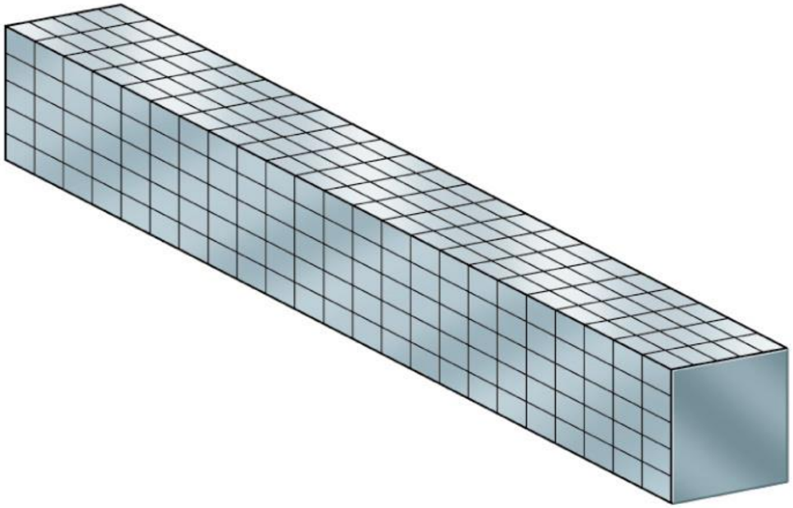
Module-4
Stresses in beams

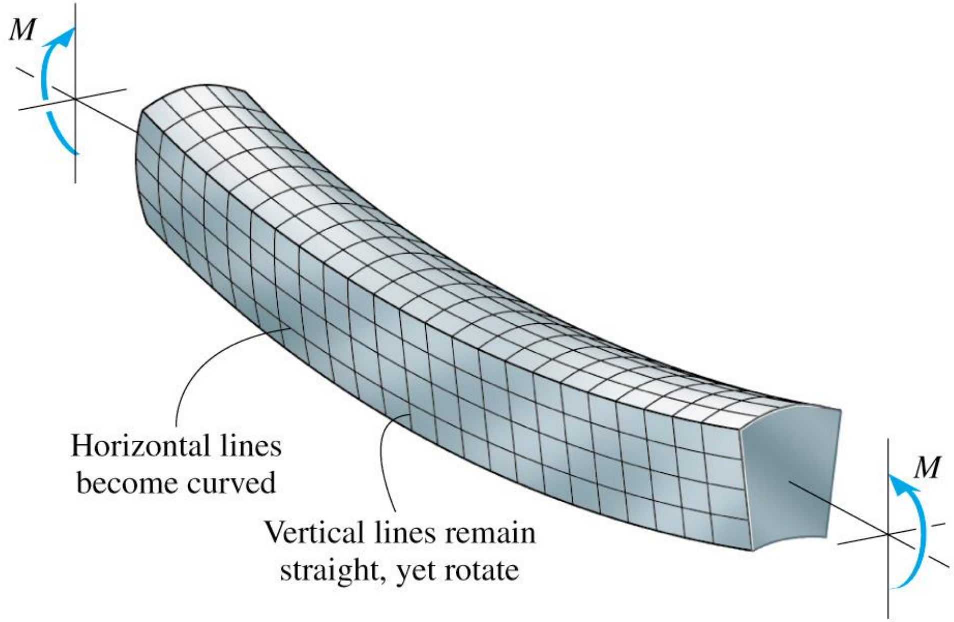
Theory of simple bending – Assumptions – Derivation of bending equation - Neutral axis – Determination of bending stresses – section modulus of rectangular and circular sections (Solid and Hollow), I, T, Angle and Channel sections – Design of simple beam sections, Shear Stresses: Derivation of formula – Shear stress distribution across various beams sections like rectangular, circular, triangular, I, T angle sections.

Bending in beams



Before deformation

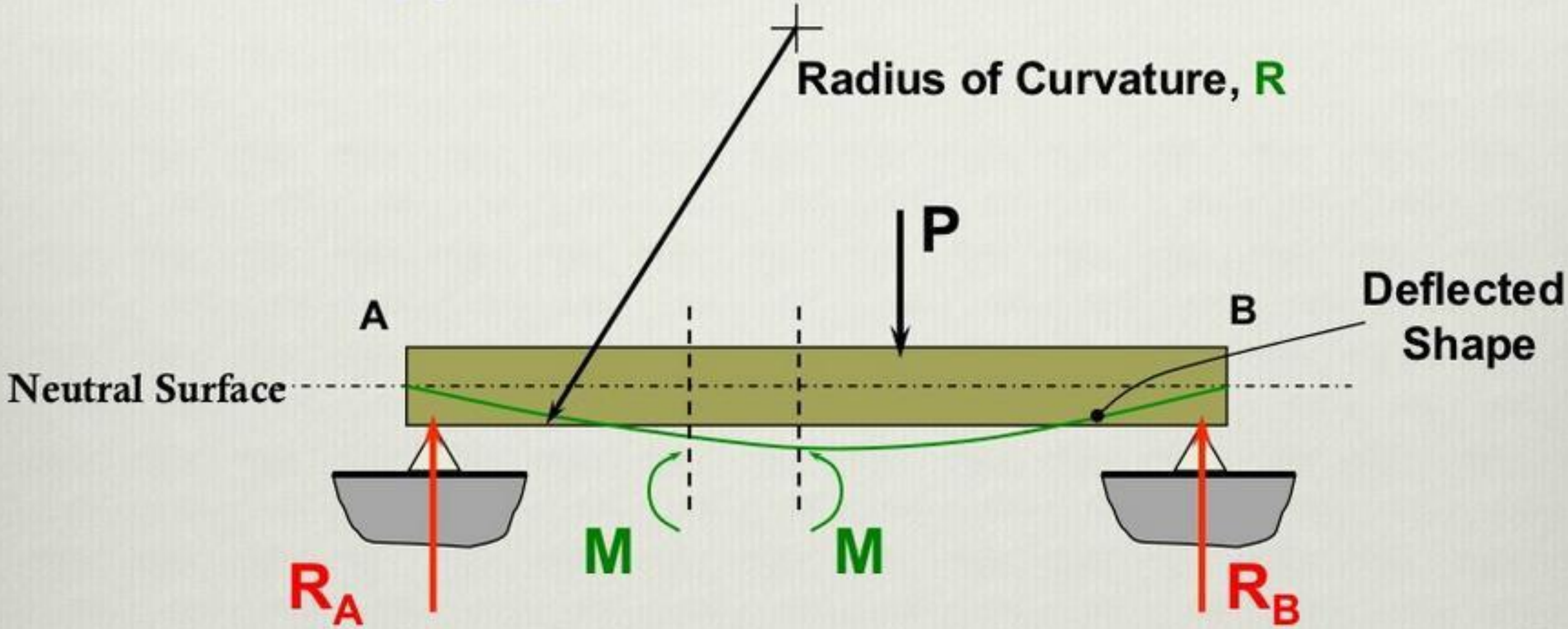
(a)

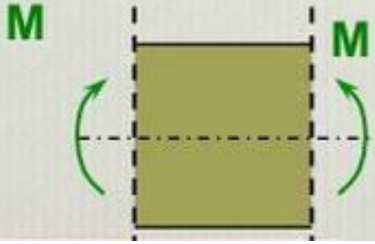


After deformation

(b)

Consider the simply supported beam below:





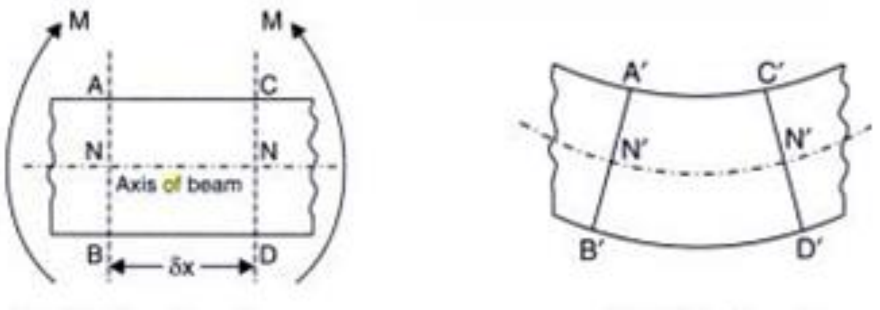
What stresses are generated within, due to bending?

SIMPLE BENDING OR PURE BENDING

- When some **external force** acts on a beam, the shear force and bending moments are set up at all the sections of the beam
- Due to **shear force and bending moment**, the beam undergoes **deformation**. The material of the beam offers resistance to deformation
- Stresses introduced by bending moment are known as **bending stresses**
- Bending stresses are indirect normal stresses

THEORY OF SIMPLE BENDING

- Consider a beam subjected to simple bending. Consider an infinitesimal element of length dx which is a part of this beam. Consider two transverse sections AB and CD which are normal to the axis of the beam and parallel to each other.

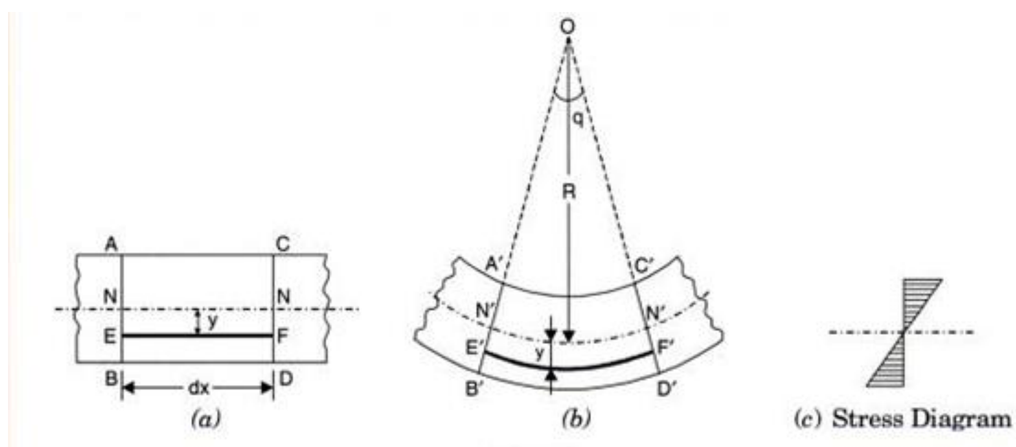


- Due to the bending action the element ABCD is deformed to A'B'C'D' (concave curve).
- The layers of the beam are not of the same length before bending and after bending .

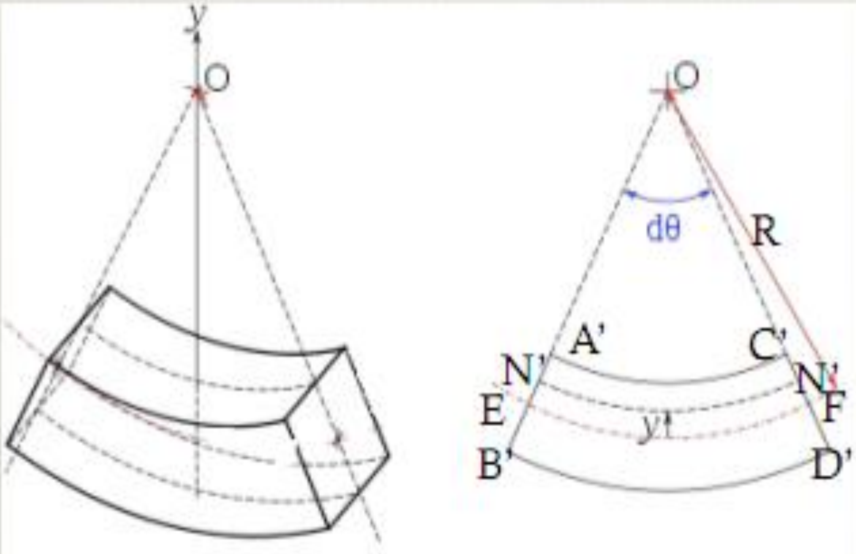
- The layer AC is shortened to A’C’. Hence it is subjected to **compressive stress**
- The layer BD is elongated to B’D’. Hence it is subjected to **tensile stresses**.
- Hence the amount of shortening decrease from the top layer towards bottom and the amount of elongation decreases from the bottom layer towards top
- Therefore there is a layer in between which neither elongates nor shortens. This layer is called **neutral layer** .

NEUTRAL AXIS

- For a beam subjected to a pure bending moment, the stresses generated on the **neutral layer is zero**.
- Neutral axis is the line of intersection of neutral layer with the transverse section
- Consider the cross section of a beam subjected to pure bending. The stress at a distance y from the neutral axis is given by σ/y=E/R



Stresses due to bending



$$\text{Strain in layer EF} = \frac{y}{R}$$

$$E = \frac{Stress\_in\_the\_layer\_EF}{Strain\_in\_the\_layer\_EF}$$

$$E = \frac{\sigma}{\left(\frac{y}{R}\right)}$$

$$\frac{\sigma}{y} = \frac{E}{R} \Rightarrow \sigma = \frac{E}{R} y$$

Flexure Formula

$$\frac{M}{I} = \frac{E}{R} = \frac{\sigma}{y}$$

Section Modulus

Section modulus is defined as ratio of moment of inertia about the neutral axis to the distance of the outermost layer from the neutral axis

$$Z = \frac{I}{y_{\max}}$$

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\frac{M}{I} = \frac{\sigma_{\max}}{y_{\max}}$$

$$M = \sigma_{\max} \frac{I}{y_{\max}}$$

$$\boxed{M = \sigma_{\max} Z}$$

Section Modulus (Z)

- It is the ratio of moment of inertia of a section about the neutral axis to the distance of the outermost layer from the neutral axis.

$$Z = \frac{I}{y_{max}}$$

I = Moment of Inertia about neutral axis.

y_{max} = Distance of the outermost layer from the neutral axis.

Hence, $M = \sigma_{max} . Z$

- Thus, moment of resistance offered by the section is maximum when Z is maximum. Hence, Z represents the strength of the section.

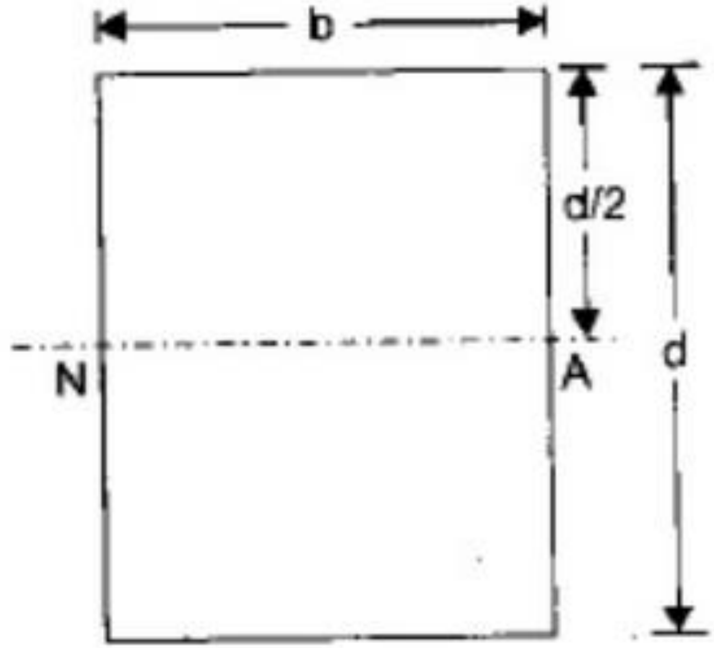
Section Modulus (Z)

1. Rectangular Section

$$I = \frac{bd^3}{12}$$

$$y_{max} = \frac{d}{2}$$

$$Z = \frac{bd^2}{6}$$



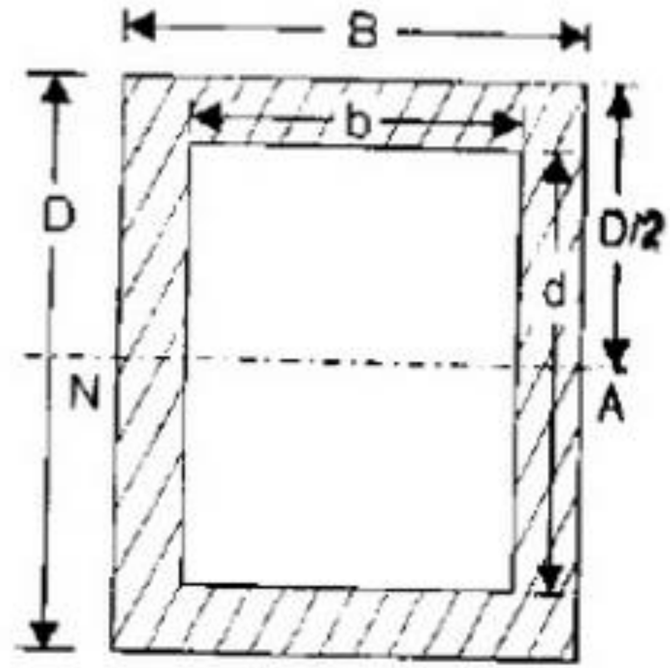
Section Modulus (Z)

2. Rectangular Hollow Section

$$I = \frac{BD^3}{12} - \frac{bd^3}{12}$$

$$y_{max} = \left(\frac{D}{2}\right)$$

$$Z = \frac{1}{6D}\left[BD^3 - bd^3\right]$$



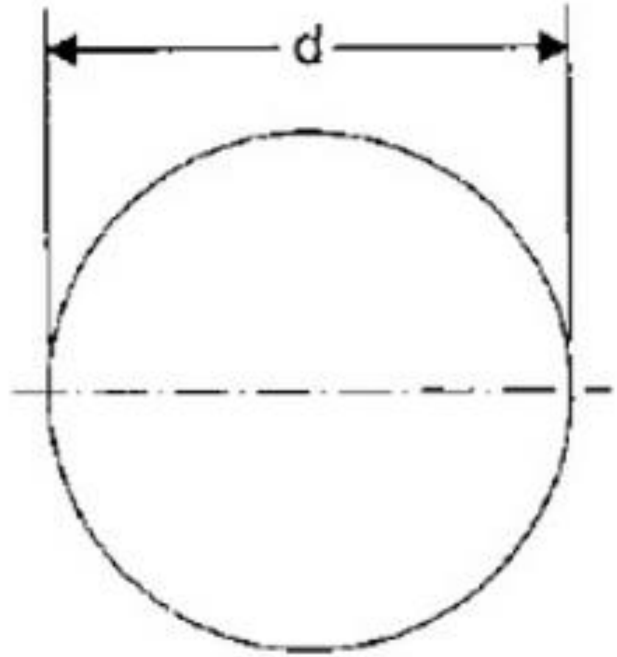
Section Modulus (Z)

3. Circular Section

$$I = \frac{\pi}{64} d^4$$

$$y_{max} = \frac{d}{2}$$

$$Z = \frac{\pi}{32} d^3$$



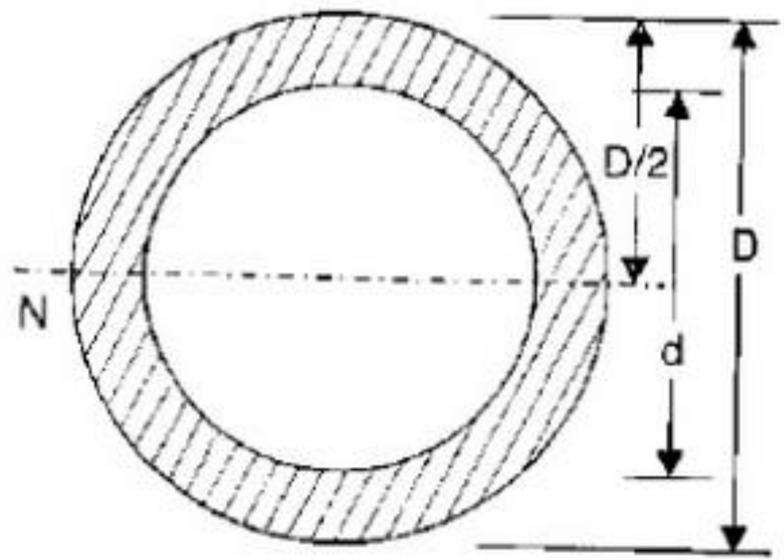
Section Modulus (Z)

4. Circular Hollow Section

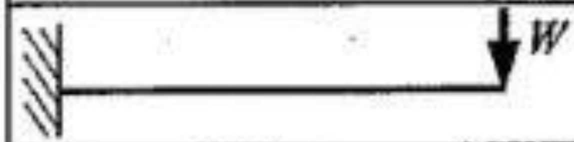
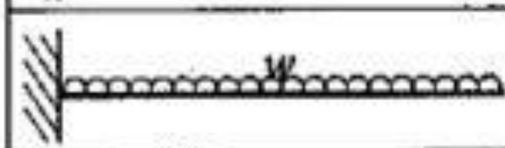
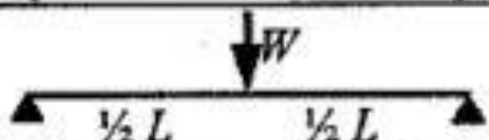
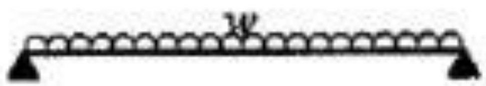
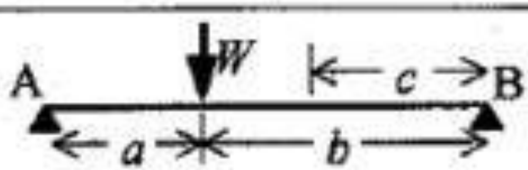
$$I = \frac{\pi}{64}\,[D^4 - d^4]$$

$$y_{max} = \frac{D}{2}$$

$$Z = \frac{\pi}{32D}\,[D^4 - d^4]$$



BEAM BENDING

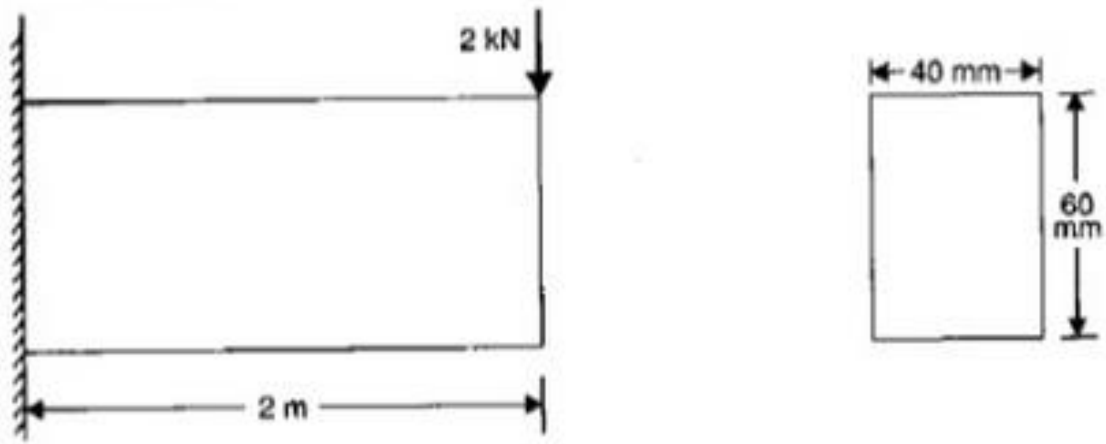
L = overall length W = point load, M = moment w = load per unit length	End Slope	Max Deflection	Max bending moment
	$\frac{ML}{EI}$	$\frac{ML^2}{2EI}$	M
	$\frac{WL^2}{2EI}$	$\frac{WL^3}{3EI}$	WL
	$\frac{wL^3}{6EI}$	$\frac{wL^4}{8EI}$	$\frac{wL^2}{2}$
	$\frac{ML}{2EI}$	$\frac{ML^2}{8EI}$	M
	$\frac{WL^2}{16EI}$	$\frac{WL^3}{48EI}$	$\frac{WL}{4}$
	$\frac{wL^3}{24EI}$	$\frac{5wL^4}{384EI}$	$\frac{wL^2}{8}$
 $a \le b,\ \ c = \sqrt{\frac{1}{3}b(L+a)}$	$\theta_B = \frac{Wac^2}{2LEI}$ $\theta_A = \frac{L+b}{L+a}\theta_B$	$\frac{Wac^3}{3LEI}$ (at position c)	$\frac{Wab}{L}$ (under load)

Problems

A cantilever of length 2m fails when a load of 2 kN is applied at the free end. If the section of the beam is 40 mm X 60 mm, find the stress at the failure.

Solution:

Problem Sketch:



$$\sigma_{max} = \frac{M}{Z} = \frac{4 \times 10^6}{24000} = \mathbf{166.67\ N/mm^2.}$$

Problem

A square beam 20mm X 20mm in section and 2m long is simply supported at the ends. The beam fails when a point load of 400N is applied at the centre of the beam. What uniformly distributed load per meter length will break a cantilever of the same material 40mm wide, 60mm deep and 3m long?

Solution:

Square c/s.: 20 mm x 20 mm; L = 2m; W = 400 N

Rectangular c/s.: 40 mm x 60 mm; L = 3m; w = ?

Equate the maximum stress in the two cases.

$$w = \frac{150 \times 24000}{4.5 \times 1000} = \mathbf{800\ N/m.}$$

A 250 mm (depth)× 150 mm(width) **rectangular beam** is subjected to maximum bending moment of 750 kNm. Determine

(a) The **maximum stress** in the beam.

(b) If the value of E for the beam materials is 200 GN/m2, find out the **radius of curvature** for that portion of the beam where the bending is maximum.

(c) The value of **longitudinal stress** at a distance of 65 mm from the top surface of the beam.

Hence, the maximum stress in the beam = **480 MN/m²** **(Ans.)**

$$R = \frac{EI}{M} = \frac{200 \times 10^9 \times 0.0001953}{750 \times 10^3} = \mathbf{52.08\ m} \quad \textbf{(Ans.)}$$

(*iii*) Longitudinal stress at a distance of 65 mm from top surface of the beam, σ_1:

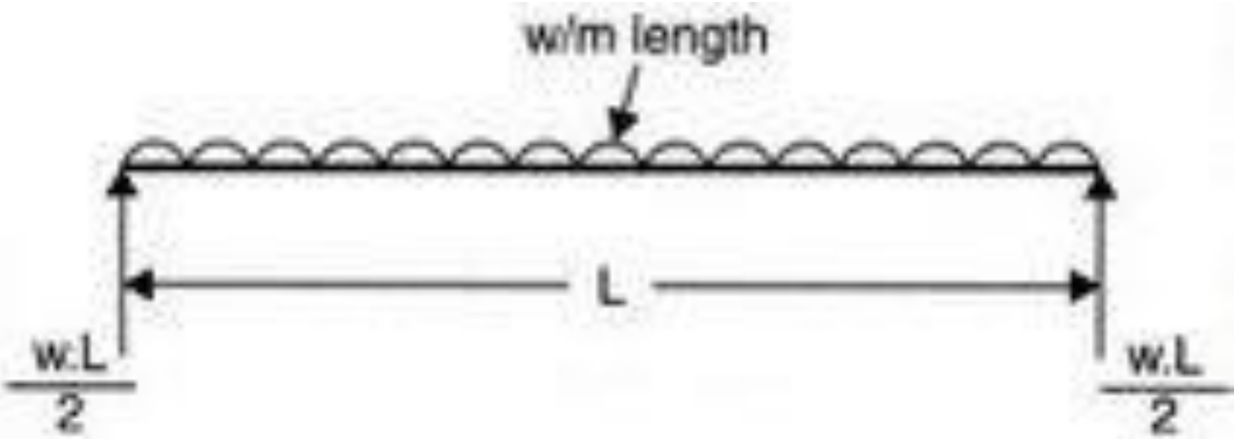
Using the relation, $\frac{M}{I} = \frac{\sigma}{y} = \frac{\sigma_1}{y_1}$, we get

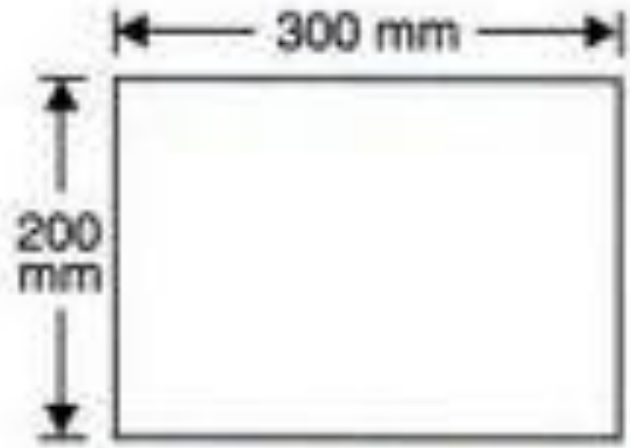
$$\sigma_1 = \frac{M \cdot y_1}{I}$$

$$= \frac{750 \times 10^3 \times (60 \times 10^{-3})}{0.0001953} \times 10^{-6}\ \text{MN/m}^2 \; (\because y_1 = 125 - 65 = 60\ \text{mm})$$

$$= \mathbf{230.4\ MN/m^2} \quad \textbf{(Ans.)}$$

Problem *A rectangular beam 200 mm deep and 300 mm wide is simply supported over a span of 8 m. What uniformly distributed load per metre the beam may carry, if the bending stress is not to exceed 120 N/mm².*





$$\therefore \quad w = \frac{120 \times 2000000}{8000} = 30 \times 1000 \text{ N/m} = \mathbf{30\ kN/m.} \quad \text{Ans.}$$

Problem *Calculate the maximum stress* induced in a cast iron pipe of external diameter 40 mm, of internal diameter 20 mm and of length 4 metre when the pipe is supported at its ends and carries a point load of 80 N at its centre.*

Sol. Given :

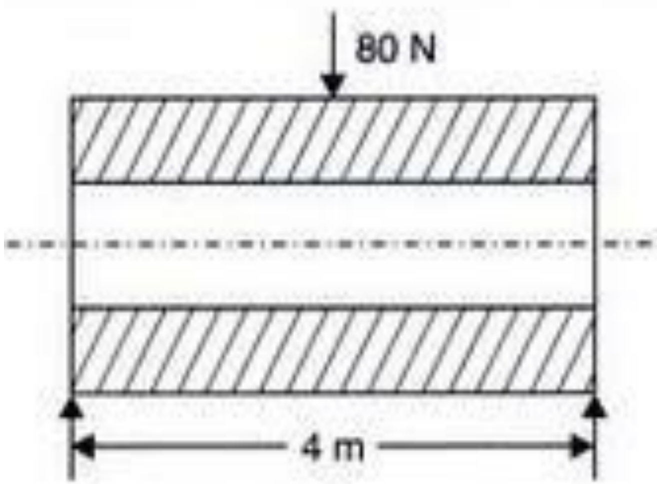
External dia., $D = 40$ mm

Internal dia., $d = 20$ mm

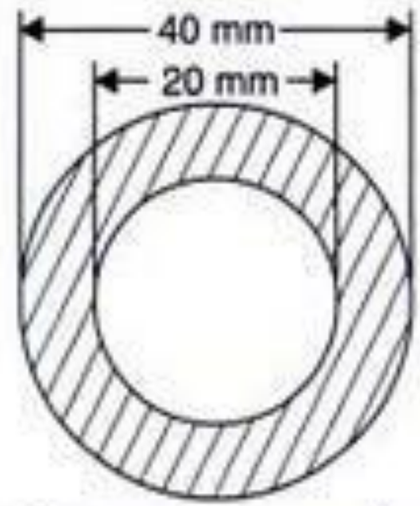
Length, $L = 4 \text{ m} = 4 \times 1000 = 4000$ mm

Point load, $W = 80$ N

In case of simply supported beam carrying a point load at the centre, the maximum bending moment is at the centre of the beam.



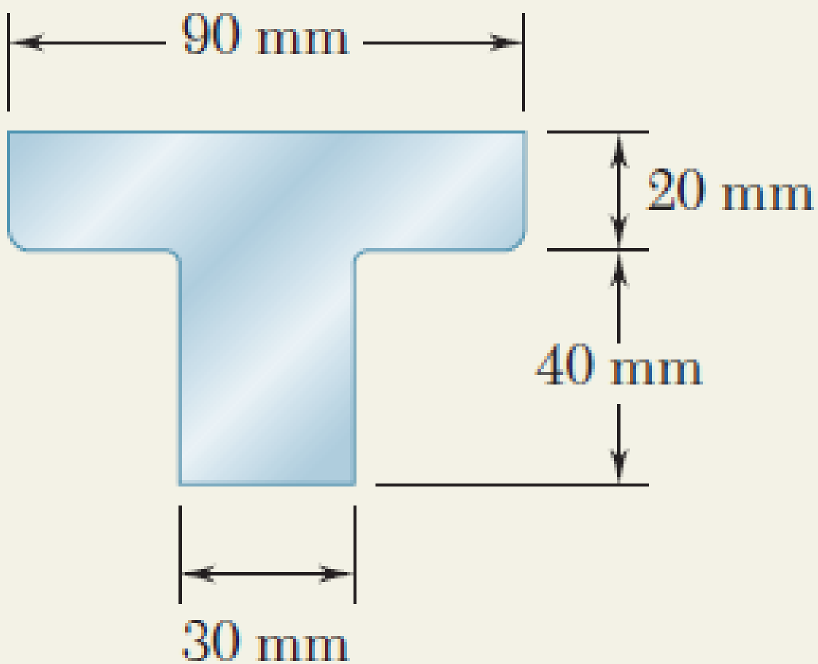
(*a*)

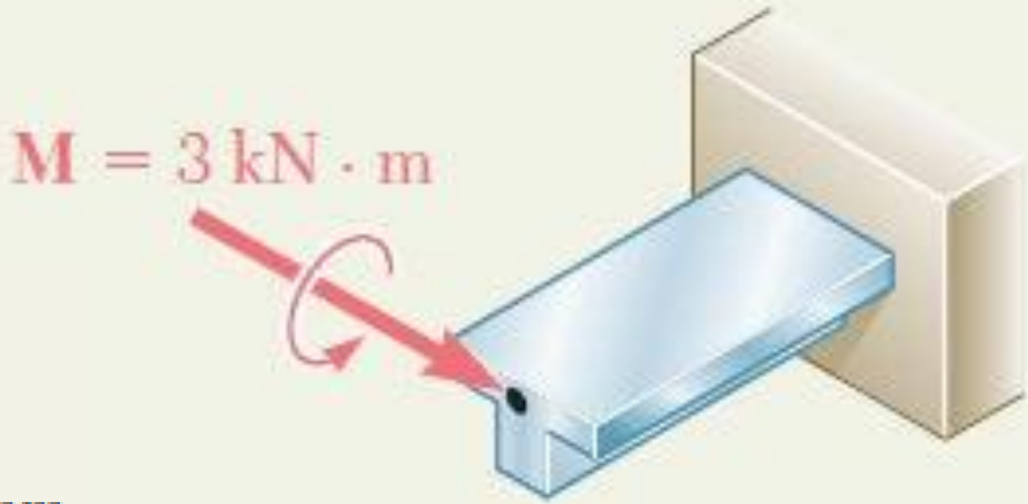


(*b*) Area of cross-section

$$= \frac{8 \times 10^4 \times 20}{117809.7} = \mathbf{13.58\ N/mm^2}. \quad \text{Ans.}$$

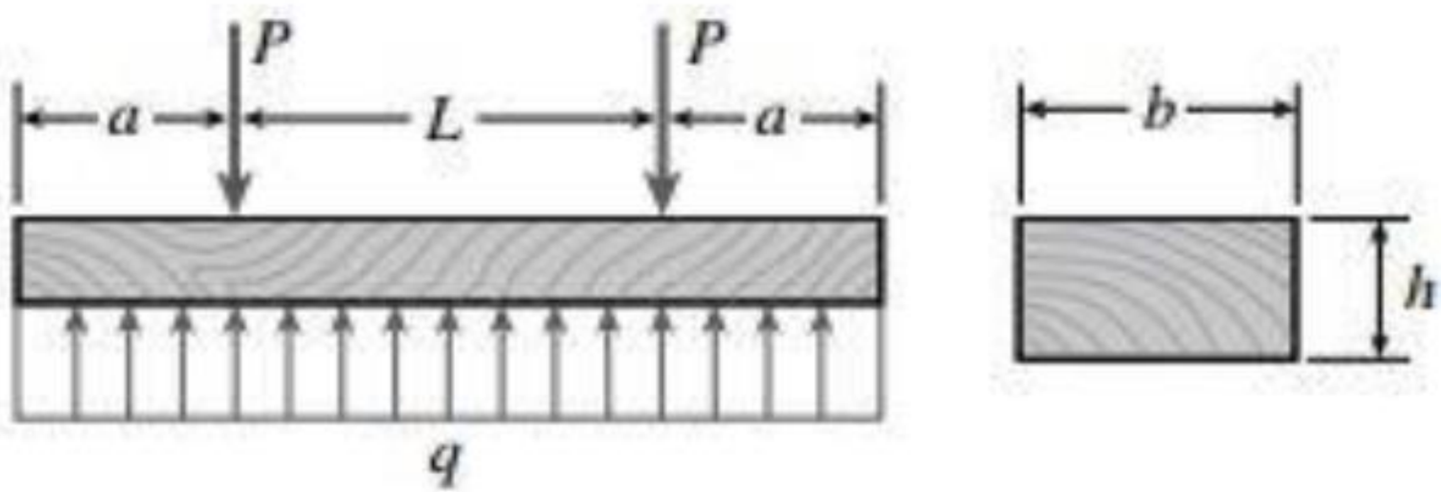
A cast-iron machine part is acted upon by the 3 kN · m couple shown. Knowing that E = 165 GPa and neglecting the effect of fillets, determine (a) the maximum tensile and compressive stresses in the casting, (b) the radius of curvature of the casting.

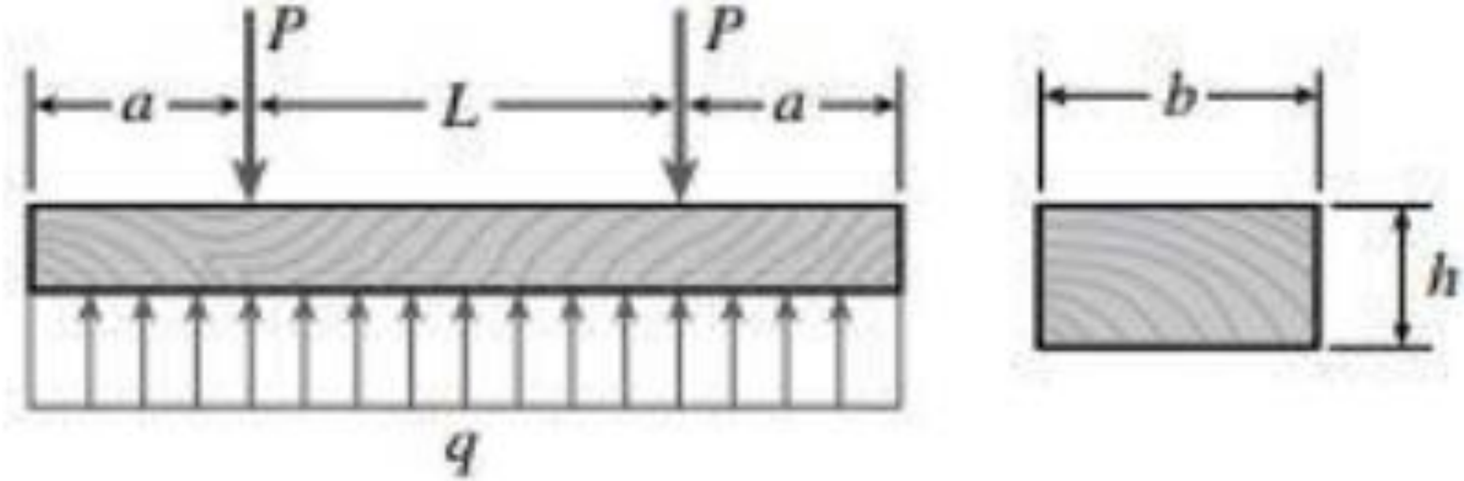


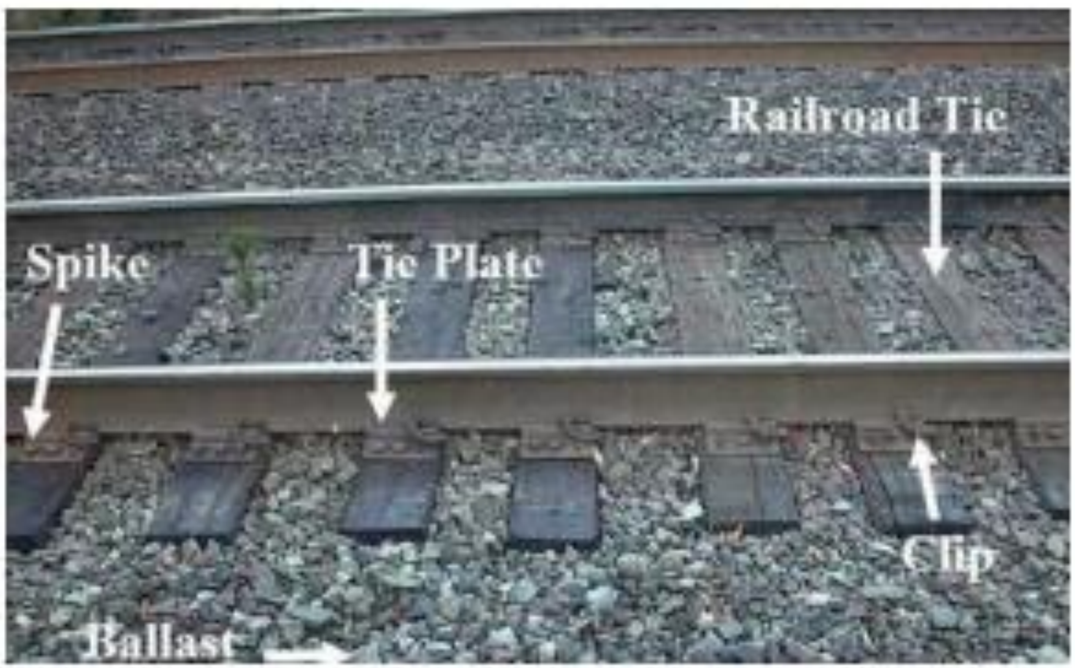


Problem A railroad tie (or *sleeper*) is subjected to two rail loads, each of magnitude $P = 175$ kN, acting as shown in the figure. The reaction q of the ballast is assumed to be uniformly distributed over the length of the tie, which has cross-sectional dimensions $b = 300$ mm and $h = 250$ mm.

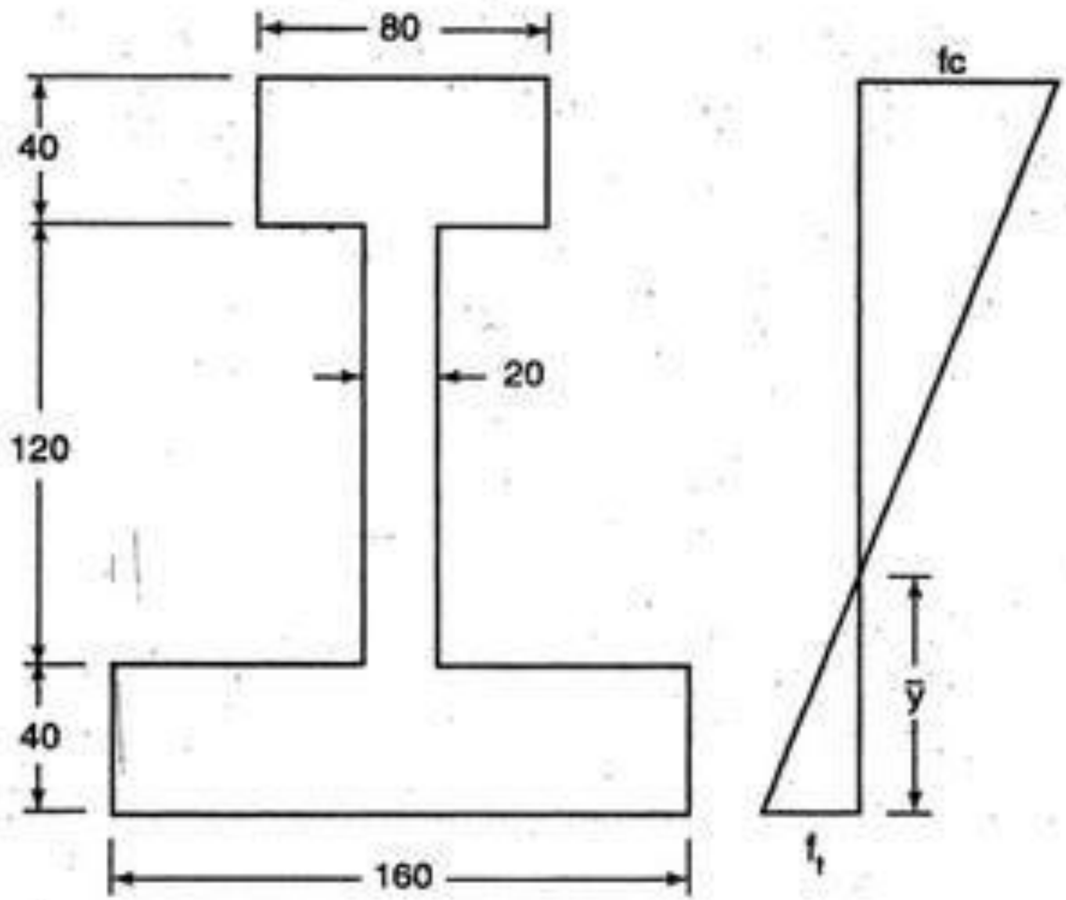
Calculate the maximum bending stress σ_{max} in the tie due to the loads P, assuming the distance $L = 1500$ mm and the overhang length $a = 500$ mm.




P
P
a
L
a
b
h
q

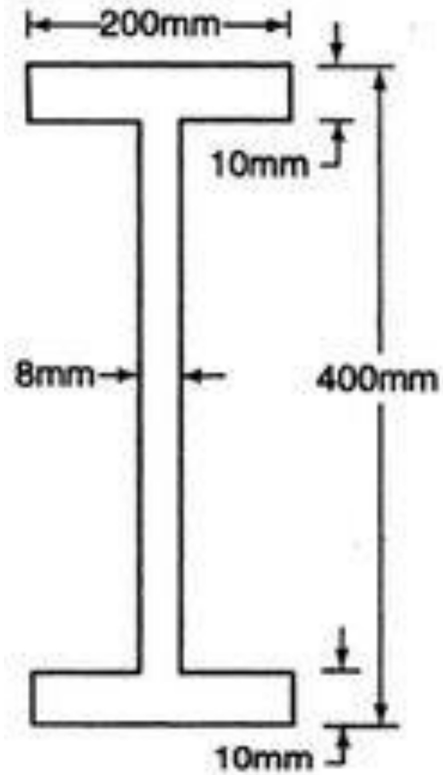

Railroad Tie
Spike
Tie Plate
Clip
Ballast

Example A cast iron beam has an I-section with top flange 80 mm × 40 mm, web 120 mm × 20 mm and bottom flange 160 mm × 40 mm. If tensile stress is not to exceed 30N/mm^2 and compressive stress 90N/mm^2, what is the maximum uniformly distributed load the beam can carry over a simply supported span of 6 m if the larger flange is in tension?

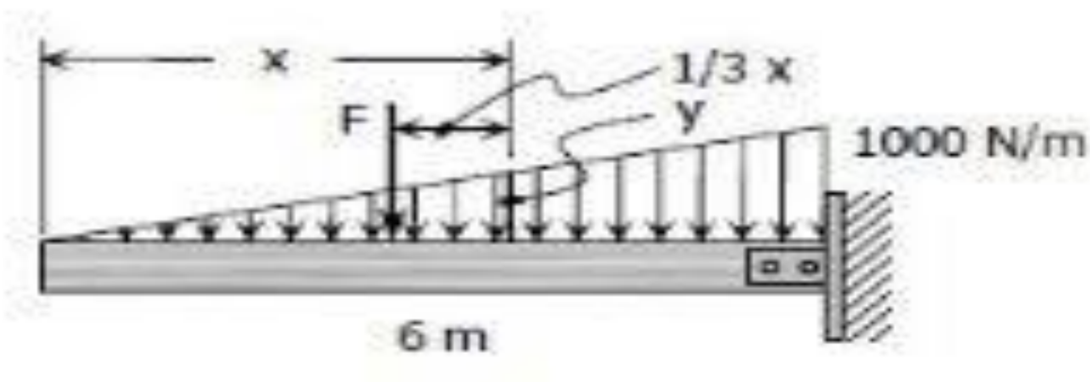


$w = 5.0965 kN/m$ **(Ans)**

Example The cross-section of a beam is as shown in Fig. 4.19. If permissible stress is 150N/mm^2, find its moment of resistance. Compare it with equivalent section of same area but (*a*) Square section (*b*) rectangular section with depth twice the width and (*c*) a circular section.

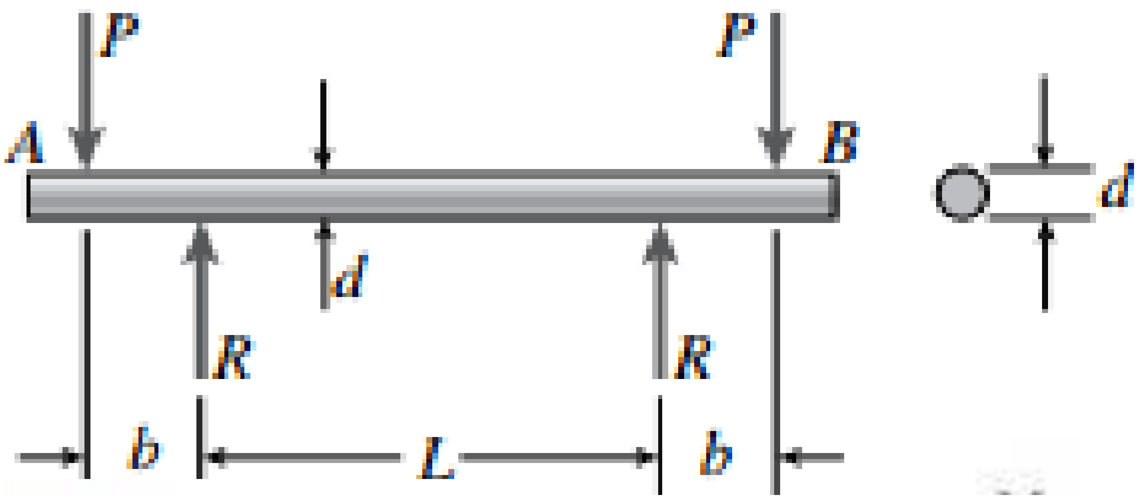


A cantilever beam, 50 mm wide by 150 mm high and 6 m long, carries a load that varies uniformly from zero at the free end to 1000 N/m at the wall. (a) Compute the magnitude and location of the maximum flexural stress. (b) Determine the type and magnitude of the stress in a fiber 20 mm from the top of the beam at a section 2 m from the free end.



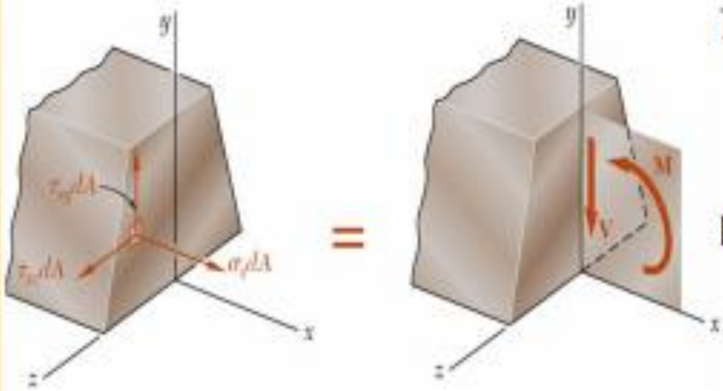
Problem A freight-car axle AB is loaded approximately as shown in the figure, with the forces P representing the car loads (transmitted to the axle through the axle boxes) and the forces R representing the rail loads (transmitted to the axle through the wheels). The diameter of the axle is $d = 80$ mm, the distance between centers of the rails is L, and the distance between the forces P and is R is $b = 200$ mm.

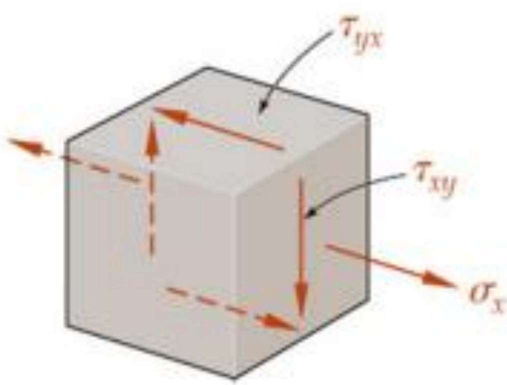
Calculate the maximum bending stress σ_{max} in the axle if $P = 47$ kN.



Shearing Stresses in Beams

Introduction





Transverse loading applied to beam results in normal and shearing stresses in transverse sections.

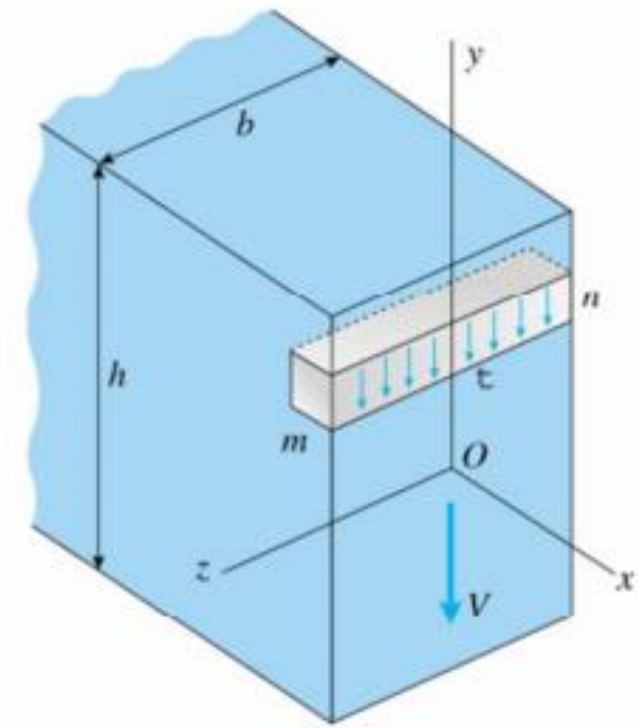
Distribution of normal and shearing stresses satisfies (from equilibrium)

$$F_x = \int \sigma_x dA = 0 \qquad M_x = \int \left(y\tau_{xz} - z\tau_{xy} \right) dA = 0$$
$$F_y = \int \tau_{xy} dA = -V \qquad M_y = \int z\sigma_x dA = 0$$
$$F_z = \int \tau_{xz} dA = 0 \qquad M_z = \int (-y\sigma_x) = M$$

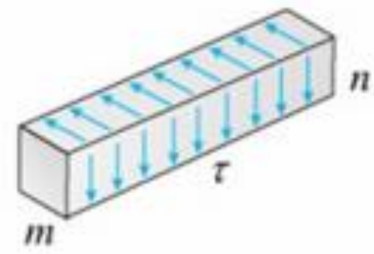
When shearing stresses are exerted on vertical faces of an element, equal stresses exerted on horizontal faces

Longitudinal shearing stresses must exist in any member subjected to transverse loading.

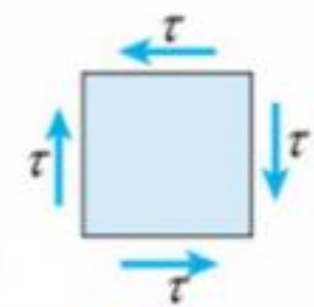
Vertical and Horizontal Shear Stresses

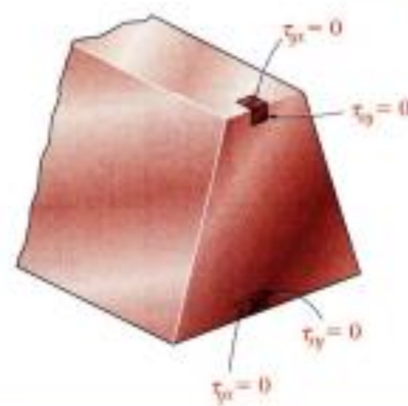


(a)

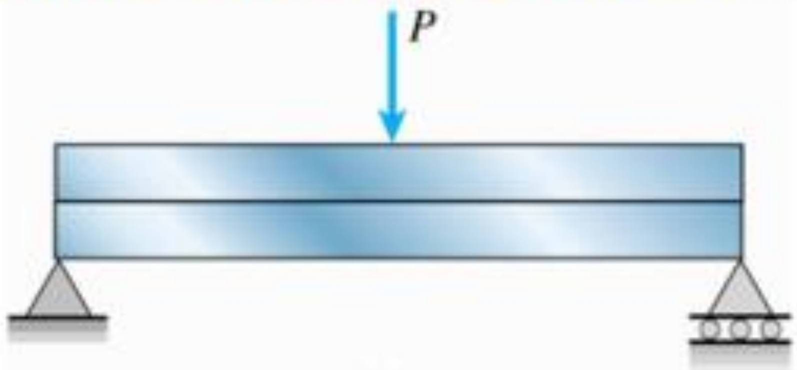


(b)



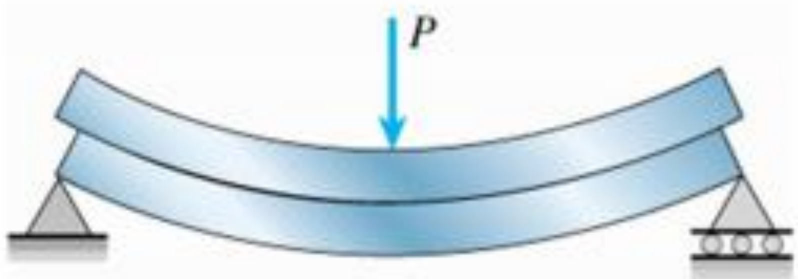


Shear Stress in Beams



(a)

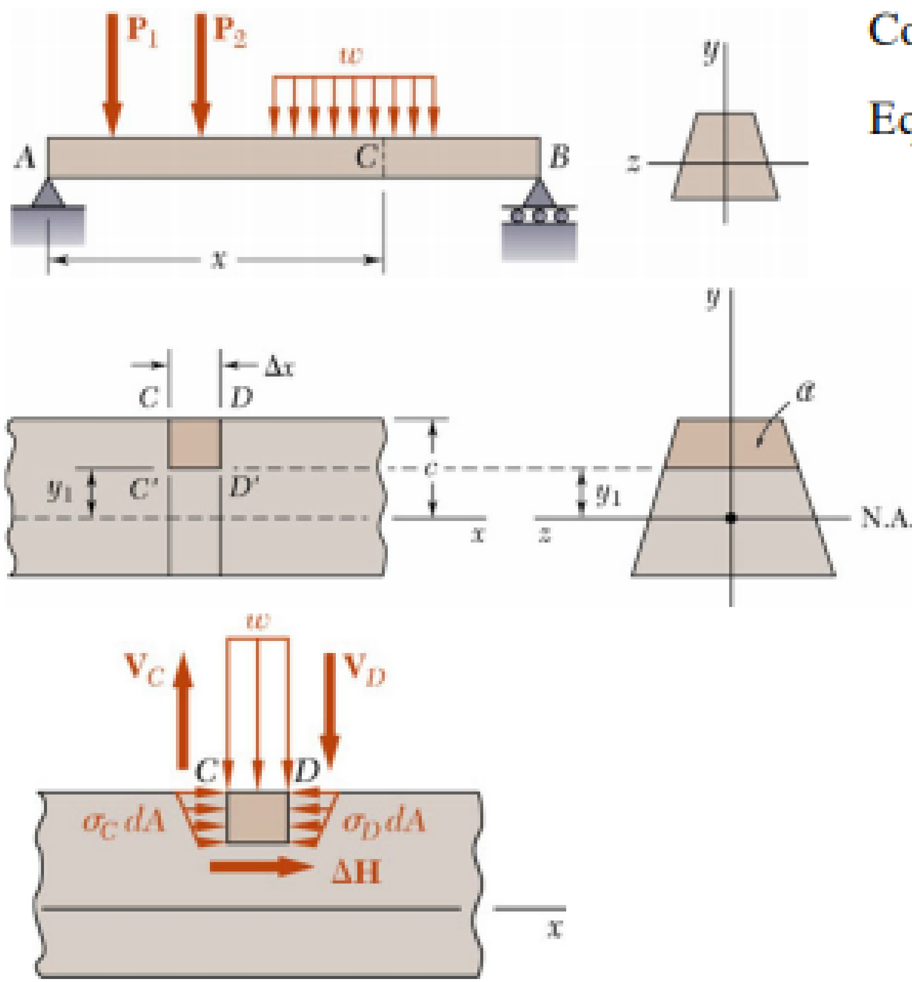
Two beams glued together along horizontal surface



(b)

When loaded, horizontal shear stress must develop along glued surface **in order to prevent sliding between** the beams.

Shear on Horizontal Face of Beam Element



Consider prismatic beam

Equilibrium of element $CDC'D'$

$$\sum F_x = 0 = \Delta H + \int_A (\sigma_D - \sigma_C) dA$$

$$\Delta H = \frac{M_D - M_C}{I} \int_A y \, dA$$

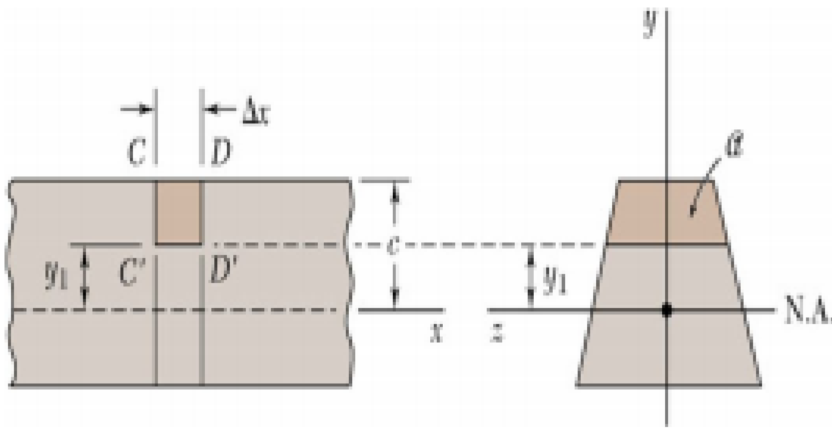
Let, $Q = \int_A y \, dA$

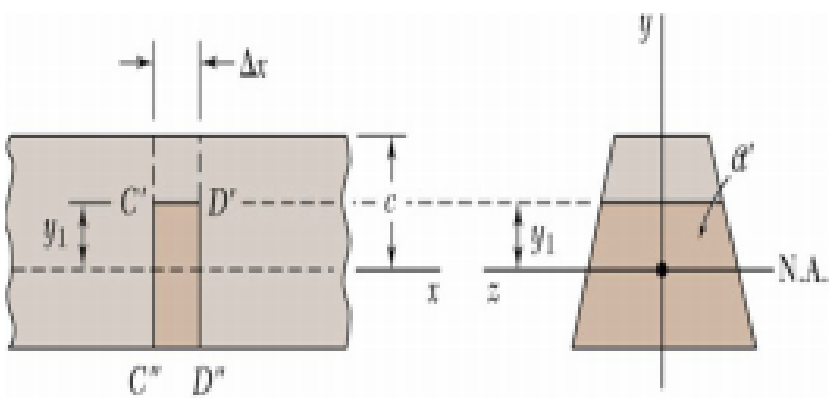
$$M_D - M_C = \frac{dM}{dx} \Delta x = V \, \Delta x$$

$$\Delta H = \frac{VQ}{I} \Delta x$$

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} = \text{shear flow}$$

Shear on Horizontal Face of Beam Element





Shear flow,

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} = \text{shear flow}$$

where

$$Q = \int_A y\, dA$$

$= \text{first moment of area above } y_1$

$$I = \int_{A+A'} y^2 dA$$

$= \text{second moment of full cross section}$

Same result found for lower area

$$Q + Q' = \int_{A_1+A_2} y\, dA = 0$$

($\because$ first moment of area wrt NA is zero)

$$q' = \frac{\Delta H'}{\Delta x} = \frac{VQ'}{I} = \frac{V(-Q)}{I} = -q$$

$$\Delta H' = -\Delta H$$

EXAMPLE 6.01

A beam is made of three planks, 20 by 100 mm in cross section, nailed together (Fig. 6.8). Knowing that the spacing between nails is 25 mm and that the vertical shear in the beam is $V = 500$ N, determine the shearing force in each nail.

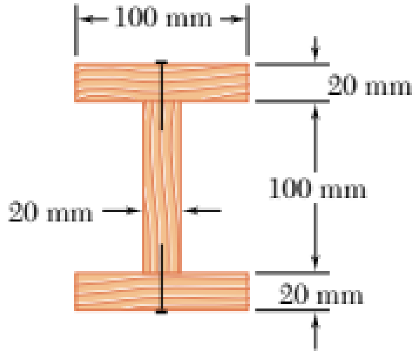


Fig. 6.8

We first determine the horizontal force per unit length, q, exerted on the lower face of the upper plank. We use Eq. (6.5), where Q represents the first moment with respect to the neutral axis of the shaded area A shown in Fig. 6.9a, and where I is the moment of inertia about the same axis of the entire cross-sectional area (Fig. 6.9b). Recalling that the first moment of an area with respect to a given axis is equal to the product of the area and of the distance from its centroid to the axis,† we have

$$Q = A\bar{y} = (0.020\text{ m} \times 0.100\text{ m})(0.060\text{ m})$$
$$= 120 \times 10^{-6}\text{ m}^3$$
$$I = \tfrac{1}{12}(0.020\text{ m})(0.100\text{ m})^3$$
$$+2[\tfrac{1}{12}(0.100\text{ m})(0.020\text{ m})^3$$
$$+(0.020\text{ m} \times 0.100\text{ m})(0.060\text{ m})^2]$$
$$= 1.667 \times 10^{-6} + 2(0.0667 + 7.2)10^{-6}$$
$$= 16.20 \times 10^{-6}\text{ m}^4$$

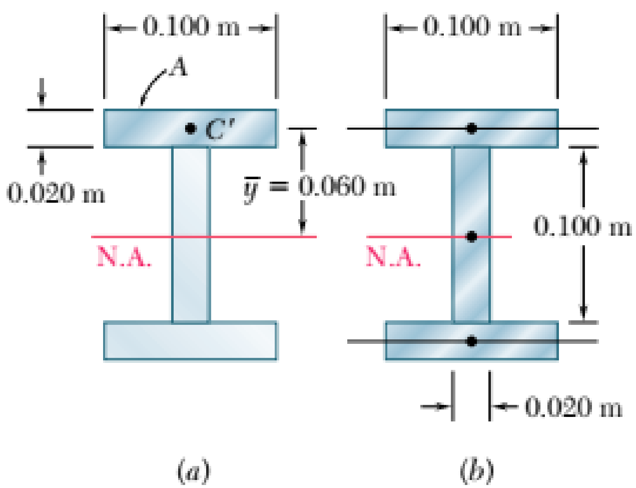


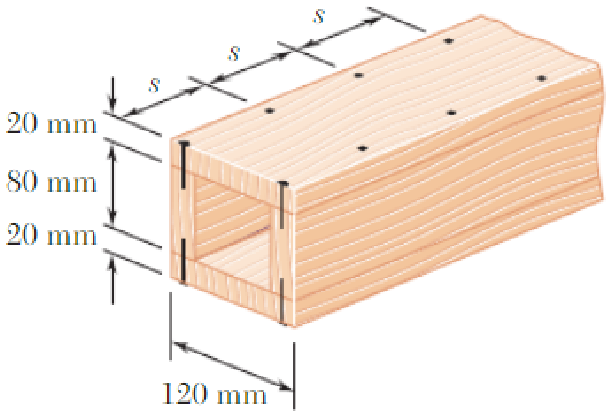
Fig. 6.9

Substituting into Eq. (6.5), we write

$$q = \frac{VQ}{I} = \frac{(500\text{ N})(120 \times 10^{-6}\text{ m}^3)}{16.20 \times 10^{-6}\text{ m}^4} = 3704\text{ N/m}$$

Since the spacing between the nails is 25 mm, the shearing force in each nail is

$$F = (0.025\text{ m})q = (0.025\text{ m})(3704\text{ N/m}) = 92.6\text{ N}$$



A square box beam is made of two 20×80-mm planks and two 20×120-mm planks nailed together as shown. Knowing that the spacing between the nails is $s = 30$ mm and that the vertical shear in the beam is $V = 1200$ N, determine (*a*) the shearing force in each nail, (*b*) the maximum shearing stress in the beam.

SOLUTION

$$I = \frac{1}{12} b_2 h_2^3 - \frac{1}{12} b_1 h_1^3$$

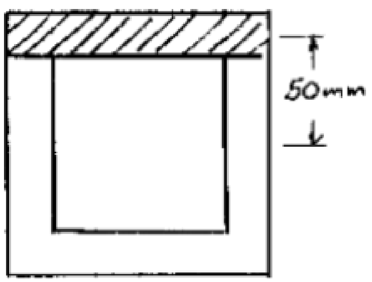
$$= \frac{1}{12}(120)(120)^3 - \frac{1}{12}(80)(80)^3 = 13.8667 \times 10^6 \text{ mm}^4$$

$$= 13.8667 \times 10^{-6} \text{m}^4$$

(*a*) $A_1 = (120)(20) = 2400 \text{ mm}^2$

$\bar{y}_1 = 50 \text{ mm}$

$Q_1 = A_1 \bar{y}_1 = 120 \times 10^3 \text{ mm}^3 = 120 \times 10^{-6} \text{ m}^3$



$$q = \frac{VQ}{I} = \frac{(1200)(120 \times 10^{-6})}{13.8667 \times 10^{-6}} = 10.3846 \times 10^3 \text{N/m}$$

$qs = 2F_{\text{nail}}$

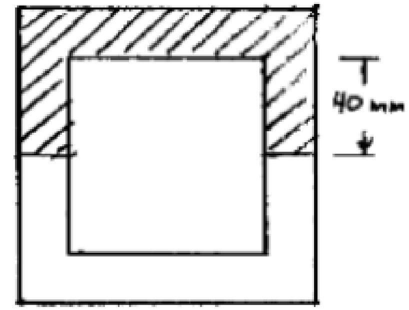
$$F_{\text{nail}} = \frac{qs}{2} = \frac{(10.3846 \times 10^3)(30 \times 10^{-3})}{2}$$

$F_{\text{nail}} = 155.8 \text{ N}$ ◀

(*b*) $Q = Q_1 + (2)(20)(40)(20)$

$= 120\times10^3 + 32\times10^3 = 152\times10^3\ \text{mm}^3$

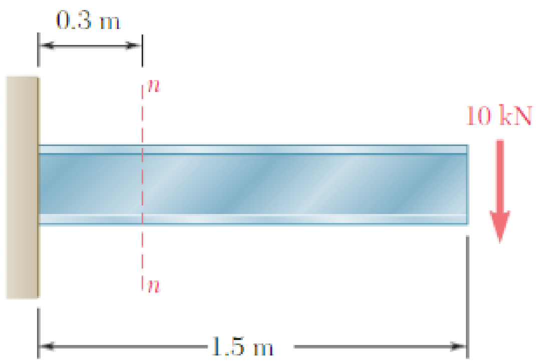
$= 152\times10^{-6}\ \text{m}^3$

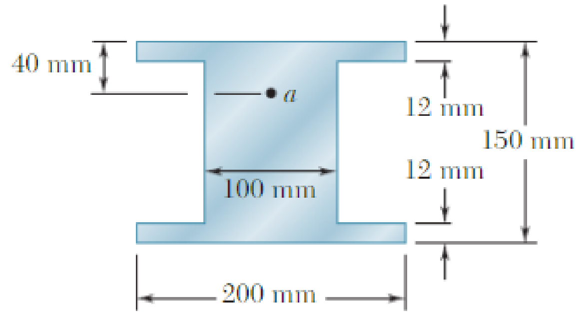


$$\tau_{\max} = \frac{VQ}{It} = \frac{(1200)(152\times10^{-6})}{(13.8667\times10^{-6})(2\times20\times10^{-3})}$$

$= 329\times10^3\ \text{Pa}$

$\tau_{\max} = 329\ \text{kPa}$ ◀

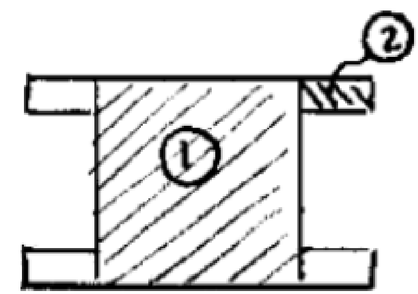




For the beam and loading shown, consider section *n-n* and determine (*a*) the largest shearing stress in that section, (*b*) the shearing stress at point *a*.

SOLUTION

At section *n-n*, $V = 10$ kN.



$$I = I_1 + 4I_2$$

$$= \frac{1}{12}b_1h_1^3 + 4\left(\frac{1}{12}b_2h_2^3 + A_2d_2^2\right)$$

$$= \frac{1}{12}(100)(150)^3 + 4\left[\left(\frac{1}{12}\right)(50)(12)^3 + (50)(12)(69)^2\right]$$

$$= 28.125\times10^6 + 4\left[0.0072\times10^6 + 2.8566\times10^6\right]$$

$$= 39.58\times10^6\ \text{mm}^4 = 39.58\times10^{-6}\ \text{m}^4$$

(*a*) $Q = A_1\bar{y}_1 + 2A_2\bar{y}_2$

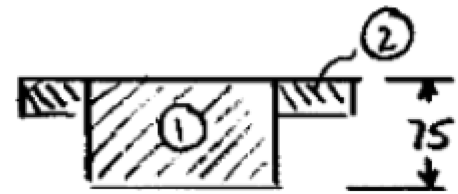
$= (100)(75)(37.5) + (2)(50)(12)(69)$

$= 364.05 \times 10^3\ \text{mm}^3 = 364.05 \times 10^{-6}\ \text{m}^3$

$t = 100\ \text{mm} = 0.100\ \text{m}$

$$\tau_{\max} = \frac{VQ}{It} = \frac{(10\times 10^3)(364.05\times 10^{-6})}{(39.58\times 10^{-6})(0.100)} = 920 \times 10^3\ \text{Pa}$$

$\tau_{\max} = 920\ \text{kPa}$ ◀



(*b*) $Q = A_1\bar{y}_1 + 2A_2\bar{y}_2$

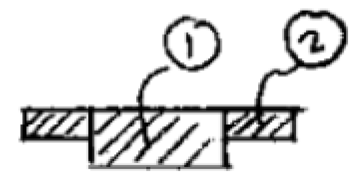
$= (100)(40)(55) + (2)(50)(12)(69)$

$= 302.8 \times 10^3\ \text{mm}^3 = 302.8 \times 10^{-6}\ \text{m}^3$

$t = 100\ \text{mm} = 0.100\ \text{m}$

$$\tau_a = \frac{VQ}{It} = \frac{(10\times 10^3)(302.8\times 10^{-6})}{(39.58\times 10^{-6})(0.100)} = 765 \times 10^3\ \text{Pa}$$

$\tau_a = 765\ \text{kPa}$ ◀



Thank You