Chapter 2
Inference Theory

An argument is a sequence of propositions $H_1, H_2, \ldots, H_n$ called premises (or hypotheses) followed by a proposition C called conclusion. An argument is usually written:

$$\begin{array}{c} H_1 \\ H_2 \\ \vdots \\ H_n \\ \text{implies} \quad C \end{array}$$

or

$$H_1 \wedge H_2 \wedge \ldots \wedge H_n \Rightarrow C$$

The argument is valid if C is true whenever $H_1, H_2, \ldots, H_n$ are true; otherwise it is invalid.

Example: $H_1 : p$ and $H_2 : p \rightarrow q$ then $C : q$ (Modus Ponens)

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$(p \wedge (p \rightarrow q)) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Notice that $(p \wedge (p \rightarrow q)) \rightarrow q$ is a tautology. Therefore it is valid.

Thus $p,\ p \rightarrow q \Rightarrow q$.

Example: $H_1 : q$ and $H_2 : p \to q$ then $C : p$

p	q	$p \to q$	$q \wedge (p \to q)$	$(q \wedge (p \to q)) \to p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Notice that $q \wedge (p \to q) \to p$ is a condigency. Therefore it is invalid.

Implication Table:

S.No	Formula	Name
1	$p \wedge q \Rightarrow p$	simplification
	$p \wedge q \Rightarrow q$	
2	$p \Rightarrow p \vee q$	addition
	$q \Rightarrow p \vee q$	
3	$p,\ q \Rightarrow p \wedge q$	
4	$p,\ p \rightarrow q \Rightarrow q$	modus ponens
5	$\neg p,\ p \vee q \Rightarrow q$	disjunctive syllogism
6	$\neg q,\ p \rightarrow q \Rightarrow \neg p$	modus tollens
7	$p \rightarrow q\ , q \rightarrow r \Rightarrow p \rightarrow r$	

Rules of inference:

Rule P: A premises can be introduced at any step of derivation.

Rule T: A formula can be introduced provided it is Tautologically implied by previously introduced formulas in the derivation.

Rule CP: If the conclusion is of the form $r \rightarrow s$ then we include r as an additional premises and derive s.

Indirect method: We use negation of the conclusion as an additional premise and try to arrive a contradiction.

Inconsistent: A set of premises are inconsistent provided their conjunction implies a contradiction.

Example: Show that $\neg q$ and $p \to q$ implies $\neg p$.

Solution: (A formal proof is as follows):

Step 1. $p \to q$ Rule P

Step 2. $\neg q \to \neg p$ Rule T

Step 3. $\neg q$ Rule P

Step 4. $\neg p$ Combined $\{2, 3\}$ and apply Modus Ponens

Example: Show that r is a valid inference from the premises $p \to q$, $q \to r$ and p.

Solution:

Step	Derivation	Rule
1	p	P
2	$p \to q$	P
3	q	$\{1, 2\}$, $\mathcal{I}_4$
4	$q \to r$	P
5	r	$\{3, 4\}$, $\mathcal{I}_4$

Example: Show that $s \vee r$ is tautologically implied by $p \vee q$, $p \rightarrow r$ and $q \rightarrow s$.

Solution:

Step	Derivation	Rule
1	$p \vee q$	P
2	$\neg p \rightarrow q$	T
3	$q \rightarrow s$	P
4	$\neg p \rightarrow s$	$\{2, 3\}$, $\mathcal{I}_7$
5	$\neg s \rightarrow p$	T
6	$p \rightarrow r$	P
7	$\neg s \rightarrow r$	$\{5, 6\}$, $\mathcal{I}_7$
8	$s \vee r$	T

Example: Prove by indirect method that $p \to q$, $p \vee r, \neg q$ implies r.
Solution: The desired result is r. Include $\neg r$ as a new premise.

Step	Derivation	Rule
1	$p \vee r$	P
2	$\neg r \to p$	T
3	$\neg r$	P(additional premise)
4	p	$\{2, 3\}$, $\mathcal{I}_4$
5	$p \to q$	P
6	q	$\{4, 5\}$, $\mathcal{I}_4$
7	$\neg q$	P
8	$q \wedge \neg q$	$\{6, 7\}$, $\mathcal{I}_3$

The new premise together with the given premises, leads to a contradiction. Thus $p \to q$, $p \vee r, \neg q$ implies r.

Example: Prove that $p \to q$, $p \to r$, $q \to \neg r$ and p are inconsistent.
Solution: The desired result is false.

Step	Derivation	Rule
1	p	P
2	$p \to q$	P
3	q	$\{1, 2\}$, $\mathcal{I}_4$
4	$q \to \neg r$	P
5	$\neg r$	$\{3, 4\}$, $\mathcal{I}_4$
6	$p \to r$	P
7	$\neg p$	$\{5, 6\}$, $\mathcal{I}_6$
8	F	$\{1, 7\}$,$\mathcal{I}_3$

III. Problems:

(i) Show that $r \vee s$ is tautologically implied by $c \vee d$, $(c \vee d) \to \neg h, \neg h \to (a \wedge \neg b)$ and $(a \wedge \neg b) \to (r \vee s)$.

(ii) Show that $r \wedge (p \vee q)$ is tautologically implied by $p \vee q$, $q \to r$, $p \to m$ and $\neg m$.

(iii) Show that $r \to s$ is tautologically implied by $\neg r \vee p$, $p \to (q \to s)$ and q.

(iv) Show that $p \to s$ is tautologically implied by $\neg p \vee q$, $\neg q \vee r$ and $r \to s$.

(v) Show that $p \to (q \to s)$ is tautologically implied by $p \to (q \to r)$ and $q \to (r \to s)$ using CP rule.

(vi) Show that the following premises are inconsistent. $v \to l$, $l \to b$, $m \to \neg b$ and $v \wedge m$.

(vii) Show that $p \to \neg s$ logically follows from the premises $p \to (q \vee r)$, $q \to \neg p$, $s \to \neg r$ and p by indirect method.

(viii) Show that r logically follows from the premises $p \to q$, $\neg q$ and $p \vee r$ by indirect method.

Example: Consider the following statements: 'I take the bus or I walk. If I walk I get tired. I do not get tired. Therefore I take the bus.' We can formalize this by calling $B=$ I take the bus, $W=$ I walk and $J=$ I get tired. The premises are $B \vee W$, $W \to J$ and $\neg J$, and the conclusion is B. The argument can be described in the following steps:

step	statement	reason
1	$W \to J$	P
2	$\neg J$	P
3	$\neg W$	$\{1, 2\}$, Modus Tollens
4	$B \vee W$	P
5	B	$\{3, 4\}$, Disjunctive Syllogism

(i) Show that the following set of premises is inconsistent.

1. If Jack misses many classes through illness, then he fails high school.
2. If Jack fails high school, then he is uneducated.
3. If Jack reads a lot of books, then he is not uneducated.
4. Jack misses many classes through illness and reads a lot of books.

Let us consider,

E : Jack misses many classes through illness

S : Jack fails high school

A : Jack reads a lot of books

H : Jack is uneducated.

The premises are,

Let us consider,
E : Jack misses many classes through illness
S : Jack fails high school
A : Jack reads a lot of books
H : Jack is uneducated.
The premises are,

- $E \to S$

Let us consider,

E : Jack misses many classes through illness

S : Jack fails high school

A : Jack reads a lot of books

H : Jack is uneducated.

The premises are,

- $E \rightarrow S$
- $S \rightarrow H$

Let us consider,
E : Jack misses many classes through illness
S : Jack fails high school
A : Jack reads a lot of books
H : Jack is uneducated.
The premises are,

- $E \rightarrow S$
- $S \rightarrow H$
- $A \rightarrow \neg H$ and

Let us consider,
E : Jack misses many classes through illness
S : Jack fails high school
A : Jack reads a lot of books
H : Jack is uneducated.
The premises are,

- $E \to S$
- $S \to H$
- $A \to \neg H$ and
- $E \wedge A$

Step	Derivation	Rule
1	$E \wedge A$	P
2	E	T
3	A	T
4	$E \to S$	P
5	S	$\{2,4\}$, $\mathcal{I}_4$
6	$S \to H$	P
7	H	$\{5,6\}$, $\mathcal{I}_4$
8	$A \to \neg H$	P
9	$\neg H$	$\{3,8\}$, $\mathcal{I}_4$
10	$H \wedge \neg H$	$\{7,9\}$, $\mathcal{I}_3$

II. Problems:

(i) Show that the following argument is valid.
My father praises me only if i can be proud of myself. Either I do well in sports
or I cannot be proud of myself. If study hard, then I cannot do well in sports.
Therefore, if father praises me, then I do not study well.
(ii) Show that the following set of premises is inconsistent.
If the contract is valid, then John is liable for penalty. If John is liable for penalty,
he will go bankrupt. If the bank will loan him money, he will not go bankrupt.
As a matter of fact, the contract is valid and the bank will loan him money.