$\rightarrow (1-\lambda)(1 - 3\lambda + \lambda^2) - 1 + \lambda = 0$$

$$\rightarrow -\lambda^3 + 4\lambda^2 - 3\lambda = 0$$

$$\rightarrow \lambda(\lambda^2 - 4\lambda + 3) = 0 \rightarrow \lambda = 0 \text{ or } \lambda^2 - 4\lambda + 3 = 0$$

$$\lambda = 1, 3$$

$\lambda = 0, 1, 3$

Q) $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$

ans) characteristic eqn $\rightarrow |A-\lambda I|=0$

$$\begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$$

$\rightarrow (1-\lambda)\left[(2-\lambda)(-1-\lambda)+1\right] + 1(-3-3\lambda+2)$
$+ 4(3-4+2\lambda) = 0$

$\rightarrow \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

$\lambda = 1, -2, 3$

for $\lambda = 1 \rightarrow (A-\lambda I)X = 0$

$\rightarrow (A-\lambda)X = 0$

$$\begin{bmatrix} 0 & -1 & 4 \\ 3 & 2-1 & -1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-y + 4z = 0$

$3x + y - z = 0$

$2x + y - 2z = 0$

to solve $\rightarrow X_1 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \begin{matrix} \rightarrow x \\ \rightarrow y \\ \rightarrow z \end{matrix}$ (ans)

Gauss elimination $\rightarrow$

$\rightarrow$ write augmented matrix of system of linear eqn

$\rightarrow$ use elementary row operations to rewrite the augmented matrix to row echelon form

$\rightarrow$ write system of linear equations corresponding to matrix in row echelon form and use back substitution to find soln

$\rightarrow$ changing all entries directly below leading 1's to zeros

Q) solve the system of linear equations $\rightarrow$

$x_2 + x_3 - 2x_4 = -3$

$x_1 + 2x_2 - x_3 = 2$

$2x_1 + 4x_2 + x_3 - 3x_4 = -2$

$x_1 - 4x_2 - 7x_3 - x_4 = -19$

ans $\longrightarrow$

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right] = [A|B]$$

$R_1 \leftrightarrow R_2$.

then $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$ gives →

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{array}\right]$$

then $R_3 \to \frac{R_3}{3}$ → then $R_4 + 6R_2 \to R_4$ gives us →

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -13 & -39 \end{array}\right]$$ then $R_4 \to \frac{R_4}{-13}$

which gives → $$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & +13 & +39 \end{array}\right]$$

$x_4 = 3$, $x_1 + 2x_2 - x_3 = 2$,

$x_2 + x_3 - 2x_4 = -3$, $x_3 - x_4 = -2$

solving all we get → $x_4 = 3$, $x_3 = 1$,

$x_2 = 2$, $x_1 = -1$

Q) $3x + y - z = 1$, $x - y + z = -3$, $2x + y + z = 0$
use GE

ans) $$[A|B] = \left[\begin{array}{ccc|c} 3 & 1 & -1 & 1 \\ 1 & -1 & 1 & -3 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_2$ $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$ and $R_3 \to R_3 - 3R_2$

$R_3 \to \frac{R_3}{2}$ gives $$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -3 \\ 0 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 1 & \frac{-3}{4} \end{array}\right]$$

$x - y + z = -3$

$y - z = \frac{5}{2}$, so $y = \frac{7}{4}$, $x = \frac{-1}{2}$, $z = \frac{-3}{4}$

Online class {continued with eigen stuff}

Q) find eigen values and eigen vectors of

$$\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$$

ans) $|A-\lambda I| = 0$

$\lambda^3 - 2\lambda^2 - 5\lambda + 6$

$\lambda = +1, -2, 3 \rightarrow$ eigen values

for $\lambda = 1 \rightarrow$

$(A - I)X = 0$

$$\begin{bmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-y + 4z = 0$

$3x + y - z = 0$

$2x + y - 2z = 0$

$\rightarrow X_1 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$

for $\lambda = -2 \rightarrow$

$(A + 2I)X = 0$

$$\begin{bmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

find x, y, z and get X_2

for $\lambda = 3 \rightarrow$

$(A - 3I)X = 0$

$$\begin{bmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x - y + 4z = 0$

$3x - y - z = 0$

$2x + y - 4z = 0$

solving all gets $x = y = z = 0$

so $X_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$ but eigen vector is non zero
so not that.

$\rightarrow$ put $z = 0$

this gives $x = 1, y = -2$

so $X_3 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

2) find E values, E vectors

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

ans) $|A - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$\rightarrow (-2-\lambda)[(1-\lambda)(-\lambda)+12] - 2[2(0-\lambda)-6] - 3(-4+1(1-\lambda)]$

$\rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

so $\lambda = -3, -3, 5$

for $\lambda = -3 \rightarrow$

$(A + 3I)X = 0$

$\rightarrow (A+3I)$

$$\rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 2y - 3z = 0 \rightarrow 1$

$2x + 4y - 6z = 0 \rightarrow 2$

$-x - 2y + 3z = 0 \rightarrow 3$ so contradiction

gives $X = Y = Z = 0$

putting $x = 0 \rightarrow$ gives $X_1 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$ for $\boxed{\lambda = -3}$

put $z = 0 \rightarrow$

$X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

for $\lambda = -5 \rightarrow$

$$(A-5I)X = 0$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x + 2y - 3z = 0$

$2x - 4y - 6z = 0$

$-x - 2y - 5z = 0$

$X_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ (ans)

Gauss jordan elimination method

Q) use G-J to solve system of eqn $\rightarrow$

$x - 2y + 3z = 9$

$-x + 3y = -4$

$2x - 5y + 5z = 17$

ans) $[A|B] = \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right]$

$R_2 \rightarrow R_2 + R_1$
$R_3 \rightarrow R_3 - 2R_1$ $\rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right]$

$R_3 \rightarrow R_3 + R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right]$

$R_3 \rightarrow \frac{R_3}{2} \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$

aim accomplished

now $R_1 \rightarrow R_1 + 2R_2$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$

$R_1 \rightarrow R_1 - 9R_3$ then $R_2 \rightarrow R_2 - 3R_3$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

so $x = 1$, $y = -1$, $z = 2$

total aim for gauss jordan

aim for jordan is to get upper and lower triangle as 0

Module 6

→ linear product spaces

dot product of vectors → the dot product of 2 vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in R^3$ is defined by $x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3$

length of vector →

$$||x|| = \sqrt{\langle (x_1, x_2) \rangle}$$

$$= \sqrt{\langle (x_1, x_2, x_3), (x_1, x_2, x_3) \rangle}$$

$$||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

dist b/w 2 vectors →

$$d(x,y) = ||x - y||$$

$$d(x,y) = ||y - x||$$

$$d(x,y) = \sqrt{(x_1 - y_1, x_2 - y_2, x_3 - y_3)}$$

$$d(y,x) = \sqrt{(y_1 - x_1), (y_2 - x_2), (y_3 - x_3)}$$

angle b/w 2 vectors →

angle θ b/w 2 vectors x, y is $\cos\theta = \frac{x \cdot y}{||x|| \, ||y||}$

$$x \cdot y = ||x|| \, ||y|| \cos\theta$$

orthogonal vector →

$x \cdot y = 0$ or $x, y = x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$

or $||x||^2 + ||y||^2 = ||x + y||^2$

Note → $x = (x_1, x_2, x_3) \in R^3$ then $||x|| = \sqrt{\langle x, x \rangle}$

$||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$= \sqrt{\langle (x_1, x_2, x_3), (x_1, x_2, x_3) \rangle}$

Q) $u_1 = (5, 5, 8, 8)$, $v = (1, 2, 3, 4)$
$w = (4, -3, 2, -1)$ in R^4

ans) $d(u, v) = ||u - v||$

$= ||(4, 3, 5, 4)||$

$= \sqrt{16 + 9 + 25 + 16} = \sqrt{66}$

then so on for $d(u, w)$, $d(v, w)$

Gauss Schmidt orthogonalization process

$B = \{u_1, u_2, u_3 \ldots\}$ be basis of vector space, we can find an orthogonal basis for V using u_1, u_2, u_3 which can be formed by G S O process

let $B' = \{v_1, v_2, v_3\}$ be orthogonal basis for V,

then → $v_1 = u_1$, $v_2 = u_2 - \frac{\langle u_2, v_1 \rangle v_1}{\langle v_1, v_1 \rangle}$

$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$

and so on

Q) apply GSO to $\{(1,1,0), (1,2,0), (0,1,2)\}$

ans) $B = \{(1,1,0), (1,2,0), (0,1,2)\}$

$u_1 = (1, 1, 0)$
$u_2 = (1, 2, 0)$
$u_3 = (0, 1, 2)$

$\boxed{v_1 = u_1 = (1,1,0)}$

$$v_2 = u_2 - \frac{\langle (1,2,0),(1,1,0)\rangle}{\langle (1,1,0),(1,1,0)\rangle}(1,1,0)$$

$$v_2 = u_2 - \frac{(1 \times 1 + 2 \times 1 + 0)}{1^2+1^2+0^2}(1,1,0)$$

$$v_2 = u_2 - \frac{3}{2}(1,1,0)$$

$$v_2 = (1,2,0) - \left(\frac{3}{2},\frac{3}{2},0\right)$$

$$\boxed{v_2 = \left(-\frac{1}{2},\frac{1}{2},0\right).}$$

$$v_3 = (0,1,2) - \frac{\langle (0,1,2),(1,1,0)\rangle}{\langle (1,1,0),(1,1,0)\rangle}(1,1,0) - \frac{\langle (0,1,2),(-\frac{1}{2},\frac{1}{2},0)\rangle}{\langle (-\frac{1}{2},\frac{1}{2},0),(-\frac{1}{2},\frac{1}{2},0)\rangle}\left(-\frac{1}{2},\frac{1}{2},0\right)$$

$$= (0,1,2) - \frac{(0+1+0)}{1^2+1^2+0^2}(1,1,0) - \frac{(0+\frac{1}{2}+0)}{\left(\frac{-1}{2}\right)^2+\left(\frac{1}{2}\right)^2+0^2}\left(\frac{-1}{2},\frac{1}{2},0\right)$$

$$v_3 = (0,1,2) - \left(\frac{1}{2},\frac{1}{2},0\right) - \left(\frac{-1}{2},\frac{1}{2},0\right)$$

$$\boxed{v_3 = (0,0,2).}$$

so orthogonal basis is →

$$B' = \left\{(1,1,0), \left(-\frac{1}{2},\frac{1}{2},0\right), (0,0,2)\right\}$$

to verify → should be 0

1) $\langle (1,1,0), \left(-\frac{1}{2},\frac{1}{2},0\right)\rangle$
$\rightarrow \frac{-1}{2} + \frac{1}{2} + 0 = 0$ so correct

2) $\langle \left(-\frac{1}{2},\frac{1}{2},0\right), (0,0,2)\rangle$
$\downarrow$ 0 so correct

3) $\langle (1,1,0), (0,0,2)\rangle \rightarrow 0$
so correct

condition is $\rightarrow \langle u, v\rangle = 0$ ie $\underset{\text{dot product}}{u \cdot v} = 0$

Q) GSO → {(1, 2, 2)}, (-1, 0, 2), (0, 0, 1)}

ans) $u_1 = v_1 = (1, 2, 2)$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1$$

$$= (-1, 0, 2) - \frac{\langle(-1,0,2)(1,2,2)\rangle}{\langle(1,2,2)(1,2,2)\rangle}(1,2,2)$$

$$= (-1, 0, 2) - \frac{(-1+4)}{1+4+4}(1,2,2)$$

$$v_2 = (-1, 0, 2) - \frac{3}{9}(1,2,2)$$

$$v_2 = (-1, 0, 2) - \left(\frac{3}{9}, \frac{6}{9}, \frac{6}{9}\right)$$

$$= \left(-\frac{12}{9}, \frac{-6}{9}, \frac{12}{9}\right)$$

$$\boxed{v_2 = \left(\frac{-4}{3}, \frac{-2}{3}, \frac{4}{3}\right)}$$

$$v_3 = (0, 0, 1) - \frac{\langle(0,0,1), \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right)\rangle}{\langle\left(-\frac{4}{3}, \frac{-2}{3}, \frac{4}{3}\right)\left(\frac{-4}{3}, \frac{-2}{3}, \frac{4}{3}\right)\rangle}\left(\frac{-4}{3}\right)$$

$$- \frac{\langle(0,0,1), (1,2,2)\rangle}{(1,2,2),(1,2,2)}(1,2,2)$$

$$v_3 = (0, 0, 1) - \frac{\left(\frac{4}{3}\right)\left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right)}{\frac{16}{9} + \frac{4}{9} + \frac{16}{9}} - \frac{2}{9}(1,2,2)$$

$$v_3 = (0, 0, 1) - \frac{\frac{4}{3}}{4}\left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) - \frac{2}{9}(1,2,2)$$

$v_3 = (0,0,1) - \frac{1}{3}\left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) - \frac{2}{9}(1,2,2)$

$= (0,0,1) - \left(\frac{-4}{9}, -\frac{2}{9}, \frac{4}{9}\right) - \left(\frac{2}{9}, \frac{4}{9}, \frac{4}{9}\right)$

$= \left(0 + \frac{4}{9} - \frac{2}{9}, 0 + \frac{2}{9} - \frac{4}{9}; 1 - \frac{4}{9} - \frac{4}{9}\right)$

$\boxed{v_3 = \left(\frac{2}{9}, -\frac{2}{9}, \frac{1}{9}\right)}$

Q) find orthogonal basis of basis $\{(1,1,0),(0,1,0),(0,1,1)\}$ by defined inner product $\langle x,y\rangle = x_1^2 + y_1^2 + 2x_1x_2 + 2y_1y_2$ where $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)$

ans) given that basis $B=\{(1,1,0),(0,1,0),(0,1,1)\}$

$u_1=(1,1,0)$, $u_2=(0,1,0)$, $u_3=(0,1,1)$

$B' = (v_1,v_2,v_3)$

$v_1 = u_1 = (1,1,0)$

$v_2 = u_2 - \langle u_2, v_1\rangle v_1$

$= (0,1,0) - \langle(0,1,0),(1,1,0)\rangle (1,1,0)$

$= (0,1,0) - (0^2+1^2+2(0)+2(1)(1))(1,1,0)$

$v_2 = (-3,-2,0)$

$v_3 = u_3 - \langle u_3, v_1\rangle v_1 - \langle u_3, v_2\rangle v_2$

$= (0,1,1) - \langle(0,1,1);(1,1,0)\rangle(1,1,0) - \langle(0,1,1),(-3,-2,0)\rangle(-3,2,0)$

$u_3 = (0,1,1) - (0^2+1^2+2\times0\times1+2\times1\times1)(1,1,0) - (0^2+(-3)^2+2(0)\times(1)+2(-3)(-2))(-3,-2,0)$

$= (0,1,1) - (3,3,0) - (-63,-42,0)$

$v_3 = (60,40,1)$

$B' = \{(1,1,0),(-3,-2,0),(60,40,1)\}$

orthonormal basis is $B'' \{w_1, w_2, w_3\}$

$$B'' = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{-3}{\sqrt{13}}, \frac{-2}{\sqrt{13}}, 0\right), \left(\frac{60}{\sqrt{5201}}, \frac{40}{\sqrt{5201}}, \frac{1}{\sqrt{5201}}\right) \right\}$$

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,0)}{\sqrt{1+1+0}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{(-3,-2,0)}{\sqrt{9+4+0}} = \left(\frac{-3}{\sqrt{13}}, \frac{-2}{\sqrt{13}}, 0\right)$$

$$w_3 = \frac{v_3}{\|v_3\|} = \frac{(60,40,1)}{\sqrt{60^2+40^2+1}} = \left(\frac{60}{\sqrt{5201}}, \frac{40}{\sqrt{5201}}, \frac{1}{\sqrt{5201}}\right)$$