Discrete Mathematics and Graph Theory

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Module-1 Mathematical logic and statement calculus.

Statement (or) Proposition

A proposition (or) statement is a declarative sentence that is either true or false, but not both.

Example:-

(1) Rose is a beautiful flower (T)

(2) $5+5=12$ (F)

Note:- The truth values True and False are denoted by the symbol's T and F respectively. Some times it is also denoted by 1 and 0, where 1 stands for true and 0 stands for false.

Types of statements

(1) Simple (2) Compound

(1) Simple (or) Atomic (or) Primary (or) primitive statement

The statements which do not contain any of the connectives are called Atomic statements

Ex: i) 3 is a prime number (T)
ii) Canada is a country (T)

(2) Compound statement

New statements can be formed from atomic statements through the use of sentential connectives.

The resulting statements are called compound statements.

Ex:- If p is a prime number, then the divisor of 'p' are '1' and 'p' itself.

Truth Table

A table showing all possible truth values of a compound statement is called the truth table.

Logical Connectives

* Negation ($\neg$ or $\sim$) Not
* Conjunction ($\wedge$ and)
* Disjunction ($\vee$ or)
* Conditional (Implication $\rightarrow$) if
* Bi-conditional ($\leftrightarrow$ iff)

* Negation ($\neg$ or $\sim$) Not

If P is a proposition, then not P is also a proposition.

Ex: If P : London is a city.

then $\neg P$ or $\sim P$: London is not a city.

Rule:- If P is true, then $\neg P$ is false and if P is false then $\neg P$ is true.

Truth Table

P	$\neg P$
T	F
F	T

★ Conjunction ($\wedge$ and)

The conjunction of two statements P and Q is also a statement denoted by $P \wedge Q$. We use the connective And for conjunction.

Eg: P : $2+3=5$

Q : 5 is a composite number.

So, $P \wedge Q$: $2+3=5$ and 5 is a composite number.

Rule:- $(P \wedge Q)$ is true if both P and Q are true, otherwise false.

Truth Table

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

★ Disjunction ($\vee$ or)

The disjunction of two statements P and Q, is also a statement denoted by $P \vee Q$. We use the connective or for disjunction.

Rule:- $(P \vee Q)$ is true, if either P or Q, is true and it is false when both P and Q are false.

Truth Table

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

* Conditional statement ($\rightarrow$)

"If P, then Q" is called a conditional statement.

Rule:- The statement $P \rightarrow Q$ has a truth value F when Q has the truth value F and P has the truth value T; otherwise it has the truth value T.

Truth Table

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

* Bi-conditional ($\leftrightarrow$)

"P if and only if Q" i.e., "P iff Q" is called a bi-conditional statement and is defined as

$$P \leftrightarrow Q : (P \rightarrow Q) \wedge (Q \rightarrow P)$$

Truth Table

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \wedge (Q \rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Construct the truth table

(i) $P \vee \neg Q$

P	Q	$\neg Q$	$P \vee \neg Q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

$P \vee \neg P$

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

(ii) $P \wedge \neg P$

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

(iii) $(P \vee Q) \vee \neg P$

P	Q	$P \vee Q$	$\neg P$	$(P \vee Q) \vee \neg P$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

(iv) $\neg(\neg P \vee \neg Q)$

(v) $\neg(\neg P \wedge \neg Q)$

(vi) $P \rightarrow (P \vee Q)$

(vii) $P \wedge Q \rightarrow Q$

(viii) $P \wedge (P \vee Q)$

(ix) $P \wedge (P \wedge Q)$

(x) $(P \rightarrow Q) \wedge (Q \rightarrow P)$

(xi) $\neg(P \wedge Q) \rightleftarrows (\neg P \vee \neg Q)$

Tautologies

* The proposition is said to be tautology if its truth value is T for any assignment of truth value to its components.

A statement formula which is always true whatever may be the truth values of its components, is called a tautology or a universally valid formula.

Examples

(1) $(P \wedge q) \rightarrow P$

(2) $q \rightarrow (P \vee q)$

(3) $(P \vee q) \longleftrightarrow (q \vee P)$

P	Q	$P \wedge Q$	$P \wedge Q \rightarrow P$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Contradiction

A statement that is always false is called a contraction.

Examples (1) $P \wedge \neg P$ (2) $(P \vee Q) \wedge (\neg P \wedge \neg Q)$

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

Contingency

A statement formula that is neither tautology nor contradiction is called contingency.

A statement formula that can be either true or false ie, neither a tautology nor a contradiction, is called a contingency.

Example (1) $P \Leftrightarrow q$ (2) $(P \vee \neg Q) \longrightarrow P \wedge Q$

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$P \leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Logical Equivalence

Two statement formula's are said to be logically equivalent if their truth columns are identical.

Such statements are represented by $P \equiv Q$ (or) $P \Leftrightarrow Q$.

Note:- Equivalence is transitive. Because if $A \Leftrightarrow B$ and $B \Leftrightarrow C$, then $A \Leftrightarrow C$.

Example

1. P.T $(P \to q) \Leftrightarrow (\neg P \vee q)$

P	q	$P \to q$	$\neg P$	$\neg P \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

H.W 2. $S: P \rightleftarrows Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$

P	Q	$P \to Q$	$Q \to P$	$P \rightleftarrows Q$	$(P \to Q) \wedge (Q \to P)$	S
T	T	T	T	T	T	T
T	F	F	T	F	F	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

A list of equivalent formulae

1. Idempotent laws —

$P \vee P \Leftrightarrow P$ & $P \wedge P \Leftrightarrow P$

2. Associative laws

$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ &

$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$

3. Distributive laws

$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ &

$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

4. Commutative laws

$P \vee Q \Leftrightarrow Q \vee P$ &

$P \wedge Q \Leftrightarrow Q \wedge P$

5. Identity laws

$P \vee F \Leftrightarrow P$, $P \wedge T \Leftrightarrow P$

6. Dominant laws

$P \vee T \Leftrightarrow T$, $P \wedge F \Leftrightarrow F$

7. Negation laws

$P \vee \neg P \Leftrightarrow T$, $P \wedge \neg P \Leftrightarrow F$

8. Absorption laws

$P \vee (P \wedge Q) \Leftrightarrow P$ &

$P \wedge (P \vee Q) \Leftrightarrow P$

9. De Morgan's Laws

$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ &

$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

Example

1. Without using truth table, show that

$Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.

Solution:-

$Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$

$\Leftrightarrow ((Q \vee P) \wedge (Q \vee \neg Q)) \vee (\neg P \wedge \neg Q)$ Distributive law

$\Leftrightarrow (Q \vee P \wedge T) \vee (\neg P \wedge \neg Q)$ Negation law

$\Leftrightarrow (Q \vee P) \vee (\neg P \wedge \neg Q)$

$\Leftrightarrow (Q \vee P) \vee \neg (P \vee Q)$ De Margon's law

$\Leftrightarrow (P \vee Q) \vee \neg (P \vee Q)$ commutative law

$\Leftrightarrow T$ tautology.

2. S.T $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$

Soln:-

$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$

$\Leftrightarrow (\neg P \wedge (\neg Q \wedge R)) \vee [(Q \vee P) \wedge R]$ by distributive laws.

$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R)$ by associative law

$\Leftrightarrow ((\neg P \wedge \neg Q) \vee (Q \vee P)) \wedge R$ by distributive laws

$\Leftrightarrow (\neg (P \vee Q) \vee (P \vee Q)) \wedge R$ by De Margon's Laws & Commutative laws

$\Leftrightarrow T \wedge R$ Negation laws

$\Leftrightarrow R$ Identity laws

H.W

1. $((P \vee Q) \wedge \neg (\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is a tautology

2. $P \rightleftarrows Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$

3. $(P \wedge Q) \to (P \vee Q)$ is a tautology.

Tautology Implications

A statement formula A is said to tautologically imply another statement formula B iff $A \to B$ is a tautology. In symbol, it is denoted by $A \Rightarrow B$.

Note:- $A \Rightarrow B$ means that $A \to B$ is a tautology.

Some Tautology implications

(1) $P \wedge Q \Rightarrow P$

(2) $P \wedge Q \Rightarrow Q$

(3) $P \Rightarrow P \vee Q$

(4) $\neg P \Rightarrow P \to Q$

(5) $Q \Rightarrow P \to Q$

(6) $\neg(P \to Q) \Rightarrow P$

(7) $\neg(P \to Q) \Rightarrow \neg Q$

(8) $P \wedge (P \to Q) \Rightarrow Q$

(9) $\neg Q \wedge (P \to Q) \Rightarrow \neg P$

(10) $\neg P \wedge (P \vee Q) \Rightarrow Q$

Note

(1) Connectives $\wedge$, $\vee$ and $\rightleftarrows$ are symmetric

Since,
$P \wedge Q \Leftrightarrow Q \wedge P$
$P \vee Q \Leftrightarrow Q \vee P$
$P \rightleftarrows Q \Leftrightarrow Q \rightleftarrows P$

But $\to$ is need not be symmetric.

i.e., $P \to Q$ is not equivalent to $Q \to P$

i.e; $P \to Q \not\Leftrightarrow Q \to P$.

(2) Converse For any statement formula, $P \to Q$ then the statement formula $Q \to P$ is called its converse.

(3) Inverse For $P \to Q$, $\neg P \to \neg Q$ is called its inverse.

(4) Contrapositive For $P \to Q$, $\neg Q \to \neg P$ is called its contrapositive.

~~Duality law~~

Functionally Complete sets of Connectives

Any set of connectives in which every formula can be expressed in terms of an equivalent formula containing the connectives from this set is called a functionally complete set of connectives.

Example

(1) $\{\wedge, \neg\}$ & $\{\vee, \neg\}$ are functionally complete.

(2) $\{\neg\}$, $\{\vee, \wedge\}$ are not functionally complete.

(3) Write an equivalent formula for $P \wedge (Q \rightleftarrows R) \vee (R \rightleftarrows P)$ which does not contain the biconditional.

Soln:- Given $P \wedge (Q \rightleftarrows R) \vee (R \rightleftarrows P)$

$$\Leftrightarrow P \wedge ((Q \to R) \wedge (R \to Q)) \vee ((R \to P) \wedge (P \to R))$$

Thus the resultant equivalent formula is free from the biconditional connective.

Other Connectives

Exclusive OR

Let P and Q be any two formulas. Then the formula $P \bar{\vee} Q$ is true whenever either P or Q, but not both is true. It is also called exclusive disjunction.

Truth table

P	Q	$P \bar{\vee} Q$
T	T	F
T	F	T
F	T	T
F	F	F

* $\bar{\vee}$ Satisfies the following equivalences

(1) $P \bar{\vee} Q \Leftrightarrow Q \bar{\vee} P$ (Symmetric)

(2) $(P \bar{\vee} Q) \bar{\vee} R \Leftrightarrow P \bar{\vee} (Q \bar{\vee} R)$ (associative)

(3) $P \wedge (Q \bar{\vee} R) \Leftrightarrow (P \wedge Q) \bar{\vee} (P \wedge R)$ (distributive)

(4) $(P \bar{\vee} Q) \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q)$

(5) $(P \bar{\vee} Q) \Leftrightarrow \neg (P \rightleftarrows Q)$.

<u>NAND</u> $(\uparrow)$

Let P and Q be any two formula. Then the connective NAND, denoted by $\uparrow$, is defined as

$$P \uparrow Q \Leftrightarrow \neg (P \wedge Q)$$

* The word NAND is the combination of NOT and AND.

<u>NOR</u> $(\downarrow)$

NOR is denoted and defined by

$$P \downarrow Q \Leftrightarrow \neg (P \vee Q).$$

Combination of NOT and OR.

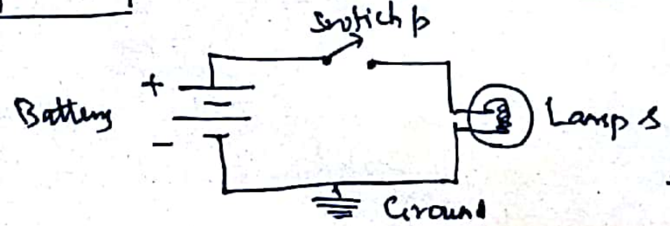
<u>Two-state Devices and Statement logic</u>

Let P : The switch p is closed

S : The Lamp s is on.

$p(P)$	$s(S)$
1	1
0	0

State of Switch p	State of light s
closed	on
open	off



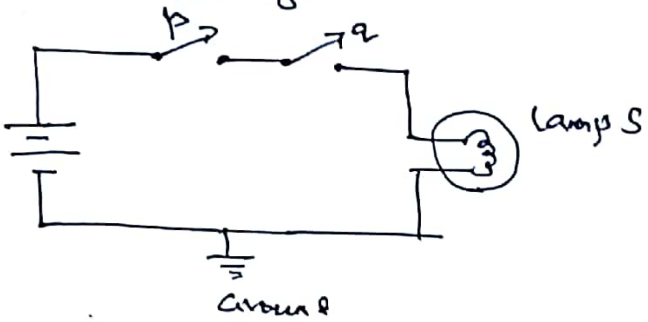
<u>A switch as a two-state device</u>

AND Gate

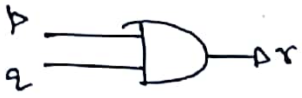
Let p : the switch p is closed

Q : The switch q is closed

S : The Light is on.



P	Q	$S = P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0



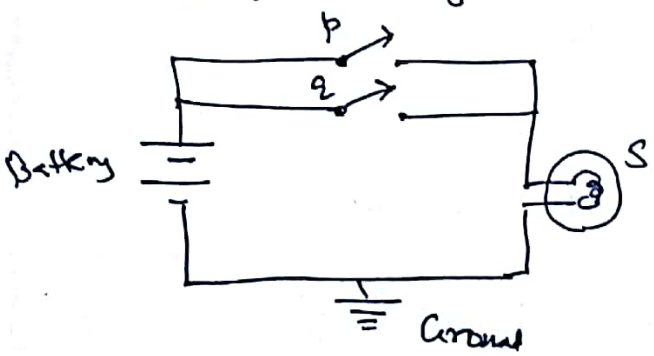
→ A two-state device for AND Logic.

OR GATE

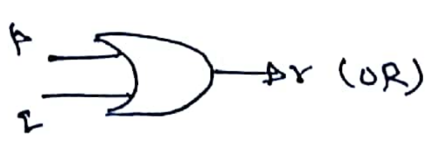
Let P : The switch 'p' is closed

Q : The switch 'q' is closed

S : The light is on



P	Q	$S = P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0



A two state device for OR Logic.

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Normal forms

- ★ Disjunctive Normal forms (DNF)
- ★ Conjunctive Normal forms (CNF)
- ★ Principal Disjunctive Normal forms
- ★ Principal conjunctive Normal form

Disjunctive Normal forms

We will use the word product in the place of conjunction and sum in the place of disjunction.

Elementary product

Let P and Q be any two atomic variables. Then

P, $\neg P \wedge Q$, $P \wedge \neg P$, $Q \wedge \neg P$ are some of the elementary product.

Elementary sum

P, $\neg P \vee Q$, $\neg Q \vee P$, $P \vee \neg Q$, $Q \vee \neg P$ are some of the elementary sums.

DNF A formula which is equivalent to a given formula and consists of a sum of elementary products is called a DNF of the given formula.

Procedure to obtain DNF

1) If the connectives $\rightarrow$ and $\leftrightarrow$ appear in the given formula, an equivalent formula can be obtained by replacing $\rightarrow$ and $\leftrightarrow$ by $\wedge$, $\vee$ and $\neg$.

2) Using De-Morgan's law, apply negations to variables.

Example

1. Obtain disjunctive normal form of

$P \wedge (P \to Q)$ $\qquad P \to Q \Leftrightarrow \neg P \vee Q$

Soln $\quad P \wedge (P \to Q) \Leftrightarrow P \wedge (\neg P \vee Q)$

$\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$

as $(P \to Q \Leftrightarrow \neg P \vee Q)$.

$\therefore$ It is DNF.

2. $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

$\Leftrightarrow (\neg(P \vee Q) \wedge P \wedge Q) \vee ((P \vee Q) \wedge \neg(P \wedge Q))$

$A \rightleftarrows B \Leftrightarrow (A \wedge B) \vee (\neg A \wedge \neg B)$

$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge (P \wedge Q)) \vee ((P \vee Q \wedge (\neg P \vee \neg Q))$

$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge \neg P) \vee ((P \vee Q) \wedge \neg Q)$

$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee (P \wedge \neg P) \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)$.

This is DNF.

Conjunctive Normal Forms (CNF)

A formula which is equivalent to a given formula consisting of a product of elementary sum is called a CNF.

Example (1) Obtain CNF of $P \wedge (P \to Q)$

$P \wedge (P \to Q) \Leftrightarrow P \wedge (\neg P \vee Q)$ which is CNF.

(2) $\neg(P \vee Q) \longleftrightarrow (P \wedge Q)$

$A \leftrightarrow B \Leftrightarrow (A \to B) \wedge (B \to A)$

$P \to Q \Leftrightarrow \neg P \vee Q$

$\Leftrightarrow (\neg(P \vee Q) \to (P \wedge Q)) \wedge ((P \wedge Q) \to \neg(P \vee Q))$

$\Leftrightarrow ((P \vee Q) \vee (P \wedge Q)) \wedge (\neg(P \wedge Q) \vee \neg(P \vee Q))$

$\Leftrightarrow (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge ((\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q))$

$\Leftrightarrow (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge (\neg P \vee \neg Q \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg Q)$ //.

Principal Disjunctive Normal forms

Minterms of P and Q are $P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$ and $\neg P \wedge \neg Q$. (NO commutative allowed)

2^n elements / 2^2 elts for P&Q.

Here $P \wedge Q$ or $Q \wedge P$ is included, but not both.

PDNF

For a given formula, an equivalent formula consisting of disjunctions of minterms only is known as its principal disjunctive NF or sum products canonical form.

Example Obtain the PDNF of (a) $P \to Q$ (b) $P \vee Q$

Soln:- The truth table of minterms are

P	Q	$P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

— (1)

and

P	Q	$P \to Q$	$P \vee Q$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	F

— (2)

(a) $P \to Q$ has 3 ~~true~~ rows with T.

∴ from (1) we have to choose 3 formulas from the 4 columns 3, 4, 5 & 6.

Thus $P \to Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

(b) $P \vee Q \Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$ //.

Prinicpal Conjuctive Normal form (PCNF)

Maxterms of P and Q are $P\vee Q,\ \neg P\vee Q,\ P\vee\neg Q,\ \neg P\vee\neg Q$.

PCNF For a given formula an equivalent formula consisting of conjunction of the maxterms only is known as its PCNF or product of sum of canonical form.

Example:- Obtain the PCNF of the formula S given by

$(\neg P \leftrightarrow R)\wedge(Q \rightleftarrows P)$.

Soln:- $(\neg P\to R)\wedge(Q\rightleftarrows P)$

$\Leftrightarrow (P\vee R)\wedge((Q\to P)\wedge(P\to Q))$ $\quad (P\rightleftarrows Q)\Leftrightarrow(P\to Q)\wedge(Q\to P)$

$P\to Q\Leftrightarrow\neg P\vee Q$

$\Leftrightarrow (P\vee R)\wedge(\neg Q\vee P)\wedge(\neg P\vee Q)$

$\Leftrightarrow \cancel{P\vee R\vee\neg Q}\ (P\vee R\vee(Q\wedge\neg Q))\wedge((\neg Q\vee P)\vee(R\wedge\neg R))$

$\wedge(\neg P\vee Q\vee(R\wedge\neg R))$ $\quad (Q\wedge\neg Q\Leftrightarrow F$

$P\vee F\Leftrightarrow P)$

$\Leftrightarrow (P\vee Q\vee R)\wedge(P\vee\neg Q\vee R)\wedge(P\vee\neg Q\vee\neg R)$

$\wedge(\neg P\vee Q\vee R)\wedge(\neg P\vee Q\vee\neg R)$.

Which is the product of sums of maxterms of 3 Variables P, Q & R. This is PCNF.

Home Works

(1) Obtain the PDNF of $P\to((P\to Q)\wedge\neg(\neg Q\vee\neg P))$

(2) " $(P\wedge Q)\vee(\neg P\wedge R)\vee(Q\wedge R)$

(3) " PCNF of $(P\wedge Q)\vee(\neg P\wedge Q\wedge R)$

(4) Obtain the DNF & CNF for

$(P\to(Q\wedge R))\wedge(\neg P\to(\neg Q\wedge\neg R))$

(5) PCNF & PDNF for

$S\Leftrightarrow(P\to(Q\wedge R))\wedge(\neg P\to(\neg Q\wedge\neg R))$.

Note:- Minterms of 3 Variables P, Q and R are

$P\wedge Q\wedge R \qquad P\wedge Q\wedge \neg R \qquad P\wedge \neg Q\wedge R \qquad \neg P\wedge Q\wedge R$

$\neg P\wedge \neg Q\wedge \neg R \qquad P\wedge \neg Q\wedge \neg R \qquad \neg P\wedge \neg Q\wedge R \qquad \neg P\wedge Q\wedge \neg R.$

Problems

① obtain the PDNF of (a) $\neg P\vee Q$

(b) $(P\wedge Q)\vee(\neg P\wedge R)\vee(Q\wedge R)$

Soln:-

(1) $\neg P\vee Q \Leftrightarrow (\neg P\wedge(Q\vee\neg Q))\vee(Q\wedge(P\vee\neg P))$

$(\because\ Q\vee\neg Q\Leftrightarrow T \ \&\ P\wedge T\Leftrightarrow P)$

$\Leftrightarrow (\neg P\wedge Q)\vee(\neg P\wedge\neg Q)$

$\vee(Q\wedge P)\vee(Q\wedge\neg P)$

(Distributive law)

$\Leftrightarrow (\neg P\wedge Q)\vee(\neg P\wedge\neg Q)\vee(P\wedge Q)$

By commutative & $P\vee P\Leftrightarrow P$

② $(P\wedge Q)\vee(\neg P\wedge R)\vee(Q\wedge R)$

$\Leftrightarrow (P\wedge Q\wedge(R\vee\neg R))\vee(\neg P\wedge R\wedge(Q\vee\neg Q))$

$\vee(Q\wedge R\wedge(P\vee\neg P))$

$\Leftrightarrow (P\wedge Q\wedge R)\vee(P\wedge Q\wedge\neg R)\vee(\neg P\wedge R\wedge Q)\vee(\neg P\wedge R\wedge\neg Q)$

$\vee(Q\wedge R\wedge P)\vee(Q\wedge R\wedge\neg P).$

$\Leftrightarrow (P\wedge Q\wedge R)\vee(P\wedge Q\wedge\neg R)\vee(\neg P\wedge R\wedge Q)\vee$

$(\neg P\wedge R\wedge\neg Q).$

③ PDNF of $P \rightarrow ((P\rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$

$P\rightarrow Q \Leftrightarrow \neg P \vee Q$

$P \rightarrow ((P\rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$

$\Leftrightarrow \neg P \vee ((P\rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$

$\Leftrightarrow \neg P \vee ((\neg P \vee Q) \wedge (Q \wedge P))$ De Morgan's law

$\Leftrightarrow \neg P \vee (\neg P \wedge Q \wedge P) \vee (Q \wedge (Q \wedge P))$

$\Leftrightarrow \neg P \vee (\neg P \wedge Q \wedge P) \vee (Q \wedge P) \Leftrightarrow \neg P \vee (\neg P \vee (Q \wedge P))$ Associative law

$\Leftrightarrow \neg P \vee (Q \wedge P)$

$\Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge P)$ $(Q \vee \neg Q \Leftrightarrow T$, $P \wedge T \Leftrightarrow P)$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$ Commutative property.

④ PCNF of $S \Leftrightarrow (P \rightarrow (Q \wedge R)) \wedge (\neg P \rightarrow (\neg Q \wedge \neg R))$

$\Leftrightarrow (\neg P \vee (Q \wedge R)) \wedge (P \vee (\neg Q \wedge \neg R))$ as $P\rightarrow Q \Leftrightarrow \neg P \vee Q$

$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee R) \wedge (P \vee \neg Q) \wedge (P \vee \neg R)$

$\Leftrightarrow ((\neg P \vee Q) \vee (R \wedge \neg R)) \wedge ((\neg P \vee R) \vee (Q \wedge \neg Q))$
$\wedge ((P \vee \neg Q) \vee (R \wedge \neg R)) \wedge ((P \vee \neg R) \vee (Q \wedge \neg Q))$

$\Leftrightarrow (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee R \vee Q) \wedge$
$(\neg P \vee R \wedge \neg Q) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R)$
$\wedge (P \vee \neg R \vee Q) \wedge (P \vee \neg R \vee \neg Q)$

$\Leftrightarrow (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R)$
$\wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (P \vee Q \vee \neg R)$

H.W PCNF & PDNF of

① $P \vee (\neg P \rightarrow (Q \vee (\neg Q \rightarrow R)))$

PCNF ② $(\neg P \rightarrow R) \wedge (Q \rightleftarrows P)$

PDNF ③ $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

The theory of Inference for statement calculus

Premises → Assumptions, axioms, hypothesis

Conclusion → consequence

Validity using Truth table

Let A and B be two statement formulas.

We say that "B logically follows from A" or "B is a valid conclusion of the premise A" iff $A \to B$ is a tautology. i.e., $A \Rightarrow B$.

Similarly, we say that from a set of premises $\{H_1, H_2, \ldots, H_m\}$, a conclusion C follows logically iff

$$H_1 \wedge H_2 \wedge \ldots \wedge H_m \Rightarrow C.$$

i.e., the argument is valid, if C is true whenever $H_1, H_2, \ldots, H_m$ are all true; otherwise it is invalid.

Example:- Determine whether the conclusion C follows logically from the premises H_1 and H_2.

(a) $H_1 : P \to Q$ $\quad H_2 : P$ $\quad C : Q$

(b) $H_1 : P \to Q$ $\quad H_2 : \neg P$ $\quad C : Q$

(c) $H_1 : P \to Q$ $\quad H_2 : \neg(P \wedge Q)$ $\quad C : \neg P$

(d) $H_1 : \neg P$ $\quad H_2 : P \rightleftarrows Q$ $\quad C : \neg(P \wedge Q)$

(e) $H_1 : P \to Q$ $\quad H_2 : Q$ $\quad C : P$

Soln:-

Truth table

P	Q	$P \to Q$	$\neg P$	$\neg Q$	$\neg(P \wedge Q)$	$P \rightleftarrows Q$
T	T	T	F	F	F	T
T	F	F	F	T	T	F
F	T	T	T	F	T	F
F	F	T	T	T	T	T

(a) $H_1: P \to Q$ $H_2: P$: $C: Q$

$H_1 \wedge H_2 \Rightarrow C$

$((P \to Q) \wedge P) \Rightarrow C$

$P \to Q \wedge P$	$((P \to Q) \wedge P) \to Q$
T	T
F	T
F	T
F	T

Tautology

Hence H_1 & H_2 implies C Logically.

(or)

The argument $H_1 \wedge H_2 \Rightarrow C$ is Valid.

(b) $H_1: P \to Q$ $H_2: Q$ $C: P$

$P \to Q \wedge Q$	$(P \to Q) \wedge Q \to P$
T	T
F	T
~~F~~	F
F	T

is not tautology.

$\therefore$ The conclusion 'C' is not valid.

(c) C is Not Valid

(d) C is Valid

(e) C is Valid.

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(5) Rules of inference

Rule P : A premise may be introduced at any point in the derivation

Rule T : A formula S may be introduced in a derivation if S is tautologically implied by any one or more of the preceeding formulas in the derivation.

Rule CP : (Deduction theorem) If we can derive S from R and a set of premises, then we can derive $R \to S$ from the set of premises alone.

Since if $(H_1 \wedge H_2 \wedge \dots \wedge H_m \wedge R) \Rightarrow S$,

then $(H_1 \wedge H_2 \wedge \dots \wedge H_m \wedge R) \Rightarrow R \to S$.

Implications

I_1 : $P \wedge Q \Rightarrow P$ } Simplification

I_2 : $P \wedge Q \Rightarrow Q$

I_3 : $P \Rightarrow P \vee Q$ } Addition

I_4 : $Q \Rightarrow P \vee Q$

I_5 : $\neg P \Rightarrow P \to Q$

I_6 : $Q \Rightarrow P \to Q$

I_7 : $\neg (P \to Q) \Rightarrow P$

I_8 : $\neg (P \to Q) \Rightarrow \neg Q$

I_9 : $P, Q \Rightarrow P \wedge Q$

I_{10} : $\neg P, P \vee Q \Rightarrow Q$

I_{11} : $P, P \to Q \Rightarrow Q$

(disjunctive syllogism)

I_{12} : $\neg Q, P \to Q \Rightarrow \neg P$

I_{11}, I_{12} - Modus ponens

I_{13} : $P \to Q, Q \to R \Rightarrow P \to R$

(hypothetical)

I_{14} : $P \vee Q, P \to R, Q \to R \Rightarrow R$

(dilemma)

Example: 1 Demonstrate that R is a valid inference from the premises $P \to Q$, $Q \to R$ and P.

Soln:-

Derivation

Step 1 :	$P \to Q$	Rule P	
2 :	P	Rule P	
3 :	Q	Rule T, and I_{11}	$\to P, P \to Q \Rightarrow Q$
4 :	$Q \to R$	Rule P	
5 :	R	Rule T & I_{11}.	

Hence R is a valid inference implied from the premises.

2. S.T $\neg Q$ and $P \to Q$ implies $\neg P$.

Step: 1 :	$P \to Q$	Rule P	
2 :	$\neg Q \to \neg P$	Rule T	$P \to Q \Leftrightarrow \neg Q \to \neg P$
3 :	$\neg Q$	Rule P	
4 :	$\neg P$	Rule T & I_{11}.	

3. S.T $S \vee R$ is tautologically implied by $P \vee Q$, $P \to R$, $Q \to S$.

Step: 1 :	$P \vee Q$	Rule P	
2 :	$\neg P \to Q$	Rule T	$P \to Q \Leftrightarrow \neg P \vee Q$ & $\neg\neg P \Leftrightarrow P$ $\neg P \to Q \Leftrightarrow \neg\neg P \vee Q$ $\Leftrightarrow P \vee Q$
3 :	$Q \to S$	Rule P	
4 :	$\neg P \to S$	Rule T	$(P \to Q, Q \to R \Leftrightarrow P \to R)$
5 :	$\neg S \to P$	Rule T	$(P \to Q \Leftrightarrow \neg Q \to \neg P$ & $\neg\neg P \Leftrightarrow P)$
6 :	$P \to R$	Rule P	
7 :	$\neg S \to R$	Rule T	$(P \to Q, Q \to R \Rightarrow P \to R)$
8 :	$S \vee R$	Rule T	$(P \to Q \Leftrightarrow \neg P \vee Q$ & $\neg\neg P \Leftrightarrow P)$

(4) S.T $R \to S$ can be derived from the Premises $P \to (Q \to S)$, $\neg R \vee P$ and Q.

Soln:- Instead of deriving $R \to S$, we shall include R as an additional premise and show S first.

step : 1 :	$\neg R \vee P$	Rule P
2 :	R	Rule P and additional premise
3 :	P	Rule T, ($\neg\neg P \Leftrightarrow P$ & $\neg P$, $P \vee Q \Rightarrow Q$)
4 :	$P \to (Q \to S)$	Rule P
5 :	$Q \to S$	Rule T, ($P \to Q$, $P \Rightarrow Q$)
6 :	Q	Rule P
7 :	S	Rule T, ($P \to Q$, $P \Rightarrow Q$)
8 :	$R \to S$	(CP)

H.W

5. S.T R is a valid inference from the premises $P \to Q$, $Q \to R$ & P.

6. $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$ & $\neg M$.

7. $\neg P$ follows logically from $\neg(P \wedge \neg Q)$, $\neg Q \vee R$, $\neg R$.

Consistency of Premises and Indirect Method of proof

A set of formula's $H_1, H_2, \ldots H_m$ is said to be consistent if their conjunction has the truth table T for some assignment of the ~~both~~ truth values of the atomic variables appearing in $H_1, \ldots H_m$.

If for every assignment of the truth values of to the atomic variables, atleast one of the formulas $H_1, \dots H_m$ is false, so that their conjunction is identically false, then the formulas $H_1, H_2, \dots, H_m$ are called <u>inconsistent</u>.

Alternatively, a set of formulas $H_1, H_2, \dots, H_m$ is <u>inconsistent</u>, if their conjunction implies a contradiction.

i.e; $H_1 \wedge H_2 \wedge \dots \wedge H_m \Rightarrow R \wedge \neg R$, for any formula R.

The notion of inconsistency is ~~as~~ used in a procedure called <u>Proof by contradiction (or) indirect method of proof</u>.

<u>Procedure</u>: In order to show that a conclusion 'C' follows logically from the premises $H_1, \dots, H_m$; we assume that 'C' is false and consider '$\neg C$' as an additional premise.

Then this new set of premises is inconsistent i.e, they imply a contradiction.

So our assumption "$\neg C$ is true" does not hold simultaneously with $H_1 \wedge H_2 \wedge \dots H_m$ being true.

OR

<u>Note</u>:- $P \to Q \Leftrightarrow \neg Q \to \neg P$.

In order to show that a conclusion C follows logically from the premises $H_1, H_2, \dots, H_n$ we assume that C is false and $\neg C$ as an additional premise. If the new set of premises is inconsistent, then our assumption $\neg C$ as an addition premise is wrong, implying that C follows logically from $H_1, H_2, \dots, H_m$

<u>Example: 1</u> S.T $\neg(P \wedge Q)$ follows from $\neg P \wedge \neg Q$

<u>Soln:-</u> Proof by contradiction (or) Indirect method

We introduce $\neg\neg(P \wedge Q)$ as an additional premise and show that this additional premise leads to a contradiction.

Step 1 :	$\neg\neg(P \wedge Q)$	Rule P (Assumed)
2 :	$P \wedge Q$	Rule T & $\neg\neg P \Leftrightarrow P$
3 :	P	Rule T & $P \wedge Q \Rightarrow P$
4 :	$\neg P \wedge \neg Q$	Rule P
5 :	$\neg P$	Rule T & $P \wedge Q \Rightarrow P$
6 :	$\underbrace{P \wedge \neg P}_{\Downarrow \text{ Contradiction}} \Leftrightarrow F$	Rule T & $P, Q \Rightarrow P \wedge Q$

(3) S.T the following premises are inconsistent.

1. If Jack misses many classes through illness, then he fails high school.
2. If Jack fails high school, then he is uneducated.
3. If Jack reads a lot of books, then he is not uneducated.
4. Jack misses many classes through illness and reads a lot of books.

<u>Soln:-</u> Let

E : Jack misses many classes P

S : Jack fails high school Q

A : Jack reads a lot of books R

H : Jack is uneducated. S

$\therefore$ The premises are

$P \to Q \quad Q \to S \quad R \to \neg S, \quad P \wedge S$

$E \to S, \quad S \to H, \quad A \to \neg H \quad \& \quad E \wedge A$

<u>Step : 1</u>	$E \to S$	P	
2	$S \to H$	P	
3	$E \to H$	T	$(P \to Q, Q \to S \Leftrightarrow P \to S)$
4	$A \to \neg H$	P	
5	$H \to \neg A$	T	$(P \to Q \Leftrightarrow \neg Q \to \neg P$ & $\neg\neg P \Leftrightarrow P)$
6	$E \to \neg A$	T	3 & 5 $(P \to Q, Q \to S \Leftrightarrow P \to S)$
7	$\neg E \vee \neg A$	T	$(P \to Q \Leftrightarrow \neg P \vee Q)$
8	$\neg(E \wedge A)$	T	DeMorgan's Law
9	$E \wedge A$	P	Rule P
10	$(E \wedge A) \wedge \neg(E \wedge A)$	T	$P, Q \Rightarrow P \wedge Q$

$$(E \wedge A) \wedge \neg(E \wedge A) \Leftrightarrow F$$

<u>HW</u>

① S.T the following sets of premises are inconsistent.

(a) $P \to Q, \ P \to R, \ Q \to \neg R, \ P$

(b) $A \to (B \to C), \ D \to (B \wedge \neg C), \ A \wedge D.$

② S.T the following (use indirect method)

(a) $R \to \neg Q, \ R \vee S, \ S \to \neg Q, \ P \to Q \Rightarrow \neg P$

(b) $S \to \neg Q, \ S \vee R, \ \neg R, \ \neg R \rightleftarrows Q \Rightarrow \neg P$

~ . ◦ . ~

Solved problems

1. Prove by indirect method $\neg Q, P \to Q, P \vee T \Rightarrow T$

Soln:-

$T \vee P$
$\neg T \Rightarrow P$

step: 1	$P \vee T$	Rule P
2	$\neg T$	Rule P (additional)
3	P	Rule T $\neg P, P \vee Q \Rightarrow Q$
4	$P \to Q$	Rule P
5	Q	Rule T $P, P \to Q \Rightarrow Q$
6	$\neg Q$	Rule P
7	$Q \wedge \neg Q \Leftrightarrow F$	Rule T

2. $\neg Q, \ P \to Q, \ P \vee R \Rightarrow R$ (same as previous pblm)

3. Prove $R \to \neg Q, \ R \vee S, \ S \to \neg Q, \ P \to Q, \ P$ are inconsistent.

Soln:-

step: 1	P	Rule P	
2	$P \to Q$	P	
3	Q	T	$P, P \to Q \Rightarrow Q$
4	$S \to \neg Q$	P	
5	$Q \to \neg S$	T	$P \to Q \Leftrightarrow \neg Q \to \neg P$
6	$\neg S$	T	steps 3 & 5 $P, P \to Q \Rightarrow Q$
7	$R \vee S$	P	
8	$\neg R \to S$	T	$\because$ $P \to Q \Leftrightarrow \neg P \vee Q$; $\neg R \to S \Leftrightarrow \neg\neg R \vee S \Leftrightarrow R \vee S$
9	$\neg S \to R$	T	$P \to Q \Leftrightarrow \neg Q \to \neg S$
10	R	T	steps (6 & 9)
11	$R \to \neg Q$	P	
12	$\neg Q$	T	
13	$Q \wedge \neg Q$	T (3 & 12)	

(4) Using indirect method of proof, derive $P \to \neg S$ from the premises $P \to (Q \vee R)$, $Q \to \neg P$, $S \to \neg R$, P.

Soln:- Consider $\neg(P \to \neg S)$ as an additional premise and prove a contradiction.

$\neg(P \to \neg S) \Leftrightarrow \neg(\neg P \vee \neg S) \Leftrightarrow P \wedge S$

Step :	Statement	Reason/Rule	
1	$R \to (Q \vee R)$	P	
2	P	P	
3	$Q \vee R$	T	$P, P \to Q \Rightarrow Q$
4	$P \wedge S$	P	
5	S	T	$P \wedge S \Rightarrow S$
6	$S \to \neg R$	P	
7	$\neg R$	T	$S, S \to \neg R \Rightarrow \neg R$
8	Q	T	(3 & 7) $\{(Q \vee R) \wedge \neg R \Rightarrow Q\}$
9	$Q \to \neg P$	P	
10	$\neg P$	T	$Q, Q \to \neg P \Rightarrow \neg P$
11.	$P \wedge \neg P \Leftrightarrow F$	T (2, 10)	

(5) P.T $a \to (b \to c)$, $d \to (b \wedge \neg c)$ & $(a \wedge d)$ are inconsistent.

Step	Statement		
1	$a \wedge d$	P	
2	a	T	$a \wedge d \Rightarrow a$
3	$a \to (b \to c)$	P	
4.	$b \to c$	T	$a, a \to (b \to c) \Rightarrow b \to c$
5.	$\neg b \vee c$	T	$P \to Q \Rightarrow \neg P \vee Q$
6.	$d \to (b \wedge \neg c)$	P	
7.	$\neg(b \wedge \neg c) \to \neg d$	T	$P \to Q \Rightarrow \neg Q \to \neg P$

8.	$\neg b \vee c \rightarrow \neg d$	T	$\neg(P \wedge Q) \Rightarrow \neg P \vee \neg Q$
9.	$\neg d$	T	$P, P \rightarrow Q \Rightarrow Q$
10.	d	T	from step 1
11.	$d \wedge \neg d \Leftrightarrow F$	T	(9 & 10)

(6) Construct an argument to show that the following premises are inconsistent.

(1) If Rama gets his degree, he will go for a job

(2) If he goes for a job, he will get married soon

(3) If he goes for higher study, he will not get married.

(4) Rama gets his degree and goes for higher study.

Soln:-

P : Rama gets his degree

Q : He will go for a job

R : He will get married soon

S : He goes for higher study

$\therefore \quad P \rightarrow Q, \quad Q \rightarrow R, \quad S \rightarrow \neg R, \quad P \wedge S.$

Steps				
1.	$P \rightarrow Q$	P		
2	$Q \rightarrow R$	P		
3	$P \rightarrow R$	T		$P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$
4	$S \rightarrow \neg R$	P		
5	$R \rightarrow \neg S$	T		$P \rightarrow Q \Rightarrow \neg Q \rightarrow \neg S$
6.	$P \rightarrow \neg S$	T	(3 & 5)	$P \rightarrow Q, Q \rightarrow S \Rightarrow P \rightarrow S$
7.	$\neg P \vee \neg S$	T		$P \rightarrow Q \Rightarrow \neg P \vee Q$
8.	$\neg(P \wedge S)$	T		De Morgan
9.	$P \wedge S$	P		
10.	$(P \wedge S) \wedge \neg(P \wedge S) \Leftrightarrow$	T	(8 & 9)	

(7) Construct an argument to show that the following premises imply the conclusion "it rained".

" If it does not rain or if there is no traffic dislocation, then the sports day will be held and the cultural programme will go on" ; " If the sports day is held, the trophy will be awarded" and " the trophy was not awarded".

Let us symbolise the statement as follows:

P : It rains

Q : There is traffic dislocation

R : Sports day will be held

S : Cultural programme will go on

Z : The trophy will be awarded.

$\therefore$ $\neg P \vee \neg Q \to R \wedge S$, $R \to Z$, $\neg Z \Rightarrow P$

Step			
1	$\neg P \vee \neg Q \to R \wedge S$	P	
2	$(\neg P \to (R \wedge S)) \wedge (\neg Q \to (R \wedge S))$	T	$(a \vee b) \to x \equiv (a \to x) \vee (b \to x)$
3.	$\neg P \to (R \wedge S)$	T	$P \wedge Q \Rightarrow P$
4.	$\neg (R \wedge S) \to P$	T	$P \to Q \Rightarrow \neg Q \to \neg S$
5	$R \to Z$	P	
6	$\neg Z \to \neg R$	T	$P \to R \Rightarrow \neg R \to \neg P$
7.	$\neg Z$	P	
8.	$\neg R$	T	$P, P \to Q \Rightarrow Q$
9.	$\neg R \vee \neg S$	T	addition $P \Rightarrow P \vee Q$
10.	$\neg (R \wedge S)$	T	De Morgan
11.	P	T	(4 & 10) $P, P \to Q \Rightarrow Q$.