Chapter 1
Propositions

Definition 1. (Proposition)

A *statement* or *proposition* is a declarative sentence that is either true or false (but not both).

For instance, the following are propositions:

1. $3 > 1$ (true).
2. $2 < 4$ (true).
3. $4 = 7$ (false)

However the following are a not propositions:

1. what is your name?.
2. x is an even number.

Definition 2. (Atomic statements)
Declarative sentences which cannot be further split into simple sentences are called *atomic statements* (also called *primary statements* or *primitive statements*).

Example: p is a prime number

Definition 3. (Compound statements)
New statements can be formed from atomic statements through the use of connectives such as "and, but, or etc..." The resulting statement are called *molecular* or *compound* (composite) statements.

Example: If p is a prime number then, the divisors are p and 1 itself

Definition 4. (truth value)
The truth or falsehood of a proposition is called its *truth value*.

Definition 5. (Truth Table)
A table, giving the truth values of a compound statement interms of its component parts, is called a *Truth Table*.

Definition 6. *(Connectives)*

Connectives are used for making compound propositions. The main ones are the following (p and q represent given two propositions):

Table 1. Logic Connectives

Name	Represented	Meaning
Negation	$\neg p$	not in p
Conjunction	$p \wedge q$	p and q
Disjunction	$p \vee q$	p or q (or both)
Implication	$p \rightarrow q$	if p then q
Biconditional	$p \leftrightarrow q$	p if and only if q

The truth value of a compound proposition depends only on the value of its components. Writing F for false and T for true, we can summarize the meaning of the connectives in the following way:

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Definition 7. (Tautology)
A proposition is said to be a *tautology* if its truth value is T for any assignment of truth values to its components.

Example: The proposition $p \vee \neg p$ is a tautology.

Definition 8. (Contradiction)
A proposition is said to be a *contradiction* if its truth value is F for any assignment of truth values to its components.

Example: The proposition $p \wedge \neg p$ is a contradiction.

Definition 9.(Contingency)
A proposition that is neither a tautology nor a contradiction is called a *contingency*.

p	$\neg p$	$p \wedge \neg p$	$p \vee \neg p$
T	F	F	T
T	F	F	T

I. Construct the truth table for the following statements:

(i) $(p \to q) \longleftrightarrow (\neg p \vee q)$

(ii) $p \wedge (p \vee q)$

(iii) $(p \to q) \to p$

(iv) $\neg(p \wedge q) \longleftrightarrow (\neg p \vee \neg q)$

(v) $(p \vee \neg q) \to q$

Solution:

(i) Let $S = (p \to q) \longleftrightarrow (\neg p \vee q)$

p	q	$\neg p$	$p \to q$	$\neg p \vee q$	S
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

(ii) Let $S = p \wedge (p \vee q)$

p	q	$p \vee q$	S
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

(iii) Let $S = (p \to q) \to p$

p	q	$p \to q$	S
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	F

(iv) Let $S = \neg(p \wedge q) \longleftrightarrow \neg p \vee \neg q$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	S
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

(v) Let $S = (p \vee \neg q) \to q$

p	q	$\neg q$	$p \vee \neg q$	S
T	T	F	T	T
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

Logical Equivalence

The compound propositions $p \rightarrow q$ and $\neg p \vee q$ have the same truth values:

p	q	$\neg p$	$p \rightarrow q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

When two compound propositions have the same truth value they are called logically equivalent.

For instance $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent, and it is denoted by

$$p \rightarrow q \Leftrightarrow \neg p \vee q$$

Definition 10. (Logically Equivalent)
Two propositions A and B are *logically equivalent* precisely when $A \Leftrightarrow B$ is a tautology.

Example: The following propositions are logically equivalent:

$$p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$$

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$	S
T	T	T	T	T	T	T
T	F	F	F	T	F	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T

Table 2. Logic equivalences

Equivalences	Name
$p \wedge T \Leftrightarrow p$ $p \vee F \Leftrightarrow p$	Identity law
$p \vee T \Leftrightarrow T$ $p \wedge F \Leftrightarrow F$	Domient law
$p \vee T \Leftrightarrow T$ $p \wedge F \Leftrightarrow F$	Idempotent law
$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	Commutative law
$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	Associative law

Table 2. Logic equivalences (Continued...)

Equivalences	Name
$(p \vee q) \wedge r \Leftrightarrow (p \wedge r) \vee (q \wedge r)$ $(p \wedge q) \vee r \Leftrightarrow (p \vee r) \wedge (q \vee r)$	Distributive law
$(p \vee q) \wedge p \Leftrightarrow p$ $(p \wedge q) \vee p \Leftrightarrow p$	Absorbtion law
$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	De morgan's law
$\neg p \wedge p \Leftrightarrow F$ $\neg p \vee p \Leftrightarrow T$ $\neg(\neg p) \Leftrightarrow p$	Negation law

Table 3. Logic equivalences involving implications

Implications
$p \rightarrow q \Leftrightarrow \neg p \vee q$
$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$
$\neg(p \rightarrow q) \Leftrightarrow p \wedge \neg q$
$p \vee q \Leftrightarrow \neg p \rightarrow q$
$p \wedge q \Leftrightarrow \neg(p \rightarrow \neg q)$
$(p \rightarrow q) \wedge (p \rightarrow r) \Leftrightarrow p \rightarrow (q \wedge r)$
$(p \rightarrow q) \vee (p \rightarrow r) \Leftrightarrow p \rightarrow (q \vee r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \Leftrightarrow (p \vee q) \rightarrow r)$
$(p \rightarrow r) \vee (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r)$

Table 4. Logic equivalences involving Biconditions

Biconditions
$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \Leftrightarrow \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \Leftrightarrow p \leftrightarrow \neg q$

Definition 11. (Converse)
The *converse* of a conditional proposition $p \rightarrow q$ is the proposition $q \rightarrow p$

Definition 12. (Inverse)
The *inverse* of a conditional proposition $p \rightarrow q$ is the proposition $\neg p \rightarrow \neg q$

Definition 13. (Contrapositive)
The *contrapositive* of a conditional proposition $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.

For example

Let us consider the statement,
"The crops will be destroyed, if there is a flood."
Let F : there is a flood & C : The crops will be destroyed
The symbolic form is, $F \to C$.
Converse $(C \to F)$
i.e., "if the crops will be destroyed then there is flood."
Inverse $(\neg F \to \neg C)$
i.e.,"if there is no flood then the crops won't be destroyed, ."
Contrapositive $(\neg C \to \neg F)$
i.e., "if the crops won't be destroyed then there is no flood."

II. Without using truth table:

(i) Show that $(p \vee q) \wedge \neg p \Leftrightarrow (\neg p \wedge q)$

(ii) Show that $(p \vee q) \wedge \neg(\neg p \wedge q) \Leftrightarrow p$

(iii) Show that $\neg(p \vee (\neg p \wedge q)) \Leftrightarrow \neg p \wedge \neg q$

(iv) Show that $\neg(\neg((p \vee q) \wedge r) \vee \neg q) \Leftrightarrow q \wedge r$

(v) Show that $(p \wedge q) \to (p \vee q)$ is a tautology.

(vi) Show that $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$.

Solution:

(i)

$S \Leftrightarrow (p \vee q) \wedge \neg p$	Reasons
$\Leftrightarrow (p \vee q) \wedge \neg p$	Given
$\Leftrightarrow (p \wedge \neg p) \vee (q \wedge \neg p)$	Distributive law
$\Leftrightarrow F \vee (\neg p \wedge q)$	Negation law, Commutative law
$\Leftrightarrow \neg p \wedge q$	Identity law

(ii)

$S \Leftrightarrow (p \vee q) \wedge \neg(\neg p \wedge q)$	Reasons
$\Leftrightarrow (p \vee q) \wedge \neg(\neg p \wedge q)$	Given
$\Leftrightarrow (p \vee q) \wedge (\neg\neg p \vee \neg q)$	De Morgan's law
$\Leftrightarrow (p \vee q) \wedge (p \vee \neg q)$	Negation law
$\Leftrightarrow p \vee (q \wedge \neg q)$	Distributive law
$\Leftrightarrow p \vee F$	Negation law
$\Leftrightarrow p$	Identity law

(iii)

$S \Leftrightarrow \neg(p \vee (\neg p \wedge q))$	Reasons
$\Leftrightarrow \neg(p \vee (\neg p \wedge q))$	Given
$\Leftrightarrow \neg p \wedge \neg(\neg p \wedge q))$	De Morgan's law
$\Leftrightarrow \neg p \wedge (p \vee \neg q))$	De Morgan's law
$\Leftrightarrow (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	Distributive law
$\Leftrightarrow F \vee (\neg p \wedge \neg q)$	Negation law
$\Leftrightarrow \neg p \wedge \neg q$	Identity law

(iv)

$S \Leftrightarrow \neg(\neg((p \vee q) \wedge r) \vee \neg q) \Leftrightarrow q \wedge r$	Reasons
$\Leftrightarrow \neg(\neg((p \vee q) \wedge r) \vee \neg q)$	Given
$\Leftrightarrow ((p \vee q) \wedge r) \wedge q$	De Morgan's law
$\Leftrightarrow (p \vee q) \wedge (r \wedge q)$	Associative law
$\Leftrightarrow (p \wedge (r \wedge q)) \vee (q \wedge (r \wedge q))$	Distributive law
$\Leftrightarrow (p \wedge (r \wedge q)) \vee (r \wedge q)$	Idempotent law
$\Leftrightarrow r \wedge q$	Absorption law

Definition 14. (Duality)
The *dual* of a compound proposition that contains only the logical operators $\vee$, $\wedge$ and $\neg$ is the proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each T by F and each F by T. The dual of proposition A is denoted by A^*.

Example. The dual of $(T \wedge p) \vee q$ is $(F \vee p) \wedge q$

Definition 15. (Functionally complete set of connectives)
Any set of connectives in which every formula can be expressed as another equivalent formula containing connectives from this set is called *functionally complete set of connectives*.

Example. The set of connectives
$\{\vee, \neg\}$ and $\{\wedge, \neg\}$ are functionally complete.
$\{\neg\}$,$\{\vee\}$ or $\{\vee, \wedge\}$ are not functionally complete.

Problem. Prove that the set $\{\vee, \neg\}$ is functionally complete.

Solution:

To prove $\{\vee, \neg\}$ is functionally complete.
We have to show that for all formulas with other connectives their exists a equivalent formula which contains $\neg$ and $\vee$ only.

$$p \leftrightarrow q \Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$$

$$p \rightarrow q \Leftrightarrow (\neg p \vee q)$$

$$p \wedge q \Leftrightarrow \neg(\neg p \vee \neg q)$$

The resultant is free from biconditional, conditional and conjunction.
Hence $\{\vee, \neg\}$ is functionally complete.