Module II Predicate calculus

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p, q, r .. Variables — Propositions
∧, ∨ → ... Connectives } Module I → Premises

functions - Predicates

Variable → function
x → $f(x)$

$P(x), Q(x)$ .... functions or Predicates.

$\underbrace{\text{Ram}}_{\text{Variable}}$ is $\underbrace{\text{listening the class}}_{\text{function}}$

x : Ram
P : is listening the class
→ $P(x)$

Saran is sleeping in the class
x : Saran
Q : is sleeping
$Q(x)$

Sai is listening the class
y : Sai
$P(y)$.

Every one is listening. P : listening

$\forall x, P(x)$ or $(x)\, P(x)$ $\forall$, (x) - for every

Every one is not listening — $\forall x\ \neg P(x)$.

Some people are listening

$\exists x\ P(x)$ $\exists$ - there exist

Some people are not listening — $\exists x\ \neg P(x)$.

I can't guess that all are listening.
(or) All need not listen.

$\neg\, \forall x\ P(x)$ — It is not the case that everyone is listening

⇓ means (or)

$\exists x\ \neg P(x)$ ← It is the case that Some people are not listening

$$\left.\begin{aligned} \neg\, \forall x\ P(x) &\equiv \exists x\ \neg P(x). \\ \neg\, \exists x\ P(x) &\equiv \forall x\ \neg P(x) \end{aligned}\right\}$$ Like demorgan's

Find the negation of the following.

(a) If the teacher is absent, then some students do not keep quiet.
(b) All the students keep quiet and the teacher is present.
(c) Some of the students do not keep quiet or the teacher is absent.

(a) If the teacher is absent, then some students do not keep quiet.
(b) All the students keep quiet and the teacher is present.
(c) Some of the students do not keep quiet or the teacher is absent.
(d) No one has done every problem in the exercise.

Connectives are for predicates also $P(x) \to Q(x)$, $P(x) \wedge \neg Q(x)$ ...

a) Teacher is absent - T
.. is present - F or $\neg T$

or P - is absent
$P(t)$ - teacher is absent
constant.

$P(t)$ - teacher is absent
Q - keep quiet.

$P(x)$ and $P(a)$
general specific value.

a) $P(t) \to \exists x \neg Q(x)$ → negation of this

$\neg(p(t) \to \exists x \neg Q(x))$

$\neg(\neg p(t) \vee \exists x \neg Q(x))$.

$\equiv p(t) \wedge \neg(\exists x \neg Q(x))$

$p(t) \wedge \forall x Q(x)$.

The teacher is absent and all the students keep quiet.

b) $\forall x Q(x) \wedge \neg p(t)$.

$\neg(\forall x Q(x) \wedge \neg P(t)) \equiv \neg($

c) $\exists x \neg Q(x) \vee P(t)$.

P : doing all the problems

$\forall x \neg P(x)$

b) $(\forall x Q(x) \wedge \neg p(t))$
negation of (")

$\equiv \neg(\forall x Q(x) \wedge \neg p(t))$

$\equiv \neg(\forall x Q(x)) \vee \neg\neg p(t)$

$\equiv \exists x \neg Q(x) \vee P(t)$ → Some students not quiet or teacher is absent

c) $\exists x \neg Q(x) \vee P(t)$.

negation : $\neg(\exists x \neg Q(x) \vee p(t))$

$\equiv \neg(\exists x \neg Q(x)) \wedge \neg P(t)$.

$\equiv \forall x Q(x) \wedge \neg p(t)$

All the student keep quiet and the teacher is present.

No one has done every problem in the exercise

x - Student.
y - problems.

$\neg \exists x$

y - problems. $\neg\exists x$

P - Solving a problem

$P(x)$ ~~x solves~~ problem

$P(y)$ - a problem is solved.

$P(x,y)$ - x solves y

$\forall x\ P(x,y)$ - all the students solve y one or more problems

$\forall x\ \forall y\ P(x,y)$ - Every student solves all the problems.

$\neg\ \forall x\ \forall y\ P(x,y)$ - It is not the case that Everyone solves all problem

d) $\neg\exists x\ \forall y\ P(x,y)$ - no one has done all the problems.

$\neg\exists x$

Negation

$$\neg(\neg\exists x\ \forall y\ P(x,y)) = \neg\neg\forall x\ \neg\exists y\ P(x,y)$$
$$= \forall x\ \neg(\exists y\ P(x,y))$$
$$= \forall x\ \forall y\ \neg P(x,y)$$

= Everyone donot solve all the problems.

Another approach

$$\neg(\neg\exists x\ \forall y\ P(x,y))$$
$$= \neg(\forall x\ \forall y\ P(x,y))$$
$$= \exists x\ \neg(\forall y\ P(x,y))$$
$$= \exists x\ \exists y\ \neg P(x,y)$$

= Some students do not solve some problems.

The meanings of the stmts are same.

Rules of Inference for Predicate calculus.

Rules of inference

$P, P\to q \Rightarrow q$

$p\to q \Rightarrow \neg q\to\neg p$

...

1 US - universal specification

$\forall x\ P(x) \Rightarrow P(c)$

1. $\forall x\ P(x)$ P
2. $P(c)$ T-US

2. ES - Existential specification

$\exists x\ q(x) \Rightarrow q(c)$

1. $\exists x\ q(x)$ P
2. $q(c)$ T ES

3 UG - Universal generalisation

1. $P(a)$ P

3 UG – Universal generalisation

$p(a) \Rightarrow \forall x\ p(x)$

1. $p(a)$ P
2. $\forall x\, p(x)$ T UG.

4 EG – Existential generalisation

$q(b) \Rightarrow \exists x\ q(x)$

1. $q(c)$ P
2. $\exists y\, q(y)$ T EG

Example

All the students of VIT are studious, Ram is a VIT student.. $\Rightarrow$ Ram is studious

P – studious

Q – is a VIT student

$\forall x\, [Q(x) \rightarrow P(x)],\ Q(a) \Rightarrow P(a)$

1 $\forall x\, [Q(x) \rightarrow P(x)]$ P

2. $Q(a)$ P .

3 $Q(a) \rightarrow P(a)$ US

4 $P(a)$ T Modus ponen 3,4

$\Rightarrow$ Ram is studious.

$[P, P \rightarrow q \Rightarrow q]$

$P, P \rightarrow q \Rightarrow q$

$P, \neg q \rightarrow \neg P \Rightarrow q$

$P, q \vee \neg P \Rightarrow q$

From the premises

$\exists x\, (P(x) \wedge Q(x)) \rightarrow \forall y\, (R(y) \rightarrow S(y))$ and $\exists y\, (R(y) \wedge \neg S(y))$.

Conclude

$\forall x (P(x) \rightarrow \neg Q(x))$

1. $\exists x\, (P(x) \wedge q(x)) \rightarrow \forall y\, (R(y) \rightarrow S(y))$ Rule P

2. $\exists y\, (R(y) \wedge \neg S(y))$ Rule P

3 $\exists y \neg(\neg R(y) \vee S(y))$ T de-morgan's law

4 $\exists y\ \neg(R(y) \rightarrow S(y))$ T

5 $\neg \forall y\, (R(y) \rightarrow S(y))$ T

$P \rightarrow q, \neg q \Rightarrow \neg P$

6 $\neg(\exists x (P(x) \wedge q(x)))$ T (1) (5)

7 $\forall x\ \neg(P(x) \wedge q(x))$ T .

8 $\forall x\ \neg P(x) \vee \neg q(x)$ T demorgan

9 $\forall x\, (P(x) \rightarrow \neg q(x)) = a$ T

for indirect method

10 $\neg \forall x\, (P(x) \rightarrow \neg q(x)) = \neg a$ ($\neg$(RHS)) - indirect method

11 $a \wedge \neg a = F$ T from (9,10)

Prove the derivation

$\exists x\, P(x) \rightarrow \forall x\, ((P(x) \vee Q(x)) \rightarrow R(x))$,
$\exists x\, P(x)$, $\exists x\, Q(x) \Rightarrow \exists x\, \exists y\, (R(x) \wedge R(y))$

1. $\exists x\, P(x) \rightarrow \forall x\, ((P(x) \vee Q(x)) \rightarrow R(x))$ P

1. $\exists x\, P(x) \Rightarrow \forall x \left((P(x) \vee Q(x)) \rightarrow R(x) \right)$ — P

2. $P(a) \rightarrow (P(b) \vee Q(b)) \rightarrow R(b)$ — T ES, US

3. $\exists x\, P(x)$ — P

4. $P(a)$ — T ES

5. $P(b) \vee Q(b) \rightarrow R(b)$ — T (2,4) Modus Ponen.

6. $\neg(P(b) \vee Q(b)) \vee R(b)$ — T

7. $(\neg P(b) \wedge \neg Q(b)) \vee R(b)$ — T demorgan

8. $(\neg P(b) \vee R(b)) \wedge (\neg Q(b) \vee R(b))$ — T distributive law

9. $\neg Q(b) \vee R(b)$ — T $\quad P \wedge q \Rightarrow P \Rightarrow q$

10. $\exists x\, Q(x)$ — P

11. $Q(b)$ — T ES.

12. $R(b)$ — T (9,10)

13. $\exists x\, R(x)$ — T EG

14. $R(a)$ — T ES

15. $\exists y\, R(y)$ — T EG

16. $\exists x\, R(x) \wedge \exists y\, R(y)$ — T

17. $\exists x\, \exists y \left[R(x) \wedge R(y) \right]$ — T.

$$\neg\left[\exists x\, \forall y\, P(x,y)\right] = \forall x\, \neg\left[\forall y\, P(x,y)\right] = \forall x\, \exists y\, \neg P(x,y)$$

Verify the validity of the following argument:
Every living thing is a plant or an animal. Rama's dog is alive and it is not a plant. All animals have hearts. Therefore Rama's dog has a heart.

P: living (alive) $\quad$ Q: is a plant $\quad$ R: is an animal $\quad$ S: have hearts

$\underbrace{\text{Every}}_{\forall x}$ $\underbrace{\text{living}}_{P}$ thing is a $\underbrace{\text{plant}}_{Q}$ or an $\underbrace{\text{animal}}_{R}$ — $\forall x \left(P(x) \rightarrow (Q(x) \vee R(x)) \right)$

$\underbrace{\text{Rama's dog}}_{a}$ is $\underbrace{\text{alive}}_{P}$ and it is $\underbrace{\text{not a plant}}_{\neg Q}$ — $P(a) \wedge \neg Q(a)$.

$\underbrace{\text{All}}_{\forall x}$ $\underbrace{\text{animals}}_{R}$ have $\underbrace{\text{hearts}}_{S}$ — $\forall x \left[R(x) \rightarrow S(x) \right]$

To prove Rama's dog has heart - $S(a)$

$\overline{\forall x}$ $\overline{R}$ S

To prove Rama's dog has heart - $S(a)$

$\forall x(p(x) \to (Q(x) \vee R(x))),\ p(a), \wedge \neg Q(a), \forall x(R(x) \to S(x)) \Rightarrow S(a).$

1	$\forall x\ (p(x) \to (Q(x) \vee R(x)))$	P
2	$p(a) \to (Q(a) \vee R(a))$	T US
3	$p(a)$	P
4	$Q(a) \vee R(a)$	T 3, 2, Modus ponen
5	$\neg Q(a)$	P
6	$R(a)$	T (4, 5)
7	$\forall x\ (R(x) \to S(x))$	P.
8	$R(a) \to S(a)$	T US
9	$S(a)$	T 6, 8 Modus ponen

Example 1.18 Show that the premises "one student in this class knows how to write programs in JAVA" and "Everyone who knows how to write programs in JAVA can get a high-paying job" imply the conclusion "Someone in this class can get a high-paying job".

P: write program in JAVA , $P(x)$ - x knows JAVA

One student knows JAVA $\Rightarrow \exists x\ P(x)$

Q . get job

Everyone who knows JAVA gets job $\Rightarrow \forall x\ (P(x) \to Q(x))$

Someone get job $\Rightarrow \exists x\ Q(x)$

$\exists x\ p(x),\ \forall x\ (p(x) \to Q(x)) \Rightarrow \exists x\ Q(x)$

1)	$\forall x\ (p(x) \to Q(x))$	Rule P
2)	$p(a) \to q(a)$	T US
3)	$\exists x\ p(x)$	Rule P
4)	$p(a)$	T ES
5	$q(a)$	T (2) (4)
6	$\exists x\ q(x)$	T EG

(i) Symbolize the sentence "If anyone can do it, Ravi can" and its negation. (5 marks)

P: can do it $p(x)$ - x can do it a : Ravi

$\forall x\ P(x) \to P(a)$

$\neg(\forall x\ P(x) \to P(a))$

$\forall x\ p(x) \wedge \neg p(a)$ Demorgan

Everyone can do it ~~but~~ and Ravi cannot do it

$\forall x\, p(x) \rightarrow p(a)$

$\neg(\forall x\, p(x) \rightarrow p(a))$ $\qquad \forall x\, p(x) \wedge \neg p(a)$

$\neg(\neg\forall x\, p(x) \vee p(a))$

Everyone can do it ~~but~~ and Ram cannot do it

Prove the following: No student is allowed in the college without got vaccinated. A person who suffered with covid-19, need not go for vaccination. Ram did not get vaccinated and allowed in college implies Ram suffered with covid-19

p : got vaccinated $\qquad p(x)$: x got vaccinated.

q : suffered with covid-19 $\qquad q(x)$: x suffers with covid-19

r : allowed in college $\qquad r(x)$: x is allowed in college.

No student is allowed in college without got vaccinated.

$\forall x\, \neg p(x) \rightarrow \neg r(x)$ – P_1

A person who suffered with covid-19 not go for vaccination

$\forall x\, q(x) \rightarrow \neg p(x)$ – P_2

Ram didn't get vaccinated $\qquad$ a : Ram – $\neg p(a)$ – P_3

Ram is allowed in college $\qquad r(a)$ – P_4

$\Rightarrow$ Ram suff. with covid $\Rightarrow q(a)$ – C

1. $\forall x\, \neg p(x) \rightarrow \neg r(x)$ $\qquad$ Rule P
2. $\forall x\, q(x) \rightarrow \neg p(x)$ $\qquad$ " P
3. $\neg p(a)$ $\qquad$ Rule P
4. $r(a)$ $\qquad$ P
5. $\neg p(a) \rightarrow \neg r(a)$ $\qquad$ (1) US
6. $q(a) \rightarrow \neg p(a)$ $\qquad$ (2) US
7. $q(a) \rightarrow \neg r(a)$ $\qquad$ T (5, 6)
8. $r(a) \rightarrow \neg q(a)$ $\qquad$ T
9. ~~r(a)~~ $\neg q(a)$

$\forall x\, q(x) \rightarrow r(x)$

$q(a) \rightarrow r(a)$

Show that the premises "A student in this class has not read the book" and "Everyone in this class passed the first examination" imply the conclusion "Someone who passed the first examination has not read the book."

to be proved

a = b

to be proved
$a \Rightarrow b$
$a \wedge b \wedge c \wedge d \Rightarrow e$
$a, b, c, d \Rightarrow e$
$a - b = 0$ - indirect method
$a \wedge b \wedge c \wedge d \wedge \neg e \Rightarrow F$

Indirect method

For the given set of premises/propositions, to prove the right hand side, we can take the negation of the right hand side as another premise with the lefthand side and to

$p \to q, q \to r, \neg(p \wedge r), p \vee r \Rightarrow r.$

Indirect method

$p \to q, q \to r, \neg(p \wedge r), p \vee r, \neg r \Rightarrow F$

1. $p \to q$ Rule P
2. $q \to r$ Rule P
3. $p \to r$ T (1, 2)
4. $\neg(p \wedge r)$ P
5. $\neg p \vee \neg r$ T De-morgan
6. $p \to \neg r$ T
7. $p \to r \wedge \neg r$ T (3) (6)
8. $p \to F$ T
9. $\neg p \vee F$ T (8)
10. $\neg p$ T
11. $p \vee r$ P
12. $\neg p \to r$ T
13. r
14. $r \wedge \neg r = F$ (-) (13)

Propositional calculus(Module 1) vs predicate calculus (Module 2)

$p, q, r \ldots$	$p(x), q(y), M(x, y) \ldots$
$p \to q \Rightarrow \neg p \vee q$	$p(x) \to q(x) \Rightarrow \neg p(x) \vee q(x)$
$p \to q, q \to r \Rightarrow p \to r$	$p(x) \to q(x), q(x) \to r(x) \Rightarrow p(x) \to r(x)$
Indirect method	Indirect method applicable
CP rule	CP rule
Truth tables	no truth tables
no rules of inference.	Rules of Inference US, UG, ES, EG

CP (Constructive premise) rule

To prove
$a, b, c, d \ldots \Rightarrow p \to q$

then we can take $a, b, c, d \ldots, p \Rightarrow q$

Use CP rule to prove the following

$p, p \to (q \to (r \wedge s)) \Rightarrow q \to s.$

$P, P \to (q \to (r \wedge s)) \Rightarrow q \to s$

$p, P \to (q \to (r \wedge s)), q \Rightarrow s$

1 P Rule P

2 $P \to q \to (r \wedge s)$ Rule P

3 $q \to (r \wedge s)$ T (1,2)

4. q Rule P

5 $r \wedge s$ T (3,4)

6 F

7 $\underline{s}$

Sheela studies better than other students in the class.

a : Sheela

$S(x,y)$ — x studies better than y

$\forall y \; S(a,y)$