5. PERMUTATIONS AND COMBINATIONS

1. INTRODUCTION

The main subject of this chapter is counting. Given a set of objects the problem is to arrange some or all of them according to some order or to select some or all of them according to some specification.

2. FUNDAMENTAL PRINCIPLE OF COUNTING

The rule of sum: If a first task can be performed in m ways, while a second task can be performed in n ways, and the tasks cannot be performed simultaneously, then performing either one of these tasks can be accomplished in any one of total m+n ways.

The rule of product: If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage and if, for each of these outcomes, there are n possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in a total of mn ways.

Illustration 1: There are three stations, A. B and C. Five routes for going from station A to station B and four routes for going from station B to station C. Find the number of different ways through which a person can go from A to C via B. **(JEE MAIN)**

Sol: This problem is an application of the Fundamental Principle of Counting. The rule of product can be used to solve this question easily.

Given there are five routes for going from A to B and four routes for going from B to C.

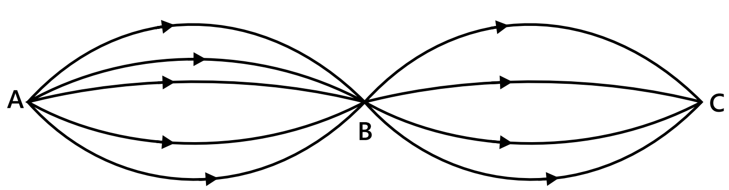

Figure 5.1

Hence, by the fundamental principle of multiplication the total number of different ways

= 5 x 4 (i.e., A to B and then B to C) = 20 ways

Illustration 2: A hall has 12 gates. In how many ways, can a man enter the hall through one gate and come out through a different gate. **(JEE MAIN)**

Sol: The rule of product can be used to solve this problem.

There are 12 ways of entering the hall. After entering the hall the man can come out through any of 11 different gates.

Hence, by the fundamental principle of multiplication, the total number of ways are 12 x 11 = 132 ways.

Illustration 3: How many numbers between 10 and 10,000 can be formed by using the digits 1, 2, 3, 4, 5 if

(i) no digit is repeated in any number. **(ii)** digits can be repeated. **(JEE MAIN)**

Sol: The numbers between 10 and 10,000 can be either two digit, three digit or four digit numbers. We consider each of these cases and try to find the number of possibilities using 1, 2, 3, 4 and 5. Finally, we add them up to get the desired result.

(i) Number of two digit numbers = 5 x 4 = 20

Number of three digit numbers = 5 x 4 x 3 = 60

Number of four digit numbers = 5 x 4 x 3 x 2 = 120

Total number of numbers = 20 + 60 +120 = 200

(ii) Number of two digit numbers = 5 x 5 = 25

Number of three digit numbers = 5 x 5 x 5 = 125

Number of four digit numbers = 5 × 5 × 5 × 5 = 625

Total number of numbers = 25 +125 +625 = 775

3. FACTORIAL NOTATION

An efficient way of writing a product of several consecutive integers is the factorial notation. The number n! (read as "n-factorial") is defined as follows :

For any positive integer n; $n! = n(n-1)(n-2) \ldots\ldots\ldots (3)(2)(1)$; For instance, $4! = 4.3.2.1 = 24$

Note: (i) $n! = n(n-1)(n-2) \ldots\ldots\ldots\ldots 3.2.1$; $n! = n.(n-1)!$; $0! = 1! = 1$; $(2n)! = 2^n.n![1.3.5.7 \ldots\ldots\ldots (2n-1)]$

(ii) $n! = n(n-1)! = n(n-1)(n-2)! = n(n-1)(n-2)(n-3)!$

(iii) $n(n-1) \ldots\ldots\ldots (n-r+1) = \dfrac{n!}{(n-r)!}$

Illustration 4: Find the sum of n terms of the series whose n^{th} term is $n!\times n$. **(JEE ADVANCED)**

Sol: Represent the general term in this question as a difference of two terms and then add them up to find the answer.

The required sum $= (1)! + 2(2)! + 3(3)! + \ldots\ldots + n(n!) = (2-1)! + (3-1)2! + (4-1)3! + \ldots\ldots\ldots + [(n+1)-1]n!$

$= (2!-1!) + (3!-2!) + (4!-3!) + \ldots\ldots\ldots + [(n+1)! - n!] = (n+1)! - 1$

4. PERMUTATION

Each of the different arrangements, which can be made by taking some or all of a number of objects is called permutation. The number of permutations of n different objects taken r at a time is represented as

$${}^nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2) \ldots\ldots\ldots (n-r+1) \text{ (where, } 0 \le r \le n)$$

Note: (i) In permutation, the order of the items plays an important role.

(ii) The number of all permutations of n distinct objects taken all at a time is n!

Illustration 5: If ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$, find ${}^{r}P_2$ **(JEE MAIN)**

Sol: Use the formula for ${}^{n}P_r$.

We have , $\dfrac{{}^{56}P_{r+6}}{{}^{54}P_{r+3}} = \dfrac{30800}{1} = \dfrac{56!}{(50-r)!} \times \dfrac{(51-r)!}{54!} = \dfrac{30800}{1} \Rightarrow 56 \text{ x } 55\ (51 - r) = 30800 \Rightarrow r = 41$

$\therefore \quad {}^{41}P_2 = 41 \text{ x } 40 = 1640$

Illustration 6: Three men have 4 coats, 5 waist coats and 6 caps. In how many ways can they wear them? **(JEE ADVANCED)**

Sol: Use the concept and understanding of Permutation, i.e. arrangement to find the answer.

The total number of ways in which three men can wear 4 coats is the number of arrangements of 4 different coats taken 3 at a time. So, three men can wear 4 coats in ${}^{4}P_3$ ways. Similarly, 5 waist coats and 6 caps can be worn by three men in ${}^{5}P_3$ and ${}^{6}P_3$ ways respectively. Hence, the required no. of ways = ${}^{4}P_3 \times {}^{5}P_3 \times {}^{6}P_3 = (4!) \times (5 \times 4 \times 3) \times (6 \times 5 \times 4) = 172800$.

Illustration 7: Suppose 8 people enter an event in a swim meet. In how many ways could the gold, silver, and bronze prizes be awarded? **(JEE ADVANCED)**

Sol: Use the formula for ${}^{n}P_r$. The required number of ways is an arrangement of 3 people out of 8 i.e.

${}^{8}P_3 = \dfrac{8!}{5!} = \dfrac{8.7.6.5.4.3.2.1}{5.4.3.2.1} = 8 . 7 . 6 = 336.$

MASTERJEE CONCEPTS

The following two steps are involved in the solution of a permutation problem:

Step 1: Recognizing the objects and the places involved in the problem.

Step 2: Checking whether the repetition of the objects is allowed or not.

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4.1 Permutation with Repetition

These kinds of problems occur with permutations of different objects in which some of the objects can be repeated. The no. of permutations of n different objects taken r at a time when each object may be repeated any number of times is n^r.

Illustration 8: A student appears in an objective test which contains 10 multiple choice questions. Each question has four choices in which one is the correct option. What maximum number of different answers can the student give? How will the answer change if each question may have more than one correct answers? **(JEE ADVANCED)**

Sol: Use the concept of Permutation with Repetition.

For the first part each question has four possible answers. So, the total possible answers = $4 \times 4 \times \ldots$ 10 times = 4^{10}.

For the second part. Suppose the choices for each question are denoted by A, B, C and D. Now the choice A is either correct or incorrect (two ways) similarly the other choices are either correct or incorrect. Thus, this particular question can have $2 \times 2 \times 2 \times 2 = 16$ possible answers. But this includes the case when all the four choices are incorrect. Thus the total number of answers = 15. Now, as there are 10 questions and each question has 15 possible answers. Therefore the total number of answers = 15^{10}.

MASTERJEE CONCEPTS

The m^n or n^m dilemma

Let us start with an example.

Q. There are 7 letters and 5 letter-boxes. In how many ways can you put the letters in the boxes?

Sol: This is the typically confusing question asked frequently from the P & C area. Let's see how you can solve this type of question.

The Exhaustive Approach: One way to solve this question is through (what we will call) The Exhaustive Approach. While solving such problems, first decide which of the items (letters and letter-boxes here) is exhaustive. Exhaustive here means the entity which is sure to be used up completely.

In this example, all the "letters" are sure to be placed in the boxes, whereas there is no such constraint as regards the "letter-boxes". Some boxes could go empty. Having decided this, just go by the options we have for all the instances of the exhaustive item and you have your answer.

As you can see here, every letter has 5 boxes to choose from. Thus the total would

be (5 x 5 x........7 times) = (5^7)

A similar question could be: In how many ways can 10 rings be worn on 5 fingers? Try it yourself.

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4.2 Permutation of Alike Objects

This kind of problems involve permutations of different objects in which some of them are similar.

The number of permutations of n objects taken all at a time in which, p are alike objects of one kind, q are alike objects of second kind & r are alike objects of a third kind and the rest (n – (p + q + r)) are all different is

$$\frac{n!}{p!q!r!}$$

Illustration 9: Determine the number of permutations of the letters of the word 'SIMPLETION' taken all at a time. **(JEE MAIN)**

Sol: In the given word the letter I occurs twice and the remaining letters occur only once. So, the concept of Permutation of Alike Objects is used to find out the answer.

There are 10 letters in the word 'SIMPLETION' and out of these 10 letters two are identical. So, just selecting all 10 objects at a time will give twice the actual result. Hence, the number of permutations of taking all the letters at a time = ${}^{10}P_{10}$ / 2! = 10! / 2! = 181440.

4.3 Permutation under Restriction

(a) The number of permutations of n different objects, taken r at a time, when a particular object is to be always included in each arrangement, is r. ${}^{n-1}P_{r-1}$

The number of permutations of n different objects, taken r at a time, when a particular object is never taken in each arrangement is ${}^{n-1}P_r$.

(b) **String method:** The number of permutations of n different objects, taken all at a time, when m specified objects always come together is m! × (n – m + 1)!.

(c) **Gap Method:** The number of permutations when no two given objects occur together.

In order to find the number of permutations when no two given objects occur together.

(a) First of all, put the m objects for which there is no restriction, in a line. These m objects can be arranged in m! ways.

(b) Then count the number of gaps between every two m objects for which there is no restriction, including the end positions. Number of such gaps will be (m + 1).

(c) If m is the number of objects for which there is no restriction and n is the number of objects, two of which are not allowed to occur together, then the required number of ways = $m! \times {}^{m+1}C_n \times n!$

The number of permutations when two types of objects are to be arranged alternately

(a) If their numbers differ by 1 put the object whose number is greater in the first, third, fifth.... places, etc. and the other object in the second, fourth, sixth.... places.

(b) If the number of two types of objects is same, consider two cases separately keeping the first type of object in the first, third, fifth places, etc. and the second type of object in the first, third, fifth places.... and then add.

4.4 Non-Consecutive Selection

The number of selections of r consecutive objects out of n objects in a row $= n - r + 1$

The number of selections of r consecutive objects out of n objects along a circle = $\begin{cases} n, & \text{when } r < n \\ 1, & \text{when } r = n \end{cases}$

Illustration 10: Find the numbers between 300 and 3000 that can be formed with the digits 0, 1, 2, 3, 4 and 5, where no digit is repeated in any number. **(JEE MAIN)**

Sol: The numbers between 300 and 3000 can either be a three digit number or a four digit number. The solution is divided into these two different cases and their sum will give us the desired result.

Any number between 300 and 3000 must be of three or four digits.

Case I: When number is of three digits: The hundreds place can be filled by any one of the three digits 3, 4 and 5 in 3 ways. The remaining two places can be filled by the remaining five digits in ${}^5P_2$, ways.

$\therefore$ The number of numbers formed in this case = $3 \times {}^5P_2 = 3 \times \frac{5!}{3!} = 60$

Case II: When number is of four digits: The thousands place can be filled by any one of the two digits 1 and 2 in 2 ways and the remaining three places can be filled by the remaining five digits in ${}^5P_3$ ways.

$\therefore$ The number of numbers formed in this case = $2 \times {}^5P_3 = 2 \times \frac{5!}{2!} = 120$

$\therefore$ Total numbers = 60 + 120 = 180

Illustration 11: How many words can be formed from the letters of the word ARTICLE, so that vowels occupy the even places? **(JEE MAIN)**

Sol: Clearly, this is an example of Permutation under Restriction. We identify the even places and the odd places and try to find the number of ways in which the vowels and consonants can fill the spaces.

There are seven places: 3 even and 4 odd in which we have to fill 3 vowels and 4 consonants.

$\therefore$ The number of words = ${}^3P_3 . {}^4P_4 = 3! \times 4! = 6 \times 24 = 144$.

Illustration 12: How many different words can be formed with the letters of the word ORDINATE so that

(a) Four vowels occupy the odd places (b) Beginning with O (c) Beginning with O and ending with E. **(JEE MAIN)**

Sol: The concept of Permutation under Restriction can be used to solve this problem.

There are 4 vowels and 4 consonants. Total 8 letters.

(a) No. of words = 4! × 4! = 24 × 24 = 576. Because 4 vowels are to be adjusted in 4 odd place and the 4 consonants in the remaining 4 even places.

(b) 7! ways, O being fixed.

(c) 6! ways, O fixed in first and E fixed in last.

Illustration 13: Find the number of ways in which 5 boys and 5 girls can be seated in a row so that

(a) No two girls may sit together.

(b) All the girls sit together and all the boys sit together

(c) All the girls are never together. **(JEE ADVANCED)**

Sol: Since the number of girls and the number of boys are equal they have to sit alternately. This can be used to solve (a). For (b), we keep the girls together and arrange the boys in five places. Also, the girls can be arranged amongst themselves in 5! ways. This gives us the number of arrangements. Use the answer of the second part to find (c).

(a) 5 boys can be seated in a row in $^5P_5$ = 5! ways. Now, in the 6 gaps between 5 boys, the 5 girls can be arranged in $^6P_5$ ways. Hence, the number of ways in which no two girls sit together = 5! × $^5P_5$ = 5! × 6!

(b) The two groups of girls and boys can be arranged in 2! ways. 5 girls can be arranged among themselves in 5! ways. Similarly, 5 boys can be arranged among themselves in 5! ways. Hence, by the fundamental principle of counting, the total number of requisite seating arrangements = 2!(5! × 5!) = $2(5!)^2$.

(c) The total number of ways in which all the girls are never together = Total number of arrangements – Total number of arrangements in which all the girls are always together = 10! – 5! × 6!

Illustration 14: The letters of the word OUGHT are written in all possible orders and these words are written out as in a dictionary. Find the rank of the word TOUGH in this dictionary. **(JEE MAIN)**

Sol: The word TOUGH will appear after all the words that start with G, H and O. Then we look at the second letter of the words starting with T and then third. Hence, the rank of the word TOUGH will be one more than the sum of all the possibilities just mentioned.

Total number of letters in the word OUGHT is 5 and all the five letters are different, the alphabetical order of these letters is G, H, O, T, U.

Number of words beginning with G = 4! = 24

Number of words beginning with H = 4! = 24

Number of words beginning with O = 4! = 24

Number of words beginning with TG = 3! = 6

Number of words beginning with TH = 3! = 6

Number of words beginning with TOG = 2! = 2

Number of words beginning with TOH = 2! = 2

Next come the words beginning with TOU and TOUH is the first word beginning with TOU.

∴ Rank of 'TOUGH' in the dictionary = 24 + 24 + 24 + 6 + 6 + 2 + 2 + 1 = 89

Illustration 15: There are 21 balls which are either white or black and the balls of same color are alike. Find the number of white balls so that, the number of arrangements of these balls in a row is maximum. **(JEE ADVANCED)**

Sol: The property of a binomial coefficient can be used to solve this question.

Let there be r white balls so that the number of arrangements of these balls in a row be maximum. Number of arrangements of these balls is

$A = \dfrac{21!}{r!(21-r)!}$ A will be maximum when $r = \dfrac{21+1}{2}$ or $\dfrac{21-1}{2}$ i.e. 10 or 11

Illustration 16: The number plates of cars must contain 3 letters of the alphabet denoting the place and area to which its owner belongs. This is to be followed by a three-digit number. How many different number plates can be formed if:

(i) Repetition of letters and digits is not allowed. (ii) Repetition of letters and digits is allowed. **(JEE ADVANCED)**

Sol: This is a simple application of Permutation with and without repetition.

There are 26 letters of alphabet and 10 digits from 0 to 9.

(i) When repetition is not allowed

3 letters selected in $26 \times 25 \times 24$ ways

3 digit numbers are = 9 x 9 x 8 (as zero can't be in the hundreds place)

$\therefore$ The Number of plates = 26 x 25 x 24 x 9 x 9 x 8 = 10108800.

(ii) When repetition is allowed

3 letters are selected $26 \times 26 \times 26$ ways

3 digit numbers are = $9 \times 10 \times 10 = 900$

$\therefore$ The number of plates = 26 x 26 x 26 x 900 = 15818400.

MASTERJEE CONCEPTS

Constraint based arrangement

Let us start with an example first:

Q. In how many ways can the word VARIETY be arranged so that exactly 2 vowels are together?

The problem could be easier if none or all vowels were to be kept together. Isn't it? Well, we will do exactly that!

Whenever question involving "constraints that has choices (2 vowels could be IE or AI or AE)" are asked, "go for the backward approach." Rather than finding the favorable cases, subtract the unfavorable ones from the total possible cases. This method is more reliable.

So, the solution to the above question would be:

(Total arrangements of VARIETY) – (Arrangements with no vowels together + Arrangements with all the vowels together)

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5. COMBINATION

Each of the different groups or selection which can be made by some or all of a number of given objects without reference to the order of the objects in each group is called a combination.

The number of all combinations of n objects, taken r at a time is generally denoted by C(n, r) or ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ $(0 \le r \le n) = \dfrac{{}^np_r}{r!}$

Note:

(a) The number of ways of selecting r objects out of n objects, is the same as the number of ways in which the remaining (n - r) can be selected and rejected.

(b) The combination notation also represents the binomial coefficient. That is, the binomial coefficient ${}^nC_r$ is the combination of n elements chosen r at a time.

(c) (a) ${}^nC_r = {}^nC_{n-r}$

(b) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(c) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(d) If n is even, then the greatest value of ${}^nC_r$ is ${}^nC_{n/2}$

(e) If n is odd, then the greatest value of ${}^nC_r$ is ${}^nC_{\frac{n+1}{2}}$

(f) ${}^nC_0 + {}^nC_r + \dots\dots\dots\dots + {}^nC_n = 2^n$

(g) ${}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \dots\dots\dots\dots + {}^{2n-1}C_n = {}^{2n}C_{n+1}$

(d) Comparison of permutation and combination

Permutations	**Combinations**
Different orderings or arrangement of the r objects are different permutations ${}^nP_r = \dfrac{n!}{(n-r)!}$ **Clue words:** arrangement, schedule, order	Each choice or subset of r object give one combination. Order within the group of r objects does not matter. ${}^nC_r = \dfrac{n!}{(n-r)!r!}$ **Clue words:** group, committee, sample, selection, subset.

Illustration 17: A basketball coach must select two attackers and two defenders from among three attackers and five defenders. How many different combinations of attackers and defenders can he select? **(JEE MAIN)**

Sol: The number of ways two attackers and two defenders can be selected is ${}^3C_2$ and ${}^5C_2$ respectively.

$\therefore$ He can select in ${}^3C_2 \times {}^5C_2 = \dfrac{5.4.3.2.1}{2.2} = 30$ different combinations.

Illustration 18: A soccer team of 11 players is to be chosen from 30 boys, of whom 4 can play only in goal, 12 can play only as forwards and the remaining 14 in any of the other positions. If the team is to include five forwards and of course, one goalkeeper, in how many ways can it be made up? **(JEE MAIN)**

Sol: Proceed according to the previous question.

There are: ${}^4C_1$ ways of choosing the goalkeeper. ${}^{12}C_5$ ways of choosing the forwards and ${}^{14}C_5$ ways of choosing the other 5 players. That is, ${}^4C_1 \times {}^{12}C_5 \times {}^{14}C_5$ combinations altogether = 4 x 792 x 2002 = 6342336.

5.1 Combinations under Restrictions

(a) Number of ways of choosing r objects out of n different objects if p particular objects must be excluded.

Consider the objects $A_1, A_2, A_3, \dots, A_p, A_{p+1}, \dots\dots\dots, A_n$. If the p objects $A_1, A_2 \dots\dots A_p$ are to be excluded then we will have to select r objects from the remaining n – p objects ($A_{p+1}, A_{p+2}, \dots\dots A_n$).

Hence the required number of ways = ${}^{(n-p)}C_r$

(b) Number of ways of choosing r objects out of n different objects if p particular objects must be included ($p \le r$).

Consider the objects $A_1, A_2, A_3 \dots\dots A_p, A_{p+1}, \dots\dots A_n$. If the p particular objects $A_1, A_2, \dots\dots A_p$ (say) must be included in the selection then to complete the selection, we must select (r – p) more objects to complete the selection. These objects are to be selected from the remaining n – p objects.

Hence, the required number of ways = ${}^{n-p}C_{r-p}$

(c) The total number of combinations of n different objects taken one or more at a time = $2^n - 1$.

5.2 Combinations of Alike Objects

(a) The number of combinations of n identical objects taking ($r \le n$) at a time is 1.

(b) The number of ways of selecting r objects out of n identical objects is n + 1.

(c) If out of (p + q + r + s) objects, p are alike of one kind, q are alike of a second kind, r are alike of the third kind and s are different, then the total number of combinations is $(p + 1)(q + 1)(r + 1)2^s - 1$

Note: The list of alike objects can be extended further

4. The number of ways in which r objects can be selected from a group of n objects of which p are identical, is $\sum_{0}^{t} {}^{n-p}C_r$, if $r \le p$ and $\sum_{r=p}^{t} {}^{n-p}C_r$ if r > p

Illustration 19: There are 4 oranges, 5 apples and 6 mangoes in a fruit basket. In how many ways can a person select fruits from among the fruits in the basket? **(JEE MAIN)**

Sol: Use the concept of Combination of Alike objects described above.

Here, we consider all fruits of the same type as identical.

Zero or more oranges can be selected out of 4 identical oranges in 4 + 1 = 5 ways.

Zero or more apples can be selected out of 5 identical apples in 5 + 1 = 6 ways.

Zero or more mangoes can be selected out of 6 identical mangoes in 6 + 1 = 7 ways

$\therefore$ The total number of selections when all the three types of fruits are selected (the number of any type of fruit may be zero) = $5 \times 6 \times 7 = 210$.

But in one of these selections number of each type of fruit is zero and hence there is no selection, this must be excluded.

$\therefore$ The required number = 210 – 1 = 209.

Caution: When all fruits of same type are different, the number of selections

$= ({}^4C_0 + {}^4C_1 + \dots\dots + {}^4C_4)({}^5C_0 + {}^5C_1 + \dots\dots + {}^5C_5)\ ({}^6C_0 + \dots\dots\dots + {}^6C_6) - 1 = 2^4 \times 2^5 \times 2^6 - 1 = 2^{15} - 1$

Illustration 20: How many four digit numbers are there whose decimal notation contains not more than two distinct digits? **(JEE MAIN)**

Sol: A four digit number can consist of either only one digit or two digits as per the question. Clearly, there are nine four digit numbers with the same digit. Similarly, calculate the number of four digit numbers with two distinct digits and hence the sum gives us the desired result.

Evidently any number so formed of four digits contains

(i) Only one digit (like 1111, 2222,...) and there are 9 numbers. **(ii)** Two digits

(a) if zero is one of the two, then the one more can be anyone of the nine, and these two digits can be arranged in ${}^9C_1\ [{}^3C_1 + {}^3C_2 + {}^3C_2 + {}^3C_3] = 63$.

(b) if zero is not one of them, then two of the digits have to be selected from 9, and these two can be arranged in ${}^9C_2[{}^4C_1 + {}^4C_2 + {}^4C_3] = 504$

Hence, the total number of required numbers = 567.

Illustration 21: In how many ways can a cricket team of eleven players be chosen out of a batch of 15 players, if

(a) there is no restriction on the selection

(b) a particular player is always chosen

(c) a particular player is never chosen **(JEE MAIN)**

Sol: Using the concept of combination of alike objects we can get the answer.

(a) The total number of ways of selecting 11 players out of 15 is = ${}^{15}C_{11}$ = ${}^{15}C_{15-11}$ = ${}^{15}C_4$ = $\frac{15\times14\times13\times12}{4\times3\times2\times1}$ = 1365

(b) A particular player is always chosen. This means that 10 players are selected out of the remaining 14 players.

$\therefore$ The required number of ways = ${}^{14}C_{10}$ = ${}^{14}C_{14-10}$ = ${}^{14}C_4$ = 1001

(c) The number of ways = ${}^{14}C_{11}$ = ${}^{14}C_3$ = $\frac{14\times13\times12}{3\times2}$ = 364

Illustration 22: In a plane there are 37 straight lines, of which 13 pass through the point A and 11 pass through the point B. Moreover, no three lines pass through one point, no line passes both points A and B. and no two are parallel. Find the number of points of intersection of the straight lines. **(JEE MAIN)**

Sol: Two non parallel straight lines give us a point of intersection. Using this idea we find the total number of points of intersection. Care should be taken that a point is not counted more than once.

The number of points of intersection of 37 straight lines is ${}^{37}C_2$. But 13 straight lines out of the given 37 straight lines pass through the same point A. Therefore, instead of getting ${}^{13}C_2$ points, we get merely one point A. Similarly, 11 straight lines out of the given 37 straight lines intersect at point B. Therefore, instead of getting ${}^{11}C_2$ points, we get only point B. Hence, the number of intersection points of the lines in ${}^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + 2 = 535$.

Illustration 23: How many five-digit numbers can be made having exactly two identical digits? **(JEE MAIN)**

Sol: Note that zero cannot occupy the first place. So we divide the solution into two cases when the common digit is 0 and otherwise. Calculate the number of possibilities in these two cases and their sum gives us the desired result.

Case I: Two identical digits are 0, 0.

The number of ways to select three more digits is ${}^{9}C_3$. The number of arrangements of these five digits is $\left(5!/2!\right)-4!=60-24=36$.

Hence, the number of such numbers is ${}^{9}C_3 \times 36 = 3024$ ... (i)

Case II: Two identical digits are (1, 1) or (2, 2) or.... or (9, 9).

If 0 is included, then number of ways of selection of two more digits is ${}^{8}C_2$.

The number of ways of arrangements of these five digits is 5! / 2! -4! / 2! =48.

Therefore, the number of such numbers is ${}^{8}C_2 \times 48$.

If 0 is not included, then selection of three more digits is ${}^{8}C_3$.

Therefore, the number of such numbers is ${}^{8}C_3 \times 5!/2! = {}^{8}C_3 \times 60$.

Hence, the total number of five-digit numbers with identical digits (1.1)......(9.9) is

$9 \times ({}^{8}C_2 \times 48 + {}^{8}C_3 \times 60) = 42336$ ... (ii)

From Eqs. (i) and (ii), the required number of numbers is 3024 + 42336 = 45360.

Illustration 24: How many words can be made with letters of the word “INTERMEDIATE” if

(i) The words neither begin with I nor end with E,

(ii) The vowels and consonants alternate in the words,

(iii) The vowels are always consecutive,

(iv) The relative order of vowels and consonants does not change,

(v) No vowel is in between two consonants,

(vi) The order of vowels does not change? **(JEE MAIN)**

Sol: This is an application of Permutation under restriction and Permutation of Alike objects. Proceed according to the given conditions.

(i) The required number of words = (the number of words without restriction) – (the number of words beginning with I) – (the number of word ending with E) + (the number of words beginning with I and ending with E) (because words beginning with I as well as words ending with E contain some words beginning with I and ending with E).

The number of words without restriction = $\frac{12!}{2!\,3!\,2!}$

($\because$ There are 12 letters in which there are two I's, three E's and two T's).

The number of words beginning with I = $\frac{11!}{2!3!}$

($\because$ With E in the extreme left place we are left to arrange 11 letters INTERMEDIATE in which there are two T's and three E's).

The number of words ending with E = $\frac{11!}{2!2!2!}$

($\because$ With E in the extreme right place we are left to arrange 11 letters INTERMEDIATE in which there are two I's. two E's and two T's.)

The number of words beginning with I and ending with E = $\frac{10!}{2!2!}$

($\because$ With I in the extreme left and E in the extreme right places we are left to arrange 10 letters INTERMEDIATE in the which there are two T's and Two E's).

$\therefore$ the required number or words = $\frac{12!}{2!3!2!}-\frac{11!}{2!3!}-\frac{11!}{2!2!2!}+\frac{10!}{2!2!}=\frac{10!}{2!3!2!}(12\times11-11\times2-11\times3+6) = \frac{83\times10!}{24}$

(ii) There are 6 vowels and 6 consonants. So, the number of words in which vowels and consonants alternate = (the number of words in which vowels occupy odd places and consonants occupy even places) + (the number of words in which consonants occupy odd places and vowels occupy even places)

$= \frac{6!}{2!3!}\times\frac{6!}{2!}+\frac{6!}{2!}\times\frac{6!}{2!3!} = 2.\frac{6!}{2!3!}.\frac{6!}{2!} = 43200$

(iii) Considering the 6 vowels IEEIAE as one object, the number of arrangements of this with 6 consonants = $\frac{7!}{2!}$ ($\because$ there are two T's in the consonants).

For each of these arrangements, the 6 consecutive vowels can be arranged among themselves in $\frac{6!}{2!3!}$.

$\therefore$ The required number of words = $\frac{7!}{2!}\times\frac{6!}{2!3!}$ (as above) = 151200

(iv) The relative order of vowels and consonants will not change if in the arrangements of letters, the vowels occupy places of vowels, i.e.. 1^{st}, 4^{th}, 7^{th}, 9^{th}, 10^{th}, 12^{th} places and consonants occupy their places, i.e., 2nd. 3rd. 5th. 6th. 8th. 11th places, the required number of words

$\frac{6!}{2!3!}\times\frac{6!}{2!} = 21600$

(v) No vowel will be between two consonants if all the consonants become consecutive

$\therefore$ the required number of words = the number of arrangements when all the consonants are consecutive

$= \frac{7!}{2!3!}\times\frac{6!}{2!}$ (as above) = 151200

(vi) The order of vowels will not change if no two vowels interchange places, i.e., in the arrangement all the vowels are treated as identical.

(For example LATE. ATLE. TLAE. etc.. have the same order of vowels A. E. But LETA, ETLA.

TLEA. etc.. have changed order of vowels A. E. So. LATE is counted but LETA is not.

If A, E, are taken as identical say V then LVTV does not give a new arrangement by interchanging V, V.

The required number of words,

= The number of arrangements of 12 letters in which 6 vowels are treated as identical

$= \frac{12!}{6!2!}$ ($\because$ there are two T's also).

Illustration 25: India and South Africa play a one day international series until one team wins 4 matches. No match ends in a draw. Find, in how many ways can the series can be won. **(JEE ADVANCED)**

Sol: The team who wins the series is the team with more number of wins. The losing team wins either 0 or 1 or 2 or 3 matches. Using this we find the number of ways in which a team can win.

Let I for India and S for South Africa. We can arrange I and S to show the wins for India and South Africa respectively

Suppose India wins the series, then the last match is always won by India.

	Wins of S	Wins of I	No. of ways
(i)	0	4	1
(ii)	1	4	4! / 3! = 4
(iii)	2	4	$\frac{5!}{2!3!} = 10$
(iv)	3	4	$\frac{6!}{3!3!} = 20$

$\therefore$ Total no. of ways = 35

In the same number of ways South Africa can win the series

$\therefore$ Total no. of ways in which the series can be won = 35 × 2 = 70

Illustration 26: There are p intermediate railway stations on a railway line from one terminal to another. In how many ways can a train stop at three of these intermediate stations, if no two of these stations (where it stops) are to be consecutive? **(JEE ADVANCED)**

Sol: The train stops only at three intermediate stations implies that the train does not stop at (p – 3) stations. Using this idea we proceed further to get the answer.

The problem then reduces to the following:

In how many ways can three objects be placed among (p – 3) objects in a row such that no two of them are next to each other (at most 1 object is to be placed between any two of these (p – 3) objects). Since there are (p – 2) positions to place the three objects, the required number of ways = ${}^{p-2}C_3$.

Illustration 27: Five balls are to be placed in three boxes. Each can hold all the five balls. In how many different ways can we place the balls so that no box remains empty if

(i) balls and boxes are all different

(ii) balls are identical but boxes are different

(iii) balls are different but boxes are identical

(iv) balls as well as boxes are identical

(v) balls as well as boxes are identical but boxes are kept in a row? **(JEE MAIN)**

Sol: Use the different cases of combination to solve the question according to the given conditions

As no box is to remain empty, boxes can have balls in the following numbers:

Possibilities 1, 1, 3 or 1, 2, 2

(i) The number of ways to distribute the balls in groups of 1, 1, 3 = ${}^5C_1 \times {}^4C_1 \times {}^3C_3$.

But the boxes can interchange their content, no exchange gives a new way when boxes containing balls in equal number interchange.

$\therefore$ the total number of ways to distribute 1, 1, 3 balls to the boxes = ${}^5C_1 \times {}^4C_1 \times {}^3C_3 \times \frac{3!}{2!}$

Similarly, the total number of ways to distribute 1, 2, 2 balls to the boxes = ${}^5C_1 \times {}^4C_2 \times {}^2C_2 \times \frac{3!}{2!}$

$\therefore$ the required number of ways = ${}^5C_1 \times {}^4C_1 \times {}^3C_3 \times \frac{3!}{2!} + {}^5C_1 \times {}^4C_1 \times {}^2C_2 \times \frac{3!}{2!} = 5 \times 4 \times 3 + 5 \times 6 \times 3 = 60 + 90 = 150$

Note: Writing the whole answer in tabular form.

Possibilities	Combinations	Permutations
1, 1, 3	${}^5C_1 \times {}^4C_1 \times {}^3C_3$	${}^5C_1 \times {}^4C_1 \times {}^3C_3 \times \frac{3!}{2!} = 5 \times 4 \times 3 = 60$
1, 2, 2	${}^5C_1 \times {}^4C_2 \times {}^2C_2$	${}^5C_1 \times {}^4C_2 \times {}^2C_2 \times \frac{3!}{2!} = 5 \times 6 \times 3 = 90$

$\therefore$ the required number of ways = 60 + 90 = 150.

(ii) When balls are identical but boxes are different the number of combinations will be 1 in each case.

$\therefore$ the required number of ways = $1 \times \frac{3!}{2!} + 1 \times \frac{3!}{2!} = 3 + 3 = 6$

(iii) When the balls are different and boxes are identical, the number of arrangements will be 1 in each case.

$\therefore$ the required number of ways = ${}^5C_1 \times {}^4C_1 \times {}^3C_3 + {}^5C_1 \times {}^4C_2 \times {}^2C_2 = 5 \times 4 + 5 \times 6 = 20 + 30 = 50$

(iv) When balls as well as boxes are identical, the number of combinations and arrangements will be 1 each in both cases.

$\therefore$ the required number of ways = $1 \times 1 + 1 \times 1 = 2$

(v) When boxes are kept in a row, they will be treated as different. So, in this case the number of ways will be the same as in (ii).

Illustration 28: There are m points on one straight line AB and n points on another straight line AC, none of them being A. How many triangles can be formed with these points as vertices? How many can be formed if point A is also included? **(JEE MAIN)**

Sol: A triangle has three vertices, so we select two points on one line and one on the other and vice versa. Also, consider the case when one point of the triangle is the intersection of the two lines.

To get a triangle, we either take two points on AB and one point on AC. or one point on AB and two points on AC. Therefore, the number of triangles, we obtain

$$= ({}^mC_2)({}^nC_1) + ({}^mC_1)({}^nC_2) = \frac{m(m-1)}{2}n + m\frac{n(n-1)}{2} = \frac{1}{2}\,mn(m - 1 + n - 1) = \frac{1}{2}mn(m + n - 2)$$

If the point A is included, we get m n additional triangles. Thus, in this case we get

$$= \frac{mn}{2}(m + n - 2) + mn = \frac{mn(m+n)}{2} \text{ triangles.}$$

5.3 Division into Groups

(a) The number of ways in which (m + n) different objects can be divided into two unequal groups containing m and n objects respectively is $\frac{(m+n)!}{m!n!}$.

If m = n, the groups are equal and in this case the number of division is $\frac{(2n)!}{n!n!2!}$; as it is possible to interchange the two groups without obtaining a new distribution.

(b) However, if 2n objects are to be divided equally between two persons then the number of ways

$= \frac{(2n)!}{n!n!2!}2! = \frac{(2n)!}{n!n!}$

(c) The number of ways in which (m + n + p) different objects can be divided into three unequal groups containing m , n and p objects respectively is $= \frac{(m+n+p)}{m!n!p!}$, $m \neq n \neq p$

If m = n = P then the number of groups = $\frac{(3n)!}{n!n!n!3!}$. However, if 3n objects are to be divided equally among three persons then the number of ways = $\frac{(3n)!}{n!n!n!3!}3! = \frac{(3n)!}{(n!)^3}$

For example, the number of ways in which 15 recruits can be divided into three equal groups is $\frac{15!}{5!5!5!3!}$ and the number of ways in which they can be drafted into three different regiments, five in each, is $\frac{15!}{5!5!5!}$

(d) The number of ways in which mn different objects can be divided equally into m groups if order of groups is not important is $\frac{mn!}{(n!)^m m!}$

(e) The number of ways in which mn different objects can be divided equally into m groups if the order of groups is important is $\frac{mn!}{(n!)^m m!} \times m! = \frac{(mn)!}{(n!)^m}$

Illustration 29: In how many ways can 12 balls be divided between 2 boys, one receiving 5 and the other 7 balls? **(JEE MAIN)**

Sol: Simple application of division of objects into groups. Since the order is important, the number of ways in which 12 different balls can be divided between two boys who each get 5 and 7 balls respectively, is

$\frac{12!}{5!7!} \times 2! = \frac{12.11.10.9.8.7!}{(5.4.3.2.1)7!} \times 2 = 1584$

Alternative:

The first boy can be given 5 balls out of 12 balls in ${}^{12}C_5$ ways. The second boy can be given the remaining 7 balls in one way. But the order is important (the boys can interchange 2 ways).

Thus, the required number of ways = ${}^{12}C_5 \times 1 \times 2! = \frac{12!}{5!7!} \times 2 = \frac{12.11.10.9.8.7!.2}{5.4.3.2.1.7!} = 1584$

Illustration 30: Find the number of ways in which 9 different toys can be distributed among 4 children belonging to different age groups in such a way that the distribution among the 3 elder children is even and the youngest one is to receive one toy more. **(JEE ADVANCED)**

Sol: Using the concept of division of objects into groups we can solve this problem very easily.

The distribution should be 2, 2, 2 and 3 to the youngest. Now, 3 toys for the youngest can be selected in ${}^{9}C_3$ ways, the remaining 6 toys can be divided into three equal groups in

$\frac{6!}{(2!)^3.3!}$ ways and can be distributed in 3 ! ways.

Thus, the required number of ways = ${}^{9}C_3.\frac{6!}{(2!)^3 3!}3! = \frac{9!}{3!(2!)^3}$

Illustration 31: Divide 50 objects in 5 groups of size 10, 10, 10, 15 and 5 objects. Also find the number of distributions? **(JEE MAIN)**

Sol: Same as the above question. Number of ways of dividing 50 objects into 5 groups as given = $\frac{50!}{(10!)^3(15)!(5)!(3)!}$

Number of ways of distributing 50 objects into above formed groups = $\frac{50!}{(10)!^3.(15)!(5)!3!}\times 5!$

MASTERJEE CONCEPTS

Identical Objects and Distinct Choices

Questions involving identical objects tend to be tricky, especially when the choices that they have are distinct.

Many choices image 1

An example would be :

Q. In how many ways can we place 10 identical oranges in 3 distinct baskets, such that every basket has at least 2 oranges each?

One method is to place 2 oranges in every basket and make cases for the rest of them. However, in such questions, the other approach results in fewer cases, and hence, simpler calculations and a more efficient solution

For this question: Divide 10 into groups of 3 rather than placing 2 in each and dividing the remaining four.

Vaibhav Gupta (JEE 2009, AIR 54)

6. CIRCULAR PERMUTATION

Let us consider that persons A, B, C and D are sitting around a round table. If all of them (A, B, C, D) are shifted one place in an anticlockwise order, then we will get fig.(b) from fig.(a). Now, if we shift A, B, C, D in anticlockwise order again, we will get fig. (c). We shift them once more and we will get fig.(d); and in the next time fig(a).

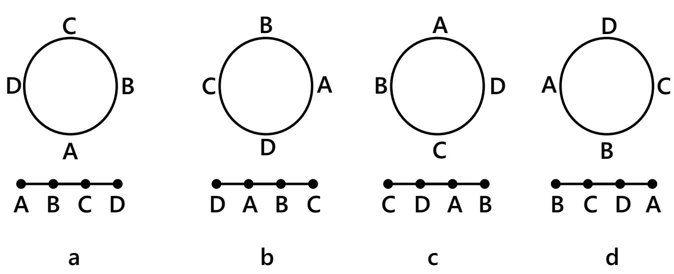


Figure 5.2

Thus, we see that if 4 persons are sitting at a round table, they can be shifted four times and the four different arrangements thus obtained will be same, because the anticlockwise order of A, B, C, D does not change. But if A, B, C and D are sitting in a row and they are shifted in such an order that the last occupies the place of first, then the four arrangements will be different.

Thus, if there are 4 objects, then for each circular arrangement number of linear arrangements is 4.

Similarly, if n different objects are arranged along a circle, for each circular arrangement the number of linear arrangements is n.

Therefore, the number of circular arrangements of n different objects = the number of linear arrangements of n different objects / n = n!/(n) = (n – 1)!

Clockwise and Anticlockwise Arrangements

Let the four persons A, B, C and D sit at a round table in anticlockwise as well as clockwise directions. These two arrangements are different. But if four flowers R (red), G (green), Y (yellow) and B (blue) are arranged to form a garland in anticlockwise and in clockwise order, then the two arrangements are same because if we view the garland from one side the four flowers R, G, Y, B will appear in anticlockwise direction and if seen from the other side the four flowers will appear in the clockwise direction. Here the two arrangements will be considered as one arrangement because the order of flowers does not change, rather only the side of observation changes. Here, two permutations will be counted as one.

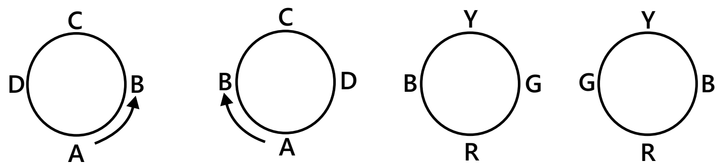


Figure 5.3

Therefore, when clockwise and anticlockwise arrangements are not different, i.e. when observations can be made from both sides, the number of circular arrangements of n different objects is $(n – 1)!/2$

Consider five persons A, B, C, D, E on the circumference of a circular table in an order which has no head. Now, shifting A, B, C, D, E one position in anticlockwise direction we will get arrangements as follows:

We see, that arrangements in all figures are different.

$\therefore$ The number of circular permutation of n different objects taken all at a time is $(n – 1)!$, if clockwise and anticlockwise orders are taken as different.

Note:

(a) The number of circular permutations of n different objects taken r at a time

$^nP_r/r$, when clockwise and anticlockwise orders are treated as different.

$^nP_r/2r$, when clockwise and anticlockwise orders are treated as same.

(b) The number of circular permutations of n different objects altogether

$^nP_n/n = (n – 1)!$, when clockwise and anticlockwise order are treated as different,

$^nP_n/2n = 1/2(n – 1)!$, when the above two orders are treated as same.

Illustration 32: In how many ways can 5 Indians and 4 Englishmen be seated at a round table if

(a) There is no restriction, **(b)** All the four Englishmen sit together,

(c) All four Englishmen don’t sit together, **(d)** No two Englishmen sit together. **(JEE MAIN)**

Sol: Clearly, this is a case of Circular Permutation. Using the formula $(n – 1)!$, we can find the answer according to the given cases.

(a) Total number of persons = 5 + 4 = 9. These 9 persons can be seated at the round table in 8! Ways.

$\therefore$ Required number of ways = 8!

(b) Regarding 4 Englishmen as one person, we have only 5 + 1 i.e. 6 persons.

These 6 persons can be seated at the round table in 5! ways. Also, the 4 Englishmen can be arranged among themselves in 4! ways.

$\therefore$ the required number of ways = 5! 4!

(c) The total number of arrangements when there is no restriction = 8!; the number of arrangements when all the four English men sit together = 5! 4!

$\therefore$ The number of arrangements when all the four Englishmen don’t sit together = 8! – 5! 4!

(d) As there is no restriction on Indians, we first arrange the 5 Indians.

Now, 5 Indians can be seated around a table in 4! ways. If an Englishman sits between two Indians, then no two Englishmen will sit together. Now, there are 5 places for 4 English men, therefore, 4 Englishmen can be seated in ${}^5P_4$ ways.

∴ The required number of ways = $4! \times {}^5P_4 = 4 \times 5!$

Illustration 33: Consider 21 different pearls on a necklace. How many ways can the pearls be placed in this necklace such that 3 specific pearls always remains together? **(JEE MAIN)**

Sol: This is the case of circular permutation when there is no distinction between clockwise and anticlockwise arrangements.

After fixing the places of three pearls. Treating 3 specific pearls = 1 unit. So we have now 18 pearls + 1 unit = 19 and the number of arrangements will be (19 – 1)! = 18!. Also the number of ways 3 pearls can be arranged between themselves is 3! = 6. As there is no distinction between the clockwise and anticlockwise arrangements, the required number of arrangements = $\frac{1}{2}18!.\ 6 = 3\ (18!)$.

Illustration 34: Six persons A, B, C, D, E and F are to be seated at a circular table. Find the number of ways this can be done if A must have either B or C on his right and B must have either C or D on his right. **(JEE MAIN)**

Sol: Fix the position of some of the persons relative to each other as per the question and arrange the remaining in the seats available.

When A has B or C to his right we have either AB or AC

When B has C or D to his right we have BC or BD.

Thus, we must have ABC or ABD or AC and BD.

For ABC, D, E, F in a circular number of ways = 3! = 6

For ABD, C, E, F in a circular number of ways = 3! = 6

For AC, BD E, F the number of ways = 3! = 6

Hence, the required number of ways = 18

7. LINEAR EQUATIONS WITH UNIT COEFFICIENTS

Excluded

Consider the equation $x_1 + x_2 + x_3 + \ldots + x_k = m$, in k variables whose sum must always be m,

The number of non-negative solutions to the above equation is given by the fictitious partition method which is stated as:

Method of fictitious partition: Number of ways in which n identical objects may be distributed among p persons if each person may receive none, one or more objects is = ${}^{n+p-1}C_n$.

Coefficient Method

(a) The number of non-negative integral solutions of equation $x_1 + x_2 + \ldots + x_r = n$

= The number of ways of distributing n identical objects among r persons when each person can get zero, one or more objects = coeff. of x^n in $[(1 + x + x^2 + \ldots + x^n)(1 + x + x^2 + \ldots + x^n)\ (1 + x + x^2 + \ldots + x^n)\ldots$upto r factors]

= coeff. of x^n in $(1 + x + x^2 + \ldots + x^n)^r$

= coeff. of x^n in $\left(\dfrac{1-x^{n+1}}{1-x}\right)^r$ = coeff. of x^n in $(1 - x^{n+1})^r\ (1 - x)^{-r}$ = coeff. of x^n in $(1 - x)^{-r}$

[leaving terms containing powers of x greater than n] = ${}^{n+r-1}C_{r-1}$

Note: If n is a positive integer, then

$$(1-x)^{-n} = 1 + \frac{(-n)}{1!}(x) + \frac{(-n)(-n-1)}{2!}(-x)^2 + \frac{(-n)(-n-1)(-n-2)}{3!}(-x)^3 + \ldots \text{ to } \infty$$

$$= 1 + \frac{n}{1!}x + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots. \text{ To } \infty = 1 + {}^{n}C_1x + {}^{n+1}C_2x^2 + {}^{2}C_3x^3 + \ldots \text{ to } \infty$$

Coeff. X^r in $(1-x)^{-n} = {}^{n+r-1}C \Rightarrow$ coeff. of x^n in $(1-x)^{-r} = {}^{n+r-1}C_n = {}^{n+r-1}C_{r-1}$

(b) The number of positive integral solutions for the equation $x_1 + x_2 + \ldots. + x_r = n$

= The number of ways of distributing n identical objects among r persons when each person can get at least one object

= coeff. of x^n in $[x + x^2 + \ldots + x^n)(x + x^2 + \ldots + x^n)(x + x^2 + \ldots + x^n)$ .... Upto r factors]

= coeff. of x^n in $(x + x^2 + \ldots + x^n)^r$ = coeff. of x^n in $x^r\left(\frac{1-x^n}{1-x}\right)^r$

= coeff. Of x^{n-r} in $(1-x^n)^r(1-x)^{-r}$ = coeff. Of x^{n-r} in $(1-x)^{-r}$

[Leaving terms containing powers of x greater than n – r]

$= {}^{n-r+r-1}C_{r-1} = {}^{n-1}C_{r-1}$.

Illustration 35: How many integral solutions are there to x + y + z + w = 29, when $x \ge 1, y \ge 2, z \ge 3$ and $w \ge 0$? **(JEE ADVANCED)**

Sol: Application of multinomial theorem.

$x + y + z + w = 29$ ....(i)

$x \ge 1, y \ge 2, z \ge 3, w \ge 0 \quad \Rightarrow \quad x-1 \ge 0, y-2 \ge 0, z-3 \ge 0, w \ge 0$

Let $x_1 = x - 1, x_2 = y - 2, x_3 = z - 3$

$\Rightarrow \quad x = x_1 + 1, y = x_2 + 2, z = x_3 + 3$ and then $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, w \ge 0$

From (i), $x_1 + 1 + x_2 + 2 + x_3 + 3 + w = 29$

$\Rightarrow \quad x_1 + x_2 + x_3 + w = 23$

Hence, the total number of solutions $= {}^{23+4-1}C_{4-1} = {}^{26}C_3 = 2600$

Illustration 36: Find the number of non-negative integral solution 3x + y + z = 24. **(JEE MAIN)**

Sol: Application of multinomial theorem.

$3x + y + z = 24, x \ge 0, y \ge 0, z \ge 0$

Let $x = k \quad \therefore y + z = 24 - 3k$ ....(i)

Here, $0 \le 24 - 3k \le 24$. Hence, $0 \le k \le 8$

The total number of integral solutions of (1) is ${}^{24-3k+2-1}C_{2-1} = {}^{25-3k}C_1 = 25 - 3k$

Hence, the total number of solutions of the original equation

$$= \sum_{k=0}^{8}(25-3k) = 25\sum_{k=0}^{8}1 - 3\sum_{k=0}^{8}k \Rightarrow 25.9 - 3.\frac{8.9}{2} = 225 - 108 = 117.$$

Illustration 37: Find the number of solutions of the equation x + y + z = 6, where $x, y, z \in W$. **(JEE MAIN)**

Sol: The number of solutions $= {}^{6+3-1}C_{3-1} = {}^{8}C_2 = 280$.

Illustration 38: How many integers are there between 1 and 1000000 having the sum of the digits as 18?

(JEE ADVANCED)

Sol: Let the digits be a_1, ..., a_6 and use multinomial theorem we get the answer.

Any number between 1 and 1000000 must be of less than seven digits. Therefore, it must be of the form $a_1\ a_2\ a_3\ a_4\ a_5\ a_6$

Where $a_1.\ a_2.a_3.a_4.a_5.a_6 \in \{0, 1, 2, \ldots 9\}$

Thus $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 18$, where $0 \le a_i \le 9$, i = 1, 2, ... 9

The required number of ways = coeff. of x^{18} in $(1 + x + x^2 + \ldots x^9)^6$

$$= \text{coeff. of } x^{18} \text{ in } \left(\frac{1-x^{10}}{1-x}\right)^6$$

$= \text{coeff. of } x^{18} \text{ in } [(1 - x^{10})^6 (1 - x)^{-6}]$; $= \text{coeff. of } x^{18} \text{ in } [(1 - {}^6C_1\, x^{10} \ldots) (1 - x)^{-6}]$

[leaving terms containing powers of x greater than 18]

$= \text{coeff. of } x^{18} \text{ in } (1 - x)^{-6} - {}^6C_1.\ \text{coeff. of } x^8 \text{ in } (1 - x)^{-6} = {}^{6+18-1}C_5 - 6.\ {}^{6+8-1}C_5 = {}^{23}C_5 - 6.{}^{13}C_5$

$$= \frac{23.22.21.20.19}{120} - 6.\ \frac{13.12.11.10.9}{120} = 33649 - 7722 = 25927$$

MASTERJEE CONCEPTS

- m different white balls and n different red balls are to be arranged in a line such that the balls of the same colour are always together = (m! n! 2!)
- m different white balls and n different red balls are to be arranged in a line such that all the red balls are together = ((m + 1)! n!)
- m different white balls and n different red balls are to be arranged in a line such that no two red balls are together $(m \ge n - 1) = ({}^{m+1}C_n\ m!\ n!)$
- m different white balls and m different red balls are to be arranged in a line such that colour of the balls is alternating $= (2 \times (m!)^2)$
- m identical white balls and n different red balls are to be arranged in a line such that no two red balls are together $(m \ge n - 1) = ({}^{m+1}C_n)$
- m identical white balls and n different red balls are to be arranged in a line such that no two red balls are together $(m \ge n - 1) = ({}^{m+1}C_n\ n!)$
- If n objects are arranged in a line the number of selections of r objects $(n \ge 2r - 1)$ such that no two objects are adjacent is same number of ways of arranging n – r identical white balls and r identical red ball in a line such that no two balls are together $= ({}^{n-r+1}C_r)$. e.g. suppose there are n stations on trains's route and a train has to stop at r stations such that no two stations are adjacent. The number of ways must be ${}^{n-r+1}C_r$.
- suppose there are N seats in a particular row of a theatre. The number of ways of making n people sit $(N \ge 2n - 1)$ such that no two people sit side by side is same as number of ways of arranging N – n identical white balls (empty seats) and n different red balls (n people) such that no two red balls are together. The required number of ways are ${}^{N-n+1}C_n \times n!$.

Nitish Jhawar (JEE 2009, AIR 7)

8. DIVISIBILITY OF NUMBERS

This is an application of previous concept.
Excluded.

The following table shows the conditions of divisibility of some numbers

Divisible by	Condition
2	Whose last digit is even
3	sum of whose digits is divisible by 3
4	whose last two digits number is divisible by 4
5	whose last digit is either 0 or 5
6	which is divisible by both 2 and 3
7	If you double the last digit and subtract it from the rest of the number, answer is a multiple of 7
8	whose last three digits number is divisible by 8
9	sum of whose digits is divisible by 9
10	Whose last digit is 0
11	If you sum every second digit and then subtract sum of all other digits, answer is a multiple of 11
25	whose last two digits are divisible by 25

Illustration 39: How many four digit numbers can be made with the digits 0, 1, 2, 3, 4, 5 which are divisible by 3 (digits being unrepeated in the same number)? How many of these will be divisible by 6? **(JEE ADVANCED)**

Sol: A number is divisible by 3 if the sum of the digits is divisible by 3. This reduces the problem to the number of non-negative integral solutions of equation $x_1 + x_2 + \ldots + x_r = n$.

Here, 0 + 1 + 2 + 3 + 4 + 5 = 15; so two digits are to be omitted whose sum is 3 or 6 or 9.

Hence, the number of four digits can be made by either

1, 2, 4, 5 or 0, 3, 4, 5 (omitting two digits whose sum is 3)

0, 1, 3, 5 or 0, 2, 3, 4 (omitting two digits whose sum 6)

0, 1, 2, 3 (omitting two digits whose sum is 9)

The number of 4-digit numbers that can be made with 1, 2, 4, 5 = ${}^4P_4 = 4!$

The number of 4-digit numbers that can be made by the digits in any one of remaining four groups (each containing 0) = $4! - 3!$

$\therefore$ The required number of 4-digit numbers divisible by 3 = $4! + 4(4! - 3!) = 24 + 4(24 - 6) = 96$

Now, a number is divisible by 6 if it is even as well as divisible by 3.

So, the number of 4-digit numbers divisible by 6 that can be made with 1, 2, 4, 5 = $2 \times 3!$ ($\because$ the number should have an even digit in the units places).

The number of numbers of 4 digits, divisible by 6, that can be made with 0, 3, 4, 5 = $(3! - 2!) + 3!$

($\because$ The number should have 4 or 0 in units place and 0 should not come in thousands place).

Similarly, the number of numbers of 4 digits, divisible by 6, that can be made with 0, 1, 2, 3 = $(3! - 2!) + 3!$

The number of 4-digit numbers divisible by 6 that can be made with the digits 0, 1, 3 = $3!$

The number of numbers of 4 digits, divisible by 6, that can be made with 0, 2, 3, 4 = $(3! - 2!) + (3! - 2!) + 3!$

($\because$ The number should have 4 or 2 or 0 in units place and 0 should not come in thousands place)

∴ the required 4-digit numbers divisible by 6

= 2 × 3! + (3! – 2!) + (3! – 2!) + 3! + 3! + (3! – 2!) + (3! – 2!) + (3! – 2!) + 3! = 12 + 4 + 6 + 4 + 6 + 6 + 4 + 4 + 6 = 52.

9. SUM OF NUMBERS

(a) For given n different digits $a_1, a_2, a_3 \ldots a_n$ the sum of the digits in the units place of all numbers formed (if numbers are not repeated) is

$(a_1 + a_2 + a_3 + \ldots + a_n)(n - 1)!$ i.e. (sum of the digits) $(n - 1)!$

(b) Sum of the total numbers which can be formed with given different digits $a_1, a_2, a_3 \ldots\ldots a_n$ is

$(a + a_2 + a_3 + \ldots + a_n)(n - 1)!$ (111. .... n times)

Illustration 40: Find the sum of all 4 digit numbers formed using the digits 1, 2, 4 and 6. **(JEE MAIN)**

Sol: Use formula, Sum = $(a_1 + a_2 + a_3 + \ldots + a_n)(n-1)!$ (111 .... N times)

Using formula, Sum = (1 + 2 + 4 + 6) 3! (1111) = 13 × 6 × 1111 = 86658

Alternate:

Here, the total 4-digit numbers will be 4! = 24. So, every digit will occur 6 times at every one of the four places. Since the sum of the given digits = 1 + 2 + 4 + 6 = 13. So, the sum of all the digits at every place of all the 24 numbers = 13 × 6 = 78.

The sum of the values of all the digits

At first place = 78

At the tens place = 780

At the hundreds place = 7800

At the thousands place = 78000

∴ The required sum 78 + 780 + 7800 + 78000 = 86658

10. FACTORS OF NATURAL NUMBERS

Excluded

Let $N = p^a \cdot q^b \cdot r^c \ldots$ where p, q, r ... are distinct primes & a, b, c ... are natural numbers, then:

(a) The total number of divisors of N including 1 and N are = (a + 1) (b + 1) (c + 1) ...

(b) The sum of these divisors is = $(p^0 + p^1 + p^2 + \ldots + p^a)(q^0 + q^1 + q^2 + \ldots + q^b)(r^0 + r^1 + r^2 + \ldots + r^c)\ldots$

(c) The number of ways in which N can be resolved as a product of two factors is

$$= \begin{cases} \frac{1}{2}(a_1+1)(a_2+1)(a_3+1)\ldots & \text{if N is not a perfect square} \\ \frac{1}{2}\left[(a_1+1)(a_2+1)(a_3+1)\ldots+1\right] & \text{if N is a perfect square} \end{cases}$$

(d) The number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N

Illustration 41: Find the number of factors of the number 38808 (excluding 1 and the number itself). Find also the sum of these divisors. **(JEE MAIN)**

Sol: Factorise 38808 into its product of primes and then use the concept of combination to find the answer.

$38808 = 2^3 \cdot 3^2 \cdot 7^2 \cdot 11$

Hence, the total number of divisors (excluding 1 and itself) = (3 + 1) (2 + 1) (2 + 1) (1 + 1) – 2 = 70

The sum of these divisors = $(2^0 + 2^1 + 2^2 + 2^3)(3^0 + 3^1 + 3^2)(7^0 + 7^1 + 7^2)(11^0 + 11^1)$

$= (15)(13)(57)(12) - 1 - 38808 = 94571$

Illustration 42: In how many ways can the number 10800 be resolved as a product of two factors? **(JEE MAIN)**

Sol: Check whether the number is a perfect square or not and accordingly use the formula to find the desired result.

$10800 = 2^4 . 3^3 . 5^2$

Here 10800 is not a perfect square ($\because$ power of 3 is odd).

Hence, the number of ways = $\frac{1}{2}(4 + 1)(3 + 1)(2 + 1) = 30$.

Illustration 43: Find the number of positive integral solutions of $x_1.x_2.x_3 = 30$. **(JEE ADVANCED)**

Sol: Factorise 30 into primes and then use combination to get the desired result.

$x_1x_2x_3 = 2 \times 3 \times 5$. If we treat 2, 3, 5 as objects and x_1, x_2, x_3 as distinct boxes then finding the number of positive integer solution is the same as finding the number of ways of distributing 3 distinct objects in 3 distinct boxes. Thus, the required number of solutions is $3^5 = 27$

(For example, if all the objects are held by x_1 the corresponding solution is $x_1 = 30$. $x_2 = 1$, $x_3 = 1$, if 2 and 3 are held by x_1 and 5 by x_3 then $x_1 = 6$, $x_2 = 1$, $x_3 = 5$ etc)

11. EXPONENT OF A PRIME P IN N!

Excluded

Consider a prime p and we want to know its exponent in n!.

The number of multiples of p in n! is given by $\left[\frac{n}{p}\right]$. Even if a number k between 1 and n has two factors of p, this formula counts it as only one. Hence we need to evaluate $\left[\frac{n}{p^2}\right]$ also. Similarly, for three and four to infinity. Hence, the exponent of prime p in n! is given by $e_p(n) = \sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right]$ e_p is called Legendre's function.

Even though this is an infinite sum result, it is finite since for all p^i greater than n, step function becomes zero.

Let's understand this method by an example of finding exponent of 2 in 100!

Sol: $\left[\frac{100}{2}\right] = 50; \left[\frac{100}{2^2}\right] = 25; \left[\frac{100}{2^3}\right] = 12; \left[\frac{100}{2^4}\right] = 6; \left[\frac{100}{2^5}\right] = 3; \left[\frac{100}{2^6}\right] = 1; \left[\frac{100}{2^7}\right] = 0$

From this result we can infer that there are 50 numbers between 1 and 100 which have a factor of 2.

Out of these 50, there are 25 numbers that have a factor of 2^2.

Out of these 25, there are 12 numbers that have a factor of 2^3.

Out of these 12, there are 6 numbers that have a factor of 2^4.

Out of these 6, there are 3 numbers that have a factor of 2^5.

Out of these 3, there is 1 number that has a factor of 2^6.

But there is no number that has a factor of 2^7 or higher.

$\Rightarrow$ 100! can be written as $r.2^n$ where $2^n = (2^6)^1 (2^5)^{(3-1)} (2^4)^{(6-3)} (2^3)^{(12-6)} (2^2)^{(25-12)} (2)^{(50-25)}$ and r being a natural number which doesn't have 2 as a factor.

$\Rightarrow 2^n = (2^5.2)^1\ (2^4.2)^{(3-1)}(2^3.2)^{(6-3)}\ (2^2.2)^{(12-6)}\ (2.2)^{(25-12)}\ (2)^{(50-25)}$

$= (2^5)^1(2^4)^{(3-1)}\ (2^3)^{(n-3)}(2^2)^{(12-6)}\ (2)^{(25-12)}\ (2)^{50}$

$= (2^4.2)^1\ (2^3.2)^{(3-1)}\ (2^2.2)^{(6-3)}\ (2.2)^{(12-6)}\ (2)^{(25-12)}\ (2)^{50}$

$= (2^3.2)^1\ (2^2.2)^{(3-1)}\ (2.2)^{(6-3)}\ (2)^{(12-6)}\ (2)^{25}\ (2)^{50}$

$= (2^2.2)^1\ (2.2)^{(3-1)}\ (2)^{(6-3)}\ (2)^{12}\ (2)^{25}\ (2)^{50}$

$= (2.2)^1\ (2)^{(3-1)}\ (2)^6\ (2)^{12}\ (2)^{25}\ (2)^{50}$

$= (2)^1\ (2)^3\ (2)^6\ (2)^{12}\ (2)^{25}\ (2)^{50}$

Hence, the exponent of 2 in 100! is = 50 + 25 + 12 + 6 + 3 + 1 = 97

MASTERJEE CONCEPTS

Following method involves part of an advanced topic in mathematics called Modular Arithmetic. Legendre's function also has another result, which is

$$e_p(n) = \sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right] = \frac{n - S_p(n)}{p-1}$$

Where $S_p(n)$ is the sum of digits of n when written in base p.

Converting n to base p is done by repeated division of n by p and by noting the remainders to form a number starting with units place.

This procedure is similar to converting a decimal number to binary. In binary, the base is equal to 2. Let us solve an example using this method.

Example: Determine the exponent of 3 in ((3!)!)!

Sol: ((3!)!)! = (6!)! = 720!

Let us convert 720 to base 3

3 | 720
3 | 240 R 0
3 | 80 R 0
3 | 26 R 2
3 | 8 R 2
3 | 2 R 2
3 | 0 R 2

Hence 720 = $(222200)_3$

$S_3(720) = 2 + 2 + 2 + 0 + 0 = 8$

$$\Rightarrow e_3(720) = \frac{720-8}{3-1} = \frac{712}{2} = 356$$

You can verify this answer using previous method.

If you are confused by base conversion then do not use this method.

Akshat Kharaya (JEE 2009, AIR 235)

Illustration 44: Find the exponent of 7 in 400!. **(JEE MAIN)**

Sol: Apply Legendre's formula.

$$e_7(400) = \left[\frac{400}{7}\right] + \left[\frac{400}{7^2}\right] + \left[\frac{400}{7^3}\right] = 57 + 8 + 1 = 66.$$

Illustration 45: Find all positive integers of n such that n! ends in exactly 1000 zeros. **(JEE ADVANCED)**

Sol: 10 is a multiple of 2 and 5. In order to get 1000 zeroes we must have 1000 as the exponent of 5 in n!. Now use the definition of the GIF to find the range of numbers satisfying the given condition.

There are clearly more 2's than 5's in the prime factorization of n!, hence it suffices to solve the equation $\left[\frac{n}{5}\right] + \left[\frac{n}{5^2}\right]$ + ... = 1000.

But $\left[\frac{n}{5}\right] + \left[\frac{n}{5^2}\right] + \ldots < \frac{n}{5} + \frac{n}{5^2} + \ldots = \frac{n}{5}\left(1 + \frac{1}{5} + \ldots\right)$ (as [x] < x) $= \frac{n}{5} \cdot \frac{1}{1 - \frac{1}{5}} = \frac{n}{4}$.

Hence, n > 4000.

On the other hand, using the inequality [x] > x – 1, we have

$$1000 > \left(\frac{n}{5} - 1\right) + \left(\frac{n}{5^2} - 1\right) + \left(\frac{n}{5^3} - 1\right) + \left(\frac{n}{5^4} - 1\right) + \left(\frac{n}{5^5} - 1\right) = \frac{n}{5}\left(1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4}\right) - 5 = \frac{n}{5} \cdot \frac{1 - \left(\frac{1}{5}\right)^5}{1 - \frac{1}{5}} - 5.$$

So, $N < \frac{1005 \cdot 4 \cdot 3125}{3124} < 4022$.

We narrowed n down to {4001, 4002, ... , 4021}. Using Legendre's formula we find that 4005 is the first positive integer with the desired property and that 4009 is the last. Hence, n = 4005, 4006, 4007, 4008, 4009.

Second solution: It suffices to solve the equation $e_5(n) = 1000$. Using the second form of Legendre's formula, this becomes $n - s_5(n) = 4000$. Hence n > 4000. We work our way upward from 4000 looking for a solution. Since $e_5(n)$ can change only at multiples of 5 (why?), we step up 5 each time:

$$e_5(4000) = \frac{4000 - 4}{5 - 1} = 999.$$

$$e_5(4005) = \frac{4005 - 5}{5 - 1} = 1000.$$

$$e_5(4010) = \frac{4010 - 6}{5 - 1} = 1001.$$

Any n > 4010 will clearly have $e_5(n) \ge e_5(4010) = 1001$. Hence the only solutions are n = 4005, 4006, 4007, 4008, 4009.

12. INCLUSION-EXCLUSION PRINCIPLE

In its general form, the principle of inclusion-exclusion states that for finite sets $A_1, \ldots A_n$. One has the identity

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n-1} |A_1 \cap \ldots \cap A_n| \cdot$$

This can be compactly written as $\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1 \le i_1 < \ldots\ldots < i_k \le n} |A_{i_1} \cap \ldots \cap A_{i_k}| \right)$

In words, to count the number of elements in a finite union of finite sets, first sum the cardinalities of the individual sets, then subtract the number of elements which appear in more than one set, then add back the number of elements which appear in more than two sets, then subtract the number of elements which appear in more than three sets, and so on. This process naturally ends since there can be no elements which appear in more than the number of sets in the union.

In applications it is common to see the principle expressed in its complementary form. That is, taking S to be a finite universal set containing all of the A_i and letting $\overline{A_i}$ denote the complement of A_i in S. By De Morgan's laws.

We have, $\left|\bigcap_{i=1}^{n}\overline{A_i}\right| = \left|S - \bigcup_{i=1}^{n} A_i\right| = |S| = \sum_{i=1}^{n}|A_i| + \sum_{1 \le i < j \le n}|A_i \cap A_j| - \ldots + (-1)^n |A_1 \cap \ldots \cap A_n|$.

Illustration 46: 105 students take an examination of whom 80 students pass in English. 75 students pass in Mathematics and 60 students pass in both subjects. How many students fail in both subjects? **(JEE MAIN)**

Sol: A simple application of Inclusion-Exclusion Principle.

Let X = the set of students who take the examination.

A = the set of students who pass in English

B = the set of students who pass in Mathematics

We are given that n(X) = 105, n(A) = 80, n(B) = 75, $n(A \cap B) = 60$

Since, $n(A \cup B) = n(A) + n(B) - n(A \cup B)$.

Therefore, $n(A \cup B) = 80 + 75 - 60 = 95$.

The required number = $n(X) - n(A \cup B) = 105 - 95 = 10$.

Thus, 10 students fail in both subjects.

Illustration 47: Find the number of permutations of the 8 letters AABBCCDD, taken all at a time, such that no two adjacent letters are alike. **(JEE ADVANCED)**

Sol: Divide the question into cases when A's are adjacent, B's are adjacent and so on. Similarly proceed to find the number of ways in which two alike objects are adjacent and so on. Then use Inclusion-Exclusion Principle to find the result.

First disregard the restriction that no two adjacent letters be alike.

The total number of permutation is then $N = \frac{8!}{2!2!2!2!} = 2250$

Now, apply the inclusion exclusion principle. Where a permutation has property α in case the A's are adjacent, property β in case the B's are adjacent, etc. It can be calculated that

$N(\alpha) = \frac{7!}{2!2!2!} = 630.$ $\quad N(\alpha, \beta) = \frac{6!}{2!2!} = 180$

$N(\alpha, \beta, \gamma) = 60.$ $\quad N(\alpha, \beta, \gamma, \delta) = 24.$

Hence, the answer is $N - 4N(\alpha) + 6N(\alpha, \beta) - 4N(\alpha, \beta, \gamma) + N(\alpha, \beta, \gamma, \delta) = 864$.

13. DERANGEMENTS THEOREM

Excluded

Derangements theorem is an important application of inclusion exclusion principle.

Suppose, there are n letters and n corresponding envelopes. The number of ways in which letters can be placed in the envelopes (one letter in each envelope) so that no letter is placed in correct envelope is $n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{(-1)^n}{n!}\right]$

Proof: n letters are denoted by 1, 2, 3, ... , n. Let, A_i denote the set of distribution of letters in envelopes (one letter in each envelope) so that the i^{th} letter is placed in the corresponding envelope. Then, $n(A_i) = (n-1)!$ [since the remaining $n-1$ letters can be placed in $n-1$ envelopes in $(n-1)!$ Ways] Then, $n(A_i \cap A_j)$ represents the number of ways where letters i and j can be placed in their corresponding envelopes. Then $N(A_i \cap A_j) = (n-2)!$

Also, $n(A_i \cap A_j \cap A_k) = (n-3)!$

Hence, the required number is $n(A'_1 \cup A'_2 \cup \ldots \cup A'_n) = n! - n(A_i \cup A_2 \cup \ldots \cup A_n)$

$= n! - \left[\Sigma n(A_i) - \Sigma n(A_i \cap A_j) + \Sigma n(A_i \cap A_j \cap A_k) + \ldots + (-1)^n \Sigma n(A_1 \cap A_2 \ldots \cap A_n)\right]$

$= n! - [({}^nC_1)(n-1)! - {}^nC_2\,(n-2)! + {}^nC_3\,(n-3)! + \ldots + (-1)^{n-1} \times {}^nC_n 1]$

$= n! - \left[\frac{n!}{1!(n-1)}(n-1)!\frac{n!}{2!(n-2)!}(n-2)! + \ldots + (-1)^{n-1}\right] = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{(-1)^n}{n!}\right]$

Remark: If r objects go to wrong place out of n object then (n – r) objects goes to original place.

$A'_r = r!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^r \frac{1}{r!})$

MASTERJEE CONCEPTS

Number of derangements $D_n = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}$; Interestingly, as $n \to \infty$, $D_n = \frac{1}{e}$

This results in an interesting relating $D_n = \left[\frac{n!}{e}\right]$.Where [x] is the nearest integer function.

Use this formula only if the given options are wide apart from one another.

ExampleYou have 6 ball in 6 different colors, and for every ball you have a box of the same color. How many derangements do you have, if no ball is in a box of the same color?

Sol: We know that e = 2.71828. To make division simple let's round it to 2.7. You have to keep in mind that we have reduced the value of e, so the result which we get is greater than the actual result.

$\therefore D_n = \left[\frac{6!}{e}\right] = \left[\frac{720}{2.7}\right] = \left[\frac{800}{3}\right] = [266.66] = 267$

Hence, the result will be close to 266.

This is a pretty good approximation as the actual answer is 265.

But, if the given options are all close to 266, then it is advised to calculate using the original formula or by rounding the value of e to the number of significant digits equal to that of n! (numerator).

So if we use e = 2.72 we get 34

$D_n = \left[\frac{6!}{e}\right] = \left[\frac{720}{2.72}\right] = \left[\frac{4500}{17}\right] = [264.70] = 265$

Vaibhav Krishnan (JEE 2009, AIR 22)

Illustration 48: A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that (i) atleast two of them are in the wrong envelopes. (ii) All the letters are in the wrong envelopes. **(JEE MAIN)**

Sol: Application of Derangement theorem.

(i) The number of ways in which at least two of them in the wrong envelopes = $\sum_{r=2}^{6} {}^6C_{6-r}D_r$

$= {}^6C_{6-2}\,D_2 + {}^6C_{6-3}D_3 + {}^6C_{6-4}\,D_4 + {}^6C_{6-5}\,D_5 + {}^6C_{6-6}\,D_6$

$= {}^6C_4.2!\left(1 - \frac{1}{1!} + \frac{1}{2!}\right) + {}^6C_3.3!\left(1 - \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}\right) + {}^6C_2.4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$

$$+{}^6C_1.5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}\right)+{}^6C_0.6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right)$$

(ii) The number of ways in which all letters be placed in wrong envelopes

$$= 6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right); = 720\left(\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720}\right); = 360 - 120 + 30 - 6 + 1 = 265$$

Excluded

14. MULTINOMIAL THEOREM

(a) If there are l objects of one kind, m objects of a second kind, n objects of a third kind and so on; then the number of ways of choosing r objects out these (i.e., l + m + n ...) is the coefficient of x^r in the expansion of

$(1 + x + x^2 + x^3 + ... + xl)\ (1 + x + x^2 + x^3 + ... + x^m)\ (1 + x + x^2 + x^3 + ... + x^n)$

Further, if one object of each kind is to be included, then the number of ways of choosing r objects out of these objects (i.e., l + m + n + ...) is the coefficient of x^r in the expansion of

$(x + x^2 + x^3 + ... + xl)\ (x + x^2 + x^3 + ... + x^m)\ (x + x^2 + x^3 + ... + x^n)...$

(b) If there are objects of one kind, m objects of a second kind, n objects of a third kind and so on; then the number of possible arrangements/permutations of r objects out of these object (i.e., l + m + m + ...) is the coefficient of x^r in the expansion of

$$r!\left(1+\frac{x}{1!}+\frac{x^2}{2!}+.....\frac{x^f}{l!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+...\frac{x^m}{m!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+...\frac{x^n}{n!}\right)$$

Illustration 49: In an examination, the maximum marks for each of three papers is n and that for the fourth paper is 2n. Prove that the number of ways in which candidate can get 3n marks is

$\frac{1}{6}$ (n + 1) (5n^2 + 10n + 6). **(JEE ADVANCED)**

Sol: The maximum marks in the four papers are n, n, n and 2n. Consider a polynomial $(1 + x + x^2 + ... + x^n)^3 (1 + x + ... + x^{2n})$. The number of ways of securing a total of 3n is equal to the co-efficient of the term containing x^{3n}.

The number of ways of getting 3n marks

= coefficient of x^{3n} in $(1 + x + x^2 + ... + x^n)^3 (1 + x + ... + x^{2n})$

= coefficient of x^{3n} in $(1 - x^{n+1})^3 (1 - x^{2n+1}) (1 - x)^{-4}$

= coefficient of x^{3n} in $(1 - 3x^{n+1} + 3x^{2n+2} - x^{3n+3}) (1 - x^{2n+1}) \times (1 + {}^4C_1 x + {}^5C_2 x^2 + {}^6C_3 x^3 + ...)$

= coefficient of x^{3n} in $(1 - 3x^{n+1} - x^{2n+1} + 3x^{2n+2}) (1 + {}^4C_1 x + {}^5C_2 x^2 ...)$

$= {}^{3n+3}C_{3n} - 3.{}^{2n+2}C_{2n-1} + 3\ .\ {}^{n+1}C_{n-2} - {}^{n+2}C_{n-1}$

$$= \frac{(3n+3)!}{3!(3n)!} - 3\ .\ \frac{(2n+2)!}{3!(2n-1)!} + 3\,\frac{(n-1)!}{3!(n-2)!} - \frac{(n+2)!}{3!(n-1)!}$$

= 1/6 (n + 1) (27n^2 + 27n + 6 – 24n^2 – 12n + 3n^2 – 3n – n^2 – 2n) = 1/6 (n + 1) (5n^2 + 10n + 6)

PROBLEM-SOLVING TACTICS

In any given problem, first try to understand whether it is a problem of permutations or combinations. Now, think if repetition is allowed and then try solving problem.

A simple method to solve these problems where repetition is not allowed is as follows -

First draw series of dashes representing the number of places you want to fill or number of items you want to select.

Now start filling dashes by the number of objects available to choose from and multiply the numbers. This is the final answer for a permutations problem.

If it is a combination problem then divide the answer with the factorial or number of items.

This calculation becomes complex if repetition is allowed.

FORMULAE SHEET

(a) Permutation (Arrangement of Objects): Each of the different arrangement, which can be made by taking some or all of a number of objects is called permutation.

(i) The number of permutations of n different objects taken r at a time is ${}^nP_r = \frac{n!}{(n-r)!}$.

(ii) The number of all permutations of n distinct objects taken all at a time is n!.

Permutation with Repetition: The number of permutations of n different objects taken r at a time when each object may be repeated any number of times is n^r.

Permutation of Alike Objects: The number of permutations of n objects taken all at a time in which, p are alike objects of one kind, q are alike objects of second kind & r are alike objects of a third kind and the rest $(n - (p + q + r))$ are all different, is $\frac{n!}{p!q!r!}$.

Permutation under Restriction: The number of permutations of n different objects, taken all at a time, when m specified objects always come together is $m! \times (n - m + 1)!$.

(b) Combination (Selection of Objects): Each of the different groups or selection which can be made by some or all of a number of given objects without reference to the order of the objects in each group is called a combination.

The number of all combinations of n objects, taken r at a time is generally denoted by C(n, r) or ${}^nC_r = \frac{n!}{r!(n-r)!}$ $(0 \le r \le n) = \frac{{}^nP_r}{r!}$

Note:

(a) The number of ways of selecting r objects out of n objects, is the same as the number of ways in which the remaining (n - r) can be selected and rejected.

(b) The combination notation also represents the binomial coefficient. That is, the binomial coefficient ${}^nC_r$ is the combination of n elements chosen r at a time.

(c) (i) ${}^nC_r = {}^nC_{n-r}$

(ii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iii) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(iv) If n is even, then the greatest value of ${}^nC_r$ is ${}^nC_{n/2}$

(v) If n is odd, then the greatest value of ${}^nC_r$ is ${}^nC_{(n+1)/2}$

(vi) ${}^nC_0 + {}^nC_r + \dots\dots\dots\dots + {}^nC_n = 2^n$

(vii) ${}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \dots\dots\dots\dots + {}^{2n-1}C_n = {}^{2n}C_{n+1}$

Combinations under Restrictions

(a) The number of ways of choosing r objects out of n different objects if p particular objects must be excluded $= {}^{(n-p)}C_r$

(b) The number of ways of choosing r objects out of n different objects if p particular objects must be included $(p \le r) = {}^{n-p}C_{r-p}$

(c) The total number of combinations of n different objects taken one or more at a time $= 2^n - 1$.

Combinations of Alike Objects

(a) The number of combinations of n identical objects taking $(r \le n)$ at a time is 1.

(b) The number of ways of selecting r objects out of n identical objects is n + 1.

(c) If out of (p + q + r + s) objects, p are alike of one kind, q are alike of a second kind, r are alike of the third kind and s are different, then total number of combinations is $(p + 1)(q + 1)(r + 1)2^s - 1$

(d) The number of ways in which r objects can be selected from a group of n objects of which p are identical, is $\sum_{0}^{t} {}^{n-p}C_r$, if $r \le p$ and $\sum_{r=p}^{t} {}^{n-p}C_r$ if $r > p$

Division into Groups

(a) The number of ways in which (m + n) different objects can be divided into two unequal groups containing m and n objects respectively is $\frac{(m+n)!}{m!n!}$.
If m = n, the groups are equal and in this case the number of divisions is $\frac{(2n)!}{n!n!2!}$; as it is possible to interchange the two groups without obtaining a new distribution.

(b) However, if 2n objects are to be divided equally between two persons then the number of ways $= \frac{(2n)!}{n!n!2!}2! = \frac{(2n)!}{n!n!}$

(c) The number of ways in which (m + n + p) different objects can be divided into three unequal groups containing m , n and p objects respectively is $= \frac{(m+n+p)}{m!n!p!}$, $m \ne n \ne p$
If m = n = P then the number of groups $= \frac{(3n)!}{n!n!n!3!}$. However, if 3n objects are to be divided equally among three persons then the number of ways $= \frac{(3n)!}{n!n!n!3!}3! = \frac{(3n)!}{(n!)^3}$

(d) The number of ways in which mn different objects can be divided equally into m groups if the order of groups is not important is $\frac{mn!}{(n!)^m m!}$

(e) The number of ways in which mn different objects can be divided equally into m groups if the order of groups is important is $\frac{mn!}{(n!)^m m!} \times m! = \frac{(mn)!}{(n!)^m}$

Circular Permutation

(a) The number of circular permutations of n different objects taken r at a time

${}^nP_r/r$, when clockwise and anticlockwise orders are treated as different.

${}^nP_r/2r$, when clockwise and anticlockwise orders are treated as same.

(b) The number of circular permutations of n different objects altogether

${}^nP_n/n = (n - 1)!$, when clockwise and anticlockwise order are treated as different

${}^{n}P_n/2n = 1/2(n-1)!$, when above two orders are treated as same

The number of non-negative integral solutions of equation $x_1 + x_2 + \ldots + x_r = n$

= The number of ways of distributing n identical objects among r persons when each person can get zero or one or more objects = ${}^{n+r-1}C_{r-1}$

The number of positive integral solutions for the equation $x_1 + x_2 + \ldots + x_r = n$

= The number of ways of distributing n identical objects among r persons when each person can get at least one object = ${}^{n-r+r-1}C_{r-1} = {}^{n-1}C_{r-1}$.

(c) For given n different digits $a_1, a_2, a_3 \ldots a_n$ the sum of the digits in the units place of all the numbers formed (if numbers are not repeated) is

$(a_1 + a_2 + a_3 + \ldots + a_n)(n-1)!$ i.e. (sum of the digits) $(n-1)!$

(d) The sum of the total numbers which can be formed with given different digits $a_1, a_2, a_3 \ldots\ldots a_n$ is

$(a + a_2 + a_3 + \ldots + a_n)(n-1)!$ (111. .... n times)

Factors of Natural Numbers

Let $N = p^a \cdot q^b \cdot r^c \ldots$ where p, q, r ... are distinct primes & a, b, c ... are natural numbers, then:

(a) The total number of divisors of N including 1 and N are $= (a + 1)(b + 1)(c + 1) \ldots$

(b) The sum of these divisors is $= (p^0 + p^1 + p^2 + \ldots + p^a)(q^0 + q^1 + q^2 + \ldots + q^b)(r^0 + r^1 + r^2 + \ldots + r^c)\ldots$

(c) The number of ways in which N can be resolved as a product of two factors is

$$= \begin{cases} \frac{1}{2}(a+1)(b+1)(c+1) & \text{if N is not a perfect square} \\ \frac{1}{2}\left[(a+1)(b+1)(c+1)\ldots+1\right] & \text{if N is a perfect square} \end{cases}$$

(d) The number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N

Exponent of a Prime P in N! = $\sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right]$

Inclusion-Exclusion Principle: The principle of inclusion-exclusion states that for finite sets $A_1,\ldots A_n$. One has the identity

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n}|A_i| - \sum_{1\le i<j\le n}|A_i \cap Aj| + \sum_{1\le i<j<k\le n}|A_i \cap Aj \cap A_k| - \ldots + (-1)^{n-1}|A_1 \cap \ldots \cap A_n|.$$

This can be compactly written as $\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}(-1)^{k+1}\left(\sum_{1\le i_1<\ldots\ldots<i_k\le n}|A_{i_1} \cap \ldots \cap A_{i_k}|\right)$

Derangements Theorem: The number of ways in which letters n can be placed in n envelopes (one letter in each envelope) so that no letter is placed in the correct envelope is $n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{(-1)^n}{n!}\right]$

If n objects are arranged at n places then the number of ways to rearrange exactly r objects at right places is =

$$\frac{n!}{r}\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^{n-r}\frac{1}{(n-r)!}\right]$$

Some Important results

(a) The number of totally different straight lines formed by joining n points on a plane of which m(<n) are collinear is ${}^nC_2 - {}^mC_2 + 1$.

(b) The number of total triangles formed by joining n points on a plane of which m(<n) are collinear is ${}^nC_3 - {}^mC_3$.

(c) The number of diagonals in a polygon of n sides is ${}^nC_2 - n$.

(d) If m parallel lines in a plane are intersected by a family of other n parallel lines. Then total number of parallelograms so formed are ${}^mC_2 \times {}^nC_2$.

(e) Given n points on the circumference of a circle, then

the number of straight lines between these points are ${}^nC_2$

the number of triangles between these points are ${}^nC_3$

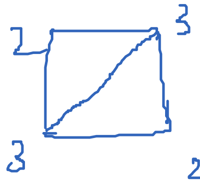
the number of quadrilaterals between these points are ${}^nC_4$

(f) If n straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. Then, the number of parts into which these lines divide the plane is = 1 + Sn

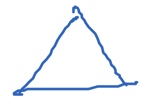
Degree Sequence

Friday, September 3, 2021 10:35 AM

If G is a gph on n vertices $v_1, v_2, \dots v_n$ with degrees $d_1, d_2, \dots d_n$ respectively, then $(d_1, d_2, \dots d_n)$ — degree Sequence



$\longrightarrow (3, 3, 2, 2)$



$\rightarrow (2, 2, 2)$

If a degree sequence is given, does $\exists$ a simple graph?

i) $(3, 3, 3, 3, 2)$

ii) $(4, 4, 4, 3, 2)$ — Sum of the degrees is odd, hence a $\nexists$

Havel - Hakimi Thm

A sequence $(d_1, d_2, \dots d_n)$ of non-negative integers is graphic iff the reduced sequence $(*, d_2 - 1, d_3 - 1, \dots d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n)$ is graphic

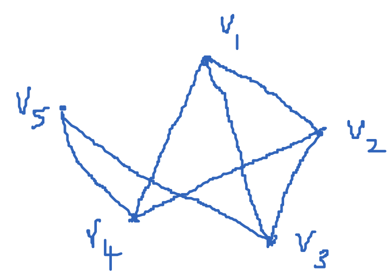
1) $(3_1, 3_2, 3_3, 3_4, 2_5)$

$(*, 2_2, 2_3, 2_4, 2_5)$

$(*, *, 1_3, 1_4, 2_5)$

$(*, *, 2_5, 1_3, 1_4)$

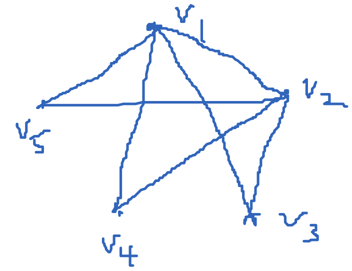
$(*, *, *, 0_3, 0_4)$



2) $(4_1, 4_2, 4_3, 3_4, 2_5)$ —

$(*, 3_2, 3_3, 2_4, 1_5)$

$(*, *, 2_3, 1_4, 0_5)$



$(\cancel{*}_1, \cancel{*}, \cancel{*}, 0_4, -1_5)$ is not graphic

3) $(5_1, 5_2, 3_3, 3_4, 2_5, 2_6, 2_7)$

$(\cancel{*}, 4_2, 2_3, 2_4, 1_5, 1_6, 2_7)$

$(\cancel{*}, 4_2, 2_3, 2_4, 2_7, 1_5, 1_6)$

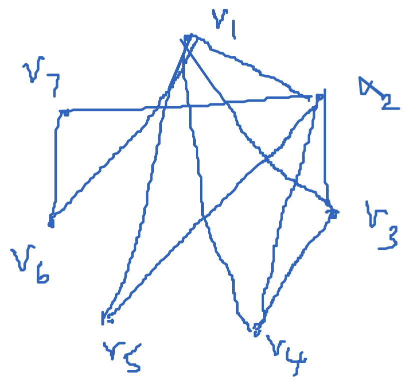
$(\cancel{*}, \cancel{*}, 1_3, 1_4, 1_7, 0_5, 1_6)$

$(\cancel{*}, \cancel{*}, 1_3, 1_4, 1_7, 1_6, 0_5)$

$(\cancel{*}_1, \cancel{*}, \cancel{*}, 0_4, 1_7, 1_6, 0_5)$

$(\cancel{*}, \cancel{*}, \cancel{*}, 1_7, 1_6, 0_4, 0_5)$

$(\cancel{*}, \cancel{*}, \cancel{*}, \cancel{*}, 0_6, 0_4, 0_5)$



Euler and Hamilton graphs

Monday, September 6, 2021 8:34 AM

Trail → A walk with no repeating edges.

Eulerian trail → A trail which visits every edge exactly once

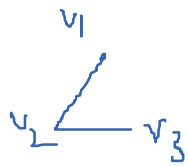
$V_1 - V_2 - V_3 - V_1$ → Eulerian trail

→ Eulerian circuit

→ Eulerian gh

Eulerian Circuit → A circuit which visits every edge exactly once

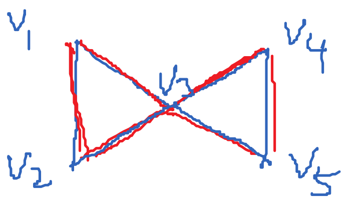
Eulerian Graph → if it contains Eulerian Circuit.



$V_1 - V_2 - V_3$ — Eulerian trail

No Eulerian circuit

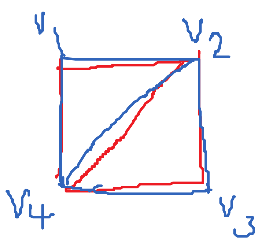
→ Not a Eulerian gh



$V_1 - V_2 - V_3 - V_4 - V_5 - V_3 - V_1$ — Eulerian trail

— " Circuit

Eulerian gh



$V_4 - V_1 - V_2 - V_4 - V_3 - V_2$ — Eulerian trail (open)

No Eulerian Circuit

→ Not Eulerian gh.

<u>Theorem 1</u>: A connected graph contains an Eulerian circuit iff each of its vertices are of even degree

Theorem 2: A connected graph contains an Eulerian trail iff if it has exactly two vertices of odd degree

Lecture-3 (Matrix representation of graphs)

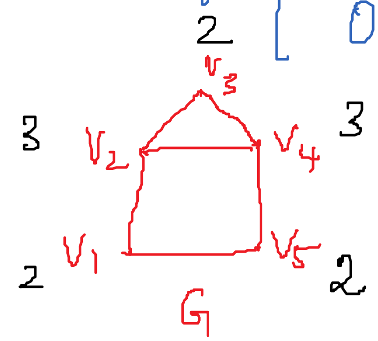
Friday, September 3, 2021 10:05 AM

Let $G = (V, E)$ be a graph $V = \{v_1, v_2, \ldots v_n\}$

$E = \{e_1, e_2, \ldots e_m\}$

Adjacency Matrix $A = (a_{ij})_{n \times n}$ → Symmetric matrix

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$



$$A = \begin{array}{c|ccccc|c} & v_1 & v_2 & v_3 & v_4 & v_5 & \\ \hline v_1 & 0 & 1 & 0 & 0 & 1 & 2 \\ v_2 & 1 & 0 & 1 & 1 & 0 & 3 \\ v_3 & 0 & 1 & 0 & 1 & 0 & 2 \\ v_4 & 0 & 1 & 1 & 0 & 1 & 3 \\ v_5 & 1 & 0 & 0 & 1 & 0 & 2 \\ \hline & 2 & 3 & 2 & 3 & 2 & \end{array}$$

Adj list

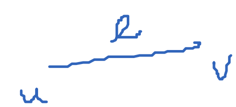
$v_1 \to$ | v_2 | v_5 |

$v_2 \to$ | v_1 | v_3 | v_4 |

$v_3 \to$ | v_2 | v_4 |

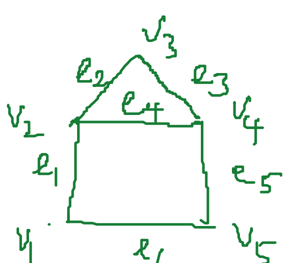
$v_4 \to$ | v_2 | v_3 | v_5 |

$v_5 \to$ | v_1 | v_4 |



Incidence Matrix $B = (b_{ij})_{n \times m}$

$$\text{where } b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident on } e_j \\ 0 & \text{otherise} \end{cases}$$

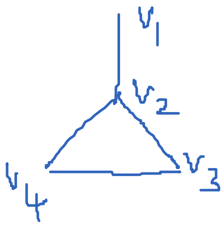


$$B = \begin{array}{c|cccccc|c} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & \\ \hline v_1 & 1 & 0 & 0 & 0 & 0 & 1 & 2 \\ v_2 & 1 & 1 & 0 & 1 & 0 & 0 & 3 \\ v_3 & 0 & 1 & 1 & 0 & 0 & 0 & 2 \\ v_4 & 0 & 0 & 1 & 1 & 1 & 0 & 3 \\ v_5 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ \hline & 2 & 2 & 2 & 2 & 2 & 2 & \end{array}$$

$$\sum_{j=1}^{n} b_{ij} = \deg v_i \qquad \sum_{j=1}^{m} b_{ij} = 2$$

Theorem: If A is the adjacency matrix of a graph G, then

$A_{ij}^{p} \rightarrow$ the number of walks of length p from v_i to v_j



$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$A^2 = A\,A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \end{array}$$

$\#$ v_3 to v_4 of length $\underline{\underline{2}} = 1$

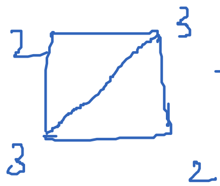
$\#$ of walks from v_2 to v_2 of length $\underline{\underline{2}} = 3$

$v_2 - v_4 - v_2$
$v_2 - v_3 - v_2$
$v_2 - v_1 - v_2$

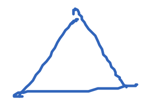
Degree Sequence

Friday, September 3, 2021 10:35 AM

If G is a gph on n vertices $v_1, v_2, \dots v_n$ with degrees $d_1, d_2, \dots d_n$ respectively, then $(d_1, d_2, \dots d_n)$ — degree Sequence



$\longrightarrow (3, 3, 2, 2)$



$\rightarrow (2, 2, 2)$

If a degree sequence is given, does $\exists$ a simple graph?

(i) $(3, 3, 3, 3, 2)$

(ii) $(4, 4, 4, 3, 2)$