Problem 2: A random variable X has the following probability distribution.

x	-2	-1	0	1	2	3
P(x)	0.1	k	0.2	2k	0.3	3k

Find (i) k, (ii) P(X<2), (iii) $P(-1 \le X < 3 / X > 0)$, (iv) cdf of X and

(v) Mean of X

(i) To find 'k' :-

W.k.T, $\sum_x P(X=x) = 1$

$0.1 + k + 0.2 + 2k + 0.3 + 3k = 1$

$6k + 0.6 = 1 \Rightarrow \boxed{k = 1/15}$

x	-2	-1	0	1	2	3
$p(x)$	0.1	$1/15$	0.2	$2/15$	0.3	$3/15 = 1/5$

(ii) $P(X<2)$

$$P(X<2) = P(X=-2) + P(X=-1) + P(X=0) + P(X=1)$$

$$= (0.1) + (1/15) + (0.2) + 2/15$$

$$\boxed{P(X<2) = 1/2}$$

x	-2	-1	0	1	2	3
$p(x)$	0.1	$1/15$	0.2	$2/15$	0.3	$3/15 = 1/5$

(iii) $P(-1 \leq X < 3 \,/\, X \geq 0)$

$$= \frac{P((-1 \leq X < 3) \cap (X \geq 0))}{P(X \geq 0)}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P\big((X=-1, X=0, X=1, X=2) \cap (X=1, X=2, X=3)\big)}{P(X=1, X=3, X=3)}$$

$$= \frac{P(X=1, X=2)}{P(X=1, X=2, X=3)}$$

$$= \frac{P(X=1) + P(X=2)}{P(X=1) + P(X=2) + P(X=3)}$$

x	-2	-1	0	1	2	3
$p(x)$	0.1	$1/15$	0.2	$2/15$	0.3	$3/15 = 1/5$

$$= \frac{P(X=1) + P(X=2)}{P(X=1) + P(X=2) + P(X=3)}$$

$$= \frac{2/15 + 3/10}{2/15 + 3/10 + 1/5}$$

$$\boxed{P(-1 \le x < 3 / X > 0) = \frac{13}{19}}$$

4.

x	-2	-1	0	1	2	3
$p(x)$	0.1	$1/15$	0.2	$2/15$	0.3	$3/15 = 1/5$

(iv) Cdf of X

Cdf is denoted by $F(x)$.

$F(x) = 0 \,, \quad x < -2.$

$F(x) = 1/10 \,, \quad -2 \le x < -1.$

$F(x) = \frac{1}{10} + \frac{1}{15} \quad (= P(X \le -1)).$

$= \frac{1}{6} \,, \quad -1 \le x < 0$

x	-2	-1	0	1	2	3
$p(x)$	0.1	$1/15$	0.2	$2/15$	0.3	$3/15 = 1/5$

$$F(x) = \frac{1}{10} + \frac{1}{15} + \frac{1}{5} \quad (= P(X \le 0))$$

$$= \frac{11}{30}, \quad 0 \le x < 1$$

$$F(x) = \frac{1}{10} + \frac{1}{15} + \frac{1}{5} + \frac{2}{15} \quad (= P(X \le 1))$$

$$= \frac{1}{2}, \quad 1 \le x < 2.$$

$$F(x) = \frac{1}{10} + \frac{1}{15} + \frac{1}{5} + \frac{2}{15} + \frac{3}{10} \quad (= P(X \le 2))$$

$$= \frac{4}{5}, \quad 2 \le x < 3.$$

x	-2	-1	0	1	2	3
$p(x)$	0.1	$1/15$	0.2	$2/15$	0.3	$3/15 = 1/5$

$$F(x) = \frac{1}{10} + \frac{1}{15} + \frac{1}{5} + \frac{2}{15} + \frac{3}{10} + \frac{1}{5}$$

$P(X \leq 3)$

$$F(x) = 1, \quad 3 \leq x = x \geq 3.$$

$\longrightarrow$ Total probability.

(v) Mean of X.

$$\text{Mean} = E(X) = \sum_x x\,p(x).$$

$$= (-2)\left(\frac{1}{10}\right) + (-1)\left(\frac{1}{15}\right) + (0)(1/5)$$

$$+ (1)(2/5) + (2)(3/10) + (3)(1/5)$$

$$\boxed{\text{Mean} = \frac{16}{15}}$$

CDF in Continuous Case

Note:-

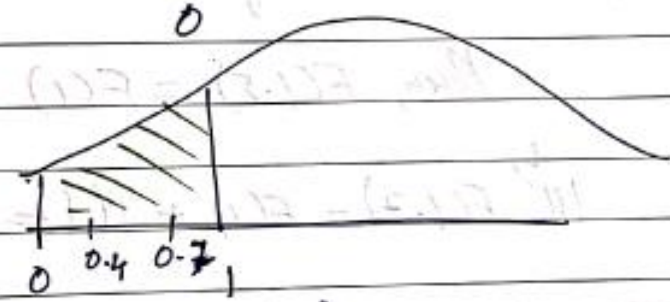
Cdf of X in continuous case.

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ ax^2, & 1 \le x \le 2 \\ 1, & 2 \le x \le 3. \end{cases}$$

(*) $F(x) = P(X \le 0) = 0.$

$$x \rightarrow ?$$

$$F(x) = \int_0^{x} (ax)\, dx, \quad 0 \le x \le 1$$



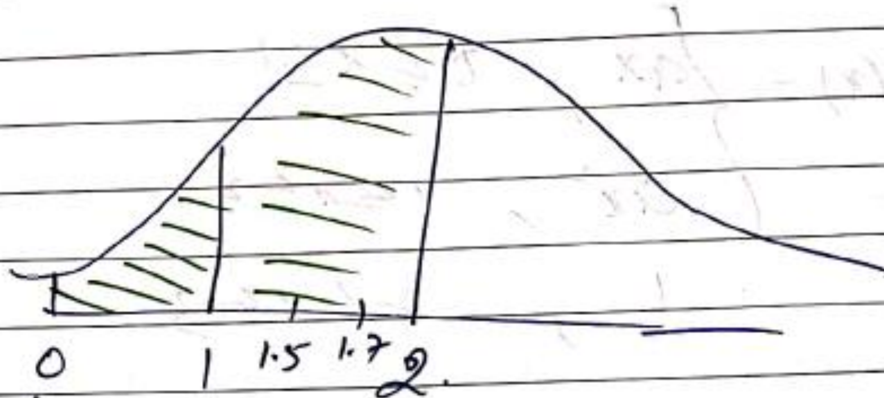
$$F(x) = \int_0^x (ax)\, dx = \frac{ax^2}{2}$$

$$F(0.4) = \frac{a(0.4)^2}{2}$$

$$F(0.7) = \frac{a(0.7)^2}{2}$$

This is the reason why we have the variable 'x' in the upper limit of the integral when we calculate cdf.

$$F(x) = \int_0^1 (ax)\,dx + \int_1^x (ax^2)\,dx, \quad 1 \le x \le 2.$$



$$\int_0^1 + \int_1^x$$

=) Because, if we want to find $F(1.5)$ or $F(1.7)$

Then, $F(1.5) = F(1) + P(1 \le X \le 1.5)$

|||ly $F(1.7) = F(1) + P(1 \le X \le 1.7)$

This why we have the variable 'x' in the second integral as an upper limit. Because it is defined in the range $1 \le x \le 2$.

Problem 3: If the density function of a continuous random variable X is given by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & elsewhere \end{cases}$$

(i) find the value of 'a', (ii) find cdf of X and (iii) If $x_1, x_2 \; and \; x_3$ are 3 independent observations of X, what is the probability that exactly one of these 3 is greater than 1.5?

Solution:

X – random variable.

$$f(x) = \left.\begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a-ax, & 2 \le x \le 3 \\ 0, & elsewhere. \end{cases}\right] \rightarrow R_x \in [0,3]$$

(i) To find 'a' :-

W.K.T, $\int_{R_x} f(x)\,dx = 1$

$\int_0^3 f(x)\,dx = 1$

$\int_0^1 (ax)\,dx + \int_1^2 (a)\,dx + \int_2^3 (3a - ax)\,dx = 1$

$\Rightarrow \boxed{a = 1/2}$

(ii) cdf of X. (i.e.) $F(x)$.

W.K.T $F(x) = P(X \le x)$

Now,

$F(x) = 0$, when $x < 0$.

$F(x) = \int_0^x \left(\frac{x}{2}\right) dx = \frac{x^2}{4}$, $0 \le x \le 1$.

$$F(x) = \int_0^1 \left(\frac{x}{2}\right) dx + \int_1^x \left(\frac{1}{2}\right) dx$$

$$= \frac{x}{2} - \frac{1}{4}, \quad 1 \le x \le 2.$$

$$F(x) = \int_0^1 \left(\frac{x}{2}\right) dx + \int_1^2 \left(\frac{1}{2}\right) dx + \int_2^x \left(\frac{3}{2} - \frac{x}{2}\right) dx$$

$$= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}, \quad 2 \le x \le 3.$$

$$F(x) = 1, \quad x > 3.$$

$\rightarrow$ Total probability.

(iii) choosing an X and observing its value can be considered as a trial and $(x>1.5)$ can be considered as a success.

Now, we can choose Bernoulli's theorem, since there are 3-independent observations.

$\therefore \boxed{n=3}$

P(Exactly one value >1.5)

$= P(x=1)$

$= 3c_1 (p)^1 (q)^{3-1}$

$$p = P(X > 1.5) = \int_{1.5}^{3} f(x)\,dx$$

$$= \int_{1.5}^{2} \left(\frac{1}{2}\right) dx + \int_{2}^{3} \left(\frac{3}{2} - \frac{x}{2}\right) dx$$

$\boxed{p = 1/2}$ $\Rightarrow$ Probability of success.

$q = 1 - p \Rightarrow \boxed{q = 1/2}$

Now,

P(Exactly one value ≥ 1.5)

$= P(X=1)$

$= 3C_1 (p)^1 (q)^{3-1}$

$= 3C_1 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2$

$= 3/8$ /.

Problem 4: A continuous random variable X that can assume any value between $x = 2 \; and \; x = 5$ has a density function given by $f(x) = k(1 + x)$. Find $P(X < 4)$.

Solution:

Given: $f(x) = k(1+x), \quad 2 < x < 5.$

W.K.T

$$\int_2^5 f(x)\,dx = 1$$

$$\int_2^5 k(1+x)\,dx = 1$$

$$\Rightarrow \boxed{k = 2/27}$$

Now,

$$P(X<4) = \int_{2}^{4} f(x)\,dx$$

$$= \int_{2}^{4} \frac{2}{27}(1+x)\,dx = \frac{16}{27} \text{ Ans.}$$