Module 3 -Counting Techniques

S. Devi Yamini

Overview

1. Basics of counting
   - Sum Rule
   - Product Rule

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2. Inclusion and Exclusion principle

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   - Product Rule
2. Inclusion and Exclusion principle
3. Pigeonhole principle
4. Recurrence Relation

Problems based on counting

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8 men and 6 women contest in an election. In how many ways can the people choose a leader? $8+6=14$ ways

Sum Rule

If the first task can be done in m ways, and the second task in n ways, and if these tasks cannot be done at the same time, then there are $m + n$ ways to do either task.
(OR)
If $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$

Product Rule

Suppose that a procedure can be broken down into two tasks. If there are m ways to do the first task and n ways to do the second task after the first task has been done, then there are mn ways to do the procedure.
(OR)
For any choice of sets A and B, $|A \text{ and } B| = |A||B|$

Problems

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3x4x2 = 24 choices

More Problems

How many positive divisors does 2000 have?

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$$2000 = 2^4 \times 5^3$$

Hence a divisor of 2000 will be of the form $2^a \times 5^b$
where $a, b \in \mathbb{Z}$ and $1 \leq a \leq 4$, $1 \leq b \leq 3$
a has 5 possibilities and b has 4 possibilities
By Product rule,
5x4 $=$ 20 divisors

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How many different bit strings are there of length seven?
Each bit has two possibilities 0 or 1.
By product rule, there are $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 128$ different bit strings of length seven.

More problems

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n^m
How many 1-1 functions are there from a set with m elements to a set with n elements?
$n(n-1)(n-2)\ldots(n-(m-1))$

Problems on combining sum and product rule

Suppose statement labels in a programming language can be either a single uppercase letter or a lowercase letter followed by a digit. Find the number of possible labels.

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Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

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Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?
Length of the password - 6 or 7 or 8
Number of passwords of length 6 $P_6 = 36^6 - 26^6$
Number of passwords of length 7 $P_7 = 36^7 - 26^7$
Number of passwords of length 8 $P_8 = 36^8 - 26^8$
Total number of passwords $P_6 + P_7 + P_8$

Principle of inclusion-exclusion

If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.
OR

$$|A \cup B| = |A| + |B| - |A \cap B|$$

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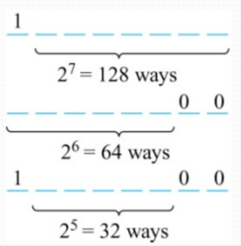

$128 + 64 - 32 = 160$

Problems

How many positive integers less than 100 is not divisible by (factor of) 2,3, and 5?

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Let $A =$ set of elements which are divisible by 2
$B =$ set of elements which are divisible by 3
$C =$ set of elements which are divisible by 5

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Number of positive integers less than 100 which are divisible by 2 or 3 or 5 is

$$= \lfloor \frac{99}{2} \rfloor + \lfloor \frac{99}{3} \rfloor + \lfloor \frac{99}{5} \rfloor - \lfloor \frac{99}{6} \rfloor - \lfloor \frac{99}{15} \rfloor - \lfloor \frac{99}{10} \rfloor + \lfloor \frac{99}{30} \rfloor$$
$$= 49 + 33 + 19 - 16 - 9 - 6 + 3 = 73$$

More problems

1. How many strings of five ASCII characters contain the character @ at least once?
2. How many strings of eight English letters are there
   (a) that contain no vowels, if letters can be repeated?
   (b) that contain no vowels, if letters cannot be repeated?
   (c) that start with a vowel, if letters can be repeated?
   (d) that start with a vowel, if letters cannot be repeated?
   (e) that contain at least one vowel, if letters can be repeated?
   (f) that contain exactly one vowel , if letters can be repeated?
   (g) that start with X and contain at least one vowel, if letters can be repeated?
   (h) that start and end with X and contain at least one vowel, if letters can be repeated?
3. How many subsets of a set with 100 elements have more than one element?
4. How many positive integers not exceeding 100 are divisible either by 4 or by 6?

Pigeonhole principle

If there are $n-1$ pigeonholes and n pigeons, then there is at least one pigeonhole which has more than one pigeon.

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Example

In any group of 27 English words, there must be at least two that begin with the same letter, since there are 26 letters in the English alphabet.

Generalized Pigeonhole principle

If N objects are placed in k boxes $(N > k)$, then there is at least one box containing at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects.

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Example

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades A, B, C, D, and F?

The minimum number of students needed to guarantee that at least six students receive the same grade is the smallest integer N such that $\lceil \frac{N}{5} \rceil = 6$. Hence, $N = 26$

More problems

1. Show that in any set of six classes there must be two that meet on the same day, assuming that no classes are held on weekends.

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   Six classes (Pigeons), 5 days (Pigeonholes). Each class must meet on a day. By PHP, at least one day must contains at least two classes.
2. Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

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   The remainders when divided by 4 are 0, 1, 2, and 3. By PHP, at least two of the five given remainders must be same.
3. Ten people are swimming in the lake. Prove that at least two of them were born on the same day of the week
   People - Pigeons and days of the week - pigeonholes.
   7 days in a week and 10 people. By PHP, at least two of them were born on the same day of the week.

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   $n_1 = 3k_1 + r, n_2 = 3k_2 + r$
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2. Twenty-five crates of apples are delivered to a store. The apples are of three different sorts, and all the apples in each crate are of the same sort. Show that among these crates there are at least nine containing the same sort of apples.

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   People - pigeons, number of friends -pigeonholes

Problems on PHP

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   $N = 50 \times 99 + 1 = 4951$
2. How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs $(a_1, b_1), (a_2, b_2)$ such that $a_1 mod 5 = a_2 mod 5$ and $b_1 mod 5 = b_2 mod 5$?

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   $N = 5 \times 1 = 6$

Recurrence relation

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Example

1. $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$
2. $a_n = a_{n-5}$
3. $a_n = a_{n-1} + a_{n-2}^2$
4. $H_n = 2H_{n-1} + 1$
5. $F_n = F_{n-1} + F_{n-2}$

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Order

Difference between greatest suffix and the least suffix.

Linear recurrence relation with constant coefficients

General equation

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} = f(n)$$

If $f(n) = 0$, then the above equation represents a linear homogeneous recurrence relation with constant coefficients; otherwise non-homogeneous.

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Example

1. $$a_n + 3a_{n-1} + 2a_{n-2} = 0$$

   is a homogeneous linear recurrence relation.

2. $$a_n + 3a_{n-1} + 2a_{n-2} = n^2$$

   is a non homogeneous linear recurrence relation.

Homogeneous linear recurrence relation of order at most 2

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} = 0$$

Procedure for finding the solution:

1. Let $a_n = x^n$
2. $c_0 x^n + c_1 x^{n-1} + c_2 x^{n-2} = 0$ is the auxillary equation.
3. Dividing throughout by x^{n-2},

$$c_0 x^2 + c_1 x + c_2 = 0$$

4. Solve for x and let x_1 and x_2 be the roots
   1. If the roots are real and distinct, then $a_n = Ax_1^n + Bx_2^n$
   2. If the roots are real and equal, then $a_n = (A + nB)x^n$
   3. If the roots are complex $\alpha + i\beta$, then $a_n = r^n(Acosn\theta + Bsinn\theta)$ where $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = tan^{-}1(\frac{\beta}{\alpha})$

Problems

Solve the following homogeneous linear recurrence relations:

1. $x_n = 5x_{n-1}, x_1 = 3$
2. $a_n = 5a_{n-1} - 6a_{n-2}$
3. $a_n = 6a_{n-1} - 9a_{n-2}$
4. $a_n = 2a_{n-1} + 15a_{n-2}, a_1 = 2, a_2 = 4$
5. $x_n = 4x_{n-1} + 5x_{n-2}, x_1 = 3, x_2 = 5$

Non homogeneous linear recurrence relation

The general non homogeneous linear recurrence relation of order 2 is

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} = f(n)$$

The general solution is

$$a_n = a_n^h + a_n^p$$

where a_n^h is the homogeneous solution and a_n^p is the particular solution.

Case 1: $f(n)$ =constant

Substitute $a_n = k$ (some constant) in the recurrence relation and find the value of k

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Problem

Solve:

1. $a_{n+2} - 5a_{n+1} + 6a_n = 2$ where $a_0 = 1, a_1 = -1$

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$a_n = (-2)^n 3^n + 2^{n+1} + 1$

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$a_n = (-2)^n 3^n + 2^{n+1} + 1$
2. $a_n - 6a_{n-1} + 8a_{n-2} = 3$
3. $a_n - 5a_{n-1} + 6a_{n-2} = 1$

Non homogeneous linear recurrence relation

Case 2: $f(n) = b^n$ where b is a constant

Assume

$$a_n = \begin{cases} A_0 b^n & \text{if b is not a root} \\ A_0 n b^n & \text{if b is a root of multiplicity one} \\ A_0 n^2 b^n & \text{if b is a double root} \end{cases} \tag{1}$$

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Solve the following:

- $a_n - 7a_{n-1} + 12a_{n-2} = 2^n$
- $a_n - 7a_{n-1} + 12a_{n-2} = 4^n$
- $a_{n+2} - 4a_{n+1} + 4a_n = 2^n$

Case 3: $f(n) = P(n)b^n$ where b is a constant, $P(n)$ is a polynomial of degree k

Assume

$$a_n = \begin{cases} P_1(n)b^n & \text{if b is not a root} \\ nP_1(n)b^n & \text{if b is a root of multiplicity one} \\ n^2P_1(n)b^n & \text{if b is a double root} \end{cases} \tag{2}$$

where $P_1(n)$ is the general polynomial of degree k.

Non homogeneous linear recurrence relation

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Solve the following:

- $a_n - 7a_{n-1} + 12a_{n-2} = n$
- $a_n - 3a_{n-1} + 2a_{n-2} = n^2$
- $a_n - 7a_{n-1} + 12a_{n-2} = n4^n$
- $a_{n+2} - 4a_{n+1} + 4a_n = n2^n$

Introduction

- Developed by English Mathematician *George Boole* in between 1815 - 1864.
- It is described as *an algebra of logic or an algebra of two values i.e True or False.*
- The term *logic* means a statement having binary decisions i.e *True/Yes* or *False/No*.

Application of Boolean algebra

- It is used to perform the logical operations in digital computer.
- In digital computer True represent by '1' (high volt) and False represent by '0' (low volt)
- Logical operations are performed by logical operators. The fundamental logical operators are:
  1. AND (conjunction)
  2. OR (disjunction)
  3. NOT (negation/complement)

AND operator

- It performs logical multiplication and denoted by (.) dot.

X	Y	X.Y
0	0	0
0	1	0
1	0	0
1	1	1

OR operator

- It performs logical addition and denoted by (+) plus.

X	Y	X+Y
0	0	0
0	1	1
1	0	1
1	1	1

NOT operator

- It performs logical negation and denoted by (-) bar. It operates on single variable.

X	$\overline{X}$	(means complement of x)
0	1	
1	0	

NAND operator

- It performs logical multiplication and then negates the overall output. It denoted by (-) bar over the (.)dot product of the inputs.

2 Input NAND gate		
A	B	$\overline{A.B}$
0	0	1
0	1	1
1	0	1
1	1	0

NOR operator

- It performs logical addition and then negates the overall output. It denoted by (-) bar over the (+)addition of the inputs.

2 Input NOR gate		
A	B	$\overline{A+B}$
0	0	1
0	1	0
1	0	0
1	1	0

XOR operator

- It performs logical operations and gives an output when the inputs are exclusively different from each other. It is denoted by a special sign as seen below.

INPUTS		OUTPUTS
A	B	$Y = A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

XNOR operator

- It performs logical operations and negates the outputs of XOR gate. Basically, it gives us the exact opposite output as compared to XOR gate.

INPUTS		OUTPUT
A	B	$Y = \overline{A \oplus B}$
0	0	1
0	1	0
1	0	0
1	1	1

Truth Table

- Truth table is a table that contains all possible values of logical variables/statements in a Boolean expression.

No. of possible combination =

2^n, where n=number of variables used in a Boolean expression.

Truth Table

- The truth table for XY + Z is as follows:

Dec	X	Y	Z	XY	XY+Z
0	0	0	0	0	0
1	0	0	1	0	1
2	0	1	0	0	0
3	0	1	1	0	1
4	1	0	0	0	0
5	1	0	1	0	1
6	1	1	0	1	1
7	1	1	1	1	1

Tautology & Fallacy

- If the output of Booean expression is always True or 1 is called Tautology.
- If the output of Boolean expression is always False or 0 is called Fallacy.

P	P′	output (PVP′)	output (PΛP′)
0	1	1	0
1	0	1	0

PVP′ is Tautology and PΛP′ is Fallacy

Implementation

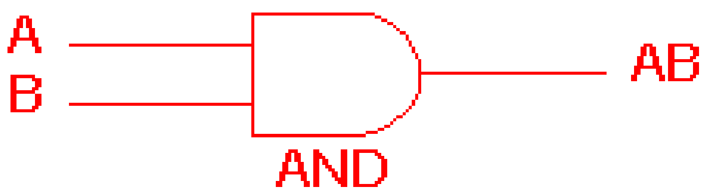
- Boolean Algebra applied in computers electronic circuits. These circuits perform Boolean operations and these are called logic circuits or logic gates.

Logic Gate

- A gate is an digital circuit which operates on one or more signals and produce single output.
- Gates are digital circuits because the input and output signals are denoted by either 1(high voltage) or 0(low voltage).
- Three type of gates are as under:
    1. AND gate
    2. OR gate
    3. NOT gate

AND gate

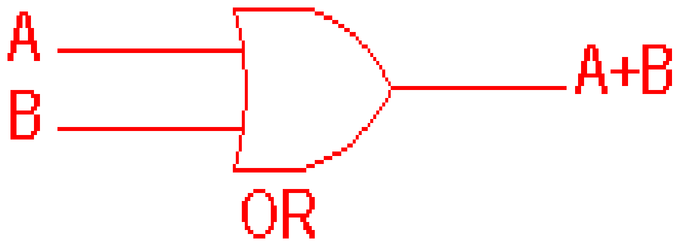
- The AND gate is an electronic circuit that gives a high output (1) only if all its inputs are high.
- AND gate takes two or more input signals and produce only one output signal.



Input A	*Input B*	*Output AB*
0	0	0
0	1	0
1	0	0
1	1	1

OR gate

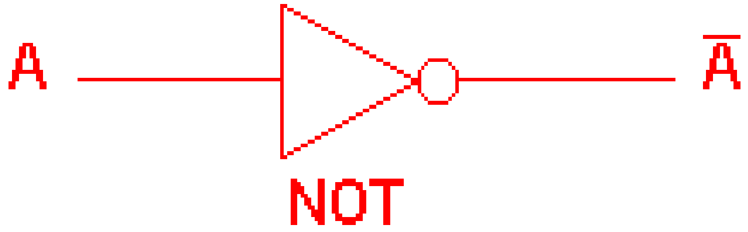
- The OR gate is an electronic circuit that gives a high output (1) if one or more of its inputs are high.
- OR gate also takes two or more input signals and produce only one output signal.



Input A	*Input B*	*Output A+B*
0	0	0
0	1	1
1	0	1
1	1	1

NOT gate

- The NOT gate is an electronic circuit that gives a high output (1) if its input is low .
- NOT gate takes only one input signal and produce only one output signal.
- The output of NOT gate is complement of its input.
- It is also called inverter.

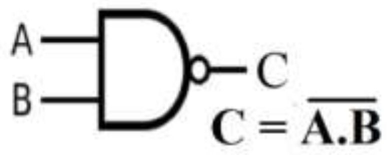


Input A	*Output $\overline{A}$*
0	1
1	0

NAND gate

- NAND gate is an electronic gate which gives an output when both of the inputs are not high(1).
- This gate is the inverse of the AND gate studied before.

NAND GATE



Truth Table

INPUT		OUTPUT
A	B	A NAND B
0	0	1
0	1	1
1	0	1
1	1	0

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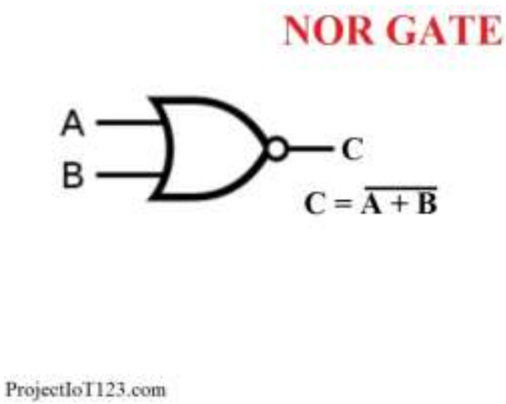
NOR gate

- NOR gate is an electronic gate which gives an output when neither of the inputs are high(1).
- This gate is the inverse of the OR gate studied before.

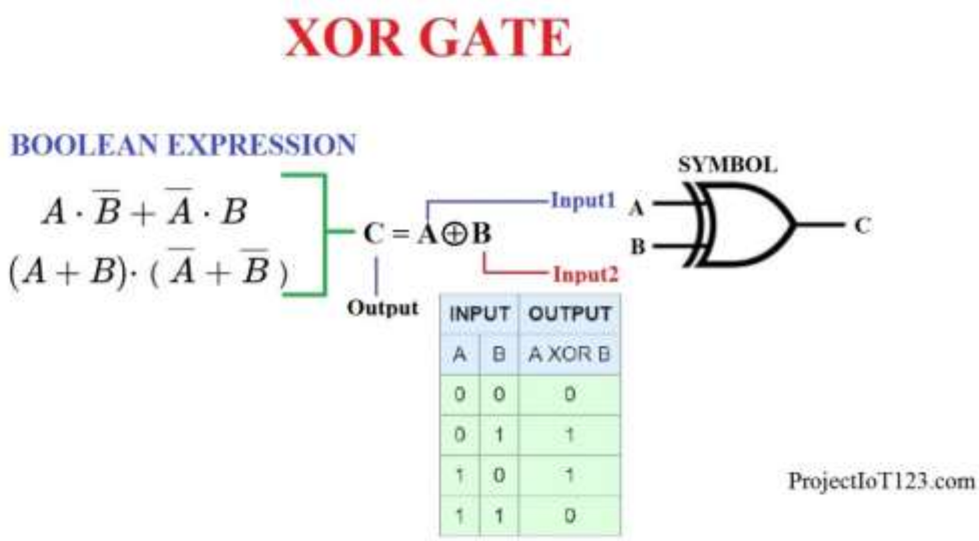


TRUTH TABLE

INPUT		OUTPUT
A	B	A NOR B
0	0	1
0	1	0
1	0	0
1	1	0

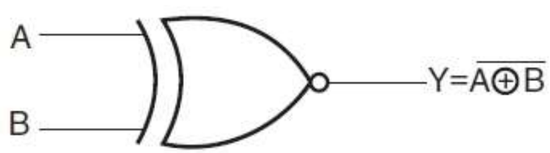
XOR gate

- XOR gate is an electronic gate which gives an output only when either of the inputs is high(1), but not both.
- It stands for “Exclusively Or” which defines the function.



XNOR gate

- XNOR gate is an electronic gate which gives an output only when both of the inputs are either high or low, but it should be the same.
- It stands for “Exclusively Nor” which defines the function.



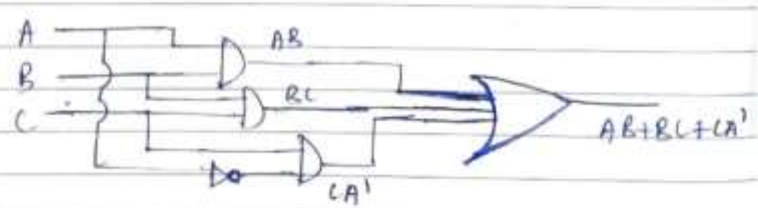
$$Y = (\overline{A \oplus B}) = (A.B + \overline{A}.\overline{B})$$

A	B	Y
0	0	1
0	1	0
1	0	0
1	1	1

Question on Logic Diagram and Truth Table

(8) Using Truth Table prove that $AB+BC+CA' = AB+CA'$.
Also draw the logic diagram for $AB+BC+CA'$.

A	B	C	A'	AB	BC	CA'	AB+BC+CA'	AB+CA'
0	0	0	1	0	0	0	0	0
0	0	1	1	0	0	1	1	1
0	1	0	1	0	0	0	0	0
0	1	1	1	0	1	1	1	1
1	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0
1	1	0	0	1	0	0	1	1
1	1	1	0	1	1	0	1	1



Principal of Duality

- In Boolean algebras the duality Principle can be is obtained by interchanging AND and OR operators and replacing 0's by 1's and 1's by 0's. Compare the identities on the left side with the identities on the right.

Example

$X.Y+Z' = (X'+Y').Z$

Basic Theorems

T1 : Properties of 0

(*a*) $0 + A = A$

(*b*) $0\,A = 0$

T2 : Properties of 1

(a) $1 + A = 1$

(b) $1\,A = A$

Basic Theorem

T3 : Commutative Law

(a) $A + B = B + A$

(b) $A B = B A$

T4 : Associate Law

(a) $(A + B) + C = A + (B + C)$

(b) $(A B) C = A (B C)$

T5 : Distributive Law

(a) $A (B + C) = A B + A C$

(b) $A + (B C) = (A + B) (A + C)$

(c) $A+A'B = A+B$

Basic Theorem

T6 : Indempotence (Identity) Law

$(a)\ A + A = A$

$(b)\ A\,A = A$

T7 : Absorption (Redundance) Law

$(a)\ A + A\,B = A$

$(b)\ A\,(A + B) = A$

Basic Theorem

T8 : Complementary Law

(a) $X+X'=1$

(b) $X.X'=0$

T9 : Involution

(a) $x'' = x$

T10 : De Morgan's Theorem

(a) $(X+Y)'=X'.Y'$

(b) $(X.Y)'=X'+Y'$

Questions on Boolean Laws

(Q) Simplify the following Boolean expression using laws.

$\overline{(A\bar{B} + \bar{A}B)}\,(A+B) = \overline{A\bar{B}}\;\overline{\bar{A}B}\,(A+B)$ {De Morgan's Law}

$= (\bar{A}+B)(A+\bar{B})(A+B) = (\bar{A}+B)(AA + AB + \bar{B}A + \bar{B}B)$ {Associative}

$= (\bar{A}+B)(A + AB + A\bar{B} + \bar{B}B)$ {$A \cdot A = A$}

$= (\bar{A}+B)(A(1 + B + \bar{B}) + B\bar{B})$ {Distributive law}

$= (\bar{A}+B)(A(1) + \bar{B}B)$ {$1 + B + \bar{B} = 1$}

$= (\bar{A}+B)(A) + 0$ {$B \cdot \bar{B} = 0$}

$= A\bar{A} + AB = AB$ {$A\bar{A} = 0$} $= \underline{AB}$

(Q) Using laws of Boolean algebra, prove:

$$A \cdot B + A' \cdot C + B \cdot C = A \cdot B + A' \cdot C$$

→ L.H.S

$$= A \cdot B + A' \cdot C + B \cdot C \cdot 1$$

$$= A \cdot B + A' \cdot C + B \cdot C(A + A') \quad \{\text{Since } A + A' = 1\}$$

$$= A \cdot B + A' \cdot C + B \cdot C \cdot A + B \cdot C \cdot A' \quad (\text{Distributive law})$$

$$= A \cdot B + A' \cdot C + A \cdot B \cdot C + A' \cdot B \cdot C \quad (\text{Associative law})$$

$$= A \cdot B + A \cdot B \cdot C + A' \cdot C + A' \cdot B \cdot C \quad (\text{Commutative law})$$

$$= A \cdot B(1 + C) + A' \cdot C(1 + B) \quad (\text{Distributive law})$$

$$= A \cdot B \cdot 1 + A' \cdot C \cdot 1 \quad (1 + B = 1,\ 1 + C = 1)$$

$$= A \cdot B + A' \cdot C = R.H.S \quad (A \cdot 1 = A).$$ Proved.

Representation of Boolean expression

Boolean expression can be represented by either

(i)Sum of Product(SOP) form or

(ii)Product of Sum (POS form)

e.g.

AB+AC □ SOP
(A+B)(A+C) □ POS

In above examples both are in SOP and POS respectively but they are not in Standard SOP and POS.

Canonical form (Standard form)

- In standard SOP and POS each term of Boolean expression must contain all the literals (with and without bar) that has been used in Boolean expression.
- If the above condition is satisfied by the Boolean expression, that expression is called Canonical form of Boolean expression.

Canonical form (Standard form) Contd..

- In Boolean expression **AB+AC** the literal C is missing in the 1st term **AB** and B is missing in 2nd term **AC**. That is why AB+AC is not a Canonical SOP.

Canonical form of Boolean Expression (Standard form) Contd..

Convert AB+AC in Canonical SOP (Standard SOP)

Sol. AB + AC

AB(C+C′) + AC(B+B′)

ABC+ABC′+ABC+AB′C **Distributive law**

ABC+ABC′+AB′C

Canonical form of Boolean Expression (Standard form) Contd..

Convert (A+B)(A+C) in Canonical POS (Standard POS)

Sol. $(A+B).(A+C)$

$(A+B)+(C.C') \, . \, (A+C)+(B.B')$

$(A+B+C).(A+B+C').(A+B+C)(A+B'+C)$ **Distributive law**

$(A+B+C).(A+B+C')(A+B'+C)$ **Remove duplicates**

Canonical form of Boolean Expression (Standard form) Contd..

Minterm and Maxterm

Individual term of Canonical Sum of Products (SOP) is called Minterm. In otherwords minterm is a product of all the literals (with or without bar) within the Boolean expression.

Individual term of Canonical Products of Sum (POS) is called Maxterm. In otherwords maxterm is a sum of all the literals (with or without bar) within the Boolean expression.

Minterms & Maxterms for 2 variables (Derivation of Boolean function from Truth Table)

x	y	Index	Minterm	Maxterm
0	0	0	$m_0 = x' y'$	$M_0 = x + y$
0	1	1	$m_1 = x' y$	$M_1 = x + y'$
1	0	2	$m_2 = x y'$	$M_2 = x' + y$
1	1	3	$m_3 = x y$	$M_3 = x' + y'$

The minterm m_i should evaluate to 1 for each combination of x and y.
The maxterm is the complement of the minterm

Minterms & Maxterms for 3 variables

x	y	z	Index	Minterm	Maxterm
0	0	0	0	$m_0 = \bar{x}\,\bar{y}\,\bar{z}$	$M_0 = x + y + z$
0	0	1	1	$m_1 = \bar{x}\,\bar{y}\,z$	$M_1 = x + y + \bar{z}$
0	1	0	2	$m_2 = \bar{x}\,y\,\bar{z}$	$M_2 = x + \bar{y} + z$
0	1	1	3	$m_3 = \bar{x}\,y\,z$	$M_3 = x + \bar{y} + \bar{z}$
1	0	0	4	$m_4 = x\,\bar{y}\,\bar{z}$	$M_4 = \bar{x} + y + z$
1	0	1	5	$m_5 = x\,\bar{y}\,z$	$M_5 = \bar{x} + y + \bar{z}$
1	1	0	6	$m_6 = x\,y\,\bar{z}$	$M_6 = \bar{x} + \bar{y} + z$
1	1	1	7	$m_7 = x\,y\,z$	$M_7 = \bar{x} + \bar{y} + \bar{z}$

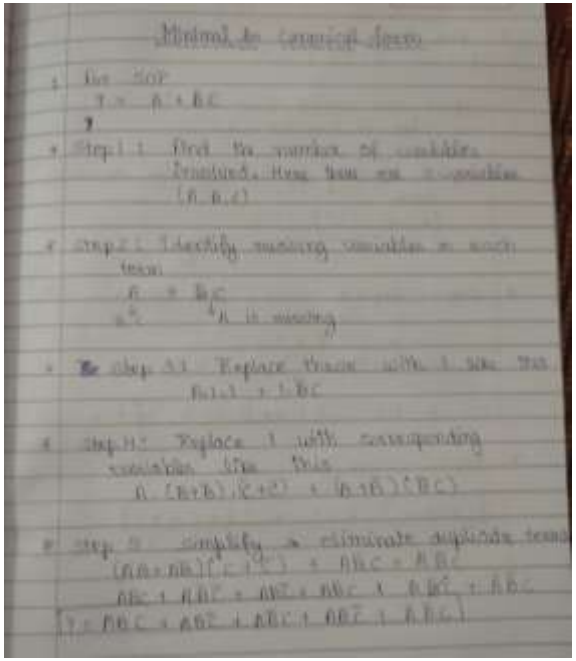
Maxterm M_i is the complement of minterm m_i

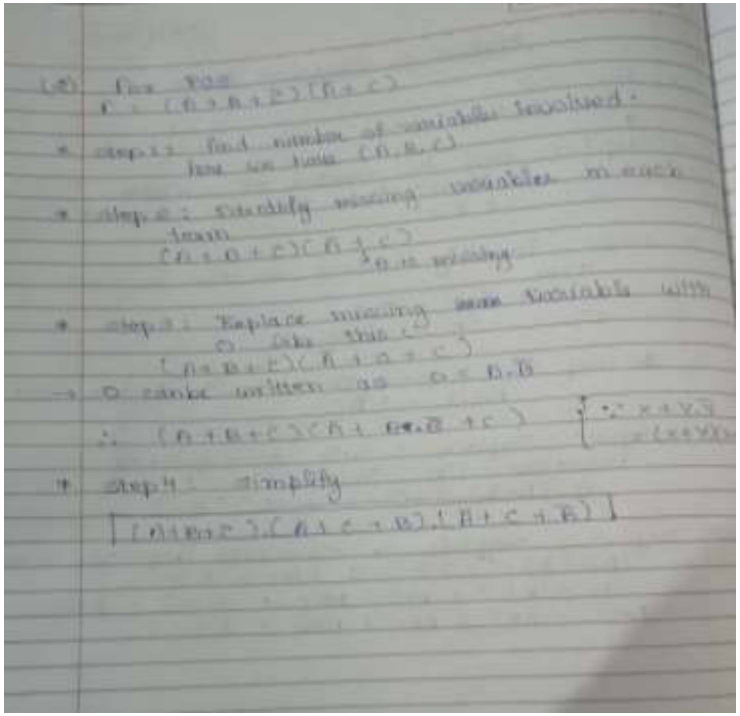
$$M_i = \overline{m_i} \text{ and } m_i = \overline{M_i}$$

Purpose of the Index

- Minterms and Maxterms are designated with an index
- The index number corresponds to a binary pattern
- The <u>index</u> for the minterm or maxterm, expressed as a binary number, is used to determine whether the variable is shown in the true or complemented form
- For Minterms:
  - ‘1’ means the variable is “Not Complemented” and
  - ‘0’ means the variable is “Complemented”.
- For Maxterms:
  - ‘0’ means the variable is “Not Complemented” and
  - ‘1’ means the variable is “Complemented”.

Questions on Canonical Form





OP and POS

Convert the given expression $f(x,y,z) = xy + xz + xyz$ into POS.

$$f = xy + xz + xyz$$

$$\bar{f} = \overline{xy + xz + xyz}$$

$$= \overline{xy} \cdot \overline{xz} \cdot \overline{xyz} \quad \text{(De Morgans Theorem)}$$

$$= [\bar{x}+\bar{y}][\bar{x}+\bar{z}][\bar{x}+\bar{y}+\bar{z}] \quad (\overline{AB} = \bar{A} + \bar{B})$$

$$= [\bar{x} + \bar{y}\bar{z}][\bar{x}+\bar{y}+\bar{z}] \quad ((A+B)(A+C) = A + BC)$$

$$= \bar{x}\bar{x} + \bar{x}\bar{y} + \bar{x}\bar{z} + \bar{y}\bar{z}\bar{x} + \bar{y}\bar{z}\bar{y} + \bar{y}\bar{z}\cdot\bar{z}$$

$$= \bar{x} + \bar{x}\bar{y} + \bar{x}\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{y}\bar{z} + \bar{y}\bar{z}$$

$$= \bar{x} + \bar{x}\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{y}\bar{z}$$

$$\bar{f} = \bar{x} + \bar{x}\bar{y}\bar{z} + \bar{y}\bar{z}$$

$$\bar{\bar{f}} = \overline{\bar{x} + \bar{x}\bar{y}\bar{z} + \bar{y}\bar{z}}$$

$$f = x \cdot (\bar{\bar{x}} + \bar{\bar{y}} + \bar{\bar{z}}) \cdot (\bar{\bar{y}} + \bar{\bar{z}})$$

$$= x \cdot (x+y+z) \cdot (y+z)$$

$$\boxed{f(x,y,z) = x(y+z)(x+y+z)}$$

Q) Convert the given expression $f(x,y,z) = x(y+z)(x+y+z)$ into SOP.

$$\rightarrow f = x[y+z][x+y+z]$$

$$= [xy + xz][x+y+z] \quad (\because A[B+C] = AB + AC)$$

$$= xy\cdot x + xy\cdot y + xy\cdot z + xz\cdot x + xzy + xz\cdot z$$

$$= xy + xy + xyz + xz + xyz + xz \quad (\because A\cdot A = A)$$

$$= xy + xyz + xz \quad (\because A + A = A)$$

$$\boxed{f(x,y,z) = xy + xz + xyz}$$

LATTICES

S. Devi Yamini

August 16, 2017

Overview

1. Partial Order Relation

Overview

1. Partial Order Relation
2. Hasse Diagram

Overview

Overview

Relation

Relation

Let A and B be two sets. The cartesian product of A and B is
$A \times B = \{(a, b) \ : \ a \in A, b \in B\}$.
A relation R is a subset of the cartesian product $A \times B$

Example

Let $A = \{1, 2, 3\}, B = \{a, b, c\}$.
$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$
$R_1 = \{(1, a), (2, b), (3, b)\}, R_2 = \{(1, b), (1, c)\}, R_3 = \{(a, 1), (1, c)\}$
R_1, R_2 are relations whereas R_3 is not.

A relation on a set A is a subset of $A \times A$.

Relation

Equivalence Relation

A relation R on a set A is called an equivalence relation if it satisfies the following conditions:

- Reflextive : $(a, a) \in R \quad \forall a \in A$
- Symmetric : If $(a, b) \in R$, then $(b, a) \in R$
- Transitive : If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$

Example

Let $A = \{1, 2, 3\}$. Then $R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$ is an equivalence relation.

Partial order Relation

A binary relation R on a set A is called Partial order relation if it satisfies the following conditions:

- Reflexive : $(a, a) \in R \quad \forall a \in A$
- Antisymmetric : If $(a, b) \in R$ and $(b, a) \in R$, then $a = b$
- Transitive : If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$

Example

Let $A = \{2, 3, 5\}$.
Consider $R = \{(2, 2), (3, 3), (5, 5), (2, 3), (3, 5), (2, 5)\}$
R is a Partial order relation

If R is a partial order relation on a set A, then (A, R) is called a POSET.

Some partial order relations

- Consider $\mathbb{R}$, the set of all real numbers. Let the relation be less than or equal to. Then, the relation is a partial order on $\mathbb{R}$. Hence, $(\mathbb{R}, \leq)$ is a POSET
- Let $\rho(A)$ denote the power set of A. The relation of set inclusion $\subseteq$ is a partial order on $\rho(A)$. Hence, $(\rho(A), \subseteq)$ is a POSET
- Let $A = \{1, 2, 4, 5, 10\}$ and the relation R be defined as follows: $R = \{(a, b) \ : \ a \text{ divides } b\}$. Then, the divisibility relation R is a partial order on A. Hence, (A, R) is a POSET. [Note: a divides b is denoted by $a|b$]

Hasse Diagram

Any POSET can be represented by a directed graph.

Procedure to obtain Hasse diagram from a directed graph

- Place the vertices from bottom to top according to the direction mentioned in the directed graph.
- Remove the loops and transitive edges from the graph
- Remove the directions

Illustration of Hasse Diagram

Example

Consider the POSET (A, R) where $A = \{1, 2, 4, 5, 10\}$ and
$R = \{(a, b) \ : \ a \text{ divides } b\}$.
Hence, $R =$
$\{(1,1),(2,2),(4,4),(5,5),(10,10),(1,2),(1,4),(1,5),(1,10),(2,4),(2,10),($
The Hasse diagram for the POSET is

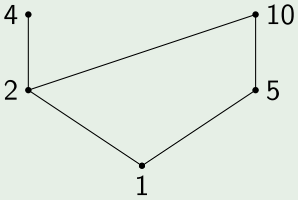
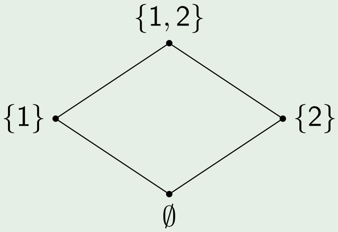


Illustration of Hasse Diagram

Example

Consider the POSET $(\rho(A), \subseteq)$ where $A = \{1, 2\}$.
$\rho(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$
$R = \{(\emptyset, \emptyset), (\{1\}, \{1\}), (\{2\}, \{2\}), (\{1, 2\}, \{1, 2\}), (\emptyset, \{1\}), (\emptyset, \{2\}), (\emptyset, \{1, 2\}), (\{1\}, \{1, 2\}), (\{2\}, \{1, 2\})\}$
The Hasse diagram for the POSET is



Try the same for $A = \{1, 2, 3\}$

Consider a POSET $(P, \leq)$ where P is any set and $\leq$ is some partial order relation. Then

Maximal

An element $a \in P$ is maximal if there is no element b in P such that $a < b$. (need not be unique)

Minimal

An element $a \in P$ is minimal if there is no element b in P such that $b < a$. (need not be unique)

Greatest

An element $a \in P$ is the greatest element of the POSET, if $b \leq a$, $\forall b \in P$. (Unique)

Least

An element $a \in P$ is the least element of the POSET, if $a \leq b$, $\forall b \in P$. (Unique)

Bounds

Consider a subset A of P.

Upper bound

$u \in P$ is an upper bound of A if $a \leq u \; \forall a \in A$. (need not be unique)

Lower bound

$u \in P$ is a lower bound of A if $u \leq a \; \forall a \in A$. (need not be unique)

Least upper bound

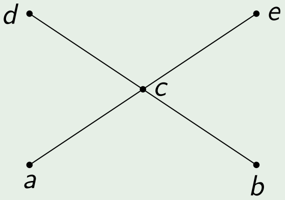
x is the least upper bound of the subset A of $(P, \leq)$ if x is an upper bound that is less than every other upper bound of A. (Unique)

Greatest lower bound

x is the greatest lower bound of the subset A of $(P, \leq)$ if x is a lower bound that is greater than every other lower bound of A. (Unique)

Illustration

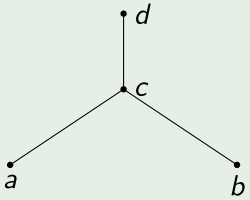
Example



Consider $A = \{a, b\}$. Upper bound is $\{c, d, e\}$. No lower bound.
LUB is {c}. No GLB.
Try to find the GLB and LUB for $A = \{a, c\}$

Illustration

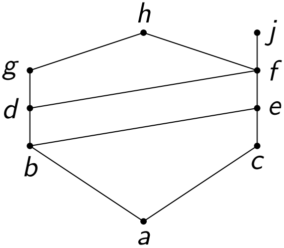
Example



Consider $A = \{c, d\}$
The lower bound of A is $\{a, b, c\}$ and upper bound is $\{d\}$
LUB is $\{d\}$ and GLB is $\{c\}$

Problems

1. Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$ and $\{a, c, d, f\}$ in the poset with the Hasse diagram. Find also the LUB and GLB of the subset $\{b, d, g\}$ if they exist.



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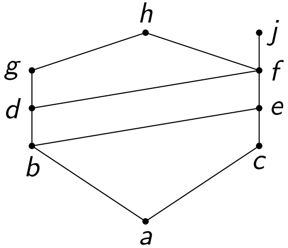

2. For the POSET $(\{3, 5, 9, 15, 24, 45\}$, divisor of $)$, find (a) maximal and minimal elements, (b) Greatest and least elements, (c) Upper and LUB of $\{3, 5\}$ and (d) Lower and GLB of $\{15, 45\}$

Lattices

Lattice

A POSET $(L, \leq)$ in which every pair of elements has a LUB and GLB.

- LUB of a subset $\{a, b\}$ is denoted by $a \vee b = a + b$ (also termed as Join)
- GLB of a subset $\{a, b\}$ is denoted by $a \wedge b = a.b$ (also termed as Meet)

A lattice is denoted by $(L, ., +, 0, 1)$[1]

[1] To check for a lattice, it is enough to consider pairwise unrelated elements

Examples of Lattices

Consider the POSET $(\{1, 2, 4, 8, 16\}, |)$. Then, the Hasse diagram is

Clearly, for every pair of elements, there is a GLB and LUB. Hence, it is a lattice.

Examples

Example of a lattice

Let n be a positive integer and S_n be the set of all divisors of n. For instance, $S_6 = \{1, 2, 3, 6\}$ and D denote the divisibility relation. Clearly, this is a lattice.

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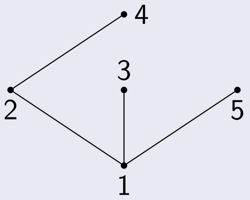
Not a lattice



This is not a lattice, as there is no LUB of $\{2, 3\}$.
Similarly LUB of $\{2, 5\}$ does not exists.

Properties of Lattices

1. Idempotency: $a + a = a$ and $a.a = a$
2. Commutativity: $a + b = b + a$ and $a.b = b.a$
3. Absorption: $a + (a.b) = a$ and $a.(a + b) = a$
4. Associativity: $a + (b + c) = (a + b) + c$

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Duality of a lattice

The POSETs $(L, \leq)$ and $(L, \geq)$ are duals of each other.
The GLB of a set with $\leq$ is same as LUB of the set with $\geq$
Similarly, the LUB of a set with $\leq$ is same as GLB of the set with $\geq$
Hence, if $(L, \leq)$ is a lattice, then $(L, \geq)$ is also a lattice.

Properties of Lattices

Let $(L, \leq)$ be a lattice. For any $a, b \in L$,

$$a \leq b \Leftrightarrow (a.b = a) \Leftrightarrow (a + b = b)$$

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Isotonic property

Let $(L, \leq)$ be a lattice. For any $a, b, c \in L$,

$$b \leq c \Rightarrow (a.b) \leq (a.c), (a + b) \leq (a + c)$$

Complete Lattice

A lattice is complete if each of its nonempty subsets has a LUB and a GLB. Every finite lattice[a] is complete.

[a] A lattice is finite if its underlying set is finite

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Bounded Lattice

The least and greatest elements of a lattice are called the bounds of the lattice. They are denoted by 0 and 1 respectively. A lattice which has both elements 0 and 1 are called Bounded lattice.

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Complements

In a bounded lattice, an element $b \in L$ is a complement of $a \in L$ if $a.b = 0$ and $a + b = 1$. An element can have more than one complement.

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Complemented Lattice

A lattice is complemented, if every element has at least one complement.

Distributive lattice

A lattice L is distributive if for any $a, b, c \in L$,

$$a.(b + c) = (a.b) + (a.c)$$

and

$$a + (b.c) = (a + b).(a + c)$$

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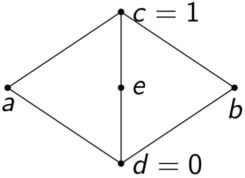
Important note

In a distributive lattice L, if an element $a \in L$ has a complement, then it is unique.
In otherwords, every element in L has either 0 or 1 complement. The converse of the statement is not true. This result can be used to prove that a lattice is not distributive.

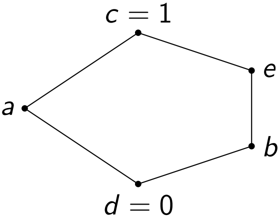
Check whether following are complemented, distributive lattices or not.
(i) S_6

Illlustrations

Check whether following are complemented, distributive lattices or not.
(i) S_6
(ii)



(iii)



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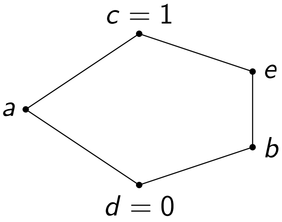

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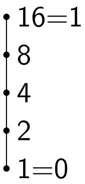


Boolean algebra

A lattice which is complemented and distributive is called a Boolean algebra.

Definition

If B is a nonempty set with two binary operations $+$ and $.$, and unary operation $'$ with two distinct elements 0 and 1, then B is called a Boolean algebra if it satisfies the following conditions $\forall a, b, c \in B$

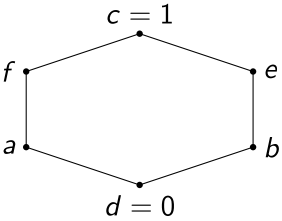
- $a + 0 = a, \quad a.1 = a$ (Identity laws)
- $a + b = b + a, \quad a.b = b.a$ (Commutative laws)
- $(a + b) + c = a + (b + c), \quad (a.b).c = a.(b.c)$ (Associative laws)
- $a + (b.c) = (a + b).(a + c), \quad a.(b + c) = (a.b) + (a.c)$ (Distributive laws)
- $a + a^{'} = 1, \quad a.a^{'} = 0$ (Negation laws)

Boolean algebra is denoted as $(B, ., +, ', 0, 1)$

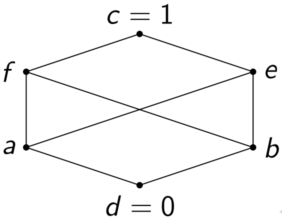
Problems

1. Check whether the following is distributive lattice or not.

(a)



(b)



Problems

2. Provide an example of (a) a POSET which is not a lattice, (b) a lattice which is not distributive, (c) a lattice which is not complemented.
3. Draw Hasse diagram for all 5 elements lattice
4. Check whether $(\rho(A), \subseteq)$ is distributive or not, where $A = \{1, 2, 3\}$
5. Let R be a relation defined on a set S of people such that $R = \{(a, b) \; : \; a, b \in S$ and a is older than $b\}$. Is (S, R) a POSET? Is (S, R) a Lattice?

Graphs - Module ~~3~~ 5

S. Devi Yamini

Overview

1. Motivation

Overview

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2. Basics

Motivation

In our campus, consider seven blocks, Academic block 1 (AB1), Academic block 2 (AB2), Medical Center (MC), Administrative block (Admin), Guest house (G), Security office (S). The problem is to design a LAN satisfying the following conditions (No multiple cables allowed)

Problem 1

- Two of the blocks are connected to exactly five of the blocks
- Two of the blocks are connected to three of the blocks
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Problem 2

- Four of the blocks are connected to five of the blocks
- Three of the blocks are connected to two of the blocks

Origin of Graph Theory

The Konigsberg bridge puzzle is the origin of Graph theory. It was due to the mathematician Leonhard Euler (1707-1783). The problem is whether one could cross all the seven bridges of the city of Konigsberg exactly once and return to the starting point.

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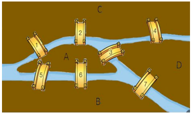

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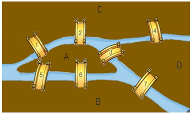

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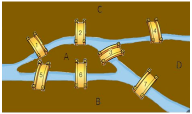

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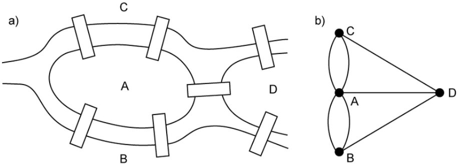

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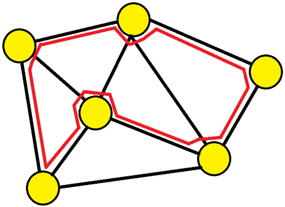

Basics

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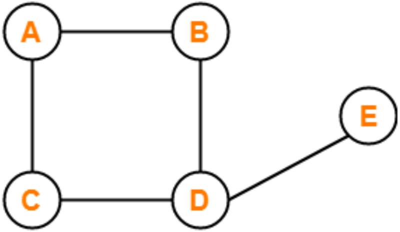
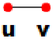
Example



Basics

- Two vertices u and v are end vertices of the edge (u, v)



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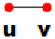
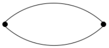


- Edges that have the same end vertices are parallel



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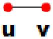

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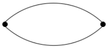
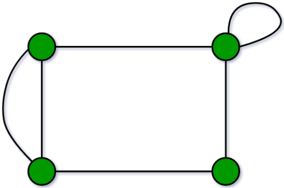
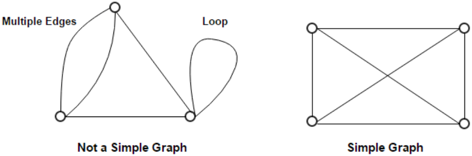


- An edge of the form (v, v) is a loop



Basics

- A graph is simple, if it has no parallel edges or loops



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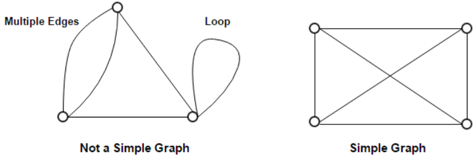
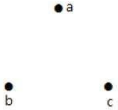


- A graph with no edges is a null graph



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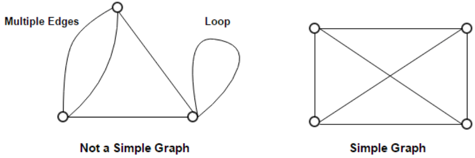

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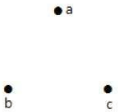

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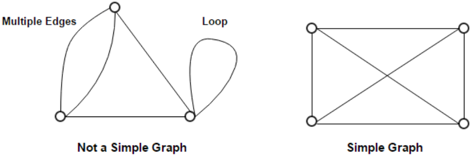

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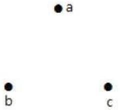

- A graph with only one vertex is trivial
- Edges are adjacent if they share a common end vertex

Contd.

- Two vertices u and v are adjacent if they are connected by an edge.

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Basics contd.

Theorem

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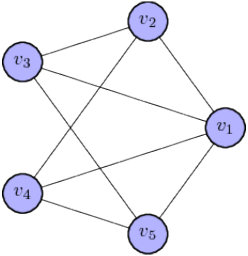

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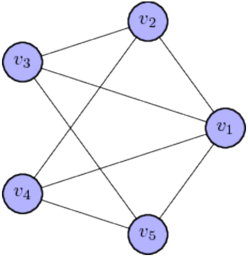

$d(v_1) = 4, d(v_2) = 3, d(v_3) = 3, d(v_4) = 3, d(v_5) = 3$

Basics contd.

Observation

Every graph has an even number of vertices of odd degree.

Basics contd.

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Verify the observation for the previous example!

Subgraph

A graph $H = (V_1, E_1)$ is a subgraph of $G = (V_2, E_2)$ if (a) $V_1 \subseteq V_2$ and (b) $E1 \subseteq E2$

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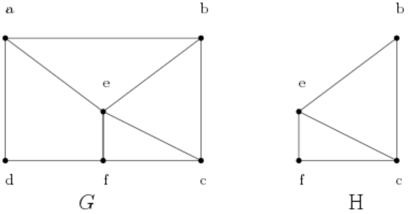

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A subgraph containing all the vertices of the original graph

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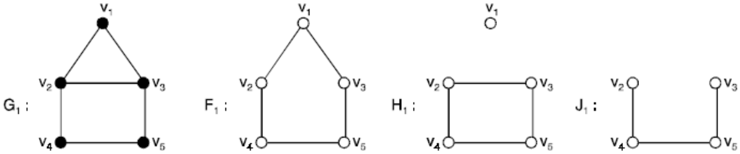

Fig. 20.

Special graphs

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Walk,trail, Path, Connectedness

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Walk,trail, Path, Connectedness

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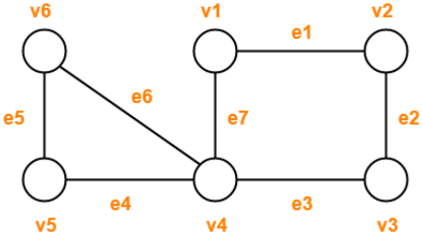

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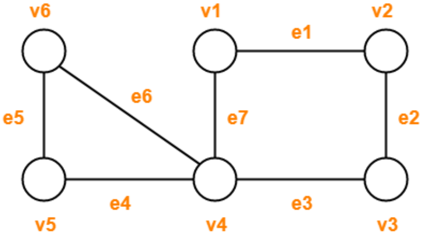

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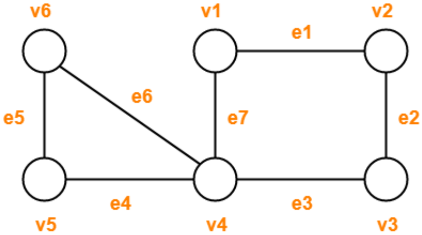

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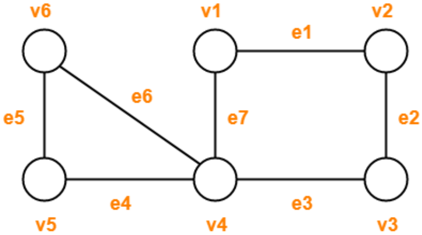

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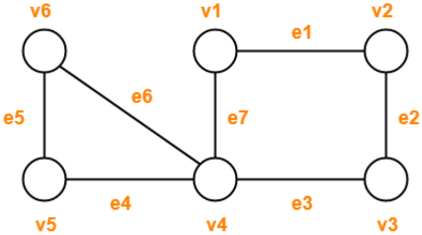

- A walk is a trail if any edge is traversed at most once. (that is, edge is not repeated)
- A closed trail is a circuit
- A trail is a path if any vertex is visited at most once. (that is, edges and vertices are not repeated)
- A closed path is a cycle

Connected graphs

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- A subgraph of a graph is a component of G if the subgraph is connected

- The complement G^c of the simple graph G is a simple graph where the edges in the complement are the non edges in the given graph.
- If the graph and its complement are same, then the graph is said to self complementary. Example: C_5 (a cycle on 5 vertices)

Complete graphs

A graph where every pair of vertices are adjacent denoted by K_n, a complete graph on n vertices.

Problem

Try to draw the complete graphs on K_n, for $n = 1, 2, 3, 4$. Also sketch the complements of these graphs.

Planar graphs

Planar graph

A graph that can be embedded on the plane, that is, it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other.

Check whether the following graphs are planar: (a) K_4, (b) K_5, (c) $K_5 - e$ where e is any arbitrary edge (d) Peterson graph