Module 5: Hypothesis Testing I

– Large Sample Test (Z test)

Testing of hypothesis –Types of errors - Critical region, Procedure for testing of hypothesis-Large sample tests- Z test for Single Proportion- Difference of Proportion- Mean and difference of means.

Basic Concepts

Population: A population consists of collection of individual units, which may be person's or experimental outcomes, whose characteristics are to be studied.

Sample: A sample is proportion of the population that is studied to learn about the characteristics of the population.

Random Sample: A random sample is one in which each item of a population has an equal chance of being selected.

Sampling:

The process of drawing a sample from a population is called sampling.

Sample size:

The number of items selected in a sample is called the sample size and it is denoted by 'n'.

- If $\boldsymbol{n} > \mathbf{30}$, then the sample is called as **large sample**.
- If $\boldsymbol{n} \leq \mathbf{30}$, then the sample is called as **small sample**.

Sampling distribution: Consider all possible samples of size $'n'$ drawn from a given population at random. We calculate mean values of these samples.

- If we group these different means according to their frequencies, the frequency distribution so formed is called sampling distribution.

$S_1, S_2, S_3, \dots, S_{10} \Rightarrow \bar{x}_1, \bar{x}_2, \bar{x}_3, \dots, \bar{x}_{10}$

Eg:

$\bar{x}_1 = 50$	$\bar{x}_5 = 45$	$\bar{x}_9 = 49$
$\bar{x}_2 = 50$	$\bar{x}_6 = 50$	$\bar{x}_{10} = 50$
$\bar{x}_3 = 45$	$\bar{x}_7 = 45$	
$\bar{x}_4 = 52$	$\bar{x}_8 = 52$	

x	50	45	52	49
f	4	3	2	1

$\Rightarrow$ Sampling distribution.

Ram

Murder case

Judge: H_0 : Ram is innocent (Null hypothesis)

H_1 : Ram is not innocent. (Alternate hypothesis)

Verdict: Based on the arguments and the evidents, ⟹ Ram is innocent ⟹ H_0

⟹ H_0 is accepted.

Case (i): Ram is really the murder.

⟹ Error in the verdict (Type-II).

Ram

Murder case

Judge: H_0 : Ram is innocent (Null hypothesis)

H_1 : Ram is not innocent. (Alternate hypothesis)

Verdict: Based on the arguments and the evidents, $\Rightarrow$ Ram is not innocent $\Rightarrow H_1$

$\Rightarrow H_0$ is rejected.

Case (ii): Ram is not the murder.

$\Rightarrow$ Error in the verdict (Type-I).

Ram

Murder case

Judge: H_0 : Ram is innocent (Null hypothesis)

H_1 : Ram is not innocent. (Alternate hypothesis)

Verdict: Based on the arguments and the evidents, $\Rightarrow$ Ram is innocent $\Rightarrow H_0$

$\Rightarrow H_0$ is accepted.

Case (i): Ram is really the murder.

Truth

$\Rightarrow$ Error in the verdict (Type-II)

Ram

Murder case

Judge: H_0 : Ram is innocent (Null hypothesis)

H_1 : Ram is not innocent. (Alternate hypothesis)

Verdict: Based on the arguments and the evidents, $\Rightarrow$ Ram is not innocent $\Rightarrow H_1$

$\Rightarrow H_0$ is rejected.

Case (ii): Ram is not the murder.

Truth

$\Rightarrow$ Error in the verdict (Type-I)

Most serious Error

Type-I > Type-II

Test of Hypothesis using Z-Test

Large Sample Test – Sample size > 30

Type I and Type II Errors

		Truth about the Population	
		H_0 is True	H_0 is False
Decision based on Sample	Reject H_0	Type I Error (α)	Correct Decision
	Accept H_0	Correct Decision	Type II Error (β)

Note:

- Type I error is more serious than Type II error. Thus, we are concerned only about α (i.e.) Type I error. So our main aim is to reduce the Type I error.
- First formulate the alternate hypothesis H_1 and then formulate the null hypothesis H_0.
- Whatever we want to test the given claim [except for equality test] then that hypothesis will come under H_1, and H_0 will be the exact opposite of H_1.
- If $\boldsymbol{H_1}$ **contains** $\neq$ **sign** then the test is a **Two tailed test** and if $\boldsymbol{H_1}$ **contains** $<$, $>$, $\leq$ $\boldsymbol{and}$ $\geq$ **signs** then the test is a **One tailed test**.

Null Hypothesis and Alternate Hypothesis

Null Hypothesis (H_0)

The hypothesis tested for possible rejection under the assumption that it is true is usually called null hypothesis. The null hypothesis is a hypothesis which reflects no change or no difference. It is usually denoted by H_0.

Alternative Hypothesis (H_1)

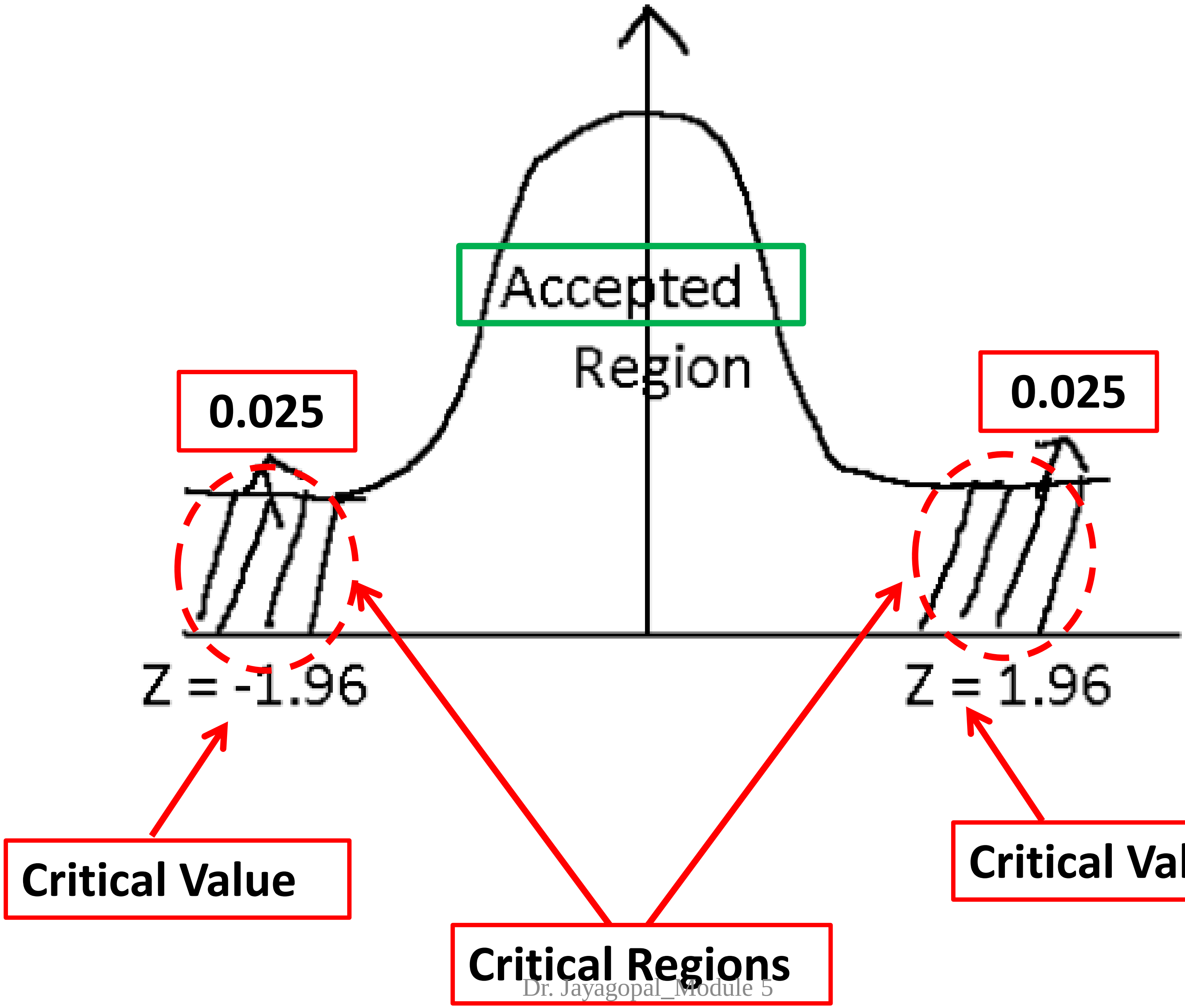
The alternative hypothesis is the statement which reflects the situation anticipated to be correct if the null hypothesis is wrong. It is usually denoted by H_1 .

Level of significance:

It is the probability level below which the null hypothesis is rejected. Generally, 1%, 5% and 10% level of significance are used.

Critical Region (or) Region of Rejection:

- The critical region of a test of statistical hypothesis is that region of the normal curve which corresponds to the rejection of null hypothesis.
- The shaded portion in the following figure is the critical region which corresponds to 5% LOS.


Accepted
Region
0.025
0.025
Z = -1.96
Z = 1.96
Critical Value
Critical Value
Critical Regions

Critical Value	$\boldsymbol{\alpha = 1\%}$	$\boldsymbol{\alpha = 5\%}$	$\boldsymbol{\alpha = 10\%}$
Z_{α} value for Two tail test	2.58	1.96	1.645
Z_{α} value for One tail test	2.33	1.645	1.28

Types in Z-test

- Single mean
- Difference of two means
- Single proportion
- Difference of two proportions

Types in Z-test

Single mean:

To test whether the difference between population mean (μ) and sample mean ($\bar{x}$) is significant (or) not.

Difference of two means:

*) Two samples taken from two different populations with mean and S.D.

*) To test whether the difference between two population (mean) is significant (or) not.

Single proportion:

To test whether the difference between population proportion (p) and sample proportion (p') is significant (or) not.

Difference of two proportion:

*) Two proportion taken from two different populations.

*) To test whether the difference between two population proportion is significant (or) not.

Notations:

- n stands for sample size.
- n_1 and n_2 stands for sizes of sample 1 and sample 2, respectively.
- μ stands for population mean and $\bar{x}$ stands for sample mean.
- σ stands for population's standard deviation and s stands for sample's standard deviation.
- $\overline{x_1}$ and $\overline{x_2}$ stands for means of sample 1 and sample 2, respectively.
- $\boldsymbol{\mu_1}$ and $\boldsymbol{\mu_2}$ stands for means of population 1 and population 2, respectively.

- σ_1 and σ_2 stands for standard deviations of population 1 and population 2, respectively.
- s_1 and s_2 stands for standard deviations of sample 1 and sample 2, respectively.
- p stands for population proportion and $q = 1 - p$.
- p' stands for sample proportion.
- p_1' and p_2' stands for proportions of sample 1 and sample 2, respectively.
- p_1 and p_2 stands for proportions of population 1 and population 2, respectively.

Type	Formula
Single mean	$$\boldsymbol{Z = \frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}}}$$ **Note:** If $'\sigma'$ is not known then replace $'\sigma'$ by sample's standard deviation $'s'$ $$\boldsymbol{Z = \frac{\bar{x} - \mu}{\sqrt{s^2/n}}}$$

Type	Formula
Difference of two means	$$Z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$ **Note:** If $'\sigma_1'$ and $'\sigma_2'$ are not known then replace $'\sigma_1'$ and $'\sigma_2'$ by samples standard deviations $'s_1'$ and $'s_2'$ $$Z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

Type	Formula
Single Proportion	$$Z = \frac{p' - p}{\sqrt{pq/n}}$$
Difference of two Proportions	$$Z = \frac{p_1' - p_2'}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$ Where, $p = \frac{n_1 p_1' + n_2 p_2'}{n_1 + n_2}$ and $q = 1 - p$

Procedure to approach the problem

1. Formulate the hypothesis H_0 & H_1 and decide whether it is a Two tailed test or an One tailed test with the help of H_1 .

2. Choose α the level of significance (LOS). [**Note:** If α is not known then assume α to be 5%]

3. Compute the test statistic Z using the formula.

4. Pick out the critical value Z_α (depending on Two tailed test or One tailed) at α level.

Critical Value	$\boldsymbol{\alpha = 1\%}$	$\boldsymbol{\alpha = 5\%}$	$\boldsymbol{\alpha = 10\%}$
Z_α value for Two tail test	2.58	1.96	1.645
Z_α value for One tail test	2.33	1.645	1.28

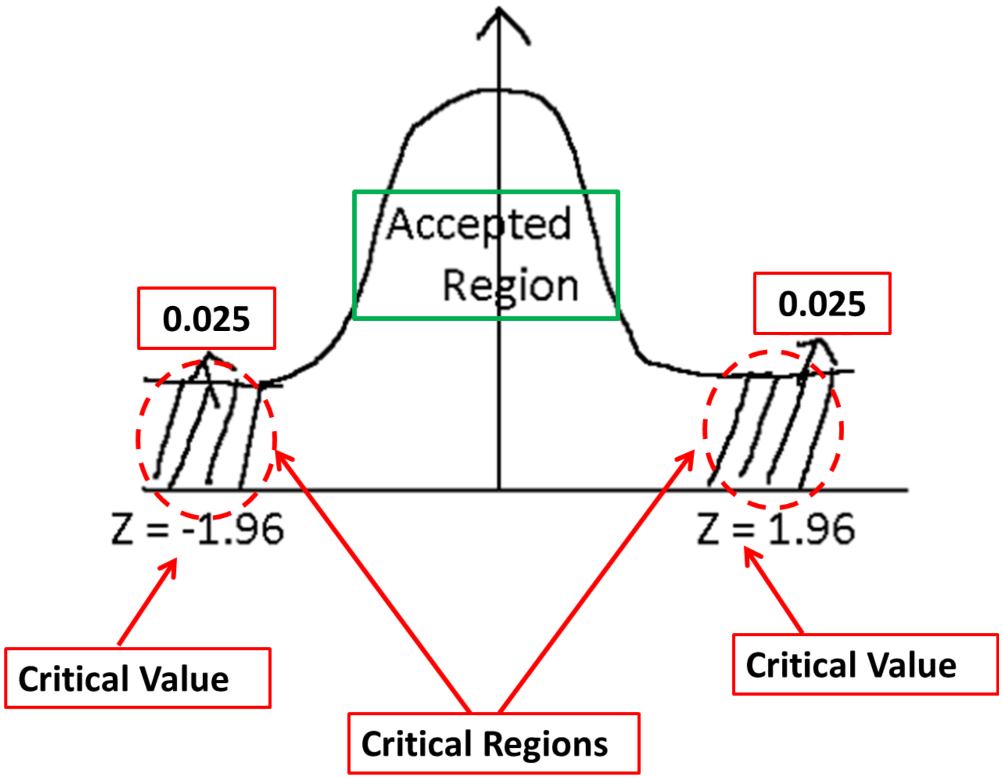
5. Drawing the conclusion:

If $|Z| \leq Z_\alpha$ then accept H_0 at α level.

If $|Z| > Z_\alpha$ then reject H_0 at α level.

Note:

- Accepting H_0 means rejecting H_1.
- Rejecting H_0 means accepting H_1.



Problems
Sample Mean (Single and Difference)

1. The mean of two large samples of 1000 and 200 members are 67.5 inches and 68 inches respectively. Can the samples be regarded as drawn from the population of standard deviation of 2.5 inches?

Sol:-

Sample 1

$\bar{x}_1 = 67.5$

$n_1 = 1000$

Sample 2

$\bar{x}_2 = 68$

$n_2 = 200$

Step 1:- H_0: Samples are drawn from the population of S.d of 2.5 (i.e.) $\boxed{\sigma_1 = \sigma_2 = 2.5}$

$H_1: \sigma_1 \neq \sigma_2$ (Two Tail test)

Step 2:- LOS, $\alpha = 5\%$ (Assume)

Step 3:- Test statistic

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{67.5 - 68}{\sqrt{\frac{2.5^2}{1000} + \frac{2.5^2}{200}}}$$

$$\boxed{Z = -2.58} \Rightarrow |Z| = 2.58$$

Step 4:- Critical value.

$Z_{5\%}$ (Two tail test) $= 1.96$.

Step 5: Conclusion

Here, $|Z| > Z_{5\%}$ (Two tail test)

$\therefore H_0$ is rejected at 5% LOS.

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2. A random sample of 200 Employee's at a large corporation showed their average age to be 42.8 years, with a S.D of 6.8 years. Test the hypothesis H_0: $\mu = 40$ versus H_1 : $\mu > 40$ at $\alpha = 0.01$ level of significance.

Sol:-

Sample

$n = 200$

$\bar{x} = 42.8$

$s = 6.8$

Step 1: H_0 : $\mu = 40$

H_1 : $\mu > 40$ - (one tail test)

Step 2:- $\alpha = 0.01 = 1\%$

Step 3: Test Statistics

$$Z = \frac{\bar{x} - \mu}{\sqrt{s^2/n}} = \frac{42.8 - 40}{\sqrt{6.8^2/200}}$$

$$\boxed{Z = 5.82}$$

Step 4: Critical value

$$Z_{1\%}(\text{one tail test}) = 2.33$$

Step 5: Conclusion

Here, $|Z| > Z_{1\%}(\text{one tail test})$

$\therefore H_0$ is rejected at 1% LOS.

Confidence Interval

- Confidence interval usually used for population purpose only. It is the interval that we are 95% (for example) confidence that this confidence interval will contain the true unknown value of the population mean (population proportion) or difference of population mean (difference of population proportion).
- If the given population mean (proportion) belongs to the confidence interval then H_0 will be accepted.
- **Confidence Interval:** $satistics \pm (critical\ value)(sampling\ error)$

Confidence Interval:

$$satistics \pm (critical\ value)(sampling\ error)$$

S.No.	Statistics	Critical Value	Standard Error
1	$\overline{x}$	Z_α	$\sqrt{s^2/n}$
2	$\overline{x_1} - \overline{x_2}$	Z_α	$\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$
3	p'	Z_α	$\sqrt{p'q'/n}$
4	$p'_1 - p'_2$	Z_α	$\sqrt{\frac{p'_1q'_1}{n_1} + \frac{p'_2q'_2}{n_2}}$

Standard Error:

The standard deviation of the sampling distribution is called the standard error.

$Z_\alpha \to$ Two tail test

Types	Range for Confidence Interval
Single mean	$\mu \in \left(\overline{x} - Z_\alpha\left(\sqrt{s^2/n}\right), \quad \overline{x} + Z_\alpha\left(\sqrt{s^2/n}\right)\right)$
Difference of two means	$\mu_1 - \mu_2 \in \left((\overline{x_1} - \overline{x_2}) - Z_\alpha\left(\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}\right), \quad (\overline{x_1} - \overline{x_2}) + Z_\alpha\left(\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}\right)\right)$
Single Proportion	$p \in \left(p' - Z_\alpha\left(\sqrt{p'q'/n}\right), \quad p' + Z_\alpha\left(\sqrt{p'q'/n}\right)\right)$
Difference of two Proportions	$p_1 - p_2 \in \left((p_1' - p_2') - Z_\alpha\left(\sqrt{\frac{p_1'q_1'}{n_1} + \frac{p_2'q_2'}{n_2}}\right), \quad (p_1' - p_2') + Z_\alpha\left(\sqrt{\frac{p_1'q_1'}{n_1} + \frac{p_2'q_2'}{n_2}}\right)\right)$

$Z\alpha \rightarrow$ Two tail test

$\boldsymbol{Z_{\alpha}}$ value for **99**% Confidence interval	2.58
$\boldsymbol{Z_{\alpha}}$ value for **95**% Confidence interval	1.96
$\boldsymbol{Z_{\alpha}}$ value for **90**% Confidence interval	1.645

Important Notes:

- $S.E.$ is the standard error which appears in the denominator of the corresponding formula.
- $\boldsymbol{Z_{\alpha}}$ is the critical value for two tailed test of the corresponding type for a given α.

4. Suppose the manufacturer claims that the mean lifetime of a light bulb is more than 10,000 hours. In a sample of 35 light bulbs, it was found that they only last 9,900 hours on average. Assume the population standard deviation is 120 hours. At 5% significance level, can we reject the claim by the manufacturer? Also find the 95% confidence limits for the population mean.

Sol:-

Sample : $n = 35$, and $\bar{x} = 9,900$.

Population: $\mu = 10,000$ and s.d $(\sigma) = 120$.

Step 1:

H_0: $\bar{x} \leq \mu$

H_1: The mean lifetime of a light bulb is more than 10,000 hours

(i.e) $\bar{x} > \mu$ (one tail test)

Step 2: los, $\alpha = 5\%$ (assume)

Step 3: Test statistic:

$$Z = \frac{\bar{x} - \mu}{\sqrt{\frac{\sigma^2}{n}}} = \frac{9{,}900 - 10{,}000}{\sqrt{\frac{120^2}{35}}}$$

$$Z = -4.93 \Rightarrow \boxed{|Z| = 4.93}$$

Step 4: Critical value

$$Z_{5\%} \text{ (one tail test)} = 1.645$$

Step 5: Conclusion: Here, $|Z| > Z_{5\%}$ (one tail test)

$\therefore H_0$ is rejected at 5% los.

Confidence interval : 95% $\boxed{Z_{5\%} = 1.96.}$ two tail test

$$\mu \in \left(\bar{x} - Z_\alpha (S.E),\ \bar{x} + Z_\alpha (S.E)\right)$$

$$\mu \in \left(\bar{x} - 1.96\left(\frac{s}{\sqrt{n}}\right),\ \bar{x} + 1.96\left(\frac{s}{\sqrt{n}}\right)\right)$$

$$\mu \in \left(9{,}900 - 1.96\left(\frac{120}{\sqrt{35}}\right),\ 9{,}900 + 1.96\left(\frac{120}{\sqrt{35}}\right)\right)$$

$$\mu \in (9860.24,\ 9939.75)$$

Note:- It is given that $\mu = 10{,}000$ and $10{,}000 \notin (9860.24, 9939.75)$. Thus H_0 is being rejected at 5% l.o.s.

5. Samples of students were drawn from two universities and from the weights is kilogram. The means and S.D.'s are calculated. Test the significance of the difference between the means of two samples. Also construct a 95% confidence interval for difference of two population mean .

	Mean	S.D.	Sample Size
University A	55	10	400
University B	57	15	100

$$\mu_1 - \mu_2 \in \left((\bar{x}_1 - \bar{x}_2) - 1.96\left(\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}\right), (\bar{x}_1 - \bar{x}_2) + 1.96\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \right)$$