Module 4 Lattices

— Dr. R. Udhayakumar

Partially ordered Relations

Partial ordering

A binary relation 'R' in a set 'P' is called a partial order relation (or) a partial ordering in P iff

(1) R is reflexive ($\forall x \in P$; xRx)

(2) R is antisymmetric ($\forall x, y \in P$, if xRy & yRx then $x = y$)

(3) R is transitive ($\forall x, y, z \in P$, if xRy, yRz then xRz).

Notation (or) symbol

It is conventional to denote a partial ordering by the symbol $\leq$. (this is not "Less than or equal to" as it is used for real numbers).

Partial ordered set or POSET

If $\leq$ is a partial ordering on P, then the ordered pair $(P, \leq)$ is called a partially ordered set or POSET.

Totally ordered set (or) ordered set (or) chain

Let $\langle P, \leq \rangle$ be a poset. If for every $x, y \in P$, we have either $x \leq y$ (or) $y \leq x$ (i.e any two elts of 'P' are related by $\leq$), then $\leq$ is called a simple ordering (or) linear ordering on P, and $\langle P, \leq \rangle$ is called a totally ordered set (or) chain.

Remarks :-

1) It is not necessary to have $x \le y$ or $y \le x$ for ~~any~~ every x, y in a poset P.

2) If x may not be related to y, in which case we say that x and y are incomparable.

3) If R is a partial ordering on P, then the converse of R (namely $\neg R$) or $\tilde{R}$ is also a partial ordering on P.

4) If R is denoted by $\le$, then $\tilde{R}$ is denoted by $\ge$.

5) If $\langle P, \le \rangle$ is a poset then $\langle P, \ge \rangle$ is also a poset.

6) $\langle P, \ge \rangle$ is a called a dual of $\langle P, \le \rangle$.

Examples of Posets

1) Let R be the set of all real numbers. Then the relation "less than or equal to" is a partial ordering on R. The converse is also a partial ordering on R (greater than or equal to).

Thus $\langle R, \le \rangle$ and $\langle R, \ge \rangle$ are posets.

Note:- $\langle R, < \rangle$ and $\langle R, > \rangle$ are not posets since $\langle < \rangle$ is not reflexive.

2) $\mathbb{N}$, $\mathbb{Z}$ are posets with the usual relation $\le$.

3) $(P(A), \subseteq)$ $\{1,2\} \not\subseteq \{1,3\}$ are not comparable.

4) $\mathbb{N}$ of positive integers with less than equal ($\le$) is a linear order of $\mathbb{N}$.

Partially orderedset: Representation and Associated Terminology

Cover (or) immediate predecessor

In a poset $\langle P, \le \rangle$, an elt $y \in P$ is said to cover an elt $x \in P$, if $x \le y$ and if there does not exist any elt $z \in P$ s.t $x \le z$ and $z \le y$; that is

$$y \text{ covers } x \Leftrightarrow \{ x \le y \wedge (x \le z \le y \Rightarrow (x = z) \vee (z = y))$$

where $\le$ is the irreflexive, antisymmetric & transitive.

Hasse Diagram (or) Partially ordered set diagram

A partial ordering $\leq$ on a set P can be represented by means of a diagram known as a "Hasse diagram" of $\langle P, \leq \rangle$.

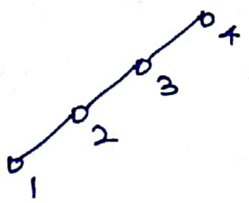
In such a diagram, each elt is represented by a small circle or a dot. The circle for $x \in P$ is drawn below the circle for $y \in P$, If $x < y$, and a line drawn between x and y if y covers x. If $x < y$ but y does not cover x, then x and y are not connected directly by a single line.

However they are connected through one or more elts of P. It is possible to obtain the set of ordered pairs in $\leq$ from such a diagram.

Examples of Hasse Diagram

① For a totally ordered set $\langle P, \leq \rangle$, the Hasse diagram consists of circles one below the other. (looks like a chain).

Let $P = \{1, 2, 3, 4\}$ and $\leq$ be the relation "less than or equal to" then the Hasse diagram is as shown below:



$$\left\{ \begin{array}{ll} * \ 1 < 2, & 2 \text{ covers } 1 \\ 2 < 3, & 3 \text{ covers } 2 \\ 3 < 4, & 4 \text{ covers } 3 \end{array} \right\}$$

Hasse Diagram.

* ✓ $\underset{x}{1} \leq \underset{y}{2}$ & $\nexists\, z \in P$ s.t $1 \leq z$ & $z \leq 2$

* ✓ $\underset{x}{1} \leq \underset{y}{3}$ & $\exists\, 2 \in P$ s.t $1 \leq 2$ & $2 \leq 3$

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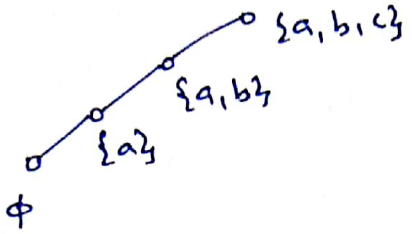
(2) Let $A = \{a, b, c\}$

let $P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\} \{a,c\} \{b,c\} \{a,b,c\} = A\}$

and the relation of inclusion $\subseteq$ on $P = \{\phi, \{a\}, \{a,b\}, \{a,b,c\}\}$.

Then the Hasse diagram is



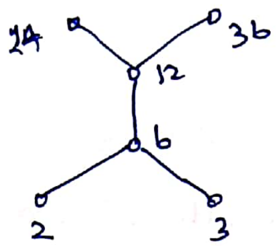
Hasse Diagram

* $\phi \subseteq \{a\}$ & $\{a\}$ covers ϕ
* $\{a\} \subseteq \{a,b\}$ & $\{a,b\}$ covers $\{a\}$.
* $\{a,b\} \subseteq A$ & A covers $\{a,b\}$

(3) Let $X = \{2, 3, 6, 12, 24, 36\}$ and the relation $\leq$ be s.t

$x \leq y$, if x divides y i.e., $x | y$.

The Hasse diagram is



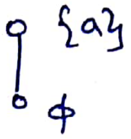
Hasse diagram of divides relation.

(4) (i) Let A be a given finite set and $P(A)$ its power set.

Let be the inclusion relation on the elts of $P(A)$.

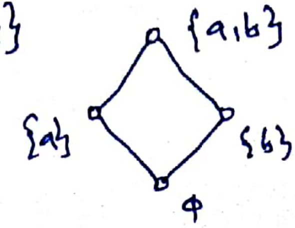
Hasse diagram of $\langle P(A), \leq \rangle$ for $A = \{a\}$.

Here $P(A) = \{\phi, \{a\}\}$ & $\leq$ is $\subseteq$.



(ii) Let $\langle P(A), \leq \rangle$ for $A = \{a, b\}$

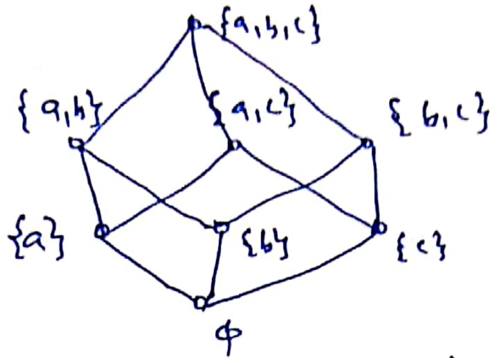
Here $P(A) = \{\phi\ \{a\}, \{b\}, \{a,b\}\}$



(iii) $\langle P(A), \leq \rangle$ for $A = \{a, b, c\}$

$P(A) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \}$

& $\leq$ is $\subseteq$.



(iv) Similarly try with $\langle P(A), \leq \rangle$ for $A = \{a, b, c, d\}$.

Least member in Poset

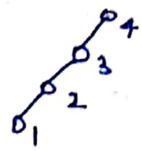
Let $\langle P, \leq \rangle$ be a poset. If there exists an elt $y \in P$ s.t $y \leq x$ for all $x \in P$, then y is called the least member in 'p' relative to the partial ordering $\leq$.

Greatest member in poset

If there exists an elt $y \in P$ s.t $x \leq y$ for all $x \in P$, then y is called the greatest member in P relative to the partial ordering $\leq$.

Note:- × the least & greatest member is unique, If it exists.

Example 1 For the poset $\langle P, \leq \rangle$, where $P = \{1, 2, 3, 4\}$ and the relation is " less than or equal to"

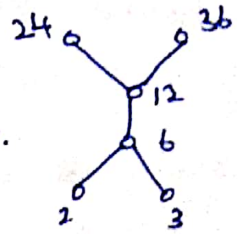


Here the least member is 1.

greatest " is 4.

2. Let $X = \{2, 3, 6, 12, 24, 36\}$ and the relation "$\leq$" is "divides". For the poset $\langle X, \leq \rangle$

There is no least & greatest member.



Upper Bound and Lower Bound of a Subset in Poset

Let $\langle P, \leq \rangle$ be a poset and let $A \subseteq P$. Any elt $x \in P$ is an upper bound for A, if for all $a \in A$, $a \leq x$.

Similarly, any elt $x \in P$ is an lower bound for A, if for all $a \in A$, $x \leq a$.

Note:- Upper and lower bounds of a subset of the poset are not necessarily unique.

Examples ① let $A = \{a, b, c\}$

Then $\langle P(A), \subseteq \rangle$ is a poset.

Consider a subset of $P(A)$, $B = \{\{b\}, \{c\}, \{b, c\}\}$.

Then $\{b, c\}$ and A are upper bounds for B and ϕ is the lower bound for B.

② Consider the poset $\langle X, \leq \rangle$, where $X = \{2, 3, 6, 12, 24, 36\}$ and $\leq$ is "divides". Let $A = \{2, 3, 6\} \subseteq X$.

Then $6, 12, 24$ & 36 are upper bounds for A and there is no lower bound for A.

Least upper Bound (LUB) or Supremum

Let $\langle P, \leq \rangle$ be a poset and let $A \subseteq P$. An elt $x \in P$ is a least upper bound for A, if 'x' is an upper bound for A and $x \leq y$, where y is any upper bound for A.

Greatest Lower Bound (GLB) or Infimum

Let $\langle P, \leq \rangle$ be a poset and let $A \subseteq P$. An elt $x \in P$ is a greatest lower bound for A, if 'x' is a lower bound for A and $y \leq x$, for all lower bounds y for A.

Example:- (i) $\langle P, \leq \rangle$ $P = \{1, 2, 3, 4\}$

"Less than or equal to"

every subset of P has a supremum and an infimum.

(Since $\langle P, \leq \rangle$ is a chain).

② For the poset $\langle X, \le \rangle$ where $X = \{2, 3, 6, 12, 24, 36\}$ and $\le$ is "divides".

The subset $A = \{2, 3, 6\} \subseteq X$ has a supremum '6' and the infimum of A does not exists.

The subset $B = \{6, 12\} \subseteq X$ has the supremum 12 and the infimum '6'.

Lattices as POSET

Definition A lattice is a partially ordered set (Poset) $\langle L, \le \rangle$ in which every pair of elts $a, b \in L$ has a greatest Lower bound (GLB) and a least upperbound (LUB).

Notation:-

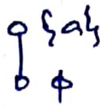
* GLB $(\{a, b\})$ will be denoted by $a * b$, ie, is called the meet or Product of a & b.

* LUB$(\{a, b\})$ will be denoted by $a \oplus b$, ie, called the join or sum of a and b.

* Sometimes $\wedge$ and $\vee$ (or) $\cdot$ and $+$ are also used to denote the meet or join of two elts.

* By the defn of Lattice $(L, \le)$ both $*$ and $\oplus$ are binary operations on L because of the uniqueness of LUB and GLB of any subset of a poset.

Examples ① Let S be any set and $P(S)$ be its power set. The poset $\langle P(S), \subseteq \rangle$ is a lattice in which the meet and join are the same as the operations $\cap$ and $\cup$ respectively.

(i) If $S = \{a\}$, $P(S) = \{\phi, \{a\}\}$ 

Hasse Diagram.

(ii) $S = \{a, b\}$, $P(S) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$.

2. Let I_+ be the set of all +ve integers and let D denotes the relation of "division" in I_+ s.t for any $a, b \in I_+$, $a D b \Leftrightarrow$ a divides b.

Then $\langle I_+, D \rangle$ is a lattice in which the join of a and b is given by the least common multiple (LCM) of a and b, $a \oplus b = LCM(a,b)$ and the meet of a & b is the greatest common divisor (GCD) of a and b, that is $a * b = GCD(a,b)$

③ Let 'n' is a +ve integer and S_n be the set of all divisors of n. For example, $n=6$, $S_6 = \{1,2,3,6\}$

$n=24$, $S_{24} = \{1,2,3,4,6,8,12,24\}$.

Let D denote the relation of "division".

ie, $a D b \Leftrightarrow$ "a divides b".

Then $\langle S_6, D \rangle$, $\langle S_{24}, D \rangle$, $\langle S_8, D \rangle$ & $\langle S_{30}, D \rangle$ are lattices.

Some Properties of Lattices

Some of the properties of the two binary operations of meet $(*)$ and join $(\oplus)$ on a lattice $(L, \leq)$ are follows.

For any, $a, b, c \in L$ we have

* Idempotent property

(L1) $a * a = a$ (L-1)' $a \oplus = a$

* Commutative property

(L2) $a * b = b * a$ (L2)' $a \oplus b = b \oplus a$

* Associative property

(L3) $(a * b) * c = a * (b * c)$ (L3)' $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

* Absorption property

(L4) $a * (a \oplus b) = a$ (L4)' $a \oplus (a * b) = a$.

Theorem ①

Let $\langle L, \leq \rangle$ be a lattice in which $*$ and $\oplus$ denote the operations of meet and join respectively.

For any $a, b \in L$, $\underbrace{a \leq b}_{①} \Leftrightarrow \underbrace{a * b = a}_{②} \Leftrightarrow \underbrace{a \oplus b = b}_{③}$.

Proof:- ①⇔② ⇒

Assume that $a \leq b$.

To prove $a * b = a$

By the defn of $\leq$, $a \leq a$.

$$a \leq b \ \& \ a \leq a \Rightarrow a \leq GLB\{a, b\} = a * b$$

$$\Rightarrow a \leq a * b \quad \text{—(I)}$$

By the defn of $*$ (meet) we have

$$a * b = GLB\{a, b\} \leq a$$

$$\Rightarrow a * b \leq a \quad \text{—(II)}$$

From (I) & (II) $\Rightarrow$ $a = a * b$

①⇔② ⇐ Assume that $a * b = a$ To prove $a \leq b$.

$$a * b = GLB\{a, b\} = a$$

By the defn of GLB, we have $a \leq a$ & $a \leq b$.

$$\Rightarrow a \leq b$$

②⇔③ ⇒ Assume that $a * b = a$.

To prove $a \oplus b = b$.

$$b \oplus (a * b) = b \oplus a = a \oplus b \text{ — (I)} \text{ (By assumption)}$$

Also $b \oplus (a * b) = b$ — (II) (By Absorbtion)

from (I) & II we have $b = a \oplus b$.

⇐ Assume that $a \oplus b = b$. To prove $a * b = a$.

$$\underset{\text{(Assumption)}}{a * (a \oplus b) = a * b}, \text{ Also } \underset{\text{(Absorbtion)}}{a * (a \oplus b) = a}$$

$\therefore$ $a * b = a$

$\therefore$ we have $a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b$.

Theorem ② Isotonicity properties in Lattices

Let $\langle L, \leq \rangle$ be a lattice. For any $a, b, c \in L$ the following properties called Isotonicity hold.

i.e., $b \leq c \Rightarrow \begin{cases} a * b \leq a * c \\ a \oplus b \leq a \oplus c. \end{cases}$

Proof:-

By theorem ①, $b \leq c \Leftrightarrow b * c = b$

To prove $a * b \leq a * c$.

By Thm ①, It is enough to show that

$$(a*b) * (a*c) = (a*b)$$

$$\begin{aligned}(a*b) * (a*c) &= a * (b*a) * c \\ &= a * (a*b) * c \\ &= (a*a) * (b*c) \\ &= a * (b*c) = a * b\end{aligned}$$

lll'y $a \oplus b \leq a \oplus c$.

Some implications in Lattice $\langle L, \leq \rangle$

$$(a \leq b) \wedge (a \leq c) \Rightarrow a \leq b \oplus c$$

$$(a \leq b) \wedge (a \leq c) \Rightarrow a \leq a * c$$

Duals of (1) & (2) are

$$(a \geq b) \wedge (a \geq c) \Rightarrow a \geq b * c$$

$$(a \geq b) \wedge (a \geq c) \Rightarrow a \geq b \oplus c$$

Theorem: ③ Distributive Inequalities in Lattices

Let $\langle L, \leq \rangle$ be a lattice. For any $a, b, c \in L$, the following inequalities, called the distributive inequalities hold:

$$a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$$

$$a * (b \oplus c) \geq (a * b) \oplus (a * c)$$

Proof:- $a, b, c \in L$

$a \oplus b$ is the LUB of $\{a, b\}$

$\Rightarrow a \leq a \oplus b$ & $a \leq a \oplus c$

$\Rightarrow a \leq (a \oplus b) * (a \oplus c)$

ie., $(a \oplus b) * (a \oplus c) \geq a$ —— ①

$(b * c) \leq b \leq (a \oplus b)$

$(b * c) \leq c \leq (a \oplus c)$

$\Rightarrow (b * c) \leq (a \oplus b) * (a \oplus c)$

$\Rightarrow (a \oplus b) * (a \oplus c) \geq (b * c)$ —— ②

$\Rightarrow (a \oplus b) * (a \oplus c) \geq a \oplus (b * c)$

$\Rightarrow a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$

Hence proved.

Theorem : 4 Modular inequality

Let $\langle L, \leq \rangle$ be a lattice. For any $a, b, c \in L$ the following is hold

$$a \leq c \Leftrightarrow a \oplus (b * c) \leq (a \oplus b) * c$$

Proof:- Assume $a \leq c$ and to prove $a \oplus (b * c) \leq (a \oplus b) * c$.

($\Rightarrow$) Since $a \leq c \Leftrightarrow a \oplus c = c$ (by thm ①),

we get the required result by substituting c for $(a \oplus c)$ in the first distributive inequality [By thm ③]

$a \oplus (b * c) \leq (a \oplus b) * (a \oplus c) = (a \oplus b) * c$

$\Rightarrow a \oplus (b * c) \leq (a \oplus b) * c$.

($\Leftarrow$) Assume $a \oplus (b * c) \leq (a \oplus b) * c$. To prove $a \leq c$.

By defn of LUB, $a \leq a \oplus (b * c)$

$\leq (a \oplus b) * c$ by assumption

$\leq c$ by defn of GLB.

$\Rightarrow a \leq c$

Hence proved.