12.12.22

Introduction

Module 1: Analytical Functions

Complex function $w = u + iv$

Module 2: Laplace Eqn

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0$$

$\boxed{\text{Laplace Transformations} \neq \text{Laplace Equations}}$

$$w = \frac{az+b}{cz+d}.$$

$z \to$ domain part

$w \to$ image part

Module 3: Complex Integral

We use cauchy Eqn. to simplify the integration.

Module 4:

If one vector is dependent on other ~~equation~~ vector,

then $\boxed{\vec{a} = \lambda \vec{b}}$.

$\{v_1, v_2, v_3\} \to$ bases, using bases entire vector can be created.

Module 6:

Orthogonal vector: $\vec{v_1} \cdot \vec{v_2} = 0$

Gram-Schmidt:

$\{v_1, v_2, v_3\} \to \{w_1, w_2, w_3\}$

Module 1 :

Geo. Representation :

$z = x + iy$

$z_1 = x_1 + iy_1$

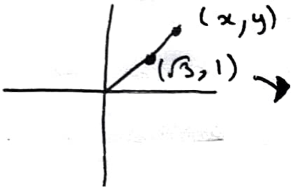
$Re(z) = x$

$Im(z) = y$

Modulus $|z| = \sqrt{x^2 + y^2}$

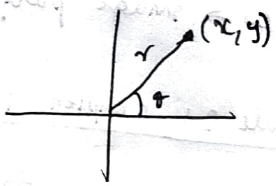
Distance between z + z_1 = $\sqrt{(x_1 - x)^2 + (y_1 - y)^2}$

Polar Form :



$\rightarrow \boxed{z = \sqrt{3} + i}$

$r, \theta \rightarrow$: Polar Form



$x = r\cos\theta$

$y = r\sin\theta$

$\theta = \tan^{-1}(y/x)$

$z = r(\cos\theta + i\sin\theta)$

Example:

1) $(\sqrt{3} + i) \rightarrow$ Polar Form

$r = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$.

$z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

2) $12\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right) \longrightarrow$ Rect. Form

$x = 12 \cos\frac{\pi}{6} = 6 \times \frac{\sqrt{3}}{2} = 6\sqrt{3}$

$y = 12 \sin\frac{\pi}{6} = 6 \times \frac{1}{2} = 6$

$z = 6\sqrt{3} + 6$

Circle: Set of all points (only boundary)

$|z - z_0| = \delta$

δ - radius

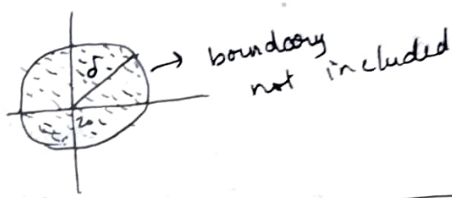
z_0 - center.

Open disk: All points inside $\boxed{\text{except boundary}}$.

$|z - z_0| < \delta$

z_0 - center

δ - radius



Closed disk: All points inside $\boxed{\text{including boundary}}$

$|z - z_0| > \delta$

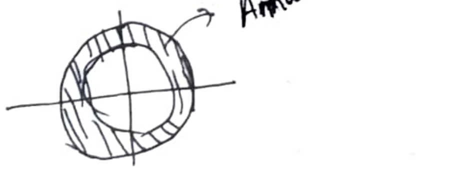
z_0 - center

δ - radius

Annulas Region: Region between 2 concentric circles

$|z - z_0| = \delta_1$

$|z - z_1| = \delta_2$



$\delta_1 \leq |z - z_0| \leq \delta_2$: boundary included : closed annulas

$\delta_1 < |z - z_0| < \delta_2$: ~~open~~ boundary excluded : open annulas

<u>δ- neighbourhood of z_0:</u>

open disk with center z_0 & radius δ is called δ- nbd of z_0.

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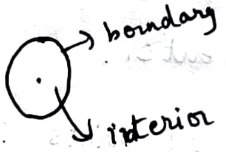
<u>Inferior Point:</u> (Boundary & Interior Point)

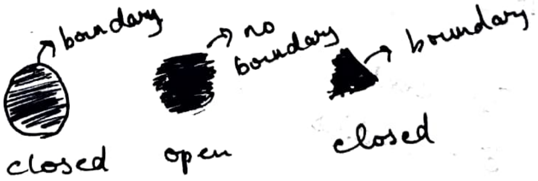
~~All~~ a ~~points of neighbour are interior~~.



→ A point $z_0 \in S$ is interior if there is $\boxed{\text{some nbd}}$ of z_0 ~~other~~ whose all element's are ~~inside~~ in S.

→ A point $z_0 \in S$ is boundary if $\boxed{\text{every nbd}}$ consists of atleast one ~~part~~ point inside S and atleast 1 point outside S.

→ A $\boxed{\text{set}}$ is open if every point is interior points.

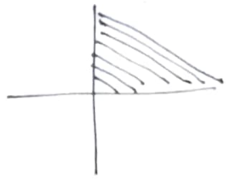


→ A $\boxed{\text{connected set}}$ is a set which is connected if any two points are connected by polygonal (on) which is entirely in S.

Bounded set :

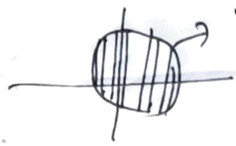
A set S is bounded if there is a positive integer M such that (all absolute values belongs to S)

$$|z| \leq M \quad \forall z \in S$$



non bounded

Domain : A set is domain if it is open and connected.



bounded.

Problems :

① $S = \{z \mid Im(z) \neq 0\}$

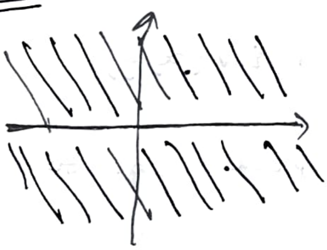
② $S = \{z \mid Im(z) > 1\}$

③ $S = \{z \mid Im(z) > Re(z)\}$

④ $S = \{z \mid Re(z) \leq 1,\ 0 \leq \theta < \frac{\pi}{4}\}$

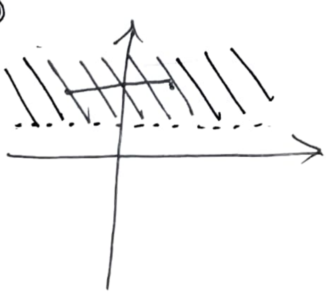
Ans :

①



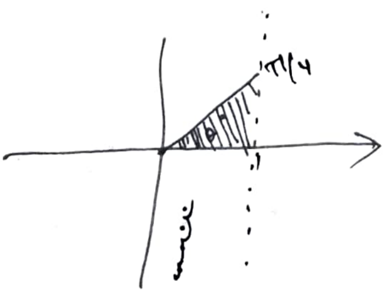
open , non-bounded.
non-connected , non-domain.

②



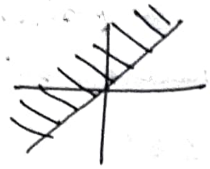
open, non-bounded, connected,
~~non~~ domain.

④

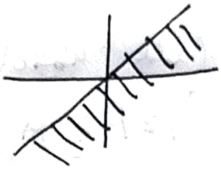


closed , bounded,
connected, non-domain.

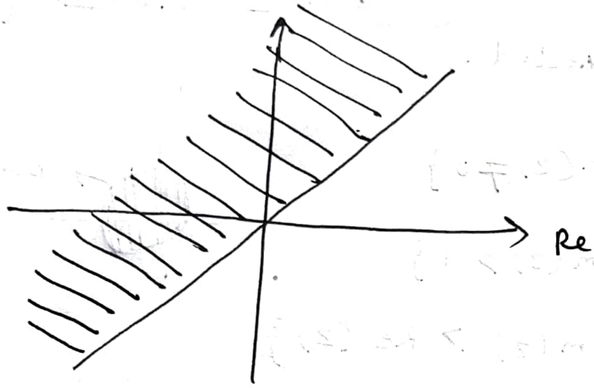
3)



$y > x$



$y < x$.



open, non bounded,
∴ connected, domain.

Complex Function:

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$

A complex function maps a complex no. to another complex no.

if, ① $f(z) = z^2$

$f(1+i) = (1+i)^2 = 2i$

② $f(z) = (x^2+y^2) + i(2xy)$

$f(2+i) = 5 + 4i$

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Limit: Let $f(z)$ be a function defines in some nbd of z_0, then if the function value tends to

l as z tends to z_0 along any direction.

$$\lim_{z \to z_0} f(z) = l.$$

① $$\lim_{z \to 0} \frac{z^2}{|z|} = \lim_{(x,y) \to (0,0)} \left(\frac{(x+iy)^2}{\sqrt{x^2+y^2}} \right)$$

$$= \lim_{(x,y) \to (0,0)} \left[\frac{x^2 - y^2 + 2xiy}{\sqrt{x^2+y^2}} \right]$$

$$= \lim_{x \to 0} \left(\lim_{y \to 0} \left(\frac{x^2 - y^2 + 2xyi}{\sqrt{x^2+y^2}} \right) \right)$$

$$= \lim_{x \to 0} \left(\frac{x^2}{x} \right) = \lim_{x \to 0} (x) = 0.$$

Similarly,

$$= \lim_{y \to 0} \left(\lim_{x \to 0} \left(\frac{x^2 - y^2 + 2xyi}{\sqrt{x^2+y^2}} \right) \right)$$

$$= \lim_{y \to 0} \left(\frac{y^2}{\cancel{y}} \right) = \lim_{y \to 0} (y) = 0.$$

$\therefore$ Limit exists.

Limit should exist along $y = mx$, $y = mx^2$, $x = my$, $x = my^2, \ldots$

② $$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{(x,y) \to (0,0)} \left(\frac{x - iy}{x + iy} \right)$$

$$\lim_{x \to 0} \left(\lim_{y \to 0} \left(\frac{x - iy}{x + iy} \right) \right) = \lim_{x \to 0} \left(\frac{x}{x} \right) = 1.$$

$$\lim_{y \to 0} \left(\lim_{x \to 0} \left(\frac{x - iy}{x + iy} \right) \right) = \lim_{y \to 0} \left(\frac{-\cancel{iy}}{\cancel{iy}} \right) = -1.$$

Since, we have different values, we say that limit doesn't exist.

(3) $$\lim_{z \to 1+i} (z^2 + i) = 3i$$

$$= \lim_{(x,y) \to (1+i, 1+i)} \cancel{z^2 +} ((x+iy)^2 + i)$$

$$= \lim_{x \to 1+i} \left(\lim_{y \to 1+i} \left(x + i(\cancel{1+i}^{\,y}) + i \right) \right)$$

$$= \lim_{x \to 1+i} \left(x + i(1+i) + i \right)$$

$$= \lim_{x \to 1+i} \left(x + i + (-1) + i \right)$$

$$= \cancel{\lim_{x \to 1+i}} \left(1+i+i-1+i \right) = 3i.$$

$$\lim_{z \to 1+i} (z^2 + i)$$

$$= \lim_{y \to 1+i} \left(\lim_{x \to 1+i} \left((x+iy)^2 + i \right) \right)$$

$$= \lim_{y \to 1+i} \left(1+i+iy + i \right)$$

$$= \lim_{y \to 1+i} \left(1+i+iy+i \right) = 1+i+i(1+i)+i$$

$$= 1+i+i-1+i = 3i$$

$\therefore$ The limit exists.

Continuity :

Let $f(z)$ be a function defined in some nbd of z_0

then the function $f(z)$ be continuous at z_0 if,

(i) $f(z_0)$ exists

(ii) $\lim_{z \to z_0} f(z)$ exists

(iii) $\lim_{z \to z_0} f(z) = f(z_0)$

Example :

① $f(z) = \begin{cases} \frac{Im(z)}{|z|} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$

$f(z_0) = f(0) = 0.$

$$\lim_{z \to 0} f(z) = \lim_{(x,y) \to (0,0)} \left(\frac{y}{\sqrt{x^2+y^2}} \right)$$

$$= \lim_{x \to 0} \left(\lim_{y \to 0} \left(\frac{y}{\sqrt{x^2+y^2}} \right) \right) = 0$$

$$\lim_{z \to 0} f(z) = \lim_{(x,y) \to (0,0)} \left(\frac{y}{\sqrt{x^2+y^2}} \right)$$

$$= \lim_{y \to 0} \left(\lim_{x \to 0} \left(\frac{y}{\sqrt{x^2+y^2}} \right) \right) = \lim_{y \to 0} \left(\frac{\cancel{y}}{\cancel{\sqrt{y^2}}} \right)$$

$$= 1.$$

$\because$ the limit value is not same it is not continuous at $z_0 = 0$.

②

$$\lim_{z \to 1-i} \left(z^2 - iz + 2 \right) = 1 - 3i$$

$$f(z_0) = f(1-i) = (1-i)^2 - i\overset{(1-i)}{\cancel{z}} + 2$$

$$= 1 - 3i$$

Differentiability:

Let $f(z)$ be a function defined in some nbd of z_0 then the function $f(z)$ is said to be differentiable at z_0 if,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

Example:

① $f(z) = x + 4iy$

(i) z_0 at $(0,0)$

$$\lim_{(x,y) \to (0,0)} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{(x,y) \to (0,0)} \frac{x + 4iy - 0}{x + iy - 0}$$

$$= \lim_{x \to 0} \left(\lim_{y \to 0} \left(\frac{x + 4iy}{x + iy} \right) \right)$$

$$= \lim_{x \to 0} \left(\frac{\cancel{x}}{\cancel{x}} \right) = 1.$$

$$\lim_{y \to 0} \left(\lim_{x \to 0} \left(\frac{x + 4iy}{x + iy} \right) \right)$$

$$= \lim_{y \to 0} \left(\frac{4i\cancel{y}}{i\cancel{y}} \right) = 4.$$

Since, the values are different the function is not differentiable.

(ii) z_0 at $(1,1)$

$$\lim_{(x,y)\to(0,0)} \left(\frac{x+4iy-(1+4i)}{x+iy-(1+i)} \right)$$

$$= \lim_{x\to 0} \left(\lim_{y\to 0} \left(\frac{x+4iy-(1+4i)}{x+iy-(1+i)} \right)\right)$$

$$= \lim_{x\to 0} \left(\frac{x+0-(1+4i)}{x+0-(1+i)} \right)$$

$$= \lim_{x\to 0} \left(\frac{x-(1+4i)}{x-(1+i)} \right)$$

$$= 0 - 1+$$

$$\lim_{(x,y)\to(1,1)} \left(\frac{x+4iy-(1+4i)}{x+iy-(1+i)} \right)$$

$$\lim_{x\to 1} \left(\lim_{y\to 1} \left(\frac{x+\cancel{4i}-1-\cancel{4i}}{x+\cancel{i}-1-\cancel{i}} \right)\right)$$

$$\lim_{x\to 1} \left(\frac{\cancel{x-1}}{\cancel{x-1}} \right) = \lim_{x\to 1} (1) = 1.$$

$$\lim_{y\to 1} \left(\lim_{x\to 1} \left(\frac{x+4iy-(1+4i)}{x+iy-(1+i)} \right)\right)$$

$$= \lim_{y\to 1} \left(\frac{\cancel{1}+4iy-\cancel{1}-4i}{\cancel{1}+iy-\cancel{1}-i} \right)$$

$$= \lim_{y\to 1} \left(\frac{4iy-4i}{iy-i} \right) = \lim_{y\to 1} \left(\frac{4\cancel{i}\cancel{(y-1)}}{\cancel{i}\cancel{(y-1)}} \right) = 4.$$

Since, the values are different. It is not differentiable.

(2) Find derivative of $z^3 - 2z^2 + 3z$.

$f'(z) = 3z^2 - 4z + 3$.

(3) Check differentiability of

$$f(z) = \begin{cases} \dfrac{x^3 y (y - ix)}{x^6 + y^2}, & z \neq 0 \\ 0 & z = 0 \end{cases}$$

at $z = 0$.

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Analytic Function:

A function $f(z)$ is said to be analytic at z_0 if it is differentiable at z_0 and in some nbd of z_0.

Necessary condition for analyticity:

(★) If a function is differentiable at z_0, then first order partial derivative exists and satisfy cauche - riemann eqn (C-R eqn)

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

(★) If a function is analytic, then it must satisfy CR equations.

CR eqn + continuity of partial Derivative → function is analytic.

Sufficient Condition for Analyticity :

If the first order partial derivative exists ~~and~~, continuous and it satisfies CR equation, then the function is analytic.

If the function is analytic at every point, then it is called $\boxed{\text{entire function}}$

Example:

① $|z|^2 = x^2 + y^2$

$$= \underbrace{(x^2+y^2)}_{u} + \underbrace{0}_{v}i$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} = \cancel{2y}\ 0 \qquad \frac{\partial u}{\partial y} = 2y$$

$$\because \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} \neq -\frac{\partial u}{\partial y}$$

The function is not analytic.

② $z^2 = (x+iy)^2$

$$= x^2 + y\,2ixy - y^2$$

$$= \underbrace{(x^2-y^2)}_{u} + \underbrace{(2xy)}_{v}\,i$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial v}{\partial x} = 2y \qquad \frac{\partial u}{\partial y} = -2y$$

$$\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

→ The function is analytic. ✪

→ Since, the function is a polynomial equation, the partial derivative ~~function~~ is continuous.

→ So, it is an entire function ✪.

Note: All polynomial equation, Exponential function, sine function & cosine functions' ~~are~~ partial derivatives are continuous

③ e^z

$$z = x + iy$$

$$e^z = e^{x+iy}$$

$$= e^x \cdot e^{iy}$$

↓ expanding this

$$= e^x(\cos y + i \sin y)$$

$$= \underbrace{(e^x \cos y)}_{u} + \underbrace{(e^x \sin y)}_{v} i$$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial v}{\partial x} = e^x \sin y \qquad \frac{\partial u}{\partial y} = -e^x \sin y.$$

→ ∵ It satisfies CR equations, it is analytic

→ Exponential functions' partial derivatives are continuous

→ ∴ It is entire function.

(4) $\bar{z} = x - iy$

$u = x, \quad v = -y$

$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial v}{\partial y} = -1 \quad \frac{\partial v}{\partial x} = 0 \quad \frac{\partial u}{\partial y} = 0$

∵ It doesn't satisfy the CR equation, It is not analytic.

So, it is not entire function.

CR equation for polar form:

$f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \; ; \; \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

Examples:

(1) $z^2 = (re^{i\theta})^2$

$= r^2 \cdot e^{2i\theta}$

$= r^2(\cos 2\theta + i\sin 2\theta)$

$= \underbrace{(r^2\cos 2\theta)}_{u} + \underbrace{(r^2\sin 2\theta)}_{v} i$

$\frac{\partial u}{\partial r} = 2r\cos 2\theta \; ; \; \frac{\partial v}{\partial \theta} = 2r^2\cos 2\theta \Rightarrow \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{2r^2\cos 2\theta}{r} = 2r\cos 2\theta$

$\frac{\partial v}{\partial r} = 2r\sin 2\theta \quad \frac{\partial u}{\partial \theta} = -2r^2\sin 2\theta \Rightarrow \frac{1}{r}\frac{\partial u}{\partial \theta} = -2r\sin 2\theta$

Since, it satisfies CR equation it is analytic.

It is an exp. function ~~and~~ so par. der is continuous.

$\therefore$ It is entire function.

(2) $\frac{1}{z^2} = \frac{1}{(re^{i\theta})^2} = (re^{i\theta})^{-2} = (\overset{u}{r^{-2}\cos 2\theta}) - (\overset{v}{r^{-2}\sin 2\theta})i$

$\frac{\partial u}{\partial r} = -2r^{-3}\cos 2\theta$; $\frac{\partial v}{\partial \theta} = -2r^{-2}\cos 2\theta \Rightarrow \frac{1}{r}\frac{\partial v}{\partial \theta} = -2r^{-3}\cos 2\theta$

$\frac{\partial v}{\partial r} = +2r^{-3}\sin 2\theta$; $\frac{\partial u}{\partial \theta} = -2r^{-2}\sin 2\theta \Rightarrow \frac{1}{r}\frac{\partial u}{\partial \theta} = -2r^{-3}\sin 2\theta$

& It satisfies CR eqn, it is analytic.

It is exp. func. So P.D of fn is Continuous. So its entire

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<u>Harmonic Function:</u> A real valued function $u(x,y)$ is said to be harmonic function if it has second order partial derivatives and satisfy Laplace eqn.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

equation – Laplace eqn
u – harmonic
function satisfies is harmonic

Ex: $u(x,y) = x^2 - y^2 - x$

~~polar form~~

$\frac{\partial u}{\partial x} = 2x - 1$ $\quad \frac{\partial^2 u}{\partial x^2} = 2 \rightarrow$ (1)

$\frac{\partial u}{\partial y} = -2y$ $\quad \frac{\partial^2 u}{\partial y^2} = -2 \rightarrow$ (2)

(1) + (2) $= 0$.

$\therefore$ It is harmonic.

Polar form of Laplace eqn for $u(r,\theta)$:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Eg: $u(r,\theta) = r^2 \cos 2\theta$

$$\frac{\partial u}{\partial r} = 2r \cos 2\theta$$

$$\frac{\partial^2 r}{\partial r^2} = 2 \cos 2\theta \rightarrow ①$$

$$\frac{1}{r}\frac{\partial u}{\partial r} = 2 \cos 2\theta \rightarrow ②$$

$$\frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta$$

$$\frac{\partial^2 u}{\partial \theta^2} = (-4r^2 \cos 2\theta) = \cancel{8r^2 \cos\theta} \rightarrow ③$$

$$① + ② + ③ = 0.$$

$\therefore$ This is harmonic.

(*) If the function is analytic, ~~it follows laplace eqn and~~ u & v are harmonic.

(*) If ~~the~~ u & v are harmonic and follows CR eqn it is analytic.

Harmonic Conjugate of a function : If 2 harmonic function $u(x,y)$ and $v(x,y)$ satisfy CR eqn in a domain and they are real and imaginary parts of an analytic function, then $v(x,y)$ is said to harmonic conjugate of $u(x,y)$ and vice versa.

Verify if z^2 is analytic by verifying CR equations and harmonic property.

Soln:

$$z = x + iy$$

$$z^2 = \underbrace{x^2 - y^2}_{u} + \underbrace{2xy}_{v}i$$

CR equation:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 - y^2) = 2x \rightarrow ①$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x \rightarrow ②$$

$$-\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 - y^2) = +2y \rightarrow ③$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y \rightarrow ④$$

$$① = ② \quad \& \quad ③ = ④$$

$\therefore$ It satisfies CR eqn.

$$\frac{\partial^2 u}{\partial x^2} = 2. \qquad \frac{\partial^2 u}{\partial y^2} = -2.$$

$$\frac{\partial^2 v}{\partial x^2} = 0 \qquad \frac{\partial^2 v}{\partial y^2} = 0$$

$\therefore$ It satisfies harmonic eqn

$\therefore$ It is analytic.

③ $u(x,y) = x^3 - 3xy^2 - 5y$ is harmonic eqn ?

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$ $\qquad$ $\frac{\partial u}{\partial x} = -6xy - 5$

$\frac{\partial u^2}{\partial x^2} = 6x$ $\qquad$ $\frac{\partial u^2}{\partial y^2} = -6x.$

$\frac{\partial u^2}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = 6x - 6x = 0.$ $\therefore$ It is harmonic

④ Verify $\sin z$ is analytic.

$u(x,y) = \sin x \cos hy$

$v(x,y) = \cos x \sin hy$

$\frac{d}{dx}(\cos hx) = \sin(hx)$

$\frac{d}{dx}(\sin hx) = \cos(hx)$

$\sin z = \sin(x+iy)$

$= (\sin x \cos iy) + (\cos x + \sin iy)$

$= (\sin x \cos hy) + i(\cos x \sin hy)$

Similarly,

$\cos z = \cos x \cos hy - i \sin x \sin hy$

~~$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\sin x \cos hy)$~~

~~$\frac{\partial u}{\partial x} = \cos x \cos hy$~~

~~$\frac{\partial^2 u}{\partial x^2} = -\sin x \cos hy$~~

~~$\frac{\partial u}{\partial y} = \sin hy$~~

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\sin x \cos hy)$

$= \cos x \cos hy \rightarrow ①$

$\frac{\partial v}{\partial y} = \cos x \cos hy \rightarrow ②$

$\frac{\partial v}{\partial x} = -\sin x \sin hy \rightarrow ③$

contd..

$\frac{\partial u}{\partial y} = \sin x \sin hy$

$① = ②$ & $③ = -④$

$\therefore$ It satisfies CR equations.

$$\frac{\partial^2 u}{\partial x^2} = -\sin x \cosh y \rightarrow (5)$$

$$\frac{\partial^2 u}{\partial y^2} = \sin x \cosh y \rightarrow (6)$$

$$\frac{\partial^2 v}{\partial x^2} = -\cos x \sinh y \rightarrow (7)$$

$$\frac{\partial^2 v}{\partial y^2} = \cos x \cosh y \rightarrow (8)$$

$(5) + (6) = 0$ & $(7) + (8) = 0$

$\therefore$ It is harmonic.

Since, ~~So,~~ it is harmonic and follow CR equations, it is analytic.

(5) V.T. $\cos z$ is analytic.

$$\cos z = \overbrace{(\cos x \cosh y)}^{u} - i\,\overbrace{(\sin x \sinh y)}^{v}$$

$$u(x,y) = \cos x \cosh y$$

$$v(x,y) = -\sin x \sinh y$$

$$\frac{\partial u}{\partial x} = -\sin x \cosh y \rightarrow (1)$$

$$\frac{\partial v}{\partial y} = -\sin x \cosh y \rightarrow (2)$$

$$\frac{\partial v}{\partial x} = -\cos x \sinh y \rightarrow (3)$$

$$\frac{\partial u}{\partial y} = \cos x \sinh y \rightarrow (4)$$

$(1) = (2)$ & $(3) = -(4)$

$\therefore$ It follows CR equations.

$$\frac{\partial^2 u}{\partial x^2} = -\cos x \cos hy \rightarrow 5$$

$$\frac{\partial^2 u}{\partial y^2} = \cos x \cos hy \rightarrow 6$$

$$\frac{\partial^2 v}{\partial x^2} = \sin x \sin hy \rightarrow 7$$

$$\frac{\partial^2 v}{\partial y^2} = -\sin x \sin hy \rightarrow 8$$

$5 + 6 = 0$

$7 + 8 = 0$

$\therefore$ It is harmonic

Since, it is harmonic and follows CR eqn it is analytic.

3) Find c_1 & c_2 such that the function $f(z)$ is analytic.

$$f(z) = \underbrace{(x^2 + c_1 y^2 - 2xy)}_{u} + i\,\underbrace{(c_2 x^2 - y^2 + 2xy)}_{v}$$

$$\frac{\partial u}{\partial x} = 2x - 2y$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = 2} \rightarrow 1$$

$$\frac{\partial u}{\partial y} = 2c_1 y - 2x$$

$$\boxed{\frac{\partial^2 u}{\partial y^2} = 2c_1} \rightarrow 2$$

$$\frac{\partial v}{\partial x} = 2c_2 x + 2y$$

$$\boxed{\frac{\partial^2 v}{\partial x^2} = 2c_2} \rightarrow 3$$

$$\frac{\partial v}{\partial y} = -2y + 2x$$

$$\boxed{\frac{\partial^2 v}{\partial y^2} = -2} \rightarrow 4$$

~~Accor~~

Since it's analytic, it is also harmonic,

So ① + ② = 0 & ③ + ④ = 0

$2 + 2c_1 = 0$

$2(1 + c_1) = 0$

$1 + c_1 = 0$

$\boxed{c_1 = -1}$

$2c_2 - 2 = 0$

$2c_2 = 2$

$\boxed{c_2 = 1}$

21.12.21

Finding Harmonic Conjugate:

Method 1: using CR Equation

① Find harmonic conjugate of $u(x,y) = x^3 - 3xy^2 - 5y$

So, we need to find $v(x,y)$.

Soln,

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y} \rightarrow ①$$

$$\frac{\partial u}{\partial y} = -6xy - 5 = -\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial x} = 6xy + 5 \rightarrow ②$$

integrating ① w.r.t. y

$$v = 3x^2y - y^3 + h(x)$$

since integrating its partial derivatives, we have constant as function of x.

partially differentiating w.r.t. x, (to find $h(x)$)

$$\frac{\partial v}{\partial x} = 6xy + h'(x) \rightarrow ③$$

Comparing ② & ③,

$$h'(x) = 5$$

$$\boxed{h(x) = 5x + c} \to ④$$

Substitute ④ in v :

$$v = 3x^2y - y^3 + 5x + c$$

② $u = x^2 - y^2 - x$ $\quad v = ?$

CR eqn :

$$\frac{\partial u}{\partial x} = 2x - 1 = \frac{\partial v}{\partial y} \to ①$$

$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial x} = 2y \to ②$$

Integrating ① w.r.t. y

$$v = 2xy - y + h(x) \to ③$$

. par. diff w.r.t.x

$$\frac{\partial v}{\partial x} = 2x + h'(x)$$

$$h'(x) = 0$$

$$h'(x) = c$$

$$\boxed{v = \cancel{3x^2y}\ 2xy - y + c}$$

(3) $v = 3x^2y - y^3$ $\quad u = ?$

CR eqn:

$$\frac{\partial v}{\partial x} = 3(2)xy = 6xy = -\frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 = \frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial y} = -6xy \rightarrow (1)$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \rightarrow (2)$$

Integrate (2) w.r.t. x

$$u = x^3 - 3xy^2 + h(y) \rightarrow (3)$$

par. diff (3) w.r.t. y

$$\frac{\partial u}{\partial y} = -6xy + h'(y) \rightarrow (4)$$

By comparing (1) & (4)

$$h'(y) = 0$$

$$h(y) = c$$

$$\boxed{u = x^3 - 3xy^2 + c}$$

(4) $u = 3x^2y + 2x^2 - y^3 - 2y^2$

CR eqn:

$$\frac{\partial u}{\partial x} = 6xy + 4x = \frac{\partial v}{\partial y} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = \cancel{2}\,3x^2 - 3y^2 - 4y = -\frac{\partial v}{\partial x} \rightarrow (2)$$

Integrate

~~Diff~~ (1) w.r.t. $\cancel{x}$ .y

$$\cancel{f} = \quad v = \int 6xy\,\partial y + \int 4x\,\partial y$$

$$= \frac{6xy^2}{2} + 4xy + h(x)$$

$$v = 3xy^2 + 4xy + h(x) \rightarrow (3)$$

Partially differentiation of (3) w.r.t x.

$$\frac{\partial v}{\partial x} = 3y^2 + 4y + h'(x)$$

$$-\frac{\partial v}{\partial x} = -3y^2 - 4y \cancel{7} - h'(x) \rightarrow (4)$$

Comparing (1) & (2), we get

$$-h'(x) = +3x^2$$

$$h(x) = -3\cancel{2}\frac{x^3}{3} = \cancel{2}-x^3.$$

$$v = 3xy^2 + 4xy + \cancel{2}(-x^3)$$

$$v = 3xy^2 + 4xy - x^3$$

Method 2 : Milne Thompson Method

$u_x = \frac{\partial u}{\partial x}$

$f' = u_x + i v_x$

$v_x = \frac{\partial v}{\partial x}$

① Take $f' = u_x - i u_y$

Take $f' = v_y + i v_x$

② Replace x by z & y by 0. (Replace r by z, θ by 0)

③ Integrate f' w.r.t. z, will give f.

If real & imaginary part asked seperately, then substitute $z = x + iy$

Example

① If $u = x^2 - y^2 - x$, find v.

Soln:

$f' = u_x - i u_y$

$f' = (2x - 1) - i(-2y)$

$f' = (2x - 1) + i2y$

replace x by z & y by 0.

$f' = 2z - 1$

$f = z^2 - z + c$

$z = x + iy$

$f = (x + iy)^2 + (x + iy) + c$

$f = x^2 - y^2 + 2xiy + x + iy + c$

$$f = \underbrace{(x^2 - y^2 - x)}_{x} + i\,\underbrace{(2xy - y)}_{v} + c$$

② $v = \dfrac{2 \sin x \quad \sin y}{\cos 2x + \cosh 2y}$

Find u.

$f' = v_y + i v_x$

Soln : $f(z) = \sec z + c$

$2 \sin A \sin B$

$= \cos(A+B) + \cos(A-B)$

$= \cos(x+y) + \cos(x-y)$

③ $u = e^x (x \cos y - y \sin y) + 2 \sin x \cdot \sin hy$

$+ x^3 - 3xy^2 + y$

Soln : $f(z) =$
$ze^z + z^3 + 2i \cos z - iz + c.$

Answers:

② $f' = v_y + i v_x$

$$\frac{\partial v}{\partial y} = \frac{[\cos 2x + \cosh(2y)][2 \sin x \cos y] - [2 \sin x \sin y][\sin h(xy)]}{[\cos 2x + \cos h(2y)]^2}$$

$$\frac{\partial v}{\partial y} = \frac{2 \cos 2x \sin x \cos y + 2 \sin x \cos y \cosh(2y) - 2 \sin x \sin y \sin h(xy)}{\cos^2 2x + \cosh^2(2y) + 2 \cos 2x \cos h(2y)}$$

22.12.22

① if $u(r, \theta) = r^2 \cos 2\theta + r \sin\theta$

find v using Milne's Method

$$f' = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right), \quad f' = e^{-i\theta}\left(\frac{\partial u}{\partial r} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right)$$

$$f' = e^{-i\theta}\Big[(2r\cos 2\theta + \sin\theta) - \frac{i}{r}(-2r^2 \sin 2\theta + r\cos\theta$$

replace r by z, θ by 0.

$$f' = 2z - i$$

$$f = z^2 - iz + c \quad (\text{integrating ①})$$

$$z = re^{i\theta}$$

$$f = (re^{i\theta})^2 - i(re^{i\theta}) + c$$

$$f = r^2 e^{2i\theta} - i(re^{i\theta}) + c$$

$$f = r^2(\cos 2\theta + i\sin 2\theta) - i(r(\cos\theta + i\sin\theta))$$

$$\boxed{v = r^2 \sin 2\theta - r\cos\theta}$$

② If $v = \left(r - \frac{1}{r}\right)\sin\theta$, find u using Method 1.

using CR equations,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = \left[1 - (-r^{-2})\right]\sin\theta = \left(1 + \frac{1}{r^2}\right)\sin\theta$$

$$\frac{\partial v}{\partial \theta} = r\cos\theta - \frac{\cos\theta}{r}$$

Applications - Orthogonal Trajectories

If the function $f = u + iv$ is analytic, then the family of curves $u(x,y) = c_1$ & $v(x,y) = c_2$ are mutually orthogonal and $u = c_1$ are said to be orthogonal trajectories of $v = c_2$.

① Find out the orthogonal trajectories $u(x,y) = x^3y - xy^3 = c$

$v = ?$

CR Eqn:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = 3x^2y - y^3$$

$$\frac{\partial u}{\partial y} = x^3 - 3y^2x$$

$$\frac{\partial v}{\partial y} = 3x^2y - y^3 \rightarrow ①$$

$$\frac{\partial v}{\partial x} = -x^3 + 3y^2x \rightarrow ②$$

~~$\partial v =$~~

Integrating ① w.r.t. y,

$$v = \frac{3x^2y^2}{2} - \frac{y^4}{4} + h(x)$$

$$\partial v/\partial x = \frac{3}{\cancel{2}}y^2(\cancel{2}x) - \frac{\cancel{4}y^3}{\cancel{4}} + h'(x)$$

$$\partial v/\partial x = 3xy^2 - y^3$$

Tutorial Sheet

5.1.23

① Find $\lim_{z \to 0} \frac{x^2 y}{x^4 + y^2}$

along x, y, mx $\lim_{z \to 0} \frac{x^2 y}{x^4 + y^2} = 0$

$\lim_{y \to mx^2} \left(\frac{m^2}{1+m^2} \right)$

∴ Limit value depends on m, so the limit doesn't exists.

② $F(z) = \begin{cases} \frac{z^2 + 3iz - 2}{z + i} & z \neq -i \\ 5 & z = -i \end{cases}$

$\lim_{z \to -i} f(z) = \frac{(-i)^2 + 3(+i)(-i) - 2}{-i-i} = \frac{-1+3-2}{0} = \frac{0}{0}$

which is undefined.

So, we are using L'hopital rule,

if $f(z) = \frac{g(z)}{h(z)}$ then $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \left(\frac{g'(z)}{h'(z)} \right)$

$\lim_{z \to -i} f(z) = \frac{\lim_{z \to -i} 2z + 3i}{1} = 2(-i) + 3i = -2i + 3i$

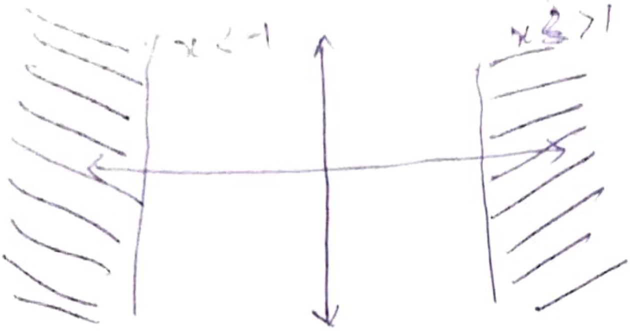
$= i.$

$f(-i) = 5$

Since, $\lim_{z \to -i} f(z) \neq f(-i)$

The function is not continuous.

⑥ $S = \{z \mid Re(z) > 1, Re(z) < -1\}$



It is open, not connected, not bounded and non-domain.

④ verify CR equation for the following:

ⓐ $z^2\bar{z}$

$f(z) = z^2\bar{z}$

$f(x+iy) = (x+iy)^2(x-iy) = (x^2 - y^2 + 2iy)(x - iy)$

$= x^3 - ix^2y - y^2x + iy^3 + 2iyx - 2i^2y$

$= (x^3 + 2y - y^2x) + i(-x^2y + y^3 + 2xy)$

$u = x^3 - xy^2 + 2y$

$v = y^3 - x^2y + 2xy.$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial x} = 3x^2 - y^2$ $\qquad \frac{\partial v}{\partial y} = 3y^2 - x^2 + 2x.$

$\frac{\partial v}{\partial x} = -2xy + 2x$ $\qquad -\frac{\partial u}{\partial y} = +2xy - 2$

CR equation satisfies only at $z=0$

Hence, it is not true.

(b) $(x-y)^2 + 2i(x+y)$

$f(x+iy) = (x-y)^2 + 2i(x+y)$

$u = (x-y)^2 = x^2 + y^2 - 2xy$

$v = 2(x+y)$

$\frac{\partial u}{\partial x} = 2x - 2y \qquad \frac{\partial v}{\partial y} = 2y$

$\frac{\partial u}{\partial y} = 2y - 2x \qquad \frac{\partial v}{\partial y} = 2x$

It satisfies only at $z = 0$.

$\therefore$ It is not true

(5) $\psi = x^2 - y^2 - 3x - 2y + 2xy$

$\phi = ?$

$\phi + i\psi$ is analytic.

Solution:

CR equations:

$\frac{\partial \psi}{\partial x} = 2x - 3 + 2y = -\frac{\partial \phi}{\partial y}$

$\frac{\partial \psi}{\partial y} = -2y - 2 + 2x = \frac{\partial \phi}{\partial x}$

$$\phi = \int (-2y - 2 + 2x)\, dx$$

$$\phi = -2xy - 2x + \frac{2x^2}{2} + c$$

$$\phi = x^2 - 2xy - 2x$$

$$\frac{\partial \phi}{\partial y} = -2x + h'(y)$$

$$\Rightarrow h'(y) = \cancel{x^2 - 2}\ 2y + 2$$

$$\phi = x^2 - 2xy - 2x + 2y + 2$$

The solution

$$\phi = x^2 - y^2 - 2x + 3y - 2xy.$$

6 Find b if $e^{bx} \cos 5y$ is harmonic and find its conjugate.

$\boxed{b = \pm 5}$

7 Find a, b if $f(z) = \cos x\, (\cosh y + a \sinh y) + i \sin x\, (\cosh y + b \sinh y)$

is analytic.

$\boxed{\text{Ans: } a = b = -1}$

8 If $u = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$, find v.

Ans: $\boxed{f = \frac{1}{z^2} + c}$

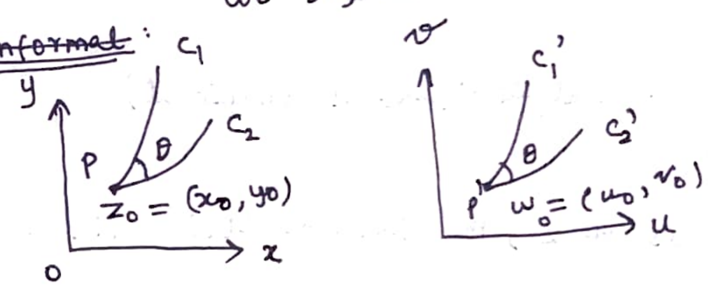
6.1.23

Module 2 : Conformal and Bilinear Transformation

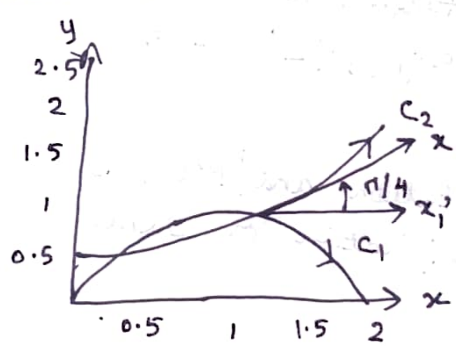
Isogonal : A transformation is said to be isogonal if 2 curves in the z-plane intersecting at the point z_0 at an angle θ are transformed into 2 corresponding curves in the w-plane intersecting at the point w_0, which corresponds to the point z_0 at the same angle θ.

We say equal in mag, if angles are same.

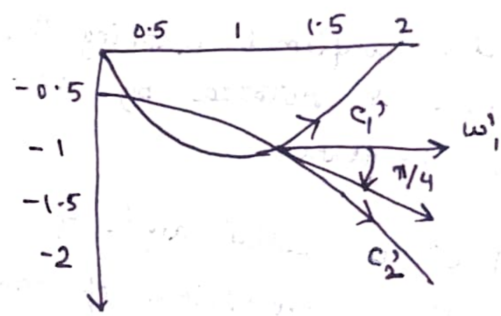
~~Conformal~~ :



Conformal : If the sense (direction) of the rotation as well as the magnitude of the angle is preserved, then the transformation is called confirmal



Curves C_1 and C_2 in z plane



Images of the curves under $w = \bar{z}$

Theorem:

If f is an analytic function in a domain D containing z_0 and if $f'(z_0) \neq 0$, then $w = f(z)$ is conformal mapping at z_0

Examples:

① e^z - analytic (has confirmal mapping for all z in $\mathbb{C}$)

$$f'(z) = e^z.$$

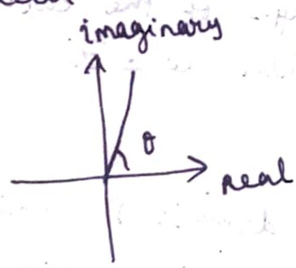
② $f(z) = z^2$ - analytic (has confirmal mapping ~~for~~ $\forall z \in \mathbb{C}$)

③ $f(z) = az + b$ - analytic (~~has~~ is confirmal)
$a \neq 0$

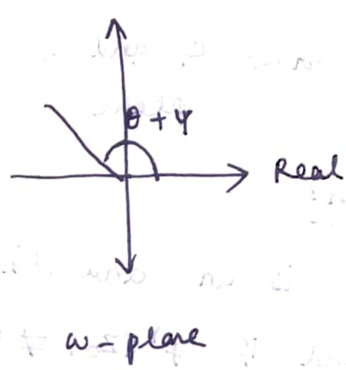
④ For what values of z, the function $f(z) = \sin z$ is confirmal?

Under confirmal mapping the tangent line of a curve is rotated by $\omega = \arg(f'(z))$ at z_0 in W- plane.

ω is called angle of rotation and $|f'(z)|$ is called co-efficient of magnification or scale factor at z_0.



$T \downarrow w = e^{i\nu} z$



Example: $\boxed{z = 2+i}$

a. The co-efficient of magnification at $z = 2+i$ is $2\sqrt{5}$

b. The angle of rotation at $z = 2+i$ is $\tan^{-1}(0.5)$

c. The co-efficient of magnification at $z = 1+i$ is $2\sqrt{2}$

d. The angle of rotation at $z = 1+i$ is $\frac{\pi}{4}$.

Standard Transformations:

- Translation
- Rotation
- Magnification
- Inversion

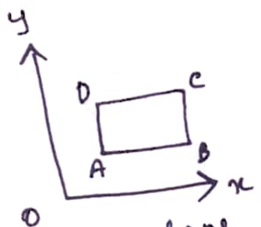
Translation:

The translation $w = z + c$ where $c = a + ib$ is called as translation.

$$w = (z + c) \quad u + iv = (x + iy) + a + ib$$

$$u = x + a \quad \text{and} \quad v = y + b$$

$$x = u - a \quad \text{and} \quad y = v - b.$$



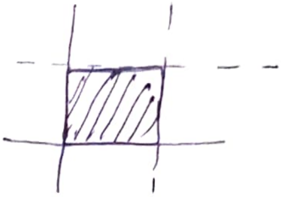
z-plane

9.1.23

Translation:

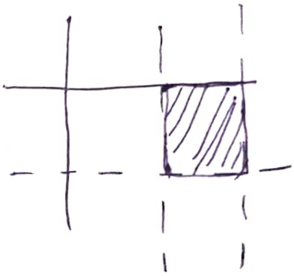
① Find image of $x = 0$, $y = 0$, $x = 1$ and $y = 1$ under $f(z) = z + 1 - i$

$f(z) = z + 1 - i = x + iy + 1 - i$



$= x + 1 + i(y - 1) = u + iv$

$u = x + 1$, $v = y - 1$



$x = 0 : u = 1$; $y = 0 \Rightarrow v = -1$

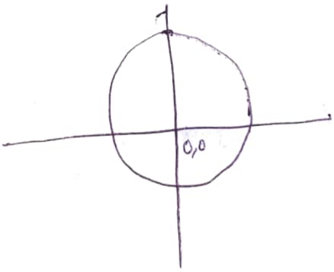
$x = 1 : u = 2$; $y = 1 \Rightarrow v = 0$

→ under translation size and shape of image does not change.

② Find image of $|z| = 2$ under $f(z) = z + 3 + 2i$

$z = x + iy$

$|z| = 2$



$\sqrt{x^2 + y^2} = 2$

$x^2 + y^2 = 4$

$f(z) = z + 3 - 2i = x + iy + 3 - 2i$

$= (x + 3) + i(y - 2)$

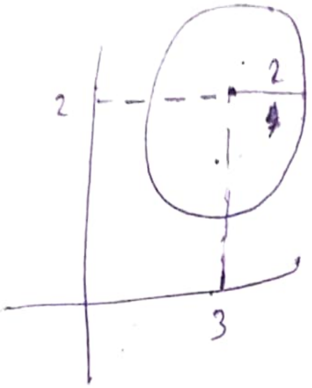
$u = x + 3$

$y = y - 2.$

~~$u = x + 3$~~,

$x = u - 3$, $y = v - 2$

$(u-3)^2 + (v-2)^2 = 4$

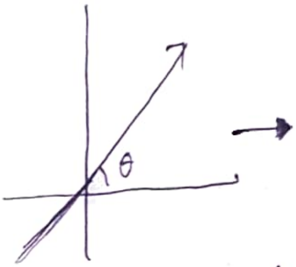


Rotation:

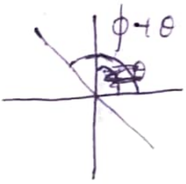
$f(z) = e^{i\phi} z$

$f(z) = e^{i\phi} \, r e^{i\theta}$

$f(z) = r e^{i(\theta+\phi)}$



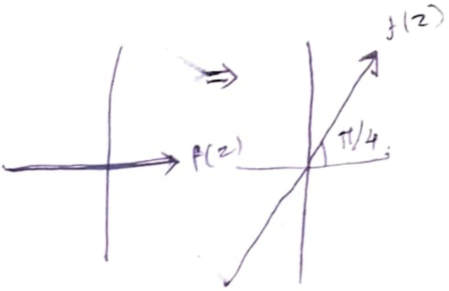
$(r, \theta) \xrightarrow[f = e^{i\phi} z]{} (r, \theta+\phi)$



① Find image of $y=0$ under $f(z) = e^{i\pi/4} z$

$f(z) = e^{\frac{i\pi}{4}} z$

$= \left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)(x+iy)$



$= \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)(x+iy)$

$= \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}} iy + i\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y$

$= \frac{1}{\sqrt{2}}(x-y) + \frac{i}{\sqrt{2}}(x+y)$

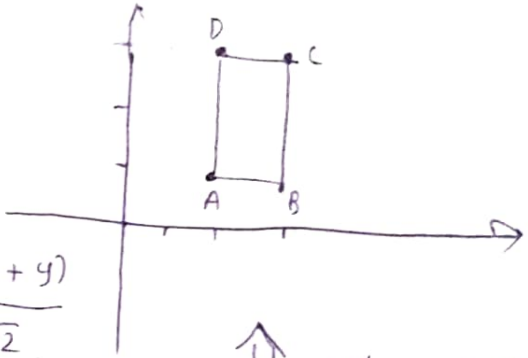
$u = \frac{1}{\sqrt{2}}(x-y) \quad v = \frac{1}{\sqrt{2}}(x+y)$

$$f(z) = \frac{x}{\sqrt{2}} + i\frac{x}{\sqrt{2}}.$$

② Find image of rectangle $A(2,1)$, $B(3,1)$, $C(3,3)$, $D(2,3)$

under $f(z) = e^{i\frac{\pi}{4}} z$.

$$f(z) = e^{i\pi/4} z$$



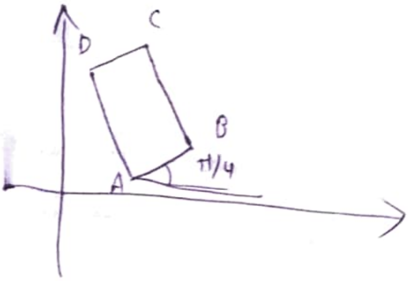
Soln:

$$f(z) = \frac{(x-y)}{\sqrt{2}} + \frac{i(x+y)}{\sqrt{2}}$$

$\Updownarrow$ $\pi/4$ rotation

$A(2,1) \to A'\left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$

$B(3,1) \to B'\left(\frac{2}{\sqrt{2}}, \frac{4}{\sqrt{2}}\right)$



$C(3,3) \to C'\left(0, \frac{6}{\sqrt{2}}\right)$

$D(2,3) \to D'\left(\frac{-1}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$

No change in size and shape of image but it gets rotated by angle ϕ.

Magnification:

$f(z) = bz$

$f(z) = bre^{i\theta}$ $\qquad z = re^{i\theta}$

$(r, \theta) \to (br, \theta)$

$f = bz$

if b is a real number.

① Find image of $x=1$, $x=3$, $y=1$, $y=3$.

$f(z) = 3z.$

$f(z) = 3(x+iy)$

$u = 3x$,

$v = 3y$

$x = 1 \Rightarrow u = 3$ $\qquad y = 1 \Rightarrow v = 3$

$x = 3 \Rightarrow u = 9$ $\qquad y = 3 \Rightarrow v = 9$

if $b > 1 \to$ image magnifies

else $\to$ image ~~magni~~ shrink.

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② Find image of $x > 0$ under $w = iz + i$.

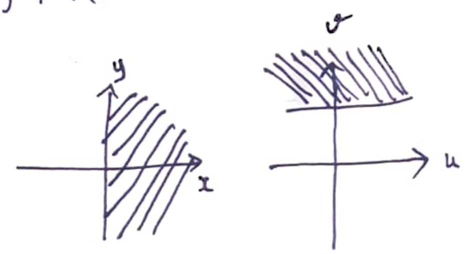
Soln: $f(z) = i(x+iy) + i = -y + i(x+1)$

$u = -y$

$v = x+1$

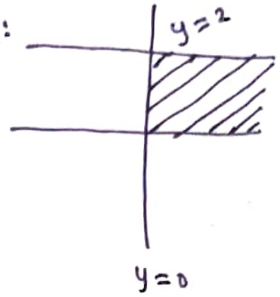
$x = v - 1$

$x > 0 \Rightarrow v - 1 > 0$ $\quad v > 1$



② Image of $x>0,\ 0<y<2$ under $f(z) = iz+1$

Soln:



$$f(z) = iz+1$$

$$f(z) = i(x+iy)+1$$

$$= ix - y + 1$$

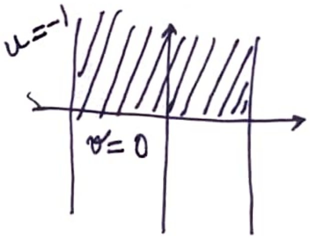
$u = -y+1$ & $v = x$

$x>0 \Rightarrow v>0,\quad 0<y<2$

$$0 < 1-u < 2$$

$$\cancel{-1<u<1} \quad -1<u<1$$

$$\cancel{-1<u<1}$$



③ $w = (1+i)z + 2 + i$, image of ~~sas~~ rectangle

A (0, 0) B (2, 0) C (2, 2) D (0, 2)

$$u = x - y + 2$$
$$v = x + y + 1$$

A' (2, 1) B' (4, 3) C' (2, 5) D' (0, 3)

~~w = (1+i)z~~

④ Inversion:

$$f(z) = \frac{1}{z}$$

$$f(z) = \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$$

$$u = \frac{x}{x^2+y^2} \qquad v = \frac{-y}{x^2+y^2}$$

$$u^2 = \frac{x^2}{(x^2+y^2)^2}$$

$$v^2 = \frac{y^2}{(x^2+y^2)^2}$$

$$u^2+v^2 = \frac{\cancel{x^2+y^2}}{(x^2+y^2)^{\cancel{2}}}$$

$$x = u(x^2+y^2)$$

$$x = \frac{u}{u^2+v^2} \qquad y = \frac{-v}{u^2+v^2}$$

① Find image of $|z-3| = 5$ under $w = \frac{1}{z}$

soln:

$$|z-3| = 5$$

$$\cancel{|x+iy3| =}$$

$$(x-3)^2 + y^2 = 25$$

$$x^2 - 6x + y^2 = 16$$

$$\frac{u^2}{(u^2+v^2)^2} - \frac{6u}{u^2+v^2} + \frac{v^2}{(u^2+v^2)^2} = 16$$

$$u^2 - 6u(u^2 - v^2) + v^2 = 16(u^2+v^2)^2$$

$$u^2 + v^2 - 6u(u^2+v^2) = 16(u^2+v^2)^2$$

$$1 - 6u = 16(u^2+v^2)$$

$$u^2 + v^2 = \frac{1}{16} - \frac{6u}{16}$$

$$u^2 + \frac{6u}{16} + v^2 = \frac{1}{16}$$

$$u^2 + \frac{6u}{16} + \frac{3^2}{16^2} + \frac{3^2}{16} + v^2 = \frac{1}{16}$$

(2) Find image of $1 < x < 2$ under $w = \frac{1}{z}$.

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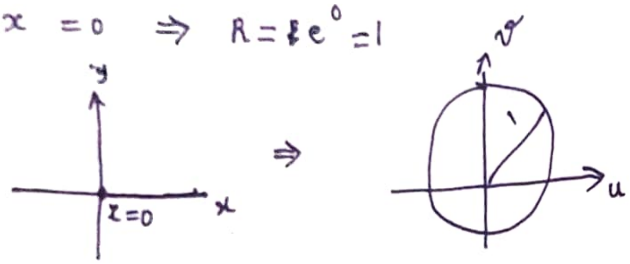
Transformation e^{z}

$$f(z) = e^{x+iy} = w = Re^{i\phi}$$

$$e^{x} \cdot e^{iy} = Re^{i\phi} \Rightarrow R = e^{x}$$

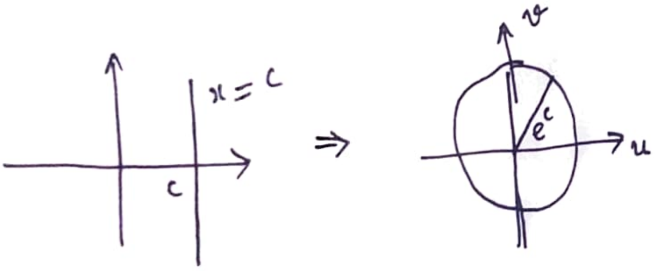
Eg:

1. $x = 0 \Rightarrow R = e^{0} = 1$

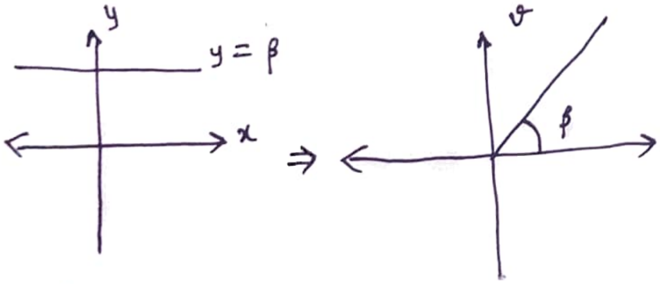


circle with radius 1.

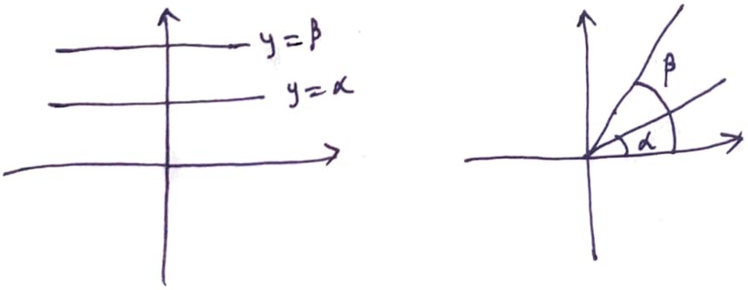
2. $x = c$ is mapped to $R = e^{c}$



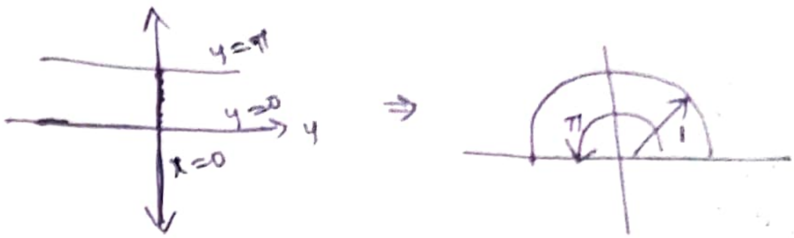
3.



4. $y = \alpha$; $y = \beta$

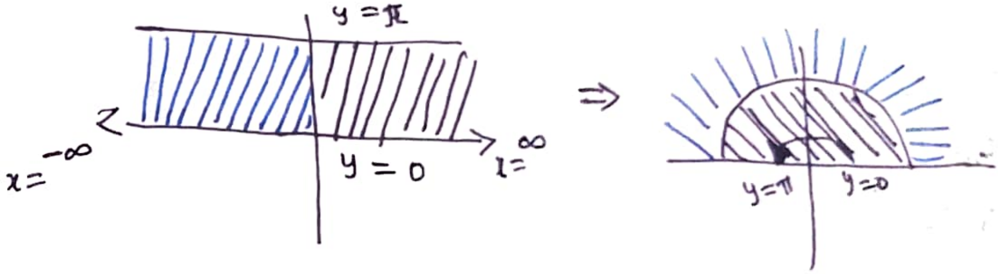


5. $x = 0, \quad 0 < y < \pi$



<u>Problems</u>:

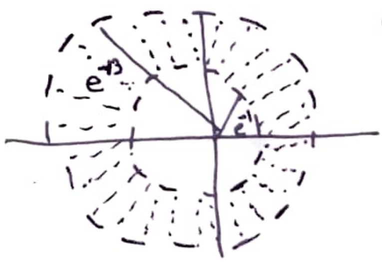
① Find image of $-\infty < x < \infty$, $0 < y < \pi$ under $w = e^{z}$.



☆ Left strip is mapped to semicircle with $R=1$

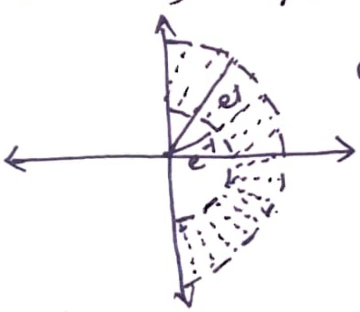
☆ Right strip is mapped to exterior of semicircle in upper half plane.

② $-1 < x < 3$, $-\pi < y < \pi$ under $w = e^{z}$.



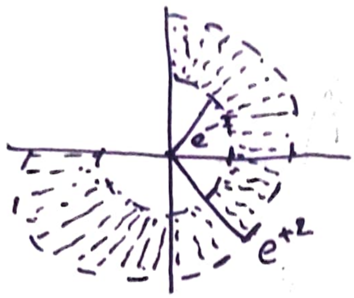
$e^{-1} < R < e^{3}$, $-\pi < \theta < \pi$

③ $-1 < x < 1$, $-\pi/2 < y < \pi/2$



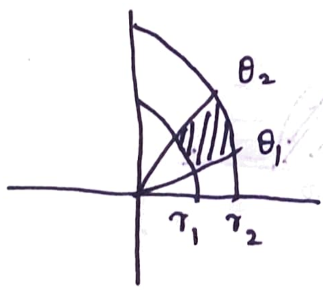
$e^{-1} < R < e^{1}$, $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$.

(4) $-2 < x < 2$, $-\pi < y < \pi/2$



Transformation :

$$w = z^2, \quad Re^{i\phi} = (re^{i\theta})^2 \Rightarrow r^2 e^{2i\theta}$$



$r_1 <$

incomplete
refer
class notes

18.1.23

① Find image of $x = 0$ under $w = \frac{z-1}{z+1}$.

$$w = \frac{az+b}{cz+d}$$

$x = 0$

$\in$ real part $= 0$

$a = 1$

$b = -1$

$c = 1$

$d = 1$

~~$z =$~~

$$(cz+d)w = az+b$$

$$czw + dw = az+b$$

$$czw - az = b - dw$$

$$z(cz-a) = b - dw$$

$$z = \frac{-dw+b}{cz-a} = \frac{-w-1}{w-1}$$

$$x+iy = z = \frac{-(u+iv)-1}{u+iv-1} = \frac{(-u-1)-iv}{(u-i)+iv} \times \frac{(u-i)-iv}{(u-i)-iv}$$

$$z = \frac{((-u-i)-iv)((u-i)-iv)}{((u-i)+iv)^2}$$

$$= \frac{(-u-i)(u-i) + (-u-i)(-iv) + (-iv)(u-i) + (-iv)(-iv)}{(u-i)^2 - v^2 - 2(u-i)(iv)}$$

$$z = \frac{-u^2 - v^2 + 2iv + 1}{u^2 - 2u + 1 + v^2}$$

Separate real and imaginary parts

$$x = \frac{-u^2 - v^2 + 1}{u^2 - 2u + 1 + v^2}$$

$$0 = -u^2 - v^2 + 1$$

$$\boxed{u^2 + v^2 = 1}$$

Cross Ratio :

Let z_1, z_2, z_3, z_4 are 4 points taken in order

then the ratio :

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

if ~~z_1, z_2~~ none of them are ∞.

If any of them are ∞ :

$$= -\frac{z_3 - z_4}{z_2 - z_3} \quad \text{if } z_1 = \infty$$

$$= -\frac{z_3 - z_4}{z_4 - z_1} \quad \text{if } z_2 = \infty$$

$$= -\frac{z_1 - z_2}{z_4 - z_1} \quad \text{if } z_3 = \infty$$

$$= -\left(\frac{z_1 - z_2}{z_2 - z_3}\right) \quad \text{if } z_4 = \infty$$

If we have ~~bim~~ bilinear transform :

$$(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4).$$

→ **Bilinear Transformation preserves the cross ratio** i.e, if the points z_1, z_2, z_3 and z_4 are mapped to w_1, w_2, w_3 and w_4 under bilinear transformation then,

$$(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4).$$

Question: Find bilinear that maps the points $z = \alpha, i, 0$ to $w = 0, i, \alpha$.

Solution :

$$(z, \alpha, i, 0) = (w, 0, i, \alpha)$$
$$(z_1, z_2, z_3, z_4) \quad (w_1, w_2, w_3, w_4)$$

$$\frac{-z_3 - z_4}{z_4 - z_1} = -\frac{i - 0}{0 - z} = \frac{w - 0}{0 - i} = \frac{-w_1 - w_2}{w_2 - w_3}.$$

$$\frac{i}{-z} = \frac{w}{-i} \Rightarrow -i^2 = -wz$$

$$wz = 1$$

$$\boxed{w = \frac{1}{z}}$$

2) Find the bilinear transformation which maps the points $(1, i, -1)$ to $(i, 0, -i)$

$$(z, 1, i, -1) = (w, i, 0, -i)$$

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} \Rightarrow \frac{(z-1)(i+1)}{(1-i)(-1-z)} = \frac{(w-i)(0+i)}{(i-0)(-i-w)}$$

~~By Com~~

$$-\frac{(z-1)(i+1)}{(1-i)(z+1)} = \frac{-(w-i)}{w+i}$$

$$\boxed{w = -\frac{(zi+1)}{zi-1}}$$

③ Find the bilinear transformations which transforms the points

$z = 2, 1, 0$

$w = 1, 0, i$

$(z, 2, 1, 0)$ to $(w, 1, 0, i)$

$$\cancel{\frac{(z_4 - z_3)(z_2 - z_1)}{(z_3 - z_2)(z_4 - z_1)}} = \cancel{\frac{(w_4 - w_3)(w_4 - w_2)}{(w_3 - w_2)(w_4}}$$

$$\Rightarrow \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)}$$

$$\Rightarrow \frac{(z-2)(1-0)}{(z-1)(0-z)} = \frac{(w-1)(0-i)}{(1-0)(i-w)}$$

$$\frac{z-2}{-z} = \frac{(w-1)(-i)}{i-w} \qquad \frac{-iw+i}{-w+i}$$

$$\boxed{\text{if } \frac{a}{b} = \frac{c}{d} \quad \frac{a+b}{a-b} = \frac{c+d}{c-d}}$$

$$\frac{\cancel{z}-2+\cancel{z}}{z-2+z} = \frac{(-iw+i)+(i-w)}{-iw+\cancel{i}-\cancel{i}+w}$$

$$\frac{-2}{2z-2} = \frac{-iw+2i-w}{-iw+w}$$

$$\frac{-1}{z-1} = \frac{-w(1+i)+2i}{w(1-i)}$$

$$-w(1-i) = (-w(1+i)+2i)(z-1)$$

$$-w(1-i) = (-w(1+i)+2i)(z-1)$$

$$-w(1-i) = (-w-wi+2i)(z-1)$$

$$-w(1-i) = -wz-wzi+2iz+w+wi-2i$$

$$\cancel{-w(1-i)}$$

$$-w-wi = -wz-wzi+2iz$$

$$+$$

$$\cancel{-w} = \cancel{-w}$$

$$\cancel{-w-wi = (-w-wi-2i)(z-i)}$$

$$\cancel{-w = -zwi+z+2z}$$

$$-w(1-i) =$$

$$\cancel{\frac{w(1-i)}{-w(1+i)+2i} = z-1}$$

19.1.23

② Find a bilinear transformation which maps $z = i, -1, 1$ to $w = 0, 1, \infty$.

③ Find Fixed pts and image ~~under~~ of $|z| = 1$ under

$$w = \frac{z-i}{1-iz}$$

Solution:

② $(z, i, -1, 1) = (w, 0, 1, \infty)$

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} = -\left(\frac{w_1 - w_2}{w_2 - w_3}\right)$$

$$\frac{(z - i)(-1-1)}{(i+1)(1-z)} = -\left(\frac{w - 0}{0 - 1}\right)$$

$$\frac{-2(z-i)}{(1+i)(1-z)} = w$$

$$\frac{-2(z-i)}{1 - z + i - zi} = w$$

$$\frac{2(z-i)}{-1 + z - i + zi} = w$$

$$\boxed{\frac{2(z-i)}{z(1+i) - (1+i)} = w}$$

(3) $w = \dfrac{z-i}{1-iz}$ $\boxed{\text{Sol : } \pm 1, \text{ image } v=0}$

$$w = \frac{z-i}{1-iz}$$

$$w = u+iv$$

$$z = x+iy$$

$$u+iv = \frac{(x+iy)-i}{1-i(x+iy)} \times$$

$$u+iv = \frac{x+i(y-1)}{1-ix+y}$$

$$u+iv =$$

Questions :

4. Find conformal points of $z^2 + \frac{1}{z^2}$

5. Find image of $|z| = 2$ under $w = \sqrt{2}\, e^{i\pi/4} z$

6. Find bilinear transformation which maps $0, -i, 2i$ into $5i, \infty, -i/3$ respectively. Find Fixed Points.