Unit - 6

Fundamentals of Graphs

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Graph

A graph $G = \langle V, E, \Phi \rangle$ consists of a non-empty set V called set of vertices (or nodes or points) of the graph; E is said to be the set of edges of the graph and Φ is mapping from the set E to a set of ordered or unordered pairs of ~~other~~ elements of V.
(ie., $\Phi : E \to V \times V$)

Assume that, the both sets V and E of a graph are finite.

Notation:- $G(V, E, \Phi)$ (or) $G(V, E)$ (or) simply G.
$\hookrightarrow$ Vertex set $\hookrightarrow$ Edge set

Remarks * If an edge $e \in E$ is associated with an ordered pair (u, v) or an unordered pair (u, v) where $u, v \in V$, then e is said to connect or join the nodes u and v.

* The edge e is said to be incident on each of the nodes u & v.

Adjacent Vertices Any pair of nodes which are connected by an edge in a graph is called adjcent nodes.

Directed graph (Digraph)

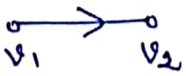
In a graph $G = \langle V, E \rangle$, an edge which is associated with an ordered pair of $V \times V$ is called a directed edge, while an edge which is associated with an unordered pair of nodes is called an undirected-edge.

* A graph in which every edge is directed is called a directed graph (or) digraph.
* A graph in which every edge is undirected is called an undirected graph.
* If some edges are directed and some are undirected in a graph, the graph is called mixed.

Examples

(1)


(2)


(3)


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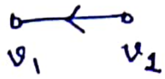

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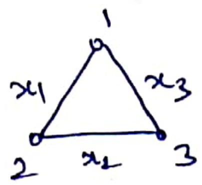

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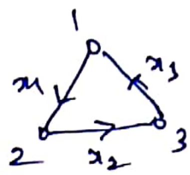

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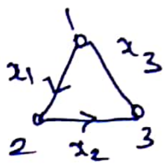


Here Example (1) is considered as either directed or undirected graph.

(2) & (5) are undirected graph.

(3), (4) & (6) are directed graph

(7) mixed graph.

Initial and Terminal Nodes

Let $G = \langle V, E \rangle$ be a graph and let $x \in E$ be a directed edge associated with the ordered pair of nodes $\langle u, v \rangle$. Then the edge 'x' is called as <u>initiating (or) originating</u> in the node 'u' and <u>terminating (or) ending</u> in the node v.

The nodes 'u' and 'v' are also called the initial and terminal nodes of the edge 'x'.

Incident on a node

An edge $x \in E$ which joins the nodes 'u' & 'v' either it be directed or undirected, is called to be incident to the nodes 'u' and 'v'.

Loop An edge of a graph which joins a node to itself is called a loop in a graph.

Parallel edges

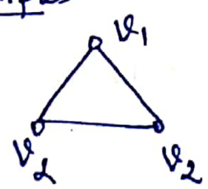
In a directed as well as undirected graphs, we may have certain pairs of nodes joined by more than one edge, such edges are called parallel edges

Multigraph

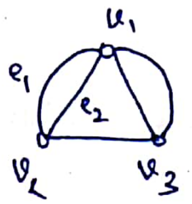
Any graph which contains some parallel edges is called multigraph.

(5) **Simple graph** If there is no loops and parallel edges then the graph is called simple graph.

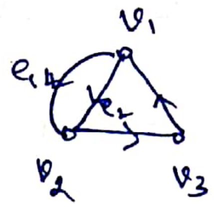
Examples



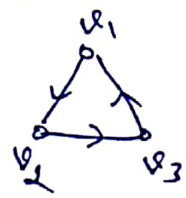
undirected & Simple



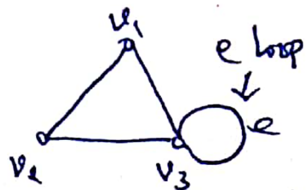
undirected Multigraph.



Directed multigraph



directed simple graph

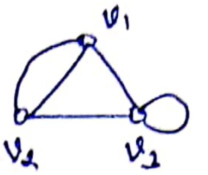


Undirected graph.

Pseudo graph:-

A graph in which loops and parallel edges are allowed is called a pseudo graph.

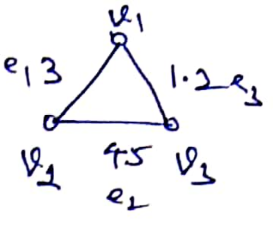
Example:-



Weighted graph

A graph in which a weight (numerical values) are assigned to every edge is called a weighted graph.

eg:

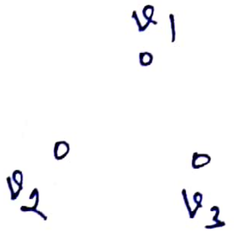


Isolated nodes and Null graph

In a graph a node which is not adjacent to any other node is called an isolated node.

A graph containing only isolated nodes is called a null graph.

Example:.



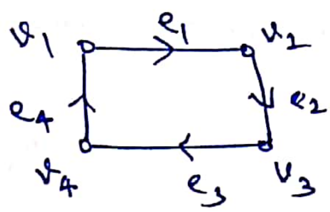
G = Null graph with isolated nodes $\{v_1, v_2, v_3\}$

Graph Isomorphic

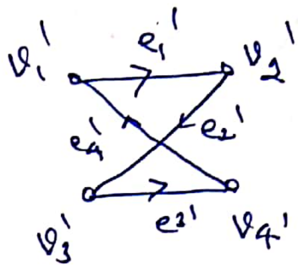
Two graphs are isomorphic if there exists a one to one correspondence between the nodes of the two graphs which preserves adjacency of the nodes as well as directions of the edges, if any.

ie, $G = \langle V_1, E_1, \phi_1 \rangle \cong G_2 = \langle V_2, E_2, \phi_2 \rangle$, if there exists a bijective function $f : V_1 \xrightarrow[\text{onto}]{1-1} V_2$ s.t which preserves the adjacency of the nodes and its direction (if any)

Examples:-

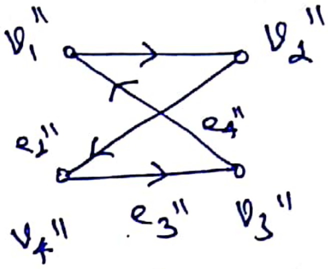


$\cong$



$G_1 \cong G_2$ $\qquad \{v_1, v_2, v_3, v_4\} \xrightarrow[\text{onto}]{1-1} \{v_1', v_2', v_4', v_3'\}$

But



G_3

not isomorphic

$G_1 \ncong G_3$.

Degree of a vertex in undirected graphs

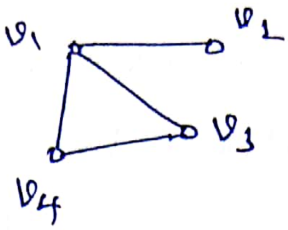
The degree of a vertex in an undirected graph is the number of edges incident with it. (only for simple undirected graph).

Note:- (1) The degree of a vertex 'v' is denoted by 'deg(v)'

(2) The degree of the isolated vertex is 'Zero'.

(3) If the deg(v) = 1 is called a pendant vertex.

Example:-



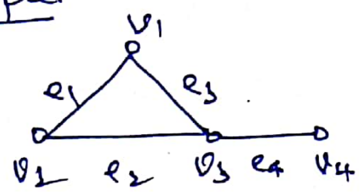
$deg(v_1) = 3$
$deg(v_2) = 1$ (pendant)
$deg(v_3) = 2 = deg(v_4)$.

Subgraph

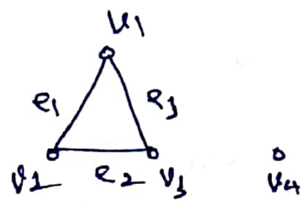
Let $G = \langle V_G, E_G, \Phi_G \rangle$ be a graph. A graph $H = \langle V_H, E_H, \Phi_H \rangle$ is called a subgraph of a graph G, if $V_H \subseteq V_G$ and $E_G \subseteq E_H$ (i.e., every edge of H is also a edge of G).

Note:-
If $V_H = V_G$, then H is called a spanning subgraph of G.
A spanning graph of G need not contain all its edges.

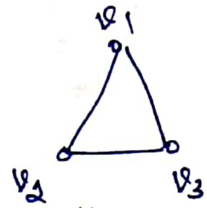
Example:-



G

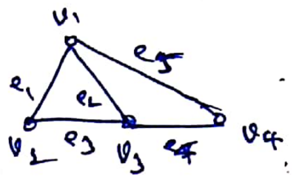


$H_1 = G - \{e_4\}$



H_2

H_1, H_2 are subgraphs
$\hookrightarrow$ spanning
H_3 is not subgraph.



H_3

Some special simple graphs

Complete graph

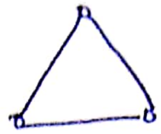
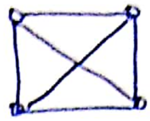
A simple graph, in which there is exactly one edge between each pair of distinct vertices, is called a complete graph.

The complete graph on 'n' vertices is denoted by K_n.

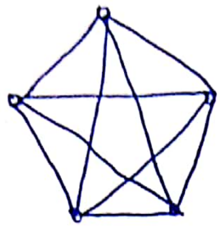
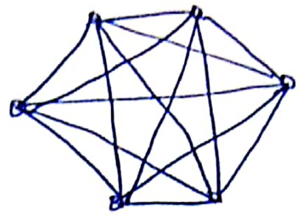
examples

K_1 K_2 K_3 K_4



K_5 K_6

Results

1) The number of edges in K_n is nC_2 or $\frac{n(n-1)}{2}$

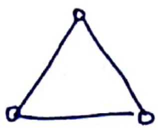
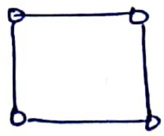
2) The maximum number of edges in a simple graph with 'n' vertices is $\frac{n(n-1)}{2}$.

Regular graph

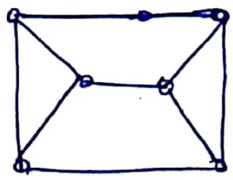
If every vertex of a simple graph has the same degree, then the graph is called a regular graph.

* If every vertex in a regular graph has degree 'n' then the graph is called n-regular.

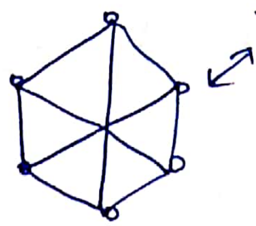
Example:-

2-regular graph



3-regular graph.



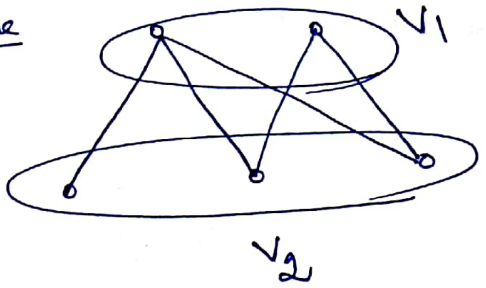
Result:- 1) Every complete graph is a regular graph.

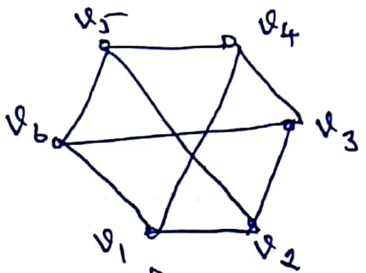
2) Every regular graph need not be a complete graph.

Bipartite graph

* If the vertex set V of a simple graph $G = (V, E)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or V_2), then G is called a bipartite graph.

* Example





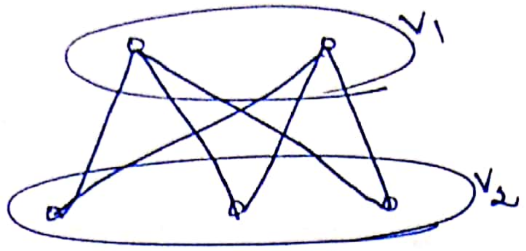
$V_1 = \{v_1, v_3, v_5\}$

$V_2 = \{v_2, v_4, v_5\}$

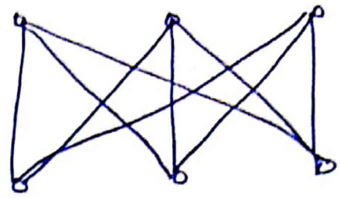
complete bipartite graph

If each vertex set V_1 is connected with every vertex of V_2 by an edge then G is called a complete bipartite graph. If V_1 have m vertices and V_2 have n vertices then the complete bipartite graph is denoted by $K_{m,n}$.

Example:-



$K_{2,3}$



$K_{3,3}$

Theorem (Fundamental theorem of Graph theory)

(The Handshaking theorem)

In any graph the sum of degrees of its vertices is equal to twice the number of edges.

i.e., $\sum_{i=1}^{n} d(v_i) = 2e$

Proof:-

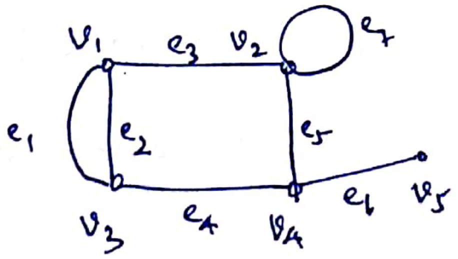
Let us consider a graph G with e edges and n vertices.

$v_1, v_2, \ldots, v_n$ are its vertices.

Since each edge contributes two degrees, the sum of the degrees of all vertices in G is twice the number of edges in G.

i.e, $\sum_i d(v_i) = 2e$.

Example: Verify the theorem



$d(v_1) = 3$ $\quad d(v_4) = 3$

$d(v_2) = 4$ $\quad d(v_5) = 1$

$d(v_3) = 3$

$\therefore \sum d(v_i) = 14$

$2e = 2 \times 7 = 14$ //.

Theorem The number of vertices of odd degree in an undirected graph is even. (or)

The number of odd vertices is always even.

Proof:- Let $G = \langle V, E \rangle$ be the undirected graph.

Let V_1 and V_2 be the sets of vertices of G of even and odd degrees respectively.

Then by previous theorem

$$2e = \sum_{v_i \in V_1} \deg(v_i) + \sum_{v_j \in V_2} \deg(v_j)$$

(even) (even)

$$\therefore \sum_{v_j \in V_2} \deg(v_j) = 2e - \sum_{v_i \in V_1} \deg(v_i)$$

$$= \text{even}$$

Since each $\deg(v_j)$ is odd, the number of terms contained in $\sum_{v_j \in V_2} \deg(v_j)$ is even.

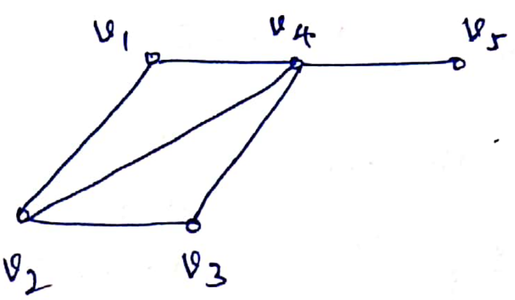
Example:-

(1)

$d(v_1) = 1 \quad d(v_2) = 1.$

$\therefore$ The no. of odd vertices is even.

(2)



$d(v_1) = 2 \quad d(v_2) = 3$

$d(v_3) = 2 \quad d(v_4) = 4$

and $d(v_5) = 1.$

$\therefore$ even (v_2, v_5).

Matrix Representation of Graphs

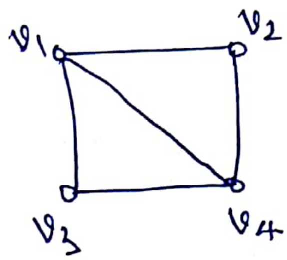
Adjacency matrix

When G is a simple graph with n vertices $v_1, v_2, \ldots, v_n$ the matrix A or $(A_G) \equiv [a_{ij}]$,

where $a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$

is called the adjacency matrix of G.

Example :



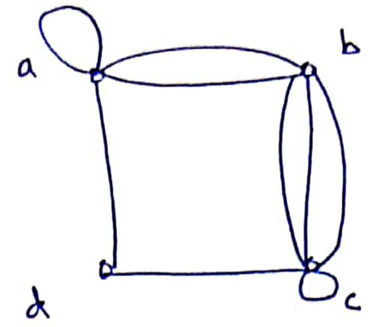
$$A = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Remarks ① Since a simple graph has no loops, each diagonal entry of A viz $a_{ii} = 0$, for $i = 1, 2, \ldots n$.

② The adjacency matrix of simple graph is symmetric.

③ $\deg(v_i)$ is equal to the number of 1's in the i^{th} row or i^{th} column.

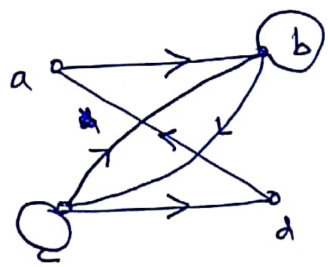
Pseudograph



$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Directed graph

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



out going vertices

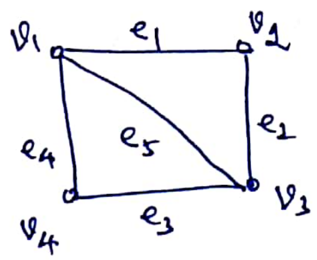
Definition   Incidence matrix

If $G = (V, E)$ is an undirected graph with n vertices and m edges $e_1, e_2, \dots e_m$, then the $(n \times m)$ matrix

$B = [b_{ij}]$ where $b_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident on } v_i \\ 0 & \text{otherwise} \end{cases}$

is called incidence matrix.

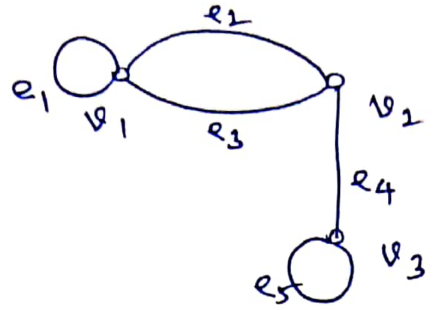
Ex:-



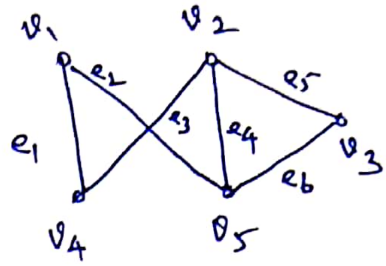
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

1. Write the incidence matrix of the graph

(i)

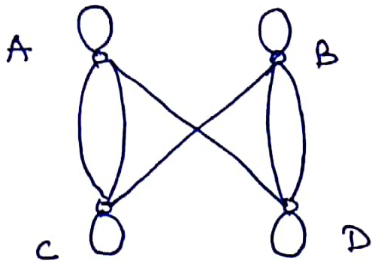


(ii)

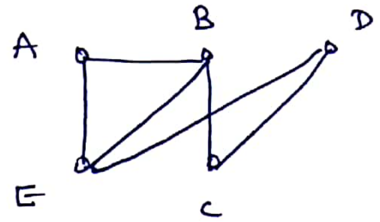


2. Write adjacency matrix

(i)



(ii)



3. Draw the graphs represented by the following adjacency matrices

(i) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

4. Draw the graphs represented by the following incidence matrix

(i)
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

(ii)
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$