Module - 5

Boolean Algebra

Definition:-

A Lattice which is complemented and distributive is called a Boolean Algebra. (or)

If B is a nonempty set with two binary operations join $\oplus$ $+$ and meet $*$ $\cdot$, two distinct elements 0 and 1 and a uniary operation $'$, then B is called a Boolean Algebra if the following basic properties hold for all $a, b, c \in B$.

B1 : $\left.\begin{array}{l} a \oplus 0 = a \\ a * 1 = a \end{array}\right\}$ Identity laws

B2 : $\left.\begin{array}{l} a \oplus b = b \oplus a \\ a * b = b * a \end{array}\right\}$ Commutative laws

B3 : $\left.\begin{array}{l} (a \oplus b) \oplus c = a \oplus (b \oplus c) \\ (a * b) * c = a * (b * c) \end{array}\right\}$ Asso. laws

B4 : $\left.\begin{array}{l} a \oplus (b * c) = (a \oplus b) * (a \oplus c) \\ a * (b \oplus c) = (a * b) \oplus (a * c) \end{array}\right\}$ Distributive laws

B5 : $\left.\begin{array}{l} a \oplus a' = 1 \\ a * a' = 0 \end{array}\right\}$ complemented laws.

Example: If $B = \{0, 1\}$ and the operations $*$, $\oplus$ and $'$ on B are given as

$*$	0	1
0	0	0
1	0	1

$\oplus$	0	1
0	0	1
1	1	1

x	x'
0	1
1	0

Hasse Diagram

here the least elt $(0) = 0$ & greatest elt $(1) = 1$.

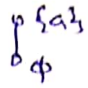
$\therefore$ The algebra (or) algebraic ~~alget~~ system $\langle B, \oplus, *, ', 0, 1 \rangle$ is a Boolean algebra.

(2) <u>Boolean algebra of power sets</u>

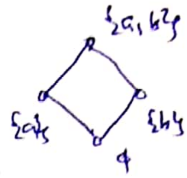
Let S be a nonempty set and $P(S)$ be its power set.

The set algebra $\langle P(S), *, \oplus, ', 0, 1 \rangle = \langle P(S), \cap, \cup, \sim, \phi, S \rangle$ is a Boolean algebra in which the complement of any subset of is $\sim A = A^c = S - A = S/A$, $|P(S)| = 2^n$ if $|S| = n$.

For (i) $S = \{a\}$, $P(S) = \{\phi, \{a\}\}$



(ii) $S = \{a, b\}$, $P(S) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$



(3) Let S = Set of all statement formulas involving n-statement Variables.

disjunction, Contradiction

Then $\langle S, \wedge, \vee, \neg, F, T \rangle$ is a Boolean algebra.

Conjunction, Negation, Tautology

(4) <u>Switching Algebra</u>

Let B_n = Set of all n-tuples whose member are either '0' or '1'.

Thus $a \in B_n \Leftrightarrow a = \langle a_1, a_2, \ldots, a_n \rangle$ where $a_i = 0$ or 1 for $i = 1, 2, \ldots n$.

Define for $a, b \in B_n$ i.e, $a = \langle a_1, \ldots a_n \rangle$ where $a_i = 0$ or 1
& $b = \langle b_1, \ldots b_n \rangle$ $b_i = 0$ or 1

$$a * b = \langle a_1 \wedge b_1, a_2 \wedge b_2, \ldots a_n \wedge b_n \rangle$$

$$a \oplus b = \langle a_1 \vee b_1, a_2 \vee b_2, \ldots, a_n \vee b_n \rangle$$

$$a' = \langle \neg a_1, \neg a_2, \ldots, \neg a_n \rangle$$

Where $\wedge \to$ Conjunction
$\vee \to$ disjunction and Logical operations on $\{0, 1\}$.
$\neg \to$ Negation

Thus the algebra $\langle B, *, \oplus, ', 0_n, 1_n \rangle$ is a Boolean algebra, which 0_n and 1_n are n-tuples whose members are all 0's and 1's resp.

This algebra is called a Switching algebra.

Subalgebra, Direct Product and Homomorphism in Boolean algebra.

Subalgebra (or) Sub-Boolean algebra

Let $\langle B, *, \oplus, ', 0, 1\rangle$ be a Boolean algebra and $S \subseteq B$. S contains the elts 0 and 1 and is closed under the operations $*$, $\oplus$ and $'$, then $\langle S, *, \oplus, ', 0, 1\rangle$ is called sub-Boolean algebra of $\langle B, *, \oplus, ', 0, 1\rangle$

Note:- (1) $a \oplus b = (a' * b')'$

(2) $(a \oplus a') = 1.$

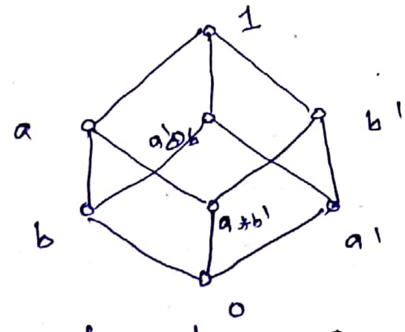
(3) $a * a' = 0$

Example:- (Trivial)

1. Let $\langle B, *, \oplus, ', 0, 1\rangle$ be a Boolean algebra. Then $\langle B, *, \oplus, ', 0, 1\rangle$ and $\langle \{0,1\}, *, \oplus, ', 0, 1\rangle$ are both sub-Boolean algebras.

2. Consider the Boolean algebra $\langle B, *, \oplus, ', 0, 1\rangle$ as given below in the form of Hasse diagram.

$B = \{0, a, b, a*b', a', b', a'\oplus b, 1\}$



Then $S_1 = \{a, a', 0, 1\}$

$S_2 = \{a'\oplus b, a*b', 0, 1\}$

$S_3 = \{a*b', b', 0, 1\}$

$S_4 = \{b', a*b', a', 0\}$

$S_5 = \{a, b', 0, 1\}.$

Here S_1 and $S_2 \to$ sub-Boolean algebras of B.

S_3 and $S_4 \to$ Boolean algebra, but not sub-Boolean algebra of B.

$S_5 \to$ not even a BA.

Direct product of two Boolean algebra

Let $\langle B_1, *_1, \oplus_1, ', 0_1, 1_1 \rangle$ & $\langle B_2, *_2, \oplus_2, '', 0_2, 1_2 \rangle$ two Boolean algebras.

The direct product of the two Boolean algebras is defined to Boolean algebra that is given by

$\langle B_1 \times B_2, *_3, \oplus_3, ''', 0_3, 1_3 \rangle$ in which the operations are defined as follows.

For any, $\langle a_1, b_1 \rangle$ and $\langle a_2, b_2 \rangle \in B_1 \times B_2$ as

$$\langle a_1, b_1 \rangle *_3 \langle a_2, b_2 \rangle = \langle a_1 *_1 a_2, b_1 *_2 b_2 \rangle$$

$$\langle a_1, b_1 \rangle \oplus_3 \langle a_2, b_2 \rangle = \langle a_1 \oplus_1 a_2, b_1 \oplus_2 b_2 \rangle$$

$$\langle a_1, b_1 \rangle''' = \langle a_1', b_1'' \rangle \quad \&$$

$$0_3 = \langle 0_1, 0_2 \rangle \quad \text{and} \quad 1_3 = \langle 1_1, 1_2 \rangle$$

Example:- Let $\langle B = \{0,1\}, *, \oplus, ', 0, 1 \rangle$ - Boolean algebra.

$B \times B = B^2$, $B \times B \times B = B^3$ . . . are Boolean algebra.

Boolean Homomorphism

Let $\langle B, *, \oplus, ', 0, 1 \rangle$ and $\langle P, \wedge, \vee, -, \alpha, \beta \rangle$ be two Boolean algebras.

A mapping $f: B \to P$ is called a Boolean homomorphism if all the operations of the Boolean algebra are preserved.

i.e, Meet $f(a * b) = f(a) \wedge f(b)$

Join $f(a \oplus b) = f(a) \vee f(b)$

Complement $f(a') = \overline{f(a)}$

$f(0) = \alpha$

$f(1) = \beta$.

Boolean Isomorphism

If a Boolean homomorphism is one-to-one and onto then it is called as Boolean Isomorphism.

Isomorphic algebra / Isomorphic Boolean algebras:

If there exist a Boolean isomorphism between two Boolean algebras, then that two algebras are called isomorphic Boolean algebras.

Theorem Stone's Representation Theorem

Any Boolean algebra is isomorphic to a power set algebra $\langle P(S), \cap, \cup, \sim, \phi, S \rangle$ for some set S.

Boolean expression (or) form (or) formula

A Boolean expression in 'n' variables $x_1, x_2, \ldots x_n$ is any finite string of symbols formed in the following manner:

(1) 0 and 1 are Boolean expressions

(2) $x_1, x_2, \ldots x_n$ are Boolean "

(3) If α_1 and α_2 are Boolean expressions, then $(\alpha_1) * (\alpha_2)$ and $\alpha_1 \oplus \alpha_2$ are also Boolean expressions

(4) If α is a Boolean expression, then α' is also.

(5) No string of symbols except those formed in accordance with rules (1) and (4) are Boolean expressions.

Notation:- $\alpha(x_1, x_2, \ldots x_n)$.

Minterm (or) Complete product (or) fundamental product (Atoms)

A Boolean form in 'n' variables $x_1, x_2, \ldots x_n$ consisting of a product of n terms such as

$$x_1^{a_1} * x_2^{a_2} * \ldots * x_n^{a_n} = \mathop{*}_{j=1}^{n} x_j^{a_j}$$

in which a_j is either '0' or '1', x_j^0 stands for x_j' and x_j^1 stands for x_j, for $i = 1, 2, \ldots n$ is called a minterm (or) complete product of the 'n' variables.

Notation:- $\min_j$ (or) $\min_j^n$ (or) m_j

Example Let $\langle B, \cdot, +, ', 0, 1\rangle$ be a Boolean algebra. Simply the Boolean expressions using Boolean algebra identity and show the following statements.

(i) $(a.b)+(a.b')+(a'.b) = a+b$

(ii) $((a'.b)+a'.b'+b')' = a.b$

(iii) $(b.(a+c)+a.b'+b.c'+c) = a+b+c$

(iv) $a+(a'.b) = a+b.$

Soln:-

$a.1 = a$
$a.0 = 0$
$a+1 = 1$
$a+0 = a$

(i) $(a.b)+(a.b')+(a'.b)$

$= a.(b+b')+(a'.b)$

$=(a.1)+(a'.b)$

$= a+(a'.b)$

$= (a+a').(a+b)$

$= 1.(a+b) = a+b$

(ii) $((a'.b)+(a'.b')+b')' = (a'.b)'.(a'.b')'.(b')'$

$= (a+b').(a+b).b$

$= (a.(b'+b)).b$

$= (a.1).b$

$= a.b$

$(a.b)' = a'+b'$
$(a+b)' = a'.b'$

(iii) $(b.(a+c)+a.b'+b.c'+c) = b.((a+c)+c')+a.b'+c$

$= b.(a+(c+c'))+a.b'+c$

$=(b.(a+1))+(a.b')+c$

$=(b.1)+(a.b')+c$

$= ((b+a).(b+b'))+c$

$= (a+b).1+c$

$= a+b+c$

(iv) $a+(a'.b) = (a+a').(a+b)$

$= 1.(a+b)$

$= a+b$

Hence proved.

Disjunctive Normal Forms (DNF)

Sum of Products canonical form (SPC)

Every Boolean expression except '0' can be expressed in an equivalent form consisting of the sums of minterms. (i.e.,) Join of minterms such an equivalent form is called the Sum-of-products canonical form.

Example 1 Write the following Boolean expression in an equivalent Sum of Products canonical form in three variables x_1, x_2 & x_3.

a) $x_1 * x_2$ b) $x_1 \oplus x_2$ c) $(x_1 \oplus x_2)' * x_3$.

Soln:-

a) $x_1 * x_2 = x_1 * x_2 * (x_3 \oplus x_3')$ $\because x_3 \oplus x_3' = 1$, $a * 1 = a$

$= (x_1 * x_2 * x_3) \oplus (x_1 * x_2 * x_3')$

$= (x_1^1 * x_2^1 * x_3^1) \oplus (x_1^1 * x_2^1 * x_3^0)$

Now, $x_1^{a_1} * x_2^{a_2} * x_3^{a_3} = x_1^1 * x_2^1 * x_3^0$ (where a_i = either 0 or 1.

Here $a_1a_2a_3 = 110$

Convert the binary number $a_1a_2a_3 = 110$ into decimal representation.

$(110)_2 = (6)_{10}$

Binary to Decimal
$0 \to 0$
$1 \to 1$
$10 \to 2$
$11 \to 3$
$100 \to 4$
$101 \to 5$
$110 \to 6$
$111 \to 7$

11ly $x_1^{a_1} * x_2^{a_2} * x_3^{a_3} = x_1^1 * x_2^1 * x_3^1$

$a_1a_2a_3 = (111)_2 = (7)_{10}$

$\therefore x_1 * x_2 = \oplus(6, 7)$

$= min_7 \oplus min_6$

c) $(x_1 \oplus x_2)' * x_3 = x_1' * x_2' * x_3 = x_1^{a_1} * x_2^{a_2} * x_3^{a_3}$

$= (001) = (1)_{10}$

$= min_1 = \oplus 1$.

(b) $x_1 \oplus x_2 = [x_1 * (x_2 \oplus x_2')] \oplus [x_2 * (x_1 \oplus x_1')]$

$\because a \oplus a' = 1$

$= [(x_1 * x_2) \oplus (x_1 * x_2')] \oplus [(x_2 * x_1) \oplus (x_2 * x_1')]$

$= \underline{(x_1 * x_2)} \oplus \underline{(x_2 * x_1)} \oplus (x_1 * x_2') \oplus (x_2 * x_1')$

$a \oplus a = a$

$= (x_1 * x_2) \oplus (x_1 * x_2') \oplus (x_2 * x_1')$

$= ((x_1 * x_2) * (x_3 \oplus x_3')) \oplus$
$((x_1 * x_2') * (x_3 \oplus x_3')) \oplus$
$((x_2 * x_1') * (x_3 \oplus x_3'))$

$= (x_1 * x_2 * x_3) \oplus (x_1 * x_2 * x_3')$
$\oplus (x_1 * x_2' * x_3) \oplus (x_1 * x_2' * x_3')$
$\oplus (x_2 * x_1' * x_3) \oplus (x_2 * x_1' * x_3')$

$= \oplus (2, 3, 4, 5, 6, 7)$

(2) Show that $(x_1' * x_2' * x_3' * x_4') \oplus (x_1' * x_2' * x_3' * x_4) \oplus$
$(x_1' * x_2' * x_3 * x_4) \oplus (x_1' * x_2' * x_3 * x_4') = x_1' * x_2'$

Soln:-

Now $(x_1' * x_2' * x_3' * x_4') \oplus (x_1' * x_2' * x_3' * x_4)$

$= (x_1' * x_2' * x_3') \oplus (x_4' * x_4)$ (Distributive Law)

$= (x_1' * x_2' * x_3') \oplus 0 = x_1' * x_2' * x_3'$ —— ①

lly $(x_1' * x_2' * x_3 * x_4) \oplus (x_1' * x_2' * x_3 * x_4') = (x_1' * x_2' * x_3)$ —— ②

Now $(x_1' * x_2' * x_3') \oplus (x_1' * x_2' * x_3)$

$= (x_1' * x_2')$

$a * a' = 0$
$a \oplus 0 = a$

Maxterms (Anti-atoms)

A Boolean form in variables $x_1, x_2, \dots x_n$ consisting of the sum (Join) of n-terms such as

$$x_1^{a_1} \oplus x_2^{a_2} \oplus \dots \oplus x_n^{a_n} = \bigoplus_{j=1}^{n} x_j^{a_j}$$

which a_j is either 0 (or) 1, x_j^0 stands for x_j & x_j^1 stands for x_j' for $j = 1, 2, \dots n$ is called the maxterms of n-variables.

Conjunctive Normal forms (CNF) (or) (PS canonical form)

Every Boolean expression in n variable is equivalent to a Boolean expression consisting of the product (meet) of maxterms only. Such a canonical form is known as product of sums canonical form.

Example:- Obtain PSC form of the Boolean expressions in three variables x_1, x_2 & x_3 given by

a) $x_1 * x_2$ b) $x_1 \oplus x_2$.

$a * a' = 0$
$a \oplus 0 = a$

Soln:- a) $(x_1 * x_2) = (x_1 \oplus 0) * (x_2 \oplus 0)$

$= x_1 \oplus (x_2 * x_2') * (x_2 \oplus (x_1 * x_1'))$

$= (x_1 \oplus x_2) * (x_1 \oplus x_2') * (x_2 \oplus x_1) * (x_2 \oplus x_1')$

$= (x_1 \oplus x_2 \oplus (x_3 * x_3')) * (x_1 \oplus x_2' \oplus (x_3 * x_3')) * (x_2 \oplus x_1' \oplus (x_3' * x_3))$

$= (x_1 \oplus x_2 \oplus x_3) * (x_1 \oplus x_2 \oplus x_3') * (x_1 \oplus x_2' \oplus x_3) * (x_1 \oplus x_2' \oplus x_3') * (x_2 \oplus x_1' \oplus x_3') * (x_2 \oplus x_1' \oplus x_3)$

$= Max_0 * Max_1 * Max_2 * Max_3 * Max_4 * Max_5$

$= *(0, 1, 2, 3, 4, 5)$

b) H.W