Chromatic number - chromatic partitioning - chromatic polynomial

chromatic number

The minimum number of colours required to colour G is called the chromatic number of G.

Note:-

(1) It is denoted by $\chi(G)$ (or) $\varphi(G)$.

(2) If $\chi(G) \leq K$, then G is said to be K-colourable.

(3) If $\chi(G) = K$, then G is said to be K-chromatic.

(4) If H is any subgraph of G then $\chi(H) \leq \chi(G)$.

(5) A graph consisting of only isolated vertices is 1-chromatic.

(6) A graph with one or more edges is atleast 2-chromatic.

(7) If G is a graph of n vertices then $\chi(G) \leq n$.

(8) The chromatic number of the complete graph K_n is n for all $n \geq 1$.

(9) Every tree with 2 or more vertices is 2-chro.

K- chromatic graph

A graph G that requires K different colours for its proper colouring and no less, is called a K-chromatic graph, and the number K is called the chromatic number of G.

THEOREM 3.1.1.

Every tree with two or more vertices is 2-chromatic.

Proof : [A.U N/D 2016 R-13]

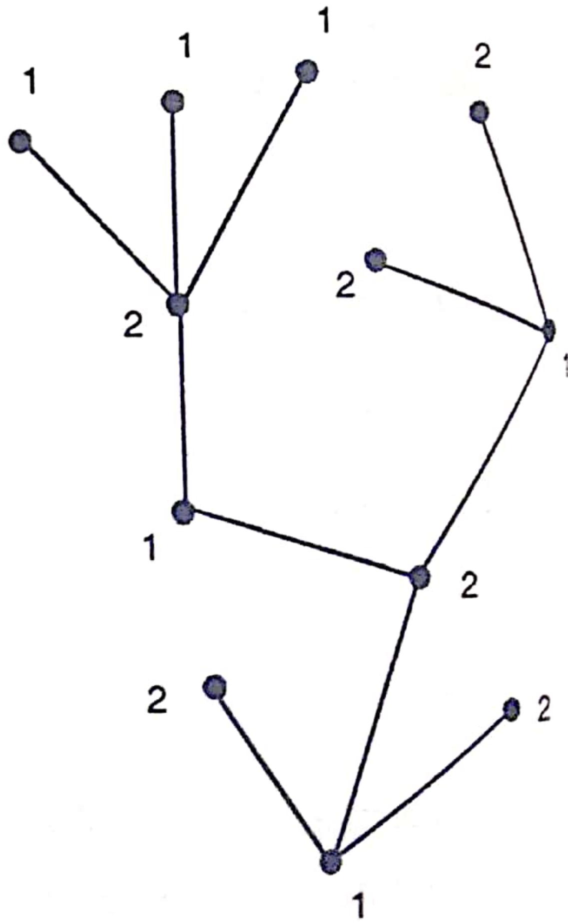
Let T be a tree with two or more vertices.

Select any vertex v in T and paint colour 1.

Paint all vertices adjacent to v with colour 2.

Paint all the vertices adjacent to those vertices which have been used colour 2 using colour 1.

Continue this process, until every vertex of T has been painted.



Now, in T we find that all vertices at odd distances from v have colour 2, while v and vertices at even distances from v have colour 1.

Since, there is one and only one path between any two vertices in a tree, no two adjacent vertices will have the same colour.

Thus T has been properly coloured with two colours. One colour would not have been enough.

Thus T is 2-chromatic.

Note : The converse of the above theorem is not true.

THEOREM 3.1.2

A graph with atleast one edge is 2-chromatic if and only if it has no circuits of odd length. (Konig's theorem)

Proof :

Let G be a connected graph with circuits of even length only.

Let T be a spanning tree from G.

We know that, every tree with two or more vertices is 2-chromatic.

$\Rightarrow$ T is 2-chromatic.

Now add the edges (chords) to T one by one.

By adding the edges circuit of even length will be created.

Since G has no odd circuit, the end vertices of every edge will be coloured with different colours.

Thus G can be properly coloured with 2 colours only.

So G is 2-chromatic.

Conversely, let G be a 2-chromatic graph. If G has a circuit of odd length then it will require 3 colours. But G is 2-chromatic, so, it cannot have a circuit of odd length.

Definition : Chromatic polynomial **[A.U A/M 2017 R-13]**

A polynomial which gives the number of different ways the graph G can be properly coloured using the minimum number of colours from λ is called chromatic polynomial of graph G and is denoted by $P_n(\lambda)$

THEOREM 3.1.5

A graph is n vertices is a complete graph if and only if its chromatic polynomial is

$$P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2) \ldots (\lambda - n + 1)$$

Proof : **[A.U N/D 2016 R-13][A.U A/M 2017 R-13]**

Let G be a complete graph with n vertices.

Let λ be the number of colours

1^{st} vertex of G can be coloured in λ ways.

2^{nd} vertex of G can be coloured in $(\lambda - 1)$ ways.

3^{rd} vertex of G can be coloured in $(\lambda - 2)$ ways.

. . . .

. . . .

n^{th} vertex of G can be coloured in $\lambda - (n - 1)$ ways.

A complete graph G can be coloured in

$$\lambda(\lambda - 1)(\lambda - 2) \ldots (\lambda - n + 1) \text{ ways.}$$

Let $P_n(\lambda)$ be the chromatic polynomial, then

$$P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2) \ldots (\lambda - n + 1)$$

THEOREM 3.1.6

An n-vertex graph is a tree if and only if its chromatic polynomial

$$P_n(\lambda) = \lambda(\lambda - 1)^{n-1}$$

Proof :

Given : n-vertex graph is a tree.

$\therefore$ 1^{st} vertex can be coloured by λ way.

2^{nd} vertex can be coloured by $\lambda - 1$ way.

3^{rd} vertex can be coloured by $\lambda - 1$ way.

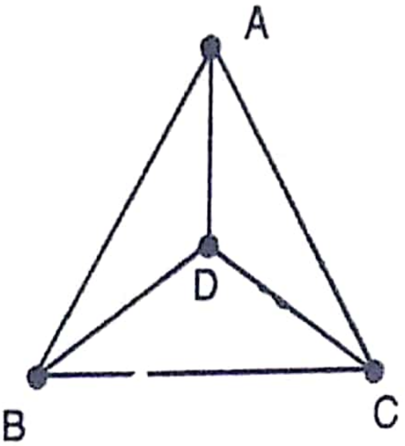
. . . .

. . . .

n^{th} vertex can be coloured by $\lambda - 1$ way.

Hence, $P_n(\lambda) = \lambda(\lambda - 1)^{n-1}$, $n \geq 2$

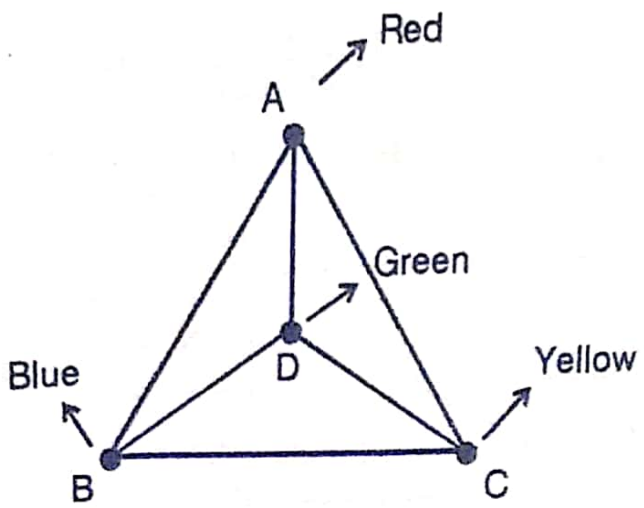
xample 3. Find the chromatic number of the following graph also ind the maximum value and minimum value of the chromatic number.



Solution :

First method :

Let G :



Minimum 4 colours required to colour G.

$\therefore$ The chromatic number $\chi(G) = 4$.

Second method :

Step 1. To find $\Delta(G)$

$$\deg(A) = 3,\ \deg(B) = 3,\ \deg(C) = 3,\ \deg(D) = 3$$

$\therefore$ The maximum degree of the vertices in a graph $G = \Delta(G) = 3$.

Step 2. We know that,

$$\chi(G) \leq 1 + \Delta(G) = 1 + 3 = 4$$

$$\Rightarrow \chi(G) \leq 4 \qquad \dots (1)$$

Step 3. The given G has a triangle subgraph

$$\therefore \chi(G) \geq 3 \qquad \dots (2)$$

From (1) & (2), we get

$$3 \leq \chi(G) \leq 4 \qquad \dots (3)$$

Step 4. Every k-chromatic graph has atleast k-vertices, such that $\deg(v) \geq k - 1$

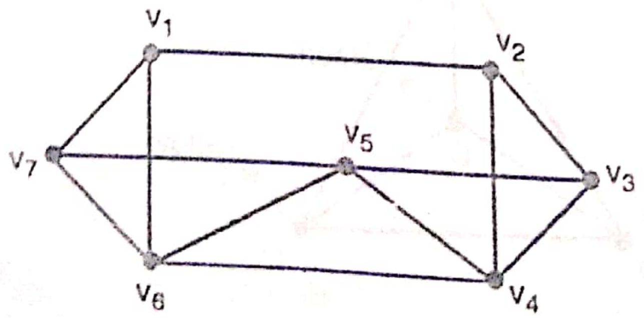
If $\chi(G) = 4$, then G should have 4 vertices with degree atleast 3.

Here, the 4 vertices (A, B, C, D) with degree atleast 3.

$$\therefore \chi(G) = 4$$

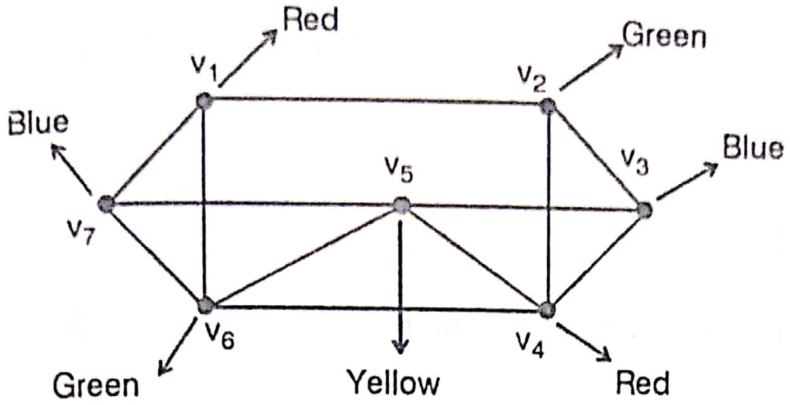
From (3) maximum value of $\chi(G) = 4$, minimum value of $\chi(G) =$

Example 4. Find the chromatic number of the following graph.



Solution :

First method :



Minimum four colour's required to colour G

$\therefore$ The chromatic number $\chi(G) = 4$

Second method :

Step 1. To find $\Delta(G)$

$\deg(v_1) = 3,\ \deg(v_2) = 3,\ \deg(v_3) = 3,\ \deg(v_7) = 3$
$\deg(v_4) = 4,\ \deg(v_5) = 4,\ \deg(v_6) = 4$

$\therefore \Delta(G) = 4$

Step 2. We know that, $\chi(G) \leq 1 + \Delta(G) = 1 + 4 = 5$

$\Rightarrow \chi(G) \leq 5$ ... (1)

Step 3. The given graph has a triangle subgraph

$\therefore \chi(G) \geq 3$ ... (2)

From (1) & (2), we get

$3 \leq \chi(G) \leq 5$

Step 4. Every k-chromatic graph has atleast k-vertices, such that $\deg(v) \geq k - 1$

If $\chi(G) = 5$, then G should have **5 vertices with degree atleast 4.**

But here, only three vertices (v_4, v_5, v_6) with degree atleast 4.

$\therefore \chi(G) \neq 5$

If $\chi(G)=4$, then G should have 4 vertices with degree atleast 3.

Here, 4 vertices (v_1, v_2, v_3, v_7) with degree atleast 3.

$\therefore \chi(G) = 4$

Third method :

Step 1. List the vertices of G in the descending order of their degree.

Step 2. Colour the first vertex in the list with colour 1.

Go along the list and colour the vertices not adjacent to vertices having colour 1 with colour 1.

Step 3. Repeat colour 2 with colour 2 for the uncoloured vertices in the order in which they appear in the list.

Step 4. Step when all the vertices have been coloured.

$\deg(v_1) = 3 \rightarrow$ Red $\qquad \deg(v_4) = 4 \rightarrow$ Red

$\deg(v_2) = 3 \rightarrow$ Green $\qquad \deg(v_5) = 4 \rightarrow$ Yellow

$\deg(v_3) = 3 \rightarrow$ Blue $\qquad \deg(v_6) = 4 \rightarrow$ Green

$\deg(v_7) = 3 \rightarrow$ Blue

Minimum 4 colours required to colour G

$$\therefore \chi(G) = 4$$

Example 5. What is the chromatic number of k_n?

Solution :

Every two vertices of k_n graph are adjacent.

A colouring of k_n can be constructed using n colours by assigning a different colour to each vertex.

No two vertices can be assigned the same colour.

Hence, the chromatic number of $k_n = n$.

$$\text{i.e., } \chi(k_n) = n$$

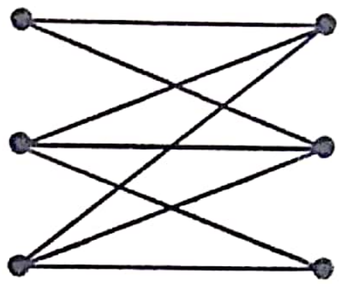
Note :

1. Chromatic number of complete graph $k_n = n$
2. Chromatic number of bipartite graph $k_{m,n} = 2$

Example 7. A graph G is bipartite if and only if any circuit in G has even length.

Proof : If Part :

Assume that G is bipartite.



To prove a circuit in G also has even length.

Since G is bipartite all of its edges must connect a left vertex with a right vertex. This means that any circuit found within G will alternate back and forth from left to right vertices.

Therefore, any circuit will contain an even number of vertices.

Since within a circuit the number of edges is equal to the number of vertices, then the number of edges must also be even.

Therefore, since the number of edges is even by definition G has even length.

Only if part :

Assume that every circuit in G has even length.

To prove : G is bipartite.

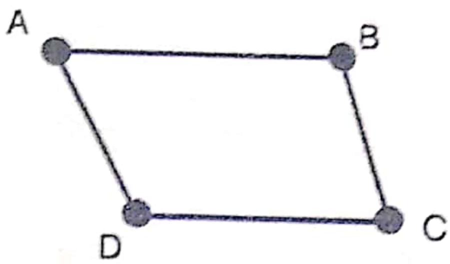
Take any vertex, lets start with a example circuit.

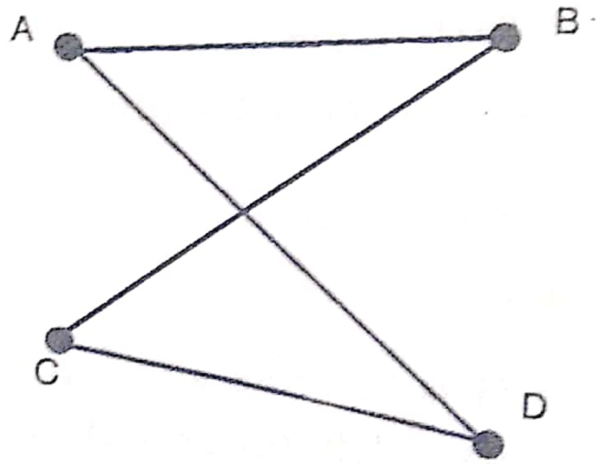
Put all vertices of odd length away from A on the right side of your graph.

Next, put all the vertices of even length away from A on the left side of your new graph near A.

Lastly use your circuit to connect the vertices in your new graph. As you can see from the example a bipartite graph G has been constructed where no two vertices on the right or left are adjacent. If two vertices on the same side were by chance connected then our circuit would have had an odd length.

Example : Graph G





♦ Working procedure for chromatic polynomial

Some fundamental cases

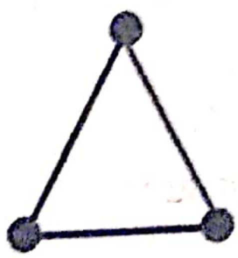
1. If the graph has only one vertex and no edge then only one colour is required, so, $C_1 = 1$ and $C_i = 0$, for $i = 1, 2, 3, \ldots, n$ and hence $P_1(\lambda) = \lambda$

2. For a graph with two vertices and one edge, at least two colours are required, so $C_1 = 0$ and the two vertices can be coloured in 2! ways. So $C_2 = 2!$, and $C_i = 0$, $i = 3, 4, \ldots, n$ and hence the Chromatic polynomial is $P_2(\lambda) = \lambda(\lambda - 1)$

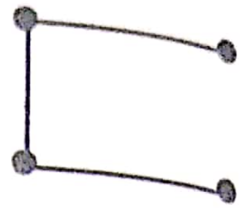
3. For a tree with 3 vertices, $P_3(\lambda) = C_3 \dfrac{\lambda(\lambda - 1)(\lambda - 2)}{3!}$, $C_3 = 3!$

   and since the chromatic number of a tree is only 2, therefore $\lambda(\lambda - 1)(\lambda - 2)$ must be positive for least value of $\lambda = 2$ and that is possible only when $P_3(\lambda) = \lambda(\lambda - 1)^2$,

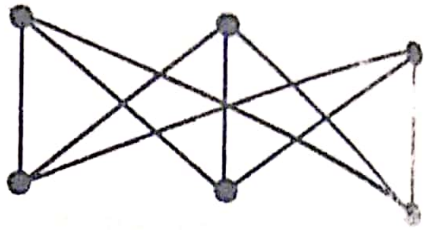
4. For a triangle with three vertices the chromatic number is 3 and hence $P_3(\lambda) = \lambda(\lambda - 1)(\lambda - 2)$



5. For 4 vertices making a tree as in figure the chromatic number is 2 and hence the chromatic polynomial is
$P_4(\lambda) = \lambda(\lambda - 1)^3$

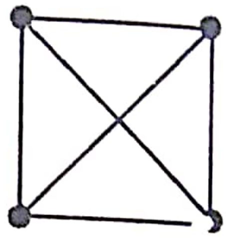


6. For K_{33} with 6 vertices, the Chromatic number is 2 and the chromatic polynomial is $\lambda(\lambda - 1)^5$



7. For K_4, the chromatic number is 4 and the chromatic polynomial is
$\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)$

This is positive for $\lambda = 4$ (Chromatic number)



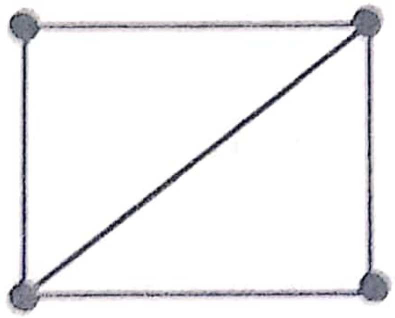
8. For $K_{2,3}$, the chromatic number is 2 and the chromatic polynomial is $P_5(\lambda) = \lambda(\lambda - 1)^4$ which is positive for $\lambda = 2$.

9. It is important to note that (i) the constant term of the chromatic Polynomial is zero (ii) the sum of the coefficients of a Chromatic polynomial is zero.

10. Chromatic polynomials for certain graphs

Triangle K_3	$t(t-1)(t-2)$
Complete graph K_n	$t(t-1)(t-2)\ldots[t-(n-1)]$
Path graph P_n	$t(t-1)^{n-1}$
Any tree on n vertices	$t(t-1)^{n-1}$
Cycle C_n	$(t-1)^n + (-1)^n(t-1)$
Petersên graph	$t(t-1)(t-2)(t^7 - 12t^6 + 67t^5 - 230t^4 + 529t^3 - 814t^2 + 775t - 352)$

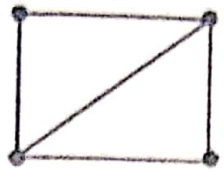
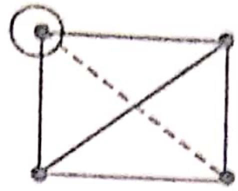
Example 12. **Find the chromatic polynomial of the graph**



Solution :

We use the recurrence formula

$$f(G,\lambda) = f(G+e,\lambda) + f(G.e\,,\,\lambda)$$

G =  =  + 

$= k_4 + k_3$

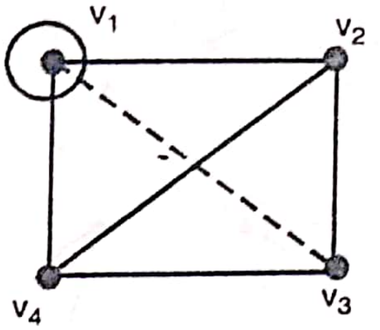
$$f(G,\lambda) = f(k_4,\lambda) + f(k_3\,,\,\lambda)$$

$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)$$

$$= \lambda(\lambda-1)(\lambda-2)[\lambda-3+1]$$

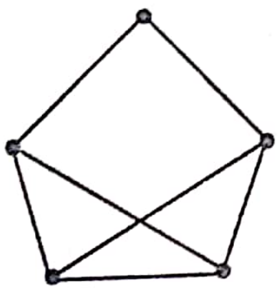
$$= \lambda(\lambda-1)(\lambda-2)^2$$

Note :



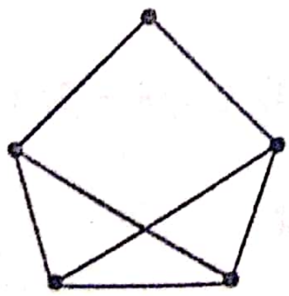
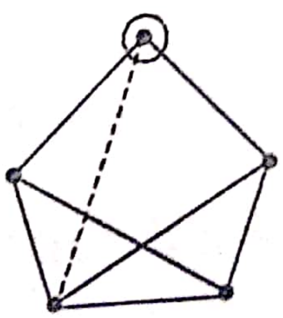
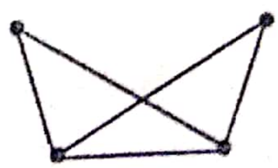
remove vertex v_1

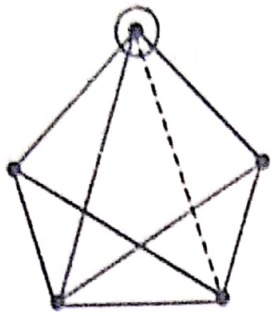
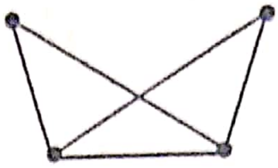
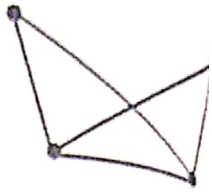
Example 13. Find the chromatic polynomial of the graph

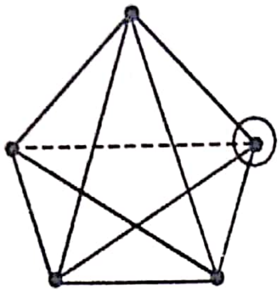
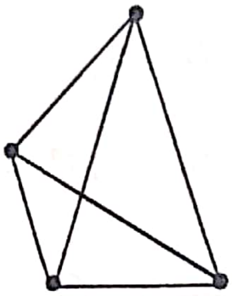
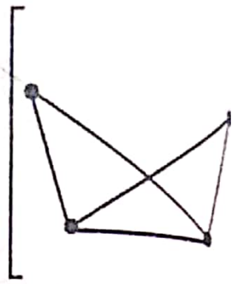


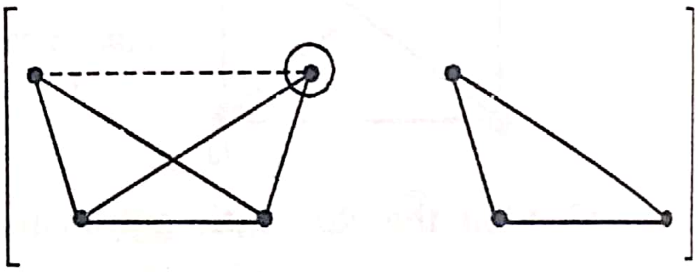
Solution : We use recurrence formula

$$f(G,\lambda) = f(G+e\,,\,\lambda) + f(G.e\,,\,\lambda)$$

G =  =  + 

$=$  $+$  $+$ 

$=$  $+$  $+\ 2$ 

$= k_5 + k_4 + 2$ 

$$= k_5 + k_4 + 2\,[k_4 + k_3]$$

$$= k_5 + 3\,k_4 + 2\,k_3$$

$$f(G,\lambda) \quad = f(k_5,\lambda) + 3f(k_4\,,\,\lambda) + 2f(k_3,\lambda)$$

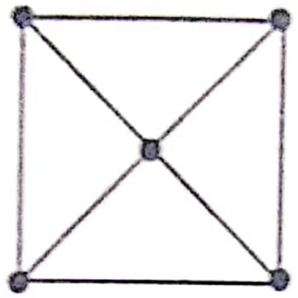
$$= \lambda\,(\lambda-1)\,(\lambda-2)\,(\lambda-3)\,(\lambda-4) + 3\lambda\,(\lambda-1)\,(\lambda-2)\,(\lambda-3$$

$$+\ 2\lambda\,(\lambda-1)\,(\lambda-2)$$

$$= \lambda\,(\lambda-1)\,(\lambda-2)\,[(\lambda-3)\,(\lambda-4) + 3\,(\lambda-3) + 2]$$

$$= \lambda^5 - 7\lambda^4 + 19\lambda^3 - 23\lambda^2 + 10\lambda$$

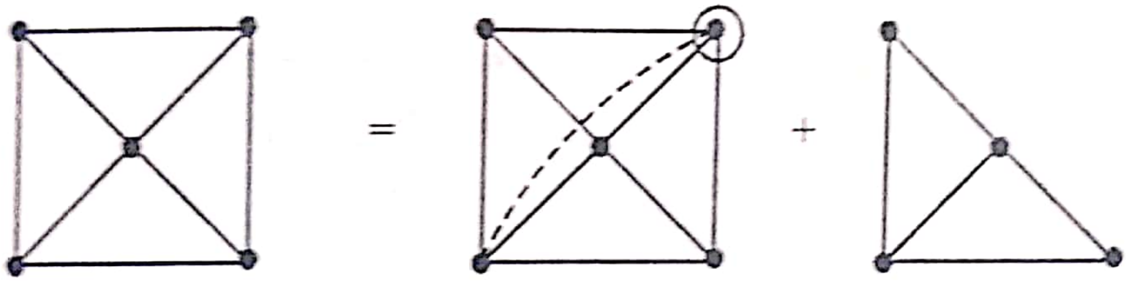
Example 14. Find the chromatic polynomial of the graph



Solution : We use the recurrence formula

$$f(G,\lambda) = f(G+e,\lambda) + f(G.e\,,\lambda)$$

G =



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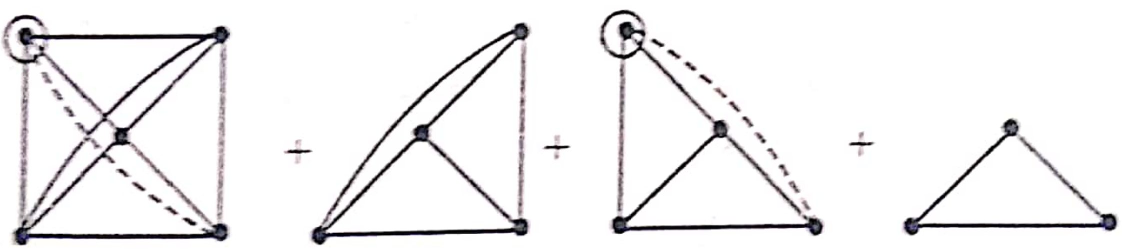


$$= k_5 + k_4 + k_4 + k_3$$

$$= k_5 + 2k_4 + k_3$$

$$f(G,\lambda) = f(k_5,\lambda) + 2f(k_4,\lambda) + f(k_3,\lambda)$$

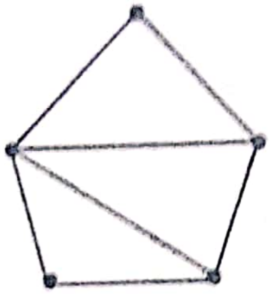
$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)$$

$$= \lambda(\lambda-1)(\lambda-2)[(\lambda-3)(\lambda-4) + 2(\lambda-3) + 1]$$

$$= \lambda(\lambda-1)(\lambda-2)[\lambda^2 - 7\lambda + 12 + 2\lambda - 6 + 1]$$

$$= \lambda(\lambda-1)(\lambda-2)[\lambda^2 - 5\lambda + 7]$$

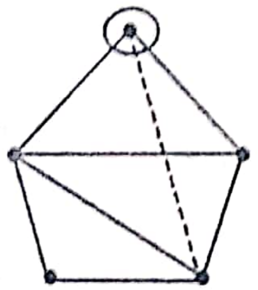
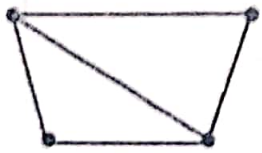
Example 15. Find the chromatic polynomial of the graph

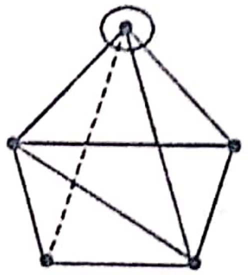
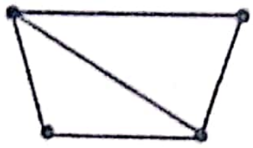
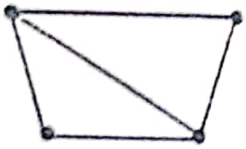


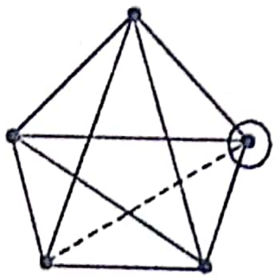
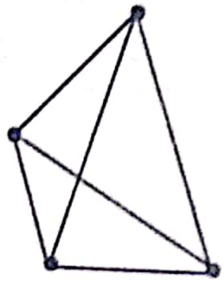
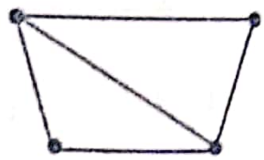
Solution :

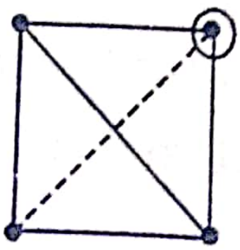
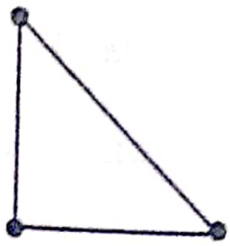
We use recurrence formula

$$f(G,\lambda) = f(G+e,\lambda) + f(G.e\,,\lambda)$$

G $=$  $+$ 

$=$  $+$  $+$ 

$=$  $+$  $+$ 

$= k_5 + k_4 + 2$ [ $+$ ]

$= k_5 + k_4 + 2\,[k_4 + k_3]$

$= k_5 + 3k_4 + 2k_3$

$$f(G,\lambda) = \lambda^5 - 7\lambda^4 + 19\lambda^3 - 23\lambda^2 + 10\lambda \text{ [see (Example 13)]}$$