Module 7 - Columns, Thin and thick cylinders

7	**Columns:** Theory of columns – Long column and short column - Euler's formula – Rankine's formula. **Thin and thick cylinders:** Thin cylinders and shells – deformation of thin cylinders and shells – thick cylindrical shell – Lame's equation.	4

Columns

- Columns are compression members.
- There are various examples of members subjected to compressive loads.

Various names of compression members as per application:

- Post is a general term applied to a compression member.
- Strut is a compression member whose lateral dimensions are small compared to it's length.
- A strut may be horizontal, inclined or vertical and this term is used in trusses. (Tie is a tension member in a truss)
- But a vertical strut, used in buildings or frames is called column.
- Columns, pillars and stanchions are vertical members used in building frames.

Classification of Columns

1.Short column: Short column fails by crushing (compressive yielding) of the material.

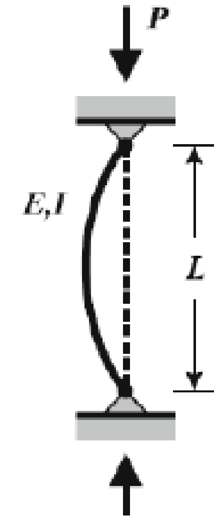
2. Long column: Long column fails by buckling or bending. (geometric or configuration failure)

3. Intermediate column: Intermediate column fails by combined buckling and crushing. (failure due to both material crushing and geometrical instability)

The column behind the steering wheel is designed to fail: it is mean to buckle during a car crash to prevent impaling the driver.

Buckling

- When a slender member is subjected to an axial compressive load, it may fail by a condition called ***buckling***.



- - -:original shape
___: Buckled shape

- Buckling is a geometric instability in which the lateral displacement of the axial member can suddenly become very large .

Buckled R.C.C. Columns

Buckled steel columns

16.3 Types of End Conditions of Columns

In actual practice, there are a number of end conditions for columns. But we shall study the Euler's column theory on the following four types of end conditions which are important from the subject point of view:

1. Both the ends hinged or pin jointed as shown in Fig. 16.1 (*a*),
2. Both the ends fixed as shown in Fig. 16.1 (*b*),
3. One end is fixed and the other hinged as shown in Fig. 16.1 (*c*), and
4. One end is fixed and the other free as shown in Fig. 16.1 (*d*).

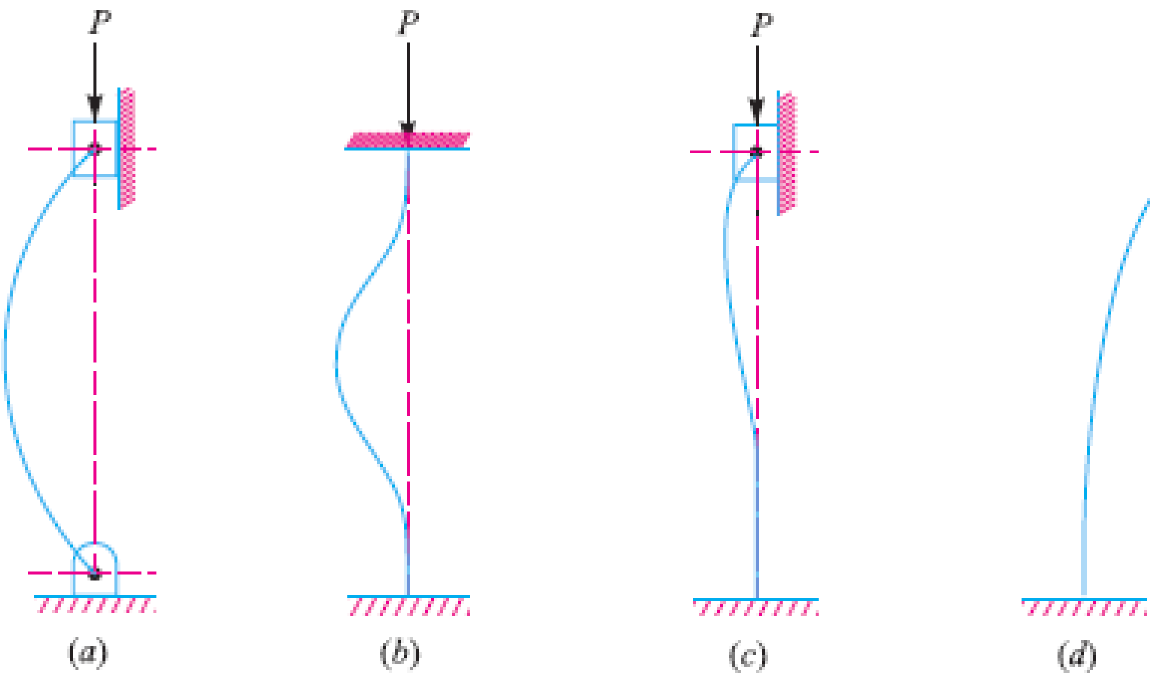
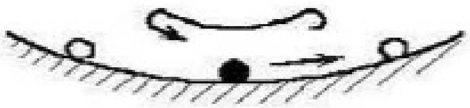


Fig. 16.1. Types of end conditions of columns.

Equilibrium States

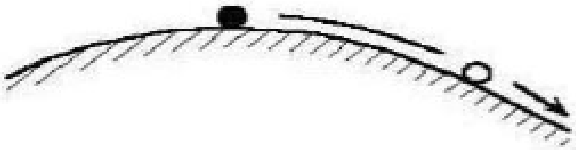
Three types of equilibrium:

1. Stable equilibrium



(a) Stable

2. Unstable equilibrium



(b) Unstable

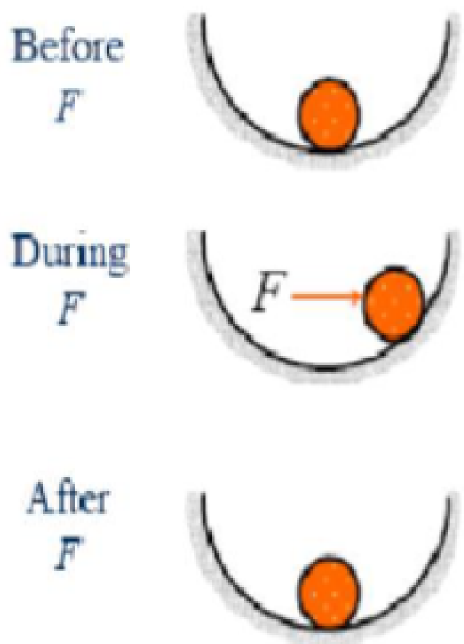
3. Neutral equilibrium

(c) Neutral

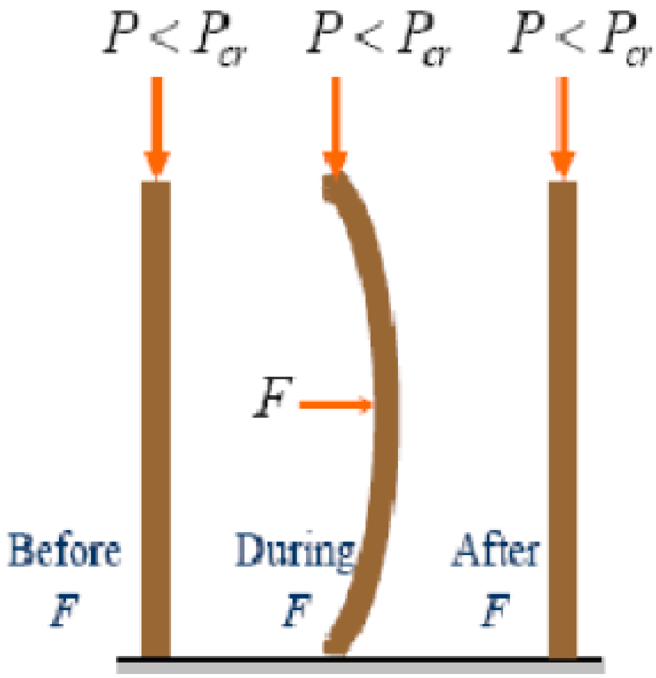
Buckling Mechanism

1. Stable equilibrium:

If the load P is sufficiently small, when the force F is removed, the column will go back to its original straight condition .



Gravity is the restoring force

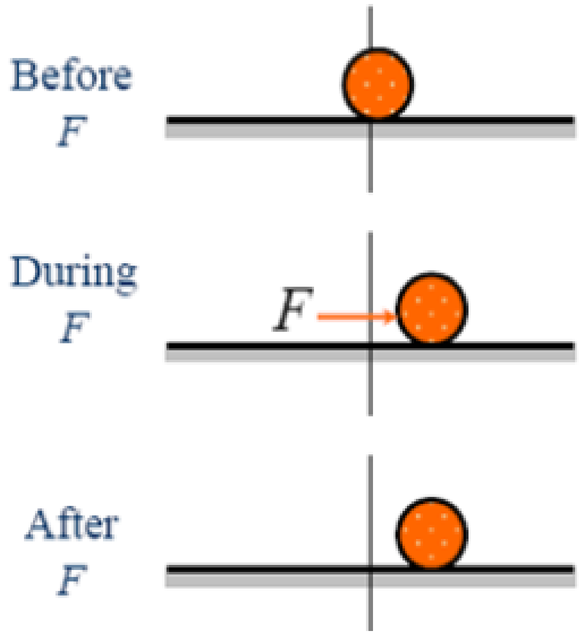


Elasticity of the column is the restoring force.

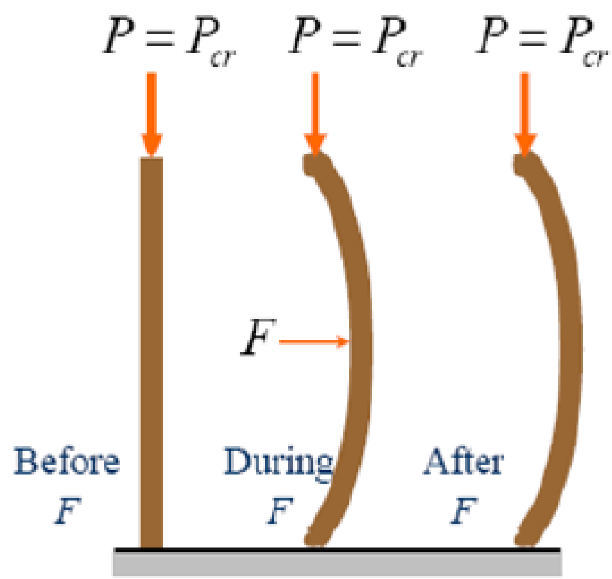
Buckling Mechanism (contd…)

2. Neutral equilibrium:

When the column carries critical load P_{cr} (Increased value of the load P) and a lateral force F is applied and removed, the column will remain in the slightly deflected position.



Deflection amount depends on magnitude of force.

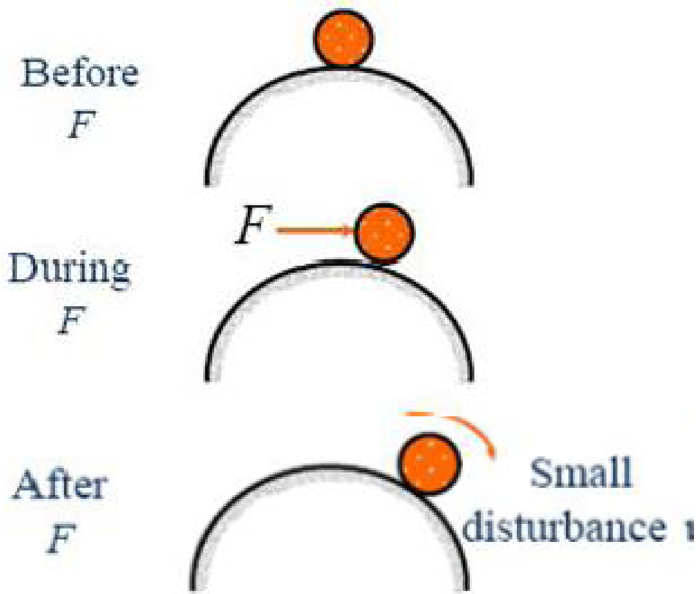


Elastic restoring force is sufficient to prevent excessive deflection.

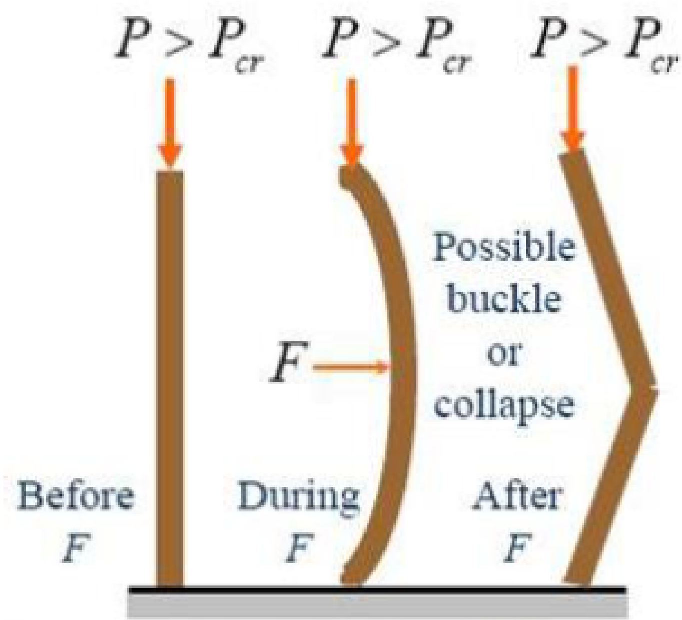
Buckling Mechanism (contd…)

3. Unstable equilibrium:

When the column carries a load which is more than critical load P_{cr} (Increased value of the load P) and a lateral force F is applied and removed, the column will bend considerably and it grows into excessively large deflection.



Even small disturbance causes unstable.



Elastic restoring force is insufficient to prevent excessive deflection.

Buckling Mechanism (contd…)

- Conclusion:

Depending on the magnitude of force P, either column remains in straight position or in slight bent position or collapse due to crack extension.

Euler's long column theory

- The direct stress f_0 due to direct load is very small compared to bending stress f_b due to buckling in long column.

- Euler derived an equation, for the buckling load of long column based on bending stress (neglecting the effect of direct stress).

- Buckling load cannot be used in short column.

Assumptions in Euler's Column Theory

The following simplifying assumptions are made in Euler's column theory :

1. Initially the column is perfectly straight, and the load applied is truly axial.
2. The cross-section of the column is uniform throughout its length.
3. The column material is perfectly elastic, homogeneous and isotropic, and thus obeys Hooke's law.
4. The length of column is very large as compared to its cross-sectional dimensions.
5. The shortening of column, due to direct compression (being very small) is neglected.
6. The failure of column occurs due to buckling alone.
7. The weight of the column itself is neglected.

16.6 Euler's Formula

According to Euler's theory, the crippling or buckling load (W_{cr}) under various end conditions is represented by a general equation,

$$W_{cr} = \frac{C\,\pi^2 E\,I}{l^2} = \frac{C\,\pi^2 E\,A\,k^2}{l^2} \qquad \ldots (\because I = A.k^2)$$

$$= \frac{C\,\pi^2 E\,A}{(l/k)^2}$$

where E = Modulus of elasticity or Young's modulus for the material of the column,

A = Area of cross-section,

k = Least radius of gyration of the cross-section,

l = Length of the column, and

C = Constant, representing the end conditions of the column or end fixity coefficient.

The following table shows the values of end fixity coefficient (C) for various end conditions.

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Table 16.1. Values of end fixity coefficient (C).

S. No.	*End conditions*	*End fixity coefficient (C)*
1.	Both ends hinged	1
2.	Both ends fixed	4
3.	One end fixed and other hinged	2
4.	One end fixed and other end free	0.25

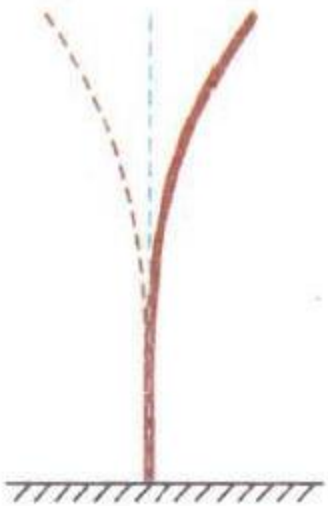
Notes : 1. The vertical column will have two moment of inertias (*viz.* I_{xx} and I_{yy}). Since the column will tend to buckle in the direction of least moment of inertia, therefore the least value of the two moment of inertias is to be used in the relation.

2. In the above formula for crippling load, we have not taken into account the direct stresses induced in the material due to the load which increases gradually from zero to the crippling value. As a matter of fact, the combined stresses (due to the direct load and slight bending), reaches its allowable value at a load lower than that required for buckling and therefore this will be the limiting value of the safe load.

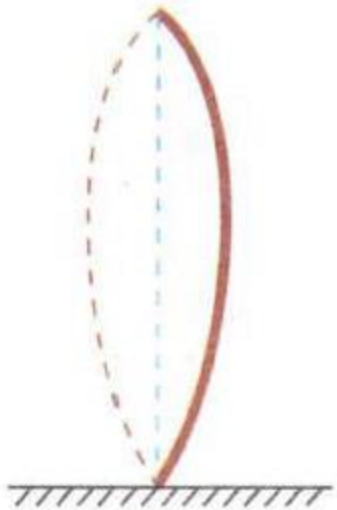
Cases of long columns based on end conditions

- 1. Both end pinned
- 2. Both ends fixed
- 3. One end fixed and the other end pinned
- 4. One end fixed and the other end free

Sign conventions for bending moments



(a) Positive



(b) Negative

Convexity towards centre line, M_x is +ve

Concavity towards centre line, M_x is -ve

End conditions of column

- Three important end conditions based on support types.

- (i) Pinned end: End is fixed in position only.
  $\therefore$ Deflection, $y = 0$.

- (ii) Fixed end: End is fixed in position and direction.
  $\therefore$ Deflection $y = 0$ and slope, $\frac{dy}{dx} = 0$.

- (iii) Free end: Neither fixed in position nor in direction.

16.7 Slenderness Ratio

In Euler's formula, the ratio l / k is known as *slenderness ratio*. It may be defined as the ratio of the effective length of the column to the least radius of gyration of the section.

It may be noted that the formula for crippling load, in the previous article is based on the assumption that the slenderness ratio l / k is so large, that the failure of the column occurs only due to bending, the effect of direct stress (*i.e.* W / A) being negligible.

This equipment is used to determine the crippling load for axially loaded long struts.

16.8 Limitations of Euler's Formula

We have discussed in Art. 16.6 that the general equation for the crippling load is

$$W_{cr} = \frac{C\,\pi^2 E\,A}{(l/k)^2}$$

$\therefore$ Crippling stress,

$$\sigma_{cr} = \frac{W_{cr}}{A} = \frac{C\,\pi^2 E}{(l/k)^2}$$

A little consideration will show that the crippling stress will be high, when the slenderness ratio is small. We know that the crippling stress for a column cannot be more than the crushing stress of the column material. It is thus obvious that the Euler's fromula will give the value of crippling stress of the column (equal to the crushing stress of the column material) corresponding to the slenderness ratio. Now consider a mild steel column. We know that the crushing stress for mild steel is 330 N/mm^2 and Young's modulus for mild steel is 0.21×10^6 N/mm^2.

Now equating the crippling stress to the crushing stress, we have

$$\frac{C\,\pi^2 E}{(l/k)^2} = 330$$

$$\frac{1 \times 9.87 \times 0.21 \times 10^6}{(l/k)^2} = 330 \qquad \text{... (Taking } C = 1)$$

or $$(l/k)^2 = 6281$$

$\therefore$ $$l/k = 79.25 \text{ say } 80$$

Hence if the slenderness ratio is less than 80, Euler's formula for a mild steel column is not valid.

Sometimes, the columns whose slenderness ratio is more than 80, are known as *long columns*, and those whose slenderness ratio is less than 80 are known as *short columns*. It is thus obvious that the Euler's formula holds good only for long columns.

16.9 Equivalent Length of a Column

Sometimes, the crippling load according to Euler's formula may be written as

$$W_{cr} = \frac{\pi^2 E I}{L^2}$$

where L is the equivalent length or effective length of the column. The equivalent length of a given column with given end conditions is the length of an equivalent column of the same material and cross-section with hinged ends to that of the given column. The relation between the equivalent length and actual length for the given end conditions is shown in the following table.

Table 16.2. Relation between equivalent length (*L*) and actual length (*l*).

S.No.	*End Conditions*	*Relation between equivalent length (L) and actual length (l)*
1.	Both ends hinged	$L = l$
2.	Both ends fixed	$L = \frac{l}{2}$
3.	One end fixed and other end hinged	$L = \frac{l}{\sqrt{2}}$
4.	One end fixed and other end free	$L = 2l$

Equivalent Length of a column

We can write the general equation for Euler's critical load as

$$P = \frac{\pi^2 EI}{{l_e}^2}$$

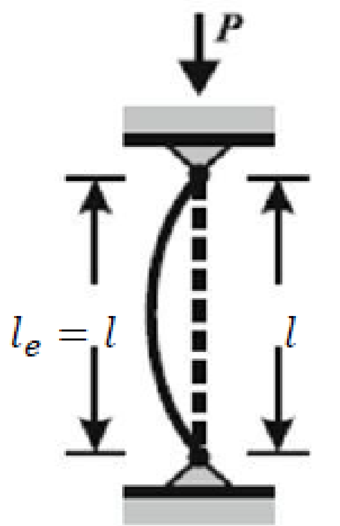
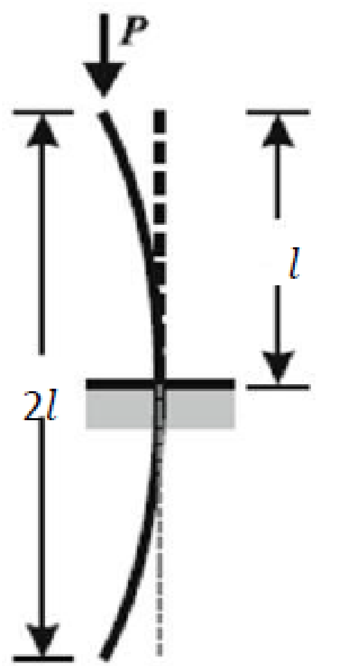
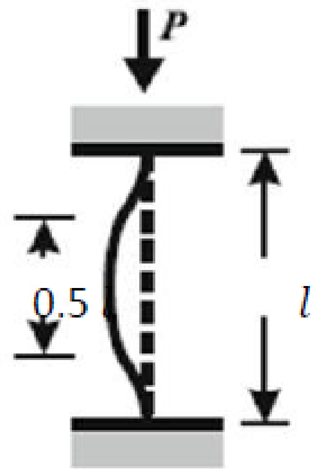
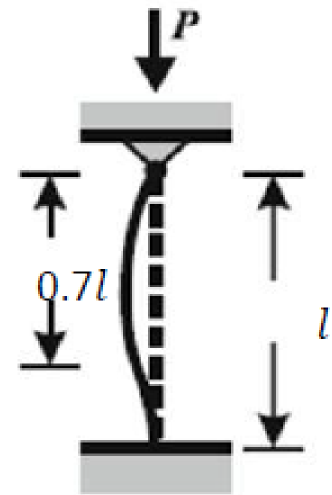
Where, l_e is the equivalent length of a column, the distance between points on the column where the moment is zero, corresponding to the end conditions of the standard pinned–pinned column.

Effective Length

- Physically, the *effective length* is the distance between points on the buckled column where the moment goes to zero, i.e., where the column is effectively pinned. Considering the *deflected shape*, the moment is zero where the curvature is zero (from beam theory).

- Zero curvature corresponds to an inflection point in the deflected shape (where the curvature changes sign).

Effective length for columns with common end conditions

Pinned–pinned	Fixed–free	Fixed–fixed	Fixed–pinned
			

S.No	End Conditions	Relation between equivalent length(L_e) and actual length(l)	Crippling Load(P)
1.	Both ends hinged	$\boldsymbol{l_e = l}$	$P = \frac{\pi^2 EI}{(l)^2} = \frac{\pi^2 EI}{l^2}$
2.	One end fixed and other free	$\boldsymbol{l_e = 2l}$	$P = \frac{\pi^2 EI}{(2l)^2} = \frac{\pi^2 EI}{4l^2}$

S.No	End Conditions	Relation between equivalent length(L_e) and actual length(l)	Crippling Load(P)
3.	Both ends fixed	$\boldsymbol{l_e = \frac{l}{2}}$	$P = \frac{\pi^2 EI}{(\frac{l}{2})^2} = \frac{4\pi^2 EI}{l^2}$
4.	One end fixed and the other hinged	$\boldsymbol{l_e = \frac{l}{\sqrt{2}}}$	$P = \frac{\pi^2 EI}{(\frac{l}{\sqrt{2}})^2} = \frac{2\pi^2 EI}{l^2}$

Example 16.1. *A T-section 150 mm × 120 mm × 20 mm is used as a strut of 4 m long hinged at both ends. Calculate the crippling load, if Young's modulus for the material of the section is 200 kN/mm².*

Solution. Given : $l = 4$ m $= 4000$ mm ; $E = 200$ kN/mm² $= 200 \times 10^3$ N/mm²

First of all, let us find the centre of gravity (G) of the T-section as shown in Fig. 16.2.

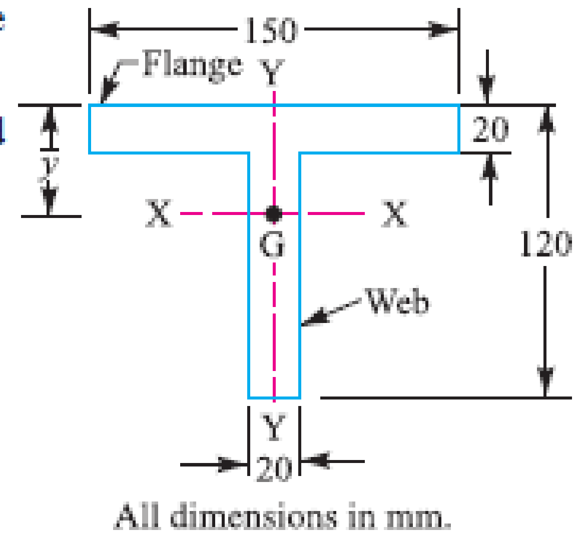


Fig. 16.2

Let $\bar{y}$ be the distance between the centre of gravity (G) and top of the flange,

We know that the area of flange,

$$a_1 = 150 \times 20 = 3000 \text{ mm}^2$$

Its distance of centre of gravity from top of the flange,

$$y_1 = 20 / 2 = 10 \text{ mm}$$

Area of web, $\quad a_2 = (120 - 20)\, 20 = 2000 \text{ mm}^2$

Its distance of centre of gravity from top of the flange,

$$y_2 = 20 + 100 / 2 = 70 \text{ mm}$$

$$\therefore \quad \bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{3000 \times 10 + 2000 \times 70}{3000 + 2000} = 34 \text{ mm}$$

We know that the moment of inertia of the section about X-X,

$$I_{XX} = \left[\frac{150\,(20)^3}{12} + 3000\,(34-10)^2 + \frac{20\,(100)^3}{12} + 2000\,(70-34)^2\right]$$

$$= 6.1 \times 10^6 \text{ mm}^4$$

and
$$I_{YY} = \frac{20\,(150)^3}{12} + \frac{100\,(20)^3}{12} = 5.7 \times 10^6 \text{ mm}^4$$

Since I_{YY} is less than I_{XX}, therefore the column will tend to buckle in Y-Y direction. Thus we shall take the value of I as $I_{YY} = 5.7 \times 10^6$ mm^4.

Moreover as the column is hinged at its both ends, therefore equivalent length,

$$L = l = 4000 \text{ mm}$$

We know that the crippling load,

$$W_{cr} = \frac{\pi^2 E\, I}{L^2} = \frac{9.87 \times 200 \times 10^3 \times 5.7 \times 10^6}{(4000)^2} = 703 \times 10^3 \text{ N} = 703 \text{ kN} \textbf{ Ans.}$$

1. A solid round bar 60mm in diameter and 2.5m long is used as a strut. One end of the strut is fixed, while its other end is hinged. Find the safe compressive load, for this strut, using Euler's formula. Assume $E=200GN/m^2$ and factor of safety =3.

Solution: end condition: one end hinged, other end fixed

effective length $Le = L/(\sqrt{2}) = 2.5/(\sqrt{2}) = 1.768m$

Euler's crippling load $= \mathbf{P_E} = (\pi^2 EI) / Le^2$

$= [\pi^2 \times 200 \times 10^9 \times \pi \times (0.06)^4 / 64] \quad / (1.768^2)$

$= 401.7 \times 10^3$ N $= 401.4$ kN

Safe compressive load $= \mathbf{P_E /3} \quad \mathbf{= 133.9kN}$

A slender pin ended aluminium column 1.8m long and of circular cross-section is to have an outside diameter of 50mm. Calculate the necessary internal diameter to prevent failure by buckling if the actual load applied is 13.6kN and the critical load applied is twice the actual load. Take $E_a = 70GN/m^2$.

Solution:

outside diameter of the column =D =50mm =0.05m;

$E = 70 \times 10^9 N/m^2$

Inside diameter = ?

16.10 Rankine's Formula for Columns

We have already discussed that Euler's formula gives correct results only for very long columns. Though this formula is applicable for columns, ranging from very long to short ones, yet it does not give reliable results. Prof. Rankine, after a number of experiments, gave the following empirical formula for columns.

$$\frac{1}{W_{cr}} = \frac{1}{W_C} + \frac{1}{W_E} \quad \text{...}(i)$$

where W_{cr} = Crippling load by Rankine's formula,

W_C = Ultimate crushing load for the column = $\sigma_c \times A$,

W_E = Crippling load, obtained by Euler's formula = $\dfrac{\pi^2 E I}{L^2}$

A little consideration will show, that the value of W_C will remain constant irrespective of the fact whether the column is a long one or short one. Moreover, in the case of short columns, the value of W_E will be very high, therefore the value of $1 / W_E$ will be quite negligible as compared to $1 / W_C$. It is thus obvious, that the Rankine's formula will give the value of its crippling load (*i.e.* W_{cr}) approximately equal to the ultimate crushing load (*i.e.* W_C). In case of long columns, the value of W_E will be very small, therefore the value of $1 / W_E$ will be quite considerable as compared to $1 / W_C$. It is thus obvious, that the Rankine's formula will give the value of its crippling load (*i.e.* W_{cr}) approximately equal to the crippling load by Euler's formula (*i.e.* W_E). Thus, we see that Rankine's formula gives a fairly correct result for all cases of columns, ranging from short to long columns.

From equation (*i*), we know that

$$\frac{1}{W_{cr}} = \frac{1}{W_C} + \frac{1}{W_E} = \frac{W_E + W_C}{W_C \times W_E}$$

$$\therefore \qquad W_{cr} = \frac{W_C \times W_E}{W_C + W_E} = \frac{W_C}{1 + \dfrac{W_C}{W_E}}$$

Now substituting the value of W_C and W_E in the above equation, we have

$$W_{cr} = \frac{\sigma_c \times A}{1 + \dfrac{\sigma_c \times A \times L^2}{\pi^2 E\, I}} = \frac{\sigma_c \times A}{1 + \dfrac{\sigma_c}{\pi^2 E} \times \dfrac{A.L^2}{A.k^2}} \qquad \ldots (\because\ I = A.k^2)$$

$$= \frac{\sigma_c \times A}{1 + a\left(\dfrac{L}{k}\right)^2} = \frac{\text{Crushing load}}{1 + a\left(\dfrac{L}{k}\right)^2}$$

where σ_c = Crushing stress or yield stress in compression,

A = Cross-sectional area of the column,

a = Rankine's constant $= \dfrac{\sigma_c}{\pi^2 E}$,

L = Equivalent length of the column, and

k = Least radius of gyration.

Table 16.3. Values of crushing stress (σ_c) and Rankine's constant (a) for various materials.

S.No.	*Material*	σ_c *in MPa*	$a = \dfrac{\sigma_c}{\pi^2 E}$
1.	Wrought iron	250	$\dfrac{1}{9000}$
2.	Cast iron	550	$\dfrac{1}{1600}$
3.	Mild steel	320	$\dfrac{1}{7500}$
4.	Timber	50	$\dfrac{1}{750}$

Eccentric Load On Columns

When the applied load is not in line with the axis of the column, the various column formulas must be modified for the effect of bending moment M (= P × e), Fig. The additional stress due to the bending moment is

$$\sigma_b = \frac{M}{Z} = \frac{My}{I} = \frac{Pey}{Ak^2}$$

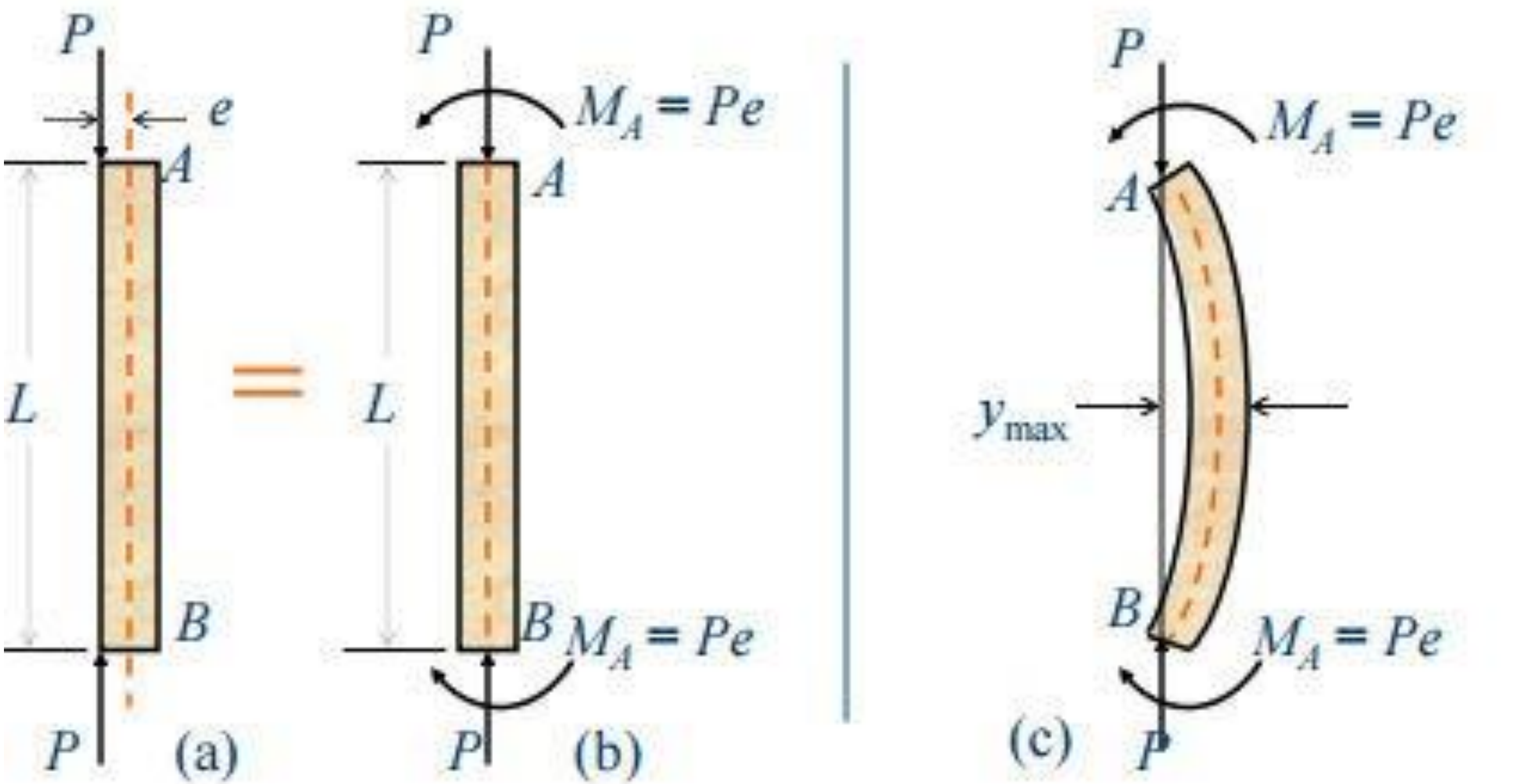
This stress must be added to the stress that would be induced if the load is along the axis of the column. Thus, the Rankine-Gordon formula is modified as

$$\frac{P_{cr}}{A} = \frac{\sigma_c}{1 + a\left(\frac{L_e}{k}\right)^2 + \frac{ey}{k^2}}$$

Another formula known as secant formula is also used for these columns, which is

$$\sigma = \frac{P}{A}\left(1 + \frac{ey}{k^2}\sec\frac{l}{2}\sqrt{\frac{P}{EI}}\right)$$

- The Secant Formula



Secant Formula

Consider a column of length L hinged at both ends and subjected to an eccentric load P as shown in Fig. 9.5. Consider a section at a distance x from the end A.

$$M_x = P \cdot y$$

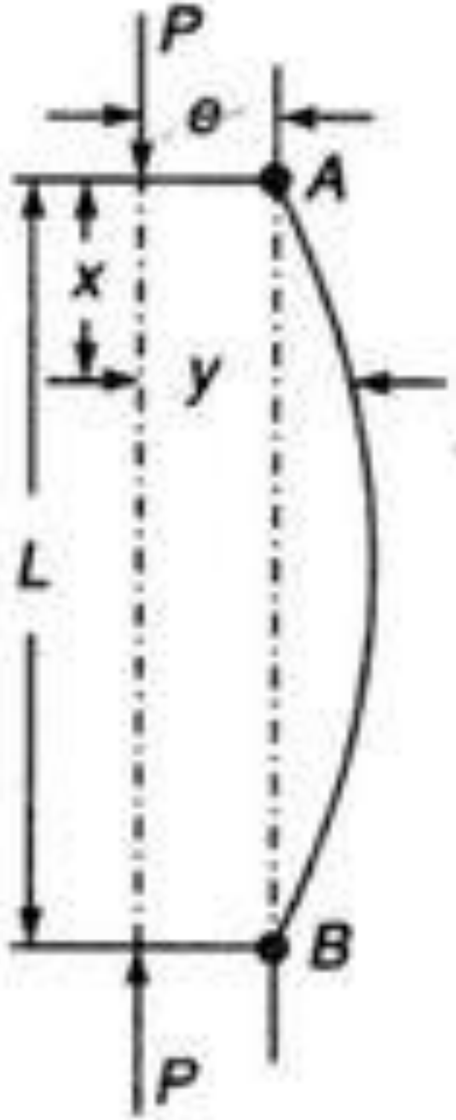
$$EI \frac{d^2y}{dx^2} = -M_x = -P \cdot y$$

$$EI \frac{d^2y}{dx^2} + P \cdot y = 0$$

$$\therefore \quad \frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = 0$$

$$\text{Let } \frac{P}{EI} = m^2$$

$$\therefore \quad \frac{d^2y}{dx^2} + m^2 y = 0$$



The general solution of Equation (i) is

$$y = C_1 \cos mx + C_2 \sin mx$$

At $\quad x = 0, y = e \quad \therefore \quad C_1 = e$

At $\quad x = L, y = e$

$\therefore \quad e = e \cos mL + C_2 \sin mL$

$$C_2 = \frac{e(1 - \cos mL)}{\sin mL} = e \tan \frac{mL}{2}$$

$\therefore$

$$y = e \cos mx + e \tan \frac{mL}{2} \sin mx$$

The expression shows that deflection is always there with an eccentric load. It is infinite when $\tan(ml/2) = \infty$ or $\alpha l = \pi$ or $\sqrt{\frac{P}{EI}}.l = \pi$ or $P = \frac{\pi^2 EI}{l^2}$ which is the same value as for crippling load . However, due to additional bending moment which is set up due to deflection , the strut always fails by compressive stress before the crippling load.

$$y_{\max(x=l/2)} = e\left(\tan\frac{ml}{2}\sin\frac{ml}{2} + \cos\frac{ml}{2}\right)$$

$$= e\left(\frac{\sin^2\frac{ml}{2} + \cos^2\frac{ml}{2}}{\cos\frac{ml}{2}}\right) = e.\frac{1}{\cos\frac{ml}{2}} = e\sec\frac{ml}{2}$$

Maximum bending moment, $M = P.y_{\max} = P.e\sec\frac{ml}{2}$

Maximum stress (compressive),

$$\sigma_{\max} = \frac{P}{A} + \frac{M.y_c}{I} = \frac{P}{A} + \frac{P.e.y_c\sec\frac{ml}{2}}{Ak^2} = \frac{P}{A}\left(1 + \frac{e.y_c}{k^2}.\sec\frac{ml}{2}\right)$$

This is known as *secant formula* for eccentric loads.

Example *An initially straight tube of 48-mm external diameter and 40-mm internal diameter in 2.4-m long and has hinged ends. It carries a compressive load of 25 kN parallel to the axis at an eccentricity of 2 mm. Determine the maximum and the minimum intensities of stresses in the tube. Also find the maximum permissible eccentricity so that no tension exists anywhere in the section. E =205 GPa.*

Solution

$$I = \frac{\pi}{64}\left(48^4 - 40^4\right) = 42\,944\,\pi \text{ mm}^4;\ A = \frac{\pi}{4}\left(48^2 - 40^2\right) = 176\pi \text{ mm}^2$$

$$k^2 = \frac{I}{A} = \frac{42\,944}{176} = 244 \text{ mm}^2$$

$$\sec\frac{\alpha l}{2} = \sec\frac{l}{2}\sqrt{\frac{P}{EI}} = \sec\frac{2400}{2}\sqrt{\frac{25\,000}{205\,000\times 42\,944\pi}} = \sec 1.141 = \sec 65.4^\circ = 2.4$$

$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{e.y_c}{k^2}.\sec\frac{\alpha l}{2}\right) = \frac{25\,000}{176\pi}\left(1 + \frac{2\times 24}{244}\times 2.4\right) \text{ MPa}$$

$$= 45.22(1 + 0.472) = 66.56 \text{ MPa} \qquad \text{(compressive)}$$

$$\sigma_{min} = \frac{P}{A}\left(1 + \frac{e.(-y_c)}{k^2}.\sec\frac{\alpha l}{2}\right)$$

$$= 45.22(1 - 0.472) = 23.88 \text{ MPa} \qquad \text{(compressive)}$$

For maximum permissible eccentricity so that no tension exists anywhere in the section

$$\sigma_{min} = \frac{25\,000}{176\pi}\left(1 + \frac{e\times(-24)}{244}\times 2.4\right) = 0 \text{ or } 1 - 0.236e = 0 \text{ or } e = 4.24 \text{ mm}$$

1.Calculate the safe compressive load on a hollow cast iron column one end fixed and other end hinged of 150mm external diameter,100mm internal diameter and 10m length. Use Euler's formula with a factor of safety of 5 and $E=95GN/m^2$

2.Bar of length 4m when used as a simply supported beam and subjected to a u.d.l of 30kN/m over the whole span., deflects 15mm at the centre. Determine the crippling loads when it is used as a column with the following end conditions:

(i) Both ends pin jointed (ii) one end fixed and other end hinged (iii) Both ends fixed

3.Determine the ratio of the buckling strengths of two columns of circular cross-section one hollow and other solid when both are made of the same material, have the same length, cross sectional area and end conditions. The internal diameter of the hollow column is half of its external diameter

4. Calculate the critical load of a strut 5m long which is made of a bar circular in section and pin jointed at both ends. The same bar when freely supported gives mid span deflection of 10mm with a load of 80N at the centre.

5. A hollow C.I column whose outside diameter is 200mm has a thickness of 20mm. It is 4.5m long and is fixed at both ends. Calculate the safe load by Rankine's formula
using a factor of safety of 4. Take $\sigma_c = 550 MN/m^2$, $a = 1/1600$

6. A hollow cylindrical cast iron column is 4m long with both ends fixed. Determine the minimum diameter of the column, if it has to carry a safe load of 250kN with a factor of safety of 5. Take the internal diameter as 0.8 times the external diameter.

$\sigma_C = 550 MN/m^2$ $a = 1/1600$

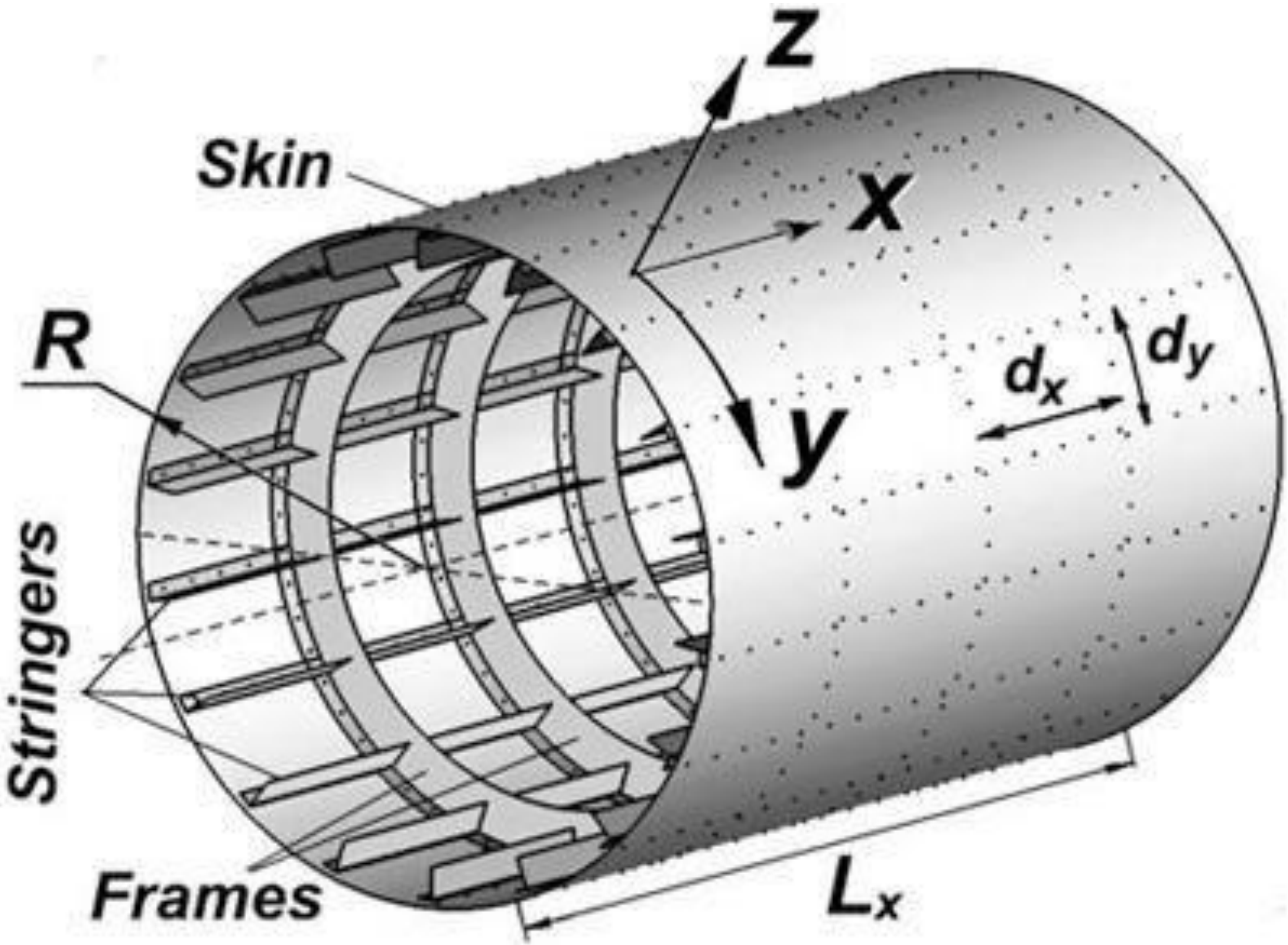
Thin cylinder and Spherical shells

Thin cylinder and Spherical shells

In many engineering applications, **cylinders** are frequently used for transporting or storing of liquids, gases or fluids.

Eg: Pipes, Boilers, storage tanks etc.

These cylinders are subjected to **fluid pressures**. When a cylinder is subjected to a internal pressure, at any point on the cylinder wall, **three types of stresses** are induced on three mutually perpendicular planes. The three principal Stresses in the Shell are Circumferential or Hoop Stress, Longitudinal Stress, and the Radial Stress

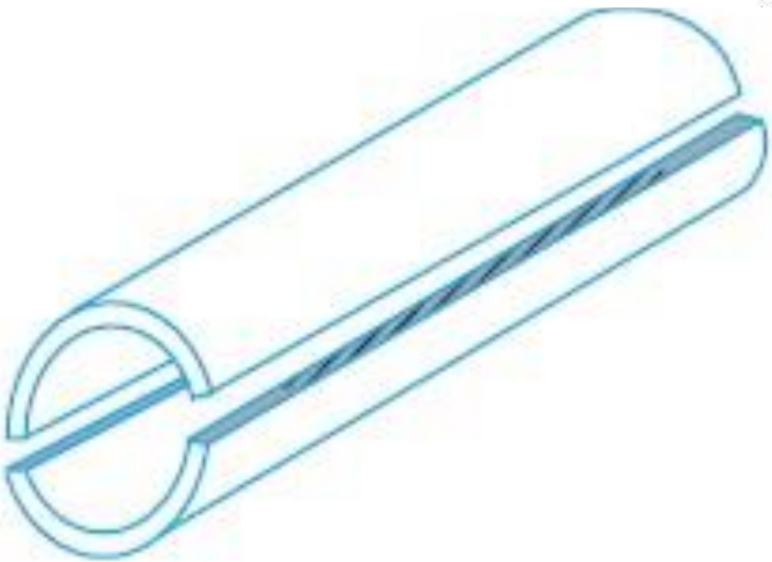


Closed orthogonally-stiffened cylindrical shell

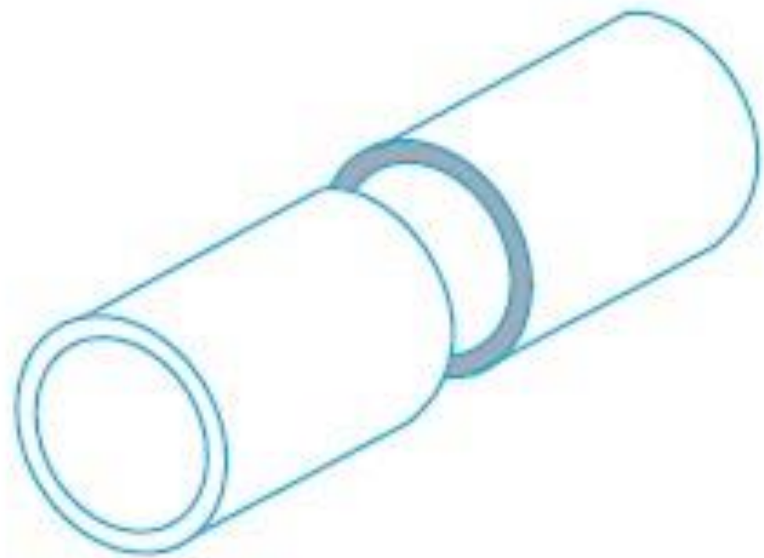
1. **Hoop or Circumferential Stress (σ_C)** – This is directed along the **tangent to the circumference** and tensile in nature. Thus, there will be increase in diameter.
2. **Longitudinal Stress (σ_L)** – This stress is directed along the **length of the cylinder**. This is also tensile in nature and tends to increase the length.

3. **Radial pressure (p_r)** – It is compressive in nature. Its magnitude is equal to fluid pressure on the inside wall and zero on the outer wall if it is open to atmosphere.

Pressure vessels.



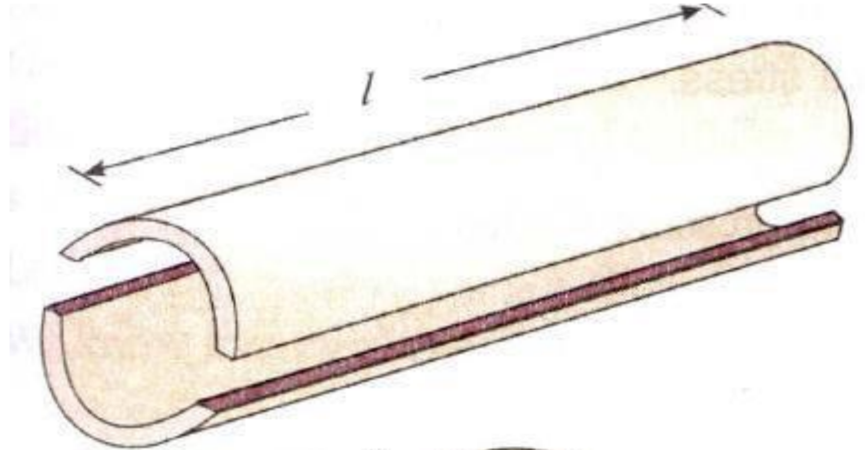
(*a*) Failure of a cylindrical shell along the longitudinal section.

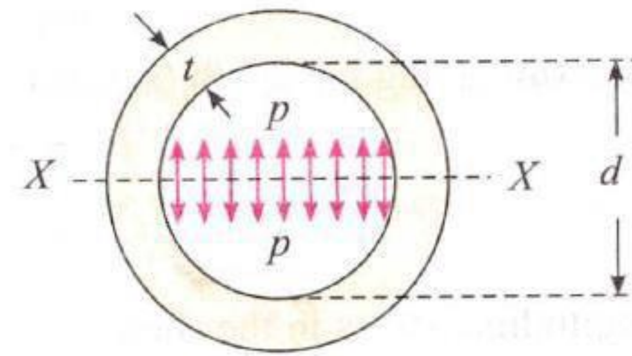


(*b*) Failure of a cylindrical shell along the transverse section.

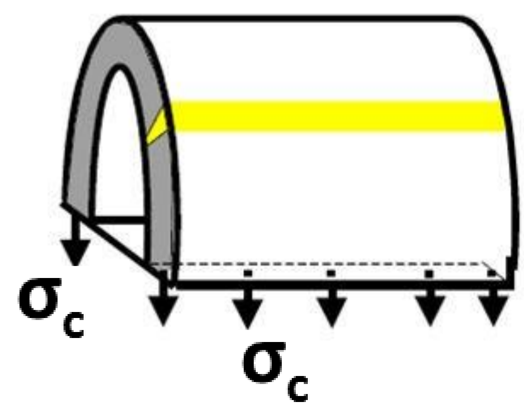
Circumferential Stress(Hoop stress)

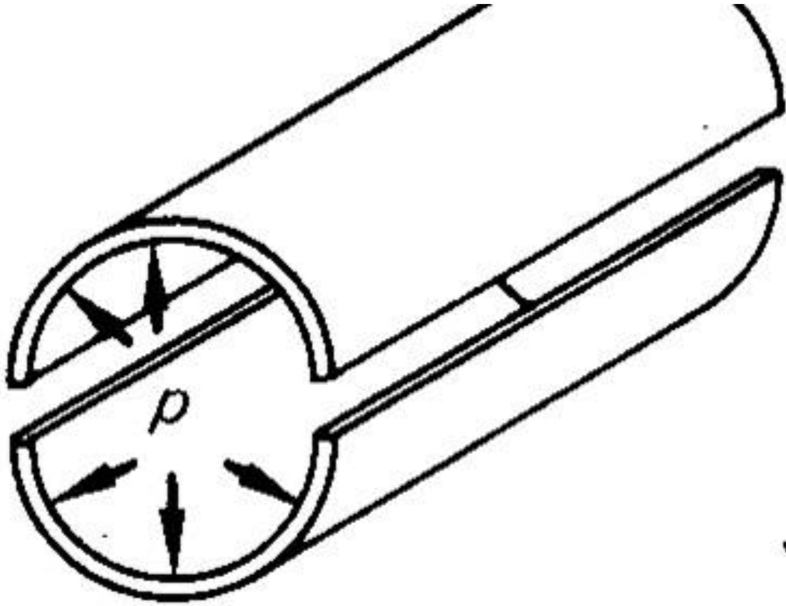
- Consider a thin cylindrical shell subjected to an internal pressure.
- As a result of the internal pressure, the cylinder has a tendency to split up into two troughs.

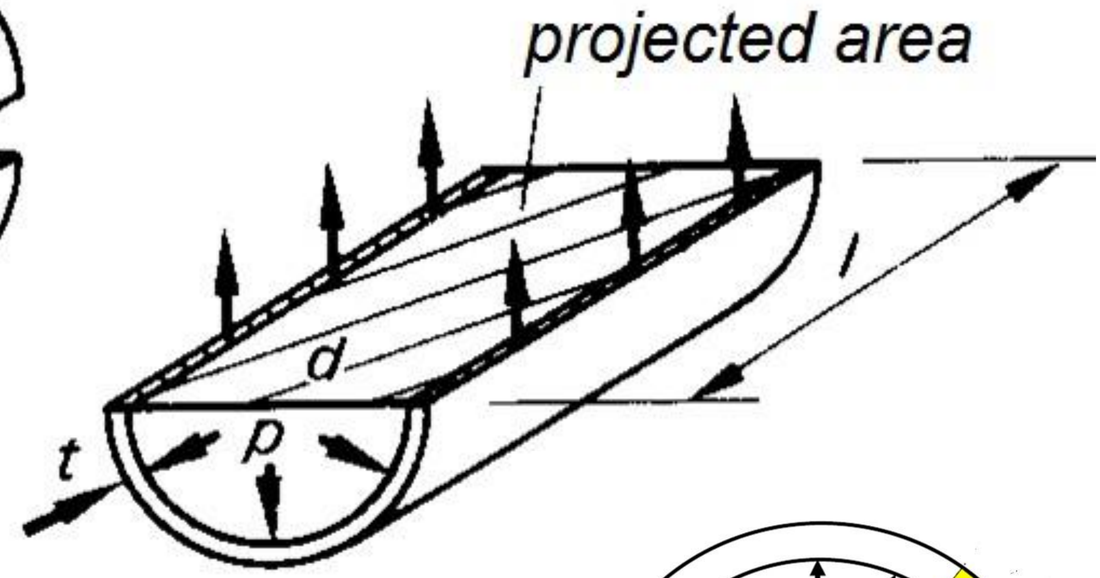




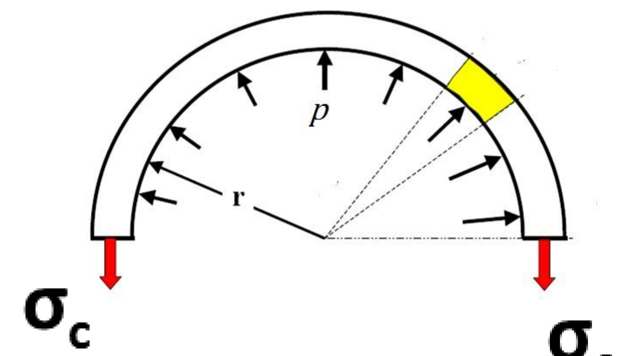
In case of thin shells, the stresses are assumed to be uniformly distributed across the wall thickness. In case of thick shells, the stresses are no longer uniformly distributed across the thickness and the problem becomes complex.







Failure along longitudinal section



Consider a thin cylindrical shell subjected to an internal Pressure as shown in fig.

σ_c = circumferential stress in the shell material.

p =internal pressure

d =internal diameter of shell

t =thickness of the shell.

The force exerted,

F = Internal Pressure x Projected area= p x (d x l)

The cross-sectional area resisting the force is A = 2tl

Therefore the circumferential stress is given by:

$$\sigma_c = \frac{Total\ pressure}{Resisting\ section} = \frac{pdl}{2tl} = \frac{pd}{2t}$$

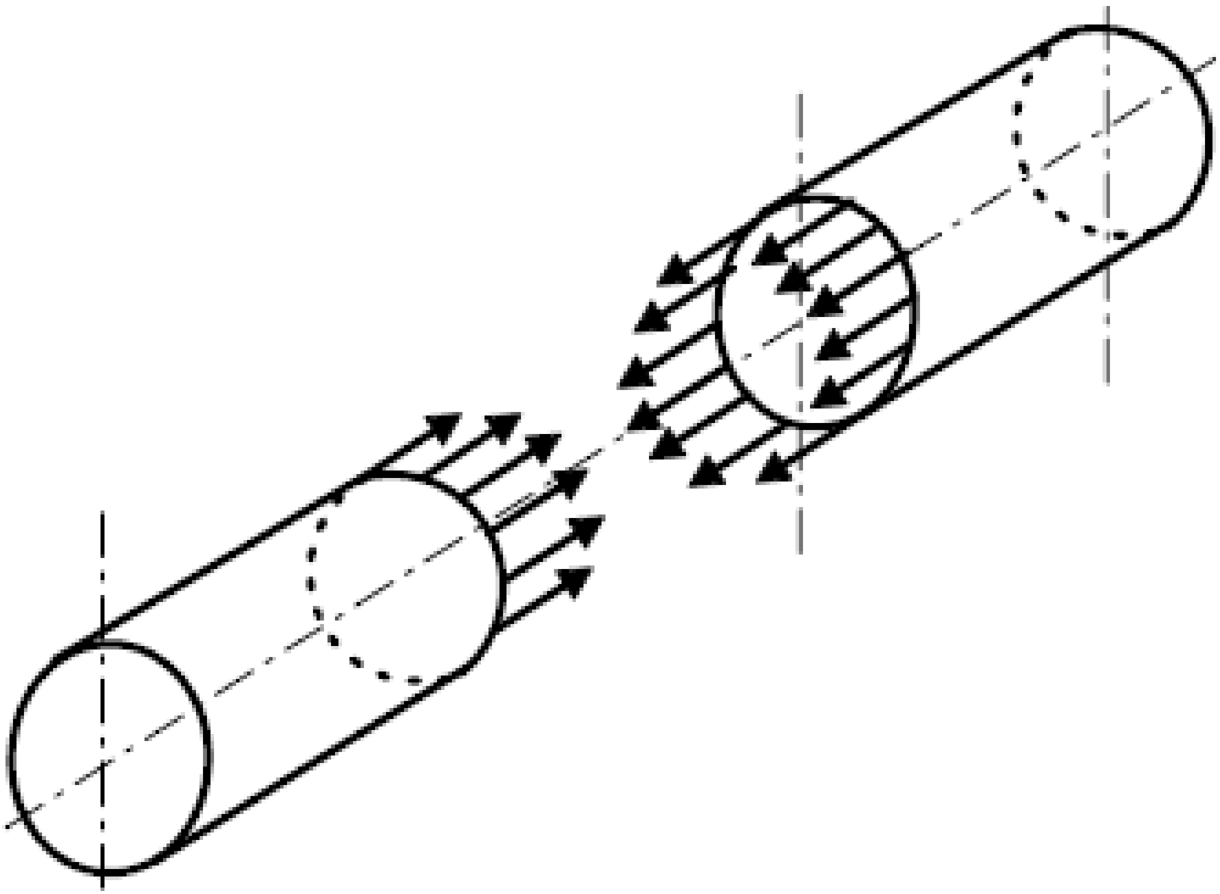
Note:

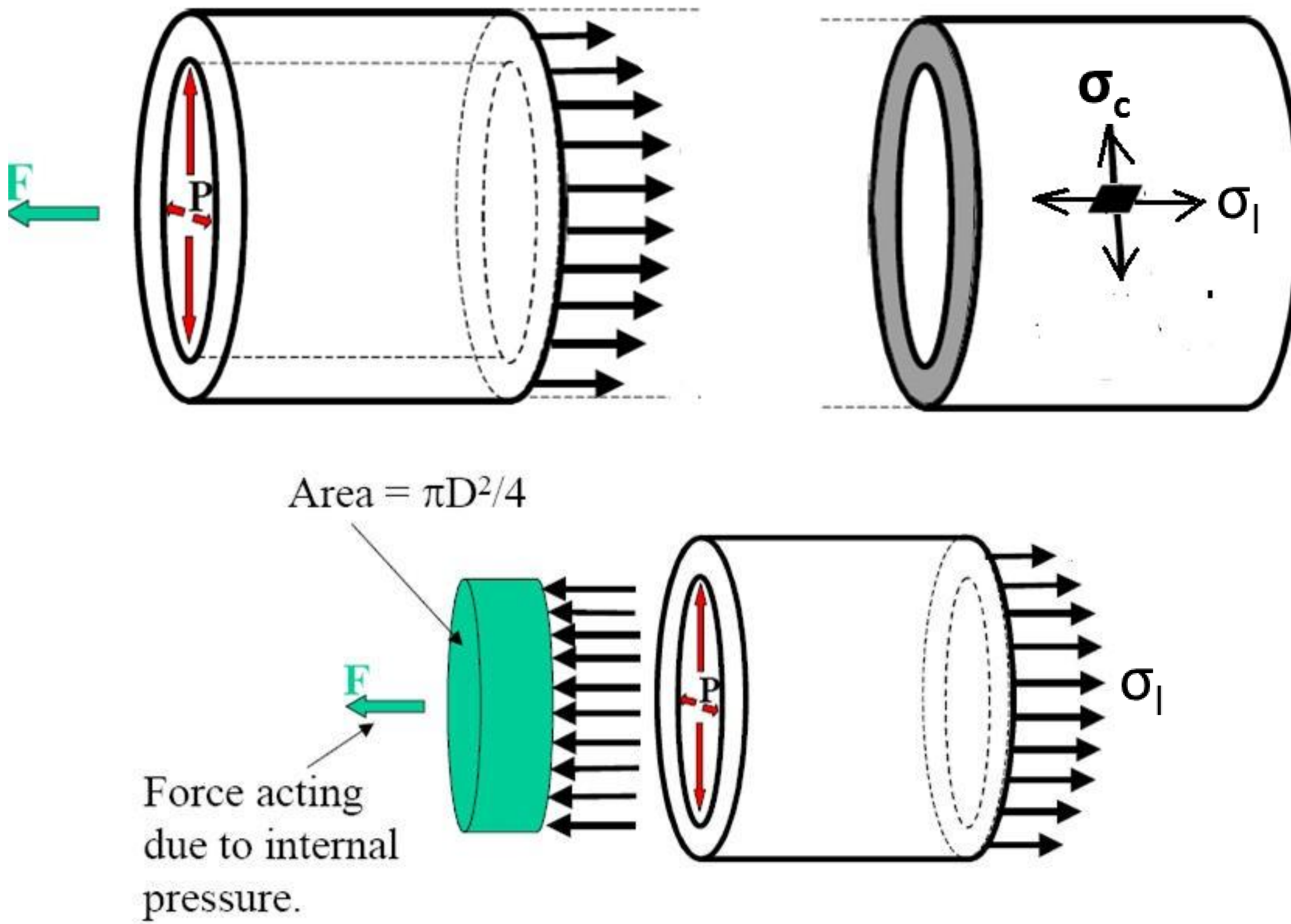
If η is the efficiency of the riveted joints of the shell, then stress,

$$\sigma_c = \frac{pd}{2t\eta}$$

Longitudinal Stress(axial stress)

- As a result of the internal pressure, the cylinder also has a tendency to split into two pieces as shown in figure.





F
P
σc
σl
Area = πD²/4
F
P
σl
Force acting
due to internal
pressure.

- The force tending to push the two parts apart,

$$F = Intensity\ of\ internal\ pressure \times Area$$

$$= p \times \frac{\pi}{4}(d)^2$$

- The cross-section area of material which sustains this force is given by:

$$A = \pi dt$$

Therefore the longitudinal stress is given by:

$$\sigma_l = \frac{Total\ pressure}{Resisting\ section} = \frac{p \times \frac{\pi}{4}(d)^2}{\pi dt} = \frac{pd}{4t}$$

Note: If η is the efficiency of the riveted joints of the shell, then the stress, $\sigma_t = \frac{pd}{4t\eta_c}$

- Since hoop stress is twice the longitudinal stress, the cylinder would fail by tearing along a line parallel to the axis, rather than on a section perpendicular to the axis.

- The equation for hoop stress is therefore used to determine the cylinder thickness.

- Allowance is made for this by dividing the thickness obtained in hoop stress equation by efficiency (i.e. tearing and shearing efficiency) of the joint.

Changes in dimension of Thin Cylindrical Shells

Now let

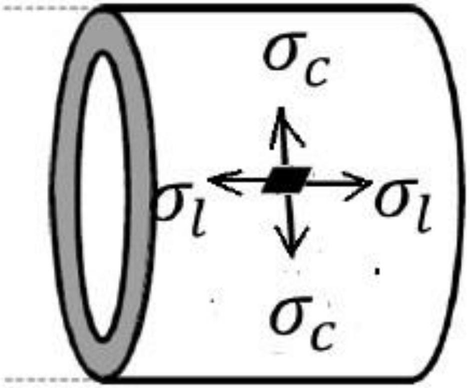
$$\delta d = change\ in\ diameter\ of\ the\ shell$$

$$\delta l\ = change\ in\ length\ of\ the\ shell\ and$$

$$\mu = Poisson's\ ratio$$

The state of stress on the shell wall is as shown in

Fig.



(a) Change in length

The change in length of the cylinder may be determined from the longitudinal strain, i.e. neglecting the radial stress.

$$\text{Longitudinal strain} = \frac{1}{E}[\sigma_L - \nu\sigma_H]$$

and

$$\text{change in length} = \text{longitudinal strain} \times \text{original length}$$

$$= \frac{1}{E}[\sigma_L - \nu\sigma_H]L$$

$$= \frac{pd}{4tE}[1 - 2\nu]L$$

(b) Change in diameter

As above, the change in diameter may be determined from the strain on a diameter, i.e. the *diametral* strain.

$$\text{Diametral strain} = \frac{\text{change in diameter}}{\text{original diameter}}$$

Now the change in diameter may be found from a consideration of the circumferential change. The stress acting around a circumference is the hoop or circumferential stress σ_H giving rise to the circumferential strain ε_H.

$$\text{Change in circumference} = \text{strain} \times \text{original circumference}$$

$$= \varepsilon_H \times \pi d$$

$$\text{New circumference} = \pi d + \pi d \varepsilon_H$$
$$= \pi d\,(1 + \varepsilon_H)$$

But this is the circumference of a circle of diameter $d\,(1 + \varepsilon_H)$

$$\therefore \quad \text{New diameter} = d\,(1 + \varepsilon_H)$$

$$\therefore \quad \text{Change in diameter} = d\varepsilon_H$$

$$\text{Diametral strain } \varepsilon_D = \frac{d\varepsilon_H}{d} = \varepsilon_H$$

i.e. **the diametral strain equals the hoop or circumferential strain**

Thus
$$\text{change in diameter} = d\varepsilon_H = \frac{d}{E}[\sigma_H - \nu\sigma_L]$$

$$= \frac{pd^2}{4tE}[2 - \nu]$$

(c) *Change in internal volume*

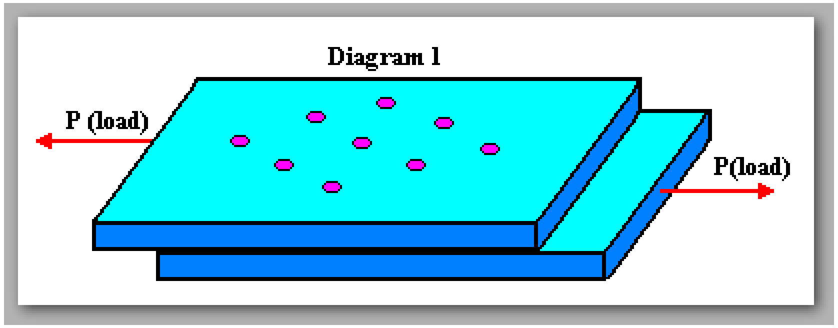
$$\text{Change in volume} = \text{volumetric strain} \times \text{original volume}$$

$$\begin{aligned}
\text{volumetric strain} &= \text{sum of three mutually perpendicular direct strains} \\
&= \varepsilon_L + 2\varepsilon_D \\
&= \frac{1}{E}[\sigma_L - \nu\sigma_H] + \frac{2}{E}[\sigma_H - \nu\sigma_L] \\
&= \frac{1}{E}[\sigma_L + 2\sigma_H - \nu(\sigma_H + 2\sigma_L)] \\
&= \frac{pd}{4tE}[1 + 4 - \nu(2 + 2)] \\
&= \frac{pd}{4tE}[5 - 4\nu]
\end{aligned}$$

Therefore with original internal volume V

$$\textbf{change in internal volume} = \frac{pd}{4tE}[5 - 4\nu]\,V$$

Seam less and Reverted Pipes



Example 1: A cylindrical shell of 1.5 m diameter is made up of 16 mm thick plates. Find the circumferential and longitudinal stress in the plates, if the boiler is subjected to an internal pressure of 2.5 MPa. Take efficiency of the joints as 80%.

Solution :

Diameter of shell (d) = 1.5 m = 1500 mm Thickness of plates (t) = 16mm

Internal pressure (P) = 2.5 MPa Efficiency (η)= 80% = 0.8

Ex 4: A cylindrical shell of 500 mm diameter is required to withstand an internal pressure of 4 MPa. Find the minimum thickness of the shell, if maximum tensile strength in the plate material is 300 MPa and efficiency of the joints is 70%. Take factor of safety as 5.

Soln :

Allowable tensile stress

(i.e., circumferential stress),

$$\sigma_c = \frac{Tensile\ strength}{Factor\ of\ safety} = \frac{300}{5} = 60\ MPa$$

and minimum thickness of shell,

$$t = \frac{pd}{2\sigma_c \eta} = \frac{4 \times 500}{2 \times 60 \times 0.7}$$

$$= 23.81\ mm\ (say\ 24\ mm)$$

EXAMPLE 1: A thin cylindrical shell, 2m long has 200 mm diameter and thickness of metal 10 mm. It is filled completely with a fluid atmospheric pressure. If an additional 25 000 mm^3 fluid is pumped in, find the pressure developed and hoop stress developed. Find also the changes in diameter and length.

Take E = $2 \times 10^5 N/mm^2$ and $\mu = 0.3$

A cylinder has an internal diameter of 230 mm, has walls 5 mm thick and is 1 m long. It is found to change in internal volume by $12.0 \times 10^{-6}\ \text{m}^3$ when filled with a liquid at a pressure p. If $E = 200\ \text{GN/m}^2$ and $\nu = 0.25$, and assuming rigid end plates, determine:

(a) the values of hoop and longitudinal stresses;
(b) the modifications to these values if joint efficiencies of 45% (hoop) and 85% (longitudinal) are assumed;
(c) the necessary change in pressure p to produce a further increase in internal volume of 15%. The liquid may be assumed incompressible.

$$\text{change in internal volume} = \frac{pd}{4tE}(5 - 4\nu)V$$

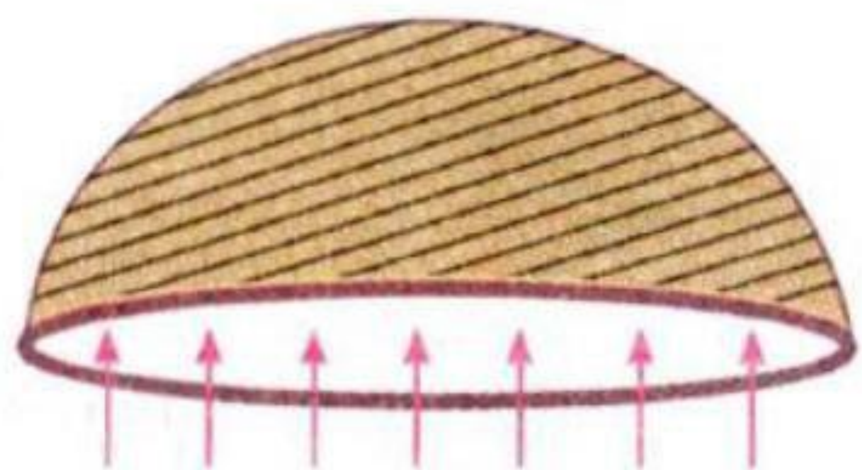
$$\text{original volume } V = \tfrac{\pi}{4} \times 230^2 \times 10^{-6} \times 1 = 41.6 \times 10^{-3}\ \text{m}^3$$

$$\text{hoop stress} = \frac{pd}{2t}$$

$$\text{longitudinal stress} = \frac{pd}{4t} =$$

THIN SPHERICAL SHELLS

- Consider a thin spherical shell subjected to an internal pressure as shown



Spherical shell

Because of the symmetry of the sphere the stresses set up owing to internal pressure will be two mutually perpendicular hoop or circumferential stresses of equal value and a radial stress. As with thin cylinders having thickness to diameter ratios less than 1:20, the radial stress is assumed negligible in comparison with the values of hoop stress set up. The stress system is therefore one of equal biaxial hoop stresses.

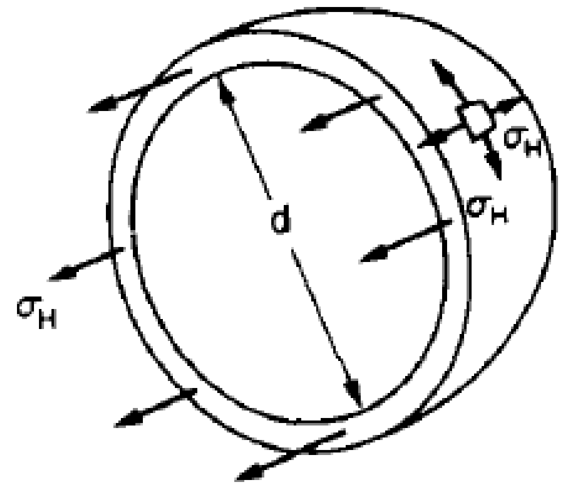
Consider, therefore, the equilibrium of the half-sphere shown in Fig. 9.4.

Force on half-sphere owing to internal pressure

$$= \text{pressure} \times \text{projected area}$$

$$= p \times \frac{\pi d^2}{4}$$

$$\text{Resisting force} = \sigma_H \times \pi d t \quad \text{(approximately)}$$



Half of a thin sphere subjected to internal pressure showing uniform hoop stresses acting on a surface element.

$$p \times \frac{\pi d^2}{4} = \sigma_H \times \pi d t$$

$$\sigma_H = \frac{pd}{4t}$$

$$\textbf{circumferential or hoop stress} = \frac{pd}{4t}$$

Change in internal volume

As for the cylinder,

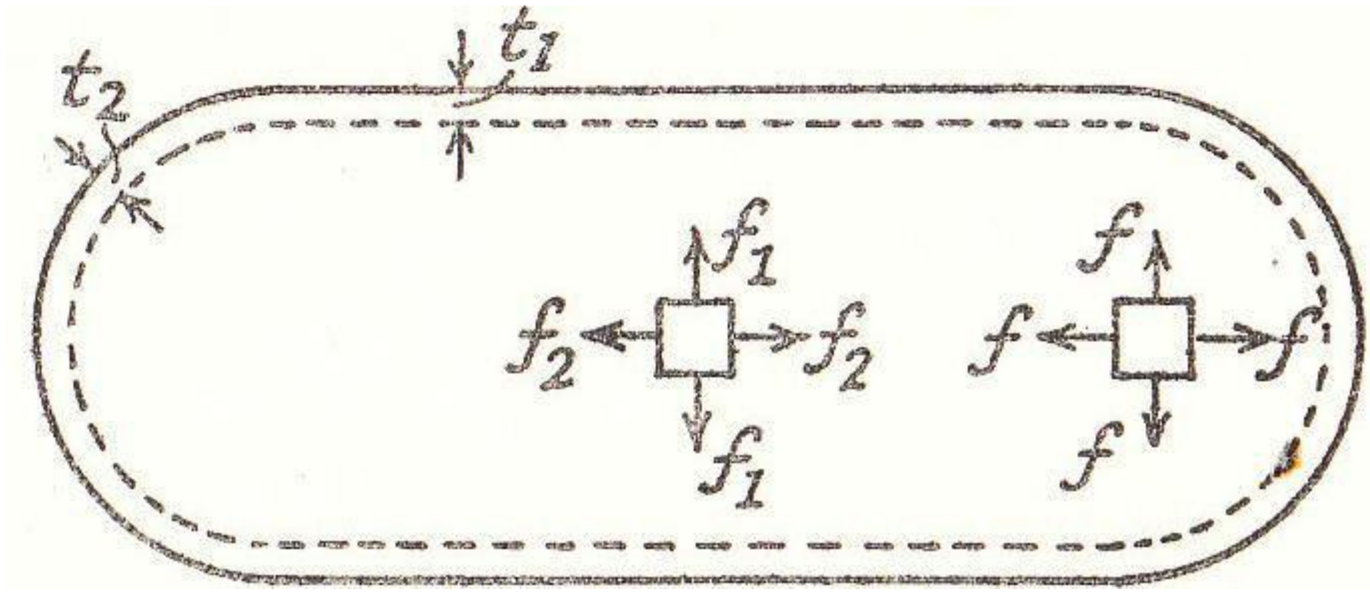
change in volume = original volume × volumetric strain

but

$$\begin{aligned} \text{volumetric strain} &= \text{sum of three mutually perpendicular strains (in this case all equal)} \\ &= 3\varepsilon_D = 3\varepsilon_H \\ &= \frac{3}{E}[\sigma_H - \nu\sigma_H] \\ &= \frac{3pd}{4tE}[1-\nu] \end{aligned}$$

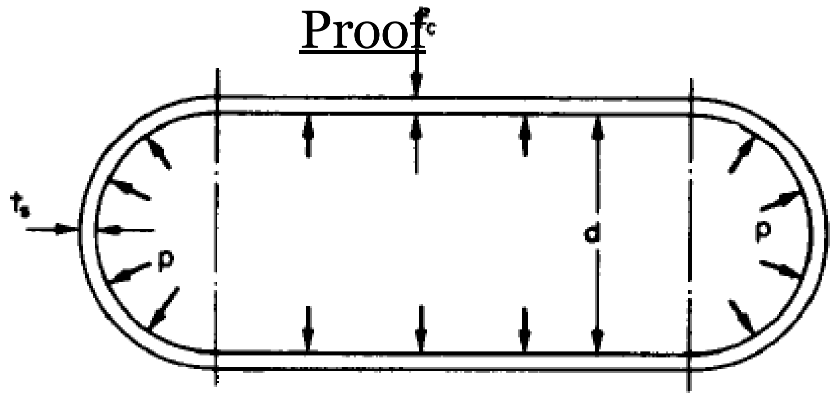
$\therefore$ **change in internal volume** $= \dfrac{3pd}{4tE}[1-\nu]V$

Cylindrical shell with hemispherical ends



$$t_1 = 2 \times t_2$$

Proof



Cross-section of a thin cylinder with hemispherical ends.

(a) *For the cylindrical portion*

$$\text{hoop or circumferential stress} = \sigma_{H_c} = \frac{pd}{2t_c}$$

$$\text{longitudinal stress} = \sigma_{L_c} = \frac{pd}{4t_c}$$

$$\text{hoop or circumferential strain} = \frac{1}{E}[\sigma_{H_c} - \nu\sigma_{L_c}]$$

$$= \frac{pd}{4t_c E}[2 - \nu]$$

(b) *For the hemispherical ends*

$$\text{hoop stress} = \sigma_{H_s} = \frac{pd}{4t_s}$$

$$\therefore \quad \text{hoop strain} = \frac{1}{E}[\sigma_{H_s} - \nu\sigma_{H_s}]$$

$$= \frac{pd}{4t_s E}[1 - \nu]$$

Thus equating the two strains in order that there shall be no distortion of the junction,

$$\frac{pd}{4t_c E}[2 - \nu] = \frac{pd}{4t_s E}[1 - \nu]$$

i.e.

$$\frac{t_s}{t_c} = \frac{(1 - \nu)}{(2 - \nu)}$$

With the normally accepted value of Poisson's ratio for general steel work of 0.3, the thickness ratio becomes

$$\frac{t_s}{t_c} = \frac{0.7}{1.7}$$

i.e. the thickness of the cylinder walls must be approximately 2.4 times that of the hemispherical ends for no distortion of the junction to occur. In these circumstances, because of the reduced wall thickness of the ends, the maximum stress will occur in the ends. For *equal maximum stresses* in the two portions the thickness of the cylinder walls must be twice that in the ends but some distortion at the junction will then occur.

Ex 6: At atmospheric pressure, a thin spherical shell has diameter 750 mm and thickness 8 mm. Find the stress introduced and change in diameter and volume when the fluid pressure is increased to 2.5 N/mm^2.

Take E = $2 \times 10^5 N/mm^2$ and $\mu = 0.3$

Thank You