# Outline

- Mass moment of inertia
- Physical Significance
- Applications
- Mathematical representation
- Radius of gyration
- Transfer formula or Parallel axis theorem
- Determination of mass moment of inertia for
    - Thin plates
    - Solids
    - Composite bodies

# Mass Moment of Inertia

## Inertia

- Objects tend to "keep on doing what they're doing."
- In fact, it is the natural tendency of objects to resist changes in their state of motion.
- This tendency to resist changes in their state of motion is described as **inertia**.
- Mass is the measure of inertia.

## Moment of Inertia

- It is the measure of an object's resistance to change its state of rotation.

## Mass Moment of Inertia

- It characterizes the angular acceleration undergone by a solid when subjected to a torque.

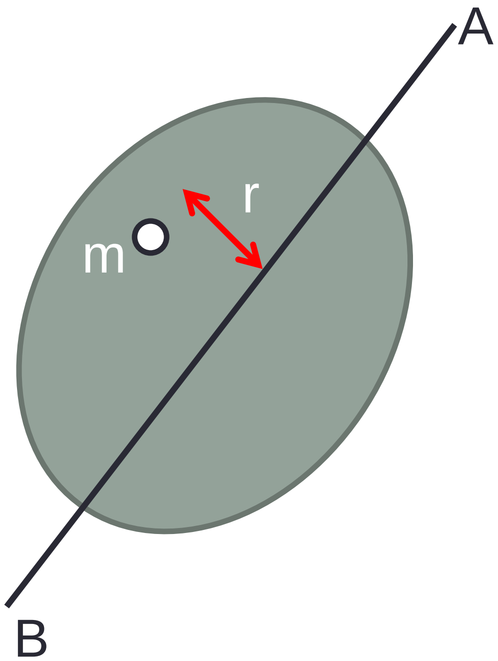
# Physical Significance

- Mass moment of inertia indicates resistance of the body to angular acceleration.
- A body with large mass moment of inertia means the body offers high resistance to angular acceleration. So for a given moment or torque on a body, lower will be its angular acceleration.
- Mass moment of inertia for a body depends on the body's mass and the location of the mass.
- The greater the distance the mass is from the axis of rotation, the larger mass moment of inertia will be.

# Applications

- Flywheel
- Stability of four wheel drive moving in a curved path
- Gyroscope concepts

# Mathematical Expression




$$\text{The radial acceleration of the particle} = \frac{v^2}{r} = \omega^2 r$$

$$\text{The radial force} = m\omega^2 r$$

$$\text{The tangential acceleration of the particle} = \frac{dv}{dt}$$

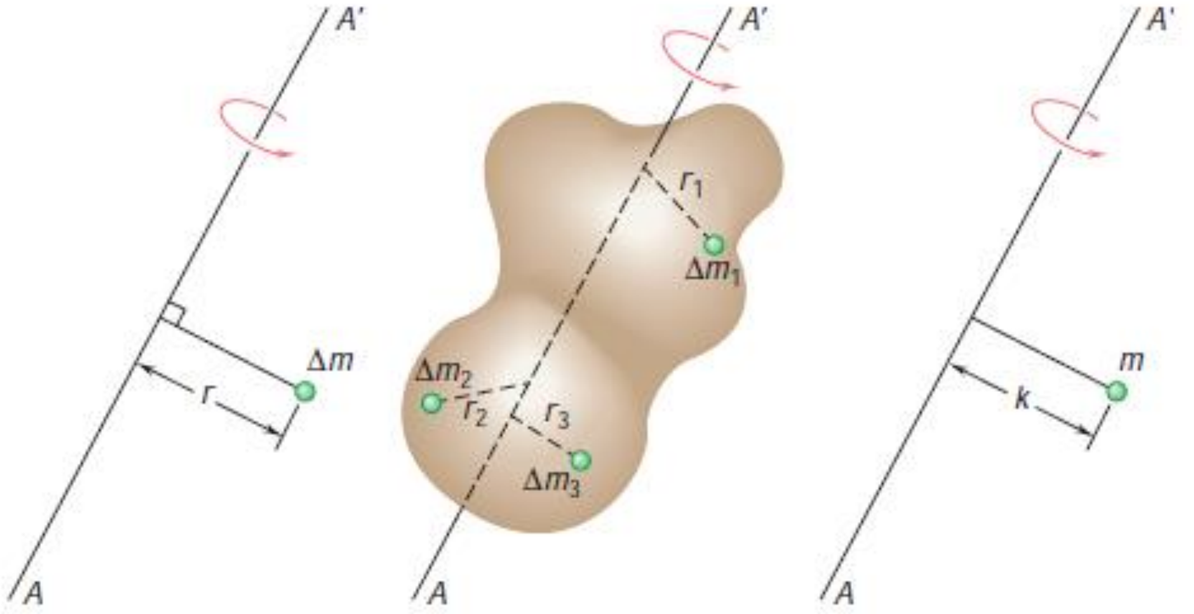
$$\text{The tangential force} = m\frac{dv}{dt} = mr\frac{d\omega}{dt} = mr\alpha$$

$$\text{The torque acting on the particle} = mr^2\alpha$$

$$\text{The total torque acting on the body} = \sum_i m_i {r_i}^2 \alpha = I\alpha$$

$$I = \sum_i m_i {r_i}^2$$

# *I and k*

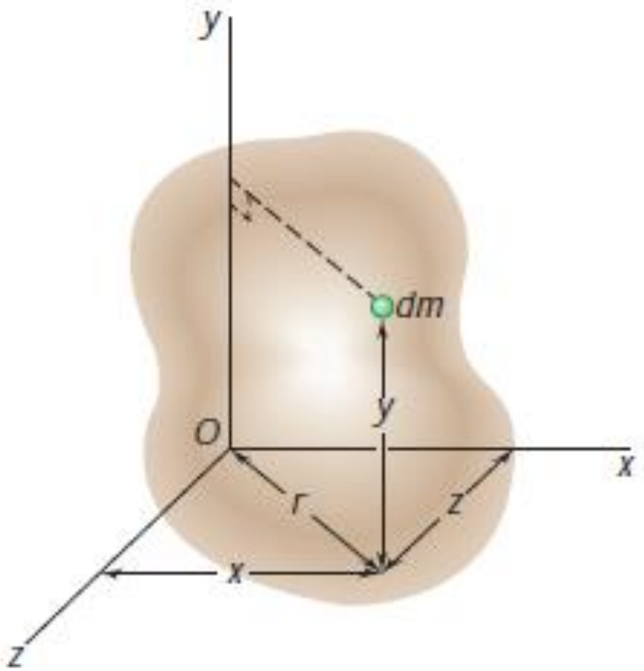



$$I = \int r^2\,dm$$

The radius of gyration is

$$I = k^2 m \qquad \text{or} \qquad k = \sqrt{\frac{I}{m}}$$

# *I* with respect to Coordinate Axes



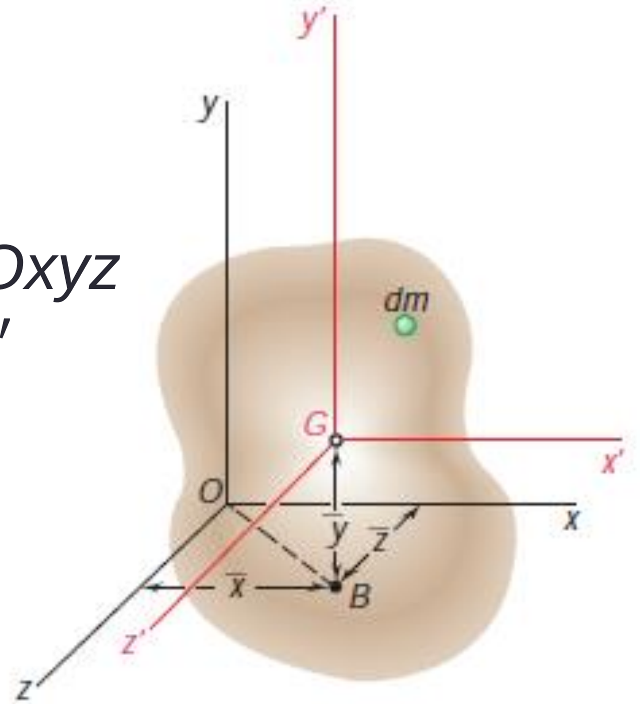

$$I_y = \int r^2\, dm = \int (z^2 + x^2)\, dm$$

$$I_x = \int (y^2 + z^2)\, dm$$

$$I_y = \int (z^2 + x^2)\, dm$$

$$I_z = \int (x^2 + y^2)\, dm$$

# Parallel Axis Theorem




$\bar{x}$, $\bar{y}$, $\bar{z}$ - co-ordinates of *G* with respect to *Oxyz*

$x'$, $y'$, $z'$- co-ordinates of *dm* w.r.to $Gx'y'z'$

$$x = x' + \bar{x} \qquad y = y' + \bar{y} \qquad z = z' + \bar{z}$$

$$I_x = \int (y^2 + z^2)\, dm = \int [(y' + \bar{y})^2 + (z' + \bar{z})^2]\, dm$$

$$= \int (y'^2 + z'^2)\, dm + 2\bar{y} \overbrace{\cancel{\int y'\, dm}}^{0} + 2\bar{z} \overbrace{\cancel{\int z'\, dm}}^{0} + (\bar{y}^2 + \bar{z}^2) \int dm$$

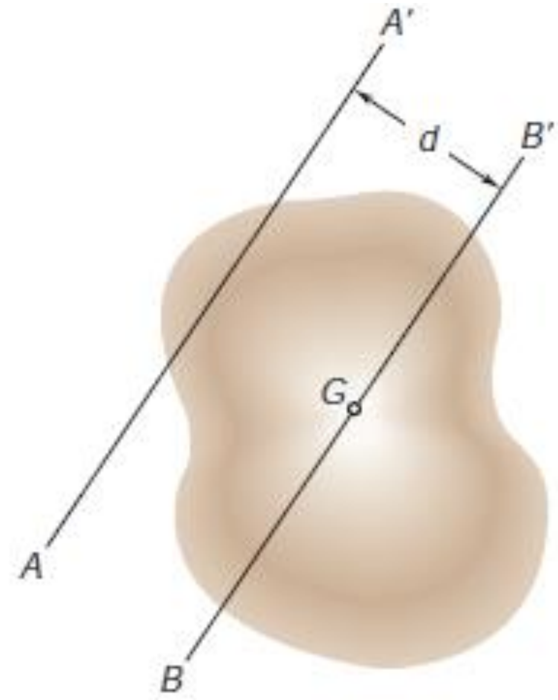
$$\boxed{I_x = \bar{I}_{x'} + m(\bar{y}^2 + \bar{z}^2)}$$

and, similarly,

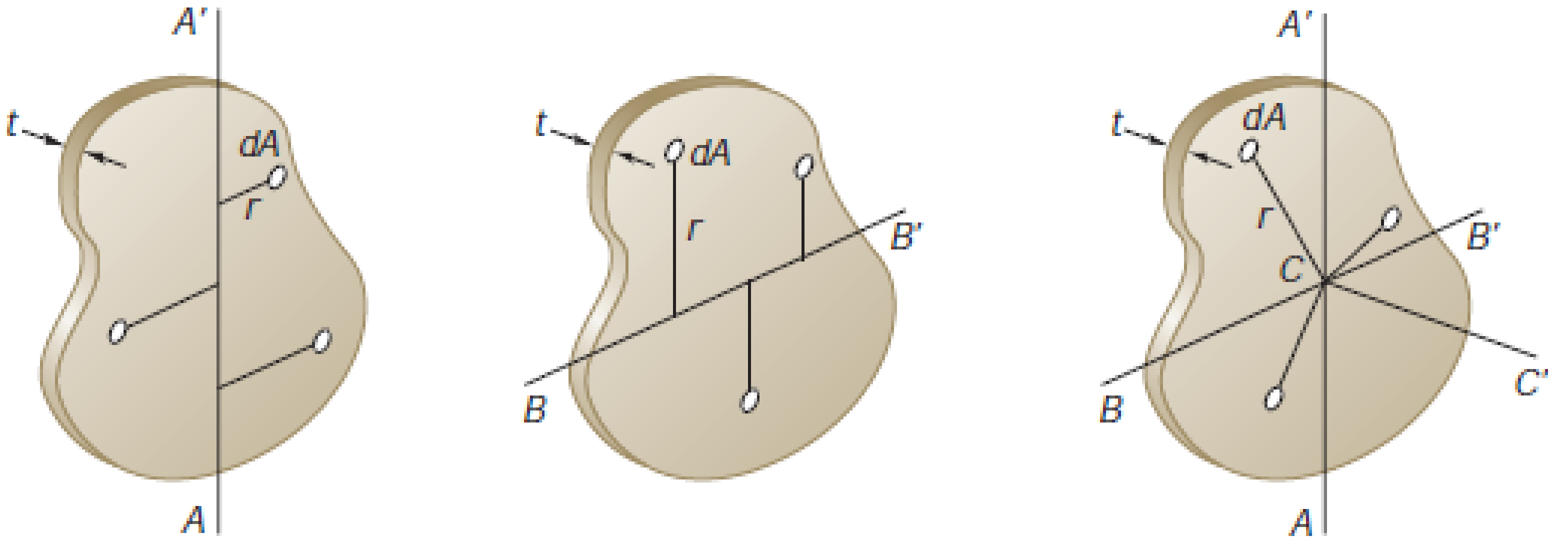
$$\boxed{I_y = \bar{I}_{y'} + m(\bar{z}^2 + \bar{x}^2) \qquad I_z = \bar{I}_{z'} + m(\bar{x}^2 + \bar{y}^2)}$$

$$I = \bar{I} + md^2$$

$$k^2 = \bar{k}^2 + d^2$$



# Moment of Inertia of Thin Plates




Consider a thin plate of uniform thickness *t*, which is made of a homogeneous material of density ρ

$$I_{AA'} = \int r^2 dm$$

$$dm = \rho t dA$$

$$I_{AA',mass} = \rho t \int r^2 dA$$

$$\boxed{I_{AA',\,mass} = \rho t I_{AA',\,area}}$$

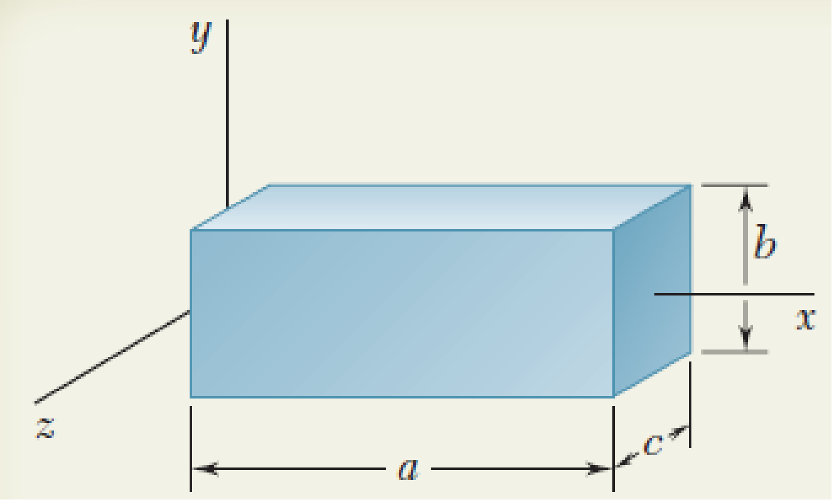
$$\boxed{I_{BB',\,mass} = \rho t I_{BB',\,area}}$$

$$\boxed{I_{CC',\,mass} = \rho t J_{C,\,area}}$$

$$J_C = I_{AA'} + I_{BB'}$$

# Problem 1

For the homogeneous rectangular prism shown, determine the moment of inertia with respect to the *z* axis.




$$I_z = \frac{1}{12}m(4a^2 + b^2)$$

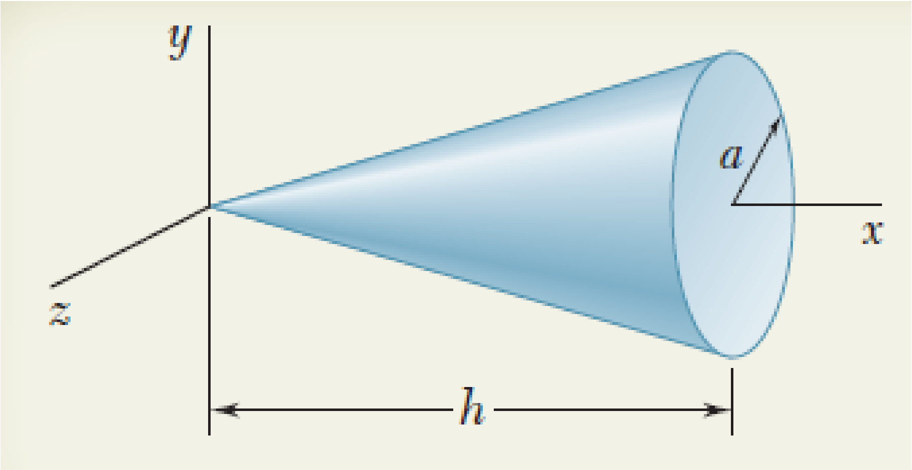
# Problem 2

Determine the moment of inertia of a right circular cone with respect to

(a) its longitudinal axis,

(b) an axis through the apex of the cone and perpendicular to its longitudinal axis,

(c) an axis through the centroid of the cone and perpendicular to its longitudinal axis.




$$I_x = \frac{3}{10} m a^2$$

$$I_y = \frac{3}{5} m \left(\frac{1}{4} a^2 + h^2\right)$$

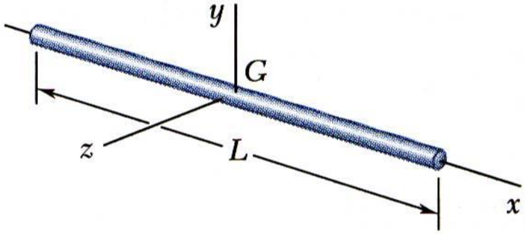
$$\bar{I}_{y''} = \frac{3}{20} m \left(a^2 + \frac{1}{4} h^2\right)$$

# Problem 3

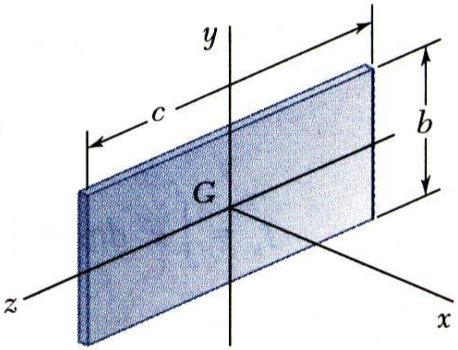
Derive the mass moment of Inertia of a sphere from the first principle.

$$\boxed{I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} m R^2}$$

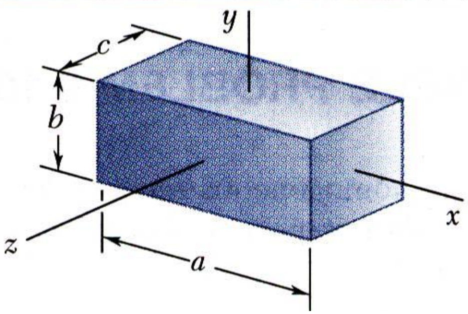
# Moments of Inertia of Common Geometric Shapes



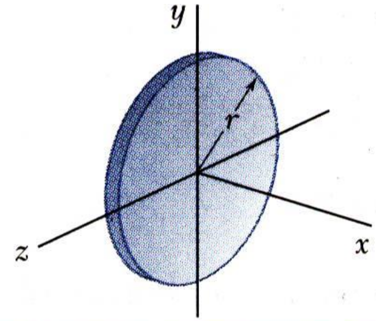

$$I_y = I_z = \frac{1}{12} mL^2$$



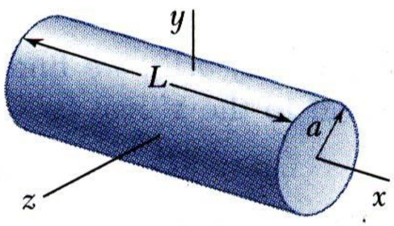

$$I_x = \frac{1}{12} m(b^2 + c^2)$$
$$I_y = \frac{1}{12} mc^2$$
$$I_z = \frac{1}{12} mb^2$$



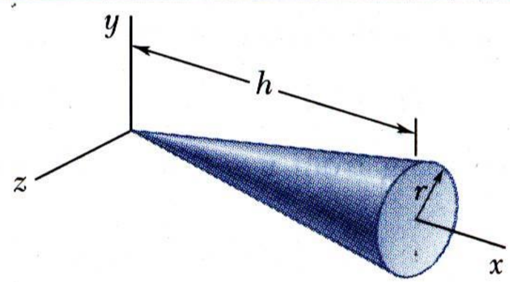

$$I_x = \frac{1}{12} m(b^2 + c^2)$$
$$I_y = \frac{1}{12} m(c^2 + a^2)$$
$$I_z = \frac{1}{12} m(a^2 + b^2)$$



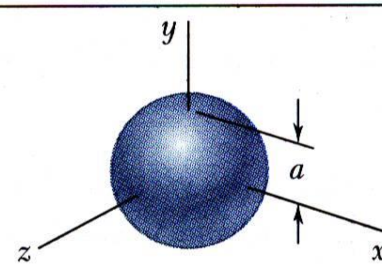

$$I_x = \frac{1}{2} mr^2$$
$$I_y = I_z = \frac{1}{4} mr^2$$




$$I_x = \frac{1}{2} ma^2$$
$$I_y = I_z = \frac{1}{12} m(3a^2 + L^2)$$




$$I_x = \frac{3}{10} ma^2$$
$$I_y = I_z = \frac{3}{5} m(\frac{1}{4} a^2 + h^2)$$

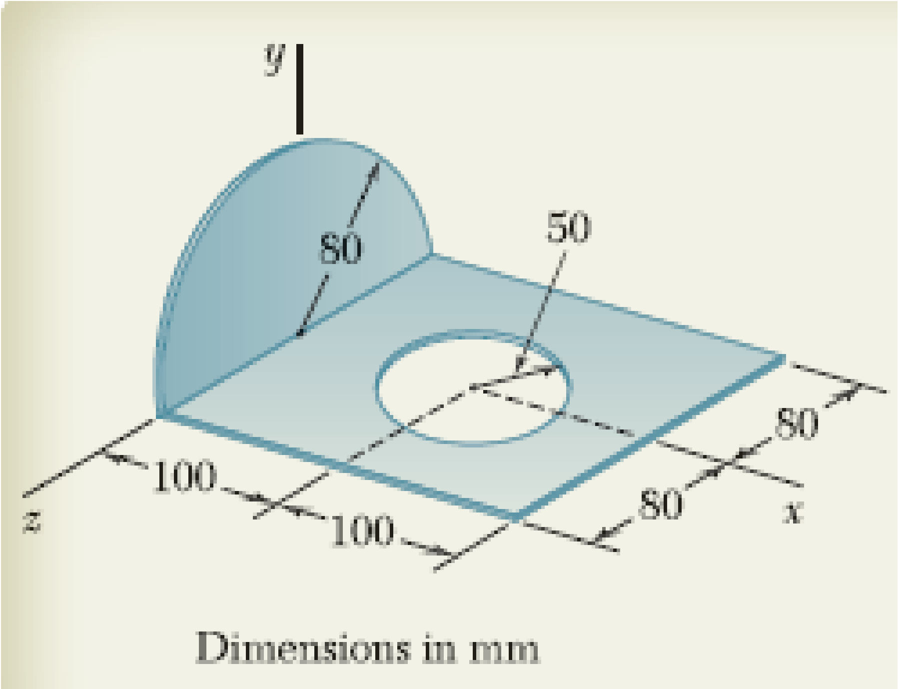


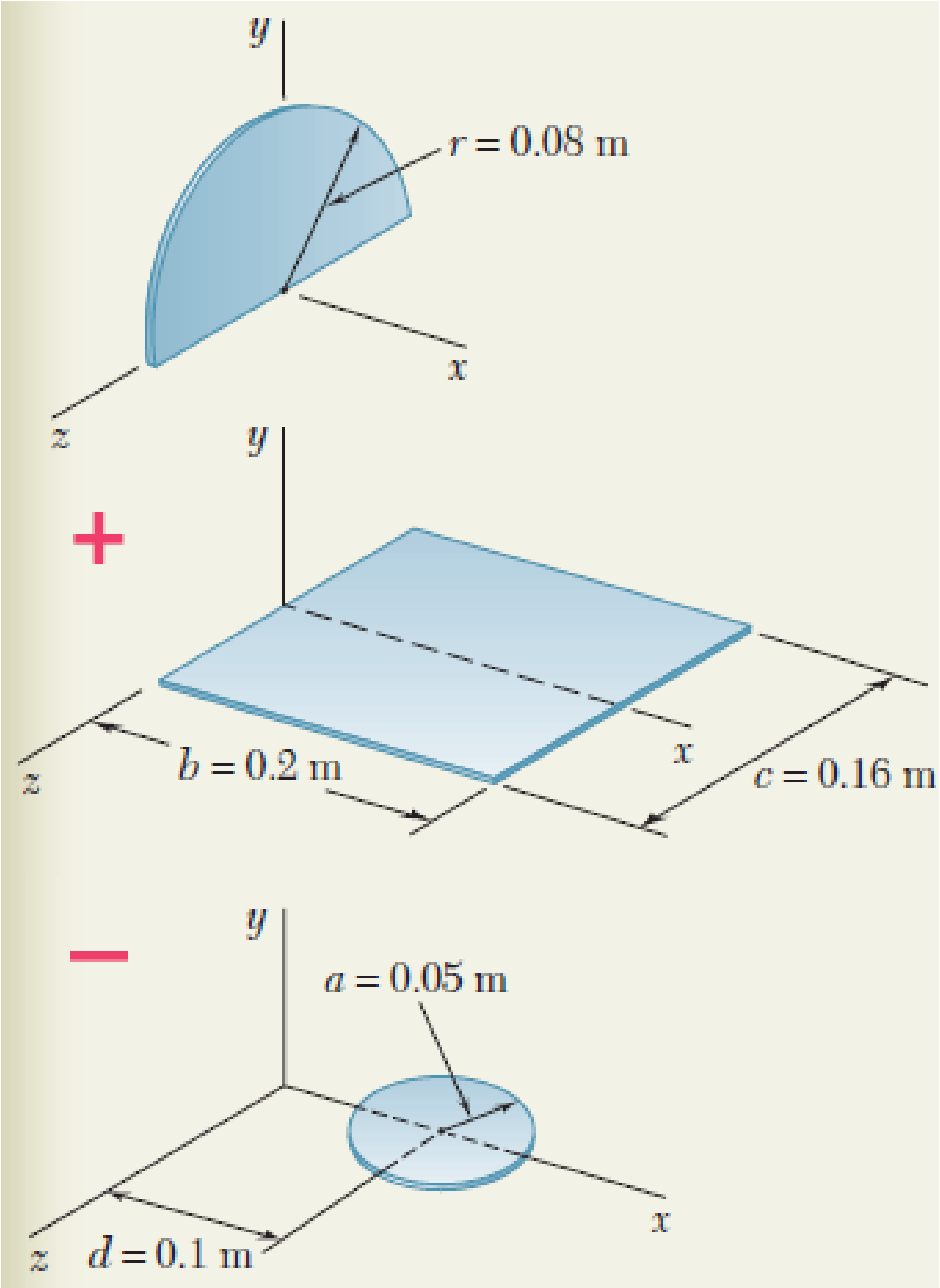

$$I_x = I_y = I_z = \frac{2}{5} ma^2$$

# Problem 4

A thin steel plate which is 4 mm thick is cut and bent to form the machine part shown. Knowing that the density of steel is 7850 kg/m³, determine the moments of inertia of the machine part with respect to the coordinate axes.

$$I_x = 3.00 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$
$$I_y = 13.28 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$
$$I_z = 11.29 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$




y
r = 0.08 m
x
z
y
+
b = 0.2 m
x
c = 0.16 m
z
y
−
a = 0.05 m
x
z
d = 0.1 m

**Computation of Masses.** ***Semicircular Plate***

$$V_1 = \tfrac{1}{2}\pi r^2 t = \tfrac{1}{2}\pi(0.08\ \text{m})^2(0.004\,\text{m}) = 40.21 \times 10^{-6}\ \text{m}^3$$

$$m_1 = \rho V_1 = (7.85 \times 10^3\ \text{kg/m}^3)(40.21 \times 10^{-6}\ \text{m}^3) = 0.3156\ \text{kg}$$

***Rectangular Plate***

$$V_2 = (0.200\ \text{m})(0.160\ \text{m})(0.004\ \text{m}) = 128 \times 10^{-6}\ \text{m}^3$$

$$m_2 = \rho V_2 = (7.85 \times 10^3\ \text{kg/m}^3)(128 \times 10^{-6}\ \text{m}^3) = 1.005\ \text{kg}$$

***Circular Plate***

$$V_3 = \pi a^2 t = \pi(0.050\ \text{m})^2(0.004\ \text{m}) = 31.42 \times 10^{-6}\ \text{m}^3$$

$$m_3 = \rho V_3 = (7.85 \times 10^3\ \text{kg/m}^3)(31.42 \times 10^{-6}\ \text{m}^3) = 0.2466\ \text{kg}$$

***Semicircular Plate.*** From Fig. 9.28, we observe that for a circular plate of mass $m$ and radius $r$

$$I_x = \tfrac{1}{2}mr^2 \qquad I_y = I_z = \tfrac{1}{4}mr^2$$

Because of symmetry, we note that for a semicircular plate

$$I_x = \tfrac{1}{2}(\tfrac{1}{2}mr^2) \qquad I_y = I_z = \tfrac{1}{2}(\tfrac{1}{4}mr^2)$$

Since the mass of the semicircular plate is $m_1 = \tfrac{1}{2}m$, we have

$$I_x = \tfrac{1}{2}m_1r^2 = \tfrac{1}{2}(0.3156\text{ kg})(0.08\text{ m})^2 = 1.010 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$
$$I_y = I_z = \tfrac{1}{4}(\tfrac{1}{2}mr^2) = \tfrac{1}{4}m_1r^2 = \tfrac{1}{4}(0.3156\text{ kg})(0.08\text{ m})^2 = 0.505 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

### Rectangular Plate

$$I_x = \tfrac{1}{12}m_2c^2 = \tfrac{1}{12}(1.005\text{ kg})(0.16\text{ m})^2 = 2.144 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$
$$I_z = \tfrac{1}{3}m_2b^2 = \tfrac{1}{3}(1.005\text{ kg})(0.2\text{ m})^2 = 13.400 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$
$$I_y = I_x + I_z = (2.144 + 13.400)(10^{-3}) = 15.544 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

### Circular Plate

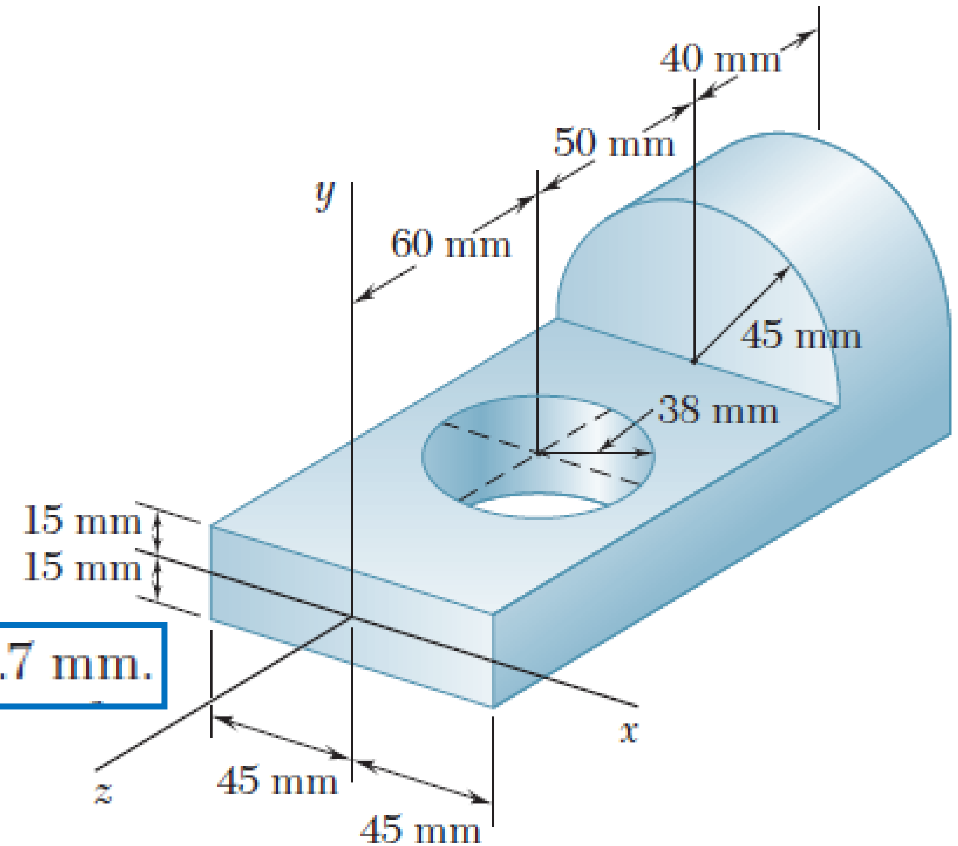
$$I_x = \tfrac{1}{4}m_3a^2 = \tfrac{1}{4}(0.2466\text{ kg})(0.05\text{ m})^2 = 0.154 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$
$$I_y = \tfrac{1}{2}m_3a^2 + m_3d^2$$
$$= \tfrac{1}{2}(0.2466\text{ kg})(0.05\text{ m})^2 + (0.2466\text{ kg})(0.1\text{ m})^2 = 2.774 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$
$$I_z = \tfrac{1}{4}m_3a^2 + m_3d^2 = \tfrac{1}{4}(0.2466\text{ kg})(0.05\text{ m})^2 + (0.2466\text{ kg})(0.1\text{ m})^2$$
$$= 2.620 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

### Entire Machine Part

$$I_x = (1.010 + 2.144 - 0.154)(10^{-3})\text{ kg}\cdot\text{m}^2 \qquad I_x = 3.00 \times 10^{-3}\text{ kg}\cdot\text{m}^2 \blacktriangleleft$$
$$I_y = (0.505 + 15.544 - 2.774)(10^{-3})\text{ kg}\cdot\text{m}^2 \qquad I_y = 13.28 \times 10^{-3}\text{ kg}\cdot\text{m}^2 \blacktriangleleft$$
$$I_z = (0.505 + 13.400 - 2.620)(10^{-3})\text{ kg}\cdot\text{m}^2 \qquad I_z = 11.29 \times 10^{-3}\text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

# Problem 5

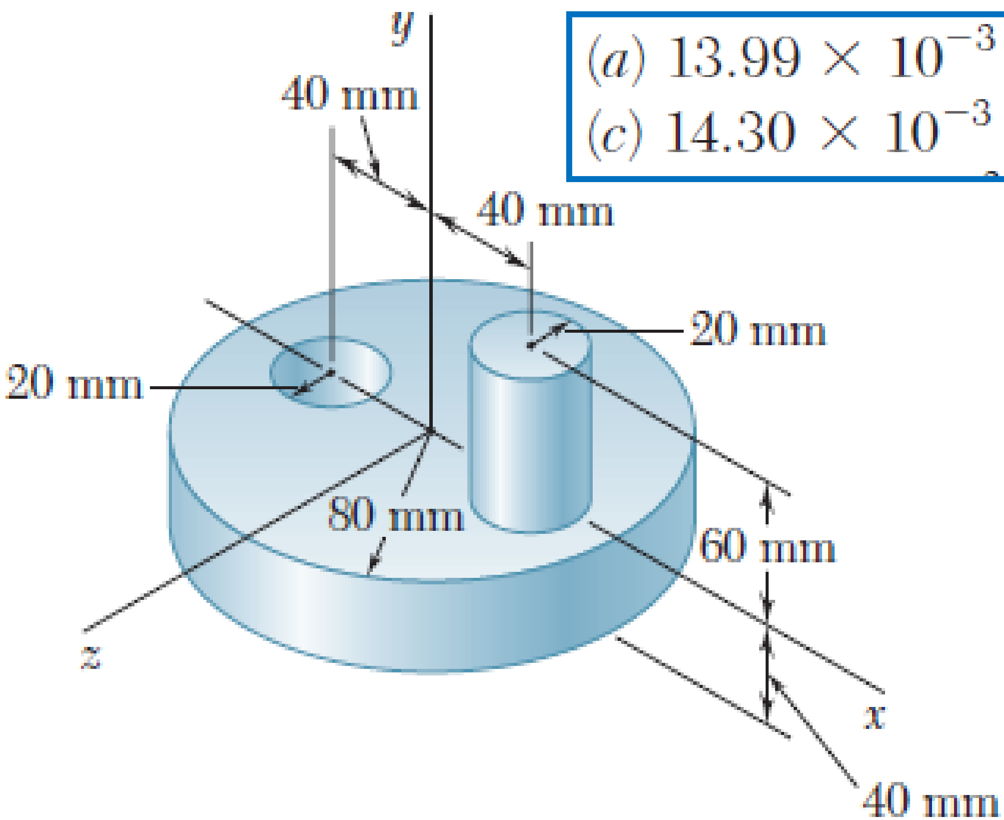
Determine the mass moment of inertia and the radius of gyration of the steel machine element shown with respect to the *x* axis. (The density of steel is 7850 kg/m$^3$.)



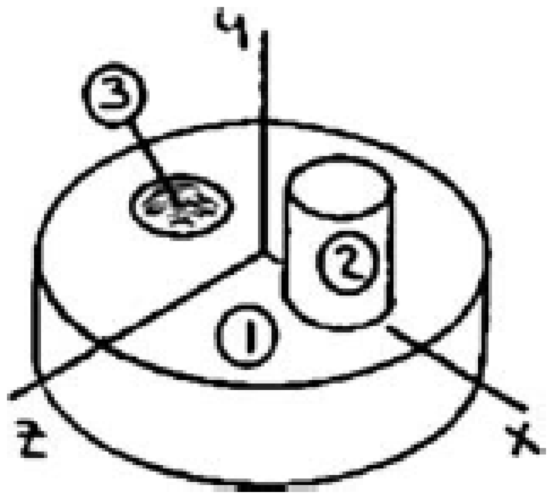

$\boxed{I_x = 38.1 \times 10^{-3} \text{ kg} \cdot \text{m}^2;\ k_x = 110.7 \text{ mm.}}$

# Problem 6

The machine element shown is fabricated from steel. Determine the mass moment of inertia of the assembly with respect to (*a*) the *x* axis, (*b*) the *y* axis, (*c*) the *z* axis. (The density of steel is 7850 kg/m$^3$.)




(*a*) $13.99 \times 10^{-3}$ kg · m$^2$. (*b*) $20.6 \times 10^{-3}$ kg · m$^2$.
(*c*) $14.30 \times 10^{-3}$ kg · m$^2$.




$$m_1 = (7850\ \text{kg/m}^3)(\pi(0.08\ \text{m})^2(0.04\ \text{m})]$$
$$= 6.31334\ \text{kg}$$

$$m_2 = (7850\ \text{kg/m}^3)[\pi(0.02\ \text{m})^2(0.06\ \text{m})] = 0.59188\ \text{kg}$$

$$m_3 = (7850\ \text{kg/m}^3)[\pi(0.02\ \text{m})^2(0.04\ \text{m})] = 0.39458\ \text{kg}$$

$$I_x = (I_x)_1 + (I_x)_2 - (I_x)_3$$

$$= \left\{\frac{1}{12}(6.31334\ \mathrm{kg})[3(0.08)^2 + (0.04)^2]\ \mathrm{m}^2 + (6.31334\ \mathrm{kg})(0.02\ \mathrm{m})^2\right\}$$

$$+ \left\{\frac{1}{12}(0.59188\ \mathrm{kg})[3(0.02)^2 + (0.06)^2]\ \mathrm{m}^2 + (0.59188\ \mathrm{kg})(0.03\ \mathrm{m})^2\right\}$$

$$- \left\{\frac{1}{12}(0.39458\ \mathrm{kg})[3(0.02)^2 + (0.04)^2]\ \mathrm{m}^2 + (0.39458\ \mathrm{kg})(0.02\ \mathrm{m})^2\right\}$$

$$= [(10.94312 + 2.52534) + (0.23675 + 0.53269)$$

$$- (0.09207 + 0.15783)] \times 10^{-3}\ \mathrm{kg \cdot m^2}$$

$$= (13.46846 + 0.76944 - 0.24990) \times 10^{-3}\ \mathrm{kg \cdot m^2}$$

$$= 13.98800 \times 10^{-3}\ \mathrm{kg \cdot m^2}$$

$$I_y = (I_y)_1 + (I_y)_2 - (I_y)_3$$

$$= \left[\frac{1}{2}(6.31334 \text{ kg})(0.08 \text{ m})^2\right]$$

$$+ \left[\frac{1}{2}(0.59188 \text{ kg})(0.02 \text{ m})^2 + (0.59188 \text{ kg})(0.04 \text{ m})^2\right]$$

$$- \left[\frac{1}{2}(0.39458 \text{ kg})(0.02 \text{ m}^2) + (0.39458 \text{ kg})(0.04 \text{ m})^2\right]$$

$$= [(20.20269) + (0.11838 + 0.94701)$$

$$- (0.07892 + 0.63133)] \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$= (20.20269 + 1.06539 - 0.71025) \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$= 20.55783 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_z = (I_z)_1 + (I_z)_2 - (I_z)_3$$

$$= \left\{\frac{1}{12}(6.31334\text{ kg})[3(0.08)^2 + (0.04)^2]\text{ m}^2 + (6.31334\text{ kg})(0.02\text{ m})^2\right\}$$

$$+\left\{\frac{1}{12}(0.59188\text{ kg})[3(0.02)^2 + (0.06)^2]\text{ m}^2 + (0.59188\text{ kg})[(0.04)^2 + (0.03)^2]\text{ m}^2\right\}$$

$$-\left\{\frac{1}{12}(0.39458\text{ kg})[3(0.02)^2 + (0.04)^2]\text{ m}^2 + (0.03958\text{ kg})[(0.04)^2 + (0.02)^2]\text{ m}^2\right\}$$

$$= [(10.94312 + 2.52534) + (0.23675 + 1.47970)$$

$$- (0.09207 + 0.78916)] \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= (13.46846 + 1.71645 - 0.88123) \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= 14.30368 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$