LATTICES

Definitions

A partially ordered set $\{L, \leq\}$ in which every pair of elements has a least upper bound and a greatest lower bound is called a *lattice*.

The LUB (supremum) of a subset $\{a, b\} \subseteq L$ is denoted by $a \vee b$ [or $a \oplus b$ or $a + b$ or $a \cup b$] and is called the *join* or *sum* of a and b.

The GLB (infemum) of a subset $\{a, b\} \subseteq L$ is denoted by $a \wedge b$ [or $a * b$ or $a \bullet b$ or $a \cap b$] and is called the *meet* or *product* of a and b.

Note Since the LUB and GLB of any subset of a poset are unique, both $\wedge$ and $\vee$ are binary operations on a lattice.

For example, let us consider the poset $(\{1, 2, 4, 8, 16\} |)$, where $|$ means 'divisor of'. The Hasse diagram of this poset is given in Fig. 2.26.

The LUB of any two elements of this poset is obviously the larger of them and the GLB of any two elements is the smaller of them. Hence this poset is a lattice.

Fig. 2.26

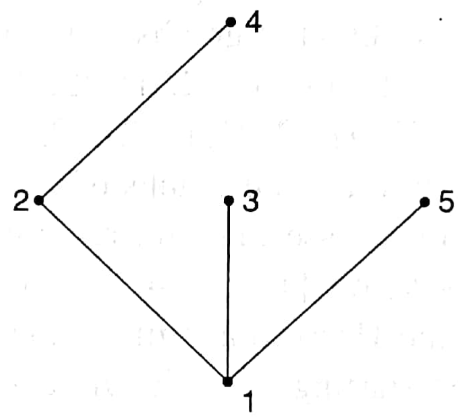


Fig. 2.27

Note All partially ordered sets are not lattices, as can be seen from the following example.

Let us consider the poset $(\{1, 2, 3, 4, 5\}, |)$ whose Hasse diagram is given in Fig. 2.27.

The LUB's of the pairs (2, 3) and (3, 5) do not exist and hence they do not have LUB. Hence this poset is not a Lattice.

PRINCIPLE OF DUALITY

When $\leq$ is a partial ordering relation on a set S, the converse $\geq$ is also a partial ordering relation on S. For example if $\leq$ denotes 'divisor of', $\geq$ denotes 'multiple of'.

The Hasse diagram of $(S, \geq)$ can be obtained from that of $(S, \leq)$ by simply turning it upside down. For example the Hasse diagram of the poset $(\{1, 2, 4, 8, 16\}$, multiple of), obtained from Fig. 2.26 will be as given in Fig. 2.28.

From this example, it is obvious that LUB(A) with respect to $\leq$ is the same as GLB(A) with respect to $\geq$ and vice versa, where $A \subseteq S$. viz. LUB and GLB are interchanged, when $\leq$ and $\geq$ are interchanged.

In the case of lattices, if $\{L, \leq\}$ is a lattice, so also is $\{L, \geq\}$. Also the operations of join and meet on $\{L, \leq\}$ become the operations of meet and join respectively on $\{L, \geq\}$.

From the above observations, the following statement, known as *the principle of duality* follows:

Any statement in respect of lattices involving the operations $\vee$ and $\wedge$ and the relations $\leq$ and $\geq$ remains true, if $\vee$ is replaced by $\wedge$ and $\wedge$ is replaced by $\vee$, $\leq$ by $\geq$ and $\geq$ by $\leq$.

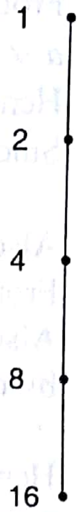


Fig. 2.28

The lattices $\{L, \leq\}$ and $\{L, \geq\}$ are called the *duals* of each other. Similarly the operations $\vee$ and $\wedge$ are duals of each other and the relations $\leq$ and $\geq$ are duals of each other.

PROPERTIES OF LATTICES

Property 1

If $\{L, \leq\}$ is a lattice, then for any $a, b, c \in L$,

L_1: $a \vee a = a$ $\quad$ $(L_1)'$: $a \wedge a = a$ $\quad$ (Idempotency)

L_2: $a \vee b = b \vee a$ $\quad$ $(L_2)'$: $a \wedge b = b \wedge a$ $\quad$ (Commutativity)

L_3: $a \vee (b \vee c) = (a \vee b) \vee c$ $\quad$ $(L_3)'$: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ $\quad$ (Associativity)

L_4: $a \vee (a \wedge b) = a$ $\quad$ $(L_4)'$: $a \wedge (a \vee b) = a$ $\quad$ (Absorption)

Proof

(i) $a \vee a = \text{LUB}(a, a) = \text{LUB}(a) = a$. Hence L_1 follows.

(ii) $a \vee b = \text{LUB}(a, b) = \text{LUB}(b, a) = b \vee a$ $\{\because$ LUB (a, b) is unique.$\}$ Hence L_2 follows.

(iii) Since $(a \vee b) \vee c$ is the LUB $\{(a \vee b), c\}$, we have

$$a \vee b \leq (a \vee b) \vee c \qquad (1)$$

and

$$c \leq (a \vee b) \vee c \qquad (2)$$

Since $a \vee b$ is the LUB $\{a, b\}$, we have

$a \leq a \vee b$ $\qquad$ (3)

and $b \leq a \vee b$ $\qquad$ (4)

From (1) and (3), $a \leq (a \vee b) \vee c$ $\qquad$ by transitivity $\quad$ (5)

From (1) and (4), $b \leq (a \vee b) \vee c$ $\qquad$ by transitivity $\quad$ (6)

From (2) and (6), $b \vee c \leq (a \vee b) \vee c$ $\qquad$ by definition of join $\quad$ (7)

From (5) and (7), $a \vee (b \vee c) \leq (a \vee b) \vee c$ $\qquad$ by definition of join $\quad$ (8)

Similarly, $a \leq a \vee (b \vee c)$ $\qquad$ (9)

$b \leq b \vee c \leq a \vee (b \vee c)$ $\qquad$ (10)

and $c \leq b \vee c \leq a \vee (b \vee c)$ $\qquad$ (11)

From (9) and (10), $a \vee b \leq a \vee (b \vee c)$ $\qquad$ (12)

From (11) and (12), $(a \vee b) \vee c \leq a \vee (b \vee c)$ $\qquad$ (13)

From (8) and (13), by antisymmetry of $\leq$, we get
$a \vee (b \vee c) = (a \vee b) \vee c.$
Hence L_3 follows.

(iv) Since $a \wedge b$ is the GLB $\{a, b\}$, we have

$$a \wedge b \leq a \quad (1)$$

Also
$$a \leq a \quad (2)$$

From (1) and (2), $a \vee (a \wedge b) \leq a$ (3)

Also
$$a \leq a \vee (a \wedge b) \quad (4)$$

by definition of LUB

$\therefore$ From (3) and (4), by antisymmetry, we get $a \vee (a \wedge b) = a$.

Hence L_4 follows.

Now the identities $(L_1)'$ to $(L_4)'$ follow from the principle of duality.

Property 2

If $\{L, \leq\}$ is a lattice in which $\vee$ and $\wedge$ denote the operations of join and meet respectively, then for $a, b \in L$,

$$a \leq b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a.$$

In other words,

(i) $a \vee b = b$, if and only if $a \leq b$.
(ii) $a \wedge b = a$, if and only if $a \leq b$.
(iii) $a \wedge b = a$, if and only if $a \vee b = b$.

Proof

(i) Let $a \leq b$.

Now $b \leq b$ (by reflexivity).

$a \leq b$

$\therefore \quad a \vee b \leq b \quad (1)$

$b \leq b$

Since $a \vee b$ is the LUB (a, b),

$a * b \leq b$

$$b \leq a \vee b \quad (2)$$

From (1) and (2), we get $a \vee b = b$ (3)

Let $a \vee b = b$.

Since $a \vee b$ is the LUB (a, b),

$$a \leq a \vee b$$

i.e., $a \leq b$, by the data (4)

From (3) and (4), result (i) follows. Result (ii) can be probed in a way similar to the proof (i).

From (i) and (ii), result (iii) follows.

Note Property (2) gives a connection between the partial ordering relation $\leq$ and the two binary operations $\vee$ and $\wedge$ in a lattice $\{L, \leq\}$.

Property 3 (Isotonic Property)

If $\{L, \leq\}$ is a lattice, then for any $a, b, c, \in L$, the following properties hold good:

If $b \leq c$, then (i) $a \vee b \leq a \vee c$ and (ii) $a \wedge b \leq a \wedge c$.

Proof

Since $b \leq c$, $b \vee c = c$, by property 2(i).

Also $a \vee a = a$, by idempotent property

Now $a \vee c = (a \vee a) \vee (b \vee c)$, by the above steps
$= a \vee (a \vee b) \vee c$, by associativity
$= a \vee (b \vee a) \vee c$, by commutativity
$= (a \vee b) \vee (a \vee c)$, by associativity

This is of the form $x \vee y = y$. $\therefore$ $x \leq y$, by property 2(i).
i.e. $a \vee b \leq a \vee c$, which is the required result (i).
Similarly, result (ii) can be proved.

Property 4 (Distributive Inequalities)

If $\{L, \leq\}$ is a lattice, then for any $a, b, c, \in L$,

(i) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

(ii) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.

Proof

Since $a \wedge b$ is the GLB(a, b), $a \wedge b \leq a$ (1)

Also $a \wedge b \leq b \leq b \vee c$ (2)

since $b \vee c$ is the LUB of b and c.

From (1) and (2), we have $a \wedge b$ is a lower bound of $\{a, b \vee c\}$

$\therefore$ $a \wedge b \leq a \wedge (b \vee c)$ (3)

Similarly $a \wedge c \leq a$

and $a \wedge c \leq c \leq b \vee c$

$\therefore$ $a \wedge c \leq a \wedge (b \vee c)$ (4)

From (3) and (4), we get

$$(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$$

i.e. $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$, which is result (i).
Result (ii) follows by the principle of duality.

Property 5 (Modular Inequality)

If $\{L, \leq\}$ is a lattice, then for any $a, b, c, \in L$, $a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$. (1)

Proof

Since $a \leq c$, $a \vee c = c$ (1), by property 2(i)

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ (2), by property 4(ii)

i.e. $a \vee (b \wedge c) \leq (a \vee b) \wedge c$ (3), by (1)

Now $a \vee (b \wedge c) \leq (a \vee b) \wedge c$

$\therefore$ $a \leq a \vee (b \wedge c) \leq (a \vee b) \wedge c \leq c$, by the definitions of LUB and GLB

i.e. $a \leq c$ (4)

From (3) and (4), we get

$$a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c.$$

LATTICE AS ALGEBRAIC SYSTEM

A set together with certain operations (rules) for combining the elements of the set to form other elements of the set is usually referred to as an *algebraic system*. Lattice L was introduced as a partially ordered set in which for every pair of elements $a, b \in L$, $\text{LUB}(a, b) = a \vee b$ and $\text{GLB}(a, b) = a \wedge b$ exist in the set. That is, in a Lattice $\{L, \leq\}$, for every pair of elements a, b of L, the two elements $a \vee b$ and $a \wedge b$ of L are obtained by means of the operations $\vee$ and $\wedge$. Due to this, the operations $\vee$ and $\wedge$ are considered as binary operations on L. Moreover we have seen that $\vee$ and $\wedge$ satisfy certain properties such as commutativity, associativity and absorption. The formal definition of a lattice as an algebraic system is given as follows:

Definition

A lattice is an algebraic system $(L, \vee, \wedge)$ with two binary operations $\vee$ and $\wedge$ on L which satisfy the commutative, associative and absorption laws.

Note We have not explicity included the idempotent law in the definition, since the absorption law implies the idempotent law as follows:

$a \vee a = a \vee [a \wedge (a \vee a)]$, by using $a \vee a$ for $a \vee b$ in $(L_4)'$ of property 1

$= a$, by using $a \vee a$ for b in L_4 of property 1.

$a \wedge a = a$ follows by duality.

Though the above definition does not assume the existence of any partial ordering on L, it is implied by the properties of the operations $\vee$ and $\wedge$ as explained below:

Let us assume that there exists a relation R on L such that for $a, b \in L$,

$$aRb \text{ if and only if } a \vee b = b$$

For any $a \in L$, $a \vee a = a$, by idempotency

$\therefore$ aRa or R is reflexive.

Now for any $a, b, \in L$, let us assume that aRb and bRa.

$\therefore$ $a \vee b = b$ and $b \vee a = a$

Since $a \vee b = b \vee a$ by commutativity, we have $a = b$ and so R is antisymmetric.

Finally let us assume that aRb and bRc

$\therefore$ $a \vee b = b$ and $b \vee c = c$.

Now $a \vee c = a \vee (b \vee c) = (a \vee b) \vee c = b \vee c = c$

viz. aRc and so R is transitive.

Hence R is a partial ordering.

Thus the two definitions given for a lattice are equivalent.

SUBLATTICES

Definition

A non-empty subset M of a lattice $\{L, \vee, \wedge\}$ is called a *sublattice* of L, iff M is closed under both the operations $\vee$ and $\wedge$. viz. if $a, b, \in M$, then $a \vee b$ and $a \wedge b$ also $\in M$.

From the definition, it is obvious that the sublattice itself is a lattice with respect to $\vee$ and $\wedge$.

For example if aRb whenever a divides b, where $a, b \in Z^+$ (the set of all positive integers) then $\{Z^+, R\}$ is a lattice in which $a \vee b = \text{LCM}(a, b)$ and $a \wedge b = \text{GCD}(a, b)$.

If $\{S_n, R\}$ is the lattice of divisors of any positive integer n, then $\{S_n, R\}$ is a sublattice of $\{Z^+, R\}$.

LATTICE HOMOMORPHISM

Definition

If $\{L_1, \vee, \wedge\}$ and $\{L_2, \oplus, *\}$ are two lattices, a mapping $f: L_1 \to L_2$ is called *a lattice homomorphism* from L_1 to L_2, if for any $a, b \in L_1$,

$$f(a \vee b) = f(a) \oplus f(b) \text{ and } f(a \wedge b) = f(a) * f(b).$$

If a homomorphism $f: L_1 \to L_2$ of two lattices $\{L_1, \vee, \wedge\}$ and $\{L_2, \oplus *\}$ is objective, i.e. one-to-one onto, then f is called an *isomorphism.* If there exists an isomorphism between two lattices, then the lattices are said to be *isomorphic.*

SOME SPECIAL LATTICES

(a) A lattice L is said to have a *lower bound* denoted by 0, if $0 \le a$ for all $a \in L$. Similarly L is said to have an *upper bound* denoted by 1, if $a \le 1$ for all $a \in L$. The lattice L is said to be *bounded,* if it has both a lower bound 0 and an upper bound 1.

The bounds 0 and 1 of a lattice $\{L, \vee, \wedge, 0, 1\}$ satisfy the following identities, which are seen to be true by the meanings of $\vee$ and $\wedge$.

For any $a \in L$, $a \vee 1 = 1$; $a \wedge 1 = a$ and $a \vee 0 = a$; $a \wedge 0 = 0$.

Since $a \vee 0 = a$ and $a \wedge 1 = a$, 0 is the identity of the operation $\vee$ and 1 is the identity of the operation $\wedge$.

Since $a \vee 1 = 1$ and $a \wedge 0 = 0$, 1 and 0 are the zeros of the operations $\vee$ and $\wedge$ respectively.

Note 1 If we treat 1 and 0 as duals of each other in a bounded lattice, the principle of duality can be extended to include the interchange of 0 and 1. Thus the identities $a \vee 1 = 1$ and $a \wedge 0 = 0$ are duals of each other; so also are $a \vee 0 = a$ and $a \wedge 1 = a$.

Note 2 If $L = \{a_1, a_2, \ldots, a_n\}$ is a finite lattice, then $a_1 \vee a_2 \vee a_3 \ldots \vee a_n$ and $a_1 \wedge a_2 \wedge a_3 \wedge \ldots \wedge a_n$ are upper and lower bounds of L respectively and hence we conclude that every finite lattice is bounded.

(ii) A lattice $\{L, \vee, \wedge\}$ is called a *distributive lattice*, if for any elements $a, b, c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \text{ and}$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

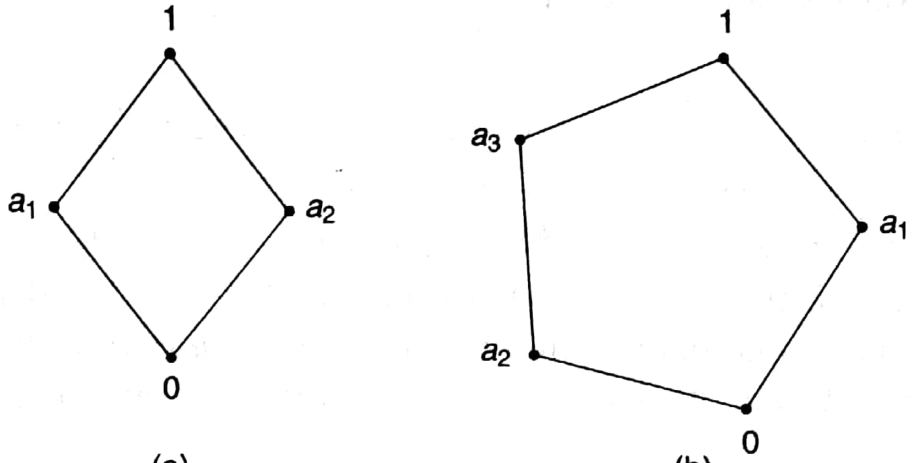
In other words if the operations $\vee$ and $\wedge$ distribute over each other in a lattice, it is said to be distributive. Otherwise it is said to be *non distributive.*

(iii) If $\{L, \vee, \wedge, 0, 1\}$ is a bounded lattice and $a \in L$, then an element $b \in L$ is called a *complement* of a, if

$$a \vee b = 1 \text{ and } a \wedge b = 0$$

Since $0 \vee 1 = 1$ and $0 \wedge 1 = 0$, 0 and 1 are complements of each other. When $a \vee b = 1$, we know that $b \vee a = 1$ and when $a \wedge b = 0$, $b \wedge a = 0$. Hence when b is the complement of a, a is the complement of b.

An element $a \in L$ may have no complement. Similarly an element, other than 0 and 1, may have more than one complement in L as seen from Fig. 2.28.



(a) (b)

Fig. 2.28

In Fig. 2.28(a), complement of a_1 is a_2, whereas in (b), complement of a_1 is a_2 and a_3. It is to be noted that 1 is the only complement of 0. If possible, let $x \neq 1$ be another complement of 0, where $x \in L$.

Then $\quad 0 \vee x = 1 \quad$ and $\quad 0 \wedge x = 0$

But $\quad 0 \vee x = x \quad \therefore \quad x = 1$, which contradicts the assumption $x \neq 1$.

Similarly we can prove that 0 is the only complement of 1.

Now a lattice $\{L, \vee, \wedge, 0, 1\}$ is called a *complemented lattice* if every element of L has at least one complement.

The following property holds good for a distributive lattice.

Property

In a distributive lattice $\{L, \vee, \wedge\}$ if an element $a \in L$ has a complement, then it is unique.

Proof

If possible, let b and c be the complements of $a \in L$.

Then $\quad a \vee b = a \vee c = 1 \quad (1)$

and $\quad a \wedge b = a \wedge c = 0 \quad (2)$

Now
$$
\begin{aligned}
b &= b \vee 0 = b \vee (a \wedge c), \text{ by (2)} \\
&= (b \vee a) \wedge (b \vee c), \text{ since } L \text{ is distributive} \\
&= 1 \wedge (b \vee c), \text{ by (1)} \\
&= b \vee c \qquad (3)
\end{aligned}
$$

Similarly,
$$
\begin{aligned}
c &= c \vee 0 = c \vee (a \wedge b), \text{ by (2)} \\
&= (c \vee a) \wedge (c \vee b), \text{ since } L \text{ is distributive} \\
&= 1 \wedge (c \vee b), \text{ by (1)} \\
&= c \vee b \qquad (4)
\end{aligned}
$$

From (3) and (4), since $b \vee c = c \vee b$, we get $b = c$.

[redacted] From the definition of complemented lattice and the previous property, it follows that every element a of a complemented and distributive lattice has a unique complement denoted by a'.

Example 2.1 Determine whether the posets represented by the Hasse diagrams given in Fig. 2.40 are lattices.

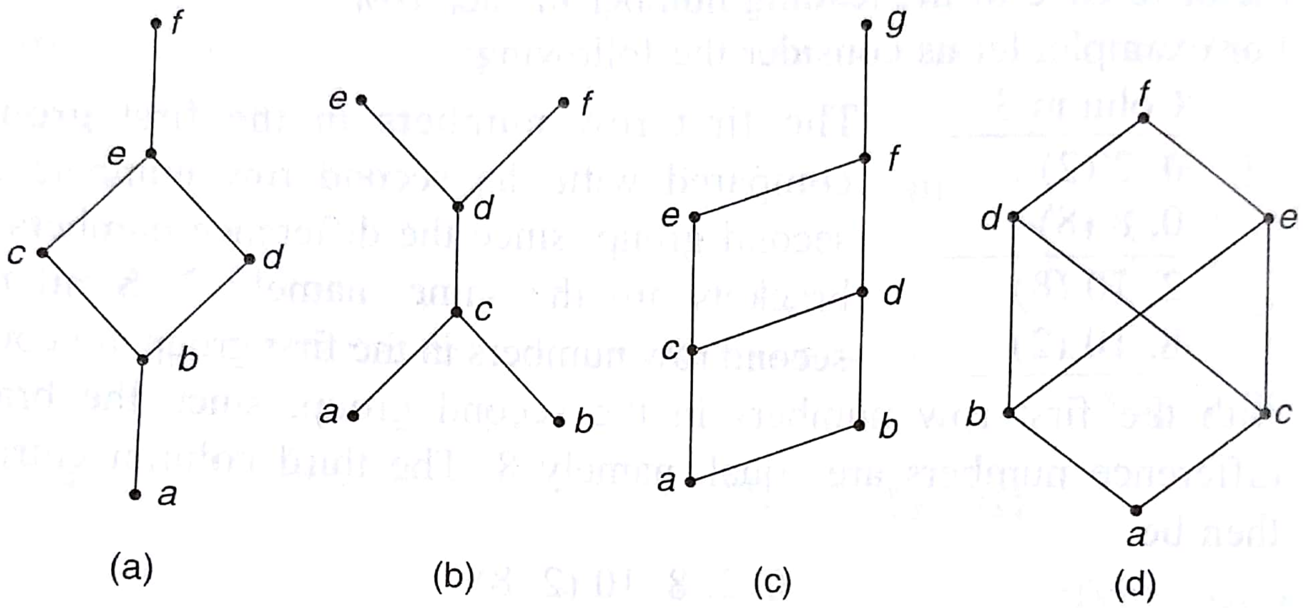


Fig. 2.40

(a) The poset represented by the Hasse diagram in Fig. 2.40(a) is a lattice, since every pair of elements of this poset has both an LUB and a GLB.

(b) The pair of elements a, b does not have a GLB and the pair e, f does not have an LUB. Hence the poset in Fig. 2.40(b) is not a lattice.

(c) Since every pair of elements of the poset in Fig. 2.40(c) has both an LUB and a GLB, it is a lattice.

(d) Though the pair of elements $\{b, c\}$ has 3 upper bounds d, e, f, none of these precedes the other two i.e. $\{b, c\}$ does not have an LUB. Hence the poset in Fig. 2.40(d) is not a lattice.

Example 2.2 If $P(S)$ is the power set of *a* set S and $\cup$ and $\cap$ are taken as the join and meet, prove that $\{P(S), \subseteq\}$ is a lattice.

Let A and B be any two elements of $P(S)$, i.e. any two subsets of S.

Then an upper bound of $\{A, B\}$ is a subset of S that contains both A and B and the least among them is $A \cup B \in P(S)$, as can be seen from the following:

We know $A \subseteq A \cup B$ and $B \subseteq A \cup B$. i.e. $A \cup B$ is an upper bound of $\{A, B\}$. If we assume that $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

Thus the LUB $\{A, B\} = A \cup B$.

Similarly $A \cap B \subseteq A$ and $A \cap B \subseteq B$

i.e. $A \cap B$ is a lower bound of $\{A, B\}$.

If we assume that $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$.

Thus the GLB $\{A, B\} = A \cap B$.

i.e. every pair of elements of $P(S)$ has both an LUB and a GLB under set inclusion relation.

Hence $\{P(S), \subseteq\}$ is a lattice.

Note Refer to the example 20 of the previous section in which the Hasse diagram of $\{P(S), \subseteq\}$, where $S \equiv \{a, b, c\}$ is given.

Example 2.3 If L is the collection of 12 partitions of $S = \{1, 2, 3, 4\}$ ordered such that $P_i \leq P_j$ if each block of P_i is a subset of a block P_j, show that L is a bounded lattice and draw its Hasse diagram.

The 12 partitions of $S = \{1, 2, 3, 4\}$ are

$P_1 = \{(1), (2), (3), (4)\}$ i.e. $[1, 2, 3, 4]$, $P_2 = \{(1, 2), (3), (4)\}$ i.e. $[12, 3, 4]$
$P_3 = [13, 2, 4]$, $P_4 = [14, 2, 3]$, $P_5 = [23, 1, 4]$, $P_6 = [24, 1, 3]$, $P_7 = [34, 1, 2]$,
$P_8 = [123, 4]$, $P_9 = [124, 3]$, $P_{10} = [134, 2]$, $P_{11} = [234, 1]$ and $P_{12} = [1234]$.

Using the ordering relation, the Hasse diagram of L has been drawn as in Fig. 2.41.

Since $P_1 \leq P_j$, for $j = 2, 3, \ldots, 12$, P_1 is a lower bound of the lattice.

Similarly since $P_j \leq P_{12}$ for $j = 1, 2, \ldots, 11$, P_{12} is an upper bound of the lattice.

Since L has both a lower bound and an upper bound, it is a bounded lattice.

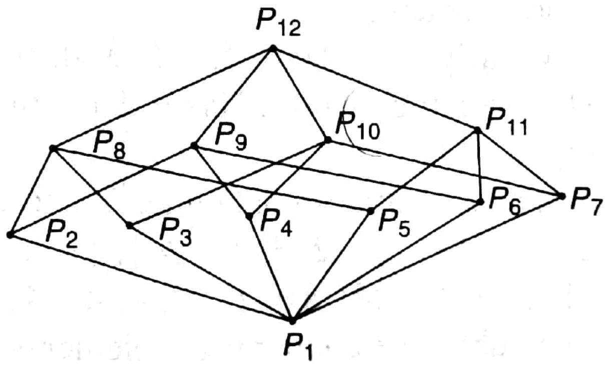


Fig. 2.41

Example 2.4 Draw the Hasse diagram of the lattice $\{P(S), \subseteq\}$ in which the join and meet are the operations $\cup$ and $\cap$ respectively, where $S = \{a, b, c\}$.

Identify *a* sublattice of this lattice with 4 elements and a subset of this lattice with 4 elements which is not a sublattice.

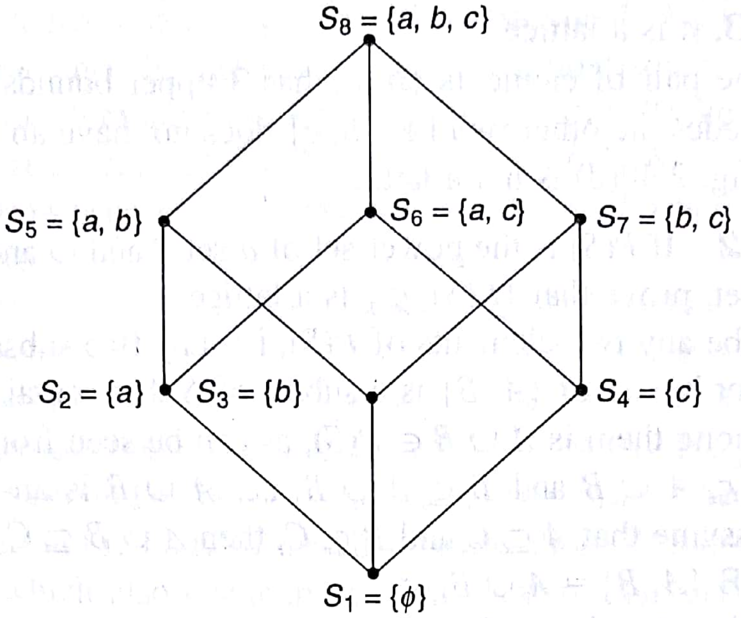


Fig. 2.42

$L_1 = \{S_1, S_2, S_4, S_6\}$ is a sublattice of L, by the argument given below:

$$S_1 \cup S_2 = S_2 \in L_1,\ S_1 \cup S_4 = S_4 \in L_1,\ S_1 \cup S_6 = S_6 \in L_1,$$
$$S_2 \cup S_4 = S_6 \in L_1,\ S_2 \cup S_6 = S_6 \in L_1 \text{ and } S_4 \cup S_6 = S_6 \in L_1$$

Thus L_1 is closed under the operation $\cup$.

Now $$S_1 \cap S_2 = S_1 \in L_1,\ S_1 \cap S_4 = S_1 \in L_1,\ S_1 \cap S_6 = S_1 \in L_1,$$
$$S_2 \cap S_4 = S_1 \in L_1,\ S_2 \cap S_6 = S_2 \in L_1,\ S_4 \cap S_6 = S_4 \in L_1.$$

Thus L_1 is closed under the operation $\cap$.

Let us now consider $L_2 = \{S_1, S_5, S_7, S_8\}$.

$S_5 \cap S_7 = b = S_3 \notin L_2$. Hence L_2 is not a sublattice of L.

Example 2.5 If S_n is the set of all divisors of the positive integer n and D is the relation of 'division', viz., aDb if and only if a divides b, prove that $\{S_{24}, D\}$ is a lattice. Find also all the sublattices of D_{24} [$= \{S_{24}, D\}$] that contain 5 or more elements.

Clearly $\{S_{24}, D\} = \{(1, 2, 3, 4, 6, 8, 12, 24), D\}$ is a lattice whose Hasse diagram is given in Fig. (2.43).

The sublattices containing 5 elements are $\{1, 2, 3, 6, 12\}$, $\{1, 2, 3, 12, 24\}$, $\{1, 2, 6, 12, 24\}$, $\{1, 3, 6, 12, 24\}$ and $\{1, 2, 4, 8, 24\}$

The sublattice containing 6 elements is $\{1, 2, 3, 6, 12, 24\}$.

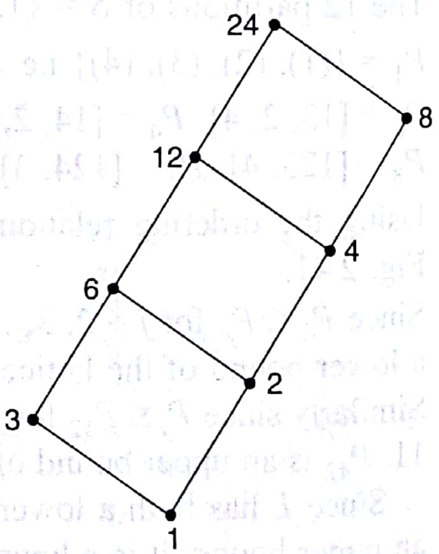


Fig. 2.43

Example 2.6 If a and b are elements of a lattice L such that $a \le b$ and if the interval $[a, b]$ is defined as the set of all $x \in L$ such that $a \le x \le b$, show that $[a, b]$ is a sublattice of L.

Let x, y be in $[a, b]$. Then $x, y \in L$.

$\therefore$ $x \vee y$ and $x \wedge y \in L$, since L is a lattice.

Now $a \le x \le x \vee y \le b$ $\quad \therefore x \vee y \in [a, b]$

Also $a \le x \wedge y \le x \le b$ $\quad \therefore x \wedge y \in [a, b]$

Hence $[a, b]$ is a sublattice.

Example 2.7 Verify whether the lattice given by the Hasse diagram in Fig. 2.44 is distributive.

$a \wedge (b \vee c) = a \wedge b = 0$

Also $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$

$\therefore$ $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (1)

Now $c \wedge (a \vee b) = c \wedge 1 = c$

Also $(c \wedge a) \vee (c \wedge b) = 0 \vee c = c$

$\therefore$ $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$ (2)

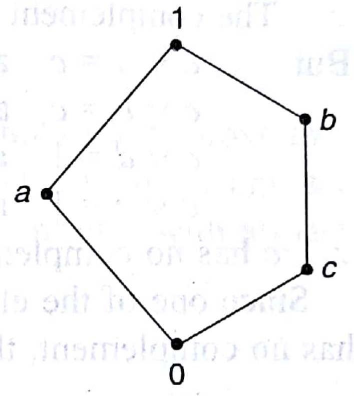


Fig. 2.44

Steps (1) and (2) do not mean that the lattice is distributive.

Now let us consider

$b \wedge (c \vee a) = b \wedge 1 = b$

But $(b \wedge c) \vee (b \wedge a) = c \vee 0 = c$

This means that $b \wedge (c \vee A) \ne (b \wedge c) \vee (b \wedge a)$

Hence the given lattice is not distributive.

Example 2.8 Prove that $D_{42} \equiv \{S_{42}, D\}$ is a complemented lattice by finding the complements of all the elements.

$$D_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$$

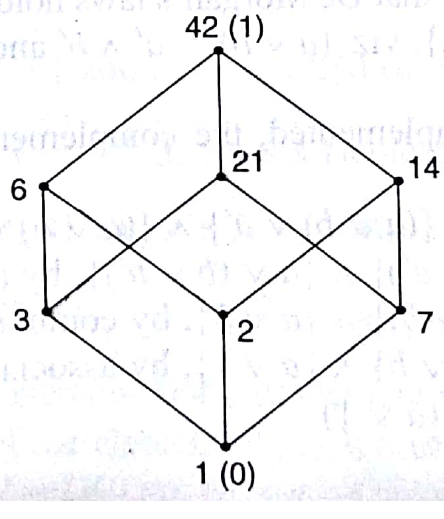
The Hasse diagram of D_{42} is given in Fig. 2.45.

The zero element of the lattice is 1 and the unit element of the lattice is 42.

$1 \vee 42 = \text{LCM}\{1, 42\} = 42 \equiv \text{‘1’}$

and $1 \wedge 42 = \text{GCD}\{1, 42\} = 1 \equiv \text{‘0’}$

$\therefore$ $1' = 42$



Similarly we can find that $2' = 21$, $3' = 14$, $6' = 7$, $7' = 6$, $14' = 3$, $21' = 2$ and $42' = 1$.

Since every element of D_{42} has a complements, it is a complemented lattice.

Example 2.9 Find the complements, if they exist, of the elements a, b, c of the lattice, whose Hasse diagram is given in Fig. 2.46. Can the lattice be complemented?

From the Hasse diagram, it is seen that $a \vee e = 1$ and $a \wedge e = 0$.

$\therefore$ The complement of a is e.

Similarly $b \vee d = 1$ and $b \wedge d = 0$

$\therefore$ The complement of b is d.

But $\quad c \vee a = c$ and $c \wedge a = a$

$\quad c \vee b = c$ and $c \wedge b = b$

$\quad c \vee d = 1$ and $c \wedge d = a$

$\quad c \vee e = 1$ and $c \wedge e = b$

$\therefore$ c has no complement.

Since one of the elements of the lattice, namely c has no complement, the lattice is not complemented.

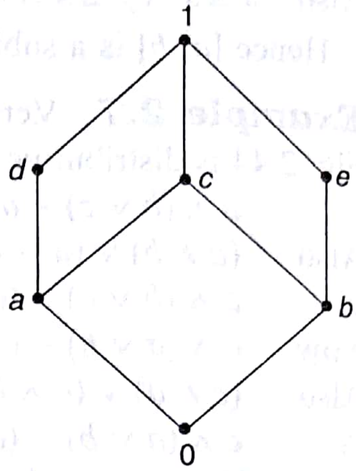


Fig. 2.46

Example 2.10 Prove that cancellation law holds good in a distributive lattice, viz. if $\{L, \vee, \wedge\}$ is a distributive lattice such that $a \vee b = a \vee c$ and $a \wedge b = a \wedge c$, where $a, b, c \in L$, then $b = c$.

$$\begin{aligned}
c \wedge (a \vee b) &= (c \wedge a) \vee (c \wedge b), \text{ since } L \text{ is distributive}\\
&= (a \wedge c) \vee (c \wedge b), \text{ by commutativity}\\
&= (a \wedge b) \vee (c \wedge b), \text{ given}\\
&= (b \wedge a) \vee (b \wedge c), \text{ by commutativity}\\
&= b \wedge (a \vee b), \text{ by distributivity}\\
&= b \wedge (b \vee a), \text{ by commutativity}\\
&= b, \text{ by absorption law} \qquad (1)
\end{aligned}$$

Also

$$\begin{aligned}
c \wedge (a \vee b) &= c \wedge (a \vee c), \text{ given}\\
&= c \wedge (c \vee a), \text{ by commutativity}\\
&= c, \text{ by absorption law} \qquad (2)
\end{aligned}$$

From (1) and (2), it follows that $b = c$.

Example 2.11 Prove that De Morgan's laws hold good for a complemented distributive lattice $\{L, \vee, \wedge\}$, viz. $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' \vee b'$, where $a, b \in L$.

Since the lattice is complemented, the complements of a and b exist. Let them be a' and b'.

Now

$$\begin{aligned}
(a \vee b) \vee (a' \wedge b') &= \{(a \vee b) \vee a'\} \wedge \{(a \vee b) \vee b'\}, \text{ by distributivity}\\
&= \{a \vee (b \vee a')\} \wedge \{a \vee (b \vee b')\}, \text{ by associativity}\\
&= \{a \vee (a' \vee b)\} \wedge \{a \vee 1\}, \text{ by commutativity}\\
&= \{(a \vee a') \vee b\} \wedge \{a \vee 1\}, \text{ by associativity}\\
&= (1 \vee b) \wedge (a \vee 1)\\
&= 1 \wedge 1 \qquad (1)\\
&= 1
\end{aligned}$$

$$
\begin{aligned}
(a \vee b) \wedge (a' \wedge b') &= \{a \wedge (a' \wedge b')\} \vee \{b \wedge (a' \wedge b')\}, \text{ by distributivity} \\
&= \{(a \wedge a') \wedge b'\} \vee \{b \wedge (b' \wedge a')\}, \\
&\qquad \text{by associativity and commutativity} \\
&= \{(a \wedge a') \wedge b'\} \vee \{(b \wedge b') \wedge a'\}, \text{ by associativity} \\
&= (0 \wedge b') \vee (0 \wedge a') \\
&= 0 \vee 0 \\
&= 0 \qquad (2)
\end{aligned}
$$

From (1) and (2), we get

$a' \wedge b'$ is the complement of $a \vee b$

or
$$(a \vee b)' = a' \wedge b' \qquad (3)$$

By principle of duality, it follows from (3) that
$$(a \wedge b)' = a' \vee b'.$$

Example 2.12 If $P(S)$ is the power set of a non-empty set S, prove that $\{P(S), \cup, \cap, \backslash, \phi, S\}$ is a Boolean algebra, where the complement of any set $A \subseteq S$ is taken as $S \backslash A$ or $S - A$ that is the relative complement of A with respect to S.

Let X, Y and Z be any three elements of $P(S)$.

Now $X \cup \phi = X$ and $X \cap S = X$

Thus ϕ and S play the roles of 0 and 1 and the identity laws are satisfied (1)

$$X \cup Y = Y \cup X \text{ and } X \cap Y = Y \cap X$$

i.e. the commutative laws are satisfied (2)

$$(X \cup Y) \cup Z = X \cup (Y \cup Z) \text{ and } (X \cap Y) \cap Z = X \cap (Y \cap Z)$$

i.e. the associative laws hold good (3)

$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z) \text{ and}$$
$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

i.e. the distributive laws hold good (4)

$$X \cup (S - X) = S \text{ and } X \cap (S - X) = \phi$$

i.e. the complement laws hold good (5)

Thus all the 5 axioms of Boolean algebra hold good.

Hence $\{P(S), \cup, \cap, \backslash, \phi, S\}$ is a Boolean algebra.

Example 2.13

(i) If $a, b \in S = \{1, 2, 3, 6\}$ and $a + b = \text{LCM}(a, b)$, $a \cdot b = \text{GCD}(a, b)$ and $a' = \dfrac{6}{a}$, show that $\{S, +, \cdot, ', 1, 6\}$ is a Boolean algebra.

(ii) If $a, b \in S = \{1, 2, 4, 8\}$ and $a + b = \text{LCM}(a, b)$, $a, b = \text{GCD}(a, b)$ and $a' = \dfrac{8}{a}$, show that $\{S, +, \cdot, ', 1, 8\}$ is not a Boolean algebra.

(i) 1 and 6 are the zero element and unit element of $\{S, +, \cdot, ', 1, 6\}$

If a represents any of the elements 1, 2, 3, 6 of S, clearly a + '0' = LCM $(a, 1) = a$ and $a \cdot$ '1' = GCD$(a, 6) = a$

i.e. identity laws hold good.