1. In a certain city, the daily consumption of electric power in millions of kilowatt hours can be treated as a random variable having an **Erlang distribution** with parameters $\lambda = 1/2$ and $\alpha = 3$. If the power plant of this city has a daily capacity of 12 million kilowatt hours, what is the probability that this power supply will be inadequate on any given day?

① Given:- $\lambda = 1/2$ and $\alpha = 3$.

Let X be the daily consumption of electric power in millions of kilowatt hours having an Erlang distribution.

Erlang distbt

$$f(x) = \frac{e^{-\lambda x} \lambda^{\alpha} x^{\alpha-1}}{\Gamma\alpha}$$

$$= \frac{(1/2)^3 x^2 e^{-x/2}}{\Gamma 3}$$

p.d.f, $\boxed{f(x) = \frac{x^2 e^{-x/2}}{16}}$

P(power supply is inadequate)

$$= P(X > 12) = \int_{12}^{\infty} f(x)\, dx$$

$$= \int_{12}^{\infty} \frac{x^2 e^{-x/2}}{16} dx$$

$$P(X > 12) = 25 e^{-6}$$

$$\boxed{P(X > 12) = 0.0625}$$

2. The time (in hours) required to repair a machine is **exponentially distributed** with a mean 2.

(a) What is the probability that the repair time exceeds 2 hrs?

(b) What is the conditional probability that a repair takes at least 10 hrs given that its duration exceeds 9 hrs?

Let X be the time to repair the machine, having an exponential distribution.

Exponential distbt $\left[\text{Mean} = 1/\lambda \Rightarrow \lambda = 1/2\right]$.

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{2} e^{-x/2}$$

$$\boxed{f(x) = \frac{1}{2} e^{-x/2}}, \quad x > 0$$

(a) $P(X > 2) = \int_2^\infty f(x)\,dx$

$= \int_2^\infty \frac{1}{2} e^{-x/2}\,dx = e^{-1} = 0.3679.$

$$\boxed{P(X > 2) = 0.3679}$$

(b) $P(X \geq 10 \mid X > 9)$

$$= P(X \geq 9+1 \mid X > 9)$$

$$= P(X > 1). \quad \text{(Memoryless property)}.$$

$$= \int_1^\infty f(x)\,dx$$

$$= \int_1^\infty \frac{1}{2} e^{-x/2}\,dx$$

$$= e^{-0.5}$$

$$\boxed{P(X \geq 10 \mid X > 9) = 0.6065}$$

3. The length of time for one individual to be served at a cafeteria is a random variable having an **exponential distribution** with a mean of 4 minutes. **What is the probability** that a person is served in less than 3 minutes on at least 4 of the next 6 days?

Let X denotes the no. of days a person is served in less than 3 minutes.

$$P(X \geq 4) = \sum_{x=4}^{6} 6C_x \, p^x \, q^{6-x}$$

Here, n=6 and p = ?, q = 1-p.

p → probability that a person is served in less than 3 minutes.

Now, let Y denotes the length of time to be served for a person.

It is given that, Y follows an exponential distribution with mean 4.

(i.e) mean $= 1/\lambda$

$$4 = 1/\lambda \Rightarrow \boxed{\lambda = 1/4}$$

Now, $p = P(y < 3) = \int_0^3 f(x).dx.$

exponential distribution pdf.

$$= \int_0^3 \lambda e^{-\lambda x}.dx.$$

$$= \int_0^3 \frac{1}{4} e^{-x/4} \, dx = 0.5276.$$

$$\therefore \boxed{p = 0.5276}.$$

Now,

$$P(X \geq 4) = \sum_{x=4}^{6} 6C_x \, (p)^x (q)^{6-x}.$$

$$= \sum_{x=4}^{6} 6C_x (0.5276)^x (0.4724)^{6-x}$$

$$\boxed{P(X \geq 4) = 0.3968.}$$

Problem 4: Let X be a R.V with exponential distribution with parameter λ.

(i) Find 'k' such that $\frac{P(X>k)}{P(X\le k)} = t$.

(ii) Find $Var(X)$ if $P(X\le 1) = P(X>1)$.

Sol:

(i) Given: $\frac{P(X>k)}{P(X\le k)} = t$ —— (1)

$f(x) = \lambda e^{-\lambda x}$ for exponential distribution

$$P(X>k) = t \cdot P(X\le k)$$

$$\int_k^\infty \lambda \cdot e^{-\lambda x} \cdot dx = t \int_0^k \lambda \cdot e^{-\lambda x} \cdot dx$$

$$\cancel{\lambda}\left[\frac{e^{-\lambda x}}{-\cancel{\lambda}}\right]_k^\infty = t \cdot \cancel{\lambda}\left[\frac{e^{-\lambda x}}{-\cancel{\lambda}}\right]_0^k$$

$$e^{-\infty} - e^{-\lambda k} = t\left[e^{-\lambda k} - e^{0}\right].$$

$$0 - e^{-\lambda k} = t\,e^{-\lambda k} - t.$$

$$t = e^{-\lambda k}(t+1) \Rightarrow e^{-\lambda k} = \frac{t}{t+1}.$$

$$e^{\lambda k} = \frac{t+1}{t} \Rightarrow \lambda k = \log\left(\frac{t+1}{t}\right).$$

$$\boxed{k = \frac{1}{\lambda}\left[\log\left(\frac{t+1}{t}\right)\right]}$$

(ii) Given: $P(X \le 1) = P(X > 1)$

$$1 = \frac{P(X>1)}{P(X \le 1)} \quad \text{——— (2)} \qquad \textbf{and} \qquad \frac{P(X>k)}{P(X \le k)} = t. \quad \text{——— (1)}$$

By comparing (1) and (2), we have $t=1$ and $k=1$.

From (i), we have; $k = \frac{1}{\lambda}\log\left(\frac{t+1}{t}\right)$.

$$\Rightarrow \lambda = \frac{1}{k}\log\left(\frac{t+1}{t}\right).$$

Now, $\lambda = \frac{1}{1}\log\left(\frac{1+1}{1}\right)$ $\left[\because t=1,\ k=1\right]$.

$\lambda = \log(2)$.

$$\boxed{Var(x) = \frac{1}{\lambda^2} = \frac{1}{(\log(2))^2}}$$

4.

X

Problem 5:

In an engineering examination, a student is considered to have failed, secured second class, first class and distinction, according as he scores less than 45%, between 45% and 60%, between 60% and 75% and above 75% respectively. In a particular year 10% of the students failed in the examination and 5% of the students got distinction. Find the percentages of students who have got first class and second class. **(Assume normal distribution of marks)**

Given:

Let $X \to$ % of marks scored by the student in the exam. which follows a normal distribution.

(i.e) $X \sim N(\mu, \sigma)$.

To find: ~~(*)~~ μ and σ.

(*) P(First class student)

P(Second class student).

Now, we have, $P(x < 45) = 0.10$

and

$P(x > 75) = 0.05$

First:-

$P(x < 45) = 0.10.$

$P\left(z < \frac{45-\mu}{\sigma}\right) = 0.10.$

$$\frac{45-\mu}{\sigma} = -1.26$$

$$\mu - 1.26\sigma = 45 \quad \text{———} \quad (1)$$

Second:-

$P(x > 75) = 0.05$

$1 - P(x < 75) = 0.05$

$1 - 0.05 = P(x < 75)$

$P(x < 75) = 0.95$

$P\left(z < \frac{75-\mu}{\sigma}\right) = 0.95$

$$\frac{75-\mu}{\sigma} = 1.64$$

$$\mu + 1.64\sigma = 75 \quad \text{———} \quad (2)$$

From Equation 1 and 2, We have

$$\mu - 1.26\sigma = 45 \quad \text{———} \quad 1$$

$$\mu + 1.64\sigma = 75 \quad \text{———} \quad 2$$

$$\mu = 58.15$$

$$\sigma = 10.28$$

(x) P(First class student)

$$= P(60 < X < 75)$$

$$= P(X < 75) - P(X < 60).$$

$$= P(Z < 1.64) - P(Z < 0.18)$$

$$= 0.9495 - 0.5714$$

$$= 0.3781$$

$$Z = \frac{x-\mu}{\sigma} = \frac{75 - 58.15}{10.28}$$

$$Z = 1.64$$

$$Z = \frac{60 - 58.15}{10.28}$$

$$Z = 0.18$$

(*) P(second class student)

$= 100 - [10 + 38 + 5]$

(%) & (%)

Sum of % of students who have failed, got first class and got distinction.

$= 100 - (53)$

$= 47$ (approx). $= 0.47$