Module 2:
Random Variables

Introduction to Probability - Conditional Probability –Bayes Theorem- Random variables-Probability mass Function, distribution and density functions - joint Probability distribution and joint density functions- Marginal, conditional distribution and density functions- Mathematical expectation, and its properties Covariance , moment generating function – characteristic function.- Managing Financial Risk -Modelling Sampling Variation(one dimension)

Experiments

Random

Outcome is determined by chance. In other way, the outcomes are known but can not predict the exact outcome

Deterministic

Outcome may be predicted with certainty beforehand, such as combining Hydrogen and Oxygen, or adding two numbers such as 2+3.

Eg: $\boxed{\text{Type 1:}}$ No. of telephone calls received in 1hour has a numerical value.

$\boxed{\text{Type 2}}$ Tossing a coin has a non-numerical value to its outcome.

In this case, we assign a numerical value to its outcomes. This process is called random variable.

(i.e) it is a function that assign a real number to every possible outcomes of an experiment.

(*) The assigned number itself can be thought of as the outcome of the experiment. Thus, $R_x \rightarrow$ sample space instead of sample space S.

(*) Therefore, hereafter we will talk about a random variable X taking the value x. and $P(X=x)$.

$$\boxed{\text{Actually, } P(X=x) = P\{s : X(s) = x\}.}$$

Random variables (R.V) : One-dimensional R.V. (X).

and

Two-dimensional R.V. (X and Y).

One-dimensional R.V $\rightarrow$ Discrete case.

and

Continuous case.

$X \rightarrow$ Random variable.

Discrete random variable	Continuous random Variable
1) $X \to$ Finite number (or) Countably infinite number.	1) $X \to$ Takes values in an interval, I.
Eg: Number shown when a die is thrown.	Eg: length of time, measuring heights etc.
2) $P(X = x_i)$ (or) simply $P(X = x)$ is the probability mass function. (pmf)	2) $f(x) \to$ probability density function. (pdf).

3) pmf satisfies followings:

(i) $P(X=x) \geq 0$.

(ii) $\sum P(X=x) = 1$.

3) pdf satisfies followings:

(i) $f(x) \geq 0, \forall x \in R_x$.

(ii) $\int_{R_x} f(x)\,dx = 1$.

$R_x \in (-\infty, \infty)$.

4) Probability distribution of the R.V. X

specific value for x.

$X=x$	$P(X=x)$
x_1	$P(X=x_1)$
x_2	$P(X=x_2)$
$\vdots$	$\vdots$
x_n	$P(X=x_n)$
$\vdots$	$\vdots$

4) Probability distribution of a continuous R.V can't be represented by a table as in discrete R.V.

* Probability distribution is known, if either its pdf (or) cdf is given.
[For specific intervals we can find its probability].

5) Cumulative distribution function (Cdf)

$$F(x) = \sum_i P(X = x_i).$$

(or)

$$\boxed{F(x) = P(X \leq x).}$$

5) $\boxed{\text{Cdf}}$

$$F(x) = P(-\infty < X < x)$$

$$= \int_{-\infty}^{x} f(x) \cdot dx.$$

$$\boxed{f(x) = \frac{d}{dx}(F(x))}$$

6) Properties:

(i) $F(x)$ is non-decreasing function of x.

(i.e) If $x_1 < x_2$ then $F(x_1) < F(x_2)$

(ii) Same as in continuous case.

(iii) $P(X = x_i) = F(x_i) - F(x_{i-1})$

If $x_1 < x_2 < \dots < x_{i-1} < x_i < \dots$

6) Properties

(i) Same as in discrete case.

(ii) $F(-\infty) = 0$ ~~F(∞) = 0~~.

and $F(\infty) = 1$.

↳ Total probability.

(iii) $f(x) = \frac{d}{dx}(F(x))$

$F(x)$, is diff at all 'x'.

Note: $P(a \le X \le b) = P(a < X \le b)$

$= P(a \le X < b) = P(a < X < b)$.

Note:

1) $X \to$ Continuous R.V then it is impossible that a continuous RV assumes a specific value.

(i.e) $P(X=a) = P(a \leq X \leq a) = \int_a^a f(x)\,dx = 0.$

Hence, $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b)$
$= P(a < X < b).$

2) Probability distribution for interval

Intervals	Probability
$a \le X \le b$.	$P(a \le X \le b)$
$c \le X \le d$.	$P(c \le X \le d)$
$X \le e$	$P(X \le e)$
$X \ge m$	$P(X \ge m)$

3) Mean $= E(X) = \sum x\, p(x)$.

Variance $= E(X^2) - (E(X))^2$

$= \sum x^2 p(x) - \left(\sum x\, p(x)\right)^2$.

→ Discrete RV.

Mean $= E(X) = \int_{R_x} x f(x)\, dx$

$E(X^2) = \int_{R_x} x^2 f(x)\, dx$.

→ Continuous RV.

Variance $= E(X^2) - (E(X))^2$

4) $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

5) Bernoulli's trials:

$n \to$ Independent repetitions (trials) of a random experiment E.

$A \to$ a event in the experiment E. such that $P(A)$ remains the same for every trials, then the trials are called "Bernoulli's trials."

$$nC_r \, p^r q^{n-r}.$$

where,

$r \to$ times out of n-trials. $0 \le r \le n$

$n \to$ Indpt trials.

$p \to$ success prob

$q \to$ failure prob.

where, $p + q = 1$.

(*) Two coins $= \{HH, HT, TH, TT\}$.

$P(\text{exactly one head}) = \frac{2}{4} = \frac{1}{2}$

$P(\text{at least one head}) = \frac{3}{4}$

$P(X \geq 1) = P(X=1) + P(X=2)$

$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

Logic: 1) First take X to be heads outcome.
2) Find probability of getting head.
3) Then find $P(X=x)$.

Let X be the no. of heads in tossing two coins.

Then, $X = \{2, 1, 0\}$.

$p = \frac{1}{2}$ (probability of getting a head).

$P(X=1) = 2C_1 (\frac{1}{2})^1 (\frac{1}{2})^{2-1}$

$= 2 \times \frac{1}{4} = \frac{1}{2}$

$P(X=2) = 2C_2 (\frac{1}{2})^2 (\frac{1}{2})^{2-2}$

$= 1 \times \frac{1}{4} = \frac{1}{4}$.

exactly two heads.

Problem 1: From a lot containing 25 items, 5 of which are defective, 4 items are chosen at random. If X is the number of defectives found, then obtained probability distribution of X, when the items are chosen with replacement and without replacement.

(1)

Case (i): With replacement

Let X → the no. of defective items.

Here we can use Bernoulli's trials since probability for an event remains same for every trials.

$p = \frac{5}{25} \Rightarrow \boxed{p = 1/5}$ then $q = 1 - p$.

$\boxed{n = 4}$ [∵ 4-items are chosen]. $\boxed{q = 4/5}$.

Bernoulli's trial.

$$P(X=x) = nC_x \, p^x \, q^{n-x}, \quad \underbrace{x = 0,1,2,3,4.}_{\text{4-trials (i.e.) 4-items are chosen.}}$$

$$= 4C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{4-x}$$

Probability distribution table:-

$P(X=x)$	$nC_x \, p^x \, q^{n-x}$.
$P(X=0)$	$4C_0 (1/5)^0 (4/5)^4 = 0.4096$
$P(X=1)$	$4C_1 (1/5)^1 (4/5)^3 = 0.4096$.
$P(X=2)$	$4C_2 (1/5)^2 (4/5)^2 = 0.1536$
$P(X=3)$	$4C_3 (1/5)^3 (4/5)^1 = 0.0256$
$P(X=4)$	$4C_4 (1/5)^4 (4/5)^0 = 0.0016$

Case (ii):- Without replacement.

In this case, all the 4-items are chosen simultaneously.

$$\therefore P(X=x) = P(\text{choosing exactly } x\text{-defective items}).$$

$$= P(\text{choosing } x\text{-defective items and } (4-x)\text{-good items}).$$

$$P(X=x) = \frac{[5C_x][20C_{4-x}]}{25C_4} \begin{matrix} \rightarrow \text{Favourable} \\ \rightarrow \text{Total} \end{matrix}, \quad x=0,1,\dots,4.$$

$$P(X=x) = \frac{\overbrace{[5C_x][20C_{4-x}]}^{\rightarrow \text{Favourable}}}{\underbrace{25C_4}_{\rightarrow \text{Total}}}, \quad x=0,1,\cdots,4.$$

Prob. distrib. table

x	$P(X=x)$
0	0.3830
1	0.4505
2	0.1502
3	0.0158
4	0.0004

Problem 2: A random variable X has the following probability distribution.

x	-2	-1	0	1	2	3
P(x)	0.1	k	0.2	2k	0.3	3k

Find (i) k, (ii) P(X<2), (iii) $P(-1 \le X < 3 / X > 0)$, (iv) cdf of X and

(v) Mean of X

(i) To find 'k' :-

W.K.T, $\sum_x P(X=x) = 1$

$0.1 + k + 0.2 + 2k + 0.3 + 3k = 1$

$6k + 0.6 = 1 \Rightarrow \boxed{k = 1/15}$

x	-2	-1	0	1	2	3
p(x)	0.1	1/15	0.2	2/15	0.3	3/15 = 1/5

(ii) $P(X<2)$

$$P(X<2) = P(X=-2) + P(X=-1) + P(X=0) + P(X=1)$$

$$= (0.1) + (1/15) + (0.2) + 2/15$$

$$\boxed{P(X<2) = 1/2}$$

x	-2	-1	0	1	2	3
$p(x)$	0.1	$1/15$	0.2	$2/15$	0.3	$3/15 = 1/5$