Moment generating function:

$$M_X(t) = e^{\mu t + \frac{t^2\sigma^2}{2}}$$

Method to find $N(\mu, \sigma)$:

Step 1: Get the distribution and the range.

Step 2: Change normal distribution into Standard normal distribution.

(i.e) $N(\mu, \sigma) \rightarrow N(0, 1)$.

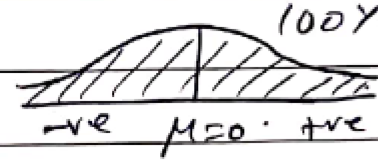
Step 3: Look up the probability using the Standard Normal Distribution table

Normal Distribution to Standard Normal Distribution

1. Normal distribution ⟶ Standard normal distribution.

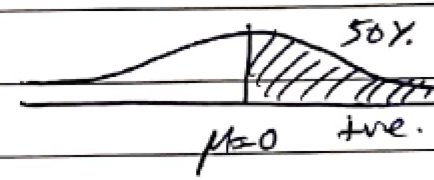
2. Standard normal distribution.

↳ Full table ⇒



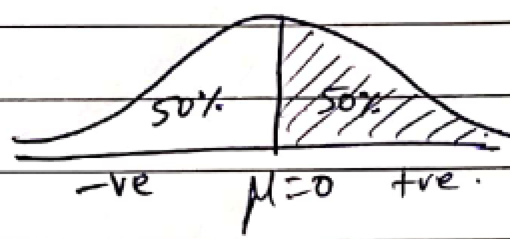
$\int_{-z}^{z}$

⟶ Half table. ⇒



$\int_{0}^{z}$

3. Using Half table.



Standard Normal Table

- Standard Normal Full table
- Standard Normal Half table

1. Most graduate schools of business require applicants for admission to take the Graduate Management Admission Council's GMAT examination. Scores on the GMAT are roughly normally distributed with a mean of 527 and a standard deviation of 112.

(i) What is the probability of an individual scoring above 500 on the GMAT?

(ii) How high must an individual score on the GMAT in order to score in the highest 5%?

Solution:

(i). Given: $\mu = 527$ and $\sigma = 112$.

Let X denotes the GMAT scores that are normally distributed.

(i.e.) $X \sim N(527, 112)$.

(1). Given: $\mu = 527$ and $\sigma = 112$.

Let X denotes the GMAT scores that are normally distributed.

(i.e.) $X \sim N(527, 112)$.

(i) $P(X > 500) = P(Z > -0.24)$.
(Normal) (Standard normal).

St. Normal

$Z = \frac{x - \mu}{\sigma}$

$= \frac{500 - 527}{112}$

$Z = -0.24$

$= 1 - P(Z < -0.24)$

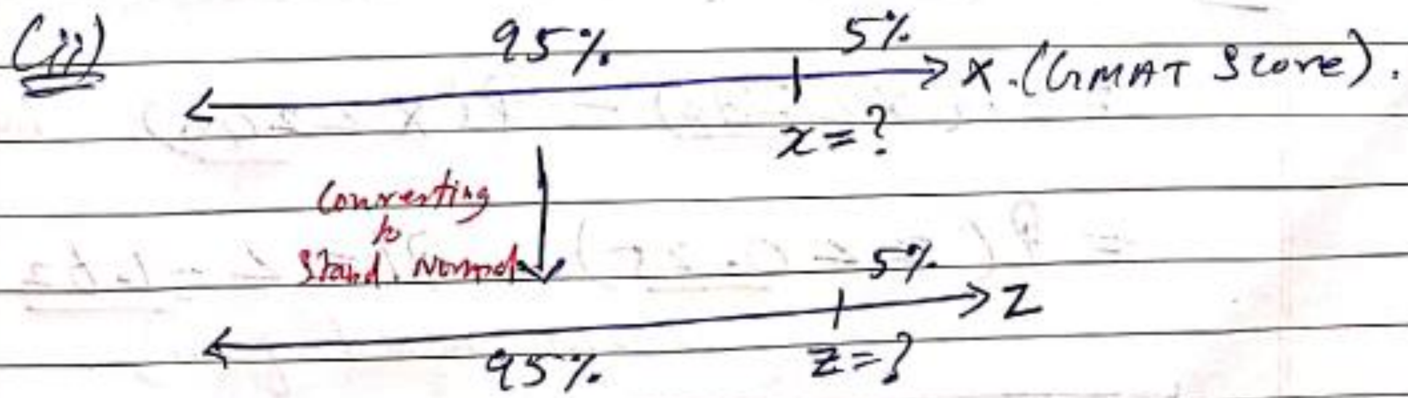
$= 1 - 0.4052$

$= 0.5948$

$\therefore \boxed{P(X > 500) = 0.5948}$.

(ii) How high must an individual score on the GMAT in order to score in the highest 5%?

(ii)



$$P(X > x) = 0.05$$

$$\Rightarrow P(Z > z) = 0.05$$

$$\Rightarrow P(Z < z) = 1 - 0.05 = 0.95$$

$$P(Z < z) = 0.95$$

$$z = 1.645 \quad \text{(from the table)}$$

W.K.T, $z = \dfrac{x-\mu}{\sigma}$

$$1.65 = \frac{x - 527}{112}$$

$$\Rightarrow x = 711.8.$$

$\therefore$ One must score more than 711.8 marks more to be in the highest 5% level.

2. The length of human pregnancies from conception to birth approximates a normal distribution with a mean of 266 days and a standard deviation of 16 days.

(i) What proportion of all pregnancies will last between 240 and 270 days (roughly between 8 and 9 months)?

(ii) What length of time marks the shortest 70% of all pregnancies?

(2). Given: $\mu = 266$ and $\sigma = 16$.

Let X denote the length of human pregnancies that are normally distributed.

(i.e.) $X \sim N(266, 16)$.

(i) $P(240 < X < 270)$

$= P(X < 270) - P(X < 240)$. [Normal form]

$= P(Z < 0.25) - P(Z < -1.62)$.

[Standard normal form].

$Z = \frac{x-\mu}{\sigma} = \frac{270-266}{16}$.	$Z = \frac{x-\mu}{\sigma} = \frac{240-266}{16}$.
$Z = 0.25$	$Z = -1.62$

$= P(Z < 0.25) - P(Z < -1.62)$ [Standard normal form].

$= 0.5987 - 0.0526$

$= 0.5461$

(ii) $P(X < x) = 0.70$. (Normal form)

$\Rightarrow P(Z < z) = 0.70$ (st. normal form).

$z = 0.53$ (from the table).

w.k.t, $z = \frac{x - \mu}{\sigma}$

$0.53 = \frac{x - 266}{16}$

$\boxed{x = 274.48}$

α

3. The independent random variables X and Y have distributions N(45, 2) and N(44, 1.5) respectively. What is the probability that randomly chosen values of X and Y differ by 1.5 or more ?

Sol:-

X	Y
N(45, 2)	N(44, 1.5)
$\mu_x = 45$ $\sigma_x = 2$	$\mu_y = 44$ $\sigma_y = 1.5$

$\sigma_x = 2 \Rightarrow \sigma_x^2 = 4.$

$\sigma_y = 1.5 \Rightarrow \sigma_y^2 = 2.25$

Let, $U = X - Y.$

$E(U) = E(X-Y) = E(X) - E(Y).$

$= 45 - 44$

$E(U) = 1 \Rightarrow \boxed{\mu_U = 1}$

$var(U) = var(X-Y).$

$= var(X + (-1)Y).$

$= var(X) + (-1)^2 var(Y).$

$= 4 + (1)\, 2.25$

$Var(U) = 6.25$

$\Rightarrow \sigma_U = \sqrt{6.25} = 2.5$

$\therefore U \sim N(1, 2.5)$.

Now, $P(|x-y| \geq 1.5) = P(|U| \geq 1.5)$

$= 1 - P(|U| \leq 1.5)$.

$= 1 - [P(-1.5 \leq U \leq 1.5)]$.

$= 1 - [P(U \leq 1.5) - P(U \leq -1.5)]$.

$= 1 - \left[P\left(\frac{U-\mu_U}{\sigma_U} \leq \frac{1.5-\mu_U}{\sigma_U}\right) - P\left(\frac{U-\mu_U}{\sigma_U} \leq \frac{-1.5-\mu_U}{\sigma_U}\right)\right]$

$= 1 - \left[P\left(Z \leq \frac{1.5-1}{2.5}\right) - P\left(Z \leq \frac{-1.5-1}{2.5}\right)\right]$

$= 1 - [P(Z \leq 0.2) - P(Z \leq -1)]$.

$= 1 - P(Z \leq 0.2) + P(Z \leq -1)$.

$= 1 - P(Z \leq 0.2) + P(Z \leq -1)$.

$= 1 - 0.5793 + 0.1587$

$= 0.5794$.

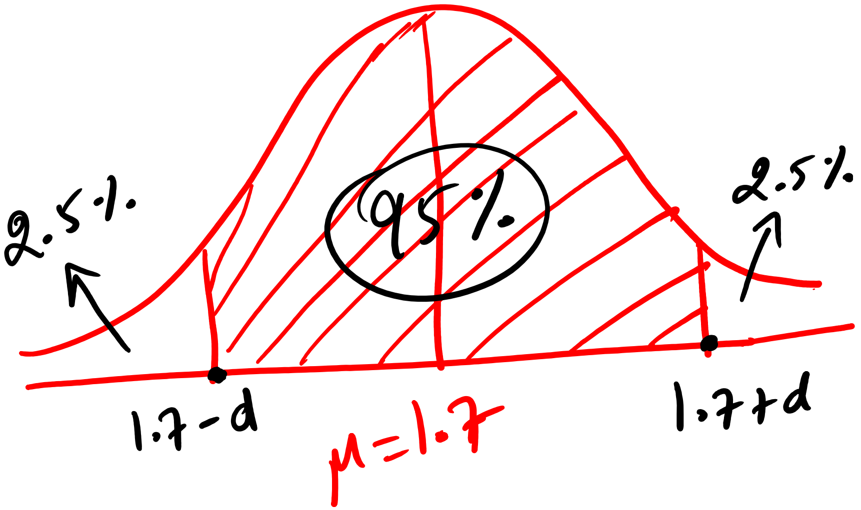
$$\boxed{P(|x-y| \geq 1.5) = 0.5794}$$

α

4. A machine used to reject all components for which a certain dimension is not within the specification $1.70 \pm d$. It is known that this measurement is normally distributed with mean 1.70 and s.d.=0.4. Determine the value $'d'$, such that the specifications cover 95% of the measurements.

Sol:-

$X \sim N(\mu = 1.7, \sigma = 0.4)$



$P(X < 1.7 - d) = 0.025$

$$z = \frac{x-\mu}{\sigma} = \frac{(1.7-d) - 1.7}{0.4}$$

$$\boxed{z = \frac{-d}{0.4}}$$

$$P(X < 1.7 - d) = 0.025$$

$$\Rightarrow P\left(Z < -\frac{d}{0.4}\right) = 0.025$$

Table value.

$$\Rightarrow P\left(Z < -\frac{d}{0.4}\right) = P(Z < -1.96)$$

$$\Rightarrow \frac{\cancel{-}d}{0.4} = \cancel{-}1.96$$

$$\Rightarrow d = (0.4)(1.96)$$

$$\boxed{d = 0.784}$$

5. If X and Y are independent random variables following N(8, 2) and N(12, $4\sqrt{3}$) respectively, find the value of λ such that $P(2X - Y \leq 2\lambda) = P(X + 2Y \geq \lambda)$.

Solution:

Given :-	$N(8, 2)$	$N(12, 4\sqrt{3})$
	$\mu_X = 8$	$\mu_Y = 12$
	s.d, $\sigma_X = 2$	$\sigma_Y = 4\sqrt{3}$

Let $\boxed{U = 2X - Y}$ and $\boxed{V = X + 2Y}$

Mean	S.d
$E(2X - Y)$	$Var(2X - Y) = 4\,Var(X) + Var(Y)$
$= 2E(X) - E(Y)$	$= 4(4) + (4\sqrt{3})^2$
$= 2(8) - 12$	$= 64$
$\boxed{\mu_U = 4}$	S.d, $\boxed{\sigma_U = 8}$

$$\boxed{U \sim N(4, 8)}$$

$$\boxed{V = x + 2y}$$

Mean, μ_V

$$E(x+2y)$$
$$= E(x) + 2E(y)$$
$$= 8 + 2(12)$$

$$\boxed{\mu_V = 32}$$

sd, σ_V

$$Var(x+2y) = Var(x) + 4\,Var(y)$$
$$= 2^2 + 4(4\sqrt{3})^2$$
$$= 196$$

$$\boxed{\sigma_V = 14}$$

$$\therefore \boxed{V \sim (32, 14)}$$

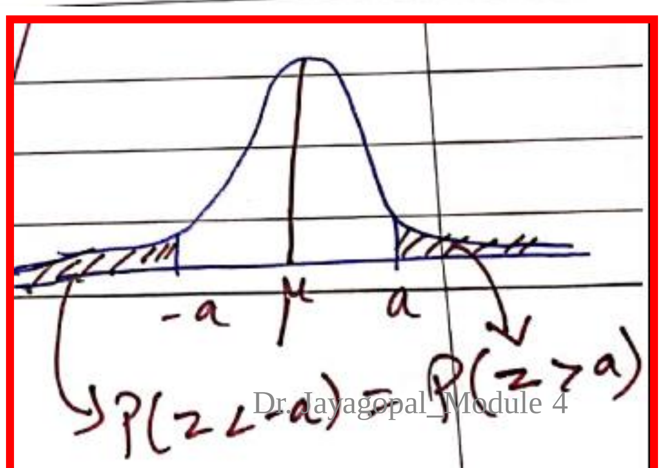
Now, $P(2x - y \le 2\lambda) = P(x + 2y \ge \lambda)$.

$$P(U \le 2\lambda) = P(V \ge \lambda).$$

$$P\left(Z \le \frac{2\lambda - \mu_U}{\sigma_U}\right) = P\left(Z \ge \frac{\lambda - \mu_V}{\sigma_V}\right).$$

$$P\left(Z \le \frac{2\lambda - 4}{8}\right) = P\left(Z \ge \frac{\lambda - 32}{14}\right).$$

$$P\left(Z \le \frac{2\lambda - 4}{8}\right) = P\left(Z \le -\left(\frac{\lambda - 32}{14}\right)\right).$$



$$P\left(Z \le \frac{2d-4}{8}\right) = P\left(Z \le -\left(\frac{d-32}{14}\right)\right).$$

$$\Rightarrow \frac{2d-4}{8} = -\left(\frac{d-32}{14}\right)$$

$$\Rightarrow \boxed{d = \frac{26}{3}}$$

Gamma Function $\Gamma(n)$

Definition : For each real number $\boldsymbol{n} > \mathbf{0}$,

the improper integral $\boxed{\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx}$

converges and its value is denoted by $\Gamma(n+1)$.

Properties of the Gamma function

- $\Gamma(n+1) = n!$, if n is a +ve integer
- $\Gamma(n+1) = n\,\Gamma(n)$, if n is a +ve fraction
- $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$, if n is a -ve fraction
- $\Gamma(1) = 1$ and $\Gamma(0) = \infty$
- $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$
- For all negative integers n, $\Gamma(n) =$ undefined

Gamma distribution and Exponential distribution

First let us see about Erlang distribution (or) general gamma distribution.

Erlang distribution (or) general gamma distribution

A continuous random variable X having the following density function is said to follow generalised gamma distribution with parameters α and λ,

$$f(x) = \begin{cases} \dfrac{e^{-\lambda x}\,\lambda^{\alpha}\,x^{\alpha-1}}{\Gamma\alpha}, & \alpha>0,\ \lambda>0,\ 0<x<\infty \\ 0 & ; \text{ otherwise.} \end{cases}$$

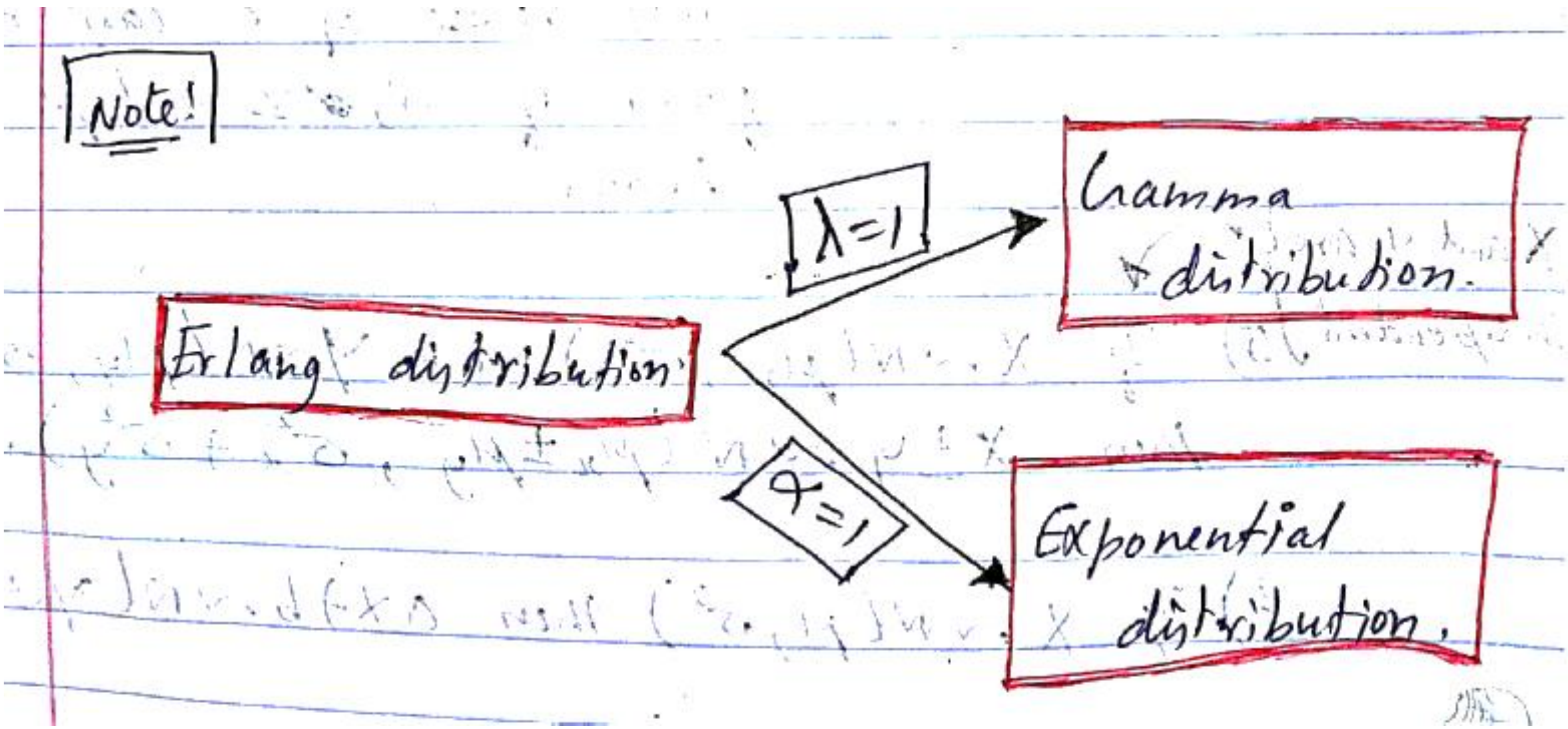
Erlang distribution

M.G.F : $M_X(t) = \left[1 - \frac{t}{\lambda}\right]^{-\alpha}$, $|t/\lambda| < 1$.

$$M_X(t) = \left[\frac{\lambda}{\lambda - t}\right]^{\alpha}$$

Mean $= \frac{\alpha}{\lambda}$, Variance $= \frac{\alpha}{\lambda^2}$

Moments ; $\mu'_r = \frac{\Gamma(\alpha+r)}{\lambda^r \Gamma(\alpha)}$


Note!
Erlang distribution
λ=1
Gamma distribution.
α=1
Exponential distribution.

Gamma distribution

*) p.d.f, $f(x) = \begin{cases} \dfrac{e^{-x} x^{\alpha-1}}{\Gamma\alpha}, & \alpha > 0,\ 0 < x < \infty \\ 0, & \text{otherwise.} \end{cases}$

Exponential distribution

*) p.d.f., $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \lambda > 0. \\ 0, & \text{otherwise.} \end{cases}$

*) M.G.F : $M_x(t) = \dfrac{\lambda}{\lambda - t}$.

*) mean $= \dfrac{1}{\lambda}$, variance $= \dfrac{1}{\lambda^2}$

*) Moments, $\mu'_r = \dfrac{\Gamma(r+1)}{\lambda^r} = \dfrac{r!}{\lambda^r}$

*) $F(x) = 1 - e^{-\lambda x}$ [cumulative distribution].

→ Memory less Property of exponential distribution

If X is a exponentially distributed with parameter λ, then for any two positive integers s and t;

$$P[(X > s+t)/(X > s)] = P[X > t].$$