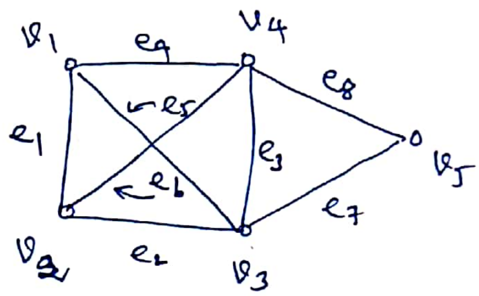
Paths, cycles and connectivity

A path in a graph is a finite alternative sequence of vertices and edges, beginning and ending with ~~edges~~ vertices such that each edge is incident on the vertices preceding and following it.

If the edges in a path are distinct, it is called a simple path.

Eg:-



* $V_1 e_1 V_2 e_2 V_3 e_5 V_1 e_1 V_2$ is a path since (e_1 twice)
* $V_1 e_4 V_4 e_6 V_2 e_2 V_3 e_7 V_5$ is a simple path.
* The number of edges in a path (simple or general) is called the length of the path.
* Cycle or Circuits

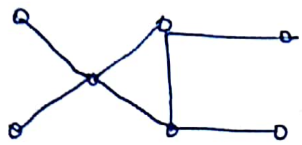
If the initial and final vertices of a path (of non-zero length) are the same, the path is called a circuit or cycle.

* If the initial and final vertices of a simple path of non-zero length are ~~the~~ same, the path is called a simple ~~path~~ cycle or simple circuits.

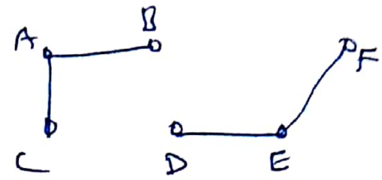
Connectedness in undirected graphs

An undirected graph is said to be connected if a path between every pair of distinct vertices.

A graph is not connected is called disconnected.



G_1 Connected



G_2 not connected.

* Clearly a disconnected graph is the union of two or more connected subgraphs, each pair of which has no vertex in common. These disjoint connected subgraphs are called the connected components of the graph.

Theorem:- 1. If a graph G (either connected or not) has exactly two vertices of odd degree, there is a path joining these two vertices.

Theorem : 2 If G is a simple graph with n vertices and K-components, then it can have atleast $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof:- Let G be a simple graph with n vertices and K-components $G_1, G_2, \ldots G_K$. Let the vertices of these components be $n_1, n_2, \ldots n_k$.

So that $n_1 + n_2 + \cdots + n_K = n$.

i.e, $\sum_{i=1}^{K} n_i = n$

Now, the component G_i is a simple graph & n_i vertices. So the maximum no. of edges in

$$G_i = \frac{n_i(n_i-1)}{2}$$

$$\therefore \; \varepsilon(G_i) \leq \frac{n_i(n_i-1)}{2} \quad [\varepsilon(G) = \text{The no. of edges in } G]$$

Now $$\varepsilon(G) = \sum_{i=1}^{K} \varepsilon(G_i)$$

$$\Rightarrow \; \varepsilon(G) \leq \sum_{i=1}^{K} \frac{n_i(n_i-1)}{2}$$

Consider the component G_i. Even if the remaining $K-1$ components are isolated vertices, the no. of vertices of G_i cannot exceed $n-(K-1) = n-K+1$.

$$\therefore \; n_i \leq n-K+1$$

Now $$\varepsilon(G) \leq \sum_{i=1}^{K} \frac{(n-K+1)(n_i-1)}{2}$$

$$= \frac{(n-K+1)}{2} \sum_{i=1}^{K} (n_i-1)$$

$$= \frac{(n-K+1)}{2} \left(\sum_{i=1}^{K} n_i - K \right) = \frac{(n-K+1)}{2}(n-K)$$

$$\varepsilon(G) \leq \frac{(n-K)(n-K+1)}{2} \quad //.$$

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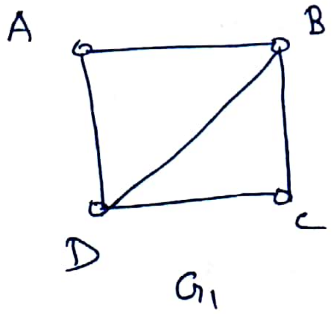
Eulerian and Hamiltonian graphs

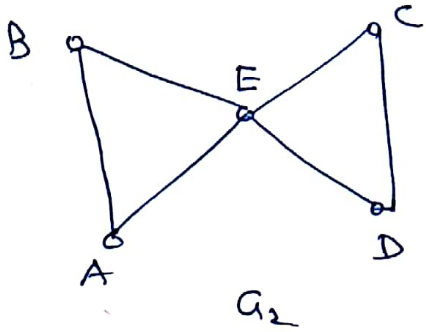
A path of graph G is called an Eulerian path, if it includes each edge of G exactly once.

A circuit or cycle of a graph G is called an Eulerian circuit if it includes each edge of G exactly once.

A graph containing an Eulerian circuit is called an Eulerian graph.

Example:-





* Graph G_1 contains an Eulerian ~~cycle~~ path between B & D namely $B \to D \to C \to B \to A \to D$ since it includes each of the edges exactly once.

* G_2 contains an Eulerian circuit namely, $A \to E \to C \to D \to E \to B \to A$ since it includes each of the edges exactly once.

$\therefore$ G_2 is Euler graph

Necessary & sufficient conditions for existence of Euler cycle & Euler paths.

1. Theorem:-

A connected graph contain an Euler cycle, if and only if each of its vertices is of even degree.

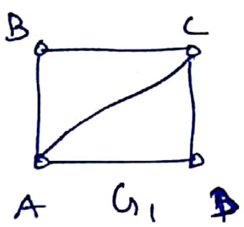
2. Theorem:-

A connected graph contains an Euler path, if and only if it has exactly two vertices of odd degree.

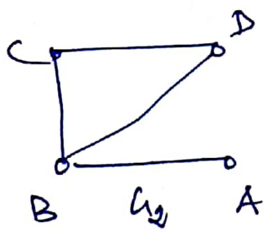
Hamiltonian graph:-

* A path of a graph G is called a Hamiltonian path, if it includes each vertex of G exactly once.
* A cycle of a graph G is called a Hamiltonian cycle if it includes each vertex of G exactly once, except the starting & end vertices which appear twice.
* A graph containing a Hamiltonian cycle is called a Hamiltonian graph.

Example



G_1

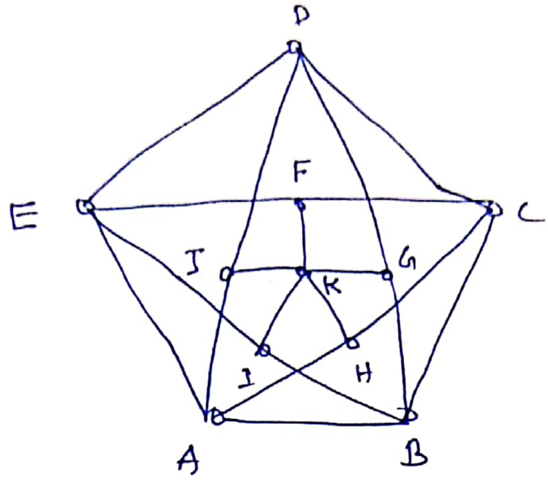


G_2

* G_1 has a Hamiltonian cycle namely $A \to B \to C \to D \to A$.
(all vertices includes only once, but not all edges)
* G_2 has a Hamiltonian path via, $A \to B \to C \to D$ but not a Hamiltonian cycle.

Problems

(1) Show that the graph is Hamiltonian



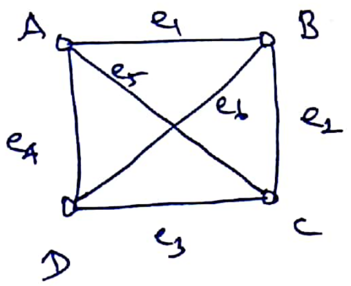
Soln:- There is a closed path $A \to H \to C \to F \to E \to D \to G \to B \to I \to K \to J \to A$.

is a Hamiltonian cycle.

$\therefore$ The given graph is Hamiltonian.

(2) Give example of a graph A Hamiltonian cycle but not an Euler cycle.

G:



$d(A) = d(B) = d(C) = d(D) = 3$

Every vertex is not of even degree.

$\therefore$ G contains no Euler cycle.

Now consider $A e_1 B e_2 C e_3 D e_4 A$ is the Hamiltonian cycle.

Try
* Euler cycle but not a Hamiltonian cycle.

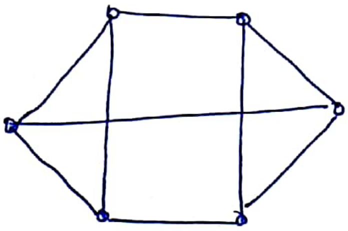
Planar graph

A graph G is said to be plannar (or) embeddable in the plane if it can be drawn in the plane so that no two edges intersect except (possibly) at their end vertices; otherwise it is said to be a non-planar graph.

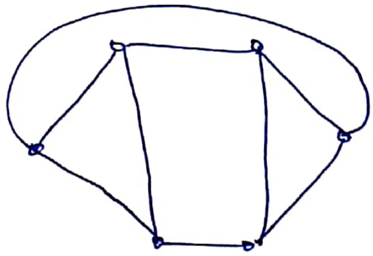
A planar graph embedded in the plane is called a plane graph.

Example:-

①

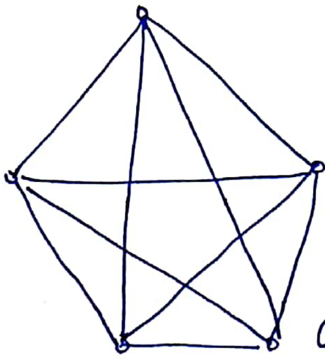


$\cong$

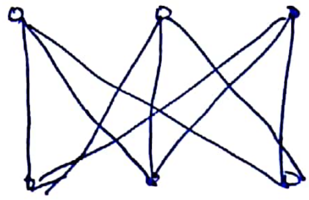


G_1 plannar

②



G_2



G_3

G_2 & G_3 are not planar

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