1.2.23

Module - 3 : Complex Integration

Power series expansion $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

where a_n - series co-efficient, $\boxed{z_0 \text{ - centre}}$

→ is convergent if summation is finite. $\left(eg = \sum_{n=1}^{\infty} 1/n \right)$

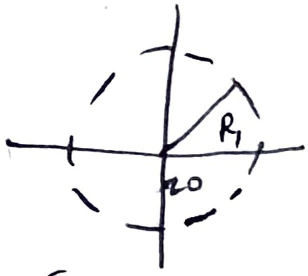
→ otherwise divergent $\left(eg = \sum_{n=1}^{\infty} n \right)$

→ Every points series expansion can be expressed as an analytic fn within region of convergence.

Region of convergence : The set of all value of z for which the series is convergent denoted as $|z - z_0| < R$ ~~whic~~ where R is radius of convergence.

Eg : $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots + z^n$

$= \dfrac{1}{1-z}$ ✓



Convergent if $|z| < 1$

→ Let $f(z)$ be the analytic fn convergent in $|z - z_0| < R$ then it can be expressed as $f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n$

where $a_n = \dfrac{f^n(z_0)}{n!}$, $a_0 = f(z_0)$, $a_1 = f'(z_0)$, ...

is called Taylor series Expansion.

→ Singular Points :

(★) For any given fn, $f(z)$ The point at which the function is undefined or derivative fails to exists.

Eg : (i) $\log z$, $z = 0$

(ii) $\frac{1}{z+3}$, $z = -3$

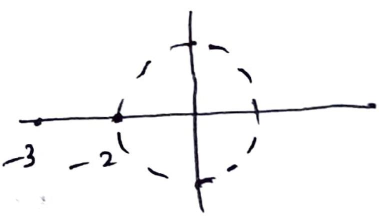
(iii) $\frac{1}{(z+i)(z-2)}$, $z = 2, i$

→ If $f(z)$ is not analytic at points z_0 power series expansion does not exists

Example:

1. $\frac{1}{(z+2)(z+3)}$ about $z_0 = 0$

$z = -2, -3$



Radius of convergence = 2

→ If $f(z)$ is analytic at z_0 and has Singular points $z_1, z_2, \ldots, z_n$, then the power series expansion about z_0 will have radius of convergence as the distance between z_0 and its $\boxed{\text{nearest singular points.}}$

Eg. 2. $\frac{1}{z(z+1)(z+4)}$, $z_0 = 2$

$z = 0, -1, 4$

Rad. of conv. = 2

① Expand $\sin z$, about $z_0 = \pi$.

$a_0 = f(\pi) = 0.$

$a_1 = f'(\pi) = \cos(\pi) = -1$

$a_2 = f''(\pi) = -\sin(\pi) = 0$

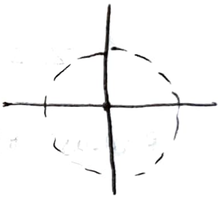
$a_3 = f'''(\pi) = -\cos(\pi) = -(-1) = 1$

$$0 - \frac{1(z-\pi)}{1!} + 0 + \frac{1(z-\pi)^3}{3!} + \dots$$

② Expand $\dfrac{z}{(z+1)(z+2)}$ about $z_0 = 0$.

$\cancel{a_0 =}$

$$\frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$



$$z = (z+2)A + B(z+1)$$

Sub, $z = -2$

$$-2 = B(-1)$$

$$\boxed{B = 2}$$

Sub $z = -1$

$$-1 = A$$

$$\boxed{A = -1}$$

Geo Series Formula

$$\boxed{\begin{aligned}(1+z)^{-1} &= 1 - z + z^2 - z^3 + \dots \\ (1-z)^{-1} &= 1 + z + z^2 + z^3 + \dots\end{aligned}}$$

$$f(z) = -1\,(1+z)^{-1} + \frac{z}{2}\left(1+\frac{z}{2}\right)^{-1}$$

$$= -1\,(1 - z + z^2 - z^3 + \dots) + \left(1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \dots\right)$$

$\cancel{z = -1\ z.}$

$|z| < 1$ ✓ $\qquad \left|\frac{z}{2}\right| < 1$

ROC $= 1$

③ $\dfrac{5z+7}{z^2+5z+6}, \quad z_0 = 0$

$$\frac{5z+7}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$$

$$5z+7 = A(z-3) + B(z-2)$$
$$= Az - 3A + Bz - 2B.$$
$$= z(A+B) - 3A - 2B.$$

$\Rightarrow A + B = 5 \qquad\qquad -3A - 2B = 7$

$\Rightarrow A = 5 - B \qquad\qquad \Rightarrow \cancel{-3A-2(}$

$\Rightarrow A = 5 + \frac{22}{5} \qquad\qquad \Rightarrow -3(5-B) - 2B = 7$

$\qquad\qquad\qquad\qquad \Rightarrow -15 - 5B = 7$

$\qquad\qquad\qquad\qquad \Rightarrow -5B = 22$

$\Rightarrow A = \frac{47}{5} \qquad\qquad \Rightarrow B = -\frac{22}{5}$

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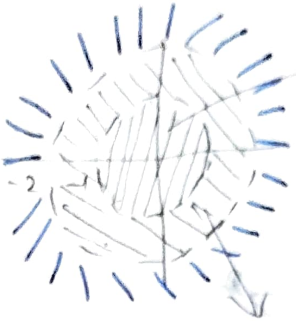
Laurent's Series Expansion :

• $\dfrac{1}{(1-z)(z-2)}$



Let c_1 & c_2 be two concentric circles with radii R_1 and R_2 ~~resp~~ respectively. $f(z)$ be analytic in annulus region $R_1 < |z - z_0| < R_2$, then Laurent's series expansion about z_0 is $f(z)$

$$= \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

→ $b_n = 0$: Taylor's

→ Radius of Convergence $R_1 < |z - z_0| < R_2$

Q) Find all possible expansion of $\dfrac{1}{(1-z)(z-2)}$ about $z_0 = 0$

case i) $|z| < 1$

case ii) $1 < |z| < 2$

iii) $|z| > 2$

Soln,

i) $\dfrac{1}{(1-z)(z-2)} = \dfrac{A}{(1-z)} + \dfrac{B}{z-2}$

$1 = A(z-2) + B(1-z)$

Sub $z = 2$

$1 = B(1-2)$

$\boxed{-1 = B}$

Sub $z = 1$

$1 = A(\cancel{z}1-2) + B(0)$

$\boxed{A = -1}$

$= \frac{-1}{(1-z)} - \frac{1}{z-2}$

$= -1(1-z)^{-1} + \frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1}$

$= -1(1+z+z^2+\ldots) + \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \ldots\right)$

$|z| < 1$, $\quad \left|\frac{z}{2}\right| < 1$

$|z| < 2$

R.O.C) min $z(1,2) = 1$.

(ii)

$$\frac{-1}{1-z} - \frac{1}{z-2} \qquad \boxed{1<|z|<2} \qquad |z|<2 \qquad |z|>1$$

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{z}\left(1-1/z\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right) + \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\dots\right)$$

R.o.c $= 1<|z-0|<2$ (Laurent's)

(iii) $|z|>2$.

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right) - \frac{1}{z}\left(1+\frac{2}{z}+\left(\frac{z^2}{4}\right)+\dots\right)$$

$$\left|\frac{1}{z}\right|<1 \qquad \left|\frac{2}{z}\right|<1$$

$$|z|>1 \qquad |z|>2$$

ROC $\max(1,2) = 2$. (Laurent's)

2(a) $\dfrac{1}{(z+1)(z+3)}$,

ⓐ $|z| < 1$ ⓑ $1 < |z| < 3$

ⓒ $|z| > 3$ ⓓ $|z+1| < 2$

$$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$$

$$1 = A(z+3) + B(z+1)$$

Sub, $z = -1$

$$1 = A(-1+3) + 0$$

$$\boxed{A = \frac{1}{2}}$$

Sub $z = -3$

$$1 = 0 + B(-3+1)$$

$$\boxed{B = \frac{-1}{2}}$$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

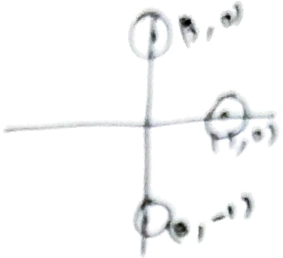
ⓐ $|z| < 1$

$$= \frac{1}{2}(1+z) - \frac{\cancel{3}}{6}($$

① Isolated singular pt:

A singular pt z_0 is said isolated singular point if there exist nbd of z_0 which doesn't contain remaining singular points

Eg: $\dfrac{1}{(z-1)(z+i)(z-3i)}$



$1, -i, 3i$

$$\frac{1}{\sin\left(\frac{\pi}{z}\right)}$$

$$\sin \frac{\pi}{z} = 0$$

$$\frac{\pi}{z} = n\pi$$

$$n = \frac{1}{z} \Rightarrow z = \frac{1}{n}, \quad \underline{z = 0}, \pm 1, \pm 1/2, \pm 1/3, \pm 1/4, \ldots$$

↓

$\boxed{\text{By sub. } n = \infty}$

Removable singular Point:

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{analytic}} + \underbrace{\sum_{n=1}^{\infty} b_n (z-z_0)^n}_{\text{principle part}}$$

↓

if PP $= 0$

then z_0 is removable singular pt

An isolated singular pt. z_0, of $f(z)$ is said to be removable singular point if principle part of Laurent's series expansion of $f(z)$ about z_0 has no terms.

Eg: $\frac{\sin(z)}{z}$, $z=0$ is singular points

$$\frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots\right)$$

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots$$

② Esssential Singular Point:

An Isolated singular pt. z_0 of $f(z)$ is said to be essential singular point of $f(z)$ if Laurent series expansion of $f(z)$ about z_0 has infinite terms in principle part.

Eg: $e^{1/z}$, $z=0$

$$e^z = 1 + z + \frac{z^2}{2!} + \dots$$

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{z^2 2!} + \dots$$

③ Pole:

An isolated singular point z_0 of $f(z)$ is said to be a pole of order m if the principle part of Laurent series expansion of $f(z)$ about z_0 has m-terms.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

$$= \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)} + \dots + \frac{b_m}{(z-z_0)}$$

__Eg:__

$$\frac{1}{z(z-1)^2} \Rightarrow 0, 1$$

$z = 0$:

$$\frac{(1-z)^{-2}}{z} = \frac{1}{z}\left(1 + 2z + 3z^2 + \dots\right)$$

$$= \frac{1}{z} + 2 + 3z + \dots$$

$\downarrow$

pole

__because__ : ~~power~~ finite no. of negative power terms.

$$\frac{1}{(z-1)^2(z+1-1)} = \frac{1}{(z-1)^2}(1+(z-1))$$

If a point z_0 is a pole of order m of $f(z)$,

then $f(z) = \dfrac{g(z)}{(z-z_0)^m}$ where $g(z_0) \neq 0$

$g(z)$ is analytic at z_0.

$g(z) = \frac{1}{z}$ is analytic at 1.

$\frac{1}{z} \neq 0$, at 1.

$z=0$ $\quad \dfrac{1}{z(z-1)^2} = \dfrac{\frac{1}{(z-1)^2}}{z}$, $g(z)$ is analytic at 0.

$g(0) \neq 0$.

Zeros of a function :

For any given function $f(z)$, the value of z at which the function value is 0

Example: $f(z) = \dfrac{(z-1)^2(z-5i)}{(z-1)^2(z+2i)^3}$

zeros = $1, 5i$

poles / singular points = $1, -2i$

order = $2, 3$.

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Note :

if $f(z) = \dfrac{g(z)}{(z-z_0)^n}$ such that $g(z)$ is analytic at z_0 and $f(z)$ has zero z_0 of order m

then i) z_0 is a removable singular points if $n=m$

ii) z_0 is a pole of order $\boxed{n-m}$ if $n>m$.

Ex : ① $\dfrac{\sin z}{z^2}$

zeros $z = n\pi$; $0, \pm\pi, \pm 2\pi$

$\boxed{m=1, \quad n=2}$, $n>m$ $(2>1)$

0 is a pole of order $2-1=1$.

Ex 2. $\dfrac{\sin z}{z}$

For,

~~zeros~~ $\cancel{z = n\pi}$ $z_0 = 0$

$m=1, \; n=1$.

$z_0 = 0$ is removable singular points

Classify Singular Points :

Ex 1. $\dfrac{e^z}{z+\sin z}$

Singular point $z_0 = 0$

$$\frac{e^z}{z + z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}$$

$$= \frac{e^z}{2z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}$$

$$= \frac{e^z}{z\left(2 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right)}$$

$$g(z) = \frac{e^z}{2 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots}$$

$g(0) \neq 0$

$g(z)$ analytic at '0'. →

Substitute 0 in z, if we get non-zero value it is analytic at 0.

Ex.2 $f(z) = \frac{(z^2-1)(z-2)}{(\sin \pi z)^3}$

Singular Points : $\sin \pi z = 0 \Rightarrow \pi z = n\pi$

$\boxed{z = n}$

$z = 0, \pm 1, \pm 2, \ldots$

	m	n	order
$z_0 = 1$	1	3	2
$z_0 = -1$	1	3	2
$z_0 = 2$	1	3	2
$z_0 = -2$	Pole	of	order 3
$z_0 = 0$	Pole	of	order 3

Remaining singular points are of order 3.

Residue:

Residue of a fn. $f(z)$ at z_0 is the co-efficient of $\frac{1}{z-z_0}$ in the Laurent Series expansion of $f(z)$ about z_0.

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

Notation : $\text{Res}\{f(z), z_0\}$

Ex:

$$\frac{1}{z^2}(\sin z) = \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)$$

$$= \frac{z}{z}\left(\frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} + \cdots\right)$$

$$\text{Res}\{f(z), 0\} = 1$$

1) Residue of a simple pole.

If z_0 is a simple pole of $f(z)$, then residue of a simple pole,

$$\boxed{\text{Res}\{f(z), z_0\} = \lim_{z \to z_0} (z-z_0) f(z)}$$

2) If z_0 is pole of order m.

$$\boxed{\text{Res}\{f(z); z_0\} = \lim_{z \to z_0} \left(\frac{1}{(m-1)!}\right) \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)}$$

Ex. 1

$$\frac{1}{(z+1)(z-2)}$$

$z=-1 \qquad z=2$

$$\text{Res}\{F(z),-1\} = \lim_{z\to -1} (z+1)\frac{1}{(z+1)(z-2)} = \frac{-1}{3}.$$

$$\text{Res}\{F(z),2\} = \lim_{z\to 2} (z-2)\frac{1}{(z+1)(z-2)} = \frac{+1}{3}.$$

Ex. 2

$$\frac{1}{z^3(z^2+1)}, \quad \text{order } 3 : \quad 0, \frac{-i}{1}, \frac{i}{1}$$

~~$z=0$~~

$$\text{Res}\{F(z), \cancel{0}\, i\} = \lim_{z\to \cancel{0} i}\left(\frac{(z-i)\times 1}{z^3(z+i)(z-i)}\right)$$

$$= 1/2$$

$$\text{Res}\{f(z),-i\} = \lim_{z\to -i}\left(\frac{(z+i)\times 1}{z^3(z+i)(z-i)}\right)$$

$$= 1/2$$

$$\text{Res}\{f(z),0\} = \lim_{z\to 0}\left(\frac{\cancel{z}\ 1}{(\cancel{z}\,3-1)!}\frac{d^2}{dz^2}\left(z^3\left(\frac{1}{z^3(z^2+1)}\right)\right)\right)$$

$$= -1$$

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Complex Integration:

① Evaluate $\int_C \bar{z}\, dz$ where $x = 3t \quad y = t^2$

$-1 \leq t \leq 4$

Soln:

$$I = \int_C (x - iy)(dx + i dy)$$

$$\boxed{\begin{aligned} z &= x + iy \\ dz &= dx + d iy \end{aligned}}$$

$x = 3t \qquad y = t^2$

$dx = 3dt \qquad dy = 2t\,dt$

$$= \int_{-1}^{4} (3t - it^2)(3dt + i \cdot 2t\,dt)$$

$$= \int_{-1}^{4} (3t - it^2)(3 + 2it)\,dt$$

$$= \int_{-1}^{4} (9t + 6it^2 - 3it^2 + 2t^3)\,dt$$

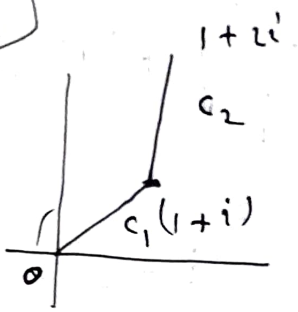
$$= \left[\frac{9t^2}{2} + \frac{i\,\cancel{6}^{2}t^3}{\cancel{3}} - \frac{\cancel{3}it^3}{\cancel{3}} + \frac{\cancel{2}t^4}{\cancel{4}_{2}}\right]_{-1}^{4}$$

$$= \left[\frac{9(4)^2}{2} + i2(4)^3 - i4^3 + 4^4/2\right] - \left[\frac{9}{2} - i2 + i - 1/2\right]$$

$$\boxed{I = 195 + 65i \longrightarrow \boxed{\text{Ans}}}$$

② $\int_C (x^2+iy^2)\,dz$ along

$dz \to \boxed{dx+diy}$



along c_1, $y=x$

$dy = dx$

x is from 0 to 1

$$I_1 = \int_0^1 (x^2+ix^2)(dx+i\,dx)$$

$$= \int_0^1 (x^2+ix^2)(1+i)\,dx$$

$$= \int_0^1 (\cancel{x^2}+ix^2+ix^2-\cancel{x^2})\,dx$$

$$= 2\int_0^1 ix^2\,dx$$

$$= 2i\left[\frac{x^3}{3}\right]_0^1$$

$$= 2i\left(\left[\frac{1}{3}\right]-\left[\frac{0}{3}\right]\right)$$

$$= \frac{2i}{3}$$

along c_2, $x = 1$ $\quad dx = 0$

$$I_2 = \int_1^2 (1 + iy^2)(i\,dy)$$

$$= \int_1^2 (i - y^2)\,dy$$

$$= \left[iy - \frac{y^3}{3}\right]_1^2$$

$$= \left[i2 - \frac{8}{3}\right] - \left[i - \frac{1}{3}\right]$$

$$I_2 = \cancel{i} + \cancel{\tfrac{9}{3}} = \cancel{i} + \cancel{3} \quad i2 - \frac{7}{3}$$

$$I = I_1 + I_2 = \frac{2}{3}i - \frac{7}{3} + i$$

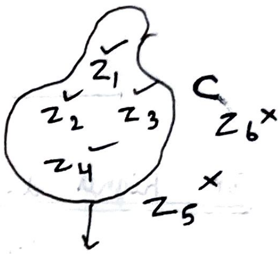
$$\boxed{I = \frac{5}{3}i - \frac{7}{3}}$$

Cauchy Residue Theorem:

Let ~~D~~ d be a connected domain and c be a contour (closed curve) lying in ~~D~~ d.

If f is analytic on and within c expect at a finite number of isolated ~~simila~~ singularity $z_1, z_2, \dots, z_n$ within c, then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}\{f(z), z_k\}$$



$\boxed{\text{find residue at each point inside curve only}}$

Ex. 1

$$I = \oint_C \frac{1}{(z-1)^2(z-3)}\,dz$$

(a) $\left.\begin{array}{l} x = 0, x = 4 \\ y = -1, y = 1 \end{array}\right\} \rightarrow$ Rectangle

Singular Points:

$z_0 = 1, 3$

↳ points lie inside closed surface

$$I = 2\pi i \left(\text{Res}\{f(z), 1\} + \text{Res}\{f(z), 3\} \right)$$

$$= 2\pi i \left(\lim_{z \to 1} \left(\frac{d}{dz}\left(\frac{1}{z-3}\right)\right) + \lim_{z \to 3} \left(\frac{1}{(z-1)^2}\right)\right)$$

$$= 2\pi i \left(\frac{-1}{4} + \frac{1}{4} \right) = 0$$

(b) $|z| = 2.$

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Cauchy Integral Formula:

Suppose that f is analytic in a simply connected domain d, and c is any contour lying within d, then for any point z_0 within c,

$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\, dz}$$

pole of order - first order

For higher derivative:

$$f^n(z_0) = \frac{n!}{2\pi i}\left(\oint \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right)$$

f^n - n^{th} derivative

Ex: 1

Qn:

$$\oint \frac{z^2 - 4z + 4}{z + i}\, dz, \quad |z| = 2.$$

Soln: The singular point lies inside

$z_0 = -i$

$f(z) = z^2 - 4z + 4$

$$\oint \frac{z^2 - 4z + 4}{z + i}\, dz = 2\pi i\, f(-i)$$

$$= 2\pi i\,(-1 + 4i + 4)$$

$$= 2\pi i\,(3 + 4i)$$

② $\oint \frac{z}{z^2+9}\,dz \;,\quad |z-2i| = 4$

$z^2+9 = 0$.

$z_0 = \pm 3i$

use diagram or substitute to find whether points ~~insid~~ inside or not

$+3i$ only lies inside.

$3i \to |3i-2i| \leq 4$ True

$-3i \to |-3i-2i| \leq 4$ ~~True~~ Not True

$= \oint \frac{z}{(z+3i)(z-3i)}\,dz$ → $f(z)$

~~$f(z) =$~~

$= 2\pi i \left(\frac{3i}{3i+3i}\right) = \cancel{2}\pi i \left(\frac{\cancel{3i}}{\cancel{6i}_{2}}\right) = \pi i$

③ $\oint \frac{z+1}{z^4+2iz^3}\,dz \;,\quad |z| = 1$

$\oint \frac{z+1}{z^3(z+2i)}\,dz = \frac{2\pi i}{n!} f^2(0)$ → $\boxed{f(z)}$

$(z-0)^3$ ~~$z_0 = 2i$~~

$\boxed{z_0 = 0}$ ✓

$z_0 = -2i$ ✗

$$f(z) = \frac{z+1}{z+2i}$$

$$\cancel{f'(z) = z+2}$$

$$f'(z) = \frac{(z+2i)(1) - (z+1)(\overset{1}{\cancel{2i}})}{(z+2i)^2}$$

$$\cancel{f'(z) = \frac{z+\cancel{2i}-2zi-2i}{(z+2i)^2}}$$

$$\cancel{f'(z) = \frac{z-2i}{(z+2i)^2}}$$

$$\cancel{f''(z) = (z+2i)^2(-2i) - (z-2}$$

$$f'(z) = \frac{\cancel{z}+2i-\cancel{z}-1}{(z+2i)^2}$$

$$f'(z) = \frac{2i-1}{(z+2i)^2}$$

$$f''(z) = \frac{\overset{0}{\cancel{(z+2i)^2(0)}} - 2(z+2i)(1)(2i-1)}{(z+2i)^4}$$

$$f''(0) = \frac{-2(2i)(2i-1)}{(2i)^4}$$

$$f''(0) = \frac{-\cancel{4}i\,(2i-1)}{\cancel{16}\,4}$$

$$\boxed{f''(0) = \frac{+2+i}{4}}$$

$$= \pi i \, \frac{d^2}{dz^2}\left(\frac{z+1}{z+2i}\right)$$

$$= \pi i \left(\frac{2+i}{4}\right) = \frac{\pi}{2} i - \frac{\pi}{4}$$

③

$$\oint \frac{e^z}{z^4 + 5z^3}\, dz \;, \quad |z| = 2$$

$$\boxed{z = 0,\ -5} \times$$

Cauchy Goursat Theorem:

Suppose that $f(z)$ is analytic in a closed contour c, then

$$\boxed{\oint f(z)\,dz = 0}$$

Eg 1) $\oint e^z \, dz$, $|z| = 5$

Eg 2) $\oint \frac{1}{z^2} \, dz$,

H.W

$$\oint \frac{z^2 e^{2z+1}}{(z+i)^2 (z^2-9)} \, dz.$$

13.2.23

Evaluation of Real Integrals:

Type I:

Integration of Trigonometric functions along unit circle.

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

$z = e^{i\theta}$ $(\because r = 1)$

$dz = ie^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{iz}$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$$

$$\sin\theta = \frac{e^{i\theta}-e^{i\theta}}{2i} = \frac{z-z^{-1}}{2i}$$

→ use CRT and evaluate
(Cauchy Residue Theorem)

Evaluate :

$$= \int_0^{2\pi} \frac{1}{(2+\cos\theta)^2}\, d\theta$$

$$d\theta = \frac{dz}{iz} \qquad \cos\theta = \frac{z+z^{-1}}{2}.$$

$$= \oint_C \frac{1}{\left[2+\left(\frac{z+z^{-1}}{2}\right)\right]^2}\left(\frac{dz}{iz}\right)$$

$$= \oint_C \frac{4z^2}{(4z+z^2+1)^2}\frac{dz}{iz}$$

$$= \frac{4}{i}\oint \frac{z}{(z^2+4z+1)^2}\, dz$$

$$z_0 = -2-\sqrt{3},\ \boxed{-2+\sqrt{3}}$$

only this lies inside

$$= \frac{4}{i}\int \frac{z}{(z^2+4z+1)}\,dz$$

$$= \frac{4}{i}\left[\int \frac{z\,dz}{[z-(-2-\sqrt{3})]^2\,[z-(-2+\sqrt{3})]^2}\right]$$

$$= \frac{4}{\cancel{i}}(2\pi \cancel{i})\ \operatorname{Res}\left\{\frac{z}{[(z-(-2-\sqrt{3}))(z-(-2+\sqrt{3}))]^2},\ -2+\sqrt{3}\right\}$$

$$= 8\pi \lim_{z\to -2+\sqrt{3}} \frac{d}{dz}\left\{\frac{z}{(z+2+\sqrt{3})^2}\right\}$$

pole of second order

$$= 8\pi \lim_{z\to -2+\sqrt{3}} \frac{d}{dz}\left(\frac{z}{(z+2+\sqrt{3})^2}\right)$$

$$= 8\pi\cdot\left(\frac{1}{6\sqrt{3}}\right) = \frac{4\pi}{3\sqrt{3}}$$

② $\displaystyle\int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta$

$d\theta = \dfrac{dz}{iz}$ $\qquad \cos\theta = \dfrac{z+z^{-1}}{2}$

$$= \oint \frac{1}{2+\left(\frac{z+z^{-1}}{2}\right)} \, dz/iz$$

$$= \oint \frac{2\cancel{z}}{4z+z^2+1} \, dz/i\cancel{z}$$

$$= \frac{2}{i} \oint \frac{dz}{4z+z^2+1}$$

$2+\dfrac{z+z^{-1}}{2}$

$= 2+\dfrac{z}{2}+\dfrac{z^{-1}}{2}$

$= \dfrac{2\cancel{z}}{2\cancel{z}}+\dfrac{z}{2}+\dfrac{1}{2z}$

$= \cancel{2+}$

$= \dfrac{4z+z^2+1}{2z}$

15-2-23

Type 2:

Let H denote upper half complex plane if f is a rational function

$f(x) = p(x)/q(x)$ where $\boxed{\deg(q(x)) \geq \deg(p(x)) + 2}$ cond ①

and $q(x)$ has $\boxed{\text{no real roots}}$ cond ②. Then, ~~integrat~~

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \text{Residues in H}}$$

③ $$\int_0^{2\pi} \cos\theta \sin\theta \, d\theta = 0$$

(0 is the pole of order 3)

④ $$\int_0^{2\pi} \frac{1}{13 + 5\sin\theta} \, d\theta$$ HW: $= \pi/6$.

① $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}$

$x \Rightarrow i, -i, 3i, -3i$

$= 2\pi i \left(\text{Res}\,(f(x), i) + \text{Res}\,(f(x), 3i) \right)$

$= 2\pi i \left(\lim_{x \to i} \frac{1}{(x+i)(x^2+9)} + \lim_{x \to 3i} \frac{1}{(x^2+1)(x+3i)} \right)$

$= \pi/12.$

② $\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^3}\, dx.$

$x \Rightarrow 2i, -2i$

$= 2\pi i \left(\text{Res} \left(\frac{1}{(x+2i)^3 (x-2i)^3}, 2i \right) \right)$

$\cancel{\frac{3\pi}{256}} = 2\pi i \left(\frac{1}{2!} \lim_{x \to 2i} \frac{d^2}{dx^2} \frac{1}{(x+2i)^3} \right)$

$= \frac{3\pi}{256}$

Bounding Theorem :

If f is continuous on a curve c and $|f(z)| \leqslant M$

$\forall z \in C$, Then

$$\left| \oint_c f(z)\,dz \right| \leqslant ML \quad \text{where } L \to \text{length of curve}$$

Example :

$\oint_c \frac{e^z}{z+1}\,dz$, $|z| = 4$ Find upper bound on

$$\left| \oint_c f(z)\,dz \right| \leqslant ML$$

$L = 2\pi(4) = 8\pi$.

use this for numerator

$|z_1 + z_2| \leqslant |z_1| + |z_2|$

$|e^z| \leqslant e^{|z|} \leqslant e^4$

$|z_1 + z_2| \geqslant |z_1| - |z_2|$

use this for denominator

$\boxed{z+1}$

$|z+1| \geqslant |z| - |1| \geqslant 4 - 1 = 3$

Ex.

$$\boxed{\int \tan z\,dz \ , \ |z| = 2}$$

$f(z) = \frac{\sin z}{\cos z}$, singular point inside $|z| = 2$ are $\pm \pi/2$.

$= 2\pi i \left(\text{Res}\left(f(z), \pi/2\right) + \text{Res}\left(f(z), -\pi/2\right) \right)$

Note:

$$f(z) = \frac{\phi(z)}{\psi'(z)}$$

$$= 2\pi i \left(\lim_{z \to \pi/2} \frac{\sin z}{-\sin z} + \lim_{z \to -\frac{\pi}{2}} \frac{\sin z}{-\sin z} \right)$$

$$= 2\pi i(-1-1)$$

16.2.23

Tutorial Sheet:

1) Evaluate $\int_0^{2+i} \bar{z}^2$ along i) $y = x/2$

(ii) on real axis from 0 to 2 and vertically to $2+i$.

(i) $\bar{z} = x - iy$, $x = 2y$, $2dy = dx$

x from $0 \to 2$

$$dz = dx + i\,dy$$
$$= dx + i\,2dx$$

$$\to \int_0^2 (x-iy)^2 \left(1 + \frac{i}{2}\right) dx$$

$$= \frac{5}{3}(2-i)$$

(ii)

$$\int_{C_1} + \int_{C_2}$$

(ii) $\int_{C_1} (x-iy)^2 (dx+idy) + \int_{C_2} (x-iy)^2 (dx+idy)$

$\downarrow \boxed{y=0}$ (real axis) $\qquad \downarrow$ x is constant; $x=2$

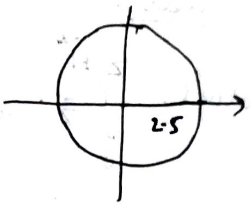
$$= \int_{x=0}^{2} (x)^2 (dx) + \int_{0}^{1} (2-iy)^2 (idy)$$

$$= \frac{1}{3}(14+11i)$$

2) $$\oint \underbrace{\boxed{\frac{z^2}{(z-1)^2(z+2)}}}_{f(z)} dz, \quad |z| = 2.5$$ (Cauchy Residue Theorem)

Singular Point:

$z=1$
$z=-2$ } Points lie inside the surface.

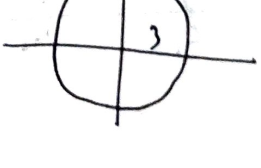


$$I = 2\pi i \left(\mathrm{Res}\{f(z), 1\} + \mathrm{Res}\{f(z), -2\} \right)$$

$$= 2\pi i \left(\lim_{z \to 1} \left(\frac{d}{dz} \left(\frac{1}{(z-1)(} \right. \right. \right.$$

$$= 2\pi i \left(\frac{5}{9} + \frac{4}{9} \right) = 2\pi i$$

(3) $$\int \frac{e^{tz}}{z^2+1} dz \qquad |z| = 3$$



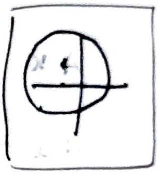
Singular points: $z^2 + 1 = 0$

$z = \pm i \rightarrow$ lies in the circle

Ans: $\cancel{2}\pi i\left(\dfrac{e^{it}-e^{-it}}{\cancel{2}}\right)$

(4) Expand $\dfrac{1}{(z-1)^2(z-3)}$, $0<|z-3|<2$

(5) Expand $\dfrac{8z+1}{z(1-z)}$, $0<|z|<1$

(6) $\displaystyle\int\frac{z-3}{z^2+2z+5}\,dz$, $|z+1-i|=2 \rightarrow$ 

(7) $\displaystyle\int_0^{2\pi}\frac{d\theta}{2-\cos\theta}$, $\oint$

6 Ans:

$\oint$ $z^2+2z+5=0$

17.2.23

Module 7: Eigen Values & Eigen Vectors:

Eigen Values & Eigen Vectors - Linear Algebra

$\boxed{\text{square matrix}}$ A x $\boxed{\text{vectors}}$ $= \lambda x$ $\boxed{\text{eigen value}}$

$\boxed{\text{direction Remains same}}$

When eigen vectors are multiplied with a square matrix ~~with a~~ it does not change direction of the vector.

$$Ax = \lambda I x$$

$$Ax - \lambda I x = 0$$

$$x(A - \lambda I) = 0$$

→ It is possible only when $[A - \lambda I]$ is singular or $\det(A - \lambda I) = 0$. This equation is called characteristic equation.
~~det (~~

→ By solving which may get Eigen values.

Example:

$$\begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$$ Find eigen values and eigen vectors.

Soln

$$A - \lambda I = 0$$

~~A~~ $A = \begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$ $\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} -5-\lambda & 2 \\ -9 & 6-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (-5-\lambda)(6-\lambda) - (2)(-9)$$

$$= \cancel{-35} \; -30 + 5\lambda - 6\lambda + \lambda^2 + 18$$

$$= \lambda^2 - \lambda - 12 = 0$$

$$\boxed{\lambda = 4, -3}$$

For $\lambda = 4$

$$(A - \lambda I)x = \begin{bmatrix} -9 & 2 \\ -9 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-9x_1 + 2x_2 = 0$$
$$-9x_1 + 2x_2 = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2/9 \; x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} 2/9 \\ 1 \end{pmatrix}$$

$(2, 9)$ $(1, 9/2)$

etc...

$\lambda = -3$

$$(A - \lambda I)x = \begin{bmatrix} -2 & 2 \\ -9 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\cancel{-9x_1 + 2x_2 = 0}$$
$$\cancel{-9x_1 - 2x_2 = 0}$$

$$\cancel{-18x_1 = 0}$$
$$\cancel{x_1 = 1}$$

$$-2x_1 + 2x_2 = 0$$

$$-9x_1 + 9x_2 = 0$$

$$\boxed{x_1 = x_2}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(2)
$$\begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix}$$

$$\begin{vmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$3-\lambda\left[(3-\lambda)(5-\lambda)\right]+2\left(-2(5-\lambda)\right)$$
$$=0$$

$$(3-\lambda)(3-\lambda)(5-\lambda)+2(\cancel{4}\ -10-2\lambda)=0$$

$$(9+\lambda^2-6\lambda)(5-\lambda)+(-20-4\lambda)=0$$

$$45-9\lambda\boxed{+5\lambda^2}-\lambda^3-\cancel{8}\,30\lambda+\boxed{6\lambda^2}-20-4\lambda$$
$$=0$$

$$-\lambda^3+11\lambda^2-35\lambda+25=0$$
$$\lambda=1,5,5$$

$\lambda=1$

$$\begin{bmatrix}2 & -2 & 0\\ -2 & 2 & 0\\ 0 & 0 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}x_1\\ x_1\\ 0\end{bmatrix}=x_1\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

$\lambda=5$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-x_2\\ x_2\\ x_3\ +0x_2\end{bmatrix}=x_2\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}+x_3\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

20.2.23

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & +1 & -1 \end{bmatrix}$$ Find eigen values and eigen vectors

$\lambda =$

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

$$\begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\boxed{\lambda = -2, 3, 1}$$

$$\boxed{-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0}$$

For $\lambda = 1$

Simplify using row operation simplification

$$\begin{bmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_2 + 4x_3 = 0$$

$$x_2 = 4x_3$$

$$3x_1 + x_2 - x_3 = 0 \Rightarrow \boxed{x_1 = -x_3}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 4x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

For $\lambda = -2$

$$\begin{bmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1 \qquad R_3 \to 3R_3 - 2R_1$

$$\begin{bmatrix} 3 & -1 & 4 \\ 0 & 5 & -5 \\ 0 & 5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x_2 - 5x_3 = 0$

$\boxed{x_2 = x_3}$

$3x_1 - x_2 + 4x_3 = 0$

$3x_1 - x_3 + 4x_3 = 0$

$x_1 = -x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$x = ?$

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$ ✓

Caley Hamilton Theorem :

Every square matrix ~~s~~ satisfies its own characteristic equation.

~~Find the~~ Find the characteristic equation.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \qquad \boxed{\lambda^2 - 5 = 0}$$

Verify, means, find the ch. eqn and substitute

$$A^2 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \rightarrow (1)$$

$$5I\ \cancel{\lambda} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \rightarrow (2)$$

$$\boxed{A^2 - 5I = 0}$$ ✓

Applications:

1. Used to find inverse of a matrix
2. Used to find integral powers of A,

i.e, A^2, A^3, A^4, etc...

Eg:

$A^2 - 5I = 0$

$A^{-1}A^2 - 5A^{-1}I = 0$

$A - 5A^{-1} = 0$

$A^{-1} = \frac{1}{5}A$

$A^2A^2 - 5A^2I = 0$

$A^4 = 5A^2 = \cancel{25}\,5(5I) = 25I.$

22.2.23

① write expression for A^4 and A^{-1} for $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

using Caley-Hamilton Theorem.

Solution:

$$|A - \lambda I| = 0$$

$$A - \lambda I =$$

$$= \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & -1 & 2 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 2-\lambda & -1 & 2 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Delta = 2-\lambda\left[(2-\lambda)(2-\lambda) - 1\right] + 1\left[2-\lambda+1\right] + 2\left[1-2+\lambda\right] = 0$$

$$0 = (2-\lambda)^3 + (3-\cancel{\lambda}) + (-2 \cancel{+2\lambda}) = 0$$

$$= \lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$$

~~For A~~

$$\rightarrow A^3 - 6A^2 + 8A - 3I = 0$$

For A^{-1},

$$= A^{-1}A^3 - 6A^{-1}A^2 + 8A^{-1}A - 3A^{-1}I$$

$$= A^2 - 6A + 8I = 3A^{-1}$$

$$A^{-1} = \frac{A^2 - 6A + 8I}{3}$$

For A^4:

$$AA^3 - 6AA^2 + 8AA - 3AI = 0$$

$$A^4 = 6A^3 - 8A^2 + 3A$$

$$= 6(6A^2 - 8A + 3I) - 8A^2 + 3A$$

$$\boxed{A^4 = 28A^2 - 45A + 18I.}$$

Properties of Eigen Values:

1) If the square matrix A is real then its eigen values are real or complex conjugate pairs.

2) For any square matrix, its trace is same as sum of eigen values.

3) For $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ → $\boxed{\text{Sum is } 6 \text{ : trace}}$

Sum of eigen values $= 6$.

3) For any square matrix, its determinants is same as product of the eigen values.

4) Eigen values of a matrix and its transpose are always same.

5) For any square matrix, if all the eigen values are non-zero then its determinant is also non-zero.

6) If atleast 1 of the eigen values ~~of~~ is zero, then the determinats is zero and vice-versa.

7) If eigen values of A are λ_i then eigen values of kA are $k\lambda_i$.

Eg: A has $\lambda = 2, 3, 4$

5A has $\lambda = 10, 15, 20$

8) The eigen values of A are λ_i then eigen values of A^k is λ_i^k.

Eg: A has $\lambda = 2, 3, 4$

then A^{100} has $\lambda = 2^{100}, 3^{100}, 4^{100}$.

A^{-1} has $\lambda = 1/2, 1/3, 1/4$

A^2 has $\lambda = 4, 9, 16$

Eg:

Find eigen values of

$$\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \Rightarrow \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$= 1/3\, A$.

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

By solving we get,

$$\boxed{\lambda^3 - 3\lambda^2 - 9\lambda + 27 = 0}$$

$$\boxed{\lambda = -3, 3, 3}$$

$$\boxed{1/3\ \lambda = +1, -1, 1}$$

② If $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

check if eigen vectors are orthogonal.

Dot product of vectors are zero

Soln:
vectors are orthogonal.

Eqn,

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\boxed{\lambda = 1, 2, 3}$$

This is not a part of sum.

For A:

Example for dot product

$(1, 2, 1)\ (2, 3, 4) = 1 \cdot 2 + 2 \cdot 3 + 1 \cdot 4$

$= 12$

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \right.$$

$\quad x_1 \qquad x_2 \qquad x_3$

23-2-23

Algebraic Multiplicity and Geometricity Multiplicity:

Ex:

$$\begin{bmatrix} 3 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

	AM	GM
1	1	1
5	2	2

$$\begin{matrix} 1 & 5 & 5 \end{matrix}$$
$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

* Order of eigen value as a root in characteristic equation is called algebraic multiplicity.

* The no. of independent eigen vectors corresponding to eigen value.

Ex:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$ Find A.M & G.M.

By solving we get,

$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$

$(\lambda - 1)^3 = 0$ Repeated 3's

$\boxed{\lambda = 1, 1, 1} \rightarrow$ A.M $= 3$

$$\text{vector} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\boxed{\text{since we get only 1 vector, GM} = 1}$

System of Equation:

$AX = B$

$[A|B] \rightarrow$ Augumented Matrix.

Echelon Form - used ~~to~~ find rank

① $\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & 0 \end{bmatrix}$ ✓ $\quad \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & e & f \end{bmatrix}$ ✗

② For any ~~leading~~ row, the leading entry (first non-zero entry) must be to the right of leading entry in the row above it.

$\begin{bmatrix} a & b & c \\ d & 0 & e \\ 0 & f & g \end{bmatrix}$ ✗ $\quad \begin{bmatrix} a & b & c \\ 0 & d & c \\ 0 & 0 & f \end{bmatrix}$ ✓

Rank: of a matrix is no. of non-zero rows in its echelon form.

Find rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 0 & -6 & -4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{Rank} = 2}$

CASE - 1

i) if rank (A) < rank (A|B), then the system is inconsistent (no solution)

$x + 2y - z = 3$

$2x + 5y = 9$

$-3x - 4y + 7z = 1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 0 & 9 \\ -3 & -4 & 7 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 10 \end{array}\right] \quad R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

Rank $(A) = 2$

Rank $(A|B) = 3$

Inconsistent.

24-2-23

CASE - 2)

if rank (A) = rank $(A|B) = n$ (no. of unknown)

then the system has unique solution.

Gauss Elimination:

① Reduce to echelon Form

② use back substitution to get solution.

Ex:

$$= \begin{bmatrix} 2 & 3 & 4 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow \sim \begin{bmatrix} 2 & 3 & 4 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & \cancel{2}1 & -2 & 5 \end{bmatrix} \quad \begin{array}{l} R_3 \to \cancel{2R_3} - \cancel{2}R_1 \\ R_2 \to R_2 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -4 & 6 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

Rank $= 3$

Rank $(A|B) = 3$

$-4z = 6$

$\boxed{z = \frac{-3}{2}}$

$-y + -2z = 1$

$-y + \cancel{2}\frac{3}{\cancel{2}} = 1$

$\boxed{y = 2}$

$2x + 3y + 4z = 1$

$2x + 3(2) + 4\left(\frac{-3}{2}\right) = 1$

$2x = 1 - 6 + 6$

$\boxed{x = 1/2}$

case - 3 :

If rank (A) = rank (A|B) < $\overset{\nearrow \text{no. of unknown}}{n}$

It has Infinitely many solution.

Ex :

$3x_1 + 6x_2 - 9x_3 = 15$

$2x_1 + 4x_2 - 6x_3 = 10$

$-2x_1 - 3x_2 + 4x_3 = -6$

$$\begin{bmatrix} 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \\ 3 & 6 & -9 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \rightarrow 2R_3 - 3R_2$$

$$\begin{bmatrix} 2 & 4 & -6 & 10 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \rightarrow R_2 + R_1$$

~~rank (A~~

$\rho(A) = \rho(A|B) = 2 < 3$

It has infinitely many solution.

System has infinitely many solution

$y - 2z = 4$

$y = 4 + 2z$

$2x + 4y - 6z = 10$

$x + 2y - 3z = 5$

$x = 5 + 3z - 2y$

$\boxed{x = 5 + 3z - 4z = -z - 3}$

$(x, y, z) = (-z - 3, 4 + 2z, z)$

For $z = 0$ $(-3, 4, 0)$

$z = 1$ $(-4, 6, 1)$

Row Reduced echelon form :

1. It is in echelon form.
2. The leading entry in each non-zero row must be 1.
3. Each column containing a leading entry 1 has zeros in all its other entries.

$$\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & b/a & c/a & d/a \\ 0 & 1 & f/e & g/e \\ 0 & 0 & 1 & i/h \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \end{bmatrix}$$

$x = p, \; y = q, \; z = r.$

<u>Gauss Jordan Method:</u>

1. Convert to reduced row echelon form
2. Find the solutions.

<u>Ex:</u>

$$2x + y + 2z = 10$$

$\longrightarrow$ $$x + 2y + z = 8$$

$$3x + y - z = 2$$

$$\begin{bmatrix} 1 & 2 & 1 & 8 \\ 2 & 1 & 2 & 10 \\ 3 & 1 & -1 & 2 \end{bmatrix} \quad -1$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 0 & -5 & -4 & -22 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & -5 & -4 & -22 \end{bmatrix} \quad R_2 \to R_2 / -3$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & -12 \end{bmatrix} \quad R_3 \to R_3 + 5R_2$$

$$\sim \begin{bmatrix} 1 & \boxed{2 \quad 1} & 8 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad R_3 \to \frac{-R_3}{4}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad R_1 \to R_1 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad R_1 \to R_1 - R_3$$

$$\boxed{x = 1, \quad y = 2, \quad z = 3}$$

2nd sum

$$\begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - 3R_1 \\ R_2 \to R_2 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 12 & 34 \end{bmatrix} \quad R_3 \to 5R_3 + \frac{R_2}{5}$$

25.2.23

Determine a & b for which

$x + 2y + 3z = 6$

$x + 3y + 5z = 9$

$2x + 5y + az = b$

has ① No solution

(ii) Unique solution

(iii) Infinite solution

Soln,

Echelon form :

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-8 & b-15 \end{bmatrix}$$

~~Solut~~

① $\rho(A) < \rho(A|B)$

$a = 8 \quad b \neq 15$

(ii) Unique soln,

$\rho(A) = \rho(A|B) = n$

$a \neq 8 \ , \ b \neq 15$

(iii) Infinite solution

$a = 8 \ , \ b = 15$

Find algebraic and geometric multiplicity of

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

$$\begin{pmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{pmatrix} = 0.$$

$\cancel{(1-\lambda)(2-\lambda)(2-\lambda) - 2(}$

$$(1-\lambda)\left[(2-\lambda)(2-\lambda) - 2\right] - 2\left[+1\right] + 2\left[2-\lambda\right] = 0$$

$$(1-\lambda)(4+\lambda^2-4\lambda)(-2+\lambda) - 2 + 4 - 2\lambda.$$

$$4+\lambda^2-4\lambda-4\lambda-\lambda^3+4\lambda^2-2+2\lambda-2+4-2\lambda = 0.$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\lambda = 1, 2, 2$$

λ	AM	GM
1	1	1
2	2	1