Example

Verify $f(z) = w = z^3$ is analytic function or not and also find $\frac{dw}{dz}$.

Sol

$$f(z) = w = u + iv = z^3 = (x+iy)^3$$

$$= x^3 + (iy)^3 + 3x^2 iy + 3x(iy)^2$$

$$u + iv = x^3 - iy^3 + i3x^2y - 3xy^2$$

$$u + iv = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$\therefore$ Real part $u = x^3 - 3xy^2$ Imaginary part $v = 3x^2y - y^3$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial v}{\partial x} = 6xy$$

$$\frac{\partial u}{\partial y} = -6xy \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\therefore$ C-R equations are satisfied so $f(z) = z^3$ is an analytic function

To find $f'(z)$ we k.t $f'(z) = \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$f'(z) = 3x^2 - 3y^2 + i6xy = 3[x^2 - y^2 + i2xy]$$

$$= 3[x^2 + (iy)^2 + i2xy] = 3[x+iy]^2$$

$$= 3z^2$$

$$\boxed{\therefore f'(z) = 3z^2}$$

Ex If $w = f(z) = \log z$, find $\frac{dw}{dz}$ and determine where w is non-analytic

Sol Given $w = u + iv = \log(x+iy)$

$x = r\cos\theta \quad y = r\sin\theta \quad r = \sqrt{x^2+y^2}$

$e^{i\theta} = \cos\theta + i\sin\theta \qquad \theta = \tan^{-1}(y/x)$

$w = \log(r\cos\theta + ir\sin\theta)$

$= \log r(\cos\theta + i\sin\theta) = \log re^{i\theta}$

$= \log r + \log e^{i\theta} = \log r + i\theta$

$= \log(x^2+y^2)^{1/2} + i\tan^{-1}(y/x)$

$u + iv = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$

Real part $u = \frac{1}{2}\log(x^2+y^2)$ Imaginary part $v = \tan^{-1}(y/x)$

$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$ $\qquad \frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)$

$= \frac{\cancel{x^2}\ -y}{x^2+y^2}$

$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$ $\qquad \frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)$

$= \frac{x}{x^2+y^2}$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence C-R equations are satisfied. So $f(z) = w = \log z$ is an analytic function everywhe

To find $\frac{dw}{dz}$ we k.t $f'(z) = \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$\therefore f'(z) = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right) = \frac{x-iy}{x^2+y^2}$$

$$= \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy} = \frac{1}{z}$$

$$\boxed{\therefore f'(z) = \frac{1}{z}}$$

Q(3)

Find $f(z) = \cos z$ is an analytic function everywhere and find its derivative.

Sol Given function $f(z) = \cos z$

$\cos iy = \cosh y$

$\sin iy = i \sinh y$

$f(z) = w = u + iv = \cos(x+iy)$

$u + iv = \cos x \cos(iy) - \sin x \sin(iy)$

$u + iv = \cos x \cosh y - i \sin x \sinh y$

Real part $u = \cos x \cosh y$ Imaginary part $v = -\sin x \sinh y$

$\dfrac{\partial u}{\partial x} = -\sin x \cosh y$ $\dfrac{\partial v}{\partial x} = -\cos x \sinh y$

$\dfrac{\partial u}{\partial y} = \cos x \sinh y$ $\dfrac{\partial v}{\partial y} = -\sin x \cosh y$

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

C-R equations are satisfied so $f(z) = \cos z$ is an analytic function

To find its derivative $f'(z) = \dfrac{\partial u}{\partial x} + i \dfrac{\partial v}{\partial x}$

$f'(z) = -\sin x \cosh y - i \cos x \sinh y$

$= -[\sin x \cos(iy) + \cos x \sin(iy)]$

$= -\sin(x+iy) = -\sin z$.

$$\boxed{f'(z) = -\sin z}$$

Ex ④ Find $f(z) = \sinh z$ is an analytic function and find $f'(z)$.

$\sinh(iy) = i\sin y$
$\cosh(iy) = \cos y$

Sol Given $f(z) = \sinh z$

$W = u + iv = \sinh(x+iy)$

$= \sinh x \cosh(iy) + \cosh x \sinh(iy)$

$u + iv = \sinh x \cos y + i \cosh x \sin y$

$\therefore$ R.P $u = \sinh x \cos y$ I.P $v = \cosh x \sin y$

$\frac{\partial u}{\partial x} = \cosh x \cos y$ $\frac{\partial v}{\partial x} = \sinh x \sin y$

$\frac{\partial u}{\partial y} = -\sinh x \sin y$ $\frac{\partial v}{\partial y} = \cosh x \cos y$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

C-R equations are satisfied so $f(z) = \sinh z$ is an analytic function.

To find $f'(z)$ $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f'(z) = \cosh x \cos y + i \sinh x \sin y$

$= \cosh x \cosh(iy) + \sinh x \sinh(iy)$

$= \cosh[x+iy] = \cosh z$

$\boxed{f'(z) = \cosh z}$

Ex (5)

"Verify $f(z) = z^n$ is analytic function or not and also find $f'(z)$.

Sol Given $f(z) = w = z^n$

$u + iv = (x+iy)^n$ Take $x = r\cos\theta$, $y = r\sin\theta$, $z = re^{i\theta}$

$= [r\cos\theta + ir\sin\theta]^n$

$= r^n[\cos\theta + i\sin\theta]^n$

$= r^n[\cos n\theta + i\sin n\theta]$

$u + iv = r^n\cos n\theta + ir^n\sin n\theta$

R.P $u = r^n\cos n\theta$ I.P $v = r^n\sin n\theta$

$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta$ $\frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta$

$\frac{\partial u}{\partial \theta} = -nr^n\sin n\theta$ $\frac{\partial v}{\partial \theta} = nr^n\cos n\theta$

$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \Rightarrow \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$

and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \Rightarrow \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

C-R equations are satisfied so $f(z) = z^n$ is an analytic function.

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To find $f'(z)$

$$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$$

$$f'(z) = e^{-i\theta}\left[n r^{n-1}\cos n\theta + i n r^{n-1}\sin n\theta\right]$$

$$= n r^{n-1} e^{-i\theta}\left[\cos n\theta + i\sin n\theta\right]$$

$$= n r^{n-1} e^{-i\theta} e^{in\theta} = n r^{n-1} e^{(n-1)i\theta} = n r^{n-1}\left(e^{i\theta}\right)^{n-1}$$

$$= n\left[r e^{i\theta}\right]^{n-1} = n z^{n-1}$$

Q. prove that the function $f(z) = \bar{z}$ is nowhere differentiable (Analytic)

Sol Let $z = x + iy$ $\quad \bar{z} = x - iy$

$f(z) = \bar{z}$

$u + iv = x - iy$

R.P $u = x$ $\qquad$ I.P $v = -y$

$\frac{\partial u}{\partial x} = 1$ $\qquad$ $\frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 0$ $\qquad$ $\frac{\partial v}{\partial y} = -1$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ $\qquad$ So $f(z) = \bar{z}$ is not differentiable nowhere.

Ex ② S.T $f(z) = |z|^2$ is differentiable nowhere

Sol $f(z) = |z|^2 = |x+iy|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2+y^2$

$u + iv = x^2 + y^2 + i0$

R.P $u = x^2 + y^2$ $\qquad$ I.P $v = 0$

$\frac{\partial u}{\partial x} = 2x$ $\qquad$ $\frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 2y$ $\qquad$ $\frac{\partial v}{\partial y} = 0$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} \neq -\frac{\partial u}{\partial y}$ So $f(z) = |z|^2$ is not analytic everywhere.

Ex Determine p such that the function

$f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{px}{y}\right)$ be an analytic function

Sol

$u+iv = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{px}{y}\right)$

$\therefore$ R.P $u = \frac{1}{2}\log(x^2+y^2)$ I.P $v = \tan^{-1}\left(\frac{px}{y}\right)$

$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$

$\frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{px}{y}\right)^2}\left(\frac{p}{y}\right)$

$= \frac{py}{y^2+p^2x^2}$

$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$

$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{px}{y}\right)^2}\left(-\frac{px}{y^2}\right)$

$= \frac{-px}{y^2+p^2x^2}$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{x}{x^2+y^2} = \frac{-px}{y^2+p^2x^2}$

Comparing $\boxed{p=-1}$

Ex Find the constants a, b, c, d if ~~f(z)=~~

$f(z) = (ax^4+bx^2y^2+cy^4+dx^2-2y^2) + i(4x^3y-exy^3+4xy)$

is analytic.

Sol $u+iv = (ax^4+bx^2y^2+cy^4+dx^2-2y^2) + i(4x^3y-exy^3+4xy)$

$u = ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2 \qquad v = 4x^3y - exy^3 + 4xy$

$\frac{\partial u}{\partial x} = 4ax^3 + 2bxy^2 + 2dx \qquad \frac{\partial v}{\partial x} = 12x^2y - ey^3 + 4y$

$\frac{\partial u}{\partial y} = 2bx^2y + 4cy^3 - 4y \qquad \frac{\partial v}{\partial y} = 4x^3 - 3exy^2 + 4x$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 4ax^3 + 2bxy^2 + 2dx = 4x^3 - 3exy^2 + 4x$

Comparing $\boxed{a=1}$, $2b = -3e \qquad 2d = 4 \Rightarrow \boxed{d=2}$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow 12x^2y - ey^3 + 4y = -2bx^2y - 4cy^3 + 4y$

$12 = -2b \Rightarrow \boxed{b=-6}$ then $e = -\frac{2}{3}(-6)$

$\boxed{e=4}$

$-e = -4c$

$-4 = -4c \Rightarrow \boxed{c=1}$

3) S.T $f(z) = z + 2\bar{z}$ is not analytic anywhere in the Complex plane.

4) S.T the function $f(z) = 2xy + i(x^2 - y^2)$ is nowhere analytic.

5) Use C-R equations in polar form, to show that $f(z) = \sqrt{r}\, e^{\frac{i\theta}{2}}$ is analytic everywhere.

Construction of an Analytic function

(1) Find the analytic function, whose real part is $e^{-x}[(x^2-y^2-2)\cos y + 2xy\sin y]$. Also find its imaginary part.

Sol Given $u(x,y) = e^{-x}[(x^2-y^2-2)\cos y + 2xy\sin y]$

$$\frac{\partial u}{\partial x} = e^{-x}[2x\cos y + 2y\sin y] - e^{-x}[(x^2-y^2-2)\cos y + 2xy\sin y]$$

$$= e^{-x}[2x\cos y + 2y\sin y - (x^2-y^2-2)\cos y - 2xy\sin y]$$

$$= e^{-x}[(2x - x^2 + y^2 + 2)\cos y + (2y - 2xy)\sin y]$$

$$\frac{\partial u}{\partial y} = e^{-x}[-(x^2-y^2-2)\sin y - 2y\cos y + 2xy\cos y + 2x\sin y]$$

$$= e^{-x}[(2x - x^2 + y^2 + 2)\sin y + (2xy - 2y)\cos y]$$

We k.t $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ $\left(\because \text{By C-R equations } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right)$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$f'(z) = e^{-x}[(2x - x^2 + y^2 + 2)\cos y + (2y - 2xy)\sin y]$$
$$- ie^{-x}[(2x - x^2 + y^2 + 2)\sin y + (2xy - 2y)\cos y]$$

By Milne-Thomson's Method, Put $x = z$, $y = 0$

$$f'(z) = e^{-z}[(2z - z^2 + 2)] - ie^{-z}[0 + 0]$$

$$f'(z) = e^{-z}[2z - z^2 + 2]$$

integrating on both sides w.r.t z

$$\int f'(z)\,dz = \int e^{-z}(2z - z^2 + 2)\,dz$$

$$f(z) = (2z - z^2 + 2)\frac{e^{-z}}{-1} - (2 - 2z)\frac{e^{-z}}{(-1)^2} + (-2)\frac{e^{-z}}{(-1)^3}$$

$$f(z) = e^{-z}\left[-2z + z^2 - 2 - 2 + 2z + 2\right]$$

$$f(z) = e^{-z}\left[z^2 - 2\right]$$ which is the required Analytic function.

To find its Imaginary part

$$f(z) = e^{-z}\left[z^2 - 2\right]$$

$$u + iv = e^{-(x+iy)}\left[(x+iy)^2 - 2\right]$$

$$= e^{-x}e^{-iy}\left[x^2 + (iy)^2 + i2xy - 2\right]$$

$$= e^{-x}(\cos y - i\sin y)\left((x^2 - y^2 - 2) + i2xy\right)$$

$$= e^{-x}\left[(x^2 - y^2 - 2)\cos y + 2xy\sin y\right] + ie^{-x}\left[2xy\cos y - (x^2 - y^2 - 2)\sin y\right]$$

$\therefore$ Imaginary part $v(x,y) = e^{-x}\left[2xy\cos y - (x^2 - y^2 - 2)\sin y\right]$

Ex Find the analytic function whose real part is $u = e^{-2xy}\sin(x^2-y^2)$ and also find its imaginary part.

Sol Given $u = e^{-2xy}\sin(x^2-y^2)$

$$\frac{\partial u}{\partial x} = e^{-2xy}\cos(x^2-y^2)\,2x + \sin(x^2-y^2)(-2y)e^{-2xy}$$

$$= 2e^{-2xy}\left[x\cos(x^2-y^2) - y\sin(x^2-y^2)\right]$$

$$\frac{\partial u}{\partial y} = e^{-2xy}\cos(x^2-y^2)(-2y) + \sin(x^2-y^2)(-2x)e^{-2xy}$$

$$= -2e^{-2xy}\left[y\cos(x^2-y^2) + x\sin(x^2-y^2)\right]$$

$\therefore$ we k.t $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ $\left(\because \text{By C-R equations } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right)$

$$f'(z) = 2e^{-2xy}\left[x\cos(x^2-y^2) - y\sin(x^2-y^2)\right] + i2e^{-2xy}\left[y\cos(x^2-y^2) + x\sin(x^2-y^2)\right]$$

By Milne-Thomson's Method, put $x = z$, $y = 0$.

$$f'(z) = 2e^0\left[z\cos z^2\right] + i2e^0\left(0 + x(0)\right)$$

$f'(z) = 2z\cos z^2$ integrate

$$\int f'(z)\,dz = \int 2z\cos z^2\,dz$$

$f(z) = \sin z^2$ which is an analytic function

Ex Determine the analytic function whose imaginary part is $\frac{x-y}{x^2+y^2}$ and also find its real part.

Sol Given $v = \frac{x-y}{x^2+y^2}$

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2)1-(x-y)2x}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2+2xy}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2+2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(-1)-(x-y)2y}{(x^2+y^2)^2} = \frac{-x^2-y^2-2xy+2y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2-2xy}{(x^2+y^2)^2}$$

we k.t $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ $\quad \left(\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right)$

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{y^2-x^2-2xy}{(x^2+y^2)^2} + i\left(\frac{y^2-x^2+2xy}{(x^2+y^2)^2}\right)$$

By Milne-Thomson's Method, put $x=z$, $y=0$

$$f'(z) = \frac{-z^2}{(z^2)^2} + i\left(\frac{-z^2}{(z^2)^2}\right)$$

$$f'(z) = -\frac{1}{z^2} - \frac{i}{z^2} = \frac{-(1+i)}{z^2}.$$

integrating on both sides w.r.t. z

$$\int f'(z)\,dz = \int -\frac{(1+i)}{z^2}\,dz$$

$f(z) = \frac{(1+i)}{z}$ which is the req'd analytic function.

<u>To find its real part $u(x,y)$</u>

$$\cancel{f}(z) = \frac{1+i}{z}$$

$$u+iv = \frac{1+i}{x+iy} = \frac{1+i}{x+iy} \cdot \frac{x-iy}{x-iy}$$

$$= \frac{x-iy+ix+y}{x^2+y^2} = \frac{x+y}{x^2+y^2} + i\,\frac{x-y}{x^2+y^2}$$

$\therefore$ Real part $u(x,y) = \frac{x+y}{x^2+y^2}$

Ex Find the analytic function $f(z) = u+iv$

if $u - v = (x-y)(x^2+4xy+y^2)$

Sol we have $f(z) = u + iv \rightarrow (i)$

$$if(z) = iu + i^2 v$$

$$if(z) = iu - v \longrightarrow (ii)$$

Add (i) & (ii)

$$\therefore f(z) + if(z) = u + iv + iu - v$$

$$(1+i) f(z) = (u-v) + i(u+v)$$

$$F(z) = U + iV$$

$\therefore$ Given $u - v = U = (x-y)(x^2+4xy+y^2)$

$$\frac{\partial U}{\partial x} = (x-y)(2x+4y) + (x^2+4xy+y^2)(1)$$

$$= 2x^2 + 4xy - 2xy - 4y^2 + x^2 + 4xy + y^2$$

$$= 3x^2 + 6xy - 3y^2$$

$$\frac{\partial U}{\partial y} = (x-y)(4x+2y) + (x^2+4xy+y^2)(-1)$$

$$= 4x^2 + 2xy - 4xy - 2y^2 - x^2 - 4xy - y^2$$

$$= 3x^2 - 6xy - 3y^2$$

we k.t $F'(z) = \frac{\partial U}{\partial x} + i \frac{\partial V}{\partial x}$ $\left(\because \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}\right)$

$$F'(z) = \frac{\partial U}{\partial x} - i \frac{\partial U}{\partial y}$$

$f'(z) = 3x^2+6xy-3y^2 - i(3x^2-6xy-3y^2)$

By Milne-Thomson's Method, put $x=z$, $y=0$

$F'(z) = 3z^2 - i3z^2 = 3(1-i)z^2$

integrating w.r.t z on both sides

$\int F'(z)dz = \int 3(1-i)z^2 dz$

$F(z) = \cancel{3}(1-i)\frac{z^3}{\cancel{3}} = (1-i)z^3$

$(1+i)f(z) = (1-i)z^3$

$f(z) = \frac{1-i}{1+i}z^3 = \frac{1-i}{1+i}\cdot\frac{1-i}{1-i}z^3 = \frac{1+i^2-2i}{1-i^2}z^3$

$f(z) = \frac{-2i}{1+1}z^3 = -\frac{2i}{2}z^3$

$f(z) = -iz^3$ is the reqd analytic function.

To find u and v

$u+iv = -i(x+iy)^3 = -i(x^3+(iy)^3+3x^2iy + 3x(iy)^2)$

$u+iv = -i(x^3 - iy^3 + i3x^2y - 3xy^2)$

$u+iv = -ix^3 - y^3 + 3x^2y + i3xy^2$

$= (3x^2y - y^3) + i(3xy^2 - x^3)$

Real part $u = 3x^2y - y^3$

Imaginary part $v = 3xy^2 - x^3$.

Ex Determine the analytic function $f(z) = u + iv$ where $u + v = x^2 - y^2 - 2xy + \frac{x+y}{x^2+y^2}$ and also find its u and v seperately.

Sn We have $f(z) = u + iv$

$$if(z) = iu - v$$

$$f(z) + if(z) = u + iv + iu - v$$

$$(1+i)f(z) = u - v + i(u+v)$$

$$F(z) = U + iV$$

Given $u + v = V = x^2 - y^2 - 2xy + \frac{x+y}{x^2+y^2}$

$$\frac{\partial V}{\partial x} = 2x - 2y + \frac{(x^2+y^2)1 - (x+y)(2x)}{(x^2+y^2)^2}$$

$$= 2x - 2y + \frac{y^2 - x^2 - 2xy}{(x^2+y^2)^2}$$

$$\frac{\partial V}{\partial y} = -2y - 2x + \frac{(x^2+y^2)1 - (x+y)2y}{(x^2+y^2)^2}$$

$$= -2y - 2x + \frac{x^2 - y^2 - 2xy}{(x^2+y^2)^2}$$

$\therefore$ we k.t $F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$

$$F'(z) = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x}$$

$$F'(z) = -2y - 2x + \frac{x^2 - y^2 - 2xy}{(x^2+y^2)^2} + i\left[2x - 2y + \frac{y^2 - x^2 - 2xy}{(x^2+y^2)^2}\right.$$

By Milne Thomson Method, put $x = z,\ y = 0$.

$$F'(z) = -2z + \frac{z^2}{(z^2)^2} + i\left[2z + \frac{-z^2}{(z^2)^2}\right]$$

$$= \cancel{-2(1-i)}\ -2z + \frac{1}{z^2} + i2z - \frac{i}{z^2}$$

$$= -2(1-i)z + \frac{1-i}{z^2}$$

$$= (1-i)\left[-2z + \frac{1}{z^2}\right] \quad \text{integrating}$$

$$\int F'(z)\,dz = \int (1-i)\left[-2z + \frac{1}{z^2}\right] dz$$

$$F(z) = (1-i)\left[-\frac{\cancel{2}z^2}{\cancel{2}} - \frac{1}{z}\right]$$

$$(1+i)f(z) = (1-i)\left[-(x+iy)^2 - \frac{1}{x+iy}\right]$$

$$f(z) = \frac{1-i}{1+i}\left[-x^2 + y^2 - i2xy - \frac{1}{x+iy}\cdot\frac{x-iy}{x-iy}\right]$$

$$= \frac{1-i}{1+i}\cdot\frac{1-i}{1-i}\left[-x^2 + y^2 - i2xy - \left(\frac{x-iy}{x^2+y^2}\right)\right]$$

$$= \frac{1+i^2-2i}{1-i^2}\left[-x^2 + y^2 - i2xy - \left(\frac{x-iy}{x^2+y^2}\right)\right]$$

$$(1+i)f(z)=(1-i)\left[-z^2-\frac{1}{z}\right]$$

$$f(z)=\frac{1-i}{1+i}\left[-z^2-\frac{1}{z}\right]=\frac{1-i}{1+i}\cdot\frac{1-i}{1-i}\left[-z^2-\frac{1}{z}\right]$$

$$=\frac{1+i^2-2i}{1+i^2}\left[-z^2-\frac{1}{z}\right]=\frac{-\cancel{2}i}{\cancel{2}}\left[-z^2-\frac{1}{z}\right]$$

$f(z)=i\left[z^2+\frac{1}{z}\right]$ is the reqd analytic function

∴ <u>To find real and imaginary parts</u>

$$f(z)=i\left[z^2+\frac{1}{z}\right]$$

$$u+iv=i\left[(x+iy)^2+\frac{1}{x+iy}\right]$$

$$=i\left[x^2+i^2y^2+i2xy+\frac{1}{x+iy}\cdot\frac{x-iy}{x-iy}\right]$$

$$=i\left[x^2-y^2+i2xy+\frac{x-iy}{x^2+y^2}\right]$$

$$=ix^2-iy^2-2xy+\frac{ix}{x^2+y^2}+\frac{y}{x^2+y^2}$$

$$u=-2xy+\frac{y}{x^2+y^2}\qquad v=x^2-y^2+\frac{x}{x^2+y^2}.$$

Harmonic functions

If the function $u(x,y)$ satisfies the Laplace equation in two variables, then $u(x,y)$ is known as Harmonic function.

ie $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

III^{ly} the function $v(x,y)$ is said to be harmonic then its satisfies Laplace equation

i.e $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

$\therefore$ If $u(x,y)$ is harmonic function then $v(x,y)$ is known as conjugate harmonic function and vice versa.

Orthogonal System

Consider the two families of curves $u(x,y) = c_1$ and $v(x,y) = c_2$

differentiating $$\frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}\Big/\frac{\partial v}{\partial x} = m_1 \text{ (say)}$$

iii) $\frac{\partial v}{\partial x}\frac{dx}{dx} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-\partial v/\partial x}{\partial v/\partial y} = m_2 \text{ (say)}$$

$$m_1 \cdot m_2 = \frac{\partial v/\partial y}{\partial v/\partial x} \cdot \frac{-\partial v/\partial x}{\partial v/\partial y} = -1$$

So the real and imaginary parts i.e $u(x,y)$ and $v(x,y)$ are form an orthogonal system

Ex ① S.t the function $u = e^{2x}(x\cos 2y - y\sin 2y)$ is a harmonic function and find its harmonic conjugate.

Sol:- Given $u = e^{2x}[x\cos 2y - y\sin 2y] \rightarrow (i)$

$$\frac{\partial u}{\partial x} = e^{2x}[\cos 2y] + 2e^{2x}[x\cos 2y - y\sin 2y]$$

$$= e^{2x}[(2x+1)\cos 2y - 2y\sin 2y]$$

$$\frac{\partial^2 u}{\partial x^2} = e^{2x}[2\cos 2y] + 2e^{2x}[(2x+1)\cos 2y - 2y\sin 2y]$$

$$= e^{2x}[(4x+4)\cos 2y - 4y\sin 2y]$$

$$= \cancel{2e^{2x}[(2x+1)\cos 2y - 2y\sin}$$

$$\frac{\partial^2 u}{\partial x^2} = 4e^{2x}[(x+1)\cos 2y - y\sin 2y] \rightarrow (ii)$$

diff. partially w.r.t y

$$\frac{\partial u}{\partial y} = e^{2x}(-2x\sin 2y - \sin 2y - 2y\cos 2y)$$

$$= -e^{2x}[(2x+1)\sin 2y + 2y\cos 2y]$$

$$\frac{\partial^2 u}{\partial y^2} = -e^{2x}[2(2x+1)\cos 2y + 2\cos 2y - 4y\sin 2y]$$

$$= -e^{2x}[(4x+4)\cos 2y - 4y\sin 2y]$$

$$\frac{\partial^2 u}{\partial y^2} = -4e^{2x}[(x+1)\cos 2y - y\sin 2y] \rightarrow (iii)$$

Adding (ii) & (iii) then we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence $u(x,y)$ is a harmonic function.

To find its Conjugate harmonic $v(x,y)$

we k.t $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$$f'(z) = e^{2x}[(2x+1)\cos 2y - 2y\sin 2y]$$
$$+ ie^{2x}[(2x+1)\sin 2y + 2y\cos 2y]$$

By Milne - Thomson's Method, put $x=z, y=0$

$$f'(z) = e^{2z}[(2z+1)]$$

integrating on both sides w.r.t z

$$\int f'(z)\,dz = \int (2z+1)\, e^{2z}\, dz$$

$$f(z) = (2z+1)\frac{e^{2z}}{2} - 2\,\frac{e^{2z}}{\cancel{4}_{2}}$$

$$= \frac{e^{2z}}{2}\left[2z+1-1\right]$$

$$= \frac{e^{2z}}{2}(2z) = z\,e^{2z}$$ which is an analytic function.

$$f(z) = z e^{2z}$$

$$u+iv = (x+iy)\, e^{2(x+iy)} = (x+iy)\, e^{2x} e^{i2y}$$

$$= e^{2x}(x+iy)(\cos 2y + i \sin 2y)$$

$$= e^{2x}\left[x\cos 2y + i x \sin 2y + i y \cos 2y - y \sin 2y\right]$$

$\therefore$ Conjugate Harmonic $v(x,y) = e^{2x}\left[x\sin 2y + y\cos 2y\right]$

Ans

Ex ② Verify whether the function $v(x,y) = y^3 - 3x^2 y$ is harmonic. If so find its harmonic conjugate $u(x,y)$ and hence express $u+iv$ as an analytic function $f(z)$.

Sol Given $v(x,y) = \cancel{y^3}\ \cancel{3x^2y}\ y^3 - 3x^2y$

$$\frac{\partial v}{\partial x} = -6xy \qquad \frac{\partial^2 v}{\partial x^2} = -6y$$

$$\frac{\partial v}{\partial y} = 3y^2 - 3x^2 \qquad \frac{\partial^2 v}{\partial y^2} = 6y$$

$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$. Hence $v(x,y)$ is harmonic function.

To find its Conjugate harmonic

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial y}$$

$$f'(z) = 3y^2 - 3x^2 - i6xy$$

By Milne Thomson's Method, put $x = z,\ y = 0$.

$f'(z) = -3z^2$ integrating

$$\int f'(z)\,dz = -3\int z^2\,dz$$

$f(z) = -\frac{3z^3}{3} \Rightarrow f(z) = -z^3$ is an analytic function.

$$u + iv = -(x+iy)^3 = -[x^3 - iy^3 + i3x^2y - 3xy^2]$$

Harmonic Conjugate $u(x,y) = -x^3 + 3xy^2$

Ex ③ Verify that the families of curves $u = c_1$ and $v = c_2$ cut orthogonally, where $w = u + iv = \frac{1}{z}$.

Sol Given $w = u + iv = \frac{1}{z} = \frac{1}{x+iy}$

$$u + iv = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy}$$

$$u + iv = \frac{x-iy}{x^2+y^2}$$

$$\therefore u = \frac{x}{x^2+y^2} \qquad v = \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(1) - x\,2x}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(-1) - (-y)2y}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\therefore m_1 m_2 = \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \cdot \frac{\partial v/\partial x}{\partial v/\partial y}$$

$$= \frac{\frac{y^2-x^2}{(x^2+y^2)^2}}{\frac{-2xy}{(x^2+y^2)^2}} \cdot \frac{\frac{2xy}{(x^2+y^2)^2}}{\frac{y^2-x^2}{(x^2+y^2)^2}} = -1$$

$\therefore$ Hence $u(x,y)$ and $v(x,y)$ cut orthogonally for the function $f(z) = \frac{1}{z}$.

Ex (4) For $w = \exp(z^2)$, find u and v and prove that the curves $u(x,y) = c_1$ and $v(x,y) = c_2$ where c_1 and c_2 are constants, cut orthogonally.

Sol $w = \exp(z^2) = e^{z^2} = e^{(x+iy)^2}$

$$u + iv = e^{x^2-y^2+i2xy} = e^{x^2-y^2} \cdot e^{i2xy}$$

$$= e^{x^2-y^2}[\cos 2xy + i\sin 2xy]$$

$\therefore u = e^{x^2-y^2}\cos 2xy \qquad v = e^{x^2-y^2}\sin 2xy$

5) If $f(z)$ is an analytic function of z, Show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Sol We have $f(z) = u+iv$ $\quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$|f(z)|^2 = u^2+v^2 = \phi \rightarrow (i) \qquad |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

diff partially (i) w.r.t x

$$\frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial^2\phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + v\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial x}\right]$$

$$\frac{\partial^2\phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right] \rightarrow (ii)$$

$|||^{ly}$ $$\frac{\partial^2\phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right] \rightarrow (iii)$$

Adding (ii) & (iii)

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right.$$
$$\left. + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$
$$= 2\left[u(0) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(-\frac{\partial v}{\partial x}\right)^2 + v(0) + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\phi = 2\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$= 4\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right] = 4|f'(z)|^2$$

$$\boxed{\therefore \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\cancel{\phi}\,|f(z)|^2 = 4|f'(z)|^2}$$

Ex② If $f(z)$ is an analytic function of z, prove that

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]|Rf(z)|^2 = 2|f'(z)|^2$$ where $Rf(z)$ is real part of $f(z)$.

Sol we have $f(z) = u+iv$

$$f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

$Rf(z) = u$

$|Rf(z)|^2 = u^2 \longrightarrow (i)$

diff. partially (i) w.r.t x

$$\frac{\partial}{\partial x}|Rf(z)|^2 = 2u\frac{\partial u}{\partial x}$$

$$\frac{\partial^2}{\partial x^2}|Rf(z)|^2 = 2\left[u\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right]$$

$$\frac{\partial^2}{\partial x^2}|Rf(z)|^2 = 2\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\right] \longrightarrow (ii)$$

$$\text{|||ly}\quad \frac{\partial^2}{\partial y^2}|Rf(z)|^2 = 2\left[u\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2\right] \longrightarrow (iii)$$

Adding (ii) & (iii) we have

$$\frac{\partial^2}{\partial x^2}|R f(z)|^2 + \frac{\partial^2}{\partial y^2}|R f(z)|^2 = 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|R f(z)|^2 = 2\left[u(0) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$\boxed{\therefore \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|R f(z)|^2 = 2|f'(z)|^2}$$

practice problem

① If $f(z)$ is an analytic function of z, prove that

a) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2|f'(z)|^2 |f(z)|^{p-2}$

b) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 0$.

Applications of analytic functions to fluid-flow and Field problems

Since the real and imaginary parts of an analytic function satisfy the Laplace equation in two variables, these conjugate functions provide solutions to a number of field and flow problems.

For example, consider the two dimensional irrotational motion of an incompressible fluid in planes parallel to xy-plane

Let $\vec{V}$ be the velocity of a fluid particle, then it can be expressed as

$$\vec{V} = v_x i + v_y j \quad \longrightarrow (1)$$

Since the motion is irrotational, there exists a scalar function $\phi(x,y)$ such that

$$\vec{V} = \nabla\phi(x,y) = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j \quad \rightarrow (2)$$

From (1) and (2) $\quad v_x = \frac{\partial \phi}{\partial x}$ and $v_y = \frac{\partial \phi}{\partial y} \quad \rightarrow (3)$

The scalar function $\phi(x,y)$ which gives the velocity components, is called the velocity potential function or simply the velocity potential.

Also the fluid being incompressible, $\operatorname{div} \vec{V} = 0$

$$\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right)(v_x i + v_y j) = 0$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \longrightarrow (4)$$

Substituting the values of v_x and v_y from (3) & (4)

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = 0 \Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Thus the function ϕ is harmonic and can be treated as real part of an analytic function

$$w = f(z) = \phi(x,y) + i\,\psi(x,y)$$

The conjugate function $\psi(x,y)$ contains the slope at any point of the curve ie $\psi(x,y) = c$ is given by

$$\frac{dy}{dx} = -\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}} = \frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial x}} \quad (\because \text{By C-R equations})$$

$$= \frac{v_y}{v_x}$$

This shows that the resultant velocity $\sqrt{v_x^2 + v_y^2}$ of the fluid particle is along the tangent to the curve $\psi(x,y) = c$. Such curves are known as stream lines and $\psi(x,y)$ is called stream function.

$\therefore$ Complex potential = Velocity potential + i Stream function

$$\boxed{W = f(z) = \phi(x,y) + i\,\psi(x,y)}$$

Ex ① In a two-dimensional fluid flow, the stream function is $\psi = \frac{y}{x^2+y^2}$, then find the velocity potential ϕ.

Sol:- Given $\psi = \frac{y}{x^2+y^2}$

$$\frac{\partial \psi}{\partial x} = \frac{-2xy}{(x^2+y^2)^2} \qquad \frac{\partial \psi}{\partial y} = \frac{(x^2+y^2)(1) - y\,2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

we k.t $f'(z) = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$ ($\because$ By C-R equati

$$= \frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x}$$

$$f'(z) = \frac{x^2-y^2}{(x^2+y^2)^2} + i\left(\frac{-2xy}{(x^2+y^2)^2}\right)$$

By Milne-Thomson's Method, put $x=z, y=0$

$$f'(z) = \frac{z^2}{(z^2)^2} + i(0) = \frac{1}{z^2}.$$

integrating on both sides w.r.t z

$\int f'(z)\,dz = \int \frac{1}{z^2}\,dz \Rightarrow f(z) = -\frac{1}{z}$ is the req'd Complex potential.

To find velocity potential ϕ

$$\phi + i\psi = \frac{-1}{x+iy}\,\frac{x-iy}{x-iy} = \frac{-x+iy}{x^2+y^2}$$

$$\boxed{\therefore \text{The Velocity potential } \phi = \frac{-x}{x^2+y^2}}$$

(ii) An electric field in the xy-plane is given by the potential function $\phi = 3x^2y - y^3$, find the stream function and hence find complex potential.

Sol Given $\phi = 3x^2y - y^3$

$$\frac{\partial \phi}{\partial x} = 6xy \qquad \frac{\partial \phi}{\partial y} = 3x^2 - 3y^2$$

we k.t $f'(z) = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$ $\quad \left(\because \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}\right)$

$$f'(z) = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}$$

$$f'(z) = 6xy - i(3x^2 - 3y^2)$$

By Milne-Thomson's Method, put $x = z$, $y = 0$

$f'(z) = -i3z^2$ integrating

$\int f'(z)\,dz = -i3\int z^2\,dz \Rightarrow f(z) = -i\cancel{3}\frac{z^3}{\cancel{3}}$

$f(z) = -iz^3$ is the reqd complex potential.

To find stream function

$$\phi + i\psi = -i(x+iy)^3 = -\cancel{i[x^3 - iy^3 + 3xy^2 - 3x]}$$

$$= -i[x^3 - iy^3 + i3x^2y - 3xy^2]$$

$$= -ix^3 - y^3 + 3x^2y + i3xy^2$$

$\boxed{\therefore \text{ stream function } \psi = 3xy^2 - x^3}$

Ex ③: If the potential function is $\log(x^2+y^2)$ find the flux function and complex potential function

Sol Given potential function $\phi = \log(x^2+y^2)$

$$\frac{\partial\phi}{\partial x} = \frac{1}{x^2+y^2}2x = \frac{2x}{x^2+y^2} \qquad \frac{\partial\phi}{\partial y} = \frac{2y}{x^2+y^2}$$

$$f'(z) = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} \qquad \left(\because \frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y} \text{ by C-R eqn}\right)$$

$$f'(z) = \frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y}$$

$$f'(z) = \frac{2x}{x^2+y^2} - i\frac{2y}{x^2+y^2}$$

$\therefore$ By Milne Thomson Method, put $x=z, y=0$

$$\therefore f'(z) = \frac{2z}{z^2} = 2/z \quad \text{integrating}$$

$$\int f'(z)dz = \int \frac{2}{z}dz = \cancel{2\log z}$$

$f(z) = 2\log z$ is a req'd Complex potential function

<u>To find flux function</u>

$$\phi + i\psi = 2\log(x+iy)$$

put $x = r\cos\theta$ $y = r\sin\theta$

$x+iy = re^{i\theta}$

where $r = (x^2+y^2)^{1/2}$

$\theta = \tan^{-1}(y/x)$

$$= 2\log(re^{i\theta})$$

$$\phi+i\psi = 2[\log r + i\theta]$$

$$= 2\log(x^2+y^2)^{1/2} + 2i\tan^{-1}(y/x)$$

$$= \cancel{2}(\tfrac{1}{\cancel{2}})\log(x^2+y^2) + i\,2\tan^{-1}y/x$$

$$\boxed{\therefore \text{flux function } \psi = 2\tan^{-1}(y/x)}$$

page-1 Module-2

Q/ Discuss the conformed mapping fn $f(z)=z^2$

A/ $\boxed{w=f(z)=z^2}$

$u+iv=(x+iy)^2$

$u+iv=x^2-y^2+i2xy$

$\therefore u=x^2-y^2 \; ; \; v=2xy$

Case(i) let us take $x=a$

$u=a^2-y^2$ $\qquad v=2ay \Rightarrow y=v/2a$

$u=a^2-(v/2a)^2=a^2-\frac{v^2}{4a^2}=\frac{4a^4-v^2}{4a^2}$

~~4a~~ $4a^2u=4a^4-v^2 \Rightarrow v^2=\cancel{4a^2u} \; 4a^4-4a^2u$

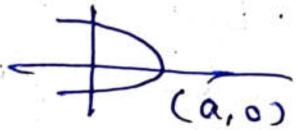
$v^2=-4a^2(u-a^2)$

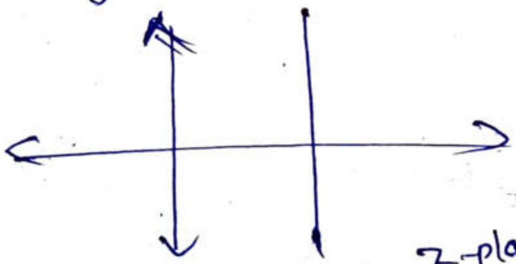
Which is a parabola with focus $(a^2,0)$ & in the Negative Direction.

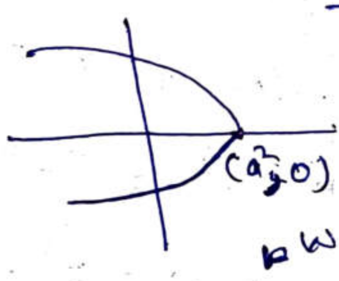
$\because y^2=4a(x-a)$

parabola eqn

$y^2=-4a(x-a)$







Thus I conclude that the straight line $x=a$ in the z-plane is transformed into a parabola with focus $(a^2,0)$ with Negative direction in the w-plane for the function $w=z^2$.

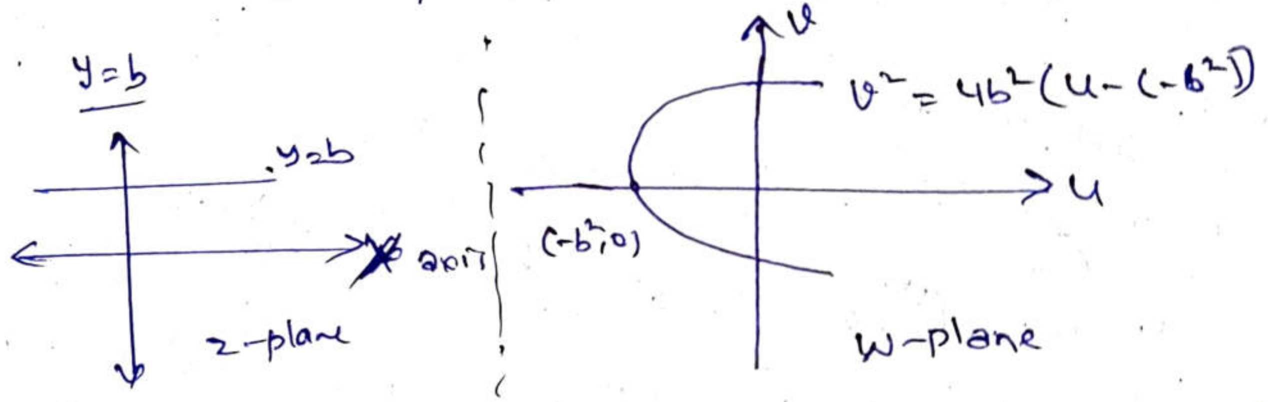
Case-ii) let us take $y=b$ (page-2

$u=x^2-b^2$; $v=2xb \Rightarrow x=v/2b$

$u=\frac{v^2}{4b^2}-b^2 = \frac{v^2-4b^4}{4b^2}$

$4ub^2+4b^4=v^2$ $\therefore v^2=4b^2[u-(-b^2)]$

Which is a parabola with focus $(-b^2,0)$



Thus I Conclude that the straight line $y=b$ in the z-plane transformed into a parabola with focus $(-b^2,0)$ in the w-plane for the transformation $w=z^2$.

2Q) Discuss the Conformal Mapping for the function $f(z)=e^z$?

Sol: Given $w=f(z)=e^z$

$u+iv=e^{x+iy} = e^x.e^{iy}$ ~~[cos~~ $=e^x[\cos y+i\sin y]$

$u=e^x\cos y$; $v=e^x\sin y$

Case(i) $u/e^x=\cos y$; $v/e^x=\sin y$

W.k.T $\sin^2y+\cos^2y=1$

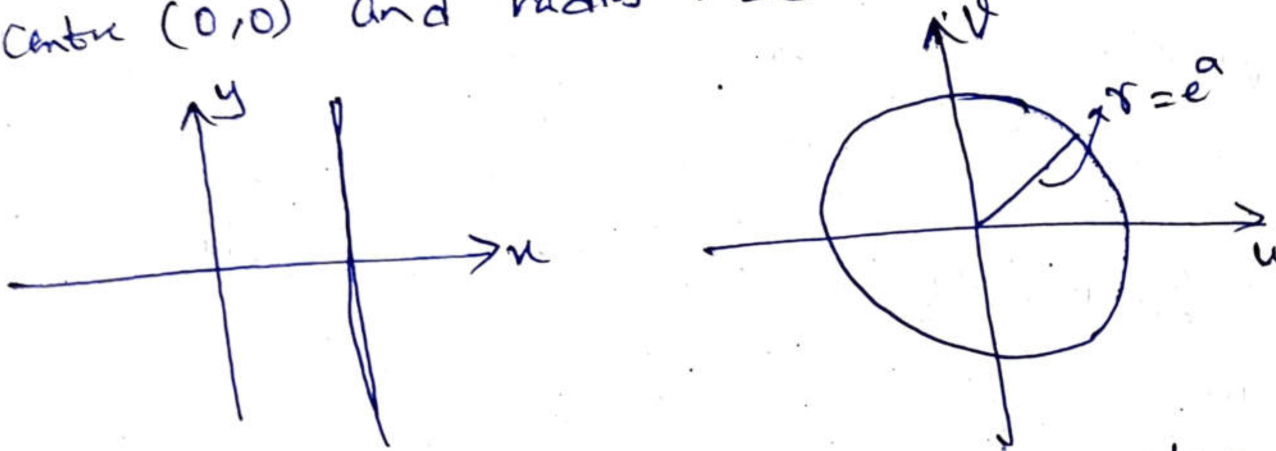
$(v/e^x)^2+(u/e^x)^2=1$ ~~$u^2+v^2=e^{2x}$~~

$\frac{u^2}{(e^x)^2}+\frac{v^2}{(e^x)^2}=1 \Rightarrow u^2+v^2=(e^x)^2$

page-3

let us take $x=a$

$u^2+v^2=(e^a)^2$ which is a circle with centre $(0,0)$ and radius $r=e^a$.



z-plane w-plane

Thus I conclude that the straight line $x=a$ in the z-plane are transformed into a circle with centre $(0,0)$ and radius ~~e^a~~ $r=e^a$ in the w-plane for the transformation $w=e^z$.

Case-ii

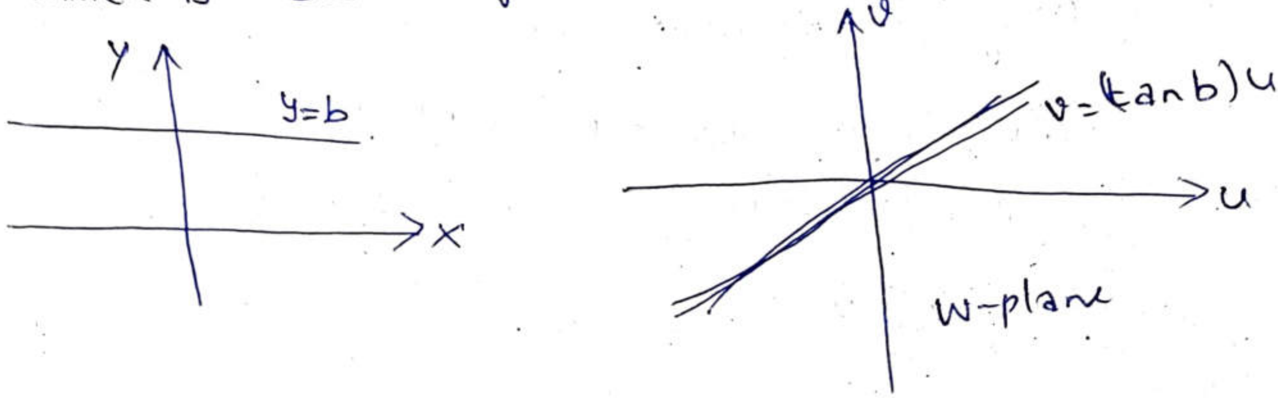
$u/\cos y = e^x$, $v/\sin y = e^x$

$\Rightarrow u/\cos y = v/\sin y \Rightarrow \sin y \cdot \frac{u}{\cos y} = v \Rightarrow v = \left(\frac{\sin y}{\cos y}\right)u$

$\Rightarrow v = \tan y \cdot u$

Take $y=b$ (let) then $v=(\tan b)u$ [$\because y=mx$ straight line]

Which is the straight line passes through the origin.



Thus I Conclude that the straight line $y = b$ in the z-plane mapped into a straight line passes through the Origin in W-plane for the transformation $W = e^z$.

$\boxed{(w = z^2, e^z) \quad \sin z, \cos z \quad \text{Translation} \quad \text{Inversion}}$

Eg: Discuss the Conformal Mapping for the function $W = 1/z$. [Inversion property]

Sol: Given $W = f(z) = 1/z$

$$u + iv = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} \Rightarrow \frac{x-iy}{x^2+y^2}$$

Real part

$$u = \frac{x}{x^2+y^2} \qquad v = \frac{-y}{x^2+y^2}$$

Case - (i)

Let us take $x = a$ then $u = \frac{a}{a^2+y^2}$; $v = \frac{-y}{a^2+y^2}$

$$\cancel{u^2 = \frac{a^2}{(a^2+y^2)^2}} \quad a^2+y^2 = \frac{a}{u} \qquad v^2 = \frac{y^2}{(a^2+y^2)^2}$$

$$y^2 = a/u - a^2 = \frac{a - a^2u}{u}$$

$$\therefore v^2 = \frac{y^2}{(a^2+y^2)^2} = \frac{\frac{a-a^2u}{u}}{(a/u)^2} = \frac{a-a^2u}{u} \cdot \frac{u^2}{a^2} \quad \cancel{= au}$$

$$v^2 = \frac{au - a^2u^2}{a^2} \Rightarrow a^2v^2 = au - a^2u^2$$

$$\Rightarrow a^2u^2 + a^2v^2 - au = 0 \Rightarrow au^2 + av^2 - u = 0$$

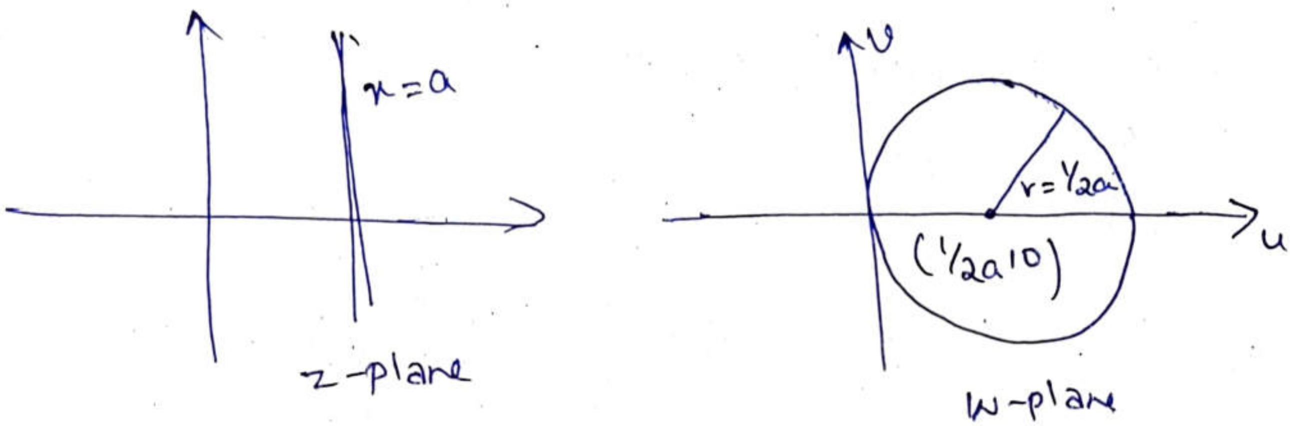
$$\Rightarrow u^2 + v^2 - \frac{1}{a}u = 0 \quad \cancel{\Rightarrow u^2 + v^2 - 2\left(\frac{1}{2a}\right)u}$$

$$\cancel{\Rightarrow u^2 + \left[v^2 - 2(1/2a)u + (1/2a)^2\right] - (1/2a)^2 = 0}$$

page-5

$\Rightarrow \left[u^2 - 2(1/2a)u + (1/2a)^2\right] - (1/2a)^2 + v^2 = 0$

$(u - 1/2a)^2 + v^2 = (1/2a)^2$ which is a circle with centre $(1/2a, 0)$ and radius $r = 1/2a$



Thus I conclude that the straight line $x = a$ in the z-plane is transformed into the circle with center $(1/2a, 0)$ and radius $r = 1/2a$ in the w-plane for the function $w = 1/z$.

Case-ii let us take $y = b$ then

$u = \dfrac{x}{x^2+b^2}$ $\qquad v = \dfrac{-b}{x^2+b^2}$ $\Rightarrow x^2 + b^2 = -b/v$

$\Rightarrow x^2 = \dfrac{-b}{v} - b^2$

$\Rightarrow x^2 = \dfrac{-b - b^2 v}{v}$

$u^2 = \dfrac{x^2}{(x^2+b^2)^2}$

$u^2 = \dfrac{\dfrac{-b-b^2v}{v}}{\cancel{(x^2+b^2)^2}\,(-b/v)^2} = \dfrac{-\cancel{b}(1+bv)}{\cancel{v}} \cdot \dfrac{v^{\cancel{2}}}{b^{\cancel{2}}}$

$\cancel{u^2 =} \; -\cancel{(1+bv)}$

$u^2 = \dfrac{-v - bv^2}{b} = \dfrac{-v}{b} - v^2$

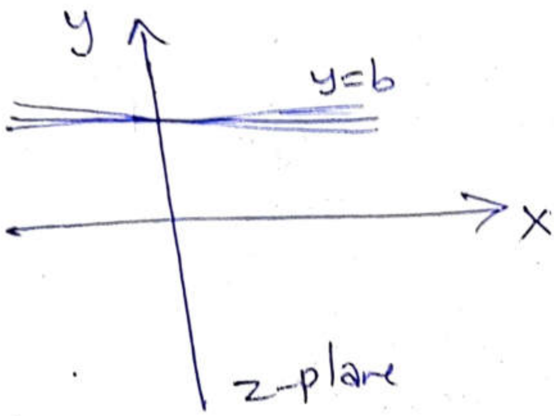
$u^2 + v^2 + v/b = 0$

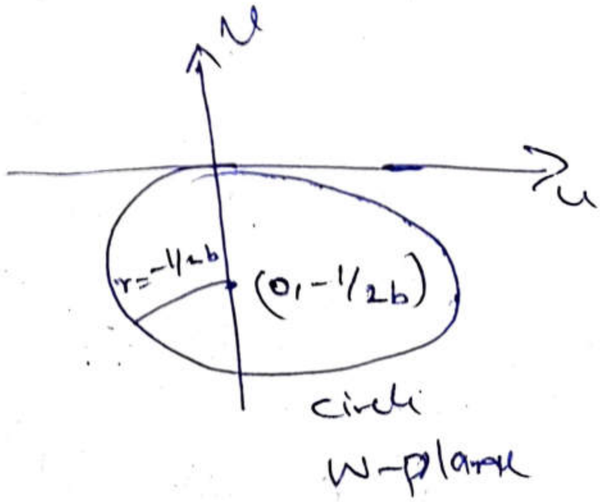
P-6

$\Rightarrow u^2 + \left[v^2 + 2(1/2b)v + \left(\frac{1}{2b}\right)^2\right] - (1/2b)^2 = 0$

$\Rightarrow u^2 + \left(v + \frac{1}{2b}\right)^2 = (1/2b)^2 \Rightarrow u^2 + \left(v - \left(\frac{-1}{2b}\right)\right)^2 = \left(\frac{1}{2b}\right)^2$

Which is a circle with centre $\left(0, \frac{-1}{2b}\right)$ and radius $r = 1/2b$.





Thus I conclude the st. line $y = b$ in the z-plane is transformed into the circle with centre $\left(0, \frac{-1}{2b}\right)$ and radius $r = -1/2b$ in the w-plane for the function $w = 1/z$.

Q) Discuss the conformal mapping for the transformation $w = z + 1/z$?

Sol:- Given $w = f(z) = z + 1/z$ { Take $z = re^{i\theta}$

$u + iv = r \cdot e^{i\theta} + \frac{1}{r \cdot e^{i\theta}} = r \cdot e^{i\theta} + \frac{1}{r} e^{-i\theta}$

$u + iv = r(\cos\theta + i\sin\theta) + 1/r(\cos\theta - i\sin\theta)$

$= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$

$\therefore u = \left(r + \frac{1}{r}\right)\cos\theta \qquad v = \left(r - \frac{1}{r}\right)\sin\theta$

P-7

Case (i)

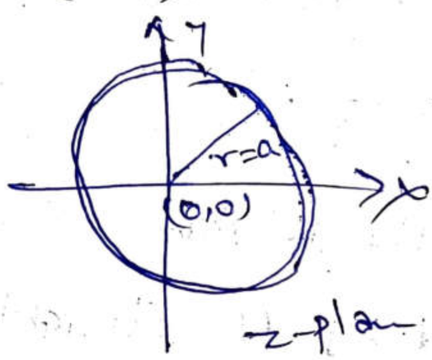
$$\frac{u}{r+\frac{1}{r}} = \cos\theta \qquad \frac{v}{r-\frac{1}{r}} = \sin\theta$$

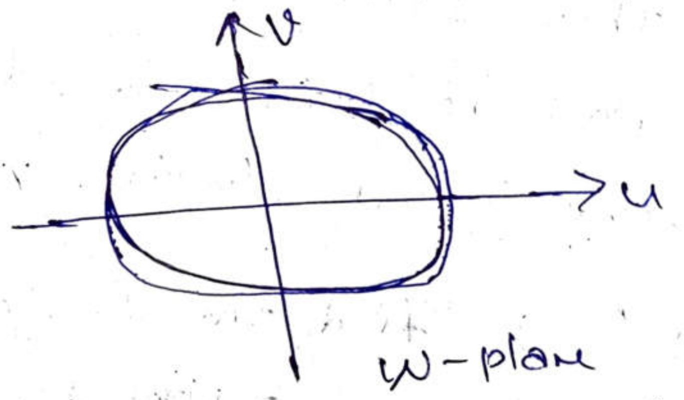
W.K.T $\cos^2\theta + \sin^2\theta = 1$

$$\left(\frac{u}{r+\frac{1}{r}}\right)^2 + \left(\frac{v}{r-\frac{1}{r}}\right)^2 = 1$$

Take 'r' = a (constant) then,

$$\frac{u^2}{\left(a+\frac{1}{a}\right)^2} + \frac{v^2}{\left(a-\frac{1}{a}\right)^2} = 1$$ which is an ellipse.





Thus I conclude that the circle $r=a$ in the z-plane is transformed into an ellipse with x-intercept $\left(a+\frac{1}{a}\right)$ and y-intercept $\left(a-\frac{1}{a}\right)$ in the w-plane for the transformation

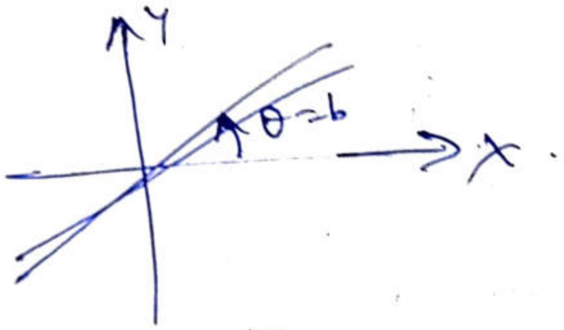
Case (ii):- $\frac{u}{\cos\theta} = r+\frac{1}{r}$, $\frac{v}{\sin\theta} = r-\frac{1}{r}$

W.K.T $\left(r+\frac{1}{r}\right)^2 - \left(r-\frac{1}{r}\right)^2 = 4$

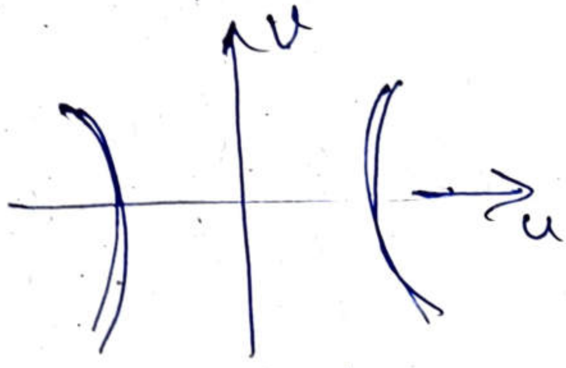
$$\frac{u^2}{\cos^2\theta} - \frac{u^2}{\sin^2\theta} = 4 \Rightarrow \frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1$$

Take $\theta = b$; $\frac{u^2}{4\cos^2 b} - \frac{v^2}{4\sin^2 b} = 1$

$$\Rightarrow \frac{u^2}{(2\cosh b)^2} - \frac{v^2}{(2\sinh b)^2} = 1$$ which is a hyperbola.



z-plane



W-plane

Thus I conclude that the Angle $\theta = b$ in the z-plane is transformed into a hyperbola in the w-plane for the function $w = z + \frac{1}{z}$.

---

pb:-1 ⊢ Find the Image in the w-plane of the region of the z-plane bounded by the straight line $x=1$, $y=1$, $x+y=1$ under the transformation $w = z^2$.

Sol:- Given $w = f(z) = z^2 \Rightarrow u+iv = (x+iy)^2$

$u+iv = x^2 - y^2 + i\,2xy$

$\therefore u = x^2 - y^2$; $v = 2xy$

$\therefore$ Given $x=1$

$v = 2y \Rightarrow y = v/2$

$u = 1 - y^2$

$u = 1 - (v/2)^2 \Rightarrow 4u = 4 - v^2 \Rightarrow v^2 = 4 - 4u$

$v^2 = -4(u-1)$

Which is a parabola with focus $(1,0)$ in the Negative direction.

P-9

$\therefore$ Given $y = 1$

$v = 2x \Rightarrow x = v/2$

$u = x^2 - 1$

$u = (v/2)^2 - 1 \Rightarrow u = \frac{v^2 - 4}{4} \Rightarrow 4u = v^2 - 4$

$\Rightarrow v^2 = 4u - 4 \Rightarrow v^2 = \cancel{4(u-1)}\ u(u-(-1))$

$\therefore v^2 = u[u - (-1)]$ which is a parabola with focus $(-1, 0)$ in the positive direction.

Given $x + y = 1 \Rightarrow y = 1 - x$

$u = x^2 - y^2$ $\qquad$ $v = 2xy$

$u = x^2 - (1-x)^2$, $v = 2x(1-x)$

$u = \cancel{x^2} - 1 - \cancel{x^2} + 2x$ $\qquad$ $v = \cancel{2}\left(\frac{u+1}{\cancel{2}}\right)\left(1 - \frac{u+1}{2}\right)$

$u + 1 = 2x$

$\Rightarrow x = \frac{u+1}{2}$

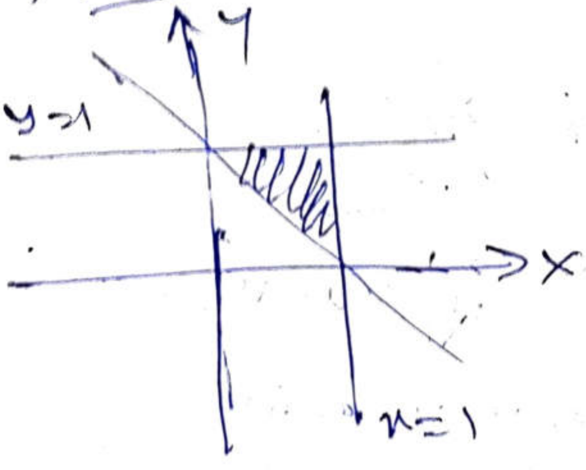
$\Rightarrow v = (u+1)\left(\frac{1-u}{2}\right) \Rightarrow 2v = 1 - u^2$

$\Rightarrow u^2 = 1 - 2v$

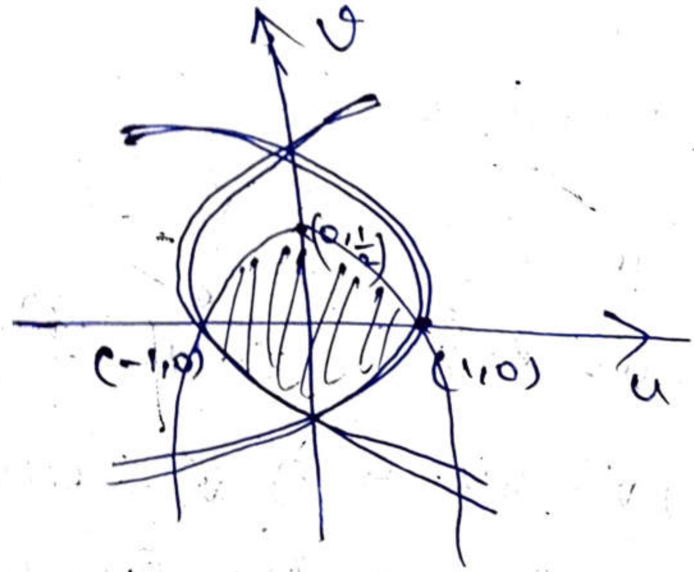
$\Rightarrow u^2 = -2\left(v - \frac{1}{2}\right)$ $\qquad$ $\therefore \left[x^2 = -4a(y - y_1)\right]$

which is a parabola parallel to x-axis with focus $(0, 1/2)$ in the negative direction

which is $\parallel$el to x-axis a parabola focus $(0, y_1)$



z-plane



W-plane

Thus I conclude that the region bounded by the straight line $x=1$, $y=1$ and $x+y=1$ ($\because$ traingle) in the z-plane is transformed as the region bounded by the parabolas $v^2=-4(u-1)$; $v^2=4(u-(-1))$ and $u^2=-2(v-1/2)$ in the W-plane for the transformation $w=z^2$.

Pb-2:- Find the Image of the Rectangular region in the z-plane bounded by the lines $x=0$, $y=0$, $x=2$ and $y=1$ under the transformation $w=(1+2i)z+(1+i)$?

Sol:- Given $w=(1+2i)z+(1+i)$

$u+iv=(1+2i)(x+iy)+(1+i)$

$u+iv = x+iy+2ix-2y+1+i$

$u+iv = x-2y+1+i(y+2x+1)$

$\therefore u = x-2y+1$; $v = 2x+y+1$

P-11

Take $x=0$, $u=-2y+1$ $\quad v=y+1$

$y=v-1$

$u=-2(v-1)+1$

$u=-2v+2+1 = -2v+3$

$\boxed{2v+u=3} \Rightarrow \boxed{\frac{u}{3}+\frac{v}{(3/2)}=1} \rightarrow Eq(1)$

It is a straight line

$\frac{x}{a}+\frac{y}{b}=1$

with x-intercept (3) and

y-intercept (3/2).

Take $y=0$

$v=2x+1$

$u=x+1$

$\Rightarrow x=u-1$

$\Rightarrow v=2(u-1)+1$

$\Rightarrow 2u-v=1$

$\Rightarrow \boxed{\frac{u}{(1/2)}+\frac{v}{(-1)}=1} \rightarrow$ It is a straight line

$\rightarrow Eq(2)$

Take $x=2$

$v=2x+y+1$

$u=x-2y+1$

$v=2(2)+y+1$

$\Rightarrow u=2-2y+1$

$v=5+y$

$u=3-2y$

$y=v-5$

$u=3-2(v-5)$

$u=3-2v+10$

$u+2v=13 \Rightarrow \boxed{\frac{u}{13}+\frac{v}{(13/2)}=1}$

It is a st line

Eq(3)

[Page-1~]

Take $y=1$

$u = x-2+1 \Rightarrow u = x-1$

$\Rightarrow x = u+1$

$v = 2x+1+1$

$v = 2x+2$

$\Rightarrow v = 2(u+1)+2$

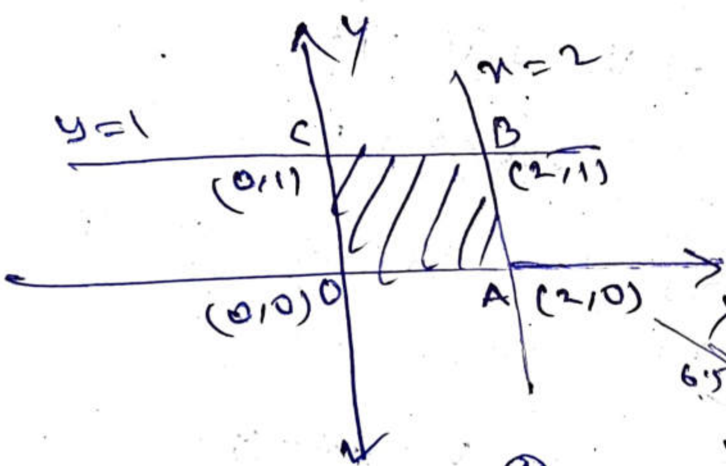
$v = 2u+4$

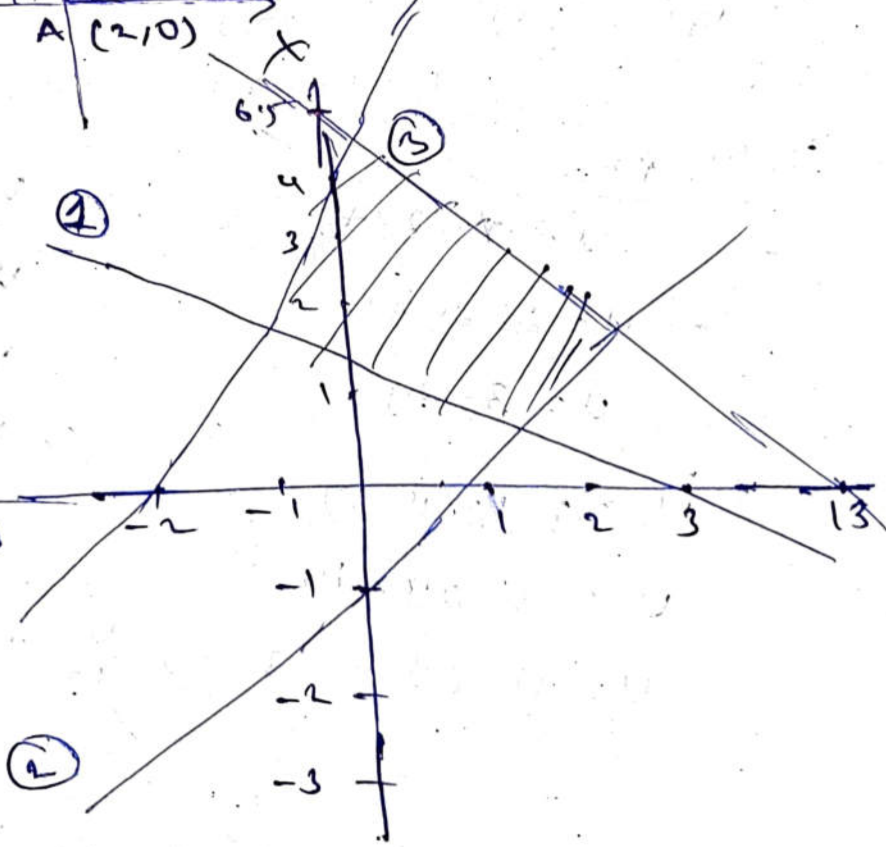
$2u - v = -4$

$\frac{u}{(-4/2)} + \frac{v}{(4)} = 1 \Rightarrow \boxed{\frac{u}{(-2)} + \frac{v}{4} = 1} \rightarrow$ eq(4)

It is a st. line

$x=0$, $y=0$, $x=2$, & $y=1$





Thus I conclude — ____

P-13

Conformal Mapping

Def :- The Mapping $w = f(z)$ is Analytic everywhere and $f'(z) \neq 0$ the function is said to be conformal.

For Ex : Take $w = f(z) = z^2 - 2z - 3$

(A) $f(z)$ is conformal at $z = 1$

(B) $f(z)$ is not conformal at $z = 1$

(C) $f(z)$ is conformal at $z = 0$

(D) $f(z)$ is not conformal at $z = 0$

$w = f(z) = z^2 - 2z - 3$

$f'(z) = 2z - 2$ $\qquad f'(z) = 0$

$\Rightarrow 2z - 2 = 0$

$\Rightarrow z = 1$

$f'(1) = 2(1) - 2 = 0$

$\boxed{\therefore f(z) \text{ is not conformal at } z = 1}$

CAT-1

15 MCQ's → 6 - Module (1) → 2 Diff, 2 Med, 2 Easy.

6 - Module (2) → " " "

3 - Module (3) → 1, 1, 1

P-14 :- 2nd part of Syllabus

Bilinear Transformation

let $w = f(z) = \dfrac{az+b}{cz+d}$ where $ad - bc \neq 0$

This is known as Bilinear Transformation.

NOTE: Bilinear Transformation is also a conformal transformation but not vice-versa.

$$w(cz+d) = az+b$$
$$wcz + wd = az+b$$
$$wcz - az = -wd + b$$

$$z(wc-a) = -wd+b$$
$$\boxed{z = \frac{-wd+b}{wc-a}}$$

which is called as inverse bilinear transformation.

Defition :-

If the image of point z under the transformation $w = f(z) = \cancel{\dfrac{az+b}{cz+d}}$ is itself, then the point is known as invariant point (or) fixed point.

for eg :- $w = \dfrac{3z-5}{z+1}$ put $w = z$

$$z = \frac{3z-5}{z+1} \Rightarrow z^2 + z = 3z - 5 \Rightarrow z^2 - 2z + 5 = 0$$

$$\Rightarrow z = \frac{2 \pm \sqrt{4 - 4(5)}}{2(1)} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$\therefore z = 1 + 2i$ and $1 - 2i$ are invariant points (or) fixed point

P-15

Bilinear Transformation

If z_1, z_2, z_3 are the points in the z-plane are mapped into w_1, w_2, w_3 in the w-plane, then the bilinear transformation is

$$\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

eg(1) :- Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$. Hence find it's invariant points.

sol :- Given $z_1 = 1$, $z_2 = i$, $z_3 = -1$

$w_1 = i$, $w_2 = 0$, $w_3 = -i$

The bilinear transformation is

$$\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

$$\Rightarrow \frac{(z-1)(i+1)}{(1-i)(-1-z)} = \frac{(w-i)(0+i)}{(i-0)(-i-w)} \quad \Rightarrow \cancel{\frac{zi+z-i-1}{-1-z+i+iz}} =$$

$$\Rightarrow \frac{zi+z-i-1}{-1-z+i+iz} = \boxed{\frac{wi+x}{1+wi}} \quad \frac{w-i}{-i-w}$$

$\frac{a}{b} = \frac{c}{d}$ by componendo and dividendo procedure

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}$$

P-16

$$\frac{zi+\cancel{z}-\cancel{1}-1-1-\cancel{z}+i+iz}{\cancel{zi}+z-i-\cancel{1}+\cancel{1}+z-1-\cancel{iz}} = \frac{\cancel{w}-i-i-\cancel{w}}{w-\cancel{1}+\cancel{1}+w}$$

$$\frac{2zi-2}{2z-2i} = \frac{-2i}{2w} \Rightarrow \frac{zi-1}{z-i} = \frac{-i}{w}$$

Reverse the functions

$$\left[\because w = \frac{az+b}{cz+d}\right]$$

$$\Rightarrow \frac{w}{-i} = \frac{z-i}{zi-1} \Rightarrow \boxed{w = \frac{-iz-1}{zi-1}}$$

Which is the bilinear Transformation.

To find it's invariant points

put $w=z \Rightarrow z = \dfrac{-iz-1}{zi-1}$

$$\Rightarrow z^2 i - z = -iz-1$$

$$iz^2 + (i-1)z + 1 = 0 \quad (\Rightarrow ax^2+bx+c \text{ form})$$

$$z = \frac{-(i-1) \pm \sqrt{(i-1)^2 - 4(i)}}{2i}$$

$$z = \frac{-i+1 \pm \sqrt{-1+1-2i-4i}}{2i} \Rightarrow$$

$$\Rightarrow \boxed{z = \frac{-i+1 \pm \sqrt{-6i}}{2i}}$$

Which are requried invariant points.

P-17

Ex:2:- Find the bilinear transformation that Maps the points $z_1 = 1$, $z_2 = i$ and $z_3 = -1$ into the points $w_1 = 0$, $w_2 = 1$ and $w_3 = \infty$. Also find the fixed points of the bilinear transformation.

Sol:- Given $z_1 = 1$, $z_2 = i$, $z_3 = -1$

$w_1 = 0$, $w_2 = 1$, $w_3 = \infty \Rightarrow w_3 = 1/w_1$ (take)

$\Rightarrow w_3 = 1/w_3 = 1/\infty = 0$.

$$\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)(w_2-1/w_3')}{(w_1-w_2)(1/w_3'-w)}$$

$$\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)\left(\frac{w_2 w_3'-1}{w_3'}\right)}{(w_1-w_2)\left(\frac{1-ww_3'}{w_3'}\right)}$$

$$\frac{(z-1)(i+1)}{(1-i)(-1-z)} = \frac{(w-0)(0-1)}{(0-1)(1-0)}$$

$$\frac{1+i}{1-i}\left(\frac{z-1}{-z-1}\right) = w \Rightarrow w = \left(\frac{z-1}{z+1}\right)\left(\frac{1+i}{-1+i}\right)$$

$$\left(\frac{i+1}{i-1}\right)\left(\frac{i+1}{i+1}\right)$$

Rationalization

$$\Rightarrow w = \frac{z-1}{z+1}\left(\frac{2i}{-2}\right) \Rightarrow \boxed{w = \frac{-iz+i}{z+1}}$$

Which is required Bilinear Transformation.

To $\boxed{\text{2nd part is in P.T.O}}$

$$\Rightarrow \frac{(z\bar{z}_1 - 1)(z_2 - z_3)}{(1 - z_2\bar{z}_1)(z_3 - z)} = \frac{(w - w_1)(w_2\bar{w}_3 - 1)}{(w_1 - w_2)(1 - w\bar{w}_3)}$$

$$\Rightarrow \frac{(0-1)(i-0)}{(1-0)(0-z)} = \frac{(w-0)(0-1)}{(0+i)(1-0)}$$

$$\Rightarrow \frac{-i}{-z} = \frac{-w}{i} \Rightarrow \frac{i^2}{z} = -w \Rightarrow \frac{i^2}{z} = -w$$

$$\Rightarrow \boxed{w = 1/z}$$

$$\boxed{w = \frac{0z+1}{z+0}} \quad \frac{az+b}{cz+d}$$

Which is the requried B.L.T.

(ii) put $w = z \Rightarrow z = 1/z \Rightarrow z^2 = 1 \Rightarrow z = \pm 1$

are fixed points of B.L.T.

(iii) $f'(z) = -1/z^2 \Rightarrow -1/z^2 = 0 \Rightarrow \boxed{z = \infty}$

$\therefore$ The function $f(z) = 1/z$ is Conformal in ~~all~~ everywhere expect at $z = \infty$.

Ex 4 :- Show that B.L.T $w = \dfrac{z-i}{1-iz}$ $\boxed{P-20}$ maps the interior of the circle $|z|=1$ onto the lower half of the w-plane.

Sol :- Given $w = \dfrac{z-i}{1-iz}$ $\Rightarrow w - iwz = z - i$

$\Rightarrow w + i = z + iwz$

$= z(1+iw)$

$\Rightarrow z = \dfrac{w+i}{1+iw}$, $\bar{z} = \dfrac{\bar{w}-i}{1-i\bar{w}}$

$|z| = 1 \Rightarrow z\bar{z} = 1$

$\dfrac{w+i}{1+iw} \cdot \dfrac{\bar{w}-i}{1-i\bar{w}} = 1$

$\Rightarrow (w+i)(\bar{w}-i) = (1+iw)$

$\Rightarrow \cancel{w\bar{w}} - iw + i\bar{w} + \cancel{1} = \cancel{1} - i\bar{w} + iw + \cancel{w\bar{w}}$

$2iw - 2i\bar{w} = 0 \Rightarrow 2i[w - \bar{w}] = 0$

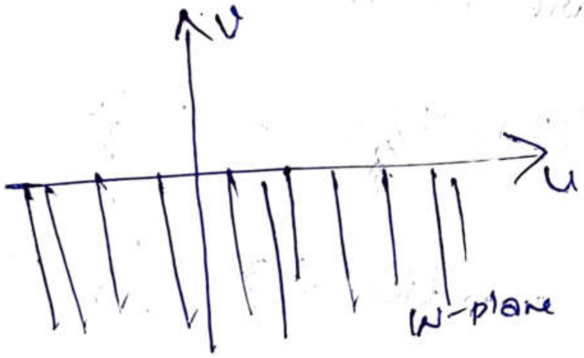
$\Rightarrow w - \bar{w} = 0$ ~~$\Rightarrow u - iv$~~ $= 0$

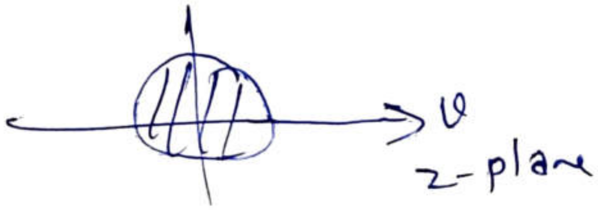
$\Rightarrow u + iv - (u - iv) = 0 \Rightarrow 2iv = 0$

$\Rightarrow (v = 0)$ which is u-axis in w-plane

interior of the circle $|z| = 1 \Rightarrow |z| < 1$

then finally we have $v < 0$





Module-3 7/8/2020

Taylor's Series

$$f(z) = f(a) + f'(a)(z-a) + f''(a)(z-a)^2 + \cdots$$

where $f(z)$ is an analytic function defined in a circle 'C' with centre $z=a$.

Eg:-(1) Find the Taylor's expansion of $f(z) = \frac{1}{(z+1)^2}$ about the point $z=-i$

Sol:- $f(z) = \frac{1}{(z+1)^2}$ $\quad f(-i) = \frac{1}{(-i+1)^2} = \frac{1}{i^2+1-2i}$

$$= \frac{-1}{2i} \cdot \left(\frac{i}{i}\right) = \frac{i}{2}$$

$f'(z) = \frac{-2}{(z+1)^3}$ $\quad f'(-i) = \frac{-2}{(-i+1)^3} = \frac{-2}{(-i)^3+1^3+3(-i)^2\cdot 1 + 3(i)}$

$$= \frac{-2}{i+1-3-3i} = \frac{-2}{-2-2i} = \frac{1}{1+i}$$

$f''(z) = \frac{(-2)(-3)}{(z+1)^4}$ $\quad f''(-i) = \frac{6}{(-i+1)^4} = \frac{6}{[(-i+1)^2]^2}$

$$\Rightarrow \cancel{6}$$

$$= \cancel{6}^{3}\left(\frac{i}{\cancel{2}}\right)\left(\frac{i}{2}\right) = \frac{-3}{2}$$

By Taylor's expansion we have

$$f(z) = f(a) + (z-a)f'(a) + (z-a)^2 \frac{f''(a)}{2!} + \cdots$$

$$\frac{1}{(z+1)^2} = \frac{i}{2} + (z+1)\frac{1}{1+i} + (z+1)^2\left(\frac{-3/2}{2}\right) + \cdots$$

$$\frac{1}{(z+1)^2} = \frac{i}{2} + \frac{z+i}{1+i} - \frac{3(z+i)^2}{4} + \frac{\lfloor P-2 \rfloor}{}$$

Laurent Series

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \cdots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + a_{-3}(z-a)^{-3} + \cdots$$

where $\boxed{a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} \, dz}$

eg (1) :- Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in the region $1 < |z| < 2$ by Laurent's expansion.

sol :- $f(z) = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$$= \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$$

$\Rightarrow 1 = A(z-2) + B(z-1)$

put "$z=1$" $\Rightarrow 1 = -A \Rightarrow \boxed{A = -1}$

put $z=2 \Rightarrow 1 = B \Rightarrow \boxed{B = 1}$

$\therefore f(z) = \frac{-1}{(z-1)} + \frac{1}{z-2}$

Given $1 < |z| < 2 \Rightarrow 1 < |z| \Rightarrow \frac{1}{|z|} < 1$

$|z| < 2 \Rightarrow \frac{|z|}{2} < 1$

$$f(z) = \frac{-1}{z(1-1/z)} + \frac{1}{2(z/2-1)} = \frac{-1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$\cancel{\because (1-x)^{n}}$

$$\left[\because (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots\right]$$

$$= \frac{-1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) - \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right)$$

$$\boxed{f(z) = \frac{-1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots - \frac{1}{2} - 3/4 - \frac{z^2}{8} - \cdots}$$

$L \rightarrow T$ but $T \nrightarrow L$

* Every Laurent Series are Taylor series but every Taylor series are not Laurent series.

* $f(z) = \underbrace{\underbrace{1 + z^2 + z^3 + \cdots}_{T} + 0 \cdot z^{-1} + 0 \cdot z^{-2}}_{L}$

* Laurent series is Applicable for both +ve & -ve but Taylor series is Applicable for only +ve.

Module-3 (Continuation)

Eg (2) :- Find the Laurant's series expansion of

$$f(z) = \frac{(z-1)(z+2)}{(z+1)(z+4)}$$ in the region

(i) $|z|<1$ (ii) $1<|z|<4$ (iii) $|z|>4$

Sol:- $$f(z) = \frac{(z-1)(z+2)}{(z+1)(z+4)} = \frac{A}{z+1} + \frac{B}{z+4}$$

$$\Rightarrow (z-1)(z+2) = A(z+4) + B(z+1)$$

put $z=-1$; $-2 = 3A \Rightarrow \boxed{A = -2/3}$.

put $z=-4$; $(-5)(-2) = -3B \Rightarrow \boxed{B = -10/3}$

$$f(z) = \frac{-2/3}{z+1} + \frac{-10/3}{z+4}$$

(i) $|z|<1 \Rightarrow f(z) = \frac{-2/3}{1+z} - \frac{10/3}{4(1+z/4)}$

$$= -2/3 (1+z)^{-1} - \frac{10}{12}(1+z/4)^{-1}$$

$$f(z) = \frac{-2}{3}(1-z+z^2-z^3+\ldots) - 5/6\left(1-\frac{z}{4}+\left(\frac{z}{4}\right)^2-\left(z/4\right)^3+\ldots\right)$$

(ii) $1<|z|<4$

$\Rightarrow 1<|z| \Rightarrow 1/|z| < 1$; $|z|<4 \Rightarrow \frac{|z|}{4} < 1$

$$f(z) = \frac{-2/3}{z(1/z+1)} - \frac{10/3}{4(z/4+1)} = \frac{-2}{3z}(1+1/z)^{-1} - \frac{5}{6}(1+z/4)^{-1}$$

$$= \frac{-2}{3z}\left[1-1/z+1/z^2-1/z^3+\ldots\right] - \frac{5}{6}\left[1-\frac{z}{4}+\frac{z^2}{16}-\frac{z^3}{64}+\ldots\right]$$

$$= \frac{-2}{3}\left[\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots \right] - \frac{5}{6}\left[1 - \frac{z}{4} + \frac{z^2}{16} - \frac{z^3}{64} + \cdots \right]$$

(iii) $|z| > 4 \Rightarrow 1 > 4/z \Rightarrow \frac{4}{|z|} < 1$

$$f(z) = \frac{-2/3}{1+z} - \frac{10/3}{z(1+4/z)} = \frac{-2}{3}(1+z)^{-1} - \frac{10}{3z}\left(1+\frac{4}{z}\right)^{-1}$$

$$= \frac{-2}{3}\left(1 - z + z^2 - z^3 + \cdots\right) - \frac{10}{3z}\left(1 - \frac{4}{z} + \frac{16}{z^2} - \frac{64}{z^3} + \cdots\right)$$

$$= \frac{-2}{3}\left(1 - z + z^2 - z^3 + \cdots\right) - \frac{10}{3}\left(\frac{1}{z} - \frac{4}{z^2} + \frac{16}{z^3} - \frac{64}{z^4} + \cdots\right)$$

Eg(3) :- Find the Laurant's series expansion of $f(z) = \frac{7z^2 + 9z - 18}{z^3 - 9z}$ in the region(

(i) $0 < |z-3| < 3$ (ii) $3 < |z-3| < 6$ (iii) $|z-3| > 6$.

Sol:- $$f(z) = \frac{7z^2 + 9z - 18}{z(z^2 - 9)} = \frac{7z^2 + 9z - 18}{z(z-3)(z+3)}$$

$$= \frac{A}{z} + \frac{B}{z-3} + \frac{C}{z+3}$$

$$7z^2 + 9z - 18 = A(z-3)(z+3) + B(z)(z+3) + C(z)(z-3)$$

put $z = 0$, $-18 = A(-9) \Rightarrow \boxed{A = 2}$

put $z = 3$, $63 + 27 - 18 = B\,3(6) \Rightarrow B = 72/18 = 4$

$\Rightarrow \boxed{B = 4}$

put $z = -3$, $63 - 27 - 18 = C(-3)(-6) \Rightarrow \boxed{C = 1}$

$$f(z) = \frac{2}{z} + \frac{4}{z-3} + \frac{1}{z+3}$$

put $z - 3 = t \Rightarrow z = t + 3$

$f(z) = \frac{2}{t+3} + \frac{4}{t} + 1/_{t+6}$

(ii) $0 < |z-3| < 3$

$\Rightarrow 0 < |t| < 3 \Rightarrow |t| < 3 \Rightarrow \frac{|t|}{3} < 1$

$f(z) = \frac{2}{3\left(\frac{t}{3}+1\right)} + \frac{4}{t} + \frac{1}{6\left(\frac{t}{6}+1\right)}$

$= \frac{2}{3}(1+t/3)^{-1} + \frac{4}{t} + 1/6\,(1+t/6)^{-1}$

$= \frac{2}{3}\left(1-\frac{t}{3}+\frac{t^2}{9} - --\right) + \frac{4}{t} + \frac{1}{6}\left(1-\frac{t}{6}+\frac{t^2}{36} - --\right)$

$= \frac{2}{3} - \frac{2t}{9} + \frac{2t^2}{27} - --- + \frac{4}{t} + \frac{1}{6} - \frac{t}{36} + \frac{t^2}{216} - ---$

$= \frac{5}{6} - \frac{1}{4}t + \frac{17}{216}t^2 - --- + \frac{4}{t}$

$= \frac{5}{6} - \frac{1}{4}(z-3) + \frac{17}{}$

Example (1):- Find all the singular poles (points) of $f(z) = \frac{1}{z^2+1}$.

Sol:- The poles are $z^2+1 = 0 \Rightarrow z^2 = -1 \Rightarrow z = \pm i$

$\therefore z = i, -i$ are simple poles.

Eg (2):- Find all the poles of $f(z) = \frac{1}{z^4+a^4}$.

Sol:- The poles are $z^4 + a^4 = 0 \Rightarrow z^4 = -a^4 = (-1)a^4$

$\Rightarrow z^4 = e^{i(2n+1)\pi}\, a^4$ $\quad \because \cos\theta + i\sin\theta = e^{i\theta}$

$\cos\pi + i\sin\pi = e^{i\pi}$

$(-1 = e^{i\pi})$

$\left(\cancel{\frac{z}{a}}\right)^{4}$ $\quad z = a e^{i(2n+1)\frac{\pi}{4}}$ $\quad \rightarrow \boxed{-1 = e^{i(2n+1)\pi}}$

(where $n = 0, 1, 2, 3$)

when, $n=0$; $z = ae^{i\pi/4}$

when $n=1$; $z = ae^{i3\pi/4}$ | when $n=2$; $z = ae^{i5\pi/4}$

when $n=3$; $z = ae^{i7\pi/4}$

$\underbrace{\qquad\qquad\qquad\qquad}$

poles of given function $f(z) = \dfrac{1}{z^4+a^4}$

Eg) Find the nature and location of singularity of the function $f(z) = \dfrac{e^{2z}}{(z-1)^4}$.

Sol: $\therefore$ The singular points are $(z-1)^4 = 0 \Rightarrow z=1$

$\therefore z=1$ is an isolated singular point with order '4'.

$$e^{2z}/(z-1)^4 = \frac{e^{2(z-1)+2}}{(z-1)^4} = \frac{e^{2(z-1)} \cdot e^2}{(z-1)^4}$$

$$= \frac{e^2 \cdot e^{2(z-1)}}{(z-1)^4} = \frac{e^2}{(z-1)^4}\left[1 + 2(z-1) + \frac{2^2(z-1)^2}{2!} + \frac{2^3(z-1)^3}{3!} + \cdots\right]$$

$$\left[\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]$$

$$= e^2\left(\frac{1}{(z-1)^4} + \frac{2}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{\frac{4}{3}}{(z-1)} + \frac{2}{3} + \cdots\right.$$

Here $f(z)$ has finite number of negative power terms.

$\therefore z=1$ is an isolated singular point (Pole) of order '4'.

Eg) If $f(z) = z^3\left[e^{-1/z^2} + \sin(1/z) + \cos(1/z)\right]$ then find the nature of singular points.

Sol.

$$f(z) = z^3\left[1 - \frac{1}{z^2} + \frac{\left(\frac{1}{z^2}\right)^2}{2!} - \frac{\left(1/z^2\right)^3}{3!} + \cdots + \frac{1}{z} - \frac{\left(\frac{1}{z^3}\right)}{3!} + \frac{\left(1/z^5\right)}{5!} + \cdots \right.$$

$$\left. \cdots + 1 - \frac{1/z^2}{2!} + \frac{1/z^4}{4!} - \cdots \right]$$

$$= z^3\left[2 + \frac{1}{z} - \frac{3}{2z^2} - \frac{1}{6z^3} + \frac{13}{24z^4} + \frac{1}{120z^5} + \cdots \right]$$

$$= 2z^3 + z^2 - \frac{3z}{2} - \frac{1}{6} + \frac{13}{24z} + \frac{1}{120z^2} + \cdots$$

$\therefore z=0$ is an essential singular point.

Procedure

1. If $z=a$ is a simple pole the $r = \operatorname{Res} f(z) = \lim\limits_{z\to a}(z-a)\cdot f(z)$

2. If $z=a$ is a pole of order 'm' then

$$r = \operatorname*{Res}_{z=a} f(z) = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

3. If $f(z) = \frac{\phi(z)}{\psi(z)}$ then $r = \underset{z=a}{\text{Res}}\, f(z) = \frac{\phi(a)}{\psi'(a)}$ if $\psi'(a) \neq 0$.

Eg(1) :- Determine the poles of fn. $f(z) = \frac{z^2}{(z-1)^2(z^2+1)}$ and residue at each pole.

Sol. Here $f(z) = \frac{z^2}{(z-1)^2(z^2+1)}$

The poles are $(z-1)^2(z^2+1) = 0 \Rightarrow (z-1)^2 = 0$ (or) $(z^2+1) = 0$

$z = 1$; $z^2 = -1 \Rightarrow z = \pm i$.

$\therefore z = i, -i$ are sample poles and $z=1$ is a pole of order '2'.

$$r_1 = \underset{z=i}{\text{Res}}(f(z)) = \lim_{z\to i} \cancel{(z-i)} \frac{z^2}{(z-1)^2 \cancel{(z-i)} (z+i)}$$

$$= \frac{i^2}{(i-1)^2 (i+i)}$$

$$= \frac{i^2}{(\cancel{i^2}+\cancel{1}-2i)(2i)} = \frac{-1}{4i} = -1/4$$

$$r_2 = \underset{z=-i}{\text{Res}}\, f(z) = \lim_{z\to -i} (z+i) \frac{z^2}{(z-1)^2(z-i)(z+i)}$$

$$= \frac{(-i)^2}{(-i-1)^2(-i-i)} = \frac{-1}{(\cancel{i^2}+\cancel{1}+2i)(-2i)}$$

$$= \frac{-1}{-4i^2} = -1/4$$

$$r_3 = \underset{z=1}{\operatorname{Res}} f(z) = \frac{1}{(m-1)!} \lim_{z \to a} \left[\frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z) \right]$$

$$= \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d}{dz} \left[\cancel{(z-1)^2} \frac{z^2}{\cancel{(z-1)^2}(z^2+1)} \right]$$

$$= \lim_{z \to 1} \frac{(z^2+1)2z - z^2(2z)}{(z^2+1)^2} = 1/2$$

$$\frac{2(2)-1(2)}{4}$$

<u>Eg(2)</u> :- Find the residue of function $f(z) = \cot z$.

Sol:- $f(z) = \cot z = \frac{\cos z}{\sin z}$. The poles are $\sin z = 0$

$$= \frac{\phi(z)}{\psi(z)}$$

$\sin z = \sin n\pi$

$z = n\pi$ when $n = 0, 1, 2.$

are simple poles

$$\phi(a) = \phi(n\pi) = \cos n\pi = (-1)^n$$

$$\psi'(z) = \cos z \Rightarrow \psi'(a) = \psi'(n\pi) = \cos n\pi = (-1)^n$$

$$\therefore r = \underset{z=n\pi}{\operatorname{Res}} f(z) = \frac{\phi(n\pi)}{\psi'(n\pi)} = \frac{(-1)^n}{(-1)^n} = 1.$$

<u>Eg(4)</u> :- Find all the poles and its residues of

$$f(z) = \frac{1}{z^4 + a^4} ?$$

Sol. The poles are $z^4 + a^4 = 0 \Rightarrow z^4 = -a^4$

$$\Rightarrow z^4 = e^{i(2n+1)\pi} a^4$$

$z = a e^{i(2n+1)\pi/4}$ where $n = 0, 1, 2, 3$

$\therefore$ When $n = 0$, $z = ae^{i\pi/4}$, when $n = 1$, $z = ae^{i\left(\frac{3\pi}{4}\right)}$

when $n = 2$, $z = ae^{i\left(\frac{5\pi}{4}\right)}$, when $n = 3$, $z = a \cdot e^{i\left(\frac{7\pi}{4}\right)}$

$$r_1 = \underset{z=ae^{i\pi/4}}{\operatorname{Res}} f(z) = \lim_{z \to ae^{i\pi/4}} (z - ae^{i\pi/4}) \frac{1}{z^4 + a^4}$$

$\left[\frac{0}{0} = \text{indetermined form} \right]$

$(\because$ L-hospital rule$)$

$$= \lim_{z \to ae^{i\pi/4}} \frac{1}{4z^3} = \frac{1}{4a^3 e^{i3\pi/4}} = \frac{1}{4a^3}\left(e^{-i\frac{3\pi}{4}}\right)$$

$$= \frac{1}{4a^3}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right] = \frac{1}{4a^3}\left[\frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right]$$

$$= \frac{-1}{4\sqrt{2}\,a^3}\left[1+i\right]$$

$$r_2 = \operatorname*{Res}_{z=ae^{i3\pi/4}} f(z) = \lim_{z \to ae^{i\frac{3\pi}{4}}} \frac{1}{4z^3} = \frac{1}{4a^3} e^{-i\left(\frac{9\pi}{4}\right)}$$

$$= \frac{1}{4a^3}\left[\cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right] =$$

$$r_3 = \operatorname*{Res}_{z=ae^{i5\pi/4}} f(z) = \lim_{z \to ae^{i\frac{5\pi}{4}}} \frac{1}{4z^3} = \frac{1}{4a^3}\left(e^{-i\left(\frac{15\pi}{4}\right)}\right)$$

$$= \frac{1}{4a^3}\left(\cos\frac{15\pi}{4} - i\sin\frac{15\pi}{4}\right) =$$

$$r_4 = \operatorname*{Res}_{z=ae^{i7\pi/4}} f(z) = \lim_{z \to ae^{i7\pi/4}} \frac{1}{4z^3} = \frac{1}{4a^3}\left(e^{-i\left(\frac{21\pi}{4}\right)}\right)$$

$$= \frac{1}{4a^3}\left(\cos\frac{21\pi}{4} - i\sin\frac{21\pi}{4}\right) = \text{---} .$$

$$\oint_C f(z)\,dz$$

3rd Residue By

Module-4 (Complex Integrals)

Cauchy's Residue Theorem (C.R-T)

If $f(z)$ is an analytic function with in and on a closed 'c' at a finite Number of singularities (poles) then

~~closed int~~

$$\oint_C f(z)\cdot dz = 2\pi i \left[\text{Sum of the residue inside the curve } c\right]$$

$$= 2\pi i \sum_{j=1}^{n} r_j$$

Evaluation of Indefinite Integrals by using C.R-T

Eg(1) Evaluate $\oint_C \frac{z}{(z-1)(z-2)^2}\,dz$ where 'c' is Circle

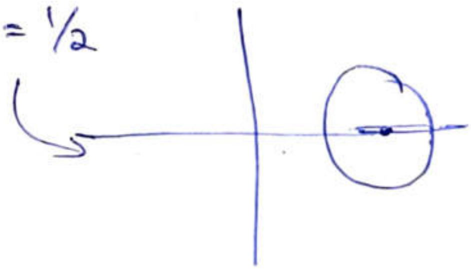
$|z-2| = 1/2$.

S/ Given circle $|z-2| = \frac{1}{2} \Rightarrow |x+iy-2| = 1/2$

$\Rightarrow |(x-2)+iy| = 1/2$

$\Rightarrow (x-2)^2 + y^2 = (1/2)^2$ which is a circle with

centre $(2,0)$ and radius 'r' $= 1/2$



The poles are $(z-1)(z-2)^2 = 0$

$z = 1$, and $z = 2$ (Pole of order-2)

$\therefore z=1$ is a simple pole which lies out the circle $|z-2| = 1/2$

$z=2$ is a pole of order (2) which lies inside the circle $|z-2| = 1/2$

$$r_1 = \underset{z=2}{\operatorname{Res}} f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$

$$= \frac{1}{(2-1)!} \lim_{z \to 2} \frac{d}{dz} \left[\cancel{(z-2)^2} \frac{z}{(z-1)\cancel{(z-2)^2}} \right]$$

$$= \lim_{z \to 2} \frac{(z-1)1 - z(1)}{(z-1)^2} = \frac{(1)(1) - 2(1)}{(2-1)^2} = -1$$

By cauchy's Residue Theorem, $\oint_C f(z) \cdot dz = 2\pi i r_1$,

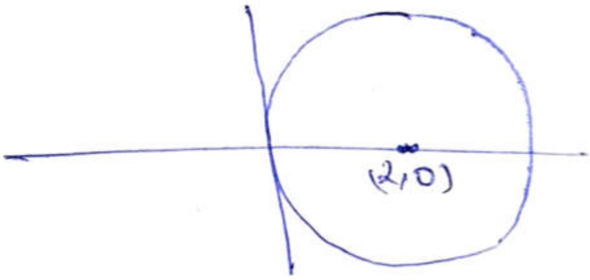
$$\oint_C \frac{z}{(z-1)(z-2)^2} dz = 2\pi i (-1) = \underline{\underline{-2\pi i}}.$$

<u>Eg(2)</u>:- Evaluate $\oint_C \frac{3z^2+2}{(z-1)(z^2+9)} dz$ where 'c' is Circle $|z-2| = 2$ by C.R.T Met

Sol. Given $|z-2| = 2 \Rightarrow |x+iy-2| = 2$

$\Rightarrow (x-2)^2 + y^2 = 2^2$ is a

circle with Centre $(2,0)$ & $r = 2$



the poles are $(z-1)(z^2+9) = 0$

$\Rightarrow z = 1, \; z = \pm 3i$

$\therefore z = 1, \; z = -3i, \; z = +3i$

$\therefore z = 1$ is a simple pole which lies inside the circle $|z-2| = 2$

$z = -3i$ and $z = 3i$ are simple poles which lies outside th circle $|z-2| = 2$.

$$r_1 = \operatorname{Res}_{z=1} f(z) = \lim_{z \to 1} \cancel{(z-1)} \frac{3z^2+2}{\cancel{(z-1)}(z^2+9)} = \frac{5}{10} = \frac{1}{2}$$

By Cauchy Residue Theorem we have

$$\oint_C f(z)\,dz = 2\pi i r_1$$

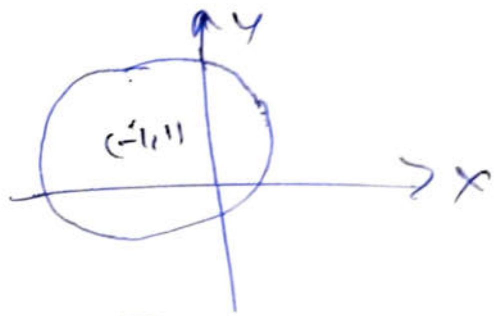
$$\oint_C \frac{3z^2+2}{(z-1)(z^2+9)}\,dz = 2\pi i\left(\tfrac{1}{2}\right) = \pi i.$$

Eg(3) :- Evaluate $\oint_C \frac{z-3}{z^2+2z+5}\,dz$ where 'c' is the circle given by (i) $|z+1-i| = 2$ (ii) $|z+1+i| = 2$

Sol: (i) Given $|z+1-i| = 2 \Rightarrow |x+iy+1-i| = 2$

$$(x+1)^2 + (y-1)^2 = 2^2$$

is a circle centre $(-1,1)$ radius $(r = 2)$



The poles are $z^2+2z+5 = 0$

$$z = \frac{-2 \pm \sqrt{4-4(5)}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$z = -1 \pm 2i \Rightarrow z = \alpha = -1+2i$

i.e $(-1,2)$ is simple pole lies inside the circle $|z+1-i| = 2$

$z = \beta = -1-2i \Rightarrow (-1,-2)$ is simple pole which lies outside the circle $|z+1-i| = 2$

$$r_1 = \operatorname*{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha}(z-\alpha)\frac{z-3}{z^2+2z+5} = \lim_{z\to\alpha}\frac{\cancel{(z-\alpha)}(z-3)}{\cancel{(z-\alpha)}(z-\beta)}$$

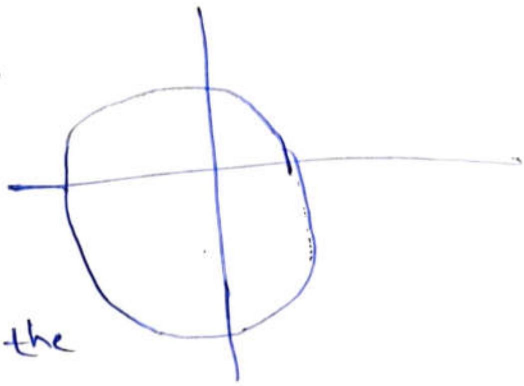
$$= \frac{\alpha-3}{\alpha-\beta} = \frac{-1+2i-3}{-1+2i-(-1-2i)} = \frac{-4+2i}{4i} = \frac{-2+i}{2i}$$

By C.R.T we have $\oint_C f(z)\,dz = 2\pi i r_1$

$$\oint_C \frac{z-3}{z^2+2z+5}\cdot dz = \cancel{2}\pi \cancel{i}\left(\frac{-2+i}{\cancel{2i}}\right) = (-2+i)\pi.$$

(ii) Given Circle is $|z+1+i| = 2 \Rightarrow |x+iy+1+i| = 2$

$\Rightarrow (x+1)^2 + (y+1)^2 = 2^2$ with



the poles are $z = \alpha = -1+2i$
i.e $(-1, 2)$

is a simple pole which lies outside the circle $|z+1+i| = 2$

$z = \beta = -1-2i$ i.e $(-1,-2)$ is a simple pole which lies inside the circle $|z+1+i| = 2$.

$$r_1 = \operatorname*{Res}_{z=\beta} f(z) = \lim_{z\to\beta} \cancel{(z-\beta)} \frac{z-3}{(z-\alpha)\cancel{(z-\beta)}} = \frac{\beta-3}{\beta-\alpha}$$

$$= \frac{-1-2i-3}{-1-2i-(-1+2i)} = \frac{-4-2i}{-4i} = \frac{2+i}{2i}$$

By C.R-T we have $\oint_C f(z)\,dz = 2\pi i r_1$,

$$\oint_C \frac{z-3}{z^2+2z+5}\,dz = 2\pi i\left(\frac{2+i}{2i}\right) = (2+i)\pi.$$

Evaluation of finite definite Integrals by C.R.T Method

The integrals of the type $\int_0^{2\pi} f(\theta)\,d\theta$, $\int_0^{\pi} f(\theta)\cdot d\theta$, $\int_{-\pi}^{\pi} f(\theta)\cdot d\theta$

this circle is unit circle $|z| = 1$

$$\Rightarrow z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta$$

$$\Rightarrow dz = iz\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$$

$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{z - z^{-1}}{2i} = \dfrac{z - \frac{1}{z}}{2i} = \dfrac{z^2 - 1}{2iz}$.

Eg(1):- Solve $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5 - 3\cos\theta} \cdot d\theta$ using contour integration.

Sol:- Let us take $z = e^{i\theta} \Rightarrow d\theta = \dfrac{dz}{iz}$

we have $\cos\theta = \dfrac{z^2+1}{2z}$ $\quad \sin\theta = \dfrac{z^2-1}{2iz}$

$$\int_0^{2\pi} \frac{\sin^2\theta}{5-3\cos\theta}\, d\theta = \oint_C \frac{\left(\frac{z^2-1}{2iz}\right)^2}{5 - 3\left(\frac{z^2+1}{2z}\right)} \cdot \frac{dz}{iz}$$

$$= \oint_C \frac{\frac{z^4+1-2z^2}{-4z^2}}{\frac{10z - 3z^2 - 3}{2z}} \left(\frac{dz}{iz}\right) = \oint_C \frac{z^4+1-2z^2}{-2iz^2(-3z^2+10z-3)} (dz)$$

$$= \frac{1}{6i} \oint_C \frac{z^4+1-2z^2}{z^2\left(z^2 - \frac{10}{3}z + 1\right)} dz$$

The poles are $z^2\left(z^2 - \frac{10}{3}z + 1\right) = 0 \Rightarrow z^2 = 0 \Rightarrow z = 0$

$z^2 - \frac{10}{3}z + 1 = 0 \Rightarrow z = \dfrac{\frac{10}{3} \pm \sqrt{\frac{100}{9} - 4}}{2} = \dfrac{\frac{10}{3} \pm \frac{8}{3}}{2}$

$\Rightarrow z = \alpha = \dfrac{\frac{10}{3} + \frac{8}{3}}{2} = \dfrac{18}{6} = 3$ is a simple pole which lies outside unit circle $|z| = 1$

$z = \beta = \dfrac{\frac{10}{3} - \frac{8}{3}}{2} = \dfrac{2}{6} = 1/3$ is a simple pole which lies inside the unit circle $|z| = 1$

$z = 0$ is a pole of order '2' which lies inside the unit circle $|z| = 1$

$$r_1 = \operatorname*{Res}_{z=\beta} f(z) = \lim_{z\to\beta} \cancel{(z-\beta)} \frac{z^4+1-2z^2}{z^2(z-\alpha)\cancel{(z-\beta)}} = \frac{\beta^4+1-2\beta^2}{\beta^2(\beta-\alpha)}$$

$$= \frac{\left(\frac{1}{3}\right)^4 + 1 - 2\left(\frac{1}{3}\right)^2}{\left(\frac{1}{3}\right)^2\left(\frac{1}{3}-3\right)} = \frac{\left(\frac{1}{81}+1-\frac{2}{9}\right)}{\frac{1}{9}\left(-\frac{8}{3}\right)} = \frac{\frac{1+81-18}{81}}{-8/27}$$

$$\therefore r_1 = -8/3$$

$$r_2 = \operatorname*{Res}_{z=0} f(z) = \frac{1}{(2-1)!} \lim_{z\to 0} \frac{d}{dz}\left[\cancel{(z-0)^2} \frac{z^4+1-2z^2}{\cancel{z^2}\left(z^2-\frac{10}{3}z+1\right)}\right]$$

$$= \lim_{z\to 0} \frac{\left(z^2-\frac{10}{3}z+1\right)(4z^3-4z)-(z^4+1-2z^2)\left(2z-\frac{10}{3}\right)}{\left(z^2-\frac{10}{3}z+1\right)^2}$$

$$r_2 = -1\left(-10/3\right) = 10/3$$

By C.R.T we have $\oint_C f(z)\cdot dz = 2\pi i\,(r_1+r_2)$

$$\oint_C \frac{z^4+1-2z^2}{z^2\left(z^2-\frac{10}{3}z+1\right)}\,dz = 2\pi i\left[\frac{-8}{3}+\frac{10}{3}\right] = \frac{4\pi i}{3}$$

$$\int_0^{2\pi} \frac{\sin\theta}{5-3\cos\theta}\,d\theta = \frac{1}{\frac{6i}{3}}\left(\frac{\overset{2}{\cancel{4}}\pi i}{3}\right) = 2\pi/9.$$

Eg(2) :- Evaluate $\int_0^{\pi} \frac{1}{5+4\cos\theta}\,d\theta$ by using calculus of Residues.

Sol. $\int_0^{\pi} \frac{1}{5+4\cos\theta}\cdot d\theta = \frac{1}{2}\int_0^{2\pi}\frac{1}{5+4\cos\theta}\cdot d\theta$

Let us take $z = e^{i\theta} \Rightarrow \frac{dz}{iz} = d\theta$, $\cos\theta = \frac{z^2+1}{2z}$

~~use~~

$$\therefore \int_0^{\pi} \frac{1}{5+4\cos\theta}\, d\theta = \frac{1}{2}\oint_C \frac{1}{5+4\left(\frac{z^2+1}{2z}\right)} \cdot \frac{dz}{iz}$$

$$= \frac{1}{2}\oint_C \frac{z}{5z+2z^2+2} \cdot \frac{dz}{iz} = \frac{1}{2i}\oint_C \frac{1}{2z^2+5z+2}\, dz$$

$$= \frac{1}{4i}\oint_C \frac{1}{z^2+\frac{5}{2}z+1} \cdot dz$$

The poles are $z^2+\frac{5}{2}z+1=0 \Rightarrow z = \dfrac{-\frac{5}{2} \pm \sqrt{\frac{25}{4}-4}}{2}$

$$z = \frac{-\frac{5}{2} \pm \frac{3}{2}}{2}$$

$z = \alpha = \dfrac{-\frac{5}{2}+\frac{3}{2}}{2} = -\frac{1}{2}$ is a simple pole which lies inside the unit circle.

$z = \beta = \dfrac{-\frac{5}{2}-\frac{3}{2}}{2} = -2$ is a simple pole which lies outside the unit circle.

$$r_1 = \operatorname*{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha} \cancel{(z-\alpha)} \frac{1}{\cancel{(z-\alpha)}(z-\beta)} = \frac{1}{\alpha-\beta} = \frac{1}{-\frac{1}{2}+2} = \frac{2}{3}$$

By Cauchy Residue Theorem we have $\oint_C f(z)\,dz = 2\pi i r_1$

$$\oint_C \frac{1}{z^2+\frac{5}{2}z+1}\, dz = 2\pi i\left(\frac{2}{3}\right) = \frac{4\pi i}{3}$$

$$\therefore \int_0^{\pi} \frac{1}{5+4\cos\theta}\, d\theta = \frac{1}{4i}\left[\frac{4\pi i}{3}\right] = \frac{\pi}{3}.$$

Eg(3):- Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} \cdot d\theta$ by interacting around the unit circle.

Sol:- let us take $z = e^{i\theta} \Rightarrow \frac{dz}{iz} = d\theta$, $\cos\theta = \frac{z^2+1}{2z}$

$$\cos 2\theta = \frac{e^{i2\theta} + e^{-i2\theta}}{2} = \frac{z^2 + z^{-2}}{2} = \frac{z^2 + \frac{1}{z^2}}{2} = \frac{z^4+1}{2z^2}$$

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta = \oint_C \frac{\frac{z^4+1}{2z^2}}{5+4\left(\frac{z^2+1}{2z}\right)} \cdot \frac{dz}{iz} = \oint_C \frac{\frac{z^4+1}{2z^2}}{\frac{5z+2z^2+2}{z}} \cdot \frac{dz}{iz}$$

$$= \frac{1}{2i} \oint_C \frac{z^4+1}{z^2(2z^2+5z+2)} dz = \frac{1}{4i} \oint_C \frac{z^4+1}{z^2(z^2+\frac{5}{2}z+1)} dz$$

The poles are $z^2(z^2+\frac{5}{2}z+1) = 0$

$$z^2 = 0 \Rightarrow z = 0, \quad z = \frac{\frac{-5}{2} \pm \sqrt{\frac{25}{4} - 4}}{2} = \frac{-1}{2}, -2$$

$\therefore z = \alpha = \frac{-1}{2}$ is a simple poles and $z = 0$ is a pole of order '2' which lies inside the unite circle

||ly $z = \beta = -2$ is a simple pole which lies outside the unit circle.

$$r_1 = \operatorname{Res}_{z \to \alpha} f(z) = \lim_{z \to \alpha} (z-\alpha) \frac{z^4+1}{z^2(z-\alpha)(z-\beta)} = \frac{\alpha^4+1}{\alpha^2(\alpha-\beta)}$$

$$= \frac{\left(\frac{-1}{2}\right)^4 + 1}{\left(\frac{-1}{2}\right)^2\left(-\frac{1}{2}+2\right)} = \frac{17/16}{1/4(3/2)} = 17/6$$

$$r_2 = \underset{z=0}{\operatorname{Res}} f(z) = \frac{1}{(2-1)!} \lim_{z\to 0} \frac{d}{dz}\left((z-0)^2 \frac{z^4+1}{z^2\left(z^2+\frac{5}{2}z+1\right)} \right)$$

$$= \lim_{z\to 0} \frac{\left(z^2+\frac{5}{2}z+1\right)(4z^3) - (z^4+1)\left(2z+\frac{5}{2}\right)}{\left(z^2+\frac{5}{2}z+1\right)}$$

$$= -(1)(5/2) = -5/2$$

By C.R.T we have $\oint_C f(z)\,dz = 2\pi i\,[r_1+r_2]$

$$\oint_C \frac{z^4+1}{z^2\left(z^2+\frac{5}{2}z+1\right)}\,dz = 2\pi i\left(\frac{17}{6}-\frac{5}{2}\right) = \frac{2\pi i}{3}$$

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = \frac{1}{4i}\oint_C \frac{z^4+1}{z^2\left(z^2+\frac{5}{2}z+1\right)}\,dz = \frac{1}{\cancel{4i}_{\,2}}\left[\frac{2\pi \cancel{i}}{3}\right]$$

$$= \pi/6$$

Eg: By Integrating around the Unit Circle, evaluate

$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta$?

sol: let us take $z = e^{i\theta} \Rightarrow \frac{dz}{iz} = d\theta$, $\cos\theta = \frac{z^2+1}{2z}$

$$\cos 3\theta = \frac{e^{i3\theta}+e^{-i3\theta}}{2} = \frac{z^3+z^{-3}}{2} = \frac{z^3+\frac{1}{z^3}}{2}$$

$$= \frac{z^6+1}{2z^3}$$

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = \oint_C \frac{\frac{z^2+1}{2z^3}}{5-4\left(\frac{z^2+1}{2z^3}\right)}\,\frac{dz}{iz}$$

$$= \oint_C \frac{\frac{z^6+1}{2z^3}}{\frac{5z-2z^2-2}{z}} \left(\frac{dz}{iz}\right) = \frac{1}{2i} \oint_C \frac{z^6+1}{z^3(-2z^2+5z-2)} dz$$

$$= \frac{-1}{4i} \oint_C \frac{z^6+1}{z^3\left(z^2-\frac{5}{2}z+1\right)} dz$$

The poles are $z^3\left(z^2-\frac{5}{2}z+1\right)=0 \Rightarrow z^3=0,$ $z=0$

$$z^2-\frac{5}{2}z+1=0 \Rightarrow z=\frac{\frac{5}{2}\pm\sqrt{\frac{25}{4}-4}}{2} = \frac{\frac{5}{2}\pm\frac{3}{2}}{2}$$

$$= 2, \tfrac{1}{2}.$$

$\therefore z=\alpha=\frac{1}{2}$ is a simple pole and

$z=0$ a pole of order '3' both are lies inside the unit circle.

$\therefore z=\beta=2$ is a simple pole which lies outside the unit circle.

$$r_1 = \underset{z=\alpha}{\operatorname{Res}} f(z) = \lim_{z\to\alpha} \cancel{(z-\alpha)} \frac{z^6+1}{z^3\cancel{(z-\alpha)}(z-\beta)} = \frac{\alpha^6+1}{\alpha^3(\alpha-\beta)}$$

$$= \frac{\left(\frac{1}{2}\right)^6+1}{\left(\frac{1}{2}\right)^3\left(\frac{1}{2}-2\right)} = \frac{65/64}{1/8\left(\frac{-3}{2}\right)} = \frac{-65}{12}$$

$$r_2 = \underset{z=0}{\operatorname{Res}} f(z) = \frac{1}{(3-1)!} \lim_{z\to 0} \frac{d^2}{dz^2}\left[(z-0)^3 \frac{z^6+1}{z^3\left(z^2-\frac{5}{2}z+1\right)}\right]$$

$$= \frac{1}{2} \lim_{z \to 0} \frac{d}{dz} \left[\frac{\left(z^2 - \frac{5}{2}z + 1\right)(6z^5) - (z^6 + 1)(2z - 5/2)}{\left(z^2 - \frac{5}{2}z + 1\right)^2} \right]$$

$$= \frac{1}{2} \lim_{z \to 0} \frac{d}{dz} \left[\frac{6z^7 - 15z^6 + 6z^5 - 2z^7 + \frac{5}{2}z^6 - 2z + \frac{5}{2}}{\left(z^2 - \frac{5}{2}z + 1\right)^2} \right]$$

$$= \frac{1}{2} \lim_{z \to 0} \frac{d}{dz} \left[\frac{4z^7 - \frac{25}{2}z^6 + 6z^5 - 2z + \frac{5}{2}}{\left(z^2 - \frac{5}{2}z + 1\right)^2} \right]$$

$$= \frac{1}{2} \lim_{z \to 0} \frac{\left(z^2 - \frac{5}{2}z + 1\right)^2 (28z^6 - 75z^5 + 30z^4 - 2) - \left(4z^7 - \frac{25}{2}z^6 + 6z^5 - 2z + \frac{5}{2}\right) 2\left(z^2 - \frac{5}{2}z + 1\right)\left(2z - \frac{5}{2}\right)}{\left(z^2 - \frac{5}{2}z + 1\right)^4}$$

$$= \frac{1}{2} \left[1(-2) - \frac{5}{2}(2)(-5/2) \right] = \frac{1}{2}\left[-2 + \frac{25}{2}\right] = 21/4$$

By C.R.T, we have $\oint_C f(z)\, dz = 2\pi i (r_1 + r_2)$

$$\oint_C \frac{z^6 + 1}{z^3\left(z^2 - \frac{5}{2}z + 1\right)} dz = 2\pi i \left[\frac{-65}{12} + \frac{21}{4}\right] = 2\pi i \left[\frac{-2}{12}\right]$$

$$= -\pi i/3.$$

$$\therefore \int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} \cdot d\theta = \frac{-1}{4i}\left(\frac{-\pi i}{3}\right) = \pi/12$$

Eg(2) :- Show that $\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2} \, d\theta = \frac{2\pi a^2}{1-a^2}$

if $a^2 < 1$

$0 < a < 1$

Sol. Let us take $z = e^{i\theta} \Rightarrow \frac{dz}{iz} = d\theta$, $\cos\theta = \frac{z^2+1}{2z}$

$$\cos 2\theta = \frac{e^{i2\theta} + e^{-i2\theta}}{2} = \frac{z^2 + z^{-2}}{2} = \frac{z^4+1}{2z^2}$$

$$\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2} \, d\theta = \oint_C \frac{\frac{z^4+1}{2z^2}}{1-2a\left(\frac{z^2+1}{2z}\right)+a^2} \cdot \frac{dz}{iz}$$

$$= \oint_C \frac{\frac{z^4+1}{2z^2}}{\frac{z-az^2-a+a^2z}{z}} \cdot \frac{dz}{iz} = \frac{1}{2i}\oint_C \frac{z^4+1}{z^2(-az^2+(1+a^2)z-a)} \cdot dz$$

$$= \frac{-1}{2ai}\oint_C \frac{z^4+1}{z^2\left(z^2-\left(\frac{1+a^2}{a}\right)z+1\right)} dz$$

The poles are $z^2 = 0$ & $z^2 - \left(\frac{1+a^2}{a}\right)z + 1 = 0$

$$z = 0, \quad z = \frac{\frac{1+a^2}{a} \pm \sqrt{\left(\frac{1+a^2}{a}\right)^2 - 4}}{2}$$

$$= \frac{\frac{1+a^2}{a} \pm \frac{1-a^2}{a}}{2}$$

$z = \alpha = \dfrac{\dfrac{1+a^2}{a} + \dfrac{1-a^2}{a}}{2} = 1/a$ is a simple pole which lies outside the unit circle

$z = \beta = a$ is a simple pole and $z=0$ is a pole of order '2' both lies inside the unit circle.

$$r_1 = \underset{z=\beta}{\text{Res}}\, f(z) = \lim_{z \to \beta} \cancel{(z-\beta)} \frac{z^4+1}{z^2(z-\alpha)\cancel{(z-\beta)}} = \frac{\beta^4+1}{\beta^2(\beta-\alpha)}$$

$$= \frac{a^4+1}{a^2\left(a-\frac{1}{a}\right)} = \frac{a^4+1}{a(a^2-1)}.$$

$$r_2 = \underset{z=0}{\text{Res}}\, f(z) = \frac{1}{(2-1)!} \lim_{z \to 0} \frac{d}{dz}\left[(z-0)^2 \frac{z^4+1}{z^2\left(z^2-\left(\frac{1+a^2}{a}\right)z+1\right)}\right]$$

$$= \lim_{z \to 0} \frac{\left(z^2-\left(\frac{1+a^2}{a}\right)z+1\right)(4z^3) - (z^4+1)\left(2z-\left(\frac{1+a^2}{a}\right)\right)}{\left(z^2-\left(\frac{1+a^2}{a}\right)z+1\right)^2}$$

$$= \frac{1+a^2}{a}$$

By C.R.T we have $\oint_C f(z)\,dz = 2\pi i\,[r_1 + r_2]$

$$\oint_C \frac{z^4+1}{z^2\left(z^2-\left(\frac{1+a^2}{a}\right)z+1\right)}\,dz = 2\pi i\left[\frac{a^4+1}{a(a^2-1)} + \frac{1+a^2}{a}\right]$$

$$= 2\pi i\left[\frac{a^4+\cancel{1}+a^4-\cancel{1}}{a(a^2-1)}\right] = 2\pi i\left[\frac{2a^3}{(a^2-1)}\right] = \frac{4\pi i a^3}{a^2-1}$$

$$\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta = \frac{-1}{2\cancel{\pi i}}\left[\frac{4\cancel{\pi i}\,a^{\cancel{3}\,2}}{a^2-1}\right] = \frac{-2\pi a^2}{a^2-1} = \frac{2\pi a^2}{1-a^2}$$

Evaluation of Infinite integrals

Eg(1) Evaluate $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}$ by using Calculus of Residue.

Sol: let us take $x = z \Rightarrow dx = dz$

$$\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx = \oint_C \frac{z^2-z+2}{z^4+10z^2+9}\,dz$$

The poles are $z^4+10z^2+9 = 0 \Rightarrow$ put $z^2 = p \Rightarrow$

$$p^2+10p+9 = 0$$

$$\Rightarrow p^2+9p+p+9 = 0$$

$$\Rightarrow p(p+9)+1(p+9) = 0$$

$$(p+1) = 0 \;;\; p+9 = 0$$

$$(\because p = z^2) \quad p = -1 \;;\; -9$$

$$z^2 = -1 \;;\; z^2 = -9$$

$$z = \pm i \;;\; z = \pm 3i$$

$\therefore z = i, 3i$ are simple poles which lies inside the upper half of the circle.

$\therefore z = -i, z = -3i$ are simple poles which lies outside the upper half of the circle.

$$r_1 = \underset{z=i}{\mathrm{Res}}\, f(z) = \lim_{z\to i} \frac{(z-i)(z^2-z+2)}{(z-i)(z+i)(z^2+9)} = \frac{i^2-i+2}{(i+i)(i+9)}$$

$$= \frac{1-i}{16i}$$

$$r_2 = \underset{z=3i}{\mathrm{Res}} f(z) = \lim_{z\to 3i} (z-3i) \frac{z^2 - z + 2}{(z^2+1)(z-3i)(z+3i)}$$

$$= \frac{(3i)^2 - 3i + 2}{((3i)^2+1)(3i+3i)} = \frac{-7-3i}{-48i} = \frac{7+3i}{48i}$$

By C.R.T we have $\oint_C f(z)\,dz = 2\pi i\,(r_1 + r_2)$

$$\oint_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}\,dz = 2\pi i\left[\frac{1-i}{16i} + \frac{7+3i}{48i}\right] = 2\pi i\left[\frac{3-\cancel{3i}+7+\cancel{3i}}{48i}\right]$$

$$= \cancel{2\pi i}\cancel{\left[\frac{10}{48i}\right]} \frac{5\pi i}{12}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\,dx = \frac{5\pi}{12}.$$

Eg(2):- Evaluate $\int_0^\infty \frac{1}{x^4+a^4}\,dx$ by Using Calculus of Residue:

Sol:- $\int_0^\infty \frac{1}{x^4+a^4}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{x^4+a^4}\,dx$

$\therefore$ Let us take $x = z \Rightarrow dx = dz$

$$\int_0^\infty \frac{1}{x^4+a^4}\,dx = \frac{1}{2}\oint_C \frac{1}{z^4+a^4}\,dz$$

$\therefore$ The poles are $z^4 + a^4 = 0 \Rightarrow z^4 = -a^4$

$$\Rightarrow z^4 = e^{i(2n+1)\pi}\, a^4$$

$\Rightarrow z = a e^{i(2n+1)\pi/4}$ where $n = 0, 1, 2, 3$

When $n=0$, $z = ae^{i\pi/4}$

$n=1$, $z = ae^{i(3\pi/4)}$

$n=2$, $z = ae^{i(5\pi/4)}$

$n=3$, $z = ae^{i(7\pi/4)}$

$\therefore z = ae^{i(\pi/4)}$, $z = ae^{i\left(\frac{3\pi}{4}\right)}$ are Simple poles which lie inside the upper half of the Circle.

$\therefore z = ae^{i(5\pi/4)}$, $z = ae^{i(7\pi/4)}$ are simple poles which lies ~~on side~~ OUTSIDE the ~~upper~~ upper half of Circle.

$$r_1 = \underset{z=ae^{i\pi/4}}{\mathrm{Res}} f(z) = \lim_{z \to ae^{i\pi/4}} (z - ae^{i\pi/4}) \frac{1}{z^4 + a^4} \quad \text{(L-H Rule)}$$

$$= \lim_{z \to ae^{i(\pi/4)}} \frac{1}{4z^3} = \frac{1}{4(ae^{i\pi/4})^3} = \frac{1}{4a^3 e^{i(3\pi/4)}}$$

$$= \frac{e^{-i\left(\frac{3\pi}{4}\right)}}{4a^3} = \frac{1}{4a^3}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$$

$$= \frac{1}{4a^3}\left[\frac{-1}{\sqrt{2}} - i\left(\frac{1}{\sqrt{2}}\right)\right].$$

$$r_2 = \underset{z \to ae^{i(3\pi/4)}}{\mathrm{Res}} f(z) = \frac{1}{4(ae^{i3\pi/4})^3} = \frac{1}{4a^3} e^{-i(9\pi/4)}$$

$$= \frac{1}{4a^3}\left(\cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right) = \frac{1}{4a^3}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

By C.R-T we have $\oint_C f(z)\,dz = 2\pi i(r_1 + r_2)$

$$\oint_C \frac{1}{z^4 + a^4}\,dz = \frac{\cancel{2}\pi i}{{}_2\cancel{4}a^3}\left[\cancel{\frac{-1}{\sqrt{2}}} - i\left(\frac{1}{\sqrt{2}}\right) + \cancel{\frac{1}{\sqrt{2}}} - i\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{\pi i}{2a^3}\left[\frac{-2i}{\sqrt{2}}\right] = \frac{\pi}{\sqrt{2}a^3}.$$

$$\therefore \int_0^\infty \frac{1}{x^4 + a^4}\,dx = \frac{1}{2}\left[\frac{\pi}{\sqrt{2}a^3}\right] = \pi/2\sqrt{2}a^3.$$

Q. Evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)} \cdot dx$ where $a>b>0$ by

Using C.R.T.

Sol:- $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)} \cdot dx$ = R.P of $\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)} \cdot dx$

$\Updownarrow$

Let us take $x = z \Rightarrow dx = dz$

The poles are $(x^2+a^2)(x^2+b^2) = 0 \Rightarrow x^2 = -a^2,\ x^2 = -b^2$

$\Rightarrow x = \pm ai,\ x = \pm bi$

$\therefore x = ai,\ x = bi$ are simple poles which are lies inside the upper half of the circle.

$x = -ai,\ x = -bi$ are simple poles which are lies outside the upper half of the circle.

$$r_1 = \underset{z=ai}{\text{Res}}\, f(z) = \lim_{z \to ai} \cancel{(z+ai)} \frac{e^{iz}}{(z-ai)\cancel{(z+ai)}(z^2+b^2)}$$

$$= \frac{e^{iai}}{(ai+ai)((ai)^2+b^2)}$$

$$= \frac{e^{-a}}{2ai(b^2-a^2)}$$

$$r_2 = \underset{z=bi}{\text{Res}}\, f(z) = \lim_{z \to bi} \frac{\cancel{(z-bi)}\, e^{iz}}{(z^2+a^2)\cancel{(z-bi)}(z+bi)}.$$

Cauchy's Integral Formula (CIF)

Statement:- If $f(z)$ is an analytic function with in and on a closed curve and a is any point inside and on the circle C

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \quad \text{(or)} \quad \oint_C \frac{f(z)}{(z-a)}\,dz = 2\pi i f(a).$$

Derivative of Analytic function CIF

$$f'(a) = \frac{1}{2\pi}\oint_C \frac{f(z)}{(z-a)^2}\,dz \Rightarrow \oint_C \frac{f(z)}{(z-a)^2}\cdot dz = 2\pi i f'(a)$$

$$f''(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}\,dz \Rightarrow \oint_C \frac{f(z)}{(z-a)^3}\,dz = \frac{2\pi i}{2!} f''(a)$$

$$f'''(a) = \frac{3!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^4}\,dz \Rightarrow \oint_C \frac{f(z)}{(z-a)^4}\,dz = \frac{2\pi i}{3!} f'''(a)$$

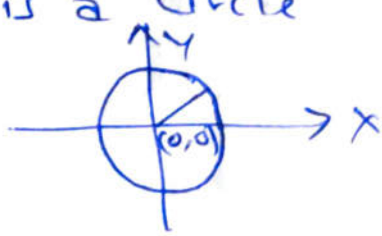
and so on

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\cdot dz \Rightarrow \oint_C \frac{f(z)}{(z-a)^{n+1}}\cdot dz = \frac{2\pi i}{n!} f^{n}(a)$$

Eg(1):- Evaluate $\oint_C \frac{z^2-z+1}{z-1}\,dz$ where c is circle

(i) $|z| = 1$ & (ii) $|z| = 1/2$.

Sol: Given $|z| = 1 \Rightarrow x^2+y^2 = 1$ is a circle with centre $(0,0)$ and $r = 1$



$\therefore$ The singular points ~~denotes~~ are

$z - 1 = 0 \Rightarrow \boxed{z=1}$

$\therefore z = 1$ lies inside the circle $|z| = 1$

$\oint_C \frac{z^2-z+1}{z-1}\cdot dz = \oint_C \frac{f(z)}{z-a}$ by comparing

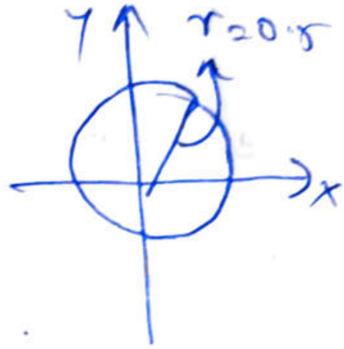
$f(z) = z^2-z+1$

$a = 1$

$f(a) = f(1) = 1^2-1+1 = 1$

By C.I.F we have $\oint_C \frac{f(z)}{z-a}\cdot dz = 2\pi i f(a)$

$\oint_C \frac{z^2+1-z}{z-1} dz = 2\pi i(1) = 2\pi i$ //.



(ii) $|z| = \frac{1}{2} \Rightarrow x^2+y^2 = (1/2)^2$ is a circle where centre $(0,0)$ & $r = \frac{1}{2} = 0.5$

$\therefore z=1$ is a singular point which lies outside the circle $|z|=1$. Hence C.I.F does not accept to calculate the integration.

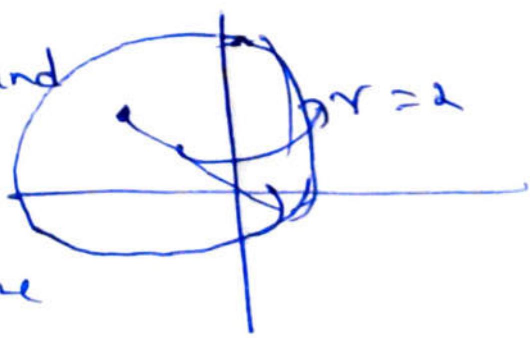
$\oint_C \frac{z^2-z+1}{z-1}\cdot dz = 0$ //

Eg(2): Evaluate $\oint_C \frac{z-3}{z^2+2z+5} dz$ (i) $|z+1-i| = 2$ (ii) $|z+1+i| = 2$

by using C.I.F.

Sol (i) Given $|z+1-i| = 2 \Rightarrow |x+iy+1-i| = 2$

$\Rightarrow |(x+1)+i(y-1)| = 2 \Rightarrow (x+1)^2+(y-1)^2 = 2^2$ is a circle with centre $(-1,1)$ and radius $r=2$



$\therefore$ The singular points are

$z^2+2z+5 = 0$

$$z = \frac{-2 \pm \sqrt{4-4(5)}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \cancel{-1 \pm \sqrt{3}} \; -1 \pm 2i$$

$z = \alpha = -1+2i$ i-e $(x,y) = (-1,2)$ lies inside the circle $|z+1-i| = 2$

$z = \beta = -1-2i$ i-e $(x,y) = (-1,-2)$ lies outside the circle $|z+1-i| = 2$

$$\therefore \oint_C \frac{z+4}{z^2+2z+5} \cdot dz = \oint_C \frac{z+4}{(z-\alpha)(z-\beta)} \cdot dz = \oint_C \frac{\left(\frac{z+4}{z-\beta}\right)}{z-\alpha} \cdot dz$$

Compare $\oint_C \frac{f(z)}{z-a} \cdot dz$

$$\exists \; f(z) = \frac{z+4}{z-\beta}, \; a = \alpha$$

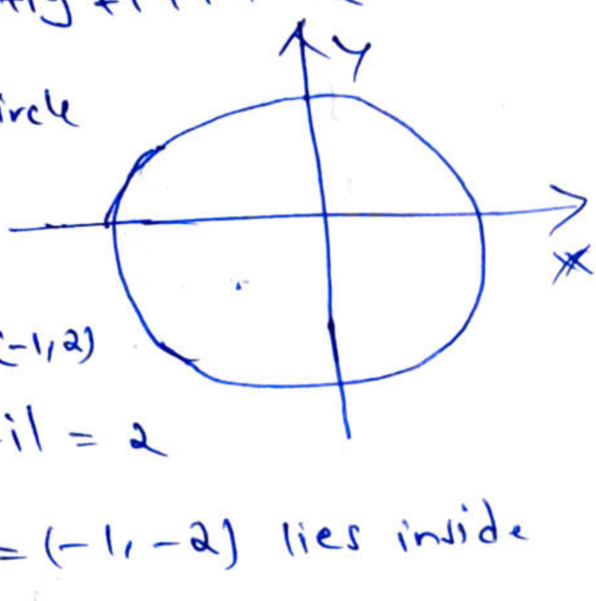
$$f(a) = f(\alpha) = \frac{\alpha+4}{\alpha-\beta} = \cancel{+\pm} \; \frac{-1+2i+4}{-1+2i+1+2i} = \frac{3+2i}{4i}$$

$\therefore$ By C.I.F we have $\oint_C \frac{f(z)}{(z-a)} \cdot dz = 2\pi i \, f(a)$

$$\oint_C \frac{z+4}{z^2+2z+5} \cdot dz = 2\pi i \left(\frac{3+2i}{4i}\right) = (3+2i)\frac{\pi}{2}.$$

(ii) Given $|z+1+i| = 2 \;\exists\; |x+iy+1+i| = 2$

$\Rightarrow (x+1)^2 + (y+1)^2 = 2^2$ is a circle with centre $(-1,-1)$ & $r = 2$.



$\therefore z = \alpha = -1+2i$ i-e $(x,y) = (-1,2)$ lies outside the circle $|z+1-i| = 2$

$\therefore z = \beta = -1-2i$ i-e $(x,y) = (-1,-2)$ lies inside the circle $|z+1+i| = 2$.

$$\oint_C \frac{z+4}{z^2+2z+5}\cdot dz = \oint_C \frac{z+4}{(z-\alpha)(z-\beta)}\cdot dz = \oint_C \frac{\left(\frac{z+4}{z-\alpha}\right)}{z-\beta}$$

compare $= \oint_C \frac{f(z)}{z-a}\cdot dz$

Hence

$$f(z) = \frac{z+4}{z-\alpha}, \quad a=\beta$$

$$f(a) = f(\beta) = \frac{\beta+4}{\beta-\alpha} = \frac{-1-2i+4}{-1-2i+1-2i} = \frac{3-2i}{-4i}$$

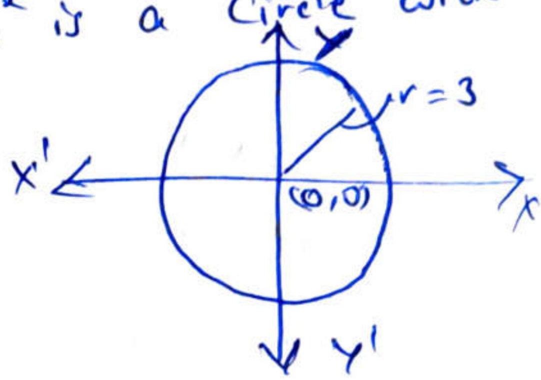
$$= -\frac{3+2i}{4i}$$

By C.I.F we have $\oint_C \frac{f(z)}{z-a}dz = 2\pi i\, f(a)$

$$\oint_C \frac{z+4}{z^2+2z+5}dz = 2\pi i\left(\frac{-3+2i}{4i}\right) = (-3+2i)\frac{\pi}{2}$$

Eg: Evaluate $\oint_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz$ where 'c' is the circle $|z|=3$.

Sol: Given $|z|=3 \Rightarrow x^2+y^2=3^2$ is a circle with centre $(0,0)$ and $r=3$.



∴ The singular points are

$(z-1)(z-2)=0$

$\Rightarrow z=1,\ z=2$

∴ $z=1$ and $z=2$ are lies inside the circle $|z|=3$.

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$1 = A(z-2) + B(z-1)$$

put $z=1$, $1=-A \Rightarrow \boxed{A=-1}$

$z=2$ $\quad 1=B \Rightarrow \boxed{B=1}$

$$\therefore \oint_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}\cdot dz = -\oint_C \frac{\sin\pi z^2+\cos\pi z^2}{z-1}\cdot dz + \oint_C \frac{\sin\pi z^2+\cos\pi z^2}{z-2}\cdot dz$$

Here $f(z) = \sin\pi z^2 + \cos\pi z^2$

$f(1) = \sin\pi + \cos\pi = -1$

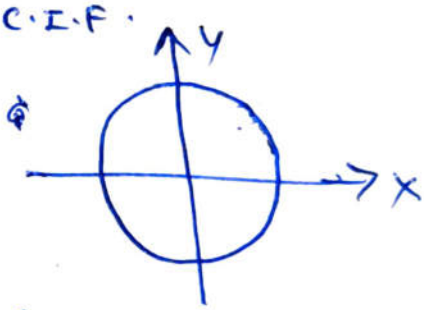
$f(2) = \sin 4\pi + \cos 4\pi = 1$

By C.I.F we have $\oint_C \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)$

$$\oint_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}\cdot dz = -\left[2\pi i(-1)\right] + 2\pi i(1) = \underline{4\pi i}.$$

Eg:- Evaluate $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\, dz$ where 'c' is $|z| = 4$ by using C.I.F.

sol:- Given $|z| = 4 \Rightarrow x^2+y^2 = 4^2$ is a circle with $c(0,0)$ & $r=4$



The singular points $(z^2+\pi^2)^2 = 0$

$\Rightarrow z^2+\pi^2 = 0 \Rightarrow z^2 = -\pi^2$

$\Rightarrow z = \pm\pi i$

$\therefore z = -\pi i$ & $z = \pi i$ lies inside the circle $|z| = 4$.

$$\frac{1}{(z^2+\pi^2)^2} = \frac{1}{(z-\pi i)^2(z+\pi i)^2} = \frac{A}{(z-\pi i)} + \frac{B}{(z-\pi i)^2} + \frac{C}{z+\pi i} + \frac{D}{(z+\pi i)^2}$$

$$1 = A(z-\pi i)(z+\pi i)^2 + B(z+\pi i)^2 + C(z+\pi i)(z-\pi i)^2 + D(z-\pi i)$$

put $z = \pi i \Rightarrow 1 = B(2\pi i)^2 \Rightarrow \boxed{B = \frac{-1}{4\pi^2}}$

put $z = -\pi i \Rightarrow 1 = D(-2\pi i)^2 \Rightarrow \boxed{D = \frac{-1}{4\pi^2}}$

Comparing 'z^3' term $\quad 0 = A + C \longrightarrow (i)$

Comparing 'z^2' term $\quad 0 = \pi i A + B - \pi i C + D$

$$\Rightarrow 0 = \pi i A - \frac{1}{4\pi^2} - \pi i C - \frac{1}{4\pi^2}$$

$$\frac{1}{2\pi^2} = \pi i [A - C] \Rightarrow A - C = \frac{1}{2\pi^3 i} \rightarrow (ii)$$

Solving (i) & (ii) $\quad A + \cancel{C} = 0$

$$A - \cancel{C} = \frac{1}{2\pi^3 i}$$

$$2A = \frac{1}{2\pi^3 i} \Rightarrow \boxed{A = \frac{1}{4\pi^3 i}}$$

then $\boxed{C = \frac{-1}{4\pi^3 i}}$

$$\oint_C \frac{e^z}{(z^2+\pi^2)^2} \cdot dz = \frac{1}{4\pi^3 i} \oint_C \frac{e^z}{z - \pi i} \cdot dz - \frac{1}{4\pi^2} \oint_C \frac{e^z}{(z-\pi i)^2} \cdot dz$$

$$- \frac{1}{4\pi^3 i} \oint_C \frac{e^z}{z + \pi i} \cdot dz - \frac{1}{4\pi^2} \oint_C \frac{e^z}{(z+\pi i)^2} \cdot dz$$

$$= \frac{1}{\cancel{4}\pi^{\cancel{3}2} i}\left[\cancel{2}\pi i f(\pi i)\right] - \frac{1}{\cancel{4}2\pi^{2}}\left[\cancel{2}\pi i f'(\pi i)\right] - \frac{1}{\cancel{4}\pi^{\cancel{3}} i}\left(\cancel{2}\pi i f(-\pi i)\right)$$

$$- \frac{1}{\cancel{4}\pi^2}\left(\cancel{2}\pi i f'(-\pi i)\right)$$

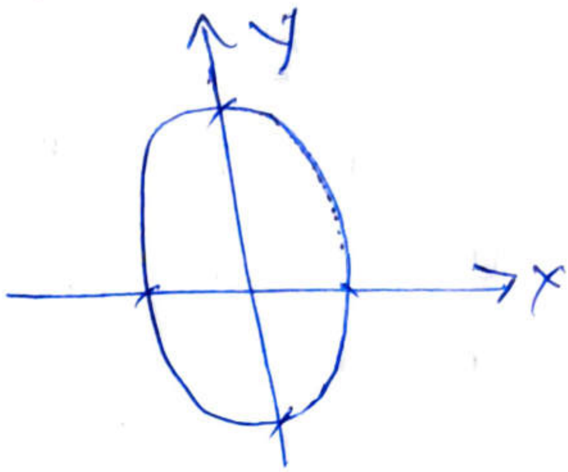
$$= \frac{1}{2\pi^2} e^{\pi i} - \frac{i}{2\pi} e^{\pi i} - \frac{1}{2\pi^2} e^{-\pi i} - \frac{i}{2\pi} e^{-\pi i}$$

$$= \frac{1}{\pi^2}\left[\frac{e^{\pi i} - e^{-\pi i}}{2}\right] - \frac{i}{\pi}\left[\frac{e^{\pi i} + e^{-\pi i}}{2}\right] = \cancel{\frac{i}{\pi^2} \sin \pi} - \frac{i}{\pi} \cos \pi$$

$$= -i/\pi\,(-1) = i/\pi$$

Eg:- Evaluate $F(\psi) = \oint_C \frac{4z^2+z+5}{z-\psi}\, dz$ where 'c' is an ellipse $\left(\frac{x^2}{4}+\frac{y^2}{9}=1\right)$

Find (i) $F(3.5)$ (ii) $F(i)$ (iii) $F'(-1)$ (iv) $F''(-i)$



Sol:-

Given ellipse

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

$$\frac{x^2}{2^2}+\frac{y^2}{3^2}=1$$

$$f(z) = 4z^2+z+5$$

(i) $F(3.5)$ Here $\psi = 3.5$ (Singular point)

$\psi = 3.5$ lies outside an ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$

By ~~CR~~ C.I.F $F(3.5) = \oint_C \frac{4z^2+z+5}{z-3.5}\, dz = 0$

(ii) $F(i)$ here $\psi = i$ which lies inside the ellipse

$f(i) = 4(i)^2+i+5 = 1+i$

By C.I.F w.r.t $\oint_C \frac{f(z)}{z-a}\, dz = 2\pi i f(a)$

$\therefore F(i) = \oint_C \frac{4z^2+z+5}{z-i}\, dz = 2\pi i(1+i)$ //.

(iii) $F'(-1)$ here $\psi = -1$ which lies inside the ellipse

$f(z) = 4z^2+z+5$ $\quad f'(z) = 8z+1$

$f'(-1) = -8+1 = -7$

By C.I.F (derivative)

$$\oint_C \frac{f(z)}{(z-a)^2} \cdot dz = 2\pi i f'(a)$$

$$F'(-1) = \oint_C \frac{4z^2+z+5}{(z+1)^2} dz = 2\pi i(-7) = -14\pi i$$

(iv) $F''(-i)$, here $\psi = -i$ which lies inside the ellipse.

$f''(z) = 8$ $\quad f''(-i) = 8$

By C.I.F (derivative) $\oint \frac{f(z)}{(z-a)^3} dz = 2\pi i f''(a)$

$$F''(-i) = \oint_C \frac{4z^2+z+5}{(z+i)^3} \cdot dz = 2\pi i(8) = 16\pi i.$$

4/09/2020 MODULE-5

PARTIAL DIFFERENTIAL EQUATIONS (PDE)

Formation of PDE

(a) Eliminating arbitary constants

$p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$, $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

Eg(1) :- Derive a P.D.E by eliminating arbitary constant from $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

Sol:- The given equation is $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \rightarrow$ (i)

diff. partially w.r.t 'x'

$$\cancel{2} \cdot \frac{\partial z}{\partial x} = \frac{\cancel{2}x}{a^2} \Rightarrow \boxed{a^2 = \frac{x}{\frac{\partial z}{\partial x}}}$$

Again diff partially (1) w.r.t 'y'

$$\cancel{2}.\frac{\partial z}{\partial y} = \frac{\cancel{2}y}{b^2} \Rightarrow \boxed{b^2 = \frac{y}{\partial z/\partial y}}$$

Substitute a^2 & b^2 values in eq(1)

$$2z = \frac{x^{\cancel{2}}}{\cancel{x}/\frac{\partial z}{\partial x}} + \frac{y^{\cancel{2}}}{\cancel{y}/\partial z/\partial y} \Rightarrow \boxed{2z = x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y}}$$

(or)

$$\boxed{2z = xp + yq}$$

is a req. PDE

Eg(2) :- From the P.D.E by eliminating orbitary constants 'a' and 'b' from $(x-a)^2 + (y-b)^2 + z^2 = c^2$.

Sol:- Given equation is $(x-a)^2 + (y-b)^2 + z^2 = c^2 \rightarrow (i)$

diff ~~parallet~~ partially (i) w.r.t 'x'

$$\cancel{2}(x-a) + \cancel{2}z\frac{\partial z}{\partial x} = 0 \Rightarrow \boxed{x-a = -z.\frac{\partial z}{\partial x}}$$

Again diff partially (1) w.r.t 'y'

$$\cancel{2}(y-b) + \cancel{2}z\frac{\partial z}{\partial y} = 0 \Rightarrow y-b = \boxed{-z.\frac{\partial z}{\partial y}}$$

Substitite $x-a$ & $y-b$ in the eq (1)

$$\left(-z.\frac{\partial z}{\partial x}\right)^2 + \left(-z.\frac{\partial z}{\partial y}\right)^2 + z^2 = c^2$$

$$\Rightarrow \boxed{z^2\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right] = c^2} \text{ (or) } \boxed{z^2(p^2+q^2+1) = c^2}$$

Eg(3) :- Form the PDE by eliminating arbitary function from $z = (x+y)\phi(x-y)$?

Sol:- Given equation is $z=(x+y)\,\phi(x-y) \rightarrow (i)$

diff partially (i) w.r.t 'x'

$$\frac{\partial z}{\partial x} = (x+y)\,\phi'(x-y) + \phi(x-y)(1) \longrightarrow (ii)$$

diff partially (ii) w.r.t 'y'

$$\frac{\partial z}{\partial y} = (x+y)\,\phi'(x-y)(-1) + \phi(x-y)(1) \rightarrow (iii)$$

Adding (ii) & (iii) we have

$$\frac{\partial z}{\partial x} = (x+y)\,\phi'(x-y) + \phi(x-y)$$

$$\frac{\partial z}{\partial y} = -(x+y)\,\phi'(x-y) + \phi(x-y)$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2\phi(x-y) \rightarrow (iv)$$

from given eqn $\phi(x-y) = \dfrac{z}{x+y}$

Substitute $\phi(x-y)$ value in eq (iv)

$$\boxed{\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{2z}{x+y}} \quad (\text{or}) \quad \boxed{p+q = \frac{2z}{x+y}}$$

which is the required P.D.E.

Eg: Obtain the P.D.E by eliminate arbitary functions from $z=f(x+at)+g(x-at)$?

Sol: Given eqn. is $z=f(x+at)+g(x-at) \longrightarrow (i)$

diff. partially (i) w.r.t 'x'

$$\frac{\partial z}{\partial x} = f'(x+at) + g'(x-at) \longrightarrow (ii)$$

diff partially (i) w.r.t 'y'

$$\frac{\partial z}{\partial y} = f'(x+at)\,a + g'(x-at)(-a) \rightarrow \text{(iii)}$$

Again diff. partially w.r.t 'x'

$$\frac{\partial^2 z}{\partial x^2} = f''(x+at) + g''(x-at) \rightarrow \text{(iv)}$$

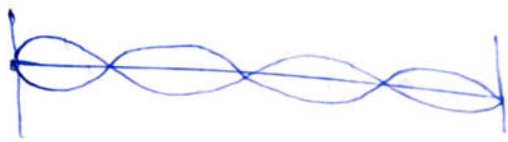
diff partially eq (iii) w.r.t 't'

$$\frac{\partial^2 z}{\partial t^2} = f''(x+at)\,a^2 + g''(x-at)(-a)^2$$

$$= a^2\left[f''(x+at) + g''(x-at)\right]$$

$$\boxed{\frac{\partial^2 z}{\partial t^2} = a^2 \cdot \frac{\partial^2 z}{\partial x^2}}$$ which is the requried P.D.E.

wave eqn.



Eg:- Form the PDE by eliminating arbitary function from $f(xyz, x^2+y^2+z^2) = 0$

Sol: Given eqn is $f(xyz, x^2+y^2+z^2) = 0$

$$\Rightarrow xyz = f(x^2+y^2+z^2) \rightarrow \text{(i)}$$

diff. partially (i) w.r.t 'x'

$$y\left[x \cdot \frac{\partial z}{\partial x} + z \cdot 1\right] = f'(x^2+y^2+z^2)\left(2x + 2z \cdot \frac{\partial z}{\partial x}\right) \rightarrow \text{(ii)}$$

diff. partially (i) w.r.t 'y'

$$x\left[y \cdot \frac{\partial z}{\partial y} + z \cdot 1\right] = f'(x^2+y^2+z^2)\left(2y + 2z \cdot \frac{\partial z}{\partial y}\right) \rightarrow \text{(iii)}$$

divide eq (ii) by (iii)

$$\frac{y\left[x\cdot\frac{\partial z}{\partial x}+z\right]}{x\left[y\cdot\frac{\partial z}{\partial y}+z\right]} = \frac{\cancel{f'(x^2+y^2+z^2)}\left(\cancel{2}x+\cancel{2}z\cdot\frac{\partial z}{\partial x}\right)}{\cancel{f'(x^2+y^2+z^2)}\left(\cancel{2}y+\cancel{2}z\cdot\partial z/\partial y\right)}$$

$$\frac{y[xp+z]}{x[yq+z]} = \frac{x+zp}{y+zq} \Rightarrow (xyp+yz)(y+zq) = (x+zp)(xyq+xz)$$

$$\Rightarrow xy^2p + \cancel{xyzpq} + y^2z + yz^2q = x^2yq + x^2z + \cancel{xyzpq} + xz^2p$$

$$\Rightarrow xy^2p - xz^2p + yz^2q - x^2yq = x^2z - y^2z$$

$$\boxed{x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)}$$

which is the required P.D.E.

Solutions of Partial Differential Equations of First Order

∴ The general form of Linear P.D.E of 1^{st} order is $P\cdot p + Q\cdot q = R$, when $p = \frac{\partial f}{\partial x}$; $q = \frac{\partial f}{\partial y}$ & P, Q, R are functions of x, y, z.

This is eqn is known as Legranges Linear P.D.

Grouping Method:-

Procedure:-

Step(1):- The Subsidiary eqns are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Step (2):

Case (i) Let us take $\frac{dx}{P} = \frac{dy}{Q}$ integrating on both sides

$\therefore$ Solution is $f(x,y) = c_1$

Case (ii) Let us take $\frac{dy}{Q} = \frac{dz}{R}$ integrate on both sides

$\therefore$ Solution is $g(y,z) = c_2$

$\therefore$ Complete Solution is $\phi(f(x,y), g(y,z)) = 0$.

Eg(1):- Solve $y^2 z p + x^2 z q = xy^2$

Sol:- The given equation is of the form $Pp + Qq = R$

Here $P = y^2 z$; $Q = x^2 z$; $R = y^2 x$

$\therefore$ The Subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{y^2 x}$$

Case (i) Take $\frac{dx}{y^2 z} = \frac{dy}{x^2 z} \Rightarrow x^2 . dx = y^2 . dy$
integrating on both Sides.

$\int x^2 . dx = \int y^2 . dy + c_1 \Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + c_1 \Rightarrow \frac{x^3}{3} - \frac{y^3}{3} = c_1$

$\Rightarrow \boxed{x^3 - y^3 = 3c_1}$

Case (ii)

Let us take $\frac{dx}{y^2 z} = \frac{dz}{y^2 x} \Rightarrow x . dx = z . dz$
integrating on both sides

$\int x . dx = \int z . dz + c_2 \Rightarrow \frac{x^2}{2} = \frac{z^2}{2} + c_2$

$\Rightarrow \boxed{x^2 + z^2 = 2c_2}$

$\therefore$ The Complete Solution is $\phi(x^3 - y^3, x^2 - z^2) = 0$
or $x^3 - y^3 = \phi(x^2 - z^2)$
(or) $x^2 - z^2 = \phi(x^3 - y^3)$.

Eg(2):- Solve $p\cdot\tan x + q\cdot\tan y = \tan z$

Sol:- The given equation is of the form

$$p\cdot P + q\cdot Q = R$$

Here $P = \tan x$, $Q = \tan y$, $R = \tan z$

The Subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} \Rightarrow \cot x\cdot dx = \cot y\cdot dy \text{ (}\because\text{ integrate)}$$

$$\int \cot x\cdot dx = \int \cot y\cdot dy + c_1 \Rightarrow \log \sin x = \log \sin y + \log c_1$$

$$\log\left(\frac{\sin x}{\sin y}\right) = \log c_1 \Rightarrow \boxed{\frac{\sin x}{\sin y} = c_1}$$

Case(ii):- Let us take $\frac{dy}{\tan y} = \frac{dz}{\tan z} \Rightarrow \int \cot y\cdot dy = \int \cot z\cdot dz + c_2$

$$\Rightarrow \log \sin y = \log \sin z + \log c_2 \Rightarrow \log\frac{\sin y}{\sin z} = \log c_2$$

$$\Rightarrow \boxed{\frac{\sin y}{\sin z} = c_2}$$

complete soln.

$\therefore$ is $\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$

Multiplier's Method

Eg(3) Solve $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

Sol:- The given equation is Linear PDE. The Subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}$$

Case (i) $\dfrac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y^2-z^2+z^2-x^2+x^2-y^2} = \dfrac{dx}{x(y^2-z^2)}$

$$\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0} = \frac{dx}{x(y^2-z^2)}$$

$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0 \Rightarrow \text{Integrating}$$

$$\int\frac{dx}{x}+\int\frac{dy}{y}+\int\frac{dz}{z}=c_1 \Rightarrow \log x+\log y+\log z=\log c_1$$

$$\Rightarrow \log xyz = \log c_1 \Rightarrow \boxed{xyz=c_1}$$

Case (ii) $\dfrac{x\,dx+y\,dy+z\,dz}{x^2(y^2-z^2)+y^2(z^2-x^2)+z^2(x^2-y^2)} = \dfrac{dx}{x(y^2-z^2)}$

$$\frac{x\,dx+y\,dy+z\,dz}{x^2y^2-x^2z^2+y^2z^2-y^2x^2+z^2x^2-z^2y^2} = \frac{dx}{x(y^2-z^2)}$$

$$\frac{x\,dx+y\,dy+z\,dz}{0} = \frac{dx}{x(y^2-z^2)} \Rightarrow x\,dx+y\,dy+z\,dz=0$$

'$\int$' on both sides $\Rightarrow \int x\cdot dx+\int y\cdot dy+\int z\cdot dz = c_2$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c_2 \Rightarrow \boxed{x^2+y^2+z^2=2c_2}$$

Complete Solution is $\phi(xyz, x^2+y^2+z^2)=0$

(or) $xyz = \phi(x^2+y^2+z^2)$

(or) $x^2+y^2+z^2=\phi(xyz)$

Eg(4):- Solve $(mz-ny)p + (nx-lz)q = ly-mx$

Sol: The given equation is a LPDE

The Subsidiary equation's are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

Case(i):- $\frac{x\cdot dx + y\cdot dy + z\cdot dz}{mxz - nxy + nxy - lzy + lyz - mzx} = \frac{dx}{mz-ny}$

$x\,dx + y\,dy + z\,dz = 0$ '$\int$' on both sides

$\int x\cdot dx + \int y\cdot dy + \int z\cdot dz = c_1 \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$

$\Rightarrow \boxed{x^2+y^2+z^2 = c_1}$.

Case (ii):- $\frac{l\cdot dx + m\cdot dy + n\cdot dz}{l\cdot mz - lny + mnx - mlz + nly - mnx} = \frac{dx}{mz-ny}$

$\Rightarrow l\cdot dx + m\cdot dy + n\cdot dz = 0$ ('$\int$' on both sides)

$\int l\cdot dx + \int m\cdot dy + \int n\cdot dz = c_2$

$\Rightarrow \boxed{lx + my + nz = c_2}$

Complete solution is $\phi(x^2+y^2+z^2, lx+my+nz) = 0$

(or) $x^2+y^2+z^2 = \phi(lx+my+nz)$

(or) $lx+my+nz = \phi(x^2+y^2+z^2)$.

Eg) Solve $(y^2+z^2)p - xyq + zx = 0$

Sol:- $\Rightarrow P.p + Q.q = R$

Here $P = y^2+z^2$, $Q = -xy$, $R = -zx$

The subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-zx}$$

Case (i) Let us take $\frac{dy}{-xy} = \frac{dz}{-zx} \Rightarrow \frac{dy}{y} = \frac{dz}{z}$ $\int$ on b.s

$\int \frac{dy}{y} = \int \frac{dz}{z} + c_1 \Rightarrow \log y = \log z + \log c_1$

$\Rightarrow \log(y/z) = \log c_1$

$\Rightarrow \boxed{y/z = c_1}$

Case (ii):- Let us take

$$\frac{x\,dx + y\,dy + z\,dz}{xy^2 + xz^2 - xy^2 - xz^2} = \frac{dz}{-zx}$$

$x.dx + y.dy + z.dz = 0$ $\quad\int$ on both sides

$\int x.dx + \int y.dy + \int z.dz = c_2$

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2 \Rightarrow \boxed{x^2+y^2+z^2 = c_2}$

$\therefore$ The Complete solution is $\phi\left(y/z, x^2+y^2+z^2\right) = 0$

(or) $\frac{y}{z} = \phi(x^2+y^2+z^2)$ (or) $x^2+y^2+z^2 = \phi(y/z)$

Eg: Solve $(x^2-yz)p + (y^2-zx)q = z^2-xy$

Sol: The Subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

Case(i): $\frac{dx-dy}{x^2-yz-y^2+zx} = \frac{dy-dz}{y^2-zx-z^2+xy}$

$$\frac{dx-dy}{x^2-y^2+z(x-y)} = \frac{dy-dz}{y^2-z^2+x(y-z)}$$

$$\frac{dx-dy}{(x-y)\cancel{[x+y+z]}} = \frac{dy-dz}{(y-z)\cancel{(y+z+x)}} \Rightarrow \frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$

$\int$ on b/s

$$\Rightarrow \int\frac{d(x-y)}{x-y} = \int\frac{d(y-z)}{y-z} + c \Rightarrow \log(x-y) = \log(y-z) + \log c$$

$$\Rightarrow \log\left(\frac{x-y}{y-z}\right) = \log c_1 \Rightarrow \boxed{\frac{x-y}{y-z} = c_1}$$

Case(ii): $\frac{x\cdot dx + y\cdot dy + z\cdot dz}{x^3-xyz+y^3-xyz+z^3-xyz} = \frac{dx+dy+dz}{x^2-yz+y^2-zx+z^2-xy}$

$$\frac{x\cdot dx + y\cdot dy + z\cdot dz}{x^3+y^3+z^3-3xyz} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{x\cdot dx + y\cdot dy + z\cdot dz}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-zx)}} = \frac{dx+dy+dz}{\cancel{x^2+y^2+z^2-xy-yz-zx}}$$

$$x\cdot dx + y\cdot dy + z\cdot dz = (x+y+z)(dx+dy+dz)$$

$$\int x\cdot dx + \int y\cdot dy + \int z\cdot dz = \int (x+y+z)\, d(x+y+z) + c_2$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x+y+z)^2}{2} + c_2$$

$x^2+y^2+z^2-(x+y+z)^2=c_2$

$\therefore$ The Complete Solution is $\phi\left(\dfrac{x-y}{y-z},\ x^2+y^2+z^2-(x+y+z)^2\right)=0$

Alternative in Case (ii)

$$\int\frac{dy-dz}{y-z}=\int\frac{dz-dx}{z-x}+c_2$$

$\Rightarrow \log(y-z)=\log(z-x)+\log c_2 \Rightarrow \log\left(\dfrac{y-z}{z-x}\right)=\log c_2$

$$\Rightarrow \boxed{\frac{y-z}{z-x}=c_2}$$

$\therefore$ The Complete Solution is

$$\phi\left(\frac{x-y}{y-z},\ \frac{y-z}{z-x}\right)=0.\ //$$

$x^2+y^2+z^2-(x+y+z)^2 = c_2$

$\therefore$ The Complete Solution is $\phi\left(\frac{x-y}{y-z},\ x^2+y^2+z^2-(x+y+z)^2\right)=0$

Alternative in Case (ii)

$$\int \frac{dy-dz}{y-z} = \int \frac{dz-dx}{z-x} + c_2$$

$\Rightarrow \log(y-z) = \log(z-x) + \log c_2 \Rightarrow \log\left(\frac{y-z}{z-x}\right) = \log c_2$

$\Rightarrow \boxed{\frac{y-z}{z-x} = c_2}$

$\therefore$ The Complete Solution is

$\phi\left(\frac{x-y}{y-z},\ \frac{y-z}{z-x}\right)=0$. //

Non-Linear Partial Differential Equation

Type (1) :- Equations of the form $f(p,q)=0$

procedure :- Let us take $p=\frac{\partial z}{\partial x}=a$, $q=\frac{\partial z}{\partial y}=b$

find 'b' values interms 'a' then the Complete Solution is $z=ax+by+c$.

Eg(1):- Solve $p^2+q^2=1$?

Sol:- This equation is of the form $f(p,q)=0$.

Let us take $p=a$, $q=b$ then $a^2+b^2=1$

$\Rightarrow b^2 = 1-a^2$

$b = \pm\sqrt{1-a^2}$

$\therefore$ The complete solution is $z = ax+by+c$

$\boxed{z = ax \pm \sqrt{1-a^2}\,y + c}$.

Eg(2) :- Solve $p^2+q^2=m^2$

Sol :- Let us take $p=a$, $q=b$ then $a^2+b^2=m^2$

$$b^2=m^2-a^2$$

$$b=\pm\sqrt{m^2-a^2}$$

$\therefore$ The complete solution is

$$z=ax+by+c$$

$$z=ax\pm(\sqrt{m^2-a^2})y+c.$$

Eg(3) :- Solve $\sqrt{p}+\sqrt{q}=1$

Sol :- Let us take $p=a$, $q=b$ then

$\sqrt{a}+\sqrt{b}=1 \Rightarrow \sqrt{b}=1-\sqrt{a} \Rightarrow b=1$

The complete solution is $z=ax+by+c$

$$z=ax+(1-\sqrt{a})^2y+c.$$

Type-II The equations of the form $f(z,p,q)=0$

procedure :- Let us take

$$u=x+ay$$

$$\frac{\partial u}{\partial x}=1 \quad \frac{\partial u}{\partial y}=a$$

$$p=\frac{\partial z}{\partial x}=\frac{dz}{du}\cdot\frac{\partial u}{\partial x}=\frac{dz}{du} \; ; \; q=\frac{\partial z}{\partial y}=\frac{dz}{du}\cdot\frac{\partial u}{\partial y}$$

$$=a.\frac{dz}{du}$$

Substitute 'p' & 'q' values in the given eqn's & use seperate variables and integrate

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Eg(1) :- Solve $p(1+q)=qz$

Sol :- let us take $u=x+ay$

$$p=\frac{\partial z}{\partial x}=\frac{dz}{du}, \quad q=a.\frac{dz}{du}$$

The given equation becomes $\cancel{\frac{dz}{du}}\cdot\left(1+a.\frac{dz}{du}\right)=a.\frac{dz}{du}.z$

$1 + a\frac{dz}{du} = az \Rightarrow a\left(\frac{dz}{du}\right) = az - 1$

Seperate Variables and integrate

$\int \frac{a\,(dz)}{az-1} = \int du + c$

$\frac{a\log(az-1)}{a} = u + c \Rightarrow \boxed{\log(az-1) = x + ay + c}$

The complete Solution is

Eg(2) :- Solve $p^2z^2 + q^2 = 1$

Sol:- Given PDE belong's to non-linear and $f(z,p,q)=0$ type

Let us take $u = x + ay$ then $p = \frac{dz}{du}$, $q = a\cdot\frac{dz}{du}$

The given equation becomes $\left(\frac{dz}{du}\right)^2 z^2 + \left(a\cdot\frac{dz}{du}\right)^2 = 1$

$\left(\frac{dz}{du}\right)^2 (z^2 + a^2) = 1 \Rightarrow \left(\frac{dz}{du}\right)^2 = \frac{1}{(z^2+a^2)}$

Square root on both sides

$\frac{dz}{du} = \frac{1}{\sqrt{a^2+z^2}}$ Seperate Variables and integrate

$\int \sqrt{a^2+z^2}\, dz = \int du + c$ $\left[\because \int\sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log(x+\sqrt{a^2+x^2})\right]$

$\frac{z}{2}\sqrt{a^2+z^2} + \frac{a^2}{2}\log\left(z + \sqrt{a^2+z^2}\right) = u + c$ $(u = x+ay)$

$z\left(\sqrt{a^2+z^2}\right) + a^2\log\left(z+\sqrt{a^2+z^2}\right) = 2x + 2ay + c.$

Eg(2) :- Solve $q^2 = z^2 p^2 (1-p^2)$

Sol> Let us take $u = x+ay$ then $p = \frac{dz}{du}$

$q = a \cdot \frac{dz}{du}$

The given equation becomes $\left(a\frac{dz}{du}\right)^2 = z^2\left(\frac{dz}{du}\right)^2\left(1-\left(\frac{dz}{du}\right)^2\right)$

$a^2 = z^2\left(1-\left(\frac{dz}{du}\right)^2\right) \Rightarrow \frac{a^2}{z^2} = 1-\left(\frac{dz}{du}\right)^2$

$\left(\frac{dz}{du}\right)^2 = 1-\frac{a^2}{z^2} = \frac{z^2-a^2}{z^2}$ square root on both sides

$\frac{dz}{du} = \frac{\sqrt{z^2-a^2}}{z}$ seperate varibles & integrate

$\int \frac{z}{\sqrt{z^2-a^2}} \cdot dz = \int du + c$

Take $z^2 - a^2 = t$

$2z \cdot dz = dt$

$\Rightarrow z \cdot dz = \frac{dt}{2}$

$\int \frac{1}{\sqrt{t}} \cdot \frac{dt}{2} = u + c \Rightarrow \frac{1}{2}\int t^{-1/2}\, dt = x+ay+c$

$\frac{1}{2}\left(\frac{t^{-1/2+1}}{-\frac{1}{2}+1}\right) = x+ay+c \Rightarrow \cancel{\frac{1}{2}}\frac{t^{1/2}}{\cancel{1/2}} = x+ay+c$

$\Rightarrow \sqrt{t} = x+ay+c \Rightarrow \boxed{\sqrt{z^2-a^2} = x+ay+c}$.

$z^2 - a^2 = (x+ay)^2 + c$ C.S. con

Type - III $f(x,y,p,q) = 0$

$f(p,x) = f(q,y) = a$ (let)

$f(p,x) = a \Rightarrow p = \phi(x)$

$f(q,y) = a \Rightarrow q = \phi(y)$

w.k.t $dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} dy$

$dz = p\cdot dx + q\cdot dy$ '$\int$' on b.s

$\int dz = \int p\cdot dx + \int q\cdot dy + c$

$\boxed{z = \int p\,dx + \int q\cdot dy + c}$ is the Complete Solution.

Eg(1) :- Solve $p^2+q^2 = x+y$

Sol:- Given P.D.E (Partial Differential equation) is $p^2+q^2 = x+y$

$\cancel{p=q}$ $p^2-x = -q^2+y = a$ (say)

Take $p^2-x = a \Rightarrow p^2 = a+x \Rightarrow p = \sqrt{a+x}$

$-q^2+y = a \Rightarrow q^2 = y-a \Rightarrow q = \sqrt{y-a}$

we have $dz = p\cdot dx + q\cdot dy$

$dz = \sqrt{a+x}\cdot dx + \sqrt{y-a}\cdot dy$ ('$\int$' on b.s)

$\int dz = \int \sqrt{a+x}\cdot dx + \int \sqrt{y-a}\cdot dy$

$z = \dfrac{(a+x)^{1/2+1}}{\frac{1}{2}+1} + \dfrac{(y-a)^{1/2+1}}{1/2+1} + c \Rightarrow 3z = 2\left[(a+x)^{3/2} + (y-a)^{3/2}\right] + c.$

Eg(2):- solve $p^2 y(1+x^2) = qx^2$

Sol:- $p^2 y(1+x^2) = qx^2 \Rightarrow \dfrac{p^2(1+x^2)}{x^2} = \dfrac{q}{y} = a$ (say)

$p^2\left(\dfrac{1+x^2}{x^2}\right) = a \Rightarrow p^2 = \dfrac{ax^2}{1+x^2} \Rightarrow \boxed{p = \dfrac{\sqrt{a}\,x}{\sqrt{1+x^2}}}$

$q/y = a \Rightarrow \boxed{q = ay}$

we have $dz = p\cdot dx + q\cdot dy$ ('$\int$' on b.s)

$\int dz = \int \dfrac{\sqrt{a}\,x}{\sqrt{1+x^2}}\cdot dx + \int ay + c$

$z = \sqrt{a}\int \dfrac{dt}{2}\cdot\dfrac{1}{\sqrt{t}} + \dfrac{ay^2}{2} + c$

$1+x^2 = t$

$2x\cdot dx = dt$

$x\cdot dx = \dfrac{dt}{2}$

$z = \frac{\sqrt{a}}{2} \frac{\sqrt{t}}{1/2} + \frac{ay^2}{2} + c$

$2z = 2\sqrt{at} + ay^2 + c$ is the required Complete Soln.

Type-IV The eqation of the form $z = px + qy + f(p,q)$

This is known as Clairaut's equation.

Eg(1): Solve $pqz = p^2(xq + p^2) + q^2(yp + q^2)$

Sol: The given equation can be rewritten as

$$z = \frac{p^2(xq + p^2)}{pq} + \frac{q^2(yp + q^2)}{pq}$$

$$z = \frac{p^2xq}{pq} + \frac{p^4}{pq} + \frac{q^2yp}{pq} + \frac{q^4}{pq}$$

$$z = px + p^3 + qy + q^3$$

$$z = px + qy + \frac{p^3}{q} + \frac{q^3}{p}$$ which is the form $z = px + qy + f(p,q)$

put $p = a$, $q = b$ then the complete solution is

$$\boxed{z = ax + by + \frac{a^3}{b} + \frac{b^3}{a}}$$

Module - 6

Applications of Partial Differential Equations

Homogenous Linear Partial Differential Equations of Second and Higher Order with Constant coeff

The general form is $\frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \cdot \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \cdot \partial y^2}$

$+ \cdots + a_{n-1} \frac{\partial^n z}{\partial x \cdot \partial y^{n-1}} + a_n \frac{\partial^n z}{\partial y^n} = f(x,y)$

where $a_1, a_2, \cdots a_{n-1}, a_n$ are constant coefficients and $f(x,y)$ is a function of 'x' and 'y'.

The Symbolic ~~to~~ from put $\frac{\partial}{\partial x} = D$; $\frac{\partial}{\partial y} = D'$

$\therefore$ The equation becomes $D^n + a_1 D^{n-1} + a_2 D^{n-2} \cdot D'^2 +$

$\cdots + a_{n-1} D \cdot D'^{n-1} + a_n D'^n] z = f(x,y)$

Complete soln $=$ C.F + P.I

$\therefore$ To find Complimentary function (CF)

Take $D = m$, $D' = 1$ then the equation becomes

$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_{n-1} m + a_n] z = f(x,y)$

Auxillary equation (AE) is

$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots a_{n-1} m + a_n = 0$

The roots are $m = m_1, m_2, \cdots m_n$

Case (I) :- If $m_1, m_2, m_3 \ldots m_n$ are real and different roots.

Then $C.F = f_1(y+m_1x) + f_2(y+m_2x) + \ldots$
$f_n(y+m_nx)$

Case (II) :- If $m_1 = m_2, m_3, m_4 \ldots m_n$ are real roots

then $C.F = f_1(y+m_1x) + x f_2(y+m_1x) +$
$f_3(y+m_3x) + \ldots + f_n(y+m_nx)$

Case (III) :- If $m_1 = m_2 = m_3$ then

$CF = f_1(y+m_1x) + x f_2(y+m_1x) + x^2 f_3(y+m_1x)$

To find particular Integral (PI)

$$PI = \frac{e^{ax+by}}{f(D,D')} = \frac{e^{ax+by}}{f(a,b)} \quad \text{If } f(a,b) \neq 0$$

Eg(1) :- Solve $\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial x \cdot \partial y} - 5\frac{\partial^2 z}{\partial y^2} = e^{x+ay}$

Sol :- put $\frac{\partial}{\partial x} = D$ and $\frac{\partial}{\partial y} = D'$ then the Symbolic form is $(D^2 + 4DD' - 5D'^2)z = e^{x+ay}$

$\therefore$ To find Complimantary function (CF)

Take $D = m$, $D' = 1$ the Auxillary equation is

$m^2 + 4m - 5 = 0$. $\left[\frac{-4 \pm \sqrt{16+20}}{2} = \frac{-4 \pm 6}{2} = -2 \pm 3 = -5, 1\right]$

The roots are $m = -5, 1$

$CF = f_1(y+x) + f_2(y-5x)$

To find particular integral (PI)

$$PI = \frac{e^{x+2y}}{D^2+4DD'-5D'^2} = \frac{e^{x+2y}}{1^2+4(1)(2)-5(2)^2}$$

$$= \frac{-e^{x+2y}}{11}$$

$$CS = CF + PI \Rightarrow \boxed{z = f_1(y+x) + f_2(y-5x) - \frac{e^{x+2y}}{11}}$$

Eg(2):- Solve $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \cdot \partial y} + 4 \cdot \frac{\partial^3 z}{\partial x \cdot \partial y^2} = e^{2x+y}$

Sol: put $\frac{\partial}{\partial x} = D$, $\frac{\partial}{\partial y} = D'$ then symbolic form is

$$[D^3+4D^2D'+4DD'^2]z = e^{2x+y}$$

Take $D=m$, $D'=1$, then the C.F is

$$m^3+4m^2+4m=0 \Rightarrow m(m^2+4m+4)=0$$

$$m = 0, -2, -2$$

$$C.F = f_1(y) + f_2(y-2x) + x f_3(y-2x)$$

To find P.I

$$PI = \frac{e^{2x+y}}{D^3+4D^2D'+4DD'^2} = \frac{e^{2x+y}}{2^3+4(2^2)(1)+4(2)(1)^2}$$

$$= \frac{e^{2x+y}}{8+16+8} = \frac{e^{2x+y}}{32}$$

$CS = CF + PI$

$$\boxed{z = f_1(y) + f_2(y-2x) + x f_3(y-2x) + \frac{e^{2x+y}}{32}}$$ //

Type - II

$$PI = \frac{\sin(ax+by) \text{ or } \cos(ax+by)}{f(D^2, DD', D'^2)}$$

$$= \frac{\sin(ax+by) \text{ or } \cos(ax+by)}{f(-a^2, -ab, -b^2)} \text{ if } f(-a^2, -ab, -b^2) \neq 0.$$

Eg (1) Solve $\frac{\partial^3 z}{\partial x^2} - 4 \cdot \frac{\partial^2 z}{\partial x \cdot \partial y} + 4 \frac{\partial^2 z}{\partial y^2} = 2\sin(3x+2y)$

Sol: put $\frac{\partial}{\partial x} = D$, $\frac{\partial}{\partial y} = D'$ then the Symbolic form is $(D^2 - 4DD' + 4D'^2) z = 2\sin(3x+2y)$

Take $D = m$, , $D' = 1$ then the A.E is

$m^2 - 4m + 4 = 0 \Rightarrow m = 2, 2$

$C.F = f_1(y+2x) + x f_2(y+2x)$

$$PI = \frac{2\sin(3x+2y)}{D^2 - 4DD' + 4D'^2} = \frac{2\sin(3x+2y)}{-3^2 - 4(-3(2)) + 4(-2^2)}$$

$$= \frac{2\sin(3x+2y)}{-1} = -2\sin(3x+2y)$$

$\therefore$ C.S = CF + P.I

$z = f_1(y+2x) + x f_2(y+2x) - 2\sin(3x+2y).$

Eg(2) :- Solve $(D^3+D^2D'-DD'^2-D'^3)z = \cos(x+2y)$

Sol: put $D=m$, $D'=1$ then A.E is

$m^3+m^2-m-1=0 \Rightarrow \cancel{m(m^2+m-} \Rightarrow \cancel{(m-1)^3}=0$

$\Rightarrow$ The roots are $m = 1, -1, -1$

$CF = f_1(y+x) + f_2(y-x) + x f_3(y-x)$

(or)

$CF = f_1(y-x) + x f_2(y-x) + f_3(y+x)$

$$PI = \frac{\cos(x+2y)}{D^3+D^2D'-DD'^2-D'^3} = \frac{\cos(x+2y)}{D^2\cdot D + D^2\cdot D' - DD'^2 - D'^2\cdot D'}$$

$$= \frac{\cos(x+2y)}{(-1^2)D + (-1^2)D' - D(-2^2) - (-2^2)\cdot D'} = \frac{\cos(x+2y)}{3D+3D'}$$

$$= \frac{\cos(x+2y)}{3(D+D')}\cdot\left(\frac{D-D'}{D-D'}\right) = \frac{(D-D')\cos(x+2y)}{3(D^2-D'^2)}$$

$$= \frac{(D-D')\cos(x+2y)}{3[(-1^2)-(-2^2)]} = \frac{[D\cos(x+2y)] - D'[\cos(x+2y)]}{9}$$

$$\left[\because D = \frac{\partial}{\partial x},\quad D' = \frac{\partial}{\partial y}\right]$$

$$= \frac{-\sin(x+2y) + 2\sin(x+2y)}{9}$$

$$= \frac{\sin(x+2y)}{9}$$

$\therefore$ CS = CF + PI

$$z = f_1(y-x) + x f_2(y-x) + f_3(y+x) + \frac{\sin(x+2y)}{9}$$

Eg(3) Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y + e^{3x+4y}$

Sol:- $\boxed{2\sin A \cos B = \sin(A+B) - \sin(A-B)}$

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{2}(2\sin x \cos 2y) + e^{3x+4y}$$

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{2}\left[\sin(x+2y) - \sin(x-2y)\right] + e^{3x+4y}$$

put $\frac{\partial}{\partial x} = D$; $\frac{\partial}{\partial y} = D'$ then the Symoblic form is

$$(D^2 - DD')z = \frac{1}{2}\sin(x+2y) - \frac{1}{2}\sin(x-2y) + e^{3x+4y}$$

Take $D = m$, $D' = 1$ then A.E is $m^2 - m = 0$

$$m(m-1) = 0$$
$$m = 0, 1$$

$$CF = f_1(y) + f_2(y+x)$$

$$PI_1 = \frac{\frac{1}{2}\sin(x+2y)}{D^2 - DD'} = \frac{\frac{1}{2}\sin(x+2y)}{(-1^2) - (-1(2))} = +\frac{1}{2}\sin(x+2y)$$

$$PI_2 = \frac{-\frac{1}{2}\sin(x-2y)}{D^2 - DD'} = \frac{-\frac{1}{2}\sin(x-2y)}{-1^2 - (-1(-2))} = +\frac{1}{6}\sin(x-2y)$$

$$PI_3 = \frac{e^{3x+4y}}{D^2 - D \cdot D'} = \frac{e^{3x+4y}}{3^2 - 3(4)} = \frac{-e^{3x+4y}}{3}$$

$$CS = CF + PI_1 + PI_2 + PI_3$$

$$z = f_1(y) + f_2(y+x) + \frac{1}{2}\sin(x+2y) + \frac{\sin(x-2y)}{6} - \frac{e^{3x+4y}}{3}$$

Type-III $PI = \dfrac{x^m y^n}{f(D,D')} = \dfrac{x^m y^n}{f\left(\frac{D}{D'}\right)}$ (if $m<n$)

$= \dfrac{x^m y^n}{f\left(\frac{D'}{D}\right)}$ if $m>n$

Eg(1): Solve $\dfrac{\partial^3 z}{\partial x^2} - 2\dfrac{\partial^3 z}{\partial x^2 \partial y} = 3x^2y + 2e^{2x}$

Sol: The Symbolic form $(D^3 - 2D^2D')z = 3x^2y + 2e^{2x}$

put $D=m,\ D'=1$ then AE is $m^3 - 2m^2 = 0$

$m^2(m-2) = 0$ ['x' power is greater than 'y']

$\Rightarrow m = 0, 0, 2$

$CF = f_1(y) + x f_2(y) + f_3(y+2x)$

$PI_1 = \dfrac{3x^2y}{D^3 - 2D^2D'} = \dfrac{3x^2y}{D^3\left(1 - \frac{2D'}{D}\right)}$ [$\because 2>1$, $m>n$]

$= \dfrac{3}{D^3}\left[1 - \dfrac{2D'}{D}\right]^{-1} x^2y$

$= \dfrac{3}{D^3}\left[1 + \dfrac{2D'}{D}\right]x^2y = \dfrac{3}{D^3}\left[x^2y + \dfrac{2D'}{D}(x^2y)\right]$

$= \dfrac{3}{D^3}\left[x^2y + \dfrac{2}{D}(x^2)\right] = \dfrac{3}{D^3}\left[x^2y + \dfrac{2x^3}{3}\right] = \dfrac{3}{D^2}\left[\dfrac{x^3y}{3} + \dfrac{2x^4}{12}\right]$ ($\frac{2x^4}{12}$ corrected to $\frac{x^4}{6}$)

$= \dfrac{3}{D}\left[\dfrac{x^4y}{12} + \dfrac{x^5}{30}\right] = 3\left[\dfrac{x^5y}{60} + \dfrac{x^6}{180}\right] = \dfrac{x^5y}{20} + \dfrac{x^6}{60}$

PI_2 $\frac{2e^{2x}}{D^3-2D^2D'} = \frac{2e^{2x}}{2^3-2(2)^2(0)} = \frac{2}{8}e^{2x} = \frac{e^{2x}}{4}$

$CS = CF + PI_1 + PI_2$

$z = f_1(y) + x f_2(y) + f_3(y+2x) + \frac{x^5 y}{20} + \frac{x^6}{60} + \frac{e^{2x}}{4}$.

Eg(2) :- Solve $[D^2 + DD' - 6D'^2]z = xy^2$

Sol: AE is $m^2+m-6=0$ where $D=m$; $D'=1$

The roots are $m = 2, -3$

$CF = f_1(y-3x) + f_2(y+2x)$

$PI = \frac{xy^2}{D^2+DD'-6D'^2} = \frac{xy^2}{D'^2\left[\frac{D^2}{D'^2}+\frac{D}{D'}-6\right]}$

$= \frac{xy^2}{-6D'^2\left[1-\left(\frac{D}{6D'}+\frac{D^2}{6D'^2}\right)\right]} = \frac{-1}{6D'^2}\left[1-\left(\frac{D}{6D'}+\frac{D^2}{6D'^2}\right)\right]^{-1} xy^2$

$= \frac{-1}{6D'^2}\left[1+\frac{D}{6D'}+\cancel{\frac{D^2}{6D'^2}}\right]xy^2 = \frac{-1}{6D'^2}\left[xy^2+\frac{y^2}{6D'}\right]$

$= \frac{-1}{6D'^2}\left[xy^2+\frac{y^3}{18}\right] = \frac{-1}{6D'}\left[\frac{xy^3}{3}+\frac{y^4}{72}\right] = \frac{-1}{6}\left[\frac{xy^4}{12}+\frac{y^5}{360}\right]$

$CS = CF + PI$

$z = f_1(y-3x) + f_2(y+2x) - \frac{1}{6}\left[\frac{xy^4}{12}+\frac{y^5}{360}\right]$

('x' power is less than 'y' power)

Common Method for finding every Model 'PI'

6/10/2020
TUESDAY

$PI = \frac{f(x,y)}{f(D,D')}$ seperate the AE into parts

$= \frac{1}{(D-mD')} f(x,y) = \int f(x, c-mx)\, dx$
where $y = c - mx$

Eg:- Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \cdot \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \cos x$

Sol:- Take $\frac{\partial}{\partial x} = D$ & $\frac{\partial}{\partial y} = D'$ then Symbolic form is

$(D^2 + DD' - 6D'^2) z = y \cos x$

AE is $D^2 + DD' - 6D'^2 = 0$ $\left[\text{here put } D = m,\ D' = 1\right]$

$m^2 + m - 6 = 0$

$m = -3, 2$

$CF = f_1(y-3x) + f_2(y+2x)$

$PI = \frac{y\cos x}{D^2 + DD' - 6D'^2} = \frac{y \cos x}{(D+3D')(D-2D')}$

$= \frac{1}{D+3D'}\left[\frac{y\cos x}{D-2D'}\right] = \frac{1}{D+3D'}\int (c-2x)\cos x \cdot dx$ $\left[\because y = c-2x,\ y = c - mx,\ (m=2)\right]$

$= \frac{1}{D+3D'}\left[(c-2x)\sin x - (-2)(-\cos x)\right]$

$= \frac{1}{D+3D'}\left[y \sin x - (2\cos x)\right] = \int \left[(c+3x)\sin x - 2\cos x\right] dx$
$(\because y = c+3x)$

$= (c+3x)(-\cos x) - 3(-\sin x) - 2\sin x$

$= -y\cos x + \sin x$

e

$CS = CF + PI$

$$\boxed{z = f_1(y-3x) + f_2(y+2x) - y\cos x + \sin x}.$$

Eg(2) :- Solve $4\cdot\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 16\log(x+2y)$

Sol:- Take $\frac{\partial}{\partial x} = D$; $\frac{\partial}{\partial y} = D'$ then symbolic form is

$(4D^2 - 4DD' + D'^2)z = 16\log(x+2y)$

AE is $4D^2 - 4DD' + D'^2 = 0$ put $D=m, D'=1$

$4m^2 - 4m + 1 = 0 \Rightarrow m = \frac{1}{2}, \frac{1}{2}$

$CF = f_1(y+\frac{1}{2}x) + x f_2(y+\frac{1}{2}x)$

$$PI = \frac{16\log(x+2y)}{4D^2 - 4DD' + D'^2} = \frac{\cancel{16}^{4}\log(x+2y)}{\cancel{4}(D-\frac{1}{2}D')(D-\frac{1}{2}D')}$$

$$= \frac{\cancel{16}\,4}{D-\frac{1}{2}D}\left[\int \log(x+2(c-\tfrac{1}{2}x))\,dx\right] \quad \begin{bmatrix} \because y = c - mx \\ y = c - \frac{1}{2}x \end{bmatrix}$$

$$= \frac{\cancel{16}\,4}{D-\frac{1}{2}D'}\int \log 2c \cdot dx = \frac{\cancel{16}^{4}\, x\log 2c}{D-\frac{1}{2}D'} \qquad c = y + \frac{1}{2}x$$

$$= \frac{\cancel{16}^{4}\, x\log 2(y+\frac{1x}{2})}{D-\frac{1}{2}D'} = \frac{4x\log(x+2y)}{D-\frac{1}{2}D'}$$

$$= 4\int x\log[x+2(c-\tfrac{1}{2}x)]\,dx \longrightarrow \begin{bmatrix} y = c - \frac{1}{2}x \\ \Rightarrow c = y + \frac{1}{2}x \end{bmatrix}$$

$$= 4\int x\log 2c \cdot dx = (\cancel{4}^{2}\log 2c)\frac{x^2}{\cancel{2}} = 2x^2\log(x+2y)$$

$CS = CF + PI$

$$\boxed{z = f_1\left(y + \tfrac{1}{2}x\right) + x f_2\left(y + \tfrac{1}{2}x\right) + 2x^2 \log(x+2y)}$$

Eg(3) :- Solve $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \cdot \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2 + x\sin y$

Sol: AE is $D^2 + 2DD' + D'^2 = 0$ (put $D = m$, $D' = 1$)

Auxiliary Eqn

$m^2 + 2m + 1 = 0 \Rightarrow m = -1, -1$

$C.F = f_1(y-x) + x f_2(y-x)$

$$PI = \frac{x^2 + xy + y^2 + x\sin y}{D^2 + 2DD' + D'^2} = \frac{x^2 + xy + y^2 + x\sin y}{(D+D')(D+D')}$$

$$= \frac{1}{D+D'}\left[\int \left[x^2 + x(c-x) + (c-x)^2 + x\sin(c-mx)\right] dx\right] \quad \begin{pmatrix} y = c - mx \\ y = c - x \\ c = y + x \end{pmatrix}$$

$$= \frac{1}{D+D'}\left[\cancel{\frac{x^3}{3} + \frac{cx^2}{2}} \int \left(x^2 + cx - x^2 + (c-x)^2 + x\sin(c-x)\right] dx\right.$$

$$= \frac{1}{D+D'}\left[\frac{cx^2}{2} + \frac{(c-x)^3}{3(-1)} + x\left(\frac{-\cos(c-x)}{-1}\right) - 1\left(\frac{-\sin(c-x)}{(-1)^2}\right)\right]$$

$$= \frac{1}{D+D'}\left[\frac{(y+x)x^2}{2} - \frac{y^3}{3} + x\cos y + \sin y\right] \quad \begin{pmatrix} y = c - x \\ c = y + x \end{pmatrix}$$

$$= \int \left[(c - \cancel{x} + \cancel{x})\frac{x^2}{2} - \frac{(c-x)^3}{3} + x\cos(c-x) + \sin(c-x)\right] dx$$

$$= \frac{cx^3}{6} - \frac{(c-x)^4}{4(3)(-1)} + \frac{x\sin(c-x)}{-1} - \frac{(-\cos(c-x))}{(-1)^2} +$$

$$\left(\frac{\cos(c-x)}{-1}\right)$$

$$= \frac{(y+x)x^3}{6} + \frac{y^4}{12} - x\sin y + 2\cos y$$

$CS = CF + PI$

$$\boxed{z = f_1(y-x) + x f_2(y-x) + \frac{(y+x)x^3}{6} + \frac{y^4}{12} - \sin y + 2\cos y.}$$

7/10

Solution of partial Differential equation by Method of Seperation of Variables :-

Eg(1): Solve PDE $\frac{\partial u}{\partial x} + 4.\frac{\partial u}{\partial y} = 3u$ given that $u(0,y) = 8e^{-3y}$ by using method of Seperation of Variables?

Sol: Given PDE is $\frac{\partial u}{\partial x} + 4.\frac{\partial u}{\partial y} = 3u$

Take $u = XY \rightarrow (i)$ be the required Solution

$$\frac{\partial u}{\partial x} = \frac{\partial X}{\partial x} Y \ ; \ \frac{\partial u}{\partial y} = X.\frac{\partial Y}{\partial y}$$

Substitute in the given equation we have

$$\frac{\partial X}{\partial x} Y + 4.\frac{\partial Y}{y} = 3XY$$

Seperate Variables

$$\frac{\partial X}{\partial x} Y = 3XY - 4X\frac{\partial Y}{\partial y}$$

$$\frac{\partial X}{\partial x} Y = X\left[3Y - 4\frac{\partial Y}{\partial y}\right]$$

$$\Rightarrow \frac{1}{X}.\frac{\partial X}{\partial x} = \frac{1}{Y}\left[3Y - 4\frac{\partial Y}{\partial y}\right] = k \text{ (say)}$$

Case(1): $\frac{1}{X}.\frac{\partial X}{\partial x} = k_1$ Seperate Variables & integrate

$$\int \frac{1}{X}.\partial X = \int k_1 . \partial x + c_1$$

$\log x = k_1 x + \log c_1$ $\qquad \left(\because x = \log e^{x}\right)$

$\Rightarrow \log x = k_1 x + \log c_1 = \log e^{k_1 x} + \log c_1$

$\cancel{\log} x = \cancel{\log} c_1 e^{k_1 x} \Rightarrow \boxed{x = c_1 e^{k_1 x}}$

Case (ii):

$\frac{1}{Y}\left[3Y - 4 \cdot \frac{\partial Y}{\partial y}\right] = k_1$

$\frac{3Y}{Y} - \frac{4}{Y}\frac{\partial Y}{\partial y} = k_1 \Rightarrow 3 - k_1 = \frac{4}{Y} \cdot \frac{\partial Y}{\partial y}$

Seperate variable and integrate

$\int \frac{3-k_1}{4} \cdot \partial y = \int \frac{\partial Y}{Y} + c_2$

$\left(\frac{3-k_1}{4}\right) y = \log Y + \log c_2 \Rightarrow \log_e^{\left(\frac{3-k_1}{4}\right) y} + \log c_2 = \log Y$

$\log Y = \log c_2 e^{\left(\frac{3-k_1}{4}\right) y} \Rightarrow \boxed{Y = c_2 e^{\left(\frac{3-k_1}{4}\right) y}}$

$\therefore$ The required Solution $u(x,y) = c_1 e^{k_1 x}\, c_2 e^{\left(\frac{3-k_1}{4}\right) y}$

$$\boxed{u(x,y) = c_1 c_2 e^{k_1 x + \left(\frac{3-k_1}{4}\right) y}}$$

Given $u(0,y) = 8e^{-3y}$

put $x = 0$ in eq(2) $u(0,y) = c_1 c_2 e^{\left(\frac{3-k_1}{4}\right) y}$

$8e^{-3y} = c_1 c_2 e^{\left(\frac{3-k_1}{4}\right) y}$

Comparing we have $c_1 c_2 = 8$, $\frac{3-k_1}{4} = -3$

$\boxed{k_1 = 15}$

$\therefore$ Comple soln. is $\boxed{u(x,y) = 8e^{15x-3y}}$

Eg(2):- Solve by Method of Seperation of Variables $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \longrightarrow (1)$

Sol:- The required Solution is $z = XY$

$\frac{\partial z}{\partial x} = \frac{\partial X}{\partial x} Y$; $\frac{\partial z}{\partial y} = X \cdot \frac{\partial X}{\partial y}$; $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 X}{\partial x^2} \cdot Y$;

Substitute in (1)

$\frac{\partial^2 X}{\partial x^2} Y - 2 \cdot \frac{\partial X}{\partial x} Y + X \cdot \frac{\partial Y}{\partial y} = 0 \Rightarrow Y\left(\frac{\partial^2 X}{\partial x^2} - \frac{2\partial X}{\partial x}\right) = -X \cdot \frac{\partial Y}{\partial y}$

$\frac{1}{X}\left(\frac{\partial^2 X}{\partial x^2} - \frac{2\partial X}{\partial x}\right) = -\frac{1}{Y} \frac{\partial Y}{\partial y} = k$ (Say)

Case (1):- $\frac{1}{X}\left[\frac{\partial^2 X}{\partial x^2} - 2\frac{\partial X}{\partial x}\right] = k$

$\Rightarrow \frac{\partial^2 X}{\partial x^2} - 2 \cdot \frac{\partial X}{\partial x} = kX \Rightarrow \frac{\partial^2 X}{\partial x^2} - 2 \cdot \frac{\partial X}{\partial x} - kX = 0$

; put $\frac{\partial}{\partial x} = D$

$[D^2 - 2D - 4] X = 0$ A.E is $D^2 - 2D - k = 0$

$D = \frac{2 \pm \sqrt{4+4k}}{2} = \frac{2 \pm 2\sqrt{1+k}}{2} = 1 \pm \sqrt{1+k}$

$X = c_1 e^{(\sqrt{1+k}+1)} + c_2 e^{(1+\sqrt{1+k})x}$

Case (ii) $\frac{1}{Y} \cdot \frac{\partial Y}{\partial y} = k$ $\Rightarrow$ Seperate Variable and integrate

$$\int \frac{1}{Y} \cdot \partial Y = -k\, dy + c_3 \Rightarrow \log Y = \log_e e^{-ky} + \log c_3$$

$$Y = \left(e^{-ky}\right) c_3.$$

$\therefore$ The req. soln. is $z = c_1 e^{1-\sqrt{1+kx}} + c_2 e^{1+\sqrt{1+kx}} + c_3 e^{-ky}$

Eg. Solve the Wave Equation in one dimension by the Method of seperation of Variables. Give all possible

$$\frac{\partial^2 y}{\partial t^2} = c^2 \cdot \frac{\partial^2 y}{\partial x^2} \text{ (wave eqn.)}$$

$$\frac{\partial u}{\partial t} = c^2 \cdot \frac{\partial^2 u}{\partial x^2} \text{ (heat eqn.)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ (laplace eqn.)}$$

Sol. The wave equation in one dimension $\frac{\partial^2 y}{\partial t^2} = c^2 \cdot \frac{\partial^2 y}{\partial x^2} \rightarrow$ (i)

let us take $y = XT \longrightarrow$ (ii)

$$\frac{\partial y}{\partial t} = X \cdot \frac{\partial T}{\partial t} \qquad \frac{\partial^2 y}{\partial t^2} = X \cdot \frac{\partial^2 T}{\partial t^2}$$

$$\frac{\partial y}{\partial x} = \frac{\partial X}{\partial x} T \quad ; \quad \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 X}{\partial x^2} T$$

eq (1) becomes $X \cdot \frac{\partial^2 T}{\partial t^2} = c^2 \cdot \frac{\partial^2 X}{\partial x^2} T$

Seperate Variables

$$\frac{1}{X} \cdot \frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2 T} \frac{\partial^2 T}{\partial t^2} = k \text{ (say)}$$

Case(i) If k is a positive integer i.e $k = p^2$

$$\frac{1}{X} \cdot \frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2 T} \frac{\partial^2 T}{\partial t^2} = p^2$$

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = p^2 \Rightarrow \frac{\partial^2 X}{\partial x^2} = p^2 X \Rightarrow \frac{\partial^2 X}{\partial x^2} - p^2 X = 0$$

put $\frac{\partial}{\partial x} = D$ then $(D^2 - p^2) X = 0$

AE is $D^2 - p^2 = 0 \Rightarrow D^2 = p^2 \Rightarrow D = \pm p$

$$X = c_1 e^{-px} + c_2 e^{px}$$

Take $\frac{1}{c^2 T} \frac{\partial^2 T}{\partial t^2} = p^2 \Rightarrow \frac{\partial^2 T}{\partial t^2} = p^2 c^2 T$

$\Rightarrow \frac{\partial^2 T}{\partial t^2} - p^2 c^2 T = 0$ then we have

$(D^2 - p^2 c^2) T = 0$

AE is $D^2 - p^2 c^2 = 0 \Rightarrow D = \pm pc$

$$T = c_3 e^{-pct} + c_4 e^{pct}$$

$\therefore$ The required Solution is $\boxed{y = (c_1 e^{-px} + c_2 e^{px})(c_3 e^{-pct} + c_4 e^{pct})}$ $\rightarrow$ (iii)

case(ii) if k is $-$ve integer ie $k = -p^2$

$$\frac{1}{X} \cdot \frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2 T} \frac{\partial^2 T}{\partial t^2} = -p^2$$

$\therefore$ Take $\frac{1}{X} \cdot \frac{\partial^2 X}{\partial x^2} = -p^2 \Rightarrow \frac{\partial^2 X}{\partial x^2} = -p^2 X$

$$\Rightarrow \frac{\partial^2 X}{\partial x^2} + p^2 X = 0$$

we have $[D^2+p^2]X=0$

AE is $D^2+p^2=0 \Rightarrow D=\pm pi$

$X = c_1\cos px + c_2\sin px$ $\boxed{\alpha \pm \beta i = e^{\alpha x}[c_1\cos\beta x + c_2\sin\beta x]}$

formation

$\frac{1}{c^2T}\frac{\partial^2 T}{\partial t^2} = -p^2 \Rightarrow \frac{\partial^2 T}{\partial t^2} = -p^2c^2 \Rightarrow \frac{\partial^2 T}{\partial t^2} + p^2c^2T = 0$

$(D^2+p^2c^2)T=0$ A.E is $D^2+p^2c^2=0$

$\Rightarrow D = \pm pci$

$T = c_3\cos pc + c_4\sin pc$

$\therefore$ The req. soln. is $\boxed{y = (c_1\cos px + c_2\sin px)(c_3\cos pct + c_4\sin pct)}$

$\rightarrow$ eq (iv)

Case (iii) :- If 'K' is a zero integer i.e $K=0$

$\frac{1}{X}\cdot\frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2T}\frac{\partial^2 T}{\partial t^2} = 0$

Take $\frac{1}{X}\cdot\frac{\partial^2 X}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 X}{\partial x^2} = 0 \Rightarrow D^2X = 0$

$A\cdot E$ is $D^2=0 \Rightarrow D=0,0$

$X = (c_1+c_2x)e^{0x} = c_1+c_2x$

$\frac{1}{c^2T}\cdot\frac{\partial^2 T}{\partial t^2} = 0 \Rightarrow \frac{\partial^2 T}{\partial t^2} = 0 \Rightarrow D^2T = 0$

$A\cdot E$ is $D^2=0 \Rightarrow D=0,0$

$T = c_3 + c_4x$

$\boxed{\text{The required Solution is } y = (c_1+c_2x)(c_3+c_4x)}$ $\rightarrow$ eq (v)

(5)

Eg(2):- Solve the laplace equation in two dimension by method of seperation of variables. Also give all possible solution?

Sol: laplace equation in two dimensions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \rightarrow (1)$$

Let us take $u = XY$

$$\frac{\partial u}{\partial x} = \frac{\partial X}{\partial x} \cdot Y \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 X}{\partial x^2} Y, \quad \frac{\partial u}{\partial y} = X \cdot \frac{\partial Y}{\partial y},$$

$$\frac{\partial^2 u}{\partial y^2} = X \cdot \frac{\partial^2 Y}{\partial y^2}$$

equation becomes $\frac{\partial^2 X}{\partial x^2} Y + X \cdot \frac{\partial^2 Y}{\partial y^2} = 0$

Seperate variables

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \frac{-1}{Y} \cdot \frac{\partial^2 Y}{\partial y^2} = k \text{ (say)}$$

Case (1)

Eg (3):- Solve the heat equation in one dimension by using the Method of Seperation of Variables. Give all possible Solutions?

Sol: Heat equation in one dimension is

$$\frac{\partial u}{\partial t} = c^2 \cdot \frac{\partial^2 u}{\partial x^2} \rightarrow (1)$$

18/10/2020

Module - 7 :- Fourier Transforms

Complex Fourier Transforms

* The Fourier transform of $f(x)$ is given by

$$F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x).e^{isx}.dx$$

* The inverse Fourier transform of $f(x)$ is given by

$$f(x) = F^{-1}\{F(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s).e^{-isx}.ds$$

* The Fourier Cosine Transform of $f(x)$ is given by

$$F_c(s) = F_c\{f(x)\} = \int_0^{\infty} f(x).\cos sx.dx$$

* The Inverse Fourier Cosine Transform of $f(x)$ is

$$F^{-1}\{F_c(s)\} = f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(s)\cos sx\, ds$$

* The Fourier Sine transform of $f(x)$ is given by

$$F_s\{f(x)\} = F_s(s) = \int_0^{\infty} f(x)\sin sx.dx$$

* The inverse Fourier Sine transform of $f(x)$ is

$$F^{-1}\{F_s(s)\} = \frac{2}{\pi}\int_0^{\infty} F_s(s)\sin sx.ds$$

Eg(1):- Find the Fourier Transform of

$$f(x) = \begin{cases} 1 & \text{for } |x|<1 \\ 0 & \text{for } |x|>1 \end{cases}$$ Hence evaluate $\int_0^{\infty} \frac{\sin x}{x}.dx$

Sol :- W.K.T the F.T of $f(x)$ is

$$F\{f(x)\} = F(s) = \int_{-\infty}^{\infty} f(x).e^{isx}.dx = \int_{-1}^{1} 1.e^{isx}.dx$$

$$= \left[\frac{e^{isx}}{is}\right]_{-1}^{1} = \frac{e^{is}}{is} - \frac{e^{-is}}{is} = \frac{e^{is} - e^{-is}}{is}$$

$$= \frac{2}{s}\left[\frac{e^{is} - e^{-is}}{2i}\right] = \frac{2}{s}\sin s$$

Inverse Fourier Transform of $f(x)$ is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\cdot e^{-isx}\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{s}\sin s\, e^{-isx}\cdot ds$$

put $x=0$

put $x=0$, $f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin s}{s}\, e^{is(0)}\cdot ds$

$$1 = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin s}{s}\,ds \Rightarrow 1 = \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin s}{s}\cdot ds$$

$$\Rightarrow \int_{0}^{\infty}\frac{\sin s}{s}\cdot ds = \frac{\pi}{2}$$

put $s=x \Rightarrow ds = dx$

$$\boxed{\therefore \int_{0}^{\infty}\frac{\sin x}{x}\cdot dx = \frac{\pi}{2}} \text{ (or) } \boxed{\int_{-\infty}^{\infty}\frac{\sin x}{x}\cdot dx = \pi}$$

Eg:- Find the Fourier Cosine transform of

$$f(x) = \begin{cases} x & \text{for } 0<x<1 \\ 2-x & \text{for } 1<x<2 \\ 0 & \text{for } x>2 \end{cases}$$

Sol:- Fourier Cosine transform of $f(x)$ is

$$F_c(s) = \int_{0}^{\infty} f(x)\cos sx\,dx = \int_{0}^{1} x\cos sx\,dx + \int_{1}^{2}(2-x)\cos sx\,dx + \int_{2}^{\infty} 0\cos sx\cdot dx$$

$$= \left\{x\frac{\sin sx}{s} - 1\left(-\frac{\cos sx}{s^2}\right)\right\}_{x=0}^{1} + \left[(2-x)\frac{\sin sx}{s} - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_{x=1}^{2}$$

$$= \frac{\cancel{\sin s}}{s} + \frac{\cos s}{s^2} - 0 - \frac{1}{s^2} + 0 - \frac{\cos 2s}{s^2} - \frac{\cancel{\sin s}}{s} + \frac{\cos s}{s^2}$$

$$= \frac{2\cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2} = \boxed{\frac{2\cos s - 1 - \cos 2s}{s^2}}$$ //.

Eg:- Find Fourier Sine and Cosine transform of

$f(x) = e^{-ax}, \ x > 0$

Sol:- (i) Fourier Sine transform of $f(x)$ is

$$F_s(s) = \int_0^\infty f(x) \cdot \sin sx \cdot dx = \int_0^\infty e^{-ax} \sin sx \cdot dx$$

$$\because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx - b \sin bx\right]$$

$$= \left\{ \frac{e^{-ax}}{(-a)^2+s^2}\left[-a \sin sx - s \cos sx\right] \right\}_0^\infty$$

$$= 0 - \frac{1}{a^2+s^2}(-s) = \frac{s}{s^2+a^2} = F_s\left[e^{-ax}\right]$$

(ii) Fourier Cosine Transform of $f(x)$ is

$$F_c(s) = \int_0^\infty f(x) \cdot \cos sx \cdot dx = \int_0^\infty e^{-ax} \cdot \cos sx \cdot dx$$

$$= \left\{ \frac{e^{-ax}}{(-a)^2+s^2}\left[-a \cos sx - s \sin sx\right] \right\}_{x=0}^\infty$$

$$= 0 - \frac{1}{a^2+s^2}(-a) = \frac{a}{a^2+s^2} = F_c\left(e^{-ax}\right)$$ //.

Eg) Find the Fourier Transform of $f(x) = e^{-2(x-3)^2}$

Sol) Fourier Transform of $f(x)$ is

$$F(s) = \int_{-\infty}^{\infty} f(x) \cdot e^{isx}\, dx = \int_{-\infty}^{\infty} e^{-2(x-3)^2} e^{isx}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-2(x-3)^2 + isx} \cdot dx = \int_{-\infty}^{\infty} e^{-[2x^2-12x}$$

Eg) Find the Fourier transform of $e^{-|x|}$

Sol) Fourier Transform of $f(x)$ is

$$F[s] = \int_{-\infty}^{\infty} f(x) \cdot e^{isx} \cdot dx = \int_{-\infty}^{\infty} e^{-|x|} e^{isx}\, dx$$

$$= \int_{-\infty}^{0} e^{x} e^{isx} \cdot dx + \int_{0}^{\infty} e^{-x} e^{isx}\, dx \qquad \boxed{\because |x| = \begin{cases} -x & x<0 \\ x & x>0 \end{cases}}$$

$$= \int_{-\infty}^{0} e^{(1+is)x} \cdot dx + \int_{0}^{\infty} e^{-(1-is)x} \cdot dx$$

$$= \left[\frac{e^{(1+is)x}}{1+is}\right]_{-\infty}^{0} + \left[\frac{e^{-(1-is)x}}{-(1-is)}\right]_{0}^{\infty}$$

$$= \frac{1}{1+is} - 0 + 0 - \frac{1}{-(1-is)} = \frac{1-\cancel{is}+1+\cancel{is}}{(1+is)(1-is)}$$

$$= \frac{2}{1+s^2} \quad //.$$

Eg1 Show that Fourier transform of $e^{-x^2/2}$ is self reciprocal.

Sol:- Fourier Transform of $f(x)$ is

$$F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x) \cdot dx$$

$$F\{e^{-\frac{x^2}{2}}\} = \int_{-\infty}^{\infty} e^{isx} \cdot e^{-x^2/2} \cdot dx = \int_{-\infty}^{\infty} e^{-1/2(x^2 - 2isx)} \cdot dx$$

$$= \int_{-\infty}^{\infty} e^{-1/2\left[x^2 - 2(is)x + (is)^2 - (is)^2\right]} \cdot dx$$

$$= \int_{-\infty}^{\infty} e^{-1/2(x-is)^2} \cdot e^{-\frac{s^2}{2}} dx = e^{-s^2/2} \int_{-\infty}^{\infty} e^{-1/2(x-is)^2} \cdot dx$$

$$\left[\frac{x - is}{\sqrt{2}} = t \Rightarrow \frac{dx}{\sqrt{2}} = dt\right]$$
$$(dx = \sqrt{2}\, dt)$$

$$= e^{-\frac{s^2}{2}} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2}\, dt$$

$$= \sqrt{2} \cdot e^{-s^2/2} \sqrt{\pi} = \sqrt{2\pi}\, e^{-s^2/2}$$

$$\left[\because \int_{-\infty}^{\infty} e^{-x^2} \cdot dx = \sqrt{\pi}\right] \qquad \Rightarrow F\{e^{-x^2/2}\} = \sqrt{2\pi}\, e^{-s^2/2} \quad (\sqrt{2\pi}\, e^{-s^2/2})$$

Self reciprocal.

Eg2 Find Fourier Cosine Transform of e^{-x^2}.

Sol: W.K.T Fourier Cosine Transform of $f(x)$ is

$$F_c(s) = F_c\{f(x)\} = \int_0^{\infty} f(x) \cos sx \cdot dx$$

$$F_c\{e^{-x^2}\} = \int_0^{\infty} e^{-x^2} \cos sx\, dx = I \text{ (say)}$$

Diff w.r.t 's' on both side

$$\frac{dI}{ds} = \int_0^{\infty} e^{-x^2}(-x \sin sx)\, dx = \frac{1}{2}\int_0^{\infty} (-2xe^{-x^2}) \sin sx \cdot dx$$

$$= \frac{1}{2}\int_0^{\infty} d(e^{-x^2}) \sin sx \cdot dx = \frac{1}{2}\int_0^{\infty} \sin sx \; d(e^{-x^2})$$

$$\left[\begin{array}{l} u = \sin sx \\ du = d(e^{-x^2}) \\ du = s\cos sx \cdot dx \\ v = e^{-x^2} \end{array}\right]$$

$$\frac{dI}{ds} = \frac{1}{2}\left\{\cancel{\left[e^{-x^2}\sin sx\right]_0^{\infty}} - \int_0^{\infty} e \; s\cos sx \cdot dx\right\}$$

$$= \frac{-s}{2}\int_0^{\infty} e^{-x^2}\cos sx \cdot dx = I \cdot \left(\frac{-s}{2}\right)$$

$\frac{dI}{ds} = I \cdot \frac{-s}{2}$ Seperate Varaibles and integrate

$$\int \frac{dI}{I} = \cancel{\int ds + c} \int \frac{s}{2} \cdot ds + c \Rightarrow \log I = \frac{-s^2}{4} + \log c$$

$$\log I = \log e^{-s^2/4} + \log c$$

$$I = c e^{-s^2/4}$$

$$\boxed{F_c\{e^{-x^2}\} = c e^{-s^2/4}}.$$

Eg:- Find the Fourier Sine transform of $\frac{e^{-ax}}{x}$?

Sol:- Fourier sine transform of $f(x)$ is

$$F_s(s) = F_s\{f(x)\} = \int_0^{\infty} f(x) \sin sx \cdot dx$$

$$F_s\left\{\frac{e^{-ax}}{x}\right\} = \int_0^{\infty} \frac{e^{-ax}}{x} \sin sx \cdot dx = I \text{ (say)}$$

diff w.r.t 's' on both sides

$$\frac{dI}{ds} = \int_0^{\infty} \frac{e^{-ax}}{x}(x\cos sx)\,dx = \left\{\frac{e^{-ax}}{(-a)^2+s^2}(-a\cos sx - s\sin sx)\right\}_0^{\infty}$$

$$\frac{dI}{ds} = 0 - \frac{1}{a^2+s^2}(-a) \Rightarrow \frac{dI}{ds} = \frac{a}{s^2+a^2}$$

Seperate variables and Integrate

$$\int dI = \int \frac{a}{s^2+a^2} \cdot ds + c$$

$$I = \cancel{a} \cdot \frac{1}{\cancel{a}} \cdot \tan^{-1}\left(\frac{s}{a}\right) + c.$$

$$\Rightarrow \boxed{F_s\left\{\frac{e^{-ax}}{x}\right\} = \tan^{-1}(s/a) + c}.$$

Eg:- Find the Fourier Cosine transform of $f(x) = \frac{1}{1+x^2}$.

sol:- Fourier Cosine transform of $f(x)$ is

$$F_c(s) = F_c\{f(x)\} = \int_0^\infty f(x) \cos sx \cdot dx$$

$$F_c\left\{\frac{1}{1+x^2}\right\} = \int_0^\infty \frac{1}{1+x^2} \cos sx \, dx = I \text{ (say)} \rightarrow (i)$$

Diff w.r.t 's' on both sides

$$\frac{dI}{ds} = \int_0^\infty \frac{1}{1+x^2} (-x \sin sx) \, dx = -\int \frac{x^2 \sin sx \, dx}{x(1+x^2)}$$

$$= -\int_0^\infty \frac{(1+x^2-1) \sin sx}{x(1+x^2)} dx$$

$$\frac{dI}{ds} = -\int_0^\infty \frac{\cancel{1+x^2}}{x\cancel{(1+x^2)}} \cdot \sin sx \cdot dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)} \cdot dx$$

$$\left[\because \int_0^\infty \frac{\sin x}{x} \cdot dx = \frac{\pi}{2}\right]$$

$$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)} \cdot dx \rightarrow (ii)$$

differentiate again w.r.t 's' on both sides

$$\frac{d^2I}{ds^2} = \int_0^\infty \frac{\cancel{x} \cos sx}{\cancel{x}(1+x^2)} \cdot dx = I \Rightarrow \frac{d^2I}{ds^2} = I \Rightarrow \frac{d^2I}{ds^2} - I = 0$$

put $\frac{d}{ds} = D$

$(D^2-1)I = 0$ Auxilary Eqn. is $D^2-1=0 \Rightarrow D^2=1$

$\boxed{D = \pm 1}$

$I = c_1 e^{-s} + c_2 e^{s} \longrightarrow$ (iii)

$\frac{dI}{ds} = -c_1 e^{-s} + c_2 e^{s} \longrightarrow$ (iv)

put $s=0$ in eq(i) & eq(iii)

From eq(i) $I = \int_0^\infty \frac{1}{1+x^2} \cdot dx = \left[\tan^{-1}\right]_0^\infty = \tan^{-1}\infty - \tan^{-1}0$
$= \pi/2$

From eq(iii) $I = c_1 + c_2$

we have $c_1 + c_2 = \frac{\pi}{2} \longrightarrow$ (v)

put $s=0$ in eq(ii) & (iv)

From eq(ii) $\frac{dI}{ds} = -\frac{\pi}{2}$

From eq(iv) $\frac{dI}{ds} = -c_1 + c_2$

$\Rightarrow -c_1 + c_2 = -\frac{\pi}{2} \longrightarrow$ (vi)

Solve eq(v) & (vi)

$c_1 + c_2 = \pi/2$
$-c_1 + c_2 = -\pi/2$

$2c_2 = 0 \Rightarrow \boxed{c_2 = 0}$

$\therefore \boxed{c_1 = \pi/2}$

from eq(iii)

$\therefore I = \frac{\pi}{2} e^{-s}$

$\boxed{F_c\left\{\frac{1}{1+x^2}\right\} = \frac{\pi}{2} e^{-s}}$.

eg: Find the Fourier Sine transform of $f(x) = \frac{1}{x(x^2+a^2)}$?

sol: F.S.T of $f(x)$ is $F_s(s) = F_s\{f(x)\}$

$$= \int_0^\infty f(x) \sin sx \cdot dx$$

$$F_s\left\{\frac{1}{x(x^2+a^2)}\right\} = \int_0^\infty \frac{1}{x(x^2+a^2)} \sin sx \cdot dx = I \text{ (say)} \rightarrow (i)$$

diff w.r.t 's' on both sides

$$\frac{dI}{ds} = \int_0^\infty \frac{1}{\cancel{x}(x^2+a^2)} \cancel{x} \cos sx \cdot dx \longrightarrow (ii)$$

diff w.r.t 's' on both sides

$$\frac{d^2I}{ds^2} = \int_0^\infty \frac{-x \sin sx}{x^2+a^2} \cdot dx = -\int_0^\infty \frac{x^2 \sin sx \cdot dx}{x(x^2+a^2)}$$

$$\frac{d^2I}{ds^2} = -\int_0^\infty \frac{x^2+a^2-a^2}{x(x^2+a^2)} \sin sx \cdot dx$$

$$= -\int_0^\infty \frac{\cancel{x^2+a^2}}{x(\cancel{x^2+a^2})} \sin sx \cdot dx + a^2 \int_0^\infty \frac{\sin sx}{x(x^2+a^2)} \cdot dx$$

$$\frac{d^2I}{ds^2} = -\frac{\pi}{2} + a^2 I \Rightarrow \frac{d^2I}{ds^2} - a^2 I = \frac{-\pi}{2}$$

put $\frac{d}{ds} = D \Rightarrow (D^2 - a^2) I = -\pi/2$

A.E is $D^2 - a^2 = 0 \Rightarrow D = \pm a$

$$CF = c_1 e^{-as} + c_2 e^{as} \quad , PI = \frac{-\frac{\pi}{2} e^{0(s)}}{D^2 - a^2} = \frac{-\frac{\pi}{2}}{-a^2}$$

$$= \pi/2a^2$$

$CS = CF + PI$

$$I = c_1 e^{-as} + c_2 e^{as} + \frac{\pi}{2a^2} \longrightarrow (iii)$$

$$\frac{dI}{ds} = \cancel{c_1 + c_2} - a c_1 e^{-as} + a c_2 e^{as} \longrightarrow (iv)$$

put $s = 0$ in eq (i) & (iii)

From eq (i) $I = \int_0^\infty \frac{\sin 0x}{x(x^2+a^2)} \cdot dx = 0$

from eq (iii) $I = c_1 + c_2 + \frac{\pi}{2a^2}$ $\Rightarrow$

$$c_1 + c_2 + \frac{\pi}{2a^2} = 0$$

$$c_1 + c_2 = \frac{-\pi}{2a^2} \rightarrow (v)$$

put $s=0$ in eq(ii) $\frac{dI}{ds} = \int_0^\infty \frac{1}{x^2+a^2} \cdot dx = \left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]_0^\infty$

$$= \pi/2a$$

in eq (iv) $= \frac{dI}{ds} = -ac_1 + ac_2$

we have $-ac_1 + ac_2 = \frac{\pi}{2a} \Rightarrow -c_1 + c_2 = \frac{\pi}{2a^2} \rightarrow (vi)$

solve eq(v) & eq(vi)

$$c_1 + c_2 = \frac{-\pi}{2a^2}$$

$$-c_1 + c_2 = \pi/2a^2$$

$$\cancel{2c_2 = \pi/a^2} \Rightarrow \boxed{\cancel{c_2 = \frac{\pi}{2a^2}}}$$

$$2c_2 = 0 \Rightarrow \boxed{\begin{array}{l} c_2 = 0 \\ c_1 = \frac{-\pi}{2a^2} \end{array}}$$

$$I = -\frac{\pi}{2a^2} \cdot e^{-as} + \frac{\pi}{2a^2}$$

$$F_s\left\{\frac{1}{x(x^2+a^2)}\right\} = \frac{\pi}{2a^2}\left[1 - e^{-as}\right]$$

27/10/2020 Tuesday

Parseval's Identity For Fourier Transforms

If the Fourier transform of $f(x)$ & $g(x)$ are $F(s)$ and $G(s)$ ~~sep~~ respectively then

(i) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,\overline{G(s)}\cdot ds = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\cdot dx$$

(ii) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(s)|^2\cdot ds = \int_{-\infty}^{\infty} |f(x)|^2\cdot dx$$

where bar indicates the conjugate of function.

Ily of the Fourier Cosine transform of $f(x)$ and $g(x)$ are $F_c(s)$ and $G_c(s)$ respectively then

(i) $$\frac{2}{\pi}\int_0^{\infty} F_c(s)\,\overline{G_c(s)}\cdot ds = \int_0^{\infty} f(x)\cdot\overline{g(x)}\cdot dx$$

(ii) $$\frac{2}{\pi}\int_0^{\infty} |F_c(s)|^2\cdot ds = \int_0^{\infty} |f(x)|^2\cdot dx$$

If the Fourier Sine transform of $f(x)$ and $g(x)$ are $F_s(s)$ and $G_s(s)$ respectively then

(i) $$\frac{2}{\pi}\int_0^{\infty} F_s(s)\cdot\overline{G_s(s)}\cdot ds = \int_0^{\infty} f(x)\cdot\overline{g(x)}\cdot dx$$

(ii) $$\frac{2}{\pi}\int_0^{\infty} |F_s(s)|^2\cdot ds = \int_0^{\infty} |f(x)|^2\cdot dx$$

Eg(1):- Using Parseval's ~~Identity~~ indentities, prove

(i) $$\int_0^{\infty}\frac{1}{(a^2+t^2)(b^2+t^2)}\cdot dt = \frac{\pi}{2ab(a+b)}$$

(ii) $$\int_0^{\infty}\frac{t^2}{(t^2+1)^2}\cdot dt = \frac{\pi}{4}$$

Sol: (i) let us take $f(x) = e^{-ax}$, $g(x) = e^{-bx}$

$$F_c(s) = \frac{a}{a^2+s^2}, \quad G_c(s) = \frac{b}{b^2+s^2}$$

By parseval's identity, we have

$$\frac{2}{\pi}\int_0^\infty F_c(s)\cdot\overline{G_c(s)}\cdot ds = \int_0^\infty f(x)\overline{g(x)}\,dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{a}{a^2+s^2}\cdot\frac{b}{b^2+s^2}\cdot ds = \int_0^\infty e^{-ax}\cdot e^{-bx}\cdot dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{ab}{(a^2+s^2)(b^2+s^2)}\cdot ds = \int_0^\infty e^{-(a+b)x}\cdot dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{ab}{(a^2+s^2)(b^2+s^2)}\cdot ds = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = 0 - \frac{1}{-(a+b)}$$

$$\int_0^\infty \frac{1}{(a^2+s^2)(b^2+s^2)}\cdot ds = \frac{\pi}{2ab(a+b)} \quad \text{put } s=t, \; ds=dt$$

$$\int_0^\infty \frac{1}{(a^2+t^2)(b^2+t^2)}\cdot dt = \frac{\pi}{2ab(a+b)}$$

(ii) Let us take $f(x) = e^{-x}$

$$F_s(s) = \frac{s}{s^2+1}$$

By parseval's identity we have

$$\frac{2}{\pi}\int_0^\infty |F(s)|^2\cdot ds = \int_0^\infty |f(x)|^2\cdot dx$$

$$\frac{2}{\pi}\int_0^\infty \left(\frac{s}{s^2+1}\right)^2\cdot ds = \int_0^\infty |e^{-x}|^2\cdot dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+1)^2}\cdot ds = \int_0^\infty e^{-2x}\,dx = \left[\frac{e^{-2x}}{-2}\right]_0^\infty = 0 - \frac{1}{+(-2)}$$

$$= 1/2$$

$$\int_0^\infty \frac{s^2}{(s^2+1)^2}\cdot ds = \frac{\pi}{4}$$ put $s = t \Rightarrow ds = dt$

$$\therefore \int_0^\infty \frac{t^2}{(t^2+1)^2}\cdot dt = \pi/4$$

Application of Transforms to Boundary Value problems

$u(x,t)$

The Fourier transform of $u(x,t)$ is

$$F(u(x,t)) = \int_{-\infty}^{\infty} u(x,t)\, e^{isx}\, dx = F[u(s)]$$

The F.T of $\frac{\partial^2 u}{\partial x^2}$ is

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{isx}\, dx = \cancel{\left[e^{isx}\frac{\partial u}{\partial x} - is\, e^{isx} u\right]_{-\infty}^{\infty}} +$$

$$(is)^2 \int_{-\infty}^{\infty} u(x,t)\, e^{isx}\, dx$$

$$\boxed{F\left[\frac{\partial^2 u}{\partial x^2}\right] = -s^2 F[u(s)]}$$

Ily $F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = s\, u(0,t) - s^2 F_s[u(s)]$

$F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = -\left(\frac{\partial u}{\partial x}(0,t)\right) - s^2 F_c[u(s)]$

Eg(1) :- Determine the distribution of temperature in the Semi-infinite medium $x \geq 0$ when the end $x=0$ is maintained at '0°' and ~~temperature and~~ initial distribution of temperature is $f(x)$?

Sol: Heat equation in the Semi infinite medium is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (x>0,\ t>0)$$

Subject to boundary Condition $u(0,t) = 0$

initial Condition is $u(x,0) = f(x)$, $x>0$.

By the given boundary Condition, let us apply Fourier Sine transform on eq(i)

$$\int_0^\infty \frac{\partial u}{\partial t} \sin sx \, dx = c^2 \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx \, dx$$

$$\frac{du_s}{dt} = c^2 \left[s u(0,t) - s^2 u_s \right] \quad \boxed{(\because u(0,t)=0)}$$

$$\frac{du_s}{dt} = -c^2 s^2 u_s$$

Seperate Variables and integrate

$$\int \frac{du_s}{u_s} = -c^2 s^2 \int dt + c_1$$

$$\log u_s = -c^2 s^2 t + \log c_1$$

$$\log u_s = \log e^{-c^2 s^2 t} + \log c_1$$

$$\log u_s = \log c_1 e^{-c^2 s^2 t} \Rightarrow \boxed{u_s = c_1 e^{-c^2 s^2 t}}.$$

~~∴ The distribution temperature is~~

Initial Condition is $u(x,0) = f(x)$.

$$u_s(x,0) = c_1 e^0 \Rightarrow \boxed{f(x) = c_1}.$$

$$\boxed{\therefore u_s = f(x) e^{-c^2 s^2 t}}.$$

The temperature distribution

$$\boxed{u(x,t) = 2/\pi \int_0^\infty f(x) e^{-c^2 s^2 t} \sin sx \, ds}$$

Eg (2) :- Using Fourier transform, solve the dimentional heat eqn. $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ for a rod with insulated sides extending from $-\infty$ to ∞ and with initial condition given by $u(x,0) = f(x)$ and $u(x,t) = 0$ at $x = \pm\infty$?

Sol: Heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$, $x = \pm\infty$

Initial condition is $u(x,0) = f(x)$

Boundary condition are $u(x,t) = 0$ & $x = \pm\infty$.

Apply Fourier transform on heat equation

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{isx} dx = \alpha^2 \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{isx} dx$$

$$\frac{du(s)}{dt} = \alpha^2 (-s^2 u(s))$$

Seperate variables and integrate

$$\int \frac{du(s)}{u(s)} = \int -\alpha^2 s^2 dt + c_1$$

$$\log u(s) = -\alpha^2 s^2 t + \log c_1$$

$$\log u(s) = \log e^{-\alpha^2 s^2 t} + \log c_1$$

$$u(s) = c_1 e^{-\alpha^2 s^2 t}$$

Initial condition is $u(x,0) = f(x)$

$$f(x) = c_1 e^0 \Rightarrow \boxed{c_1 = f(x)}$$

$$\boxed{u(s) = f(x)\, e^{-\alpha^2 s^2 t}}$$

the req. temperature distribution is

$$\boxed{u(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cdot e^{-\alpha^2 s^2 t} e^{-isx} \, ds}$$

$u(x,0) = f(x)$

(6/11/2020) Friday

Ex:- An infinite string is initially at rest and the initial displacement is $f(x)$ $(-\infty < x < \infty)$. Determine the displacement $y(x,t)$ of the string.

Sol: The wave equation is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \rightarrow (1)$

The initial condition $y_t(x,0) = 0$, $y(x,0) = f(x)$
$-\infty < x < \infty$

The boundary condition are $y(-\infty,t) = y(\infty,t) = 0$

Apply Fourier transform~~ation~~ in eq(1) on both sides

$$\int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial t^2} \cdot e^{isx} \cdot dx = c^2 \int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial x^2} \cdot e^{isx} \cdot dx$$

$$\frac{d^2 y(s)}{dt^2} = c^2\left[(is)^2 y(s)\right]$$

$$\frac{d^2}{dt^2} y(s) + c^2 s^2 Y(s) = 0$$

$$(D^2 + c^2 s^2) Y(s) = 0$$

Auxilary Equation is
$D^2 + c^2 s^2 = 0$
$D^2 = -c^2 s^2 \Rightarrow D = \pm csi$

$$Y(s) = c_1 \cos cst + c_2 \sin cst \rightarrow (iii)$$

$$\frac{\partial Y(s)}{\partial t} = \cancel{-c_1 cs \cos cst + c_2 cs}$$
$$-c_1 cs \sin cst + c_2 cs \cos cst$$

$$\left.\frac{\partial Y(s)}{\partial t}\right|_{t=0} = -c_1(0) + c_2 cs$$

$$0 = c_2 cs \Rightarrow \boxed{c_2 = 0}$$

eq(iii) becomes $Y(s) = c_1 \cos cst$

$$Y(s,0) = c_1$$

$\boxed{f(x) = c_1}$ eq(iii) becomes

$$\boxed{Y(s) = f(x) \cos cst}$$

By inverse Fourier Transform, we get the required displacement of the string is

$$\boxed{y(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \cos cst \, e^{-ist} \cdot ds}$$

Eg(2):- An infinitly long string $\cancel{f(x) = 1}$ having one end at $x=0$ is initially at rest along the x-axis. The end $x=0$ is given a displace $f(t)$, $t>0$. Find the displacement of any point of the string at any time.

Sol:- Wave equation is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \longrightarrow$ (i)

Conditions are $y(x,0) = \cancel{0}\ f(t)$; $y_t(x,0) = 0$

$y(0,t) = \cancel{f(t)}\ 0$

By Fourier Sine transform we have

$$\int_0^\infty \frac{\partial^2 y}{\partial t^2} \sin sx \, dx = c^2 \int_0^\infty \frac{\partial^2 y}{\partial x^2} \sin sx \, dx$$

$$\frac{d^2 y}{dt^2}(s) = c^2 \left[s\, y(\cancel{0},t) - s^2 y_s(s) \right]$$

$$\frac{d^2 y_s(s)}{dt^2} + c^2 s^2 y_s(s) = 0 \Rightarrow \left[D^2 + c^2 s^2 \right] y_s(s) = 0$$

AE is $D^2+c^2s^2=0 \Rightarrow D^2=-c^2s^2 \Rightarrow D=\pm csi$

$Y_s(s) = A\cos cst + B\sin cst \rightarrow$ (ii)

$$\frac{\partial Y_s(s)}{\partial t} = -Acs\sin cst + Bcs\cos cst$$

$$\frac{\partial Y_s(s,0)}{\partial t} = -A\,cs\,(0) + Bcs(1)$$

$$0 = Bcs \Rightarrow \boxed{B=0}.$$

eq(ii) becomes $Y_s(s) = A\cos cst \longrightarrow$ (iii)

$Y_s(s,0) = A$

$\boxed{f(t) = A}$ eq(iii) becomes, $\boxed{y_s(s) = f(t)\cos cst}$

$\therefore$ The displacement of any point of the string at any time is $\boxed{y(x,t) = \frac{2}{\pi}\int_0^\infty f(t)\cos cst\,\sin cst\cdot ds}$