Problems:

1). Three balls are drawn at random without replacement from a box containing 2 white, 3 red and 4 black balls. If X denotes the no. of white balls drawn and Y denotes the no. of red balls drawn, find the joint probability distribution of (X, Y).

(1) Red balls = 3
White balls = 2
Black balls = 4.

Given:
Three balls are drawn without replacement.

Let $X \rightarrow$ No. of white balls drawn.
$Y \rightarrow$ No. of red balls drawn.

Now, $X = \{0, 1, 2\}$

$Y = \{0, 1, 2, 3\}$

To find:- Joint prob. distb. of (X, Y)

(i.e.) $P(X=0, Y=0)$	$P(X=1, Y=0)$	$P(X=2, Y=0)$
$P(X=0, Y=1)$	$P(X=1, Y=1)$	$P(X=2, Y=1)$
$P(X=0, Y=2)$	$P(X=1, Y=2)$	$P(X=2, Y=2)$
$P(X=0, Y=3)$	$P(X=1, Y=3)$	$P(X=2, Y=3)$

Now,

$$P(X=0, Y=0) = \frac{4C_3}{9C_3} = \frac{1}{21}$$

$$P(X=0, Y=1) = \frac{(3C_1)(4C_2)}{9C_3} = \frac{3}{14}$$

$$P(X=0, Y=2) = \frac{(3C_2)(4C_1)}{9C_3} = \frac{1}{7}$$

$$P(X=0, Y=3) = \frac{(3C_3)(4C_0)}{9C_3} = \frac{1}{84}$$

$$P(X=1, Y=0) = \frac{(2C_1)(4C_2)}{9C_3} = \frac{1}{7}.$$

$$P(X=1, Y=1) = \frac{(2C_1)(3C_1)(4C_1)}{9C_3} = \frac{2}{7}$$

$$P(X=1, Y=2) = \frac{(2C_1)(3C_2)(4C_0)}{9C_3} = \frac{1}{14}.$$

$P(X=1, Y=3) = 0$ ($\because$ only three balls are drawn).

lly $P(X=2, Y=2) = 0$.
$P(X=2, Y=3) = 0$.

$$P(X=2, Y=0) = \frac{(2C_2)(4C_1)}{9C_3} = \frac{1}{21}$$

$$P(X=2, Y=1) = \frac{(2C_2)(3C_1)(4C_0)}{9C_3}$$

$$= \frac{1}{28}$$

Joint prob. distt. of (X, Y)

X \ Y	0	1	2	3
0	$\frac{1}{21}$	$\frac{3}{14}$	$\frac{1}{7}$	$\frac{1}{84}$
1	$\frac{1}{7}$	$\frac{2}{7}$	$1/14$	0
2	$\frac{1}{21}$	$\frac{1}{28}$	0	0

X

2). For the bivariate probability distribution of (x,y) given below, find (i) $P(x \le 1)$

(ii) $P(x \le 1, y \le 3)$

(iii) $P(x \le 1 / y \le 3)$

(iv) $P(x+y \le 4)$.

X \ Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

(i) $P(X \le 1) = P(X=0) + P(X=1)$

$$= \sum_{j=1}^{6} P(X=0, Y=j) + \sum_{j=1}^{6} P(X=1, Y=j)$$

$$= \sum_{i=0}^{1} \sum_{j=1}^{6} P(X=i, Y=j)$$

X \ Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8

X \ Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8

$$= \left(0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{2}{32} + \frac{3}{32}\right)$$

$$+ \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right).$$

$P(X \leq 1) = 7/8$

(ii) $P(X \leq 1, Y \leq 3) = \sum_{i=0}^{1} \sum_{j=1}^{3} P(X=i, Y=j)$

X \ Y	1	2	3
0	0	0	1/32
1	1/16	1/16	1/8

$$= \left(0 + 0 + \frac{1}{32}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8}\right) = \frac{9}{32}$$

(iii) $P(X \le 1 / Y \le 3) = \dfrac{P(X \le 1, Y \le 3)}{P(Y \le 3)}$

Now,

$$P(Y \le 3) = \sum_{i=0}^{2} \sum_{j=1}^{3} P(X=i, Y=j)$$

X \ Y	1	2	3
0	0	0	1/32
1	1/16	1/16	1/8
2	1/32	1/32	1/64

$$= \left(0 + 0 + \tfrac{1}{32}\right) + \left(\tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{8}\right) + \left(\tfrac{1}{32} + \tfrac{1}{32} + \tfrac{1}{64}\right).$$

$$P(Y \le 3) = \frac{23}{64}.$$

$\therefore P(X \leq 1 / Y \leq 3) = \dfrac{P(X \leq 1, Y \leq 3)}{P(Y \leq 3)}$

$= \dfrac{(9/32)}{(23/64)} = \dfrac{18}{23}.$

(iv) $P(X + Y \leq 4)$.

X \ Y	1	2	3	4	5	6
0	✓	✓	✓	✓	✗	✗
1	✓	✓	✓	✗	✗	✗
2	✓	✓	✗	✗	✗	✗

$$P(X+Y \le 4) = \sum_{j=1}^{4} P(X=0, Y=j)$$

$$+ \sum_{j=1}^{3} P(X=1, Y=j)$$

$$+ \sum_{j=1}^{2} P(X=2, Y=j).$$

$$= \left(0 + 0 + \tfrac{1}{32} + \tfrac{2}{32}\right)$$

$$+ \left(\tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{8}\right)$$

$$+ \left(\tfrac{1}{32} + \tfrac{1}{32}\right)$$

$$P(X+Y \leq 4) = \frac{13}{32}$$

Problem 3:

From the following table examine whether X and Y are independent (or) not.

X \ Y	0	1	2
0	0.1	0.04	0.06
1	0.2	0.08	0.12
2	0.2	0.08	0.12

To prove :- $p_{ij} = (p_{i*})(p_{*j}), \ \forall i, j.$

Solution:

$x \backslash y$	0	1	2	p_{i*}
0	0.1	0.04	0.06	0.2
1	0.2	0.08	0.12	0.4
2	0.2	0.08	0.12	0.4
p_{*j}	0.5	0.2	0.3	1

$p_{0*} = 0.1 + 0.04 + 0.06$

$p_{1*} = 0.2 + 0.08 + 0.12$

$p_{2*} = 0.2 + 0.08 + 0.12$

Total probability

$p_{*0} = 0.1 + 0.2 + 0.2$

$p_{*1} = 0.04 + 0.08 + 0.08$

$p_{*2} = 0.06 + 0.12 + 0.12$

To prove :- $p_{ij} = (p_{i*})(p_{*j}), \ \forall i, j.$

Now,

$\Rightarrow (p_{0*})(p_{*0}) = (0.2)(0.5)$

$= 0.1$

$(p_{0*})(p_{*0}) = p_{00}.$

$X \backslash Y$	0	1	2	p_{i*}
0	0.1	0.04	0.06	0.2
1	0.2	0.08	0.12	0.4
2	0.2	0.08	0.12	0.4
p_{*j}	0.5	0.2	0.3	1

$\Rightarrow (p_{1*})(p_{*1}) = (0.4)(0.2)$

$= 0.08$

$(p_{1*})(p_{*1}) = p_{11}$

$$\Rightarrow (p_{2*})(p_{*2}) = (0.4)(0.3)$$

$$= 0.12$$

$$(p_{2*})(p_{*2}) = p_{22}$$

$x \backslash y$	0	1	2	p_{i*}
0	0.1	0.04	0.06	0.2
1	0.2	0.08	0.12	0.4
2	0.2	0.08	0.12	0.4
p_{*j}	0.5	0.2	0.3	1

$$\Rightarrow (p_{0*})(p_{*1}) = (0.2)(0.2)$$

$$= 0.04$$

$$(p_{0*})(p_{*1}) = p_{01}$$

$$\Rightarrow (p_{0*})(p_{*2}) = (0.2)(0.3)$$

$$= 0.06$$

$$(p_{0*})(p_{*2}) = p_{02}$$

$$\Rightarrow (p_{1*})(p_{*0}) = (0.4)(0.5)$$

$$= 0.2$$

$$(p_{1*})(p_{*0}) = p_{10}$$

$$\Rightarrow (p_{1*})(p_{*2}) = (0.4)(0.3)$$

$$= 0.12$$

$$(p_{1*})(p_{*2}) = p_{12}$$

$$\Rightarrow (p_{2*})(p_{*0}) = (0.4)(0.5)$$

$$= 0.2$$

$$(p_{2*})(p_{*0}) = p_{20}$$

X \ Y	0	1	2	p_{i*}
0	0.1	0.04	0.06	0.2
1	0.2	0.08	0.12	0.4
2	0.2	0.08	0.12	0.4
p_{*j}	0.5	0.2	0.3	1

$$\Rightarrow (p_{2*})(p_{*1}) = (0.4)(0.2)$$

$$= 0.08$$

$$(p_{2*})(p_{*1}) = p_{21}$$

$x \backslash y$	0	1	2	p_{i*}
0	0.1	0.04	0.06	0.2
1	0.2	0.08	0.12	0.4
2	0.2	0.08	0.12	0.4
p_{*j}	0.5	0.2	0.3	1

$$\therefore p_{ij} = (p_{i*})(p_{*j}), \quad \forall i, j.$$

Hence, x and y are independent.

$\propto$