Predicate Calculus

Consider the statement

p : x is a prime number (the statement is not a proposition)

The truth value of p depends on the value of x.
p is true when $x = 3$, and false when $x = 10$.

In this section we extend the system of logic to include such an above statements.

Definition 1. (predicates).
A *predicate* refers to a property that the subject of the statement can have. A predicate is a sentence that contains a finite number of specific values are substituted for the variables.

That is, let $P(x)$ be a statement involving variable x and a set D. We call P as a propositional function if for each x in D, $P(x)$ is a proposition.

Definition 2. (universe of discourse)
The set D is called the *domain of discourse* (or*universe of discourse*) of P. It is the set of all possible values which can be assigned to variables in statements involving predicates.

Example: Let $p(x)$ denote the statement $x \geq 4$. What are the truth values of $p(5)$ (T) and $p(2)$ (F).

Example: Let $g(x, y)$ denote the statement $g.c.d(x, y) = 1$. What are the truth values of $g(3, 5)$ (T) and $g(2, 8)$ (F)

Definition 3. (universal quantifier)
Consider the proposition
All odd prime numbers are greater than 2. The word all in this proposition is a logical quantifier. The proposition can be translated as follows:
For every x, if x is an odd prime then x is greater than 2

Similarly, the proposition:

Every rational number is a real number may be translated as.
For every x, if x is a rational number, then x is a real number.

The phrase for every x is called a *universal quantifier*.

In symbols it is denoted by $(\forall x)$ or (x).

The phrases for every x, for all x and for each x have the same meaning and we can symbolize each by (x).

If $P(x)$ denotes a predicate (propositional function), then the universal quantification for $P(x)$, is the statement.

$$(x)\ P(x) \text{ is true.}$$

Example :
(a) Let $A = \{x : \text{x is a natural number less than } 9\}$
Here $P(x)$ is the sentence x is a natural number less than 9. The common property *is a natural number less than* 9. $P(1)$ is true, therefore, $1 \in A$ and $P(12)$ is not true, therefore $12 \notin A$.

(b) The proposition $(\forall N)\,(n + 4 > 3)$ is true.
Since $\{n|n + 4 > 3\} = \{1, 2, 3, \dots\} = N$.

(c) The proposition $(\forall N)\,(n + 2 > 8)$ is false.
Since $\{n|n + 2 > 8\} = \{7, 8, 9, \dots\} \neq N$.

Definition 4. (existential quantifier).
In some situations we only require that there be at least one value for each the predicate is true. This can be done by prefixing $P(x)$ with the phrase *there exists an*. The phrase there exists an is called an *existential quantifier*.

The existential quantification for a predicate is the statement *There exists a value of* x for which $P(x)$.

The symbol, $\exists$ is used to denote the logical quantifier there exists. The phrases There exists an x, There is a x, for some x and for at least one x have the same meaning.

The existential quantifier for $P(x)$ is denoted by $(\exists\, x)\ P(x)$

Example :

(a) The proposition there is an integer between 1 and 3 may be written as
($\exists$ an integer) (the integer is between 1 and 3)

(b) The proposition $(\exists N)\,(n+4<7)$ is true.
Since $\{n|n+4<7\} = \{1,2\} \neq \phi$.

(c) The proposition $(\exists N)\,(n+6<4)$ is false.
Since $\{n|n+6<4\} = \phi$.

IV. Problems:

(i) Show that $(x)(H(x) \longrightarrow M(x)) \wedge H(a) \Longrightarrow M(a)$.

Solution:

Step 1	$(x)(H(x) \longrightarrow M(x))$	Rule P
Step 2	$H(a) \longrightarrow M(a)$	Rule US
Step 3	$H(a)$	Rule P
Step 4	$M(a)$	$\{2, 3\}$ and apply Modus Ponens

(ii) Show that $(x)(P(x) \longrightarrow Q(x)) \wedge (x)(Q(x) \longrightarrow R(x)) \Longrightarrow (x)(P(x) \longrightarrow R(x))$.

Solution:

Step 1	$(x)(P(x) \longrightarrow Q(x))$	Rule P
Step 2	$P(a) \longrightarrow Q(a)$	Rule US
Step 3	$(x)(Q(x) \longrightarrow R(x))$	Rule P
Step 4	$Q(a) \longrightarrow R(a)$	Rule US
Step 5	$P(a) \longrightarrow R(a)$	$\{2,4\}, \mathcal{I}_7$
Step 6	$(x)P(x) \longrightarrow R(x)$	Rule UG

(iii) Show that $(\exists x)(P(x) \wedge Q(x)) \implies (\exists x)P(x) \wedge (\exists x)Q(x)$.

Solution:

Step 1	$(\exists x)(P(x) \wedge Q(x))$	Rule P
Step 2	$P(a) \wedge Q(a)$	Rule ES
Step 3	$P(a)$	$\mathcal{I}_1$
Step 4	$Q(a)$	$\mathcal{I}_1$
Step 5	$(\exists x)P(x)$	$\{3\}$,EG
Step 6	$(\exists x)Q(x)$	$\{4\}$,EG
Step 7	$(\exists x)P(x) \wedge (\exists x)Q(x)$	$\{5,6\}$, $\mathcal{I}_3$

(iv) Show that $(x)(P(x) \vee Q(x)) \Longrightarrow (x)P(x) \vee (\exists x)Q(x)$.
Solution: Proof by indirect method

Step 1	$\neg((x)(P(x) \vee Q(x)))$	Rule P
Step 2	$\neg(x)P(x) \wedge \neg(\exists x)Q(x)$	Rule T
Step 3	$\neg(x)P(x)$	$\mathcal{I}_1$
Step 4	$\neg(\exists x)Q(x)$	$\mathcal{I}_1$
Step 5	$(\exists x)\neg P(x)$	3,Rule T
Step 6	$(x)\neg Q(x)$	4,Rule T
Step 7	$\neg P(a)$	5,ES
Step 8	$\neg Q(a)$	6,US

Step 9	$\neg P(a) \wedge \neg Q(a)$	$\{7,8\}, \mathcal{I}_3$
Step 10	$\neg(P(a) \vee Q(a))$	Rule T
Step 11	$(x)(P(x) \vee Q(x))$	Rule P
Step 12	$P(a) \vee Q(a)$	US
Step 13	$\neg(P(a) \vee Q(a)) \wedge (P(a) \vee Q(a))$	$\{10,12\}, \mathcal{I}_3$
Step 14	F	Rule T

(v) Show that from

(a) $(\exists x)(F(x) \wedge S(x)) \longrightarrow (y)(M(y) \longrightarrow W(y))$

(b) $(\exists y)(M(y) \wedge \neg W(y))$

the conclusion $(x)(F(x) \longrightarrow \neg S(x))$.

Solution:

Step 1	$(\exists y)(M(y) \wedge \neg W(y))$	Rule P
Step 2	$(M(a) \wedge \neg W(a))$	ES
Step 3	$\neg(M(a) \longrightarrow W(a))$	Rule T
Step 4	$(\exists y)\neg(M(y) \longrightarrow W(y))$	EG
Step 5	$\neg(y)(M(y) \longrightarrow W(y))$	Rule T

Step 6	$(\exists x)(F(x) \wedge S(x)) \longrightarrow (y)(M(y) \longrightarrow W(y))$	Rule P
Step 7	$\neg(\exists x)(F(x) \wedge S(x))$	$\{5{,}6\}$, $\mathcal{I}_6$
Step 8	$(x)\neg(F(x) \wedge S(x))$	Rule T
Step 9	$\neg(F(a) \wedge S(a))$	US
Step 10	$F(a) \longrightarrow \neg S(a)$	Rule T
Step 11	$(x)(F(x) \longrightarrow \neg S(x))$	UG