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CHENNAI CAMPUS
Vandalur - Kelambakkam Road, Chennai - 600127

School of Mechanical and Building Sciences

MEE 2002 Strength of Materials

MODULE 5

Deflection of Beams

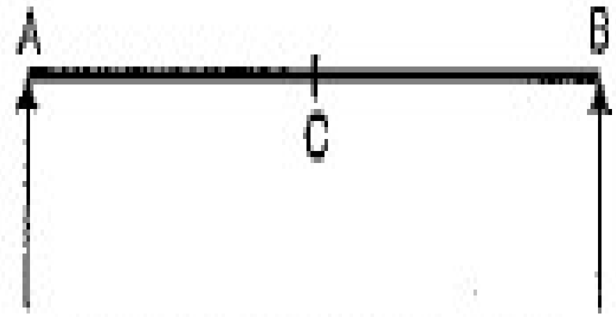
By

Dr. S.JEYANTHI

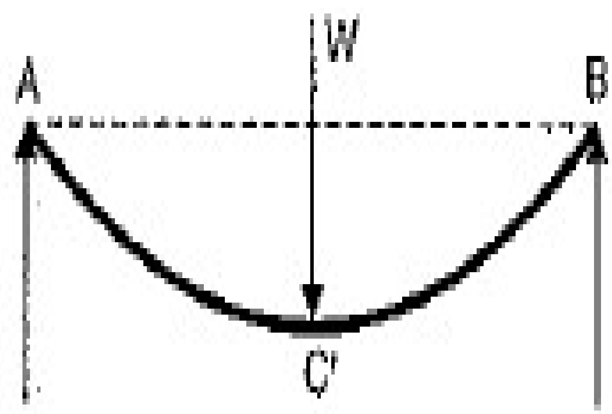
Associate Professor

If a beam carries uniformly distributed load or a point load, the beam is deflected from its original position. In this chapter, we shall study the amount by which a beam is deflected from its position. Due to the loads acting on the beam, it will be subjected to bending moment. The radius of curvature of the deflected beam is given by the equation $\frac{M}{I} = \frac{E}{R}$. The radius of curvature will be constant if $R = \frac{I \times E}{M}$ = constant. The term $(I \times E)/M$ will be constant, if the beam is subjected to a constant bending moment M. This means that a beam for which, when loaded, the value of $(E \times I)/M$ is constant, will bend in a circular arc.

Fig. 12.1 (a) shows the beam position before any load is applied on the beam whereas Fig. 12.1 (b) shows the beam position after loading.



(a) Beam position before loading



(b) Beam position after loading

Fig. 12.1

12.2. DEFLECTION AND SLOPE OF A BEAM SUBJECTED TO UNIFORM BENDING MOMENT

A beam AB of length L is subjected to a uniform bending moment M as shown in Fig. 12.1 (c). As the beam is subjected to a constant bending moment, hence it will bend into a circular arc. The initial position of the beam is shown by ACB, whereas the deflected position is shown by $AC'B$.

Let R = Radius of curvature of the deflected beam,

y = Deflection of the beam at the centre (*i.e.*, distance CC'),

I = Moment of inertia of the beam section,

E = Young's modulus for the beam material, and

θ = Slope of the beam at the end A (*i.e.*, the angle made by the tangent at A with the beam AB).

For a practical beam the deflection y is a small quantity.

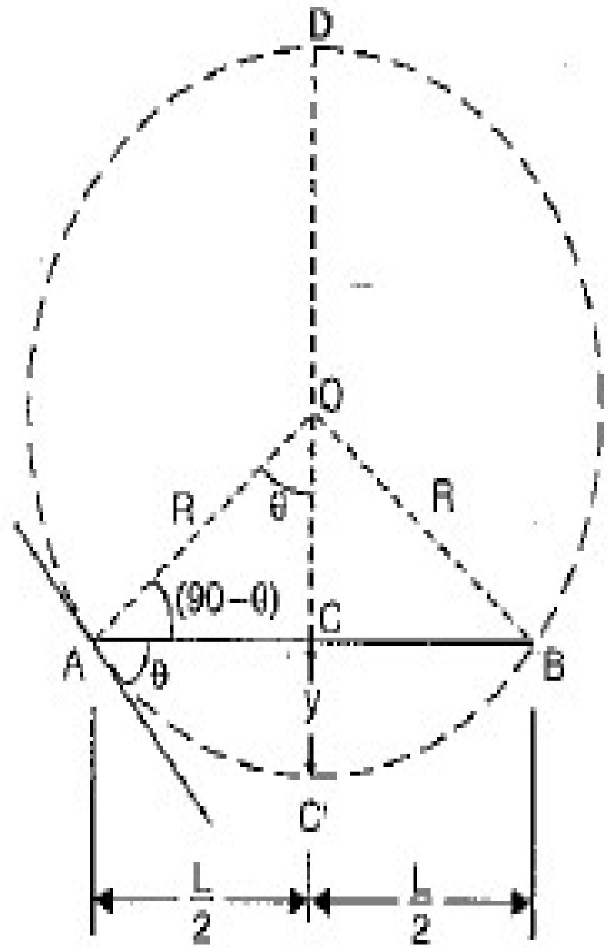


Fig. 12.1 (c)

dx

Hence, the relation between curvature, slope, deflection etc. at a section is given by :

Deflection $= y$

Slope $= \dfrac{dy}{dx}$

Bending moment $= EI \dfrac{d^2y}{dx^2}$

Shearing force $= EI \dfrac{d^3y}{dx^3}$

The rate of loading $= EI \dfrac{d^4y}{dx^4}$.

Units. In the above equations, E is taken in N/mm^2

I is taken in mm^4, y is taken in mm,

M is taken in Nm and x is taken in m.

METHODS TO DETERMINE SLOPE AND DEFLECTION

- Moment Area Method
- Double integeration method
- Maculay’s method

SIMPLY SUPPORTED BEAM WITH POINT LOAD AT CENTER DOUBLE INTEG METHOD

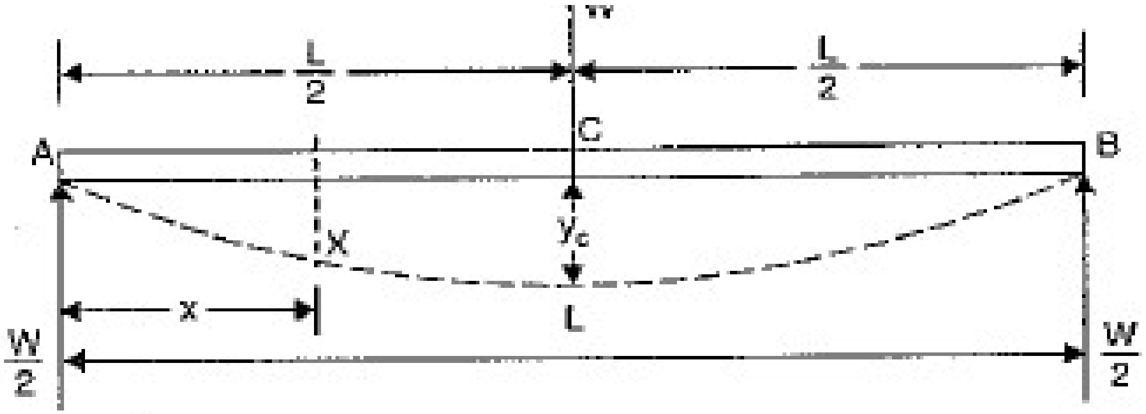


Fig. 12.3

Now $$R_A = R_B = \frac{W}{2}$$

Consider a section X at a distance x from A. The bending moment at this section is given by,

$$M_x = R_A \times x$$

$$= \frac{W}{2} \times x$$

(Plus sign is as B.M. for left portion at X is clockwise)

But B.M. at any section is also given by equation (12.3) as

$$M = EI \frac{d^2y}{dx^2}$$

Equating the two values of B.M., we get

$$EI \frac{d^2y}{dx^2} = \frac{W}{2} \times x \qquad \text{..(i)}$$

On integration, we get

$$EI \frac{dy}{dx} = \frac{W}{2} \times \frac{x^2}{2} + C_1 \qquad \text{...(ii)}$$

where C_1 is the constant of integration. And its value is obtained from boundary conditions.

The boundary condition is that at $x = \frac{L}{2}$, slope $\left(\frac{dy}{dx}\right) = 0$ (As the maximum deflection is at the centre, hence slope at the centre will be zero). Substituting this boundary condition in equation (*ii*), we get

$$O = \frac{W}{4} \times \left(\frac{L}{2}\right)^2 + C_1$$

or

$$C_1 = -\frac{WL^2}{16}$$

Substituting the value of C_1 in equation (*ii*), we get

$$EI\,\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{WL^2}{16} \qquad \text{...(iii)}$$

The above equation is known the *slope equation*. We can find the slope at any point on the beam by substituting the values of x. Slope is maximum at A. At A, $x = 0$ and hence slope at A will be obtained by substituting $x = 0$ in equation (*iii*).

$$\therefore \quad EI\left(\frac{dy}{dx}\right)_{at\ A} = \frac{W}{4}\times 0 - \frac{WL^2}{16}$$

$$\left[\left(\frac{dy}{dx}\right)_{at\ A} \text{ is the slope at } A \text{ and is represented by } \theta_A\right]$$

or
$$EI \times \theta_A = -\frac{WL^2}{16}$$

$$\therefore \quad \theta_A = -\frac{WL^2}{16EI}$$

The slope at point B will be equal to θ_A, since the load is symmetrically applied.

$$\therefore \quad \theta_B = \theta_A = -\frac{WL^2}{16EI} \qquad \ldots(12.6)$$

Equation (12.6) gives the slope in radians.

Deflection at any point

Deflection at any point is obtained by integrating the slope equation (*iii*). Hence integrating equation (*iii*), we get

$$EI \times y = \frac{W}{4}\cdot\frac{x^3}{3} - \frac{WL^2}{16}x + C_2 \qquad \ldots(iv)$$

where C_2 is another constant of integration. At A, $x = 0$ and the deflection (y) is zero.

Hence substituting these values in equation (*iv*), we get

$$EI \times 0 = 0 - 0 + C_2$$

or
$$C_2 = 0$$

Substituting the value of C_2 in equation (*iv*), we get

$$EI \times y = \frac{Wx^3}{12} - \frac{WL^2 \cdot x}{16} \qquad \ldots(v)$$

Deflection at any point

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$$EI \times 0 = 0 - 0 + C_2$$

or $$C_2 = 0$$

Substituting the value of C_2 in equation (*iv*), we get

$$EI \times y = \frac{Wx^3}{12} - \frac{WL^2 \cdot x}{16} \quad \text{...(v)}$$

The above equation is known as *the deflection equation.* We can find the deflection at any point on the beam by substituting the values of x. The deflection is maximum at centre point C, where $x = \frac{L}{2}$. Let y_c represents the deflection at C. Then substituting $x = \frac{L}{2}$ and $y = y_c$ in equation (v), we get

$$EI \times y_c = \frac{W}{12}\left(\frac{L}{2}\right)^3 - \frac{WL^2}{16} \times \left(\frac{L}{2}\right)$$

$$= \frac{WL^3}{96} - \frac{WL^3}{32} = \frac{WL^3 - 3WL^3}{96}$$

$$= -\frac{2WL^3}{96} = -\frac{WL^3}{48}$$

$$\therefore \qquad y_c = -\frac{WL^3}{48EI}$$

(Negative sign shows that deflection is downwards)

$$\therefore \quad \text{Downward deflection, } y_c = \frac{WL^3}{48EI}$$

Problem 12.1. *A beam 6 m long, simply supported at its ends, is carrying a point load of 50 kN at its centre. The moment of inertia of the beam (i.e. I) is given as equal to 78×10^6 mm^4. If E for the material of the beam $= 2.1 \times 10^5$ N/mm^2, calculate : (i) deflection at the centre of the beam and (ii) slope at the supports.*

Sol. Given :

Length, $L = 6 \text{ m} = 6 \times 1000 = 6000$ mm

Point load, $W = 50 \text{ kN} = 50{,}000$ N

M.O.I., $I = 78 \times 10^6 \text{ mm}^4$

Value of $E = 2.1 \times 10^5 \text{ N/mm}^2$

Let y_c = Deflection at the centre and

θ_A = Slope at the support.

(*i*) Using equation (12.7) for the deflection at the centre, we get

$$y_c = \frac{WL^3}{48EI}$$

$$= \frac{50000 \times 6000^3}{48 \times 2.1 \times 10^5 \times 78 \times 10^6}$$

$$= \mathbf{13.736 \text{ mm.}} \quad \textbf{Ans.}$$

(*ii*) Using equation (12.6) for the slope at the supports, we get

$$\theta_B = \theta_A = -\frac{WL^2}{16EI}$$

$$= \frac{WL^2}{16EI} \qquad \text{(Numerically)}$$

$$= \frac{50000 \times 6000^2}{16 \times 2.1 \times 10^5 \times 78 \times 10^6} \text{ radians}$$

$$= 0.06868 \text{ radians}$$

$$= 0.06868 \times \frac{180}{\pi} \text{ degree} \qquad \left(\because \; 1 \text{ radian} = \frac{180}{\pi} \text{ degree}\right)$$

$$= \mathbf{3.935^\circ.} \quad \textbf{Ans.}$$

SSB WITH UDL

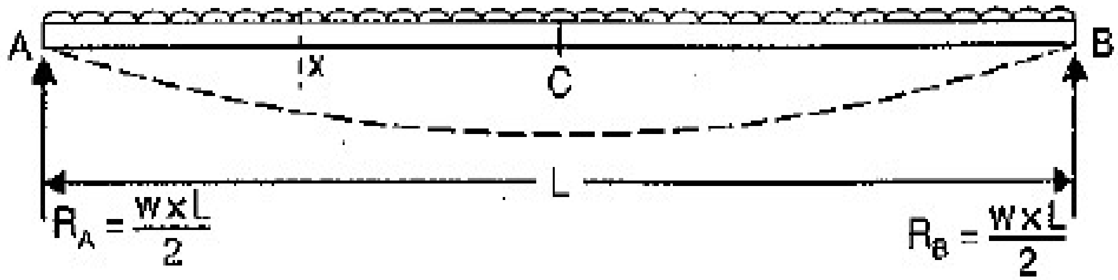

Fig. 12.6

$$\therefore \qquad R_A = R_B = \frac{w \times L}{2}$$

Consider a section X at a distance x from A. The bending moment at this section is given by,

$$M_x = R_A \times x - w \times x \times \frac{x}{2} = \frac{w \, . \, L}{2} \, . \, x - \frac{w \, . \, x^2}{2}$$

But B.M. at any section is also given by equation (12.3), as

$$M = EI \, \frac{d^2 y}{dx^2}$$

Equating the two values of B.M., we get

$$EI \, \frac{d^2 y}{dx^2} = \frac{w \, . \, L}{2} x - \frac{w \, . \, x^2}{2}$$

Integrating the above equation, we get

$$EI \, \frac{dy}{dx} = \frac{w \, . \, L}{2} \, . \, \frac{x^2}{2} - \frac{w}{2} \, . \, \frac{x^3}{3} + C_1 \qquad \ldots(i)$$

where C_1 is a constant of integration.

Integrating the above equation again, we get

$$EIy = \frac{w.L}{4}.\frac{x^3}{3} - \frac{w}{6}.\frac{x^4}{4} + C_1x + C_2 \qquad ...(ii)$$

where C_2 is another constant of integration. Thus two constants of integration (*i.e.*, C_1 and C_2) are obtained from boundary conditions. The boundary conditions are :

(*i*) at $x = 0, y = 0$ and (*ii*) at $x = L, y = 0$

Substituting first boundary condition *i.e.*, $x = 0, y = 0$ in equation (*ii*), we get

$$0 = 0 - 0 + 0 + C_2 \quad \text{or} \quad C_2 = 0$$

Substituting the second boundary condition *i.e.*, at $x = L, y = 0$ in equation (*ii*), we get

$$0 = \frac{w.L}{4}.\frac{L^3}{3} - \frac{w}{6}.\frac{L^4}{4} + C_1.L \qquad (C_2 \text{ is already zero})$$

$$= \frac{w.L^4}{12} - \frac{w.L^4}{24} + C_1.L$$

or

$$C_1 = -\frac{wL^3}{12} + \frac{wL^3}{24} = -\frac{wL^3}{24}$$

Substituting the value of C_1 in equations (*i*) and (*ii*), we get

$$EI\frac{dy}{dx} = \frac{w.L}{4}.x^2 - \frac{w}{6}x^3 - \frac{wL^3}{24} \qquad ...(iii)$$

and $$EI.y = \frac{w \cdot L}{12} x^3 - \frac{w}{24} \cdot x^4 + \left(-\frac{wL^3}{24}\right) x + 0 \qquad (\because \quad C_2 = 0)$$

or $$EIy = \frac{w \cdot L}{12} x^3 - \frac{w}{24} \cdot x^4 - \frac{wL^3}{24} x \qquad \ldots(iv)$$

The equation (*iii*) is known as *slope equation*. We can find the slope $\left(i.e., \text{the value of } \frac{dy}{dx}\right)$ at any point on the beam by substituting the different values of x in this equation.

The equation (*iv*) is known as *deflection equation*. We can find the deflection (*i.e.*, the value of y) at any point on the beam by substituting the different values of x in this equation.

Slope at the supports

Let θ_A = Slope at support A. This is equal to $\left(\frac{dy}{dx}\right)_{at\ A}$

and θ_B = Slop at support $B = \left(\frac{dy}{dx}\right)_{at\ B}$

At A, $x = 0$ and $\frac{dy}{dx} = \theta_A$.

Substituting these values in equation (*iii*), we get

$$EI.\theta_A = \frac{wL}{4} \times 0 - \frac{w}{6} \times 0 - \frac{wL^3}{24}$$

$$= \frac{wL^3}{24} = -\frac{WL^2}{24} \qquad (\because \quad w \cdot L = W = \text{Total load})$$

$$\therefore \qquad \theta_A = -\frac{WL^2}{24EI} \qquad \ldots(12.12)$$

(Negative sign means that tangent at A makes an angle with AB in the anti-clockwise direction)

By symmetry,
$$\theta_B = -\frac{WL^2}{24EI} \qquad \ldots(12.13)$$

Maximum Deflection

The maximum deflection is at the centre of the beam *i.e.*, at point C, where $x = \frac{L}{2}$. Let y_C = deflection at C which is also maximum deflection. Substituting $y = y_C$ and $x = \frac{L}{2}$ in the equation (*iv*), we get

$$EI.y_C = \frac{w.L}{12}.\left(\frac{L}{2}\right)^3 - \frac{w}{24}.\left(\frac{L}{2}\right)^4 - \frac{wL^3}{24} - \left(\frac{L}{2}\right)$$

$$= \frac{w.L^4}{96} - \frac{wL^4}{384} - \frac{wL^4}{48} = -\frac{5w.L^4}{384}$$

$$\therefore \qquad y_C = -\frac{5}{384}.\frac{wL^4}{EI} = -\frac{5}{384}.\frac{W.L^3}{EI} \qquad (\because \quad w.L = W = \text{Total load})$$

Problem 12.5. *A beam of uniform rectangular section 200 mm wide and 300 mm deep is simply supported at its ends. It carries a uniformly distributed load of 9 kN/m run over the entire span of 5 m. If the value of E for the beam material is 1 × 10⁴ N/mm², find :*

(i) the slope at the supports and *(ii) maximum deflection.*

Sol. Given :

Width, $b = 200$ mm

Depth, $d = 300$ mm

M.O.I., $I = \dfrac{bd^3}{12} = \dfrac{200 \times 300^3}{12} = 4.5 \times 10^8 \text{ mm}^4$

U.d.l., $w = 9$ kN/m $= 9000$ N/m

Span, $L = 5$ m $= 5000$ mm

∴ Total load, $W = w \,.\, L^* = 9000 \times 5 = 45000$ N

Value of $E = 1 \times 10^4$ N/mm²

Let θ_A = Slope at the support

and y_C = Maximum deflection.

(*i*) Using equation (12.12), we get

$$\theta_A = -\frac{W \,.\, L^2}{24EI}$$

$$= -\frac{45000 \times 5000^2}{24 \times 1 \times 10^4 \times 4.5 \times 10^8} \text{ radians}$$

$= \mathbf{0.0104}$ **radians. Ans.**

(*ii*) Using equation (12.14), we get

$$y_C = \frac{5}{384} \cdot \frac{W \cdot L^3}{EI}$$

$$= \frac{5}{384} \times \frac{45000 \times 5000^3}{1 \times 10^4 \times 4.5 \times 10^8}$$

$= \mathbf{16.27}$ **mm. Ans.**

SIMPLY SUPPORTRD BEAMS WITH ECCENTRIC LOADING

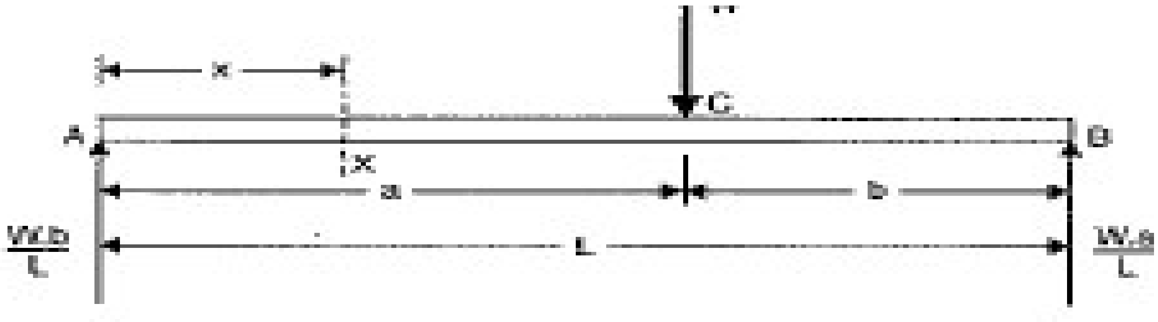

Fig. 12.4

(a) Now consider a section X at a distance x from A in length AC. The bending moment this section is given by,

$$M_x = R_A \times x$$

$$= \frac{W \times b}{L} \times x \qquad \text{(Plus sign due to sagging)}$$

But B.M. at any section is also given by equation (12.3) as

$$M = EI \frac{d^2y}{dx^2}$$

Equating the two values of B.M., we get

$$EI = \frac{d^2y}{dx^2} = \frac{W \times b}{L} \times x$$

Integrating the above equation, we get

$$EI \frac{dy}{dx} = \frac{W \times b}{L} \times \frac{x^2}{2} + C_1 \qquad \text{...(i)}$$

here C_1 is the constant of integration.

Integrating the equation (i), we get

$$EI.y = \frac{W.b}{2L} \cdot \frac{x^3}{3} + C_1.x + C_2 \qquad \text{...(ii)}$$

here C_2 is another constant of integration. The values of C_1 and C_2 are obtained from boundary conditions.

(i) At A, $x = 0$ and deflection $y = 0$

Substituting these values in equation (ii), we get

$$0 = 0 + 0 + C_2$$

$$\theta_A = \frac{-W.a.b}{6EI.L}(a+2b)$$

$\therefore$ Downward,

$$y_C = \frac{Wa^2b^2}{3EIL}$$

MACULAY'S METHOD

Problem 12.9. *A beam of length 6 m is simply supported at its ends and carries two point loads of 48 kN and 40 kN at a distance of 1 m and 3 m respectively from the left support. Find :*

(i) deflection under each load,

(ii) maximum deflection, and

(iii) the point at which maximum deflection occurs.

Given $E = 2 \times 10^5$ N/mm^2 *and* $I = 85 \times 10^6$ mm^4.

Sol. Given :

$$I = 85 \times 10^5 \text{ mm}^4 \; ; E = 2 \times 10^5 \text{ N/mm}^2$$

First calculate the reactions R_A and R_B.

Taking moments about A, we get

$$R_B \times 6 = 48 \times 1 + 40 \times 3 = 168$$

$$\therefore \quad R_B = \frac{168}{6} = 28 \text{ kN}$$

$$\therefore \quad R_A = \text{Total load} - R_B = (48 + 40) - 28 = 60 \text{ kN}$$

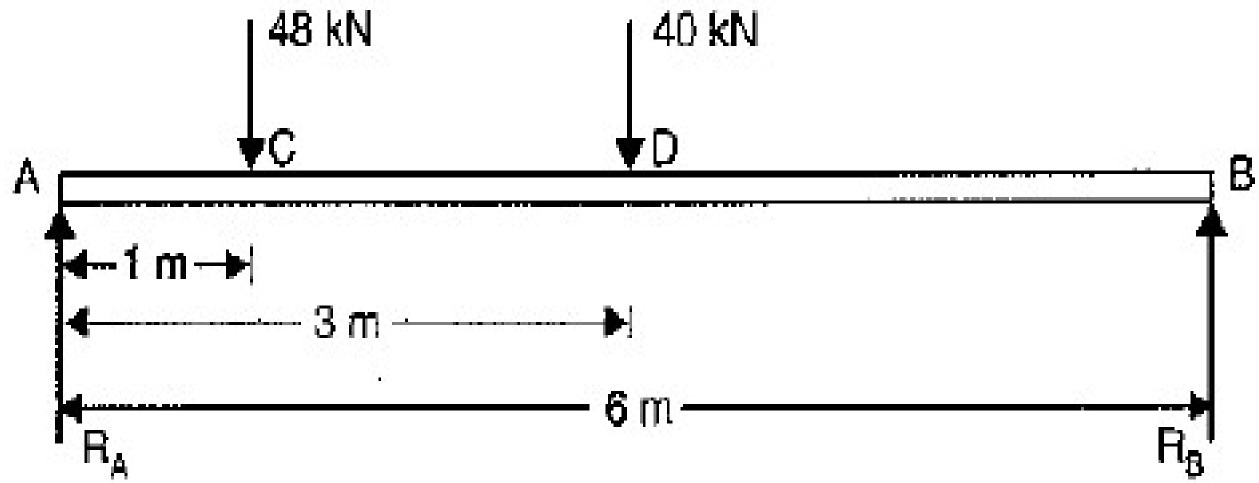

Fig. 12.8

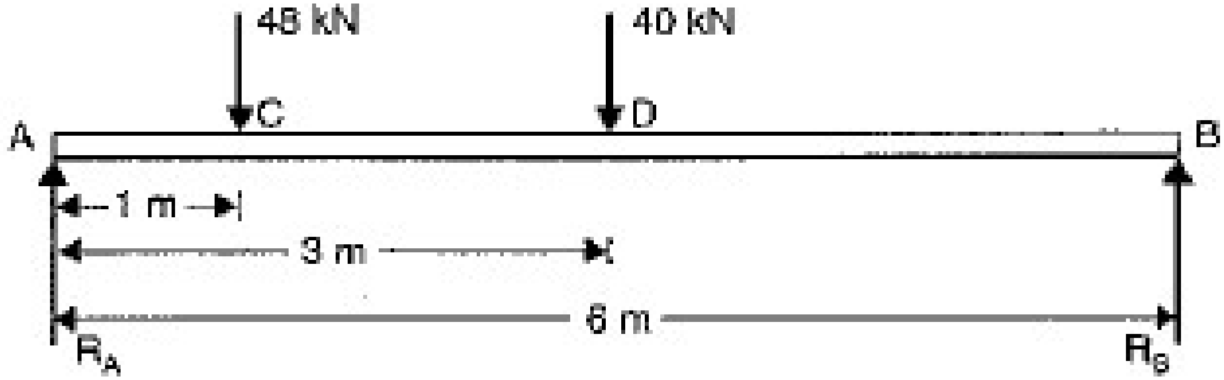

Fig. 12.8

Consider the section X in the last part of the beam (*i.e.*, in length DB) at a distance x the left support A. The B.M. at this section is given by,

$$EI\frac{d^2y}{dx^2} = R_A.x \;\vdots\; -48(x-1) \;\vdots\; -40(x-3)$$

$$= 60x \;\vdots\; -48(x-1) \;\vdots\; -40(x-3)$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = \frac{60x^2}{2} + C_1 \;\vdots\; -48\frac{(x-1)^2}{2} \;\vdots\; -40\frac{(x-3)^2}{2}$$

$$= 30x^2 + C_1 \;\vdots\; -24(x-1)^2 \;\vdots\; -20(x-3)^2 \qquad \text{...(i)}$$

Integrating the above equation again, we get

$$EIy = \frac{30x^3}{3} + C_1x + C_2 \;\vdots\; \frac{-24(x-1)^3}{3} \;\vdots\; \frac{-20(x-3)^3}{3}$$

$$= 10x^3 + C_1x + C_2 \;\vdots\; -8(x-1)^3 \;\vdots\; -\frac{20}{3}(x-3)^3 \qquad \text{...(ii)}$$

To find the values of C_1 and C_2, use two boundary conditions. The boundary conditions are :

(*i*) at $x = 0, y = 0$, and (*ii*) at $x = 6$ m, $y = 0$.

(*i*) Substituting the first boundary condition *i.e.*, at $x = 0$, $y = 0$ in equation (*ii*) and considering the equation upto first dotted line (as $x = 0$ lies in the first part of the beam), we get

$$0 = 0 + 0 + C_2 \quad \therefore \quad C_2 = 0$$

(*ii*) Substituting the second boundary condition *i.e.*, at $x = 6$ m, $y = 0$ in equation (*ii*) and considering the complete equation (as $x = 6$ lies in the last part of the beam), we get

$$0 = 10 \times 6^3 + C_1 \times 6 + 0 - 8(6-1)^3 - \frac{20}{3}(6-3)^3 \qquad (\because \quad C_2 = 0)$$

or

$$0 = 2160 + 6C_1 - 8 \times 5^3 - \frac{20}{3} \times 3^3$$

$$= 2160 + 6C_1 - 1000 - 180 = 980 + 6C_1$$

$$\therefore \qquad C_1 = \frac{\cdots}{6} = -163.33$$

Now substituting the values of C_1 and C_2 in equation (*ii*), we get

$$EIy = 10x^3 - 163.33x \;\vdots\; - 8(x-1)^3 \;\vdots\; - \frac{20}{3}(x-3)^3 \qquad \text{...(iii)}$$

(*i*) (*a*) *Deflection under first load i.e., at point C.* This is obtained by substituting $x = 1$ in equation (*iii*) upto the first dotted line (as the point C lies in the first part of the beam). Hence, we get

$$\begin{aligned} EI.\,y_c &= 10 \times 1^3 - 163.33 \times 1 \\ &= 10 - 163.33 = -153.33 \text{ kNm}^3 \\ &= -153.33 \times 10^3 \text{ Nm}^3 \\ &= -153.33 \times 10^3 \times 10^9 \text{ Nmm}^3 \\ &= -153.33 \times 10^{12} \text{ Nmm}^3 \end{aligned}$$

$$\therefore \qquad y_c = \frac{-153.33 \times 10^{12}}{EI} = \frac{-153.33 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} \text{ mm}$$

$$= -\mathbf{9.019 \text{ mm.}} \quad \textbf{Ans.}$$

(Negative sign shows that deflection is downwards).

(b) *Deflection under second load i.e. at point D*. This is obtained by substituting $x = 3$ m in equation (*iii*) upto the second dotted line (as the point D lies in the second part of the beam). Hence, we get

$$EI.y_D = 10 \times 3^3 - 163.33 \times 3 - 8(3 - 1)^3$$
$$= 270 - 489.99 - 64 = - 283.99 \text{ kNm}^3$$
$$= - 283.99 \times 10^{12} \text{ Nmm}^3$$

$$\therefore \qquad y_D = \frac{- 283.99 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} = - \mathbf{16.7 \text{ mm.}} \quad \textbf{Ans.}$$

(*ii*) *Maximum Deflection*. The deflection is likely to be maximum at a section between C and D. For maximum deflection, $\frac{dy}{dx}$ should be zero. Hence equate the equation (*i*) equal to zero upto the second dotted line.

$$\therefore \qquad 30x^2 + C_1 - 24(x - 1)^2 = 0$$

or $\quad 30x^2 - 163.33 - 24(x^2 + 1 - 2x) = 0 \qquad (\because \quad C_1 = - 163.33)$

or $\quad 6x^2 + 48x - 187.33 = 0$

The above equation is a quadratic equation. Hence its solution is

$$x = \frac{- 48 \pm \sqrt{48^2 + 4 \times 6 \times 187.33}}{2 \times 6} = 2.87 \text{ m.}$$

(Neglecting – ve root)

Now substituting $x = 2.87$ m in equation (*iii*) upto the second dotted line, we get maximum deflection as

$$EIy_{max} = 10 \times 2.87^3 - 163.33 \times 2.87 - 8(2.87 - 1)^3$$
$$= 236.39 - 468.75 - 52.31$$
$$= 284.67 \text{ kNm}^3 = - 284.67 \times 10^{12} \text{ Nmm}^3$$

$$\therefore \qquad y_{max} = \frac{- 284.67 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} = - \mathbf{16.745 \text{ mm.}} \quad \textbf{Ans.}$$

Problem 12.10. *A beam of length 8 m is simply supported at its ends. It carries a uniformly distributed load of 40 kN/m as shown in Fig. 12.9. Determine the deflection of the beam at its mid-point and also the position of maximum deflection and maximum deflection. Take E = 2 × 10⁵ N/mm² and I = 4.3 × 10⁸ mm⁴.*

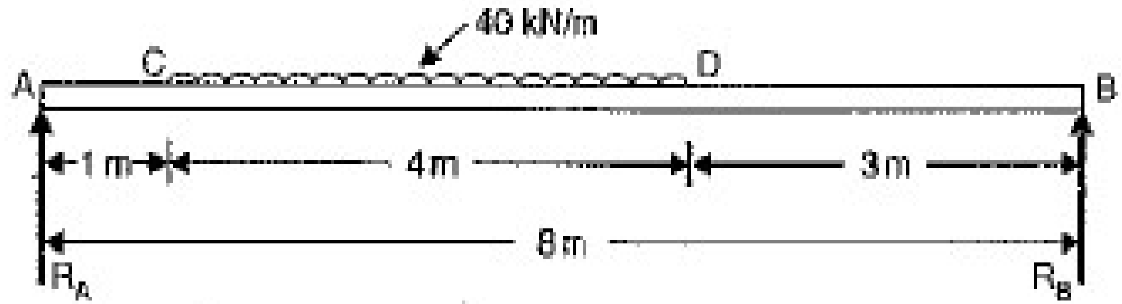

Fig. 12.9

Sol. Given :

Length, $L = 8$ m

U.d.l., $W = 40$ kN/m

Value of $E = 2 \times 10^5$ N/mm²

Value of $I = 4.3 \times 10^8$ mm⁴

First calculate the reactions R_A and R_B.

Taking moments about A, we get

$$R_B \times 8 = 40 \times 4 \times \left(1 + \frac{4}{2}\right) = 480 \text{ kN}$$

$$\therefore \quad R_B = \frac{480}{8} = 60 \text{ kN}$$

$$\therefore \quad R_A = \text{Total load} - R_B = 40 \times 4 - 60 = 100 \text{ kN}$$

In order to obtain the general expression for the bending moment at a distance x from the left end A, which will apply for all values of x, it is necessary to extend the uniformly distributed load upto the support B, compensating with an equal upward load of 40 kN/m over the span DB as shown in Fig. 12.10. Now Macaulay's method can be applied.

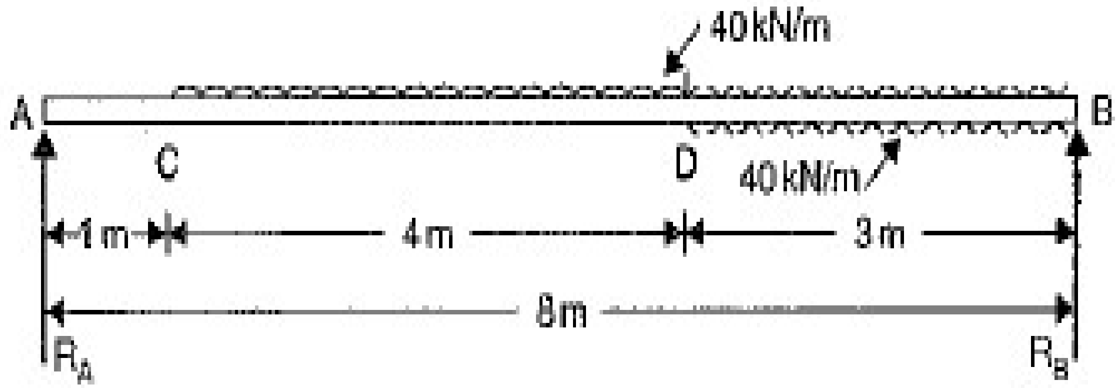

Fig. 12.10

The B.M. at any section at a distance x from end A is given by,

$$EI\frac{d^2y}{dx^2} = R_A.x \;\vdots\; -40(x-1)\times\frac{(x-1)}{2} \;\vdots\; +40\times(x-5)\times\frac{(x-5)}{2}$$

or

$$EI\frac{d^2y}{dx^2} = 100x \;\vdots\; -20\,(x-1)^2 \;\vdots\; +20\,(x-5)^2$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = \frac{100x^2}{2} + C_1 \;\vdots\; -\frac{20(x-1)^3}{3} \;\vdots\; +20\,\frac{(x-5)^3}{3} \qquad \text{...(i)}$$

Integrating again, we get

$$EIy = 50\,\frac{x^3}{3} + C_1x + C_2 \;\vdots\; -\frac{20}{3}\,\frac{(n-1)^4}{4} \;\vdots\; +\frac{20}{3}\,\frac{(x-5)^4}{3}$$

$$= 50\,\frac{x^3}{3} + C_1x + C_2 \;\vdots\; -\frac{5}{3}\,(x-1)^4 \;\vdots\; +\frac{5}{3}\,(x-5)^4 \qquad ...(ii)$$

where C_1 and C_2 are constants of integration. Their values are obtained from boundary conditions which are :

(*i*) at $x = 0,\ y = 0$ and (*ii*) at $x = 8$ m, $y = 0$

(*i*) Substituting $x = 0$ and $y = 0$ in equation (*ii*) upto first dotted line (as $x = 0$ lies in the first part AC of the beam), we get

$$0 = 0 + C_1 \times 0 + C_2 \qquad \therefore\ C_2 = 0$$

(*ii*) Substituting $x = 8$ and $y = 0$ in complete equation (*ii*) (as point $x = 8$ lies in the last part DB of the beam), we get

$$0 = \frac{50}{3} \times 8^3 + C_1 \times 8 + 0 - \frac{5}{3}\,(8-1)^4 + \frac{5}{3}\,(8-5)^4 \qquad (\because\ C_2 = 0)$$

$$= 8533.33 + 8C_1 - 4001.66 + 135$$

or
$$8C_1 = -4666.67$$

or
$$C_1 = \frac{-4666.67}{8} = -583.33$$

Substituting the value of C_1 and C_2 in equation (*ii*), we get

$$EI_y = \frac{50}{3}\,x^3 - 583.33x \;\vdots\; -\frac{5}{3}(x-1)^4 \;\vdots\; +\frac{5}{3}\,(x-4)^4 \qquad ...(iii)$$

(a) *Deflection at the centre*

By substituting $x = 4$ m in equation (*iii*) upto second dotted line, we get the deflection at the centre. [The point $x = 4$ lies in the second part (*i.e.*, *CD*) of the beam].

$$\therefore \qquad EI.y = \frac{50}{3} \times 4^3 - 583.33 \times 4 - \frac{5}{3}(4-1)^4$$

$$= 1066.66 - 2333.32 - 135 = -1401.66 \text{ kNm}^3$$

$$= -1401.66 \times 1000 \text{ Nm}^3$$

$$= -1401.66 \times 1000 \times 10^9 \text{ Nmm}^3$$

$$= -1401.66 \times 10^{12} \text{ Nmm}^3$$

$$\therefore \qquad y = \frac{-1401.66 \times 10^{12}}{EI} = \frac{-140166 \times 10^{12}}{2 \times 10^5 \times 4.5 \times 10^8}$$

$$= -16.29 \text{ mm downward.} \quad \textbf{Ans.}$$

(b) *Position of maximum deflection*

The maximum deflection is likely to lie between *C* and *D*. For maximum deflection the slope $\frac{dy}{dx}$ should be zero. Hence equating the slope given by equation (*i*) upto second dotted line to zero, we get

$$0 = 100\frac{x^2}{2} + C_1 - \frac{20}{3}(x-1)^3$$

$$0 = 50x^2 - 583.33 - 6.667(x-1)^3 \qquad \text{...(iv)}$$

The above equation is solved by trial and error method.

Let $x = 1$, then R.H.S. of equation (*iv*)

$$= 50 - 583.33 - 6.667 \times 0 = -533.33$$

Let $x = 2$, then R.H.S. $= 50 \times 4 - 583.33 - 6.667 \times 1 = -390.00$

Let $x = 3$, then R.H.S. $= 50 \times 9 - 583.33 - 6.667 \times 8 = -136.69$

Let $x = 4$, then R.H.S. $= 50 \times 16 - 583.33 - 6.667 \times 27 = +36.58$

In equation (*iv*), when $x = 3$ then R.H.S. is negative but when $x = 4$ then R.H.S. is positive. Hence exact value of x lies between 3 and 4.

Let $x = 3.82$, then R.H.S. $= 50 \times 3.82 - 583.33 - 6.667\,(3.82 - 1)^3$

$$= 729.63 - 583.33 - 149.51 = -3.22$$

Let $x = 3.83$, then R.H.S. $= 50 \times 3.83^2 - 583.33 = 6.667\,(3.83 - 1)^3$

$$= 733.445 - 583.33 - 151.1 = -0.99$$

The R.H.S. is approximately zero in comparison to the three terms (*i.e.*, 733.445, 583.33 and 151.1).

$\therefore$ Value of x = **3.83. Ans.**

Hence maximum deflection will be at a distance of 3.83 m from support A.

(*c*) *Maximum deflection*

Substituting $x = 3.83$ m in equation (*iii*) upto second dotted line, we get the maximum deflection [the point $x = 3.83$ lies in the second part *i.e.*, CD of the beam.]

$$\therefore \quad EI.y_{max} = \frac{50}{3} \times 3.83^3 - 583.33 \times 3.83 - \frac{5}{3}\,(3.83 - 1)^4$$

$$= 936.36 - 2234.15 - 106.9 = -1404.69 \text{ kNm}^3$$

$$= -1404.69 \times 10^{12} \text{ Nmm}^3$$

$$\therefore \quad y_{max} = \frac{-1404.69 \times 10^{12}}{2 \times 10^5 \times 4.3 \times 10^8} = \mathbf{-16.33 \text{ mm.}} \quad \textbf{Ans.}$$

Problem 12.11. *An overhanging beam ABC is loaded as shown in Fig. 12.11. Find the slopes over each support and at the right end. Find also the maximum upward deflection between the supports and the deflection at the right end.*

Take $E = 2 \times 10^5$ N/mm² and $I = 5 \times 10^8$ mm⁴.

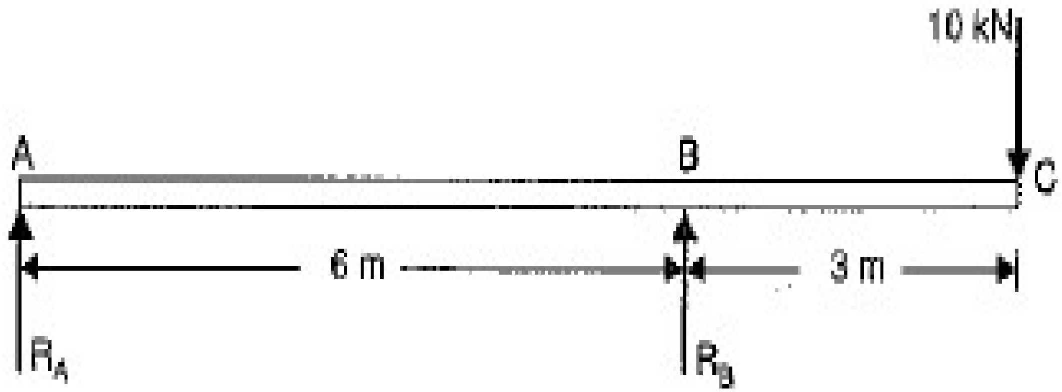

Fig. 12.11

Sol. Given :

Point load, $W = 10$ kN

Value of $E = 2 \times 10^5$ N/mm²

Value of $I = 5 \times 10^8$ mm⁴

First calculate the reaction R_A and R_B.

Taking moments about A, we get

$$R_B \times 6 = 10 \times 9$$

$$\therefore \quad R_B = \frac{10 \times 9}{6} = 15 \text{ kN}$$

$$\therefore \quad R_A = \text{Total load} - R_B = 10 - 15 = -5 \text{ kN}$$

Hence the reaction R_A will be in the downward direction. Hence Fig. 12.11 will be modified as shown in Fig. 12.12. Now write down an expression for the B.M. in the last section of the beam.

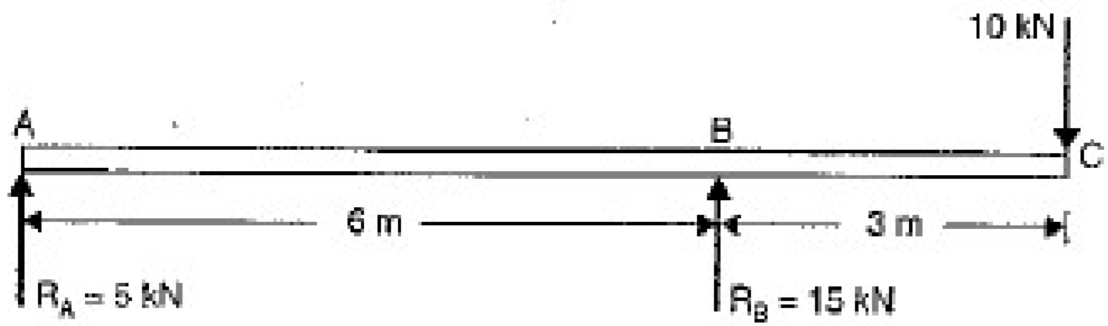

Fig. 12.12

The B.M. at any section at a distance x from the support A is given by,

$$EI \frac{d^2y}{dx^2} = -R_A \times x \;\;\vdots\;\; + R_B \times (x-6)$$

$$= -5x \;\;\vdots\;\; + 15(x-6) \qquad (\because \; R_A = 5)$$

Integrating the above equation, we get

$$EI \frac{dy}{dx} = \frac{-5x^2}{2} + C_1 \;\;\vdots\;\; + \frac{15(x-6)^2}{2} \qquad ...(i)$$

Integrating again, we get

Integrating again, we get

$$EI.y = -\frac{5}{2}\frac{x^3}{3} + C_1x + C_2 \quad \vdots \quad + \frac{15}{2}\frac{(x-6)^3}{3}$$

$$= -\frac{5}{6}x^3 + C_1x + C_2 \quad \vdots \quad + \frac{5}{2}(x-6)^3 \qquad \text{...(ii)}$$

where C_1 and C_2 are constant of integration. Their values are obtained from boundary conditions which are :

(*i*) at $x = 0$, $y = 0$ and (*ii*) at $x = 6$ m, $y = 0$.

(*i*) Substituting $x = 0$ and $y = 0$ in equation (*ii*) upto dotted line (as $x = 0$ lies in the first part *AB* of the beam), we get

$$0 = 0 + C_1 \times 0 + C_2 \qquad \therefore C_2 = 0$$

(*ii*) Substituting $x = 6$ m and $y = 0$ in equation (*ii*) upto dotted line (as $x = 6$ lies in the first part *AB* of the beam), we get

$$0 = \frac{-5}{6} \times 6^3 + C_1 \times 6 + 0 \qquad (\because \quad C_2 = 0)$$

$$\therefore \qquad = -5 \times 36 + 6C_1$$

$$\therefore \qquad C_1 = \frac{+5 \times 36}{6} = 30$$

Substituting the values of C_1 and C_2 in equations (*i*) and (*ii*), we get

$$EI\frac{dy}{dx} = -\frac{5}{2}x^2 + 30 \quad \vdots \quad + \frac{15}{2}(x-6)^2 \qquad \text{...(iii)}$$

and

$$EIy = -\frac{5}{6}x^3 + 30x \quad \vdots \quad + \frac{5}{2}(x-6)^3 \qquad \text{...(iv)}$$

(a) Slope over the support A

By substituting $x = 0$ in equation (*iii*) upto dotted line, we get the slope at support A (the point $x = 0$ lies in the first part AB of the beam).

$$\therefore \quad EI.\theta_A = -\frac{5}{2} \times 0 + 30 = 30 \text{ kNm}^2 = 30 \times 1000 \text{ Nm}^2 \quad \left(\because \frac{dy}{dx} \text{ at } A = \theta_A\right)$$

$$= 30 \times 1000 \times 10^6 \text{ Nmm}^2 = 30 \times 10^9 \text{ Nmm}^2$$

$$\therefore \quad \theta_A = \frac{30 \times 10^9}{E \times I} = \frac{30 \times 10^9}{2 \times 10^5 \times 5 \times 10^8}$$

$$= \textbf{0.0003 radians.} \quad \textbf{Ans.}$$

(*b*) *Slope at the support B*

By substituting $x = 6$ m in equation (*iii*) upto dotted line, we get the slope at support B (the point $x = 6$ lies in the first part AB of the beam).

$$EI.\theta_B = -\frac{5}{2} \times 6^2 + 30 = -90 + 30 \quad \left(\because \frac{dy}{dx} \text{ at } B = \theta_B\right)$$

$$= -60 \text{ kNm}^2 = -60 \times 10^9 \text{ Nmm}^2$$

$$\therefore \quad \theta_B = \frac{-60 \times 10^9}{E \times I} = \frac{-60 \times 10^9}{2 \times 10^5 \times 5 \times 10^8}$$

$$= \textbf{-0.0006 radians.} \quad \textbf{Ans.}$$

(*c*) *Slope at the right end i.e., at C*

By substituting $x = 9$ m in equation (*iii*), we get the slope at C. In this case, complete equation is to be taken as point $x = 9$ m lies in the last part of the beam.

$$\therefore \quad EI.\theta_C = -\frac{5}{2} \times 9^2 + 30 + \frac{15}{2}(9-6)^2 \quad \left(\because \frac{dy}{dx} \text{ at } C = \theta_C\right)$$

$$= -202.5 + 30 + 67.5 = -105 \text{ kNm}^2$$

$$= -105 \times 10^9 \text{ Nmm}^2$$

$$\therefore \quad \theta_C = \frac{-105 \times 10^9}{E \times I} = \frac{-105 \times 10^9}{2 \times 10^5 \times 5 \times 10^8}$$

$$= \textbf{-0.00105 radians.} \quad \textbf{Ans.}$$

[illegible] radians. Ans.

(d) *Maximum upward deflection between the supports*

For maximum deflection between the supports, $\frac{dy}{dx}$ should be zero. Hence equating the slope given by the equation (*iii*) to be zero upto dotted line, we get

$$0 = -\frac{5}{2}x^2 + 30 = -5x^2 + 60$$

or $$5x^2 = 60 \qquad \text{or} \quad x = \sqrt{\frac{60}{5}} = \sqrt{12} = 3.464 \text{ m}$$

Now substituting $x = 3.464$ m in equation (*iv*) upto dotted line, we get the maximum deflection as

$$EIy_{max} = -\frac{5}{6} \times 3.464^3 + 30 \times 3.464$$

$= -34.638 + 103.92 = 69.282 \text{ kNm}^3$

$= 69.282 \times 1000 \times 10^9 \text{ Nmm}^3 = 69.282 \times 10^{12} \text{ mm}^3$

$$\therefore \quad y_{max} = \frac{69.282 \times 10^{12}}{2 \times 10^5 \times 5 \times 10^8}$$

$= 0.6928$ **mm (upward). Ans.**

(e) Deflection at the right end i.e., at point C

By substituting $x = 9$ m in equation *(iv)*, we get the deflection at point C. Here complete equation is to be taken as point $x = 9$ m lies in the last part of the beam.

$$\therefore \quad EI\, y_C = -\frac{5}{6} \times 9^3 + 30 \times 9 + \frac{5}{2}(9-6)^3$$

$= -607.5 + 270 + 67.5$

$= -270 \text{ kNm}^3 = -270 \times 10^{12} \text{ Nmm}^3$

$$\therefore \quad y_C = \frac{-270 \times 10^{12}}{2 \times 10^5 \times 5 \times 10^8}$$

$= -2.7$ **mm (downwards). Ans.**