School of Mechanical and Building Sciences

MEE 2002 Strength of Materials

Unit 6

Torsion and Springs

By

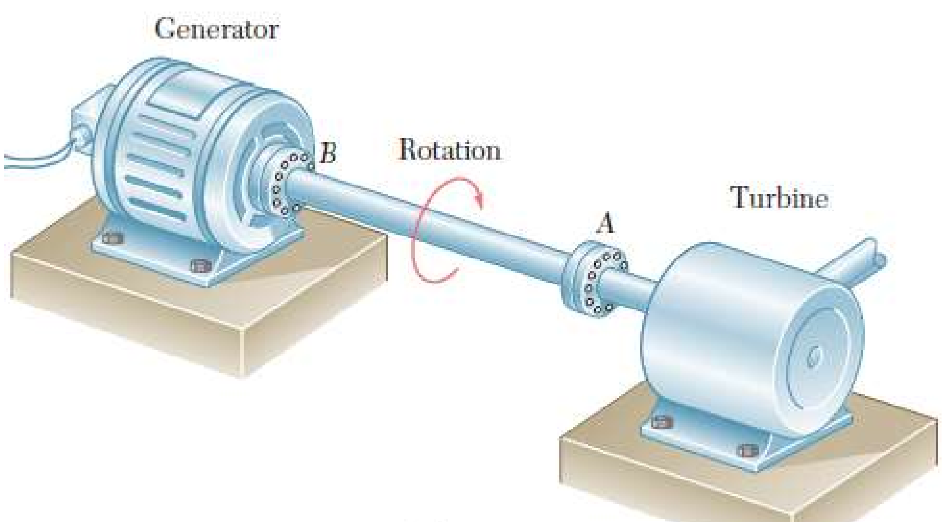
Dr. S.jeyanthi

Associate Professor

Introduction

In this chapter, structural members and machine parts that are in torsion will be analyzed

The stresses and strains in members of circular cross section (Transmission Shafts) subjected to twisting couples, or torques, will be discussed



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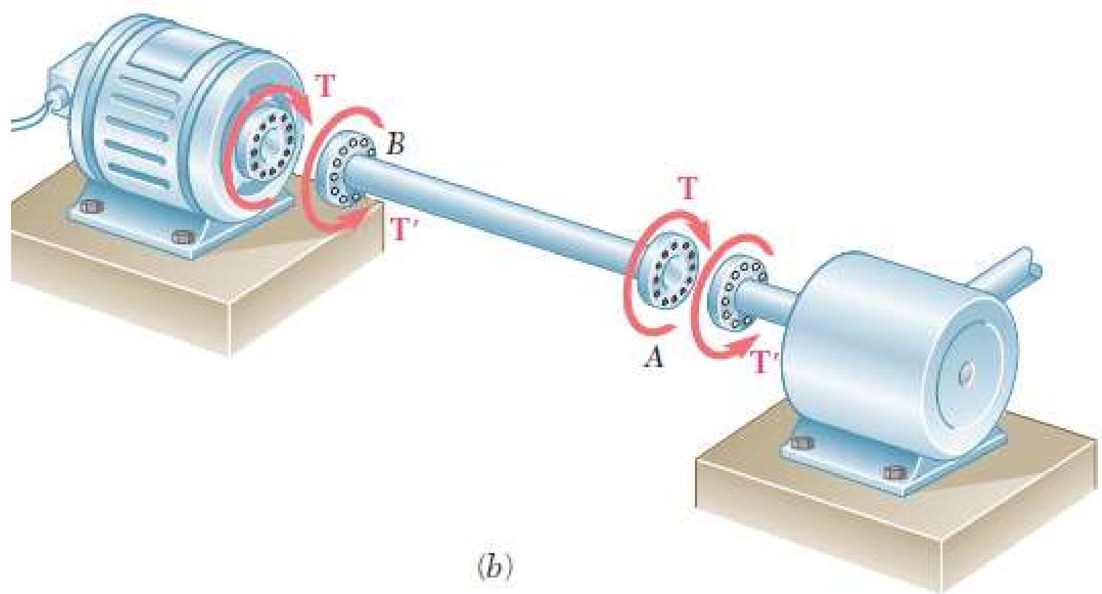

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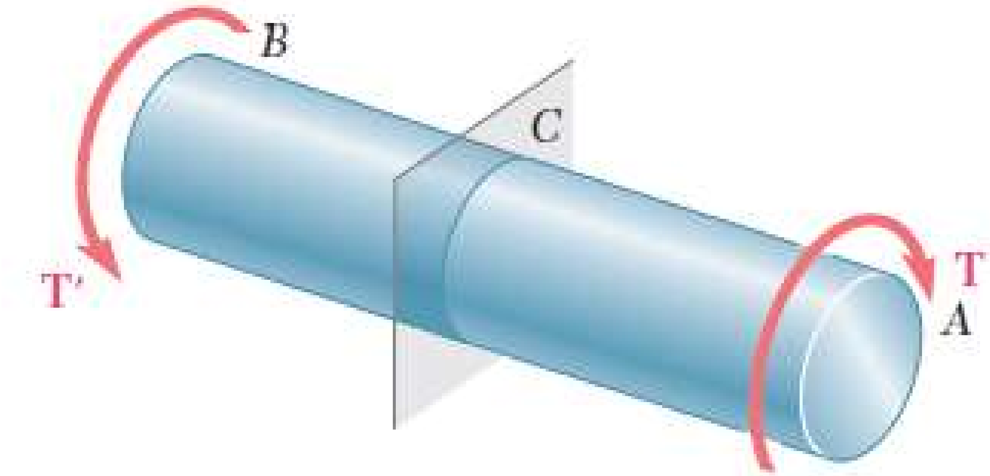
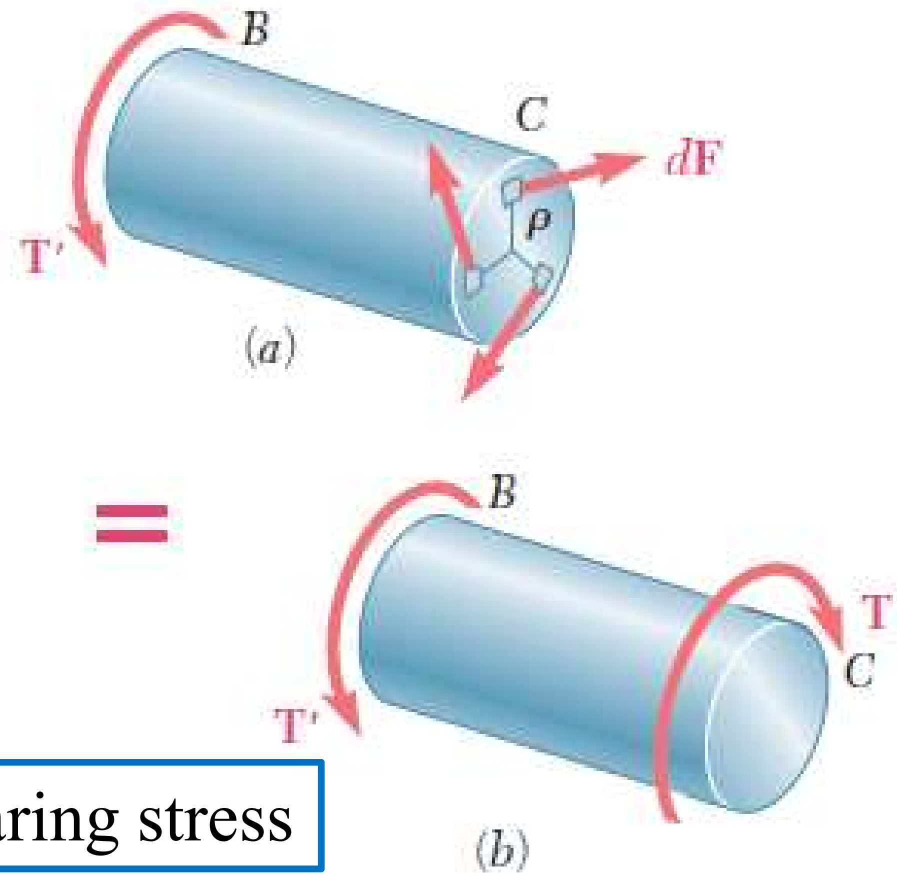
Shear Stress and Strain





$$\int \rho \, dF = T$$

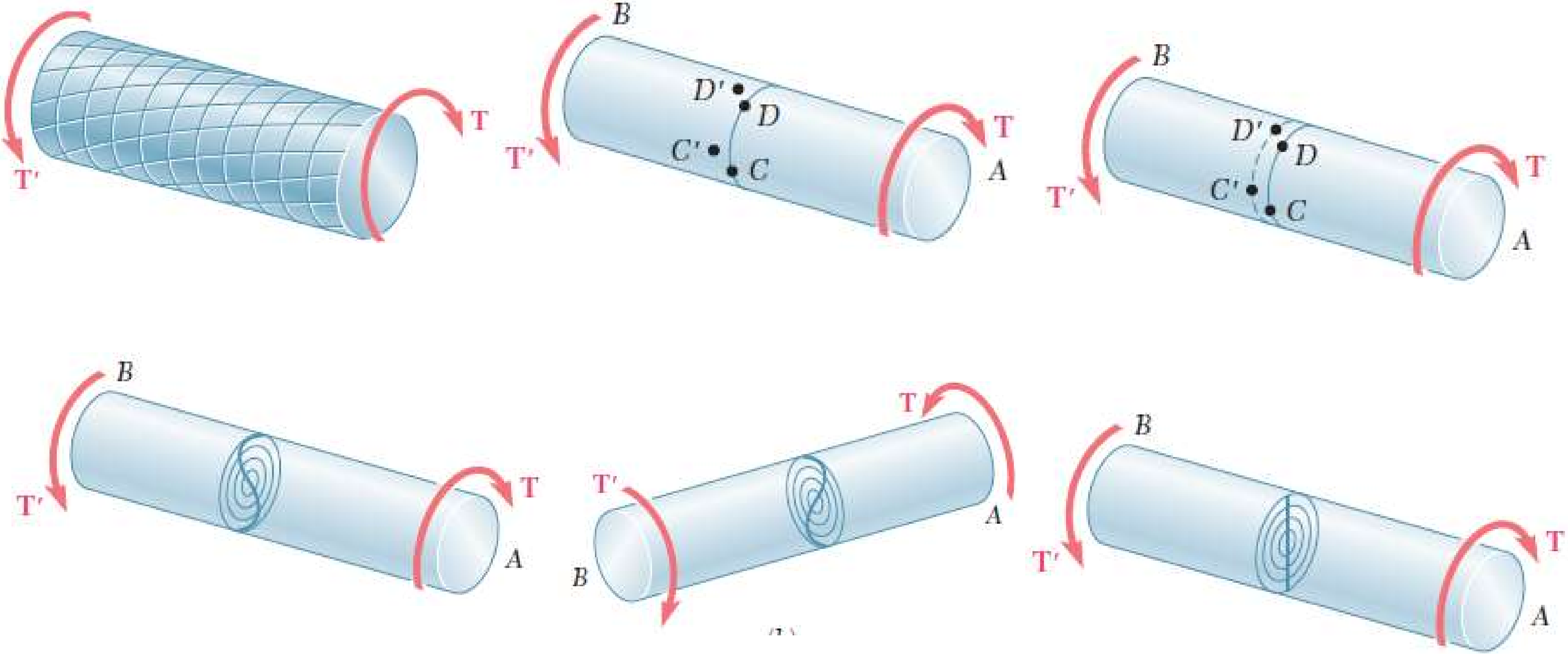
$$\int \rho (\tau \, dA) = T$$

τ is the shearing stress

The actual distribution of stresses under a given load is statically indeterminate, i.e., this distribution cannot be determined by the methods of statics.

Deformations in a Circular Shaft

When a circular shaft is subjected to torsion, every cross section remains plane and undistorted.



Let us now determine the distribution of *shearing strains* in a circular shaft of length *L* and radius *c* that has been twisted through an angle *θ*

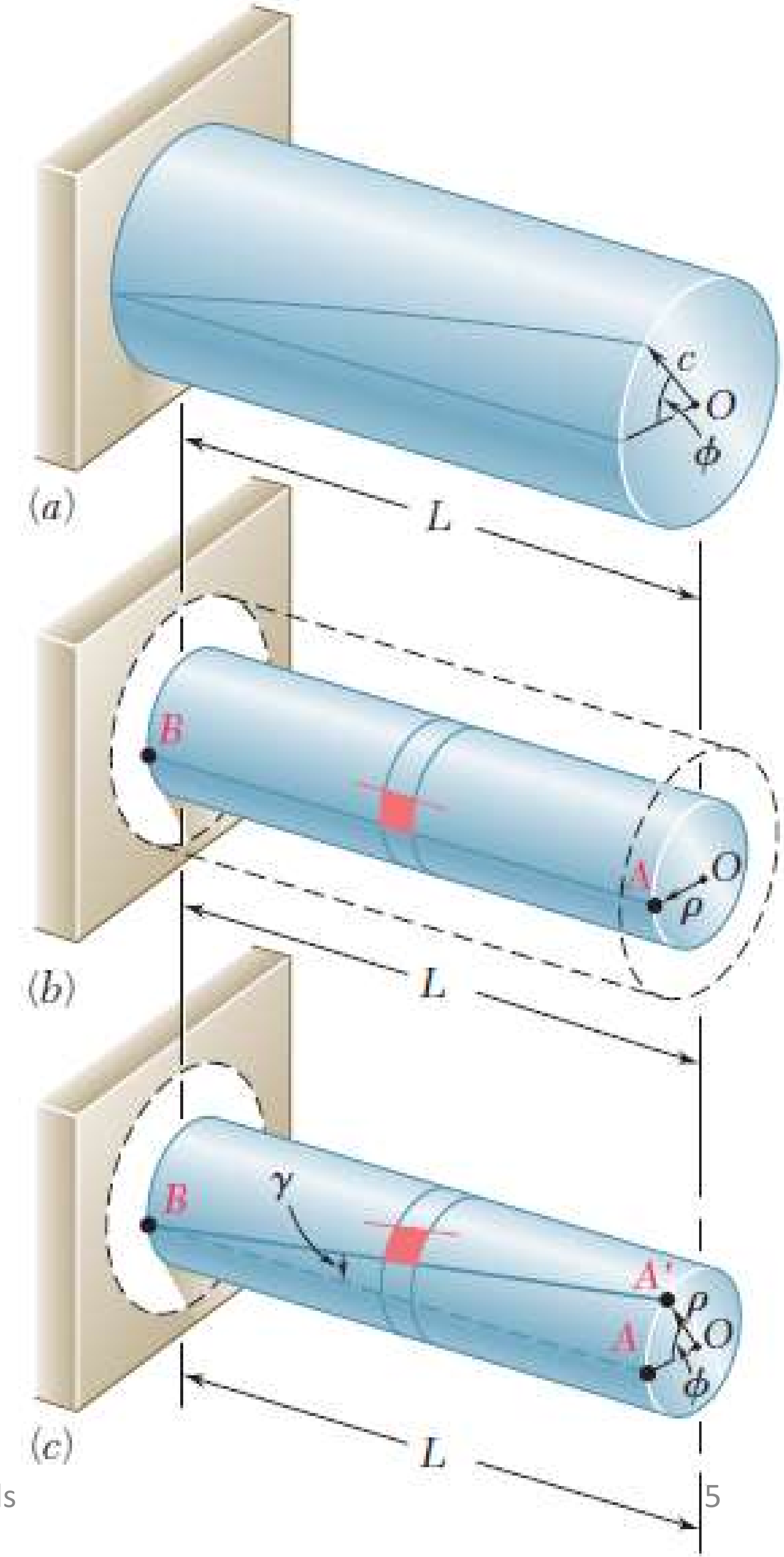
Since the circles defining two of the sides of the element considered here remain unchanged, the shearing strain γ must be equal to the angle between lines *AB* and *A'B*.

$AA' = L\gamma \qquad AA' = \rho\varphi \qquad \therefore L\gamma = \rho\varphi$

$$\gamma = \frac{\rho\phi}{L}$$

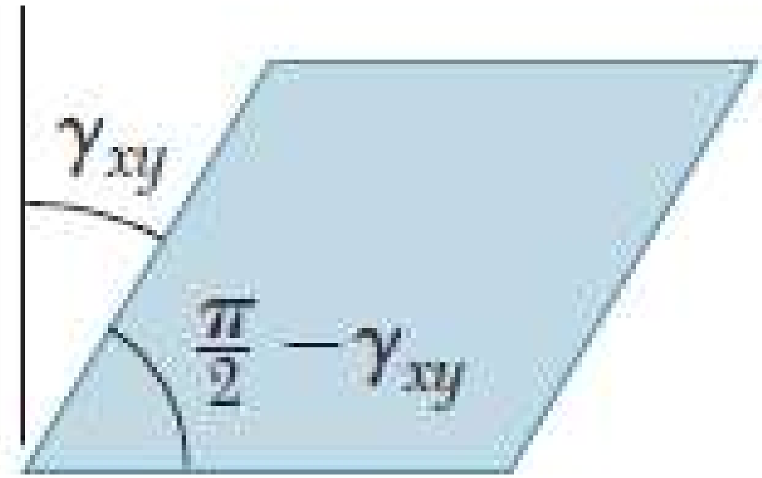
$$\gamma_{\max} = \frac{c\phi}{L}$$

$$\gamma = \frac{\rho}{c}\gamma_{\max}$$

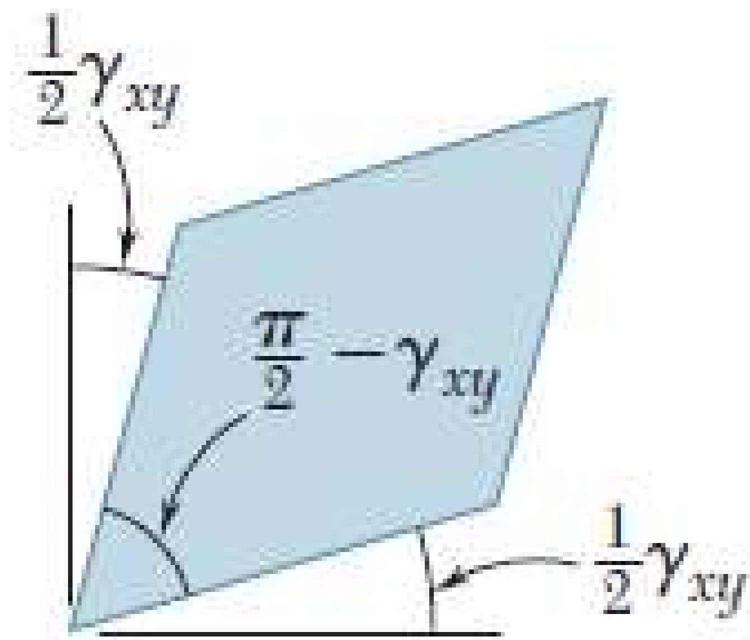


Definition of Shear Strain

Change in the right angle of the element measured in radians



Change in the angle formed by two faces



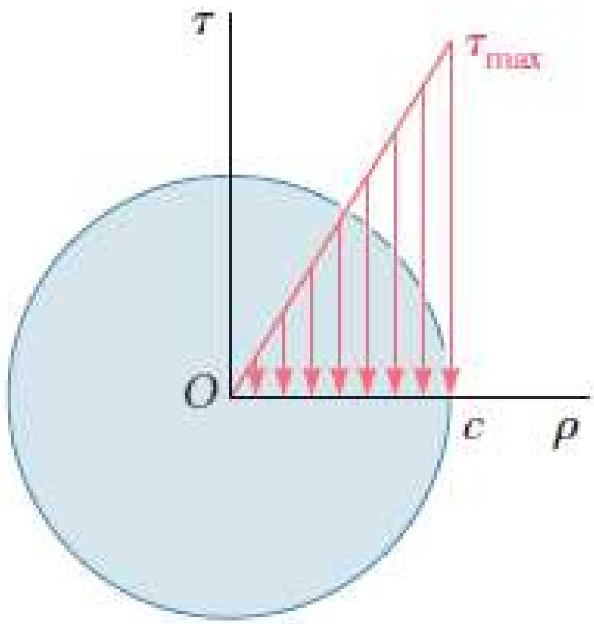
Using Hooke's law for shearing stress and strain,

$$\tau = G\gamma$$

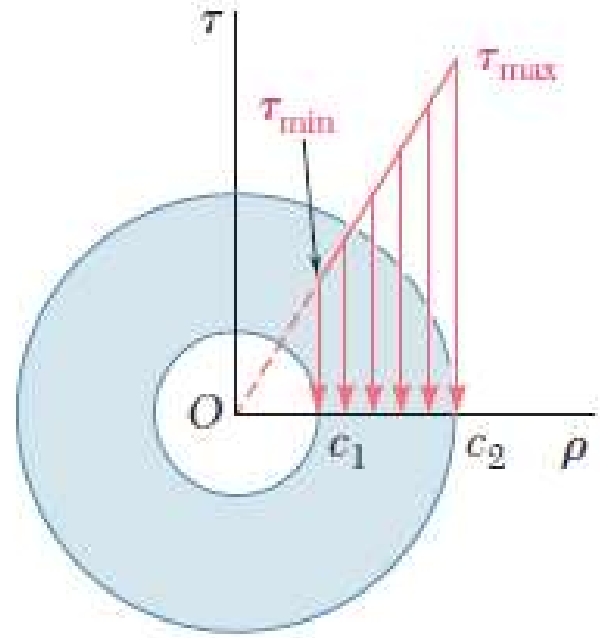
where ***G*** is the modulus of rigidity or shear modulus of the material

$$G\gamma = \frac{\rho}{c} G\gamma_{\max}$$

$$\tau = \frac{\rho}{c}\tau_{\max}$$



Solid Shaft



Hollow Shaft

The shearing stress in the shaft varies linearly with the distance ρ from the axis of the shaft

$$\tau_{\min} = \frac{c_1}{c_2}\tau_{\max}$$

$$T = \int \rho\tau \, dA = \frac{\tau_{\max}}{c} \int \rho^2 \, dA$$

$$J = \int \rho^2 \, dA$$

$$T = \frac{\tau_{\max} J}{c}$$

where J is the polar moment of inertia of the cross section

$$\tau_{\max} = \frac{Tc}{J}$$

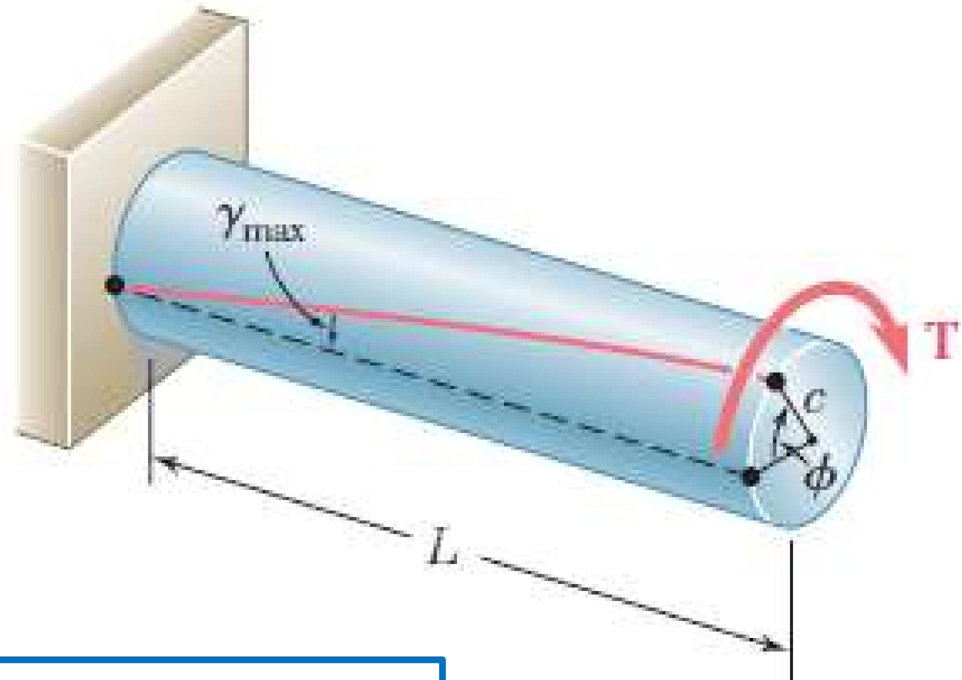
$$\tau = \frac{T\rho}{J}$$

The above equations are known as the *elastic torsion formulas*

Angle of Twist (φ) in the Elastic Range

$$\gamma_{max} = \frac{c\phi}{L}$$

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$$



$$\boxed{\phi = \frac{TL}{JG}}$$

$$\boxed{\frac{T}{J} = \frac{\tau}{\rho} = \frac{G\varphi}{L}}$$

Within the elastic range, the angle of twist **φ** is proportional to the torque T applied to the shaft

Torque

$$\frac{T}{J} = \frac{\tau}{r} \qquad T = \frac{\tau J}{r}$$

Solid Shaft

$$J = \frac{\pi}{32} d^4 \qquad T = \frac{\tau \left[\frac{\pi}{32} d^4\right]}{\frac{d}{2}} \qquad \boxed{T = \frac{\pi}{16} \tau d^3}$$

Hollow Shaft

$$J = \frac{\pi}{32} [D^4 - d^4] \qquad T = \frac{\tau \left[\frac{\pi}{32} (D^4 - d^4)\right]}{\frac{D}{2}}$$

$$\boxed{T = \frac{\pi}{16} \tau \left[\frac{(D^4 - d^4)}{D}\right]}$$

Power (kW)

$$P = T\omega \qquad \omega = \frac{2\pi N}{60}$$

$$\boxed{P = \frac{2\pi NT}{60}}$$

where
T – Twisting Moment or Torque transmitted
N – rpm

Problems

1) Stresses and deformations in circular and hollow shafts

2) Stepped shafts

3) Shafts fixed at the both ends

Stresses and deformations in circular and hollow shafts

Problem 1

A shaft transmits 100 kW of power at 150 rpm. Determine the suitable diameter of the shaft if the maximum torque transmitted exceeds the mean by 20% in each revolution. The shear stress is not to exceed 60 MPa. Also maximum angle of twist in a length of 4 m of the shaft. G = 80 GPa.

$$\boxed{d = 86.6 \text{ mm}}$$

$$\boxed{\varphi = 0.06928 \text{ rad or } 3.97^\circ}$$

Solution

$$G = 80 \text{ GPa} = 80\,000 \text{ N/mm}^2;\ \tau = 60 \text{ MPa} = 60 \text{ N/mm}^2$$

$$P_{mean} = T.\omega$$

or $$100\,000 = T_{mean} \times \frac{2\pi \times 150}{60} \quad \text{or} \quad T_{mean} = 6366 \text{ N.m}$$

$$T_{max} = 6366 \times 1.2 = 7639 \text{ N.m} \quad \text{or} \quad 7.639 \times 10^6 \text{ N.mm}$$

Now, $$\tau = \frac{16T}{\pi d^3} \quad \text{or} \quad 60 = \frac{16 \times 7.639 \times 10^6}{\pi d^3} \quad \text{or} \quad d = 86.6 \text{ mm}$$

Also $$\frac{\tau}{r} = \frac{G\theta}{l} \quad \text{or} \quad \frac{60}{86.6/2} = \frac{80\,000 \times \theta}{4000}$$

or $$\theta = 0.069\,28 \text{ rad} = (0.069\,28 \times 180/\pi)^\circ = 3.97^\circ$$

Problem 2

A hollow steel shaft transmits 200 kW of power at 180 rpm. The total angle of twist in a length of 5 m of the shaft is 3°. Find the inner and outer diameters of the shaft if the permissible shear stress is 60 MPa. G = 80 GPa.

$\boxed{D = 143.2 \text{ mm}}$

$\boxed{d = 130.7 \text{ mm}}$

Solution

$$\theta = 3 \times \frac{\pi}{180} = 0.0524 \text{ rad}$$

$$P = T.\omega \quad \text{or} \quad 200\,000 = T \times \frac{2\pi \times 180}{60}$$

or $$T = 10\,610 \text{ N.m} \quad \text{or} \quad 10.61 \times 106 \text{ N.mm}$$

$$\frac{T}{J} = \frac{G\theta}{l} \quad \text{or} \quad \frac{10.61 \times 10^6}{\pi(D^4 - d^4)/32} = \frac{80\,000 \times 0.0524}{5000}$$

or $$D^4 - d^4 = 129 \times 10^6 \text{ mm}^4 \qquad \text{(i)}$$

Also, $$T = \tau.\frac{\pi}{16}\left(\frac{D^4 - d^4}{D}\right) \quad \text{or} \quad 10.61 \times 10^6 = 60 \times \frac{\pi}{16}\left(\frac{D^4 - d^4}{D}\right)$$

or $$D^4 - d^4 = 900\,605D \qquad \text{(ii)}$$

From (i) and (ii), $900\,605D = 129 \times 10^6$ or $D = 143.2$ mm

and $143.2^4 - d^4 = 129 \times 10^6$ or $d = 130.7$ mm

Problem 3

Compare the weights of equal lengths of a solid and a hollow shaft to transmit a given torque for the same maximum stress if the inside diameter of the shaft is three fourth of the outside.

$\boxed{\text{Ratio of the weights of same length} = (W_h / W_s) = 0.564}$

Sol. Given :

Dia. of hollow shaft, $D_i = \frac{3}{4}$ Dia. at outside

$$= \frac{3}{4} D_0 = 0.75\, D_0$$

Let L = Length of both shaft (equal length),

T = Torque transmitted by each shaft (equal torque),

τ = Maximum shear stress developed in each shaft (equal max. shear stress),

D = Dia. of solid shaft,

W_s = Weight of solid shaft, and

W_h = Weight of hollow shaft.

Torque transmitted by a solid shaft is given by equation (16.4) as

$$T = \frac{\pi}{16} \times \tau \times D^3 \qquad \ldots(i)$$

Torque transmitted by a hollow shaft is given by equation (16.6) as

$$T = \frac{\pi}{16} \times \tau \times \left[\frac{D_0^4 - D_i^4}{D_0}\right] = \frac{\pi}{16} \times \tau \times \left[\frac{D_0^4 - (0.75\, D_0)^4}{D_0}\right]$$

$$= \frac{\pi}{16} \times \tau \times \frac{D_0^4}{D_0}\,[1 - (0.75)^4] = \frac{\pi}{16} \times \tau \times D_0^3 \times [1 - 0.3164]$$

$$= \frac{\pi}{16} \times \tau \times D_0^3 \times 0.6836 \qquad \text{...(ii)}$$

But torque transmitted by solid shaft

= Torque transmitted by hollow shaft.

Hence equating equations (*i*) and (*ii*), we get

$$\frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times \tau \times D_0^3 \times 0.6836 \quad \text{or} \quad D^3 = 0.6836\, D_0^3$$

$$\therefore \qquad D = (0.6836)^{1/3} \times D_0 \quad \text{or} \quad D = 0.8809\, D_0 \qquad \text{...(iii)}$$

Now weight of solid shaft,

$$W_s = \rho \times g \times \text{Volume of solid shaft}$$

$$= \rho \times g \times \left(\frac{\pi}{4} D^2 \times L\right) \qquad \text{...(iv)}$$

Weight of hollow shaft,

$$W_h = \rho \times g \times \text{Volume of hollow shaft}$$

$$= \rho \times g \times \left[\frac{\pi}{4}(D_0^2 - D_i^2) \times L\right] = \rho \times g \times \frac{\pi}{4}\,[D_0^2 - (0.75D_0)^2] \times L$$

$$= \rho \times g \times \frac{\pi}{4}\,[D_0^2 - 0.5625D_0^2] \times L = \rho \times g \times \frac{\pi}{4} \times D_0^2(1 - 0.5625) \times L$$

$$= \rho \times g \times \frac{\pi}{4} \times D_0^2 \times 0.4375 \times L \qquad \text{...(v)}$$

Dividing equation (*iv*) by equation (*v*),

$$\frac{W_s}{W_h} = \frac{\rho \times g \times \frac{\pi}{4} \times D^2 \times L}{\rho \times g \times \frac{\pi}{4} D_0^{\ 2} \times 0.4375 \times L} = \frac{D^2}{0.4375\, D_0^{\ 2}}$$

$$= \frac{(0.8809\, D_0)^2}{0.4375\, D_0^{\ 2}} \qquad [\because \text{ From } (iii),\ D = 0.8809\, D_0]$$

$$= \frac{0.776\, D_0^{\ 2}}{0.4375\, D_0^{\ 2}} = 1.7737. \quad \textbf{Ans.}$$

Problem 4

A shaft transmits 280 kW of power at 160 rpm. Determine

i. The diameter of a solid shaft to transmit the required power
ii. The inner and outer diameters of hollow circular shaft if the ratio of the inner to the outer diameter is 2/3
iii. The percentage saving in the material on using a hollow shaft instead of a solid shaft

Take the allowable stress as 80 MPa and the density of material 78 kN/m^3.

$\boxed{\text{(i) } d = 102 \text{ mm}}$ $\boxed{\text{(ii) } D = 109.8 \text{ mm; } d = 73.2 \text{ mm}}$

$\boxed{\text{(iii) } \textit{Percent saving in material} = 35.6}$

Problem 16.9. *A solid cylindrical shaft is to transmit 300 kW power at 100 r.p.m.*

(a) If the shear stress is not to exceed 80 N/mm², find its diameter.

(b) What percent saving in weight would be obtained if this shaft is replaced by a hollow one whose internal diameter equals to 0.6 of the external diameter, the length, the material and maximum shear stress being the same?

Sol. Given :

Power, $P = 300 \text{ kW} = 300 \times 10^3 \text{ W}$

Speed, $N = 100$

Max. shear stress, $\tau = 80 \text{ N/mm}^2$

(a) Let D = Dia. of solid shaft

$$P = \frac{2\pi NT}{60}$$

$$300 \times 10^3 = \frac{2\pi \times 100 \times T}{60}$$

$$\therefore \quad T = \frac{300 \times 10^3 \times 60}{2\pi \times 100} = 28647.8 \text{ Nm} = 28647800 \text{ Nmm}$$

Now using equation (16.4),

$$T = \frac{\pi}{16} \times \tau \times D^3 \quad \text{or} \quad 28647800 = \frac{\pi}{16} \times 80 \times D^3$$

$$\therefore \quad D = \left(\frac{16 \times 28647800}{\pi \times 80}\right)^{1/3} = 121.8 \text{ mm}$$

$$= \textbf{say 122.0 mm.} \quad \textbf{Ans.}$$

(b) Percent saving in weight

Let D_0 = External dia. of hollow shaft

D_i = Internal dia. of hollow shaft

$= 0.6 \times D_0$. (given)

The length, material and maximum shear stress in solid and hollow shafts are given the same. Hence torque transmitted by solid shaft is equal to the torque transmitted by hollow shaft. But the torque transmitted by hollow shaft is given by equation (16.6).

$\therefore$ Using equation (16.6),

$$T = \frac{\pi}{16} \times \tau \times \frac{(D_0^4 - D_i^4)}{D_0}$$

$$= \frac{\pi}{16} \times 800 \times \frac{[D_0^4 - (0.6\, D_0)^4]}{D_0} \qquad (\because \; D_i = 0.6\, D_0)$$

$$= \pi \times 50 \times \frac{[D_0^4 - (0.6\, D_0)^4]}{D_0}$$

But torque transmitted by solid shaft

$= 28647800$ Nmm

$\therefore$ Equating the two torques, we get

$$28647800 = \pi \times 50 \times \left(\frac{0.8704\, D_0^4}{D_0}\right) = \pi \times 50 \times 0.8704\, D_0^3$$

$$\therefore \quad D_0 = \left(\frac{28647800}{\pi \times 50 \times 0.8704}\right)^{1/3} = 127.6 \text{ mm} = \text{say } 128 \text{ mm}$$

$\therefore$ Internal dia. $D_i = 0.6 \times D_0 = 0.6 \times 128 = 76.8$ mm

Now let W_s = Weight of solid shaft,

and W_h = Weight of hollow shaft.

Then W_s = Weight density × Area of solid shaft × Length

$$= w \times \frac{\pi}{4} D^2 \times L \qquad \text{(where } w = \text{weight density)}$$

Similarly W_h = Weight density × Area of hollow shaft × Length

$$= w \times \frac{\pi}{4} [D_0^2 - D_i^2] \times L$$

($\because$ Both shafts are of same lengths and of same material)

Now percent saving in weight

$$= \frac{W_s - W_h}{W_s} \times 100$$

$$= \frac{w \times \frac{\pi}{4} D^2 \times L - w \times \frac{\pi}{4} [D_0^2 - D_i^2] \times L}{w \times \frac{\pi}{4} D^2 \times L} \times 100$$

$$= \frac{D^2 - (D_0^2 - D_i^2)}{D^2} \times 100 \qquad \left(\text{Cancelling } w \times \frac{\pi}{4} \times L\right)$$

$$= \frac{122^2 - (128^2 - 75.8^2)}{122^2} \times 100 = \frac{14884 - (16364 - 5898)}{14884} \times 100$$

$$= \frac{14884 - 10486}{14884} \times 100 = \mathbf{29.55\%}. \quad \textbf{Ans.}$$

Problem 16.13. *Determine the diameter of a solid steel shaft which will transmit 90 kW at 160 r.p.m. Also determine the length of the shaft if the twist must not exceed 1° over the entire length. The maximum shear stress is limited to 60 N/mm². Take the value of modulus of rigidity = 8 × 10⁴ N/mm².*

Sol. Given :

Power, $P = 90 \text{ kW} = 90 \times 10^3 \text{ W}$

Speed, $N = 160$ r.p.m.

Angle of twist, $\theta = 1° \quad \text{or} \quad \frac{\theta}{180} \text{ radian}$ $\left(\because \; 1° = \frac{\pi}{180} \text{ radian}\right)$

Max. shear stress, $\tau = 60 \text{ N/mm}^2$

Modulus of rigidity, $C = 8 \times 10^4 \text{ N/mm}^2$

Let D = Diameter of the shaft and

L = Length of the shaft.

(i) Diameter of the shaft

Using equation (16.7),

$$P = \frac{2\pi NT}{60}$$

or

$$90 \times 10^3 = \frac{2\pi \times 160 \times T}{60}$$

$\therefore$

$$T = \frac{90 \times 10^3 \times 60}{2\pi \times 160} = 5371.48 \text{ N-m} = 5371.48 \times 10^3 \text{ N-mm}$$

$2\pi \times 160$

Now using equation (16.4),

$$T = \frac{\pi}{16} \tau D^3$$

or $$5371.48 \times 10^3 = \frac{\pi}{16} \times 60 \times D^3$$

$\therefore$ $$D^3 = \frac{5371.48 \times 10^3 \times 16}{\pi \times 60} = 455945$$

$\therefore$ $$D = (455945)^{1/3} = \mathbf{76.8\ mm.} \quad \textbf{Ans.}$$

(*ii*) *Length of the shaft*

Using equation (16.7),

$$\frac{\tau}{R} = \frac{C\theta}{L}$$

or $$\frac{60}{\left(\frac{76.8}{2}\right)} = \frac{8 \times 10^4 \times \pi}{L \times 180} \qquad \left(\because \ R = \frac{D}{2} = \frac{76.8}{2} \text{ mm}, \theta = \frac{\pi}{180} \text{ radian}\right)$$

or $$L = \frac{8 \times 10^4 \times \pi \times 76.8}{60 \times 180 \times 2} = \mathbf{893.6\ mm.} \quad \textbf{Ans.}$$

Shafts in Series and Parallel

Shafts in Series

$$T = \frac{J_1\,\tau_1}{r_1} = \frac{J_2\,\tau_2}{r_2} \qquad T = \frac{G_1 J_1\,\varphi_1}{L_1} = \frac{G_2 J_2\,\varphi_2}{L_2}$$

$$\varphi = \frac{T L_1}{G_1 J_1} + \frac{T L_2}{G_2 J_2}$$

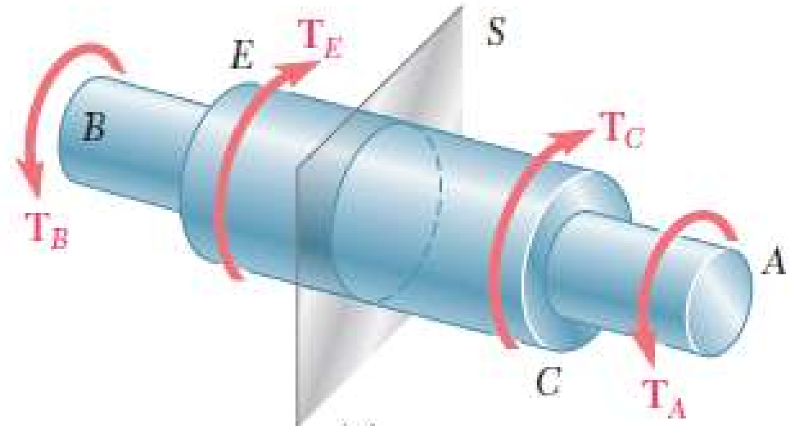
Shafts in Parallel

$$T = T_1 + T_2 = \frac{G_1 J_1\,\varphi_1}{L_1} + \frac{G_2 J_2\,\varphi_2}{L_2}$$

$$\varphi_1 = \varphi_2$$

Shafts with Variable Cross Section

- The torsion formulas are derived for a shaft of uniform circular cross section subjected to torques at its ends
- However, they can also be used for a shaft of variable cross section or for a shaft subjected to torques at locations other than its ends
- The value of *T* is obtained by drawing the free-body diagram of the portion and the sum of torques is zero



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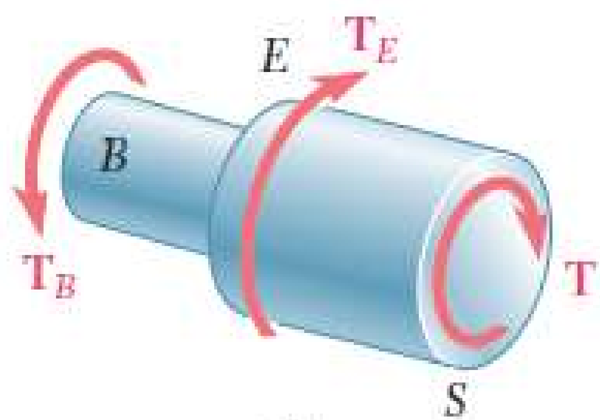
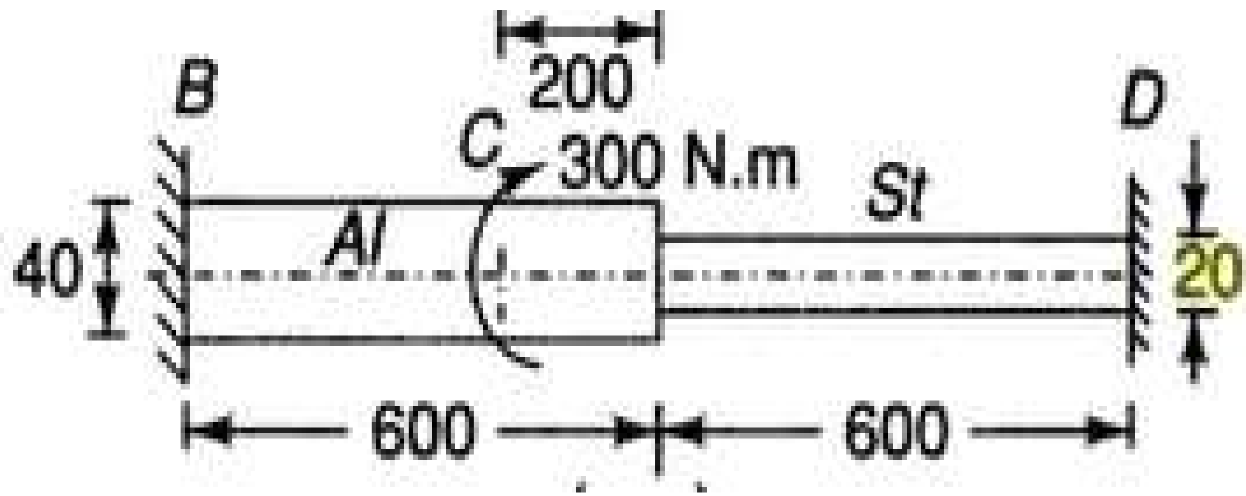


(b)

Example 10.11 *A stepped steel shaft fixed at the two ends as shown in Fig. 10.5 is subjected to a torque of 300 N.m at section C. Determine the maximum stresses in the two materials.* $G_s = 82$ *GPa and* $G_a = 27$ *GPa.*



The fixing torque at $B = (300 - T_d)$ N.m

$$J_a = \frac{\pi}{32} \times 40^4$$
$$= 80\,000\pi \text{ mm}^4$$

and
$$J_s = \frac{\pi}{32} \times 20^4 = 5000\pi \text{ mm}^4$$

Thus
$$\theta = \frac{(300 - T_d) \times (600 - 200)}{G_a J_a} = \frac{T_d \times 200}{G_a J_a} + \frac{T_d \times 600}{G_s J_s}$$

or
$$\frac{120\,000}{G_a J_a} = \frac{600 T_d}{G_a J_a} + \frac{600 T_d}{G_s J_s}$$

or
$$\frac{120\,000}{27\,000 \times 80\,000\pi} = 600\ T_d \left(\frac{1}{27\,000 \times 80\,000\pi} + \frac{1}{82\,000 \times 5000\pi} \right)$$

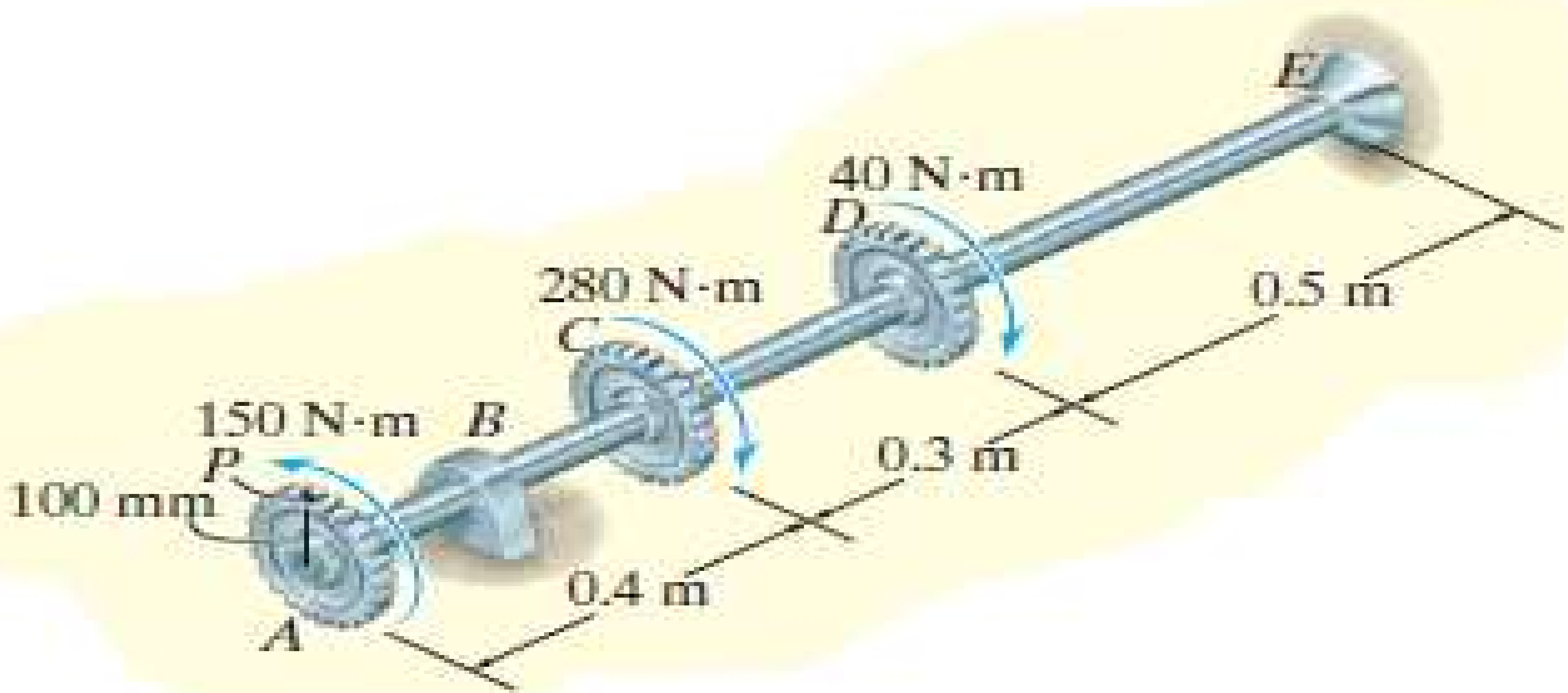
$$55.555 = 1.741\ T_d \quad \text{or} \quad T_d = 31.9 \text{ N.m}$$

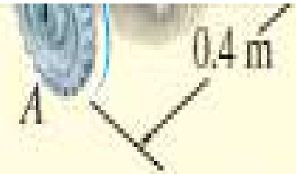
and
$$T_b = 300 - 31.6 = 268.1 \text{ N.m}$$

Now, $$\sigma_a = \frac{16T}{\pi d^3} = \frac{16 \times 268.1 \times 10^3}{\pi \times 40^3} = 21.3 \text{ MPa}$$

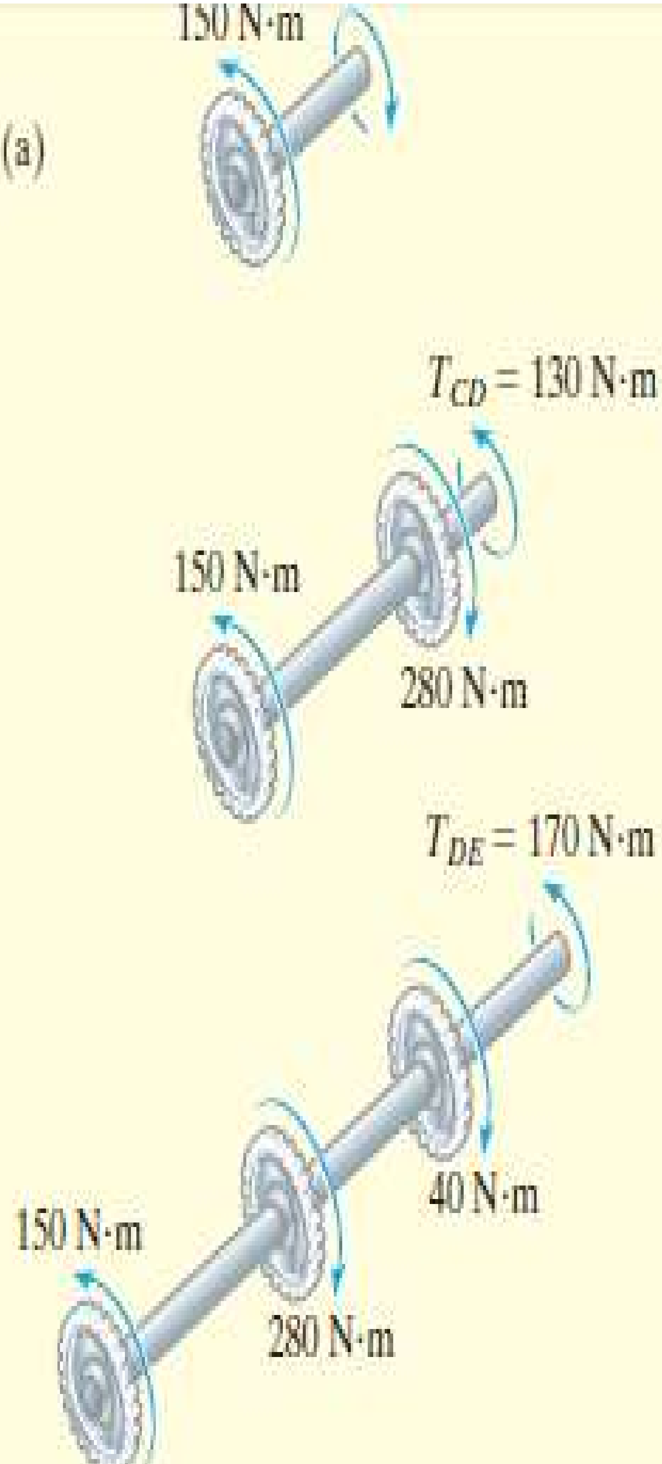
and $$\sigma_s = \frac{16 \times 31.9 \times 10^3}{\pi \times 20^3} = 20.3 \text{ MPa}$$

The gears attached to the fixed-end steel shaft are subjected to the torques shown in Fig. 5–19*a*. If the shear modulus of elasticity is 80 GPa and the shaft has a diameter of 14 mm, determine the displacement of the tooth *P* on gear *A*. The shaft turns freely within the bearing at *B*.





(a)



SOLUTION

Internal Torque. By inspection, the torques in segments AC, CD, and DE are different yet *constant* throughout each segment. Free-body diagrams of appropriate segments of the shaft along with the calculated internal torques are shown in Fig. 5–19*b*. Using the right-hand rule and the established sign convention that positive torque is directed away from the sectioned end of the shaft, we have

$$T_{AC} = +150\ \text{N}\cdot\text{m} \qquad T_{CD} = -130\ \text{N}\cdot\text{m} \qquad T_{DE} = -170\ \text{N}\cdot\text{m}$$

These results are also shown on the torque diagram, Fig. 5–19*c*.

Angle of Twist. The polar moment of inertia for the shaft is

$$J = \frac{\pi}{2}(0.007\ \text{m})^4 = 3.771(10^{-9})\ \text{m}^4$$

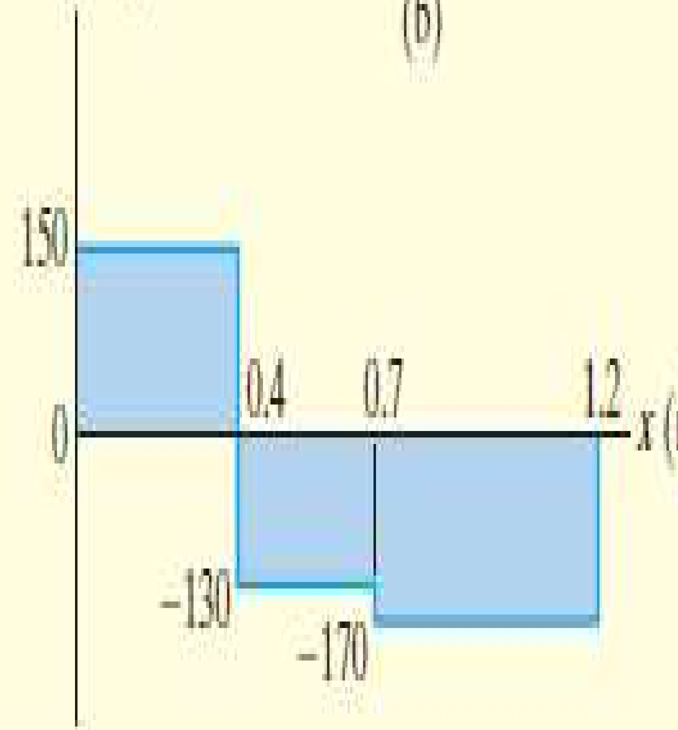
Applying Eq. 5–16 to each segment and adding the results algebraically, we have

(b)

$$\phi_A = \sum \frac{TL}{JG} = \frac{(+150\ \mathrm{N \cdot m})(0.4\ \mathrm{m})}{3.771(10^{-9})\ \mathrm{m}^4 [80(10^9)\ \mathrm{N/m}^2]}$$

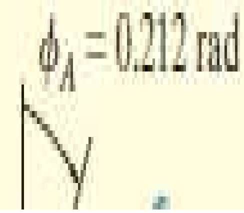
$$+ \frac{(-130\ \mathrm{N \cdot m})(0.3\ \mathrm{m})}{3.771(10^{-9})\ \mathrm{m}^4 [80(10^9)\ \mathrm{N/m}^2)]}$$

$$+ \frac{(-170\ \mathrm{N \cdot m})(0.5\ \mathrm{m})}{3.771(10^{-9})\ \mathrm{m}^4 [80(10^9)\ \mathrm{N/m}^2)]} = -0.2121\ \mathrm{rad}$$



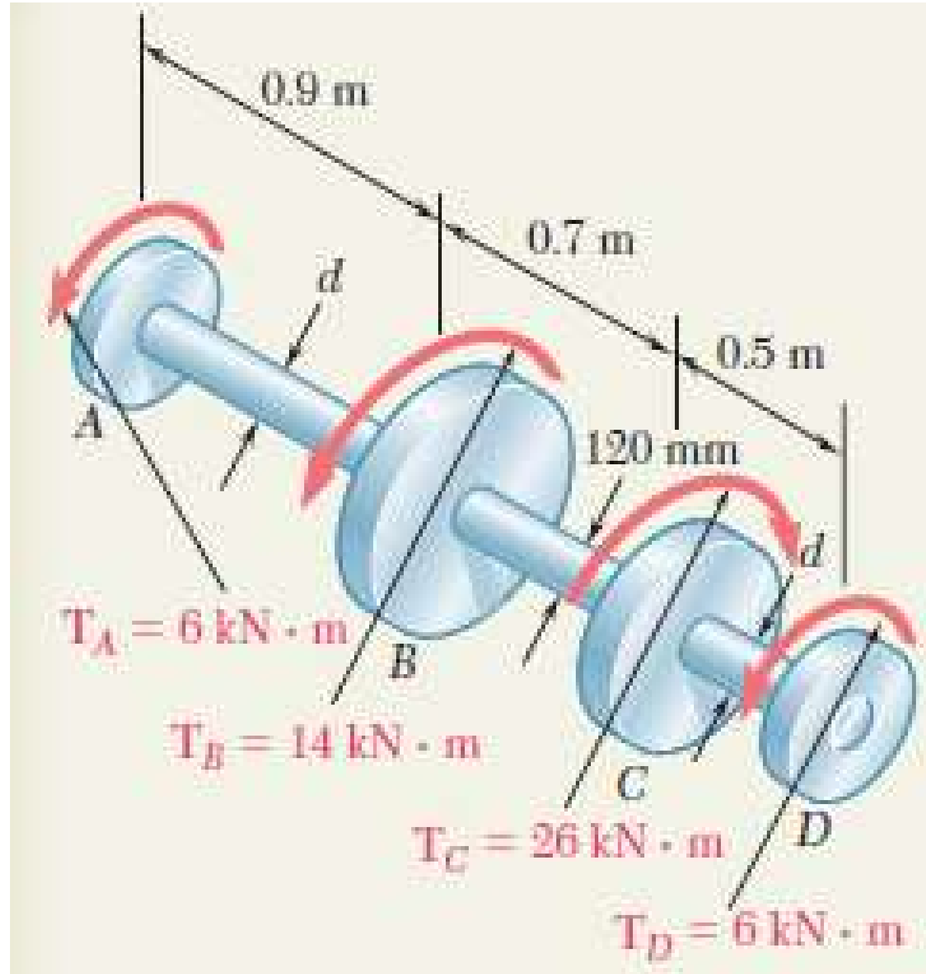
(c)

Since the answer is negative, by the right-hand rule the thumb is directed *toward* the end *E* of the shaft, and therefore gear *A* will



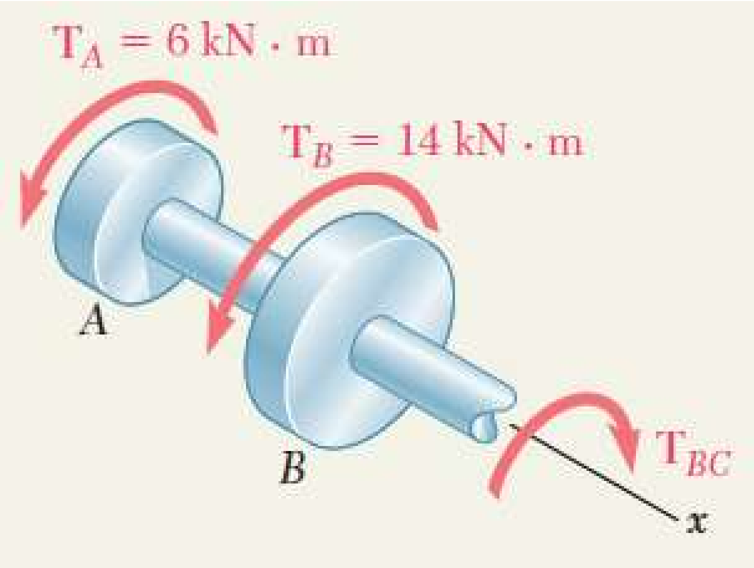
Problem 5

Shaft *BC* is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts *AB* and *CD* are solid and of diameter *d*. For the loading shown, determine (*a*) the maximum and minimum shearing stress in shaft *BC*, (*b*) the required diameter *d* of shafts *AB* and *CD* if the allowable shearing stress in these shafts is 65 MPa.



Solution

Shaft BC



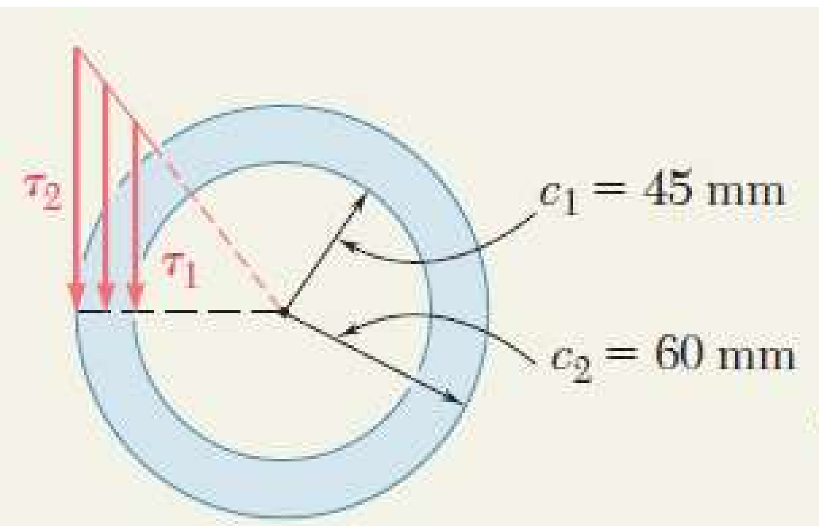
$$\Sigma M_x = 0:$$

$$(6\text{ kN}\cdot\text{m}) + (14\text{ kN}\cdot\text{m}) - T_{BC} = 0$$

$$T_{BC} = 20\text{ kN}\cdot\text{m}$$

$$J = \frac{\pi}{2}(c_2^4 - c_1^4) = \frac{\pi}{2}[(0.060)^4 - (0.045)^4]$$

$$= 13.92 \times 10^{-6}\text{ m}^4$$



Maximum Shearing Stress

$$\tau_{\max} = \tau_2 = \frac{T_{BC}c_2}{J} = \frac{(20\text{ kN}\cdot\text{m})(0.060\text{ m})}{13.92\times 10^{-6}\text{ m}^4}$$

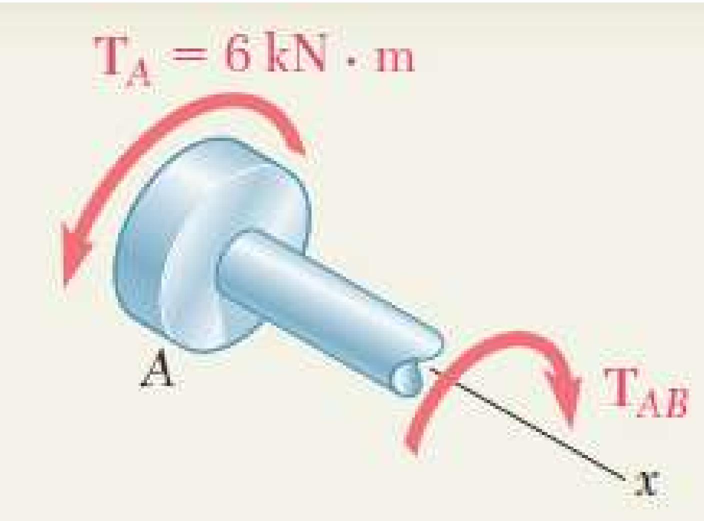
$$\tau_{\max} = 86.2\text{ MPa}$$

Minimum Shearing Stress

$$\frac{\tau_{\min}}{\tau_{\max}} = \frac{c_1}{c_2}$$

$$\frac{\tau_{\min}}{86.2\text{ MPa}} = \frac{45\text{ mm}}{60\text{ mm}}$$

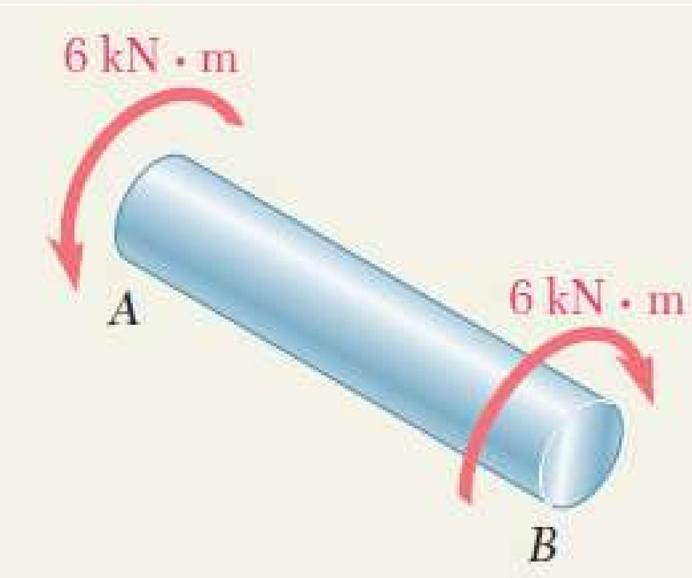
$$\tau_{\min} = 64.7\text{ MPa}$$



Shaft *AB*

$$\Sigma M_x = 0:$$

$$(6\text{ kN}\cdot\text{m}) - T_{AB} = 0 \qquad T_{AB} = 6\text{ kN}\cdot\text{m}$$



Shafts *AB* and *CD*

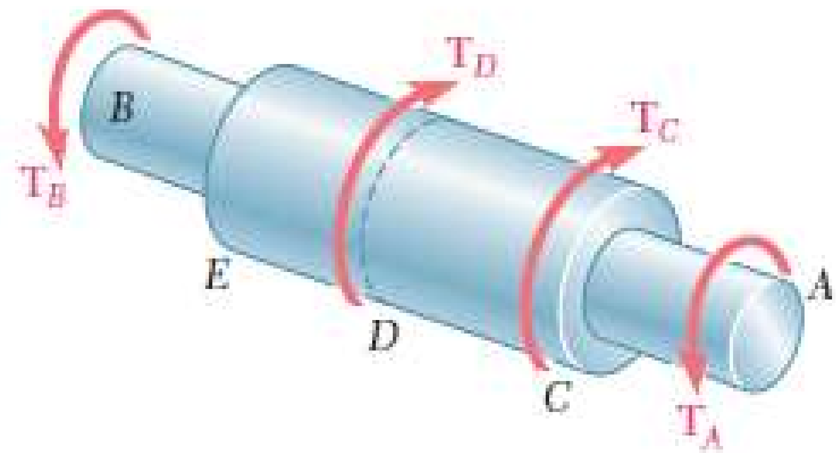
$$T = 6\text{ kN}\cdot\text{m and } \tau_{\text{all}} = 65\text{ MPa.}$$

$$\tau = \frac{Tc}{J} \qquad 65\text{ MPa} = \frac{(6\text{ kN}\cdot\text{m})c}{\frac{\pi}{2}c^4}$$

$$d = 2c = 2(38.9\text{ mm}) \qquad d = 77.8\text{ mm}$$

Angle of Twist for Multiple Sections and Multiple Torques

$$\phi = \frac{TL}{JG}$$

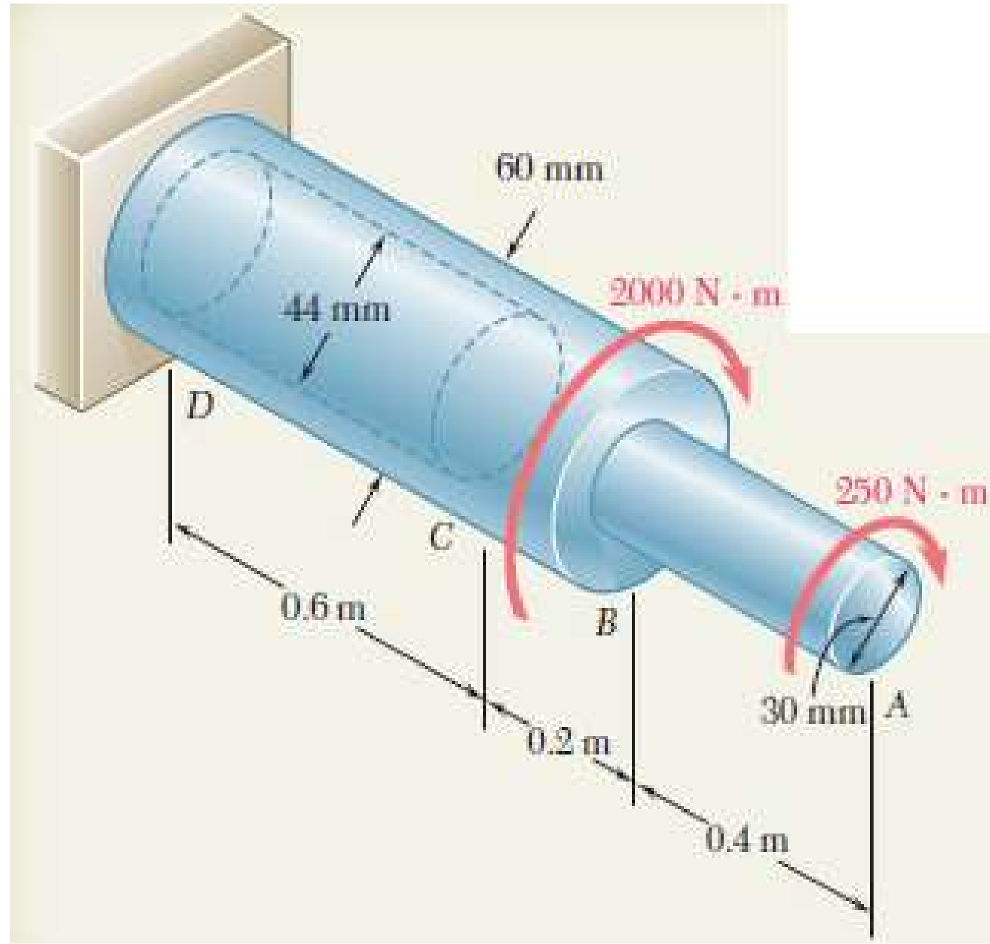


- The above formula can be used only if the shaft is homogeneous (constant *G*), has a uniform cross section, and is loaded only at its ends
- For the figure shown, we must divide it into component parts that satisfy individually the required conditions for the application of the above formula, and the angle of twist is

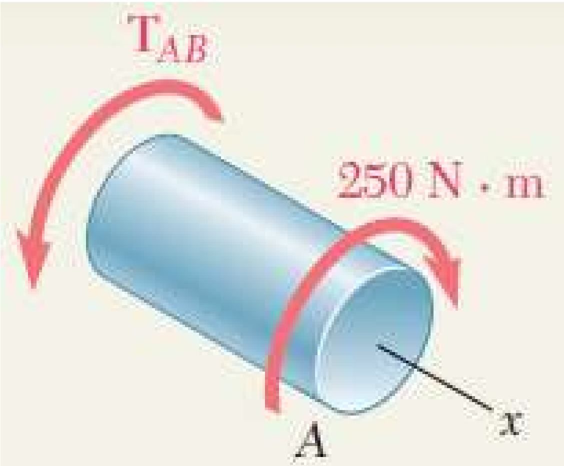
$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

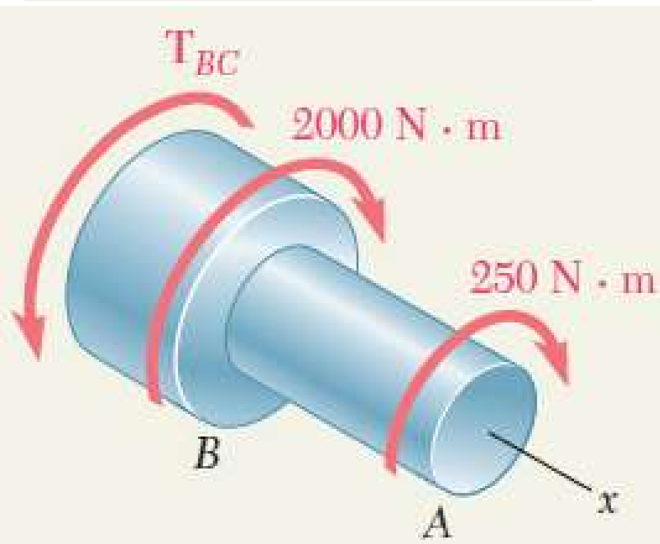
Problem 6

The horizontal shaft *AD* is attached to a fixed base at *D* and is subjected to the torques shown. A 44 mm diameter hole has been drilled into portion *CD* of the shaft. Knowing that the entire shaft is made of steel for which G = 77 GPa, determine the angle of twist at end *A*.



Solution





$$\Sigma M_x = 0:$$

$$(250\ \text{N}\cdot\text{m}) - T_{AB} = 0$$

$$T_{AB} = 250\ \text{N}\cdot\text{m}$$

$$(250\ \text{N}\cdot\text{m}) + (2000\ \text{N}\cdot\text{m}) - T_{BC} = 0$$

$$T_{BC} = 2250\ \text{N}\cdot\text{m}$$

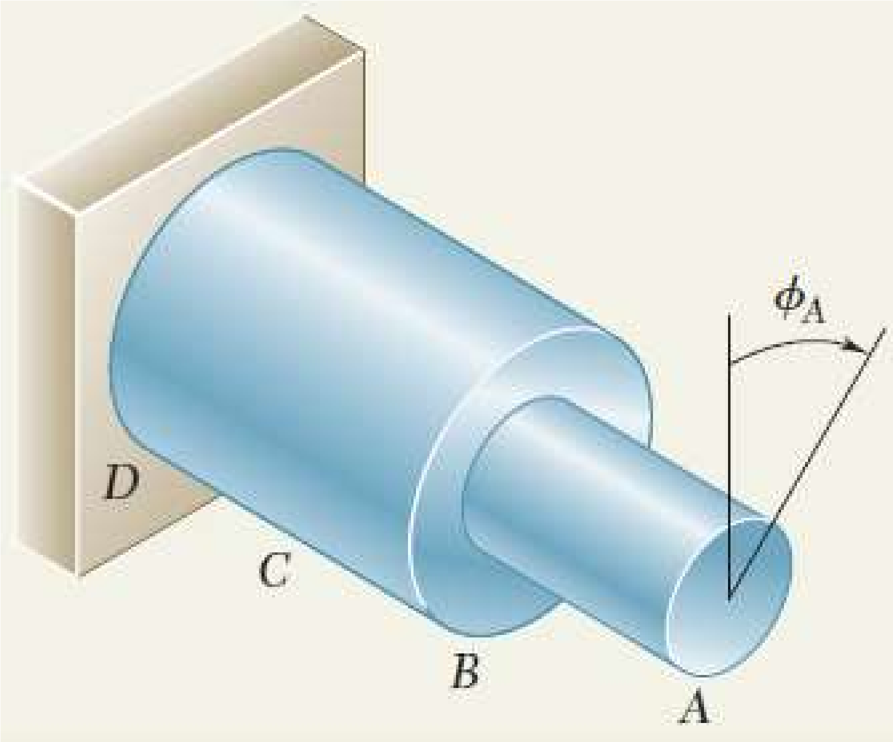
$$T_{CD} = T_{BC} = 2250\ \text{N}\cdot\text{m}$$

$$J_{AB} = \frac{\pi}{2}c^4 = \frac{\pi}{2}(0.015\ \text{m})^4 = 0.0795 \times 10^{-6}\ \text{m}^4$$

$$J_{BC} = \frac{\pi}{2}c^4 = \frac{\pi}{2}(0.030\ \text{m})^4 = 1.272 \times 10^{-6}\ \text{m}^4$$

$$J_{CD} = \frac{\pi}{2}(c_2^4 - c_1^4) = \frac{\pi}{2}[(0.030\ \text{m})^4 - (0.022\ \text{m})^4]$$

$$= 0.904 \times 10^{-6}\ \text{m}^4$$



$$\phi_A = \sum_i \frac{T_i L_i}{J_i G} = \frac{1}{G}\left(\frac{T_{AB}L_{AB}}{J_{AB}} + \frac{T_{BC}L_{BC}}{J_{BC}} + \frac{T_{CD}L_{CD}}{J_{CD}}\right)$$

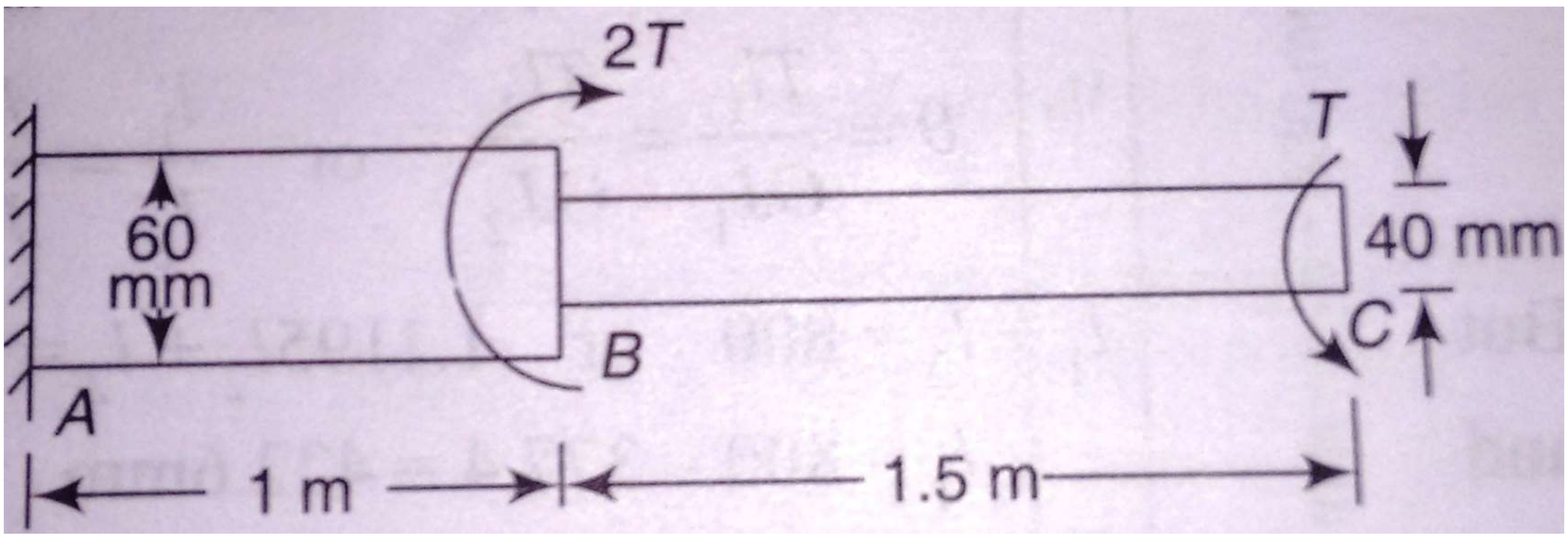
$$\phi_A = \frac{1}{77\text{ GPa}}\left[\frac{(250\text{ N}\cdot\text{m})(0.4\text{ m})}{0.0795\times 10^{-6}\text{ m}^4} + \frac{(2250)(0.2)}{1.272\times 10^{-6}} + \frac{(2250)(0.6)}{0.904\times 10^{-6}}\right]$$

$$= 0.01634 + 0.00459 + 0.01939 = 0.0403\text{ rad}$$

$$\phi_A = 2.31^\circ$$

Problem 7

Figure shows a stepped steel shaft subjected to a torque T at the free end and a torque $2T$ in the opposite direction at the junction of the two sizes. Determine the total angle of twist if the maximum shear stress is limited to 80 MPa. Take G = 80 GPa.



Angle of twist = 0.065 rad = 3.73°

Fig. 10.3

Solution Torque in the portion BC = T (counter-clockwise)

Torque in the portion $AB = T - 2T$ (counter-clockwise) or (T clockwise)

Thus the two portions of the shaft are subjected to a torque of same magnitude but in the opposite direction.

Maximum stress will reach the maximum value in the thinner portion first.

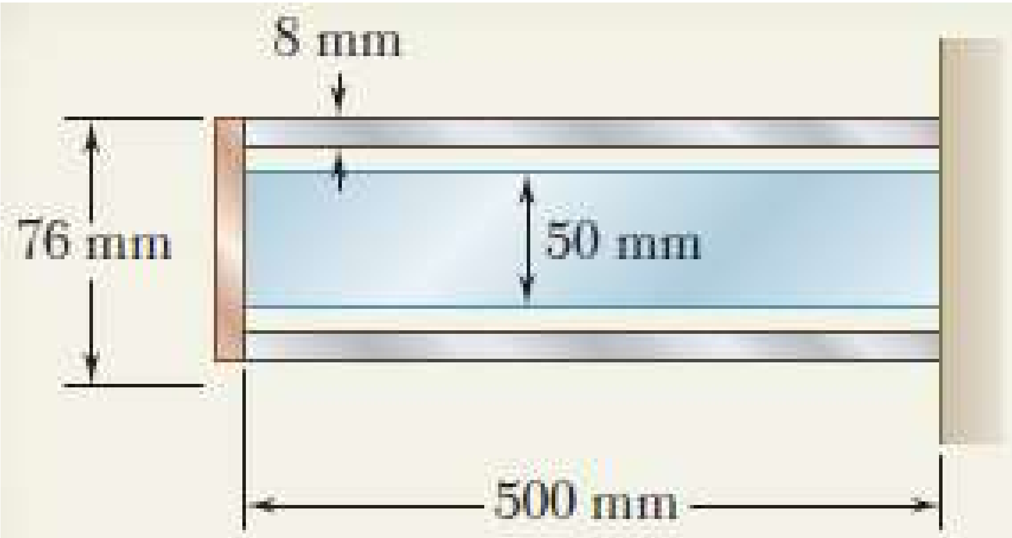
$$T = \frac{\pi d^3}{16}.\tau = \frac{\pi \times 40^3}{16} \times 80 = 1\,005\,310 \text{ N.mm}$$

Angle of twist, $$\theta = \frac{Tl_{bc}}{GJ_{bc}} - \frac{Tl_{ab}}{GJ_{ab}} = \frac{T}{G}\left(\frac{l_{bc}}{J_{bc}} - \frac{l_{ab}}{J_{ab}}\right)$$

$$= \frac{1\,005\,310}{80\,000 \times (\pi/32)}\left(\frac{1500}{40^4} - \frac{1000}{60^4}\right)$$

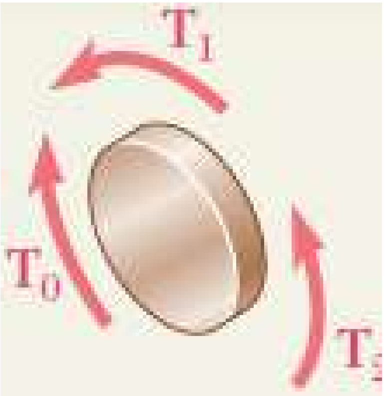
$$= 0.065 \text{ rad} = 0.065 \times \frac{180°}{\pi} = 3.73°$$

Problem 8



A steel shaft and an aluminum tube are connected to a fixed support and to a rigid disk as shown in the cross section. Knowing that the initial stresses are zero, determine the maximum torque T_0 that can be applied to the disk if the allowable stresses are 120 MPa in the steel shaft and 70 MPa in the aluminum tube. Use *G* = 77 GPa for steel and *G* 5 27 GPa for aluminum.

Solution

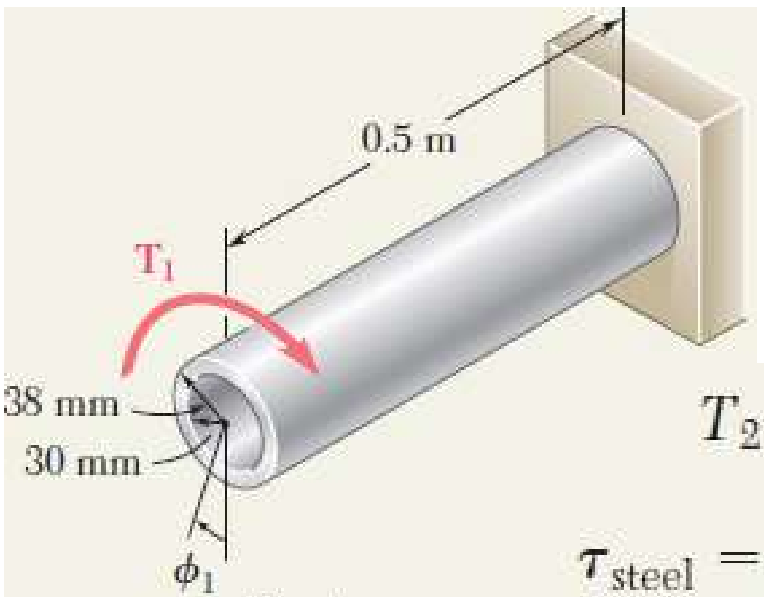


$$T_0 = T_1 + T_2$$

$$\phi_1 = \phi_2: \qquad \frac{T_1 L_1}{J_1 G_1} = \frac{T_2 L_2}{J_2 G_2}$$

$$\frac{T_1(0.5\text{ m})}{(2.003 \times 10^{-6}\text{ m}^4)(27\text{ GPa})} = \frac{T_2(0.5\text{ m})}{(0.614 \times 10^{-6}\text{ m}^4)(77\text{ GPa})}$$

$$T_2 = 0.874 T_1$$



$$T_1 = \frac{\tau_{\text{alum}} J_1}{c_1} = \frac{(70\text{ MPa})(2.003 \times 10^{-6}\text{ m}^4)}{0.038\text{ m}}$$

$$= 3225\text{ N}\cdot\text{m}$$

$$T_2 = 0.874 T_1 = 0.874(3690) = 3225\text{ N}\cdot\text{m}$$

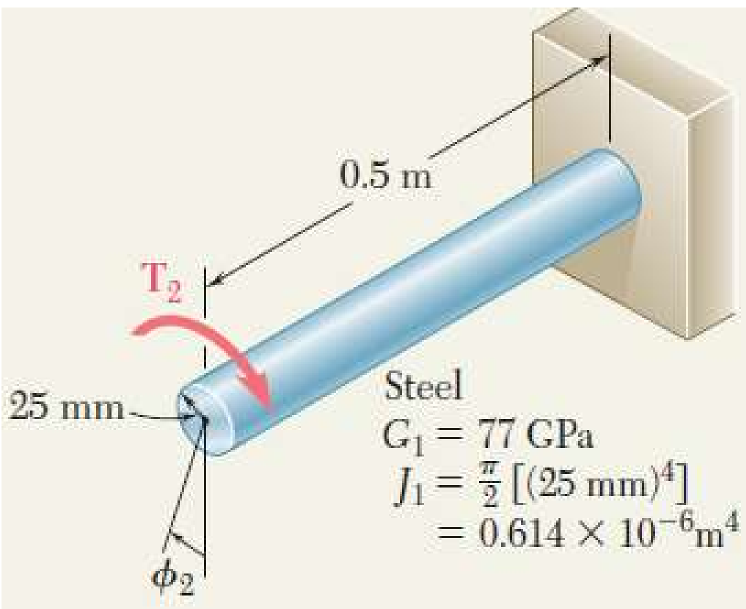
$$\tau_{\text{steel}} = \frac{T_2 c_2}{J_2} = \frac{(3225\text{ N}\cdot\text{m})(0.025\text{ m})}{0.614 \times 10^{-6}\text{ m}^4} = 131.3\text{ MPa}$$

We note that the allowable steel stress of 120 MPa is exceeded and hence our assumption was *wrong*. Thus the maximum torque **T_0** will be obtained by making τ_{steel} = 120 MPa.

We first determine the torque **T_2**,

$$T_2 = \frac{\tau_{steel} J_2}{c_2} = \frac{(120\ \text{MPa})(0.614 \times 10^{-6}\ \text{m}^4)}{0.025\ \text{m}}$$

$$= 2950\ \text{N} \cdot \text{m}$$



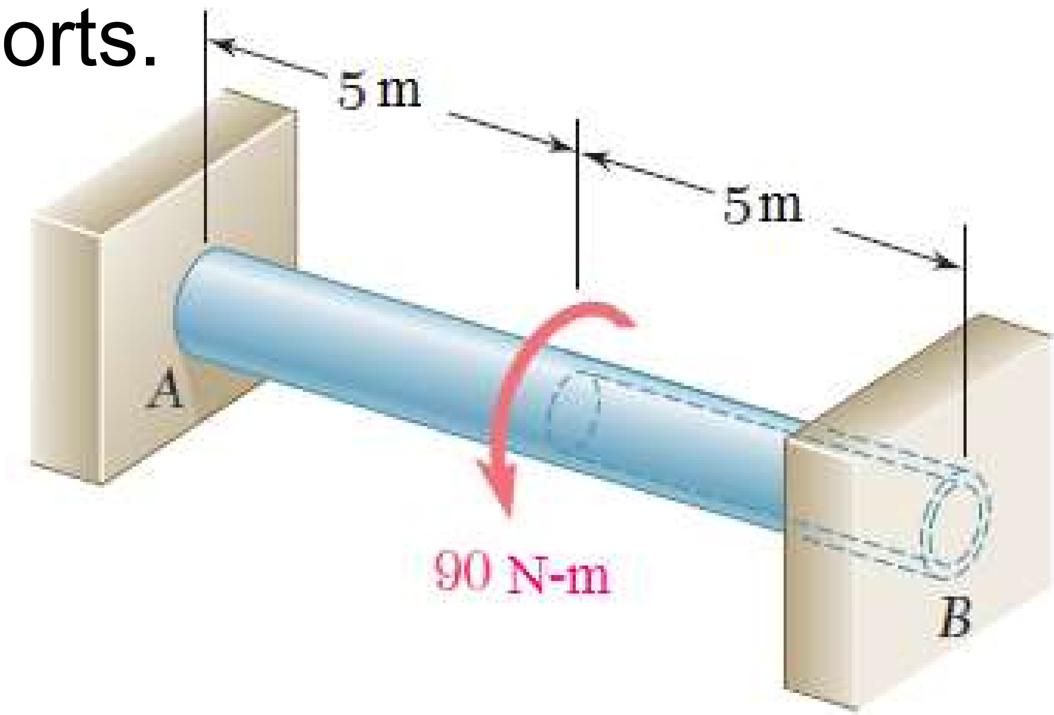
$$2950\ \text{N} \cdot \text{m} = 0.874 T_1 \qquad T_1 = 3375\ \text{N} \cdot \text{m}$$

$$T_0 = T_1 + T_2 = 3375\ \text{N} \cdot \text{m} + 2950\ \text{N} \cdot \text{m}$$

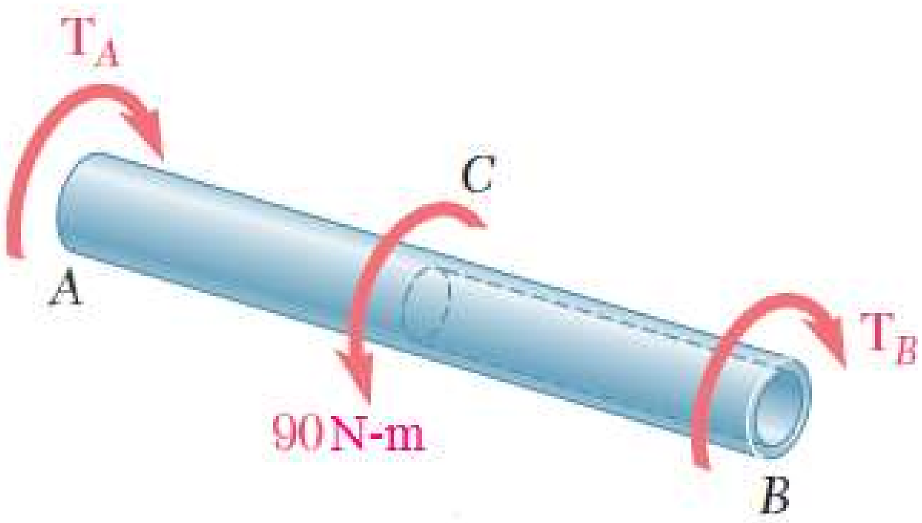
$$T_0 = 6.325\ \text{kN} \cdot \text{m}$$

Problem 9

A circular shaft *AB* consists of a 10 m long, 7/8 m diameter steel cylinder, in which a 5 m long, 5/8 m diameter cavity has been drilled from end *B*. The shaft is attached to fixed supports at both ends, and a 90 N-m torque is applied at its midsection as shown in figure. Determine the torque exerted on the shaft by each of the supports.



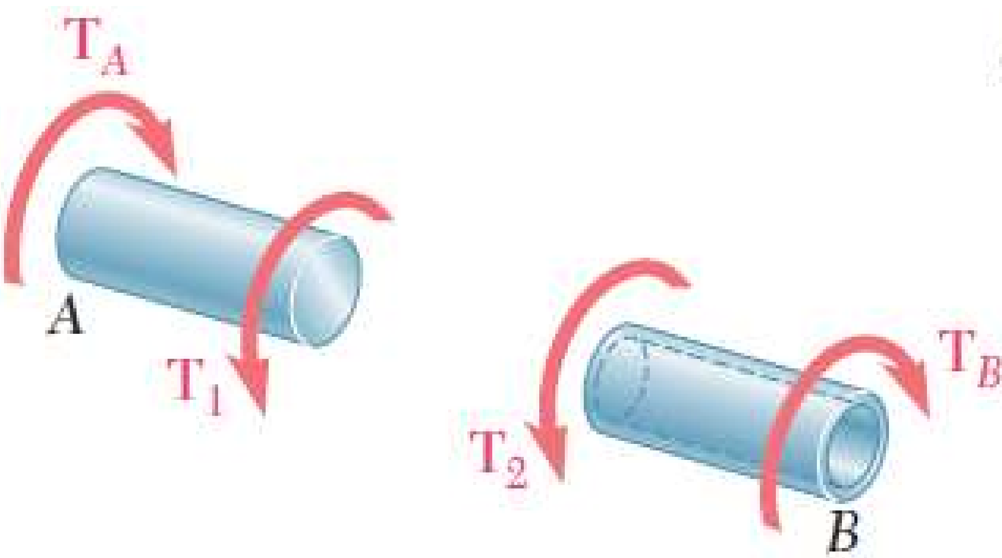
Solution



$$T_A + T_B = 90\ N - m$$

The total angle of twist of shaft *AB* must be zero, since both of its ends are restrained.

$$\phi = \phi_1 + \phi_2 = 0$$



$$\phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0$$

$$T_B = \frac{L_1 J_2}{L_2 J_1} T_A$$

$$L_1 = L_2 = 5\ m$$

$$J_1 = \frac{\pi}{32}\left(\frac{7}{8}\right)^4 = 0.0575\, m^4$$

$$J_1 = \frac{\pi}{32}\left[\left(\frac{7}{8}\right)^4 - \left(\frac{5}{8}\right)^4\right] = 0.0426\, m^4$$

$$T_B = 0.740\ T_A$$

$$T_A + 0.740\, T_A = 90$$

$$1.740\, T_A = 90$$

$$\boxed{T_A = 51.7\, N - m}$$

$$\boxed{T_B = 38.3\, N - m}$$

Springs

Introduction

Spring is defined as an elastic machine element, which deflects under the action of the load and returns to its original shape when the load is removed

Functions and Applications:

- To absorb shock and vibrations
- To store energy
- To measure forces
- To apply force and control motion

Types

Helical Compression Spring

Helical Tension Spring

Conical Spring

Disc Spring

Helical Torsion Spring

Leaf Spring

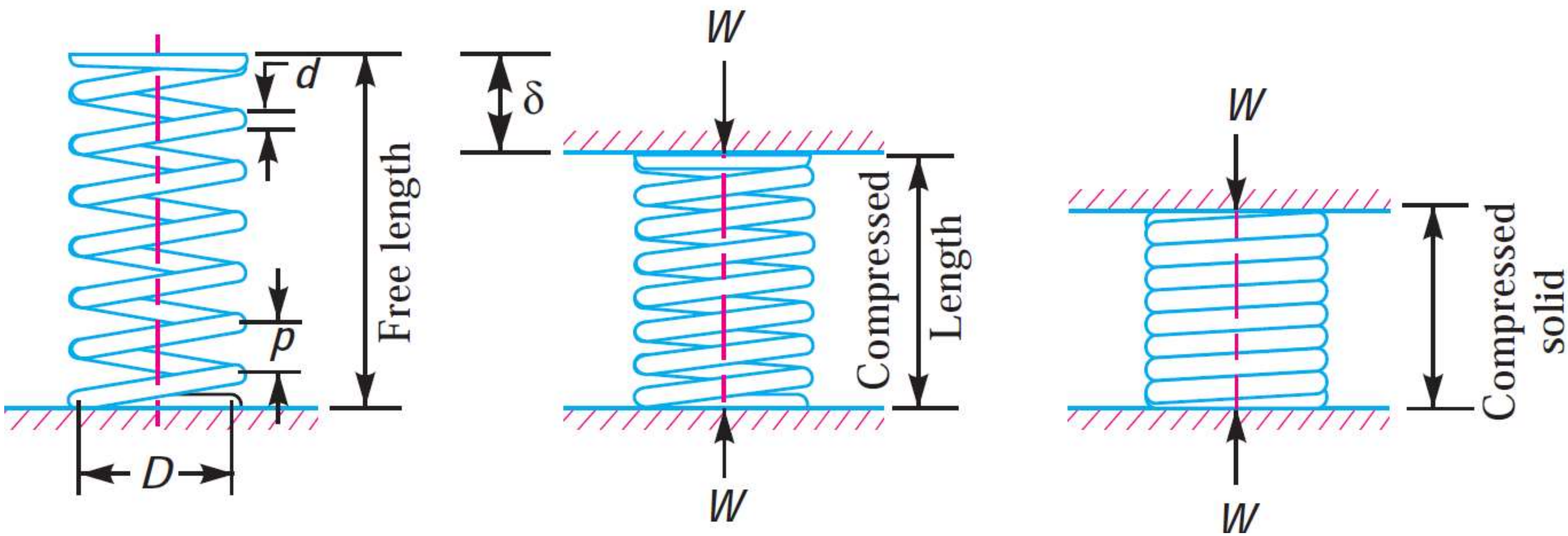
Advantages of Helical Springs

- Easy to manufacture
- Available in wide range
- Reliable
- Have constant spring rate
- Performance can be predicted more accurately
- Characteristics can be varied by changing dimensions

Helical Springs

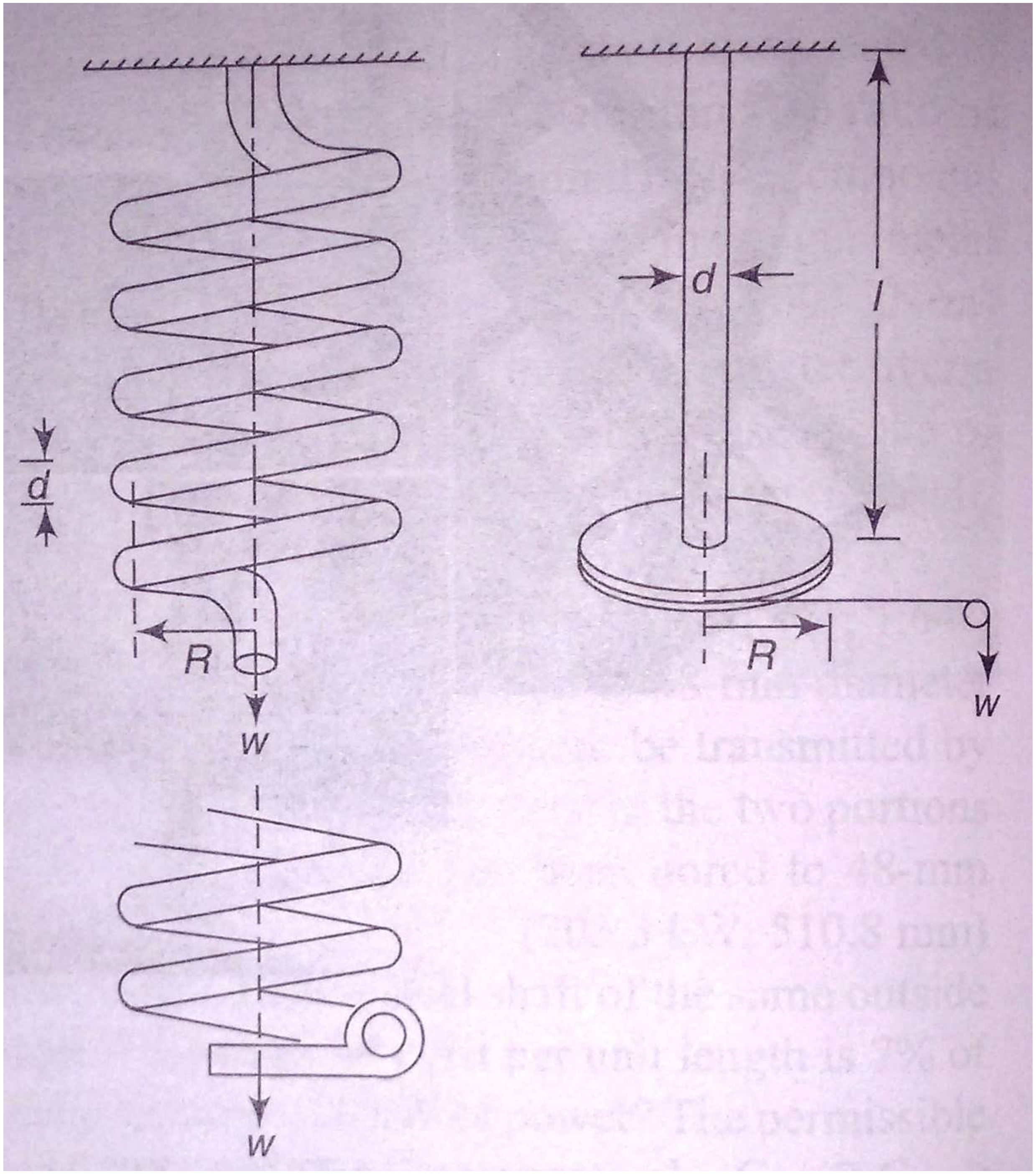
1. Close-coiled helical springs
2. Open-coiled helical springs

Close-coiled Helical Springs



d- diameter of spring wire
p- pitch of the helical spring
n- number of coils
R- Mean radius of spring coil

W- Axial load on spring
G- Modulus of rigidity
τ- Maximum shear stress induced in wire
δ- Deflection of spring due to axial load


d
R
W
W
d
l
R
W

Maximum Shear Stress

Twisting moment on the wire

$$T = W \times R$$

$$\boxed{T = \frac{\pi}{16}\tau d^3}$$

$$W \times R = \frac{\pi}{16}\tau d^3$$

$$\boxed{\tau = \frac{16W \times R}{\pi d^3}}$$

θ- Total angle of twist along the wire

l- Length of wire

Deflection

Equating the work done by the axial force and the torsional strain energy

$$\frac{1}{2}W \times \delta = \frac{1}{2}T\theta = \frac{1}{2}WR\theta$$

$$\boxed{\delta = R\theta}$$

$$\theta = \frac{Tl}{GJ} = \frac{WRl}{G(\pi d^4/32)} = \frac{32WRl}{G\pi d^4}$$

$$\boxed{l = 2\pi Rn}$$

$$\therefore \theta = \frac{32WR(2\pi Rn)}{G\pi d^4} = \frac{64WR^2n}{Gd^4}$$

$$\boxed{\delta = R\theta}$$

$$\boxed{\therefore \delta = \frac{64WR^3n}{Gd^4} = \frac{8WD^3n}{Gd^4}}$$

Stiffness

$$s = \frac{W}{\delta} = \frac{W}{\frac{64WR^3n}{Gd^4}} = \frac{Gd^4}{64R^3n} = \frac{Gd^4}{8D^3n}$$

Problem 1

A closely coiled helical spring is to carry a load of 500 N. Its mean coil diameter is to be 10 times that of the wire diameter. Calculate these diameters if the maximum shear stress in the material of the spring is to be 80 N/mm^2. If the stiffness of the spring is 20 N per mm deflection and modulus of rigidity = 8.4×10^5 N/mm^2, find the number of coils in the spring.

$$\boxed{d = 12.6\ \text{mm}}$$

$$\boxed{D = 126\ \text{mm}}$$

$$\boxed{n = 7}$$

Problem 2

The stiffness of a close-coiled helical spring is 1.5 N/mm of compression under a maximum load of 60 N. the maximum shear stress produced in the wire of the spring is 125 N/mm^2. The solid length of the spring (when the coils are touching) is given as 5 cm. Find: (i) diameter of wire. (ii) mean diameter of the coils and (iii) number of coils required. Take modulus of rigidity = 4.5×10^4 N/mm^2 .

Solution

$$s = \frac{Gd^4}{64R^3n} \qquad 1.5 = \frac{4.5 \times 10^4 \times d^4}{64R^3n}$$

$$d^4 = \frac{1.5 \times 64 \times R^3 \times n}{4.5 \times 10^4} = 0.002133R^3n$$

$$\boxed{\tau = \frac{16W \times R}{\pi d^3}} \qquad 125 = \frac{16 \times 60 \times R}{\pi d^3}$$

$$R = \frac{125 \times \pi d^3}{16 \times 60} = 0.409d^3$$

$$d^4 = 0.002133(0.409d^3)^3n = 0.00014599 \times d^9n$$

$$\frac{d^9n}{d^4} = \frac{1}{0.00014599} \qquad d^5n = \frac{1}{0.00014599}$$

$$Solid\ Length = n \times d \qquad 50 = n \times d$$

$$n = \frac{50}{d} \qquad d^5 \times \frac{50}{d} = \frac{1}{0.00014599}$$

$$d^4 = \frac{1}{0.00014599} \times \frac{1}{50}$$

$$\boxed{d = 3.42\ \text{mm}} \qquad \therefore n = \frac{50}{d} = \frac{50}{3.42} = 14.62\ say\ 15$$

$$\boxed{n = 15} \qquad R = 0.409d^3 = 0.409 \times 3.42^3 = 16.36\ \text{mm}$$

$$\boxed{\therefore D = 32.72\ \text{mm}}$$