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School of Mechanical and Building Sciences

Unit 4

Bending & Shear Stresses in Beams

By

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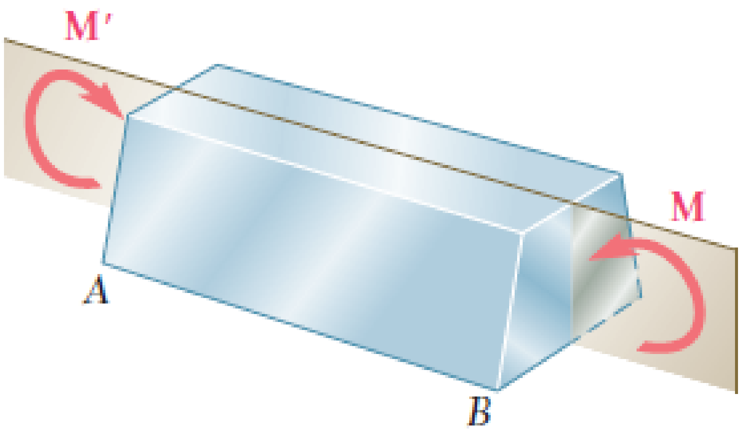
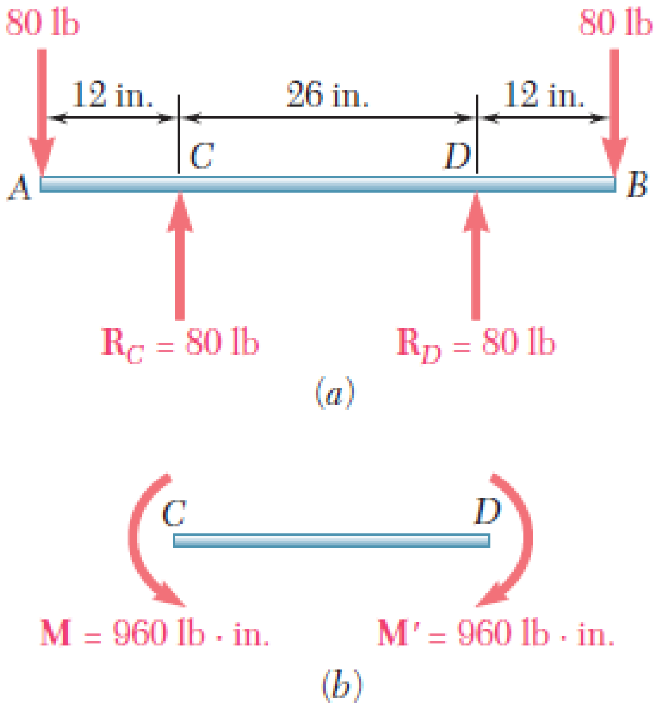
Outline

- Introduction
- Assumptions
- Bending Equation
- Problems

Introduction

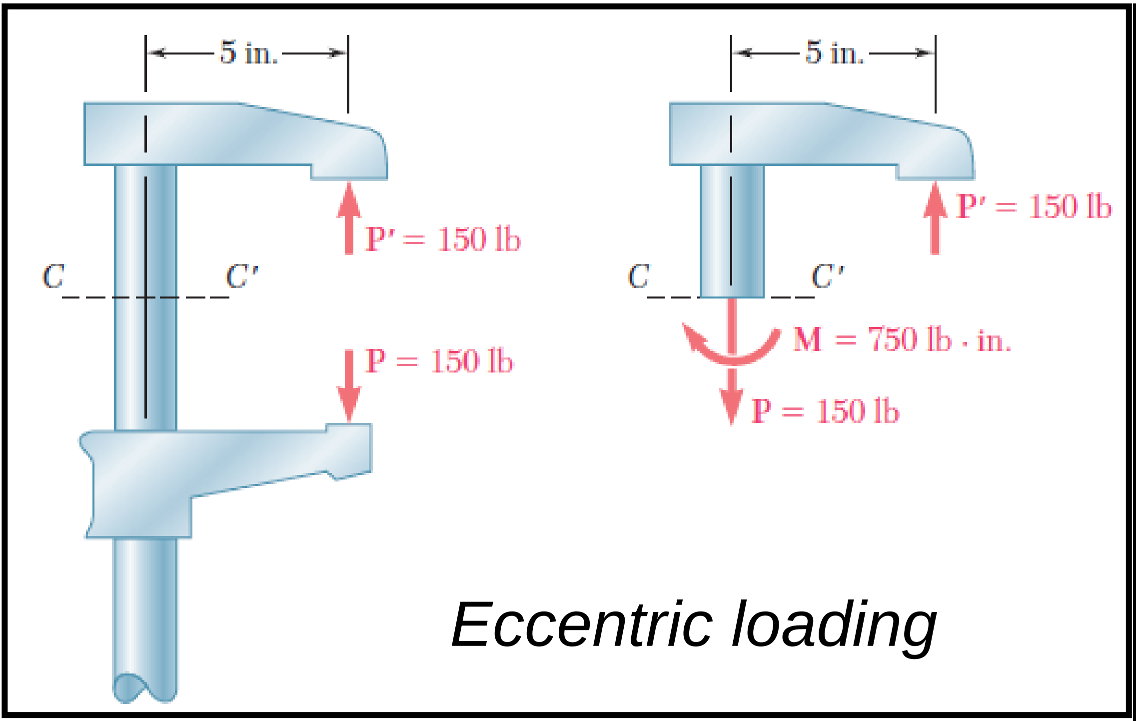
If a member is subjected to equal and opposite couples acting in the same longitudinal plane, the member is said to be in pure bending.



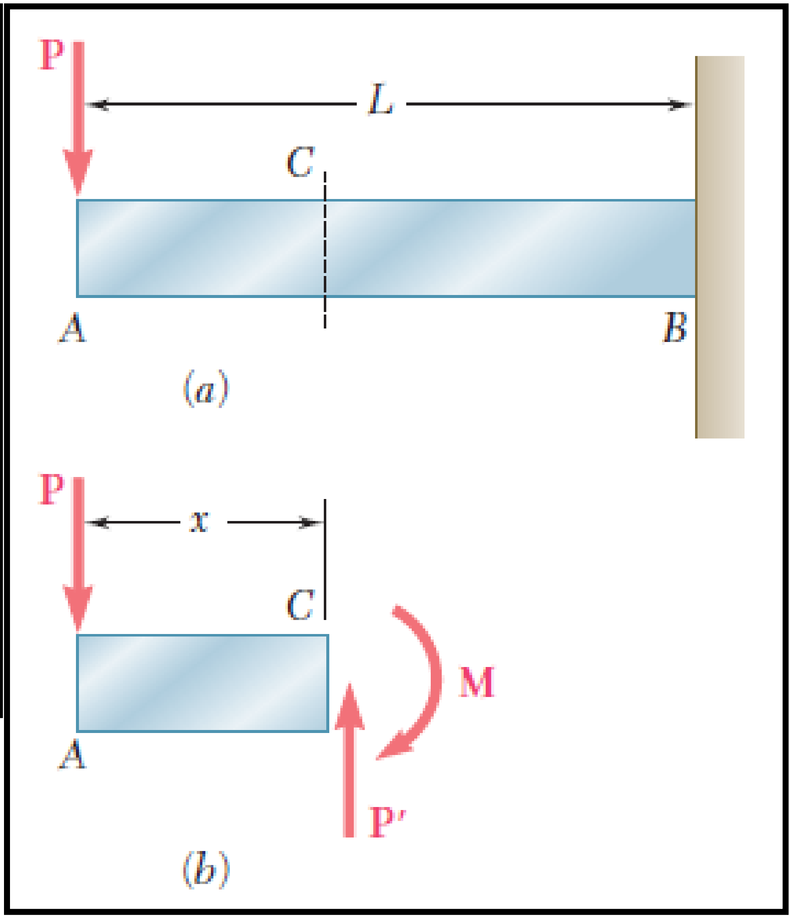


Beam in which portion *CD* is in pure bending

Bending is a major concept used in the design of many machine and structural components, such as beams and girders



Eccentric loading

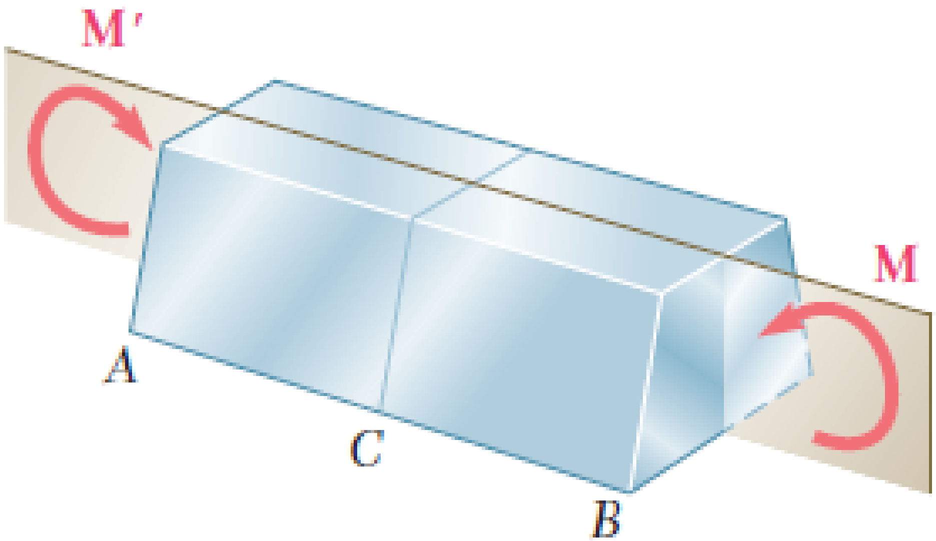


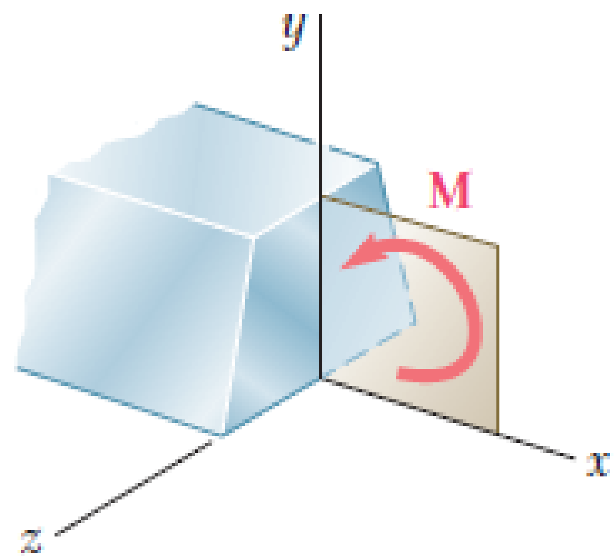
Transverse loads

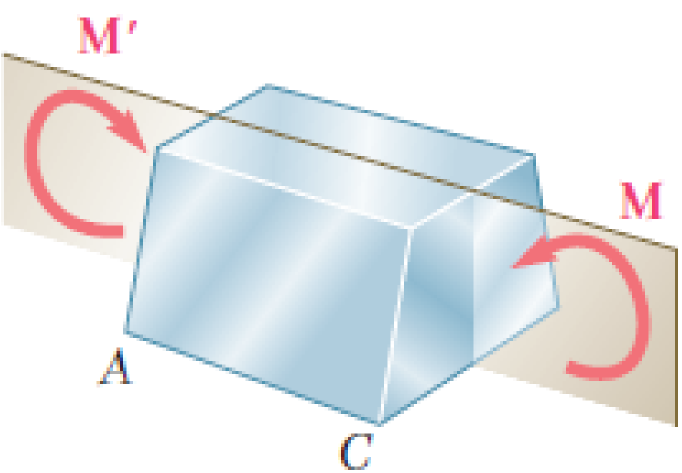
Assumptions

- The material is homogeneous and isotropic
- Transverse planes remain plane and perpendicular to the neutral surface after bending
- The longitudinal filaments are bent into circular arcs with a common centre of curvature
- The beam is symmetric about a vertical longitudinal plane passing through vertical axis of symmetry for horizontal beams
- The stress is purely longitudinal and the stress concentration effects near the concentrated loads are neglected

Symmetry

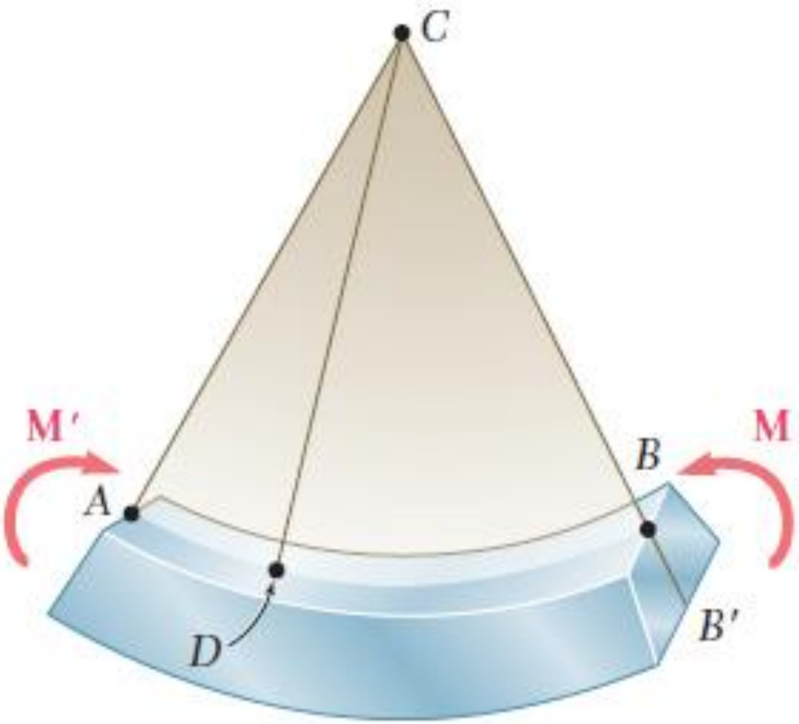






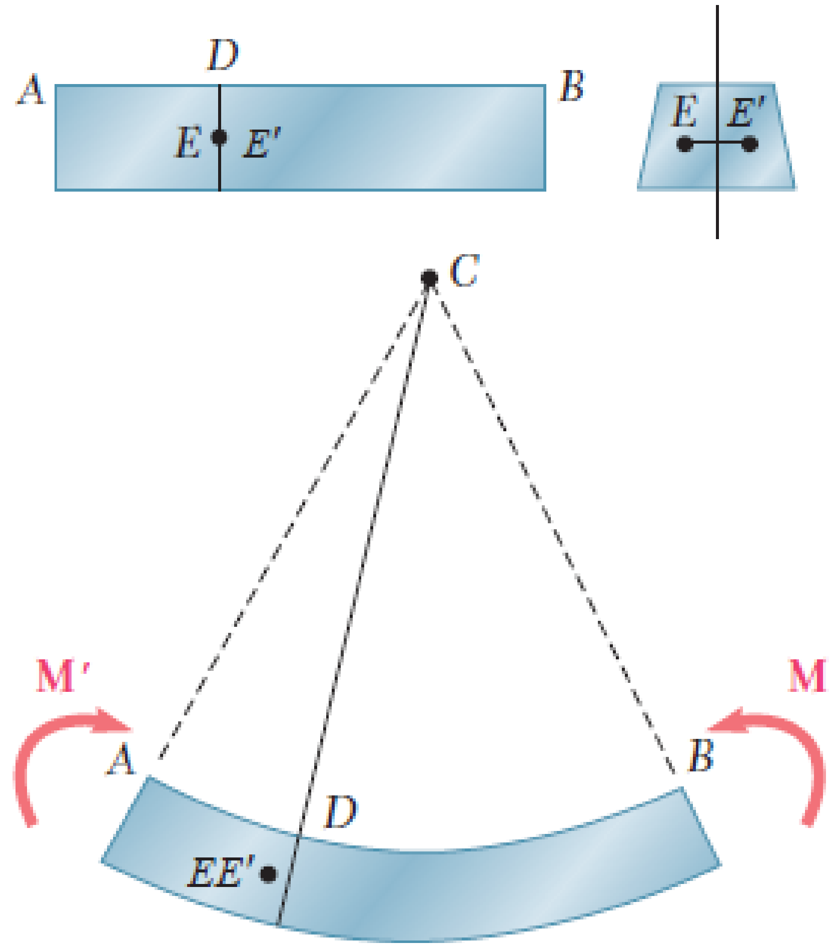
- The internal forces in any cross section of a symmetric member in pure bending are equivalent to a couple

Circular Arcs with a Common Centre of Curvature

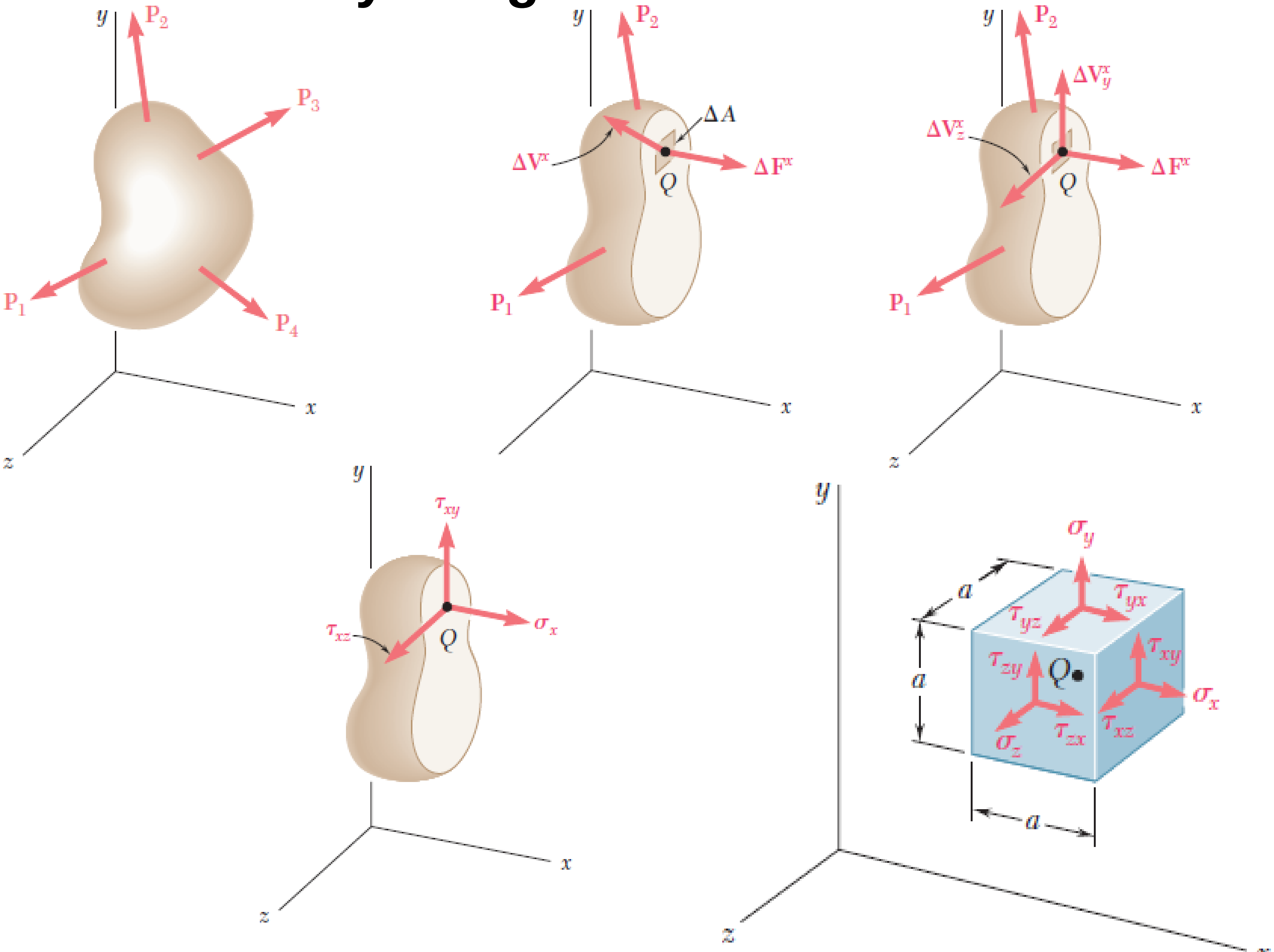


- since the bending moment *M* is the same in any cross section, the member will bend uniformly

Transverse Planes Remain Plane after Bending



Stress is Purely Longitudinal

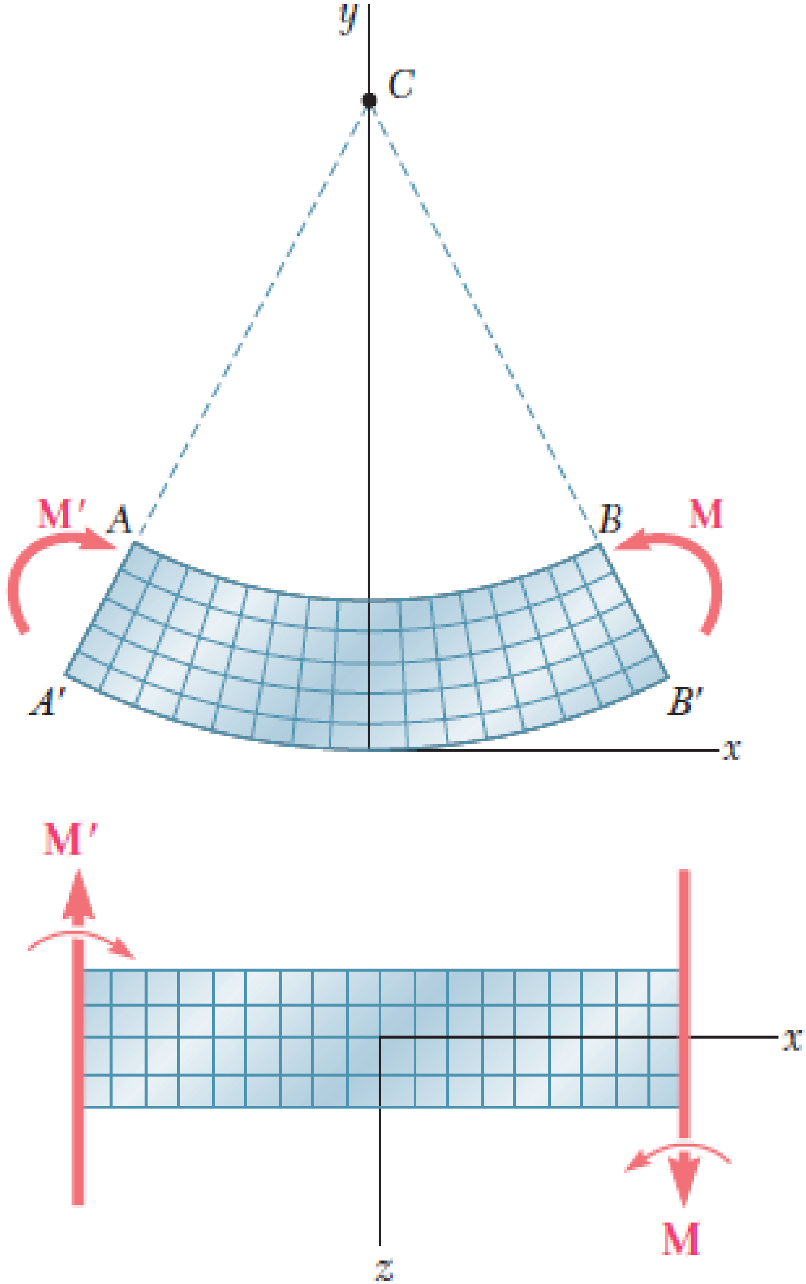


Since all the faces represented in the two projections of figure are at 90° to each other, we conclude that

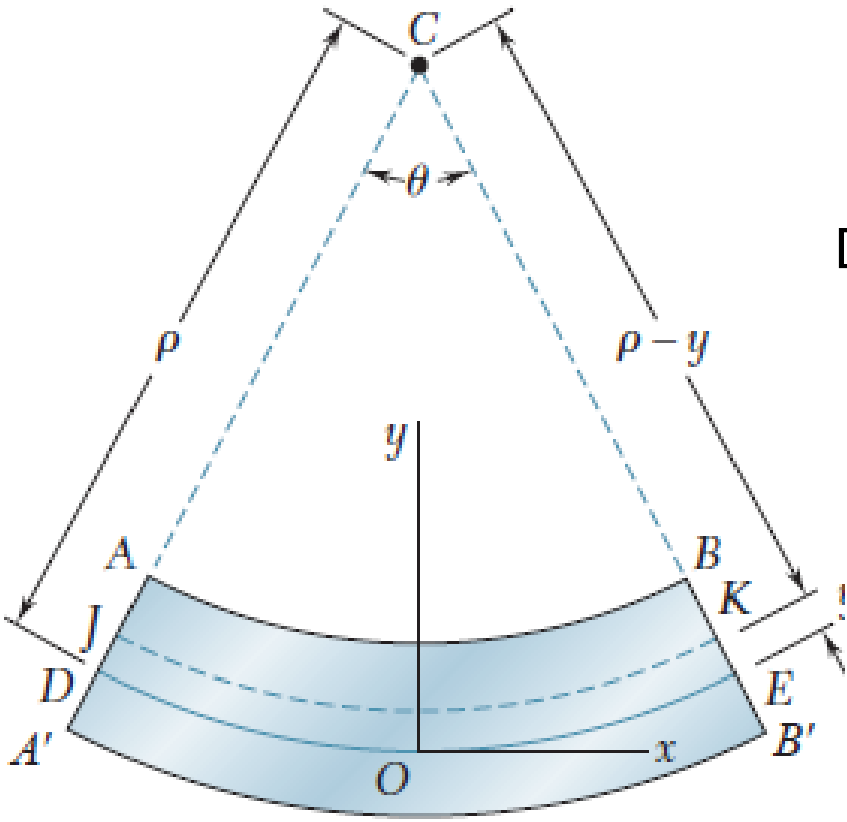
$$\gamma_{xy} = \gamma_{zx} = 0 \; and \; thus$$
$$\tau_{xy} = \tau_{zx} = 0$$
$$\sigma_y = \sigma_z = \tau_{yz} = 0$$

The only nonzero stress component exerted on any of the small cubic elements considered here is the normal component $\boldsymbol{\sigma_x}$

- The strain $\boldsymbol{\epsilon_x}$ and the stress $\boldsymbol{\sigma_x}$ are negative in the upper portion of the member (*compression*) and positive in the lower portion (*tension*)



The surface where ϵ_x and σ_x are zero is called the *neutral surface*

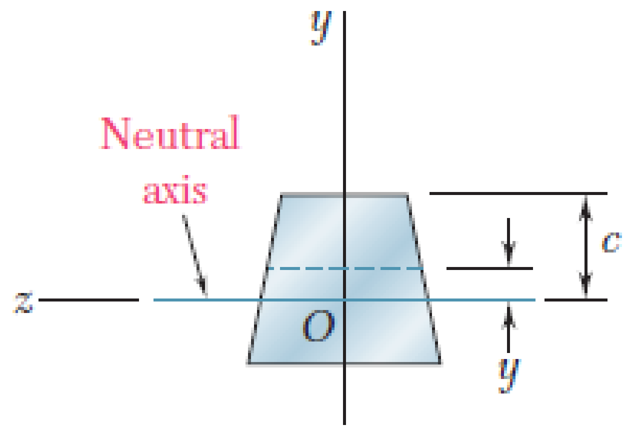


Length of arc DE, $L = \rho\theta$

Length of arc JK, $L' = (\rho - y)\theta$

Deformation of JK, $\delta = L' - L$

$$= (\rho - y)\theta - \rho\theta = -y\theta$$



The longitudinal normal strain ϵ_x varies linearly with the distance y from the neutral surface

$$\boxed{\epsilon_x = \frac{\delta}{L} = \frac{-y\theta}{\rho\theta} = \frac{-y}{\rho}}$$

Let c be the largest distance from neutral axis, the maximum absolute value of strain

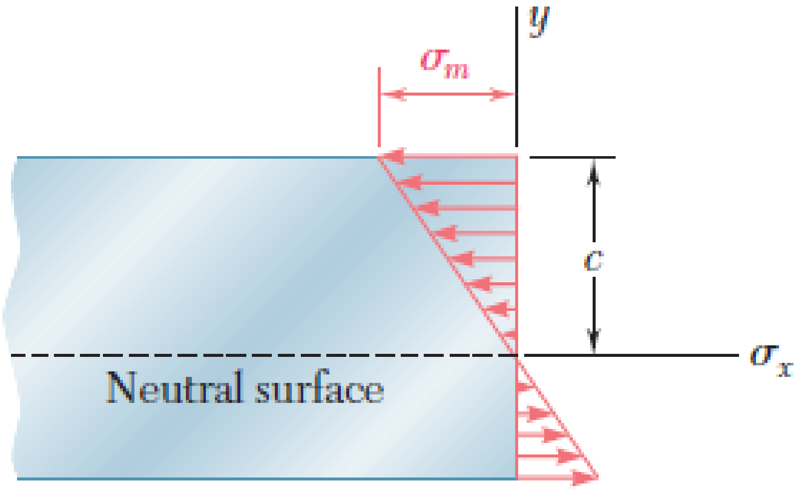
$$\boxed{\epsilon_m = \frac{c}{\rho}}$$

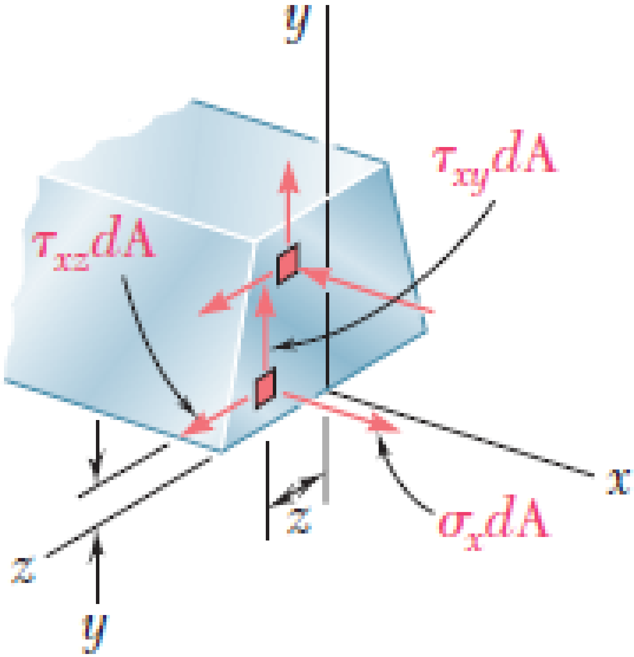
Substitute for ρ, we get $$\epsilon_x = -\frac{y}{c}\epsilon_m$$

Stress-strain relation of the material used, $\sigma_x = E\epsilon_x$

$$E\epsilon_x = -\frac{y}{c}(E\epsilon_m)$$

$$\sigma_x = -\frac{y}{c}\sigma_m$$





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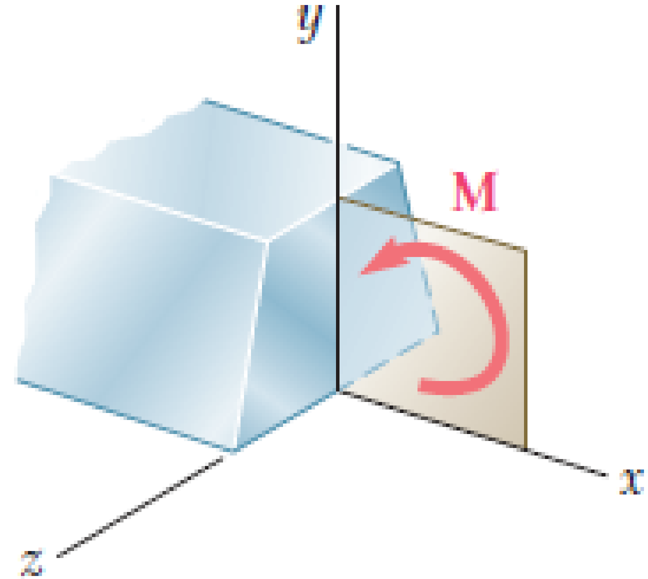


x components:	$\int \sigma_x \, dA = 0$
moments about y axis:	$\int z\sigma_x \, dA = 0$
moments about z axis:	$\int(-y\sigma_x \, dA) = M$

Location of the Neutral Axis

$$\int \sigma_x \, dA = \int \left(-\frac{y}{c} \sigma_m \right) dA = -\frac{\sigma_m}{c} \int y \, dA = 0$$

$$\int y \, dA = 0$$

- This equation shows that the first moment of the cross section about its neutral axis must be zero
- In other words, for a member subjected to pure bending, and as long as the stresses remain in the elastic range, the neutral axis passes through the centroid of the section.

Elastic Flexure Formulas

$$\int (-y\sigma_x \, dA) = M \qquad \int (-y)\left(-\frac{y}{c}\sigma_m\right) dA = M \qquad \frac{\sigma_m}{c}\int y^2 \, dA = M$$

$$\boxed{\sigma_m = \frac{Mc}{I}} \qquad \boxed{\sigma_x = -\frac{My}{I}}$$

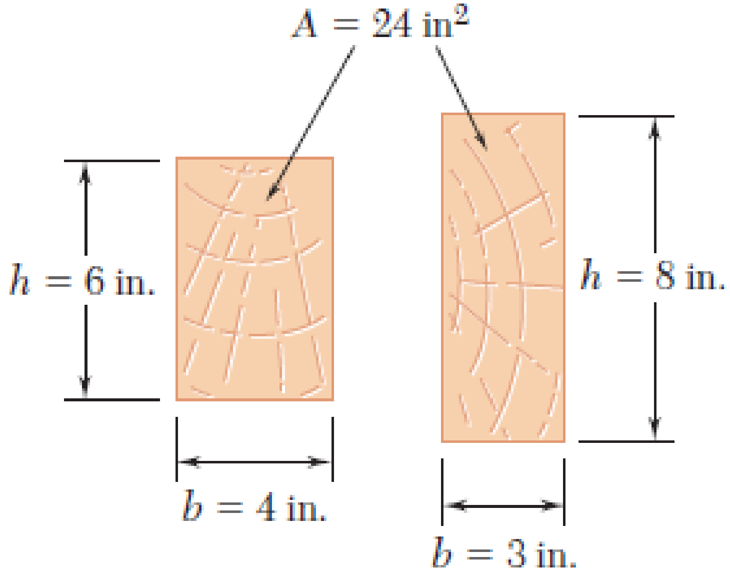
$$\text{Elastic section modulus} = S = \frac{I}{c} \qquad \boxed{\sigma_m = \frac{M}{S}}$$

- Since the maximum stress $\boldsymbol{\sigma_m}$ is inversely proportional to the elastic section modulus S, it is clear that beams should be designed with as large a value of S as practicable

Example

For example, in the case of a wooden beam with a rectangular cross section of width *b* and depth *h*, we have

$$S = \frac{I}{c} = \frac{\frac{1}{12}bh^3}{h/2} = \frac{1}{6}bh^2 = \frac{1}{6}Ah$$



This shows that, of two beams with the same cross-sectional area *A*, the beam with the larger depth *h* will have the larger section modulus and, thus, will be the more effective in resisting bending

Moment-Curvature Relationship

$$\boxed{\epsilon_m = \frac{c}{\rho}}$$

$$\frac{1}{\rho} = \frac{\epsilon_m}{c}$$

Reciprocal of radius of curvature is known as curvature

$$\frac{1}{\rho} = \frac{\sigma_m}{Ec} = \frac{1}{Ec}\frac{Mc}{I}$$

$$\frac{1}{\rho} = \frac{M}{EI}$$

$$\boxed{\frac{M}{I} = \frac{\sigma_m}{c} = \frac{E}{\rho}}$$

EI – Flexural rigidity of the beam

- Flexural rigidity is a measure of the resistance of a beam to bending, that is, the larger the flexural rigidity, the smaller the curvature for a given bending moment

Problem 1

A steel plate of width 120 mm and of thickness 20 mm is bent into a circular arc of radius 10 m. Determine the maximum stress induced and the bending moment which will produce the maximum stress. Take $E = 2 \times 10^5$ N/mm^2.

$$\boxed{\begin{array}{c} \sigma_m = 200\ N/mm^2 \\ M = 1.6\ kN - m \end{array}}$$

Width of plate, $b = 120$ mm

Thickness of plate, $t = 20$ mm

$\therefore$ Moment of inertia, $I = \dfrac{bt^3}{12} = \dfrac{120 \times 20^3}{12} = 8 \times 10^4 \text{ mm}^4$

Radius of curvature, $R = 10 \text{ m} = 10 \times 10^3$ mm

Young's modulus, $E = 2 \times 10^5 \text{ N/mm}^2$

Let σ_{max} = Maximum stress induced, and

M = Bending moment.

Using equation (7.2), $\dfrac{\sigma}{y} = \dfrac{E}{R}$

$$\therefore \quad \sigma = \frac{E}{R} \times y \qquad \text{...(}i$$

Equation (i) gives the stress at a distance y from N.A.

Stress will be maximum, when y is maximum. But y will be maximum at the top layer o bottom layer.

$$\therefore \quad y_{max} = \frac{t}{2} = \frac{20}{2} = 10 \text{ mm.}$$

Now equation (i) can be written as

$$\sigma_{max} = \frac{E}{R} \times y_{max}$$

$$= \frac{2 \times 10^5}{10 \times 10^3} \times 10 = \mathbf{200\ N/mm^2.} \quad \textbf{Ans.}$$

From equation (7.4), we have

$$\frac{M}{I} = \frac{E}{R}$$

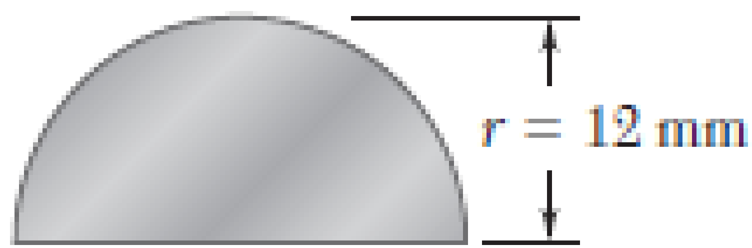
$$\therefore \quad M = \frac{E}{R} \times I = \frac{2 \times 10^5}{10 \times 10^3} \times 8 \times 10^4$$

$$= 16 \times 10^5 \text{ N mm} = \mathbf{1.6\ kNm.} \quad \textbf{Ans.}$$

Problem 2

An aluminum rod with a semicircular cross section of radius *r* = 12 mm shown in figure is bent into the shape of a circular arc of mean radius *ρ* = 2.5 m. Knowing that the flat face of the rod is turned toward the center of curvature of the arc, determine the maximum tensile and compressive stress in the rod. Use *E* = 70 GPa.

$$\bar{y} = \frac{4r}{3\pi}$$



$$\boxed{\begin{aligned} \sigma_m &= 193.4\ N/mm^2 \\ \sigma_{comp} &= -142.6\ N/mm^2 \end{aligned}}$$

An aluminum rod with a semicircular cross section of radius $r = 12$ mm (Fig. 4.16) is bent into the shape of a circular arc of mean radius $\rho = 2.5$ m. Knowing that the flat face of the rod is turned toward the center of curvature of the arc, determine the maximum tensile and compressive stress in the rod. Use $E = 70$ GPa.

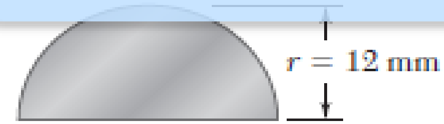


Fig. 4.16

We could use Eq. (4.21) to determine the bending moment M corresponding to the given radius of curvature ρ, and then Eq. (4.15) to determine σ_m. However, it is simpler to use Eq. (4.9) to determine ϵ_m, and Hooke's law to obtain σ_m.

The ordinate $\bar{y}$ of the centroid C of the semicircular cross section is

$$\bar{y} = \frac{4r}{3\pi} = \frac{4(12 \text{ mm})}{3\pi} = 5.093 \text{ mm}$$

The neutral axis passes through C (Fig. 4.17) and the distance c to the point of the cross section farthest away from the neutral axis is

$$c = r - \bar{y} = 12 \text{ mm} - 5.093 \text{ mm} = 6.907 \text{ mm}$$

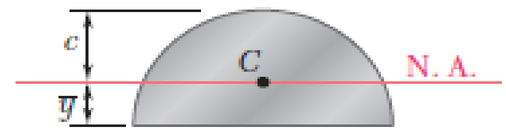


Fig. 4.17

Using Eq. (4.9), we write

$$\epsilon_m = \frac{c}{\rho} = \frac{6.907 \times 10^{-3} \text{ m}}{2.5 \text{ m}} = 2.763 \times 10^{-3}$$

and, applying Hooke's law,

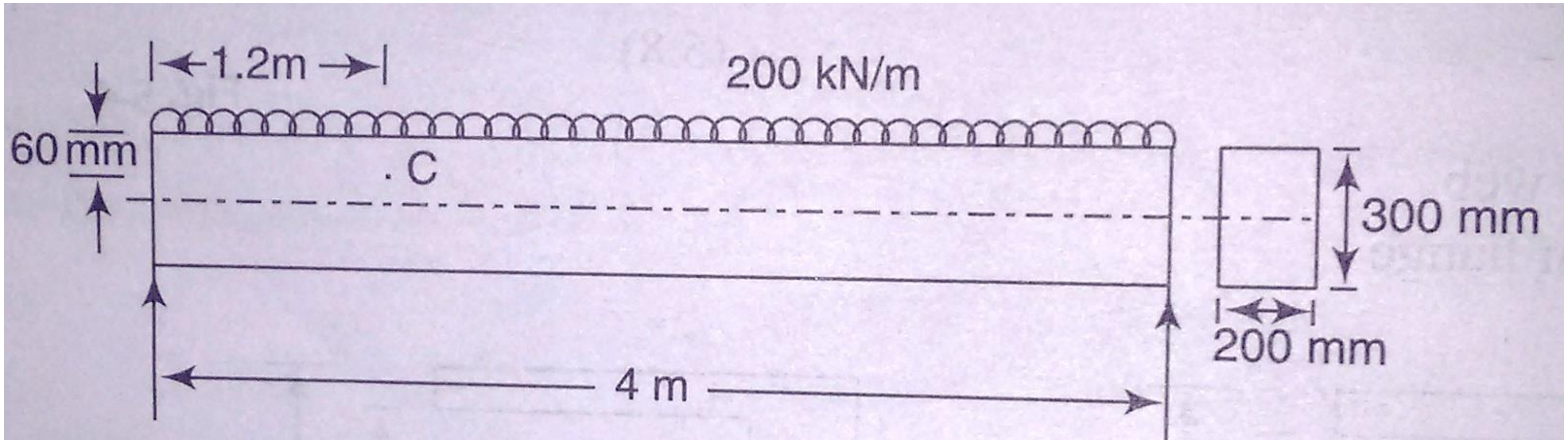
$$\sigma_m = E\epsilon_m = (70 \times 10^9 \text{ Pa})(2.763 \times 10^{-3}) = 193.4 \text{ MPa}$$

Since this side of the rod faces away from the center of curvature, the stress obtained is a tensile stress. The maximum compressive stress occurs on the flat side of the rod. Using the fact that the stress is proportional to the distance from the neutral axis, we write

$$\sigma_{\text{comp}} = -\frac{\bar{y}}{c}\sigma_m = -\frac{5.093 \text{ mm}}{6.907 \text{ mm}}(193.4 \text{ MPa})$$
$$= -142.6 \text{ MPa}$$

Problem 3

Figure shows a simply supported 200-mm wide, 300-mm deep and 4-m long beam. Determine the bending stress at the point C which is 60 mm below the top surface and 1.2 m from the left support.



$$\boxed{\sigma_b = 67.2\ N/mm^2}$$

Moment of inertia of the section = $\dfrac{200 \times 300^3}{12} = 450 \times 10^6$ mm^4

Bending moment at cross-section at $C = R \cdot x - \dfrac{wx^2}{2} = 400 \times 1.2 - \dfrac{200 \times 1.2^2}{2} = 336$ kN·m

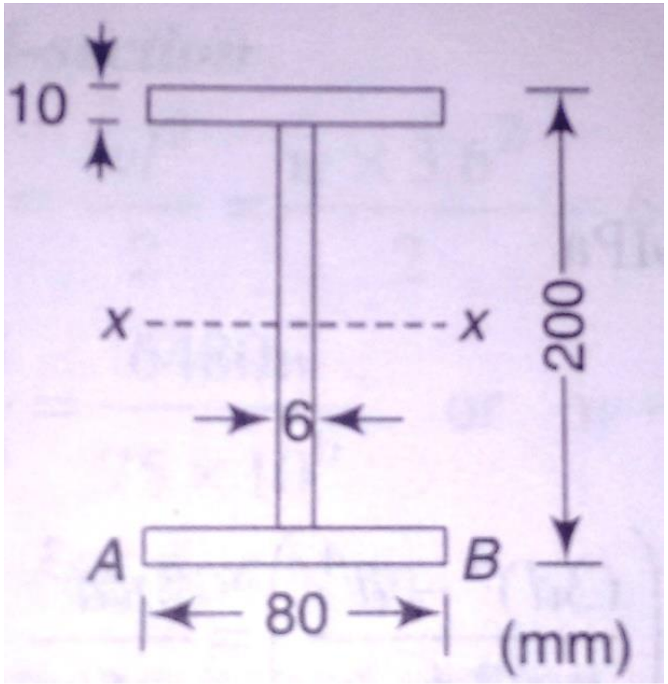
Calculation of bending stress

Distance of C from the neutral axis = 150 – 60 = 90 mm

Now, $\dfrac{\sigma}{y} = \dfrac{M}{I}$ or $\sigma = \dfrac{336 \times 10^6}{450 \times 10^6} \times 90 = 67.2$ MPa

Problem 4

A 200 mm x 80 mm I-beam is to be used as a simply supported beam of 6.75 m span. The web thickness is 6 mm and the flanges are of 10-mm thickness. Determine what concentrated load can be carried at a distance of 2.25 m from one support if the maximum permissible stress is 80 MPa.



$$\boxed{W = 9.264\ \text{kN}}$$

$$I = \frac{1}{12}(80 \times 200^3 - 74 \times 180^3) = 17.37 \times 10^6 \text{ mm}^4$$

Let W kN be the concentrated load so that the reaction at the supports are $W/3$ and $2W/3$ as shown in Fig. 5.9b.

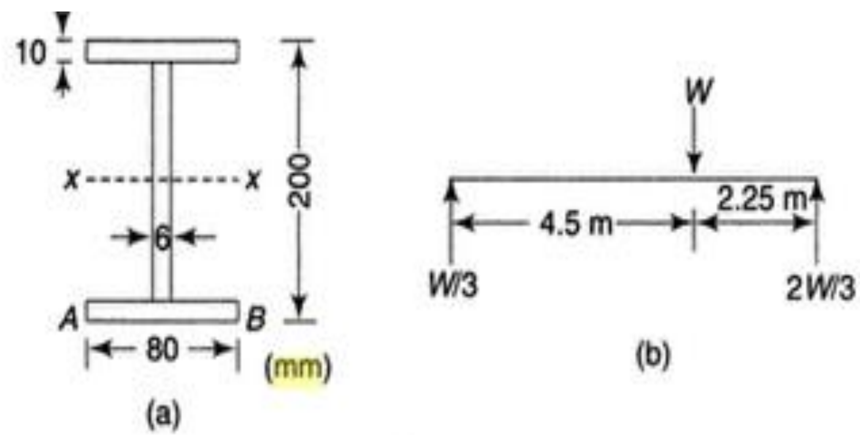


Fig. 5.10

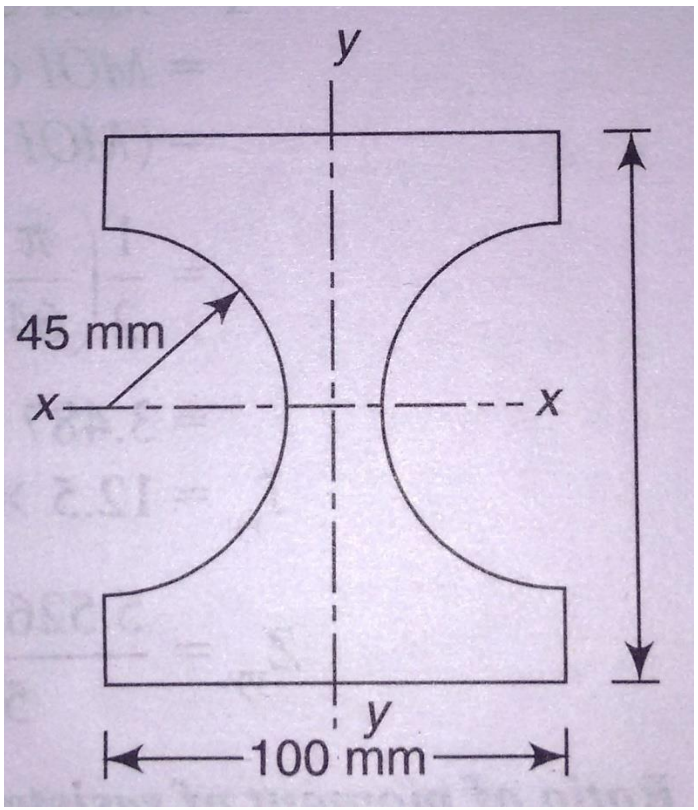
Determination of load

$$\text{Maximum bending moment} = \frac{W}{3} \times 4.5 = 1.5W \text{ kN·m} \quad \text{or} \quad 1.5W \times 10^6 \text{ N·mm}$$

Now, $$\frac{\sigma}{y} = \frac{M}{I} \text{ or } \frac{80}{100} = \frac{1.5W \times 10^6}{17.37 \times 10^6} \quad \text{or} \quad W = 9.264 \text{ kN}$$

Problem 5

Figure shows the section of a beam. Determine the ratio of its moment of resistance to bending in the y-y plane to that in the x-x plane if the maximum bending stress remains same in the two cases.



$$\boxed{\frac{Z_{yy}}{Z_{xx}} = 0.33}$$

Given A beam section as shown in Fig. 5.18

To find Ratio of moment of resistance to bending in y-plane to that in x-plane

Refer Fig. 5.19,

Section modulus in x-plane

MOI of the section about xx

= *MOI* of complete rectangle about xx – *MOI* of two semicircles about xx

$$I_{xx} = \underbrace{\frac{1}{12}\times 100\times 150^3}_{\text{(full rectangle)}} - \underbrace{2\left[\frac{1}{2}\cdot\frac{\pi}{64}(90)^4\right]}_{\text{(two semicircles)}} = 24.9\times 10^6\,\text{mm}^4$$

$$Z_{xx} = \frac{24.9\times 10^6}{75} = 0.332\times 10^6\,\text{mm}^3$$

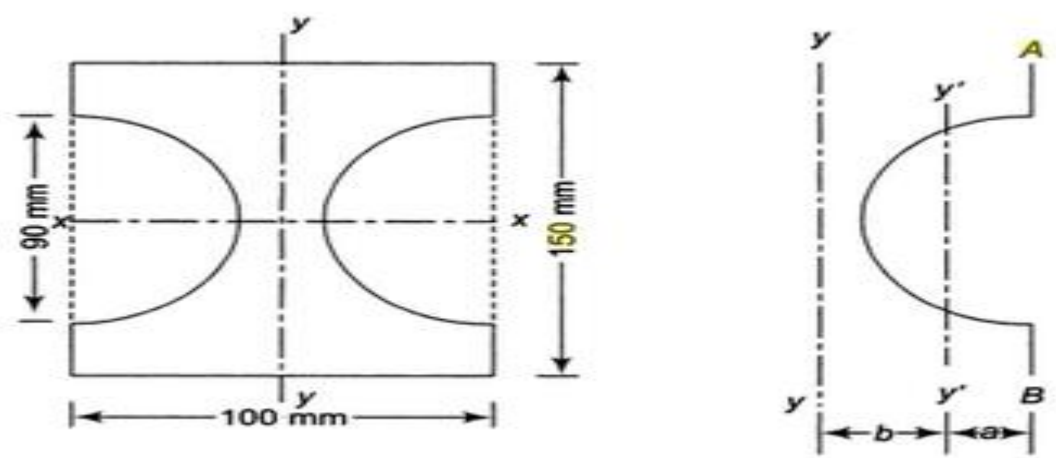


Fig. 5.19

Section modulus in y-plane

Let I = moment of inertia of each semicircle about yy

Then, $I_{yy} = \frac{1}{12} \times 150 \times 100^3 - 2I = 12.5 \times 10^6 - 2I$

Let $y'y'$ be the neutral axis of each semi-circle.

Area of each semi-circle $= \frac{1}{2} \times \frac{\pi}{4} \times 90^2 = 3181 \text{ mm}^2$

Distance a = distance of centroid of semi-circle from the centre

$$= \frac{4r}{3\pi} = \frac{4 \times 45}{3\pi} = 19.1 \text{ mm}$$

I = MOI of each semicircle about axis yy

= MOI of each semicircle about its own neutral axis + Area $\times b^2$

= (MOI of each semicircle about AB – Area $\times a^2$) + Area $\times b^2$

$$= \frac{1}{2}\left(\frac{\pi}{64} \times 90^4\right) - 3181 \times 19.1^2 + 3181 \times (50 - 19.1)^2$$

$$= 3.487 \times 10^6 \text{ mm}^4$$

$$I_{yy} = 12.5 \times 10^6 - 2 \times 3.487 \times 10^6 = 5.526 \times 10^6 \text{ mm}^4$$

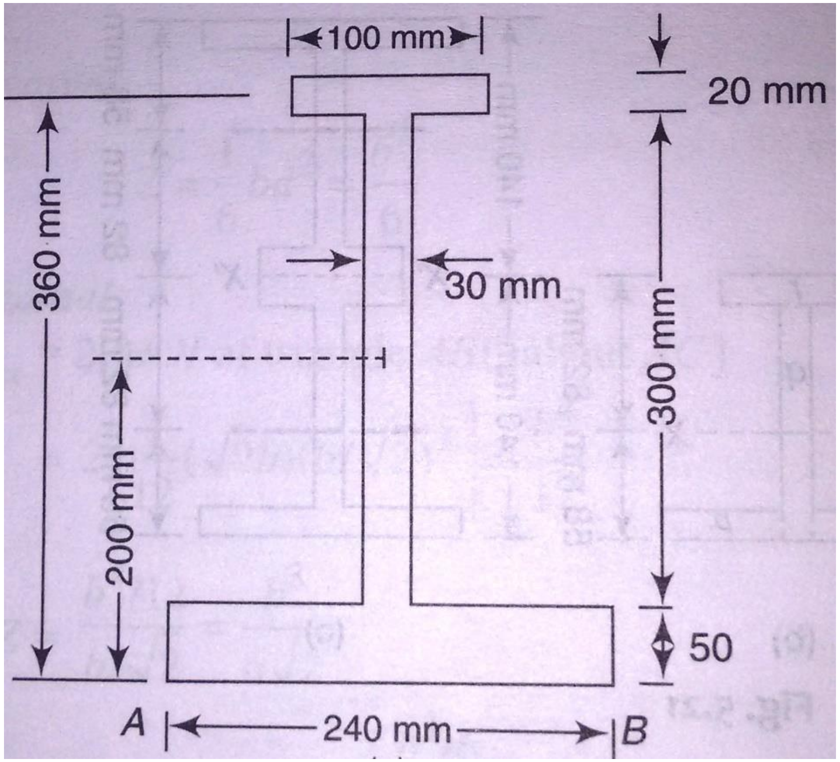
$$Z_{yy} = \frac{5.526 \times 10^6}{50} = 0.1105 \times 10^6 \text{ mm}^4$$

Ratio of moment of resistances

$$\frac{M_{yy}}{M_{xx}} = \frac{Z_{yy}}{Z_{xx}} = \frac{0.1105 \times 10^6}{0.332 \times 10^6} = 0.333$$

Problem 6

The tension flange of a cast iron I-section beam is 240 mm wide and 50 mm deep, the compression flange is 100 mm wide and 20 mm deep whereas the web is 300 mm x 30 mm. Find the load per *m* run which can be carried over a 4 m span by a simply supported beam if the maximum permissible stresses are 90 MPa in compression and 24 MPa in tension.



$$\boxed{w = 34.36 \text{ kN/m}}$$

Solution

Given An *I*-section beam as shown in Fig. 5.20a.
Permissible stresses 90 MPa in compression and 24 MPa in tension.
To find Load per m run for 4 m span

Moment of inertia of the section

$$A = 100 \times 20 + 300 \times 30 + 240 \times 50 = 23\,000 \text{ mm}^2$$

$$y = \frac{2000 \times 360 + 9000 \times 200 + 12\,000 \times 25}{23\,000} = 122.6 \text{ mm}$$

$$I_{ab} = \underbrace{\frac{1}{3} \times (240 - 30) \times 50^3}_{\text{Fig.5.20(b) (for } p)} + \underbrace{\frac{1}{3} \times 30 \times (300 + 50)^3}_{(\text{for } q)} + \underbrace{\frac{1}{12} X\, 100\, X\, 20^3 + 2000\, X\, 360^2}_{(\text{for } r)}$$

$$= 696.77 \times 10^6 \text{ mm}^4$$

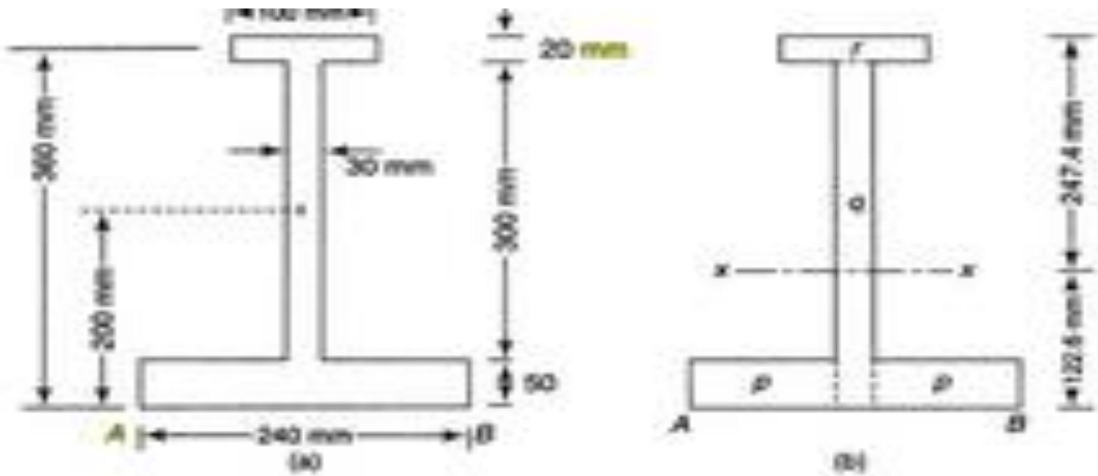


Fig. 5.20

$$I_{xx} = \text{MOI of the section about } AB - \text{Area} \times 122.6^2$$
$$= 696.77 \times 10^6 - 23000 \times 122.6^2 = 351.06 \times 10^6 \text{ mm}^4$$

Moment of resistance

Assuming that the stress in tension reaches its maximum value.

$$\frac{\sigma_c}{\sigma_t} = \frac{y_c}{y_t} \quad \text{or} \quad \frac{\sigma_c}{24} = \frac{(370 - 122.6)}{122.6}$$

or $\sigma_c = 48.43$ MPa

which is within the safe limits.

Moment of resistance $M_r = \frac{\sigma}{y} \cdot I = \frac{24}{122.6} \times 351.06 \times 10^6 = 68.72 \times 10^6$ N-mm

Calculation for uniformly distributed load

Let w be the uniformly distributed load on the beam.

$$\text{Maximum bending moment} = \frac{wl^2}{8} = \frac{w \times 4^2}{8} = 2w \text{ N-m}$$

or $2000w$ N-mm

Thus $2000w = 68.72 \times 10^6$

or $w = 34\,360$ N/m or 34.36 kN/m

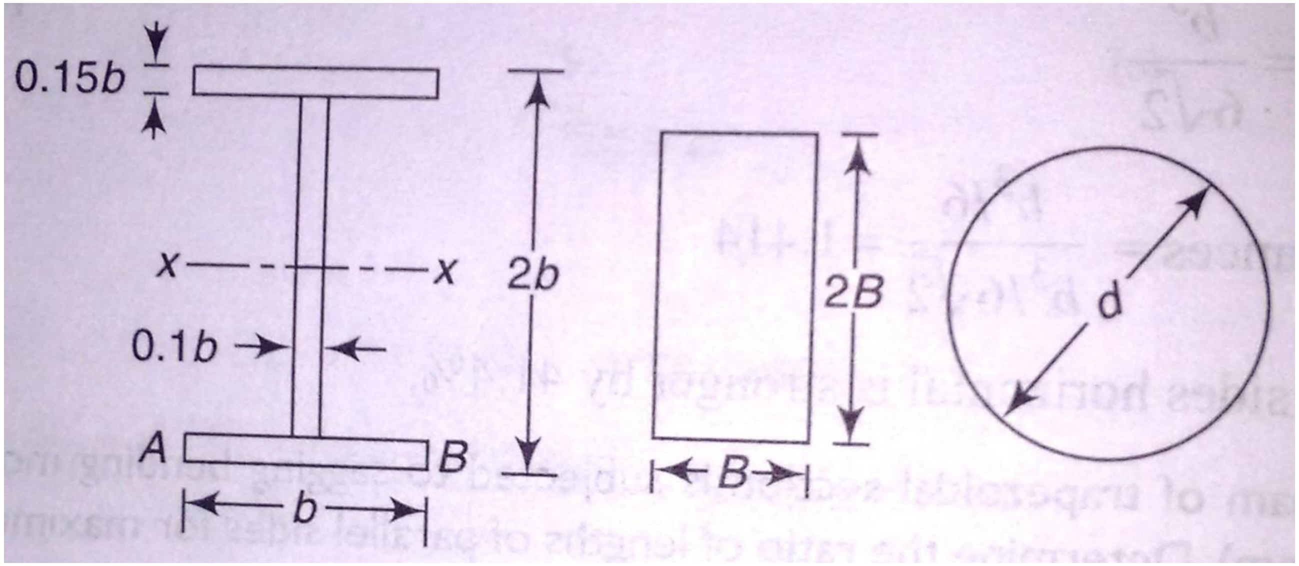
Problem 7

Compare the flexure strength of three beams of equal weight with the following specifications:

(i) I-section with length of web, thickness of each flange and the thickness of web in terms of width of *b* of the flanges being 1.7*b*, 0.15*b* and 0.1*b* respectively.

(ii) Rectangular section of depth equal to twice the width

(iii) Solid circular section

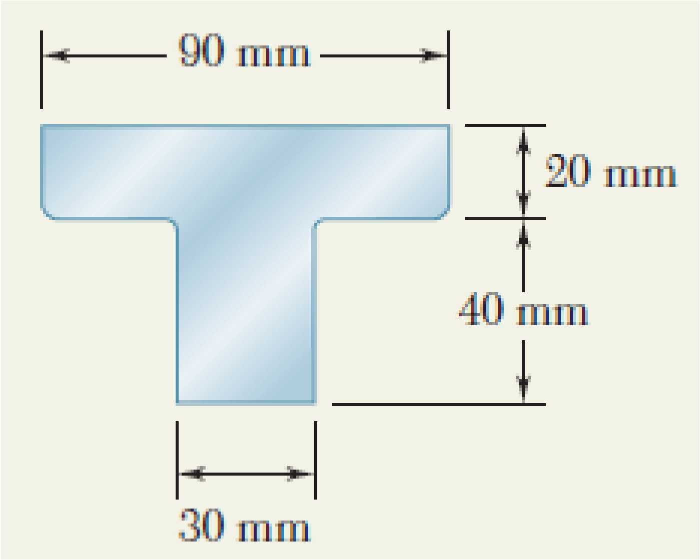


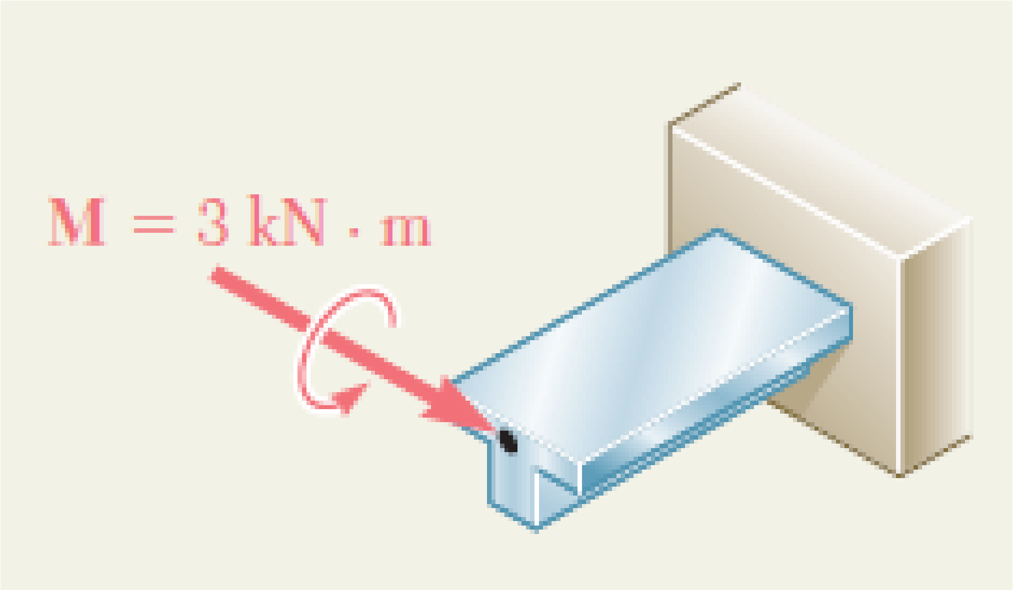
$$\boxed{\frac{Z_1}{Z_2} = 3.926}$$

$$\boxed{\frac{Z_1}{Z_3} = 6.561}$$

Problem 8

A cast-iron machine part is acted upon by the 3 kN-m couple shown. Knowing that E = 165 GPa and neglecting the effect of fillets, determine (*a*) the maximum tensile and compressive stresses in the casting, (*b*) the radius of curvature of the casting.



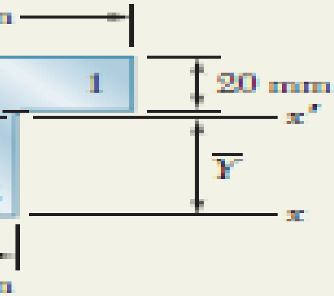


$\sigma_A = +76.0 \text{ MPa}$

$\sigma_B = -131.3 \text{ MPa}$

$\rho = 47.7 \text{ m}$

SOLUTION

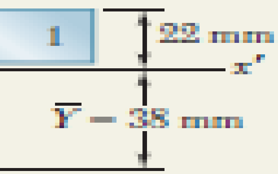


Centroid. We divide the T-shaped cross section into the two rectangles shown and write

	Area, mm²	$\bar{y}$, mm	$\bar{y}A$, mm³
1	$(20)(90) = 1800$	50	90×10^3
2	$(40)(30) = 1200$	20	24×10^3
	$\Sigma A = 3000$		$\Sigma \bar{y}A = 114 \times 10^3$

$$\bar{Y}\Sigma A = \Sigma \bar{y}A$$
$$\bar{Y}(3000) = 114 \times 10^6$$
$$\bar{Y} = 38 \text{ mm}$$

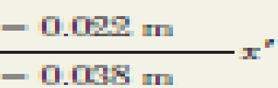
Centroidal Moment of Inertia. The parallel-axis theorem is used to determine the moment of inertia of each rectangle with respect to the axis x' that passes through the centroid of the composite section. Adding the moments of inertia of the rectangles, we write



$$\begin{aligned} I_{x'} &= \Sigma(\bar{I} + Ad^2) = \Sigma(\tfrac{1}{12}bh^3 + Ad^2) \\ &= \tfrac{1}{12}(90)(20)^3 + (90 \times 20)(12)^2 + \tfrac{1}{12}(30)(40)^3 + (30 \times 40)(18)^2 \\ &= 868 \times 10^3 \text{ mm}^4 \\ I &= 868 \times 10^{-9} \text{ m}^4 \end{aligned}$$

a. Maximum Tensile Stress. Since the applied couple bends the casting downward, the center of curvature is located below the cross section. The maximum tensile stress occurs at point A, which is farthest from the center of curvature.

$$\sigma_A = \frac{Mc_A}{I} = \frac{(3 \text{ kN} \cdot \text{m})(0.022 \text{ m})}{868 \times 10^{-9} \text{ m}^4} \qquad \sigma_A = +76.0 \text{ MPa} \blacktriangleleft$$



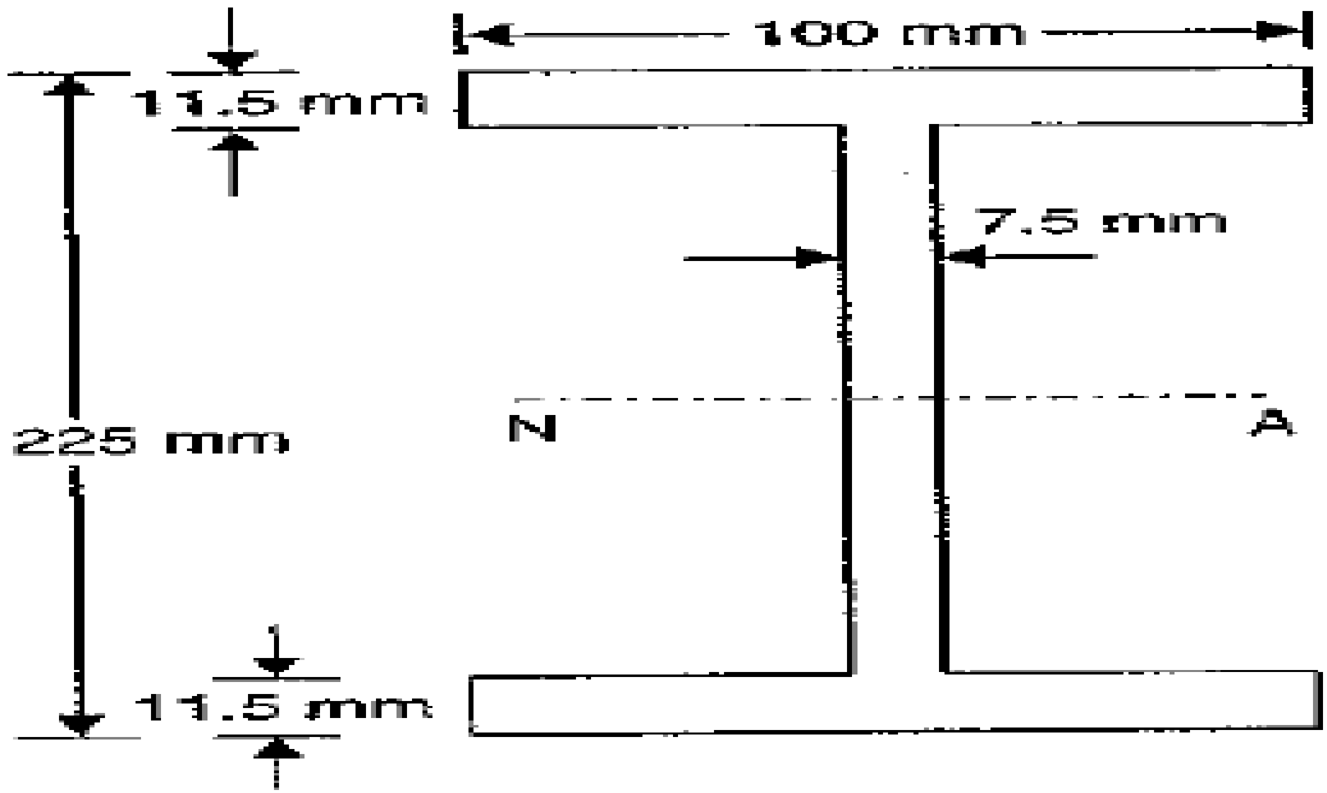
Maximum Compressive Stress. This occurs at point B; we have

$$\sigma_B = -\frac{Mc_B}{I} = -\frac{(3 \text{ kN} \cdot \text{m})(0.038 \text{ m})}{868 \times 10^{-9} \text{ m}^4} \qquad \sigma_B = -131.3 \text{ MPa} \blacktriangleleft$$

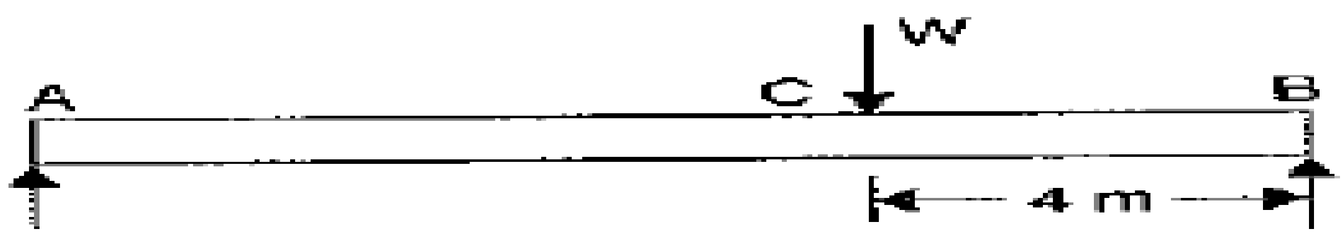
b. Radius of Curvature. From Eq. (4.21), we have

$$\frac{1}{\rho} = \frac{M}{EI} = \frac{3 \text{ kN} \cdot \text{m}}{(165 \text{ GPa})(868 \times 10^{-9} \text{ m}^4)}$$
$$= 20.95 \times 10^{-3} \text{ m}^{-1} \qquad \rho = 47.7 \text{ m} \blacktriangleleft$$

Problem 7.11. *An I-section shown in Fig. 7.16, is simply supported over a span of 12 m. If the maximum permissible bending stress is 80 N/mm², what concentrated load can be carried at a distance of 4 m from one support ?*



(a)



To find the maximum bending moment (which will be at point C where concentrated load is acting), first calculate the reactions R_A and R_B.

Taking moments about point A, we get

$$R_B \times 12 = W \times 8$$

$$\therefore \quad R_B = \frac{8W}{12} = \frac{2}{3} W$$

and

$$R_A = W - R_B = W - \frac{2}{3} W = \frac{W}{3}$$

B.M. at point C

$$= R_A \times 8 = \frac{W}{3} \times 8 = \frac{8}{3} W \text{ Nm}$$

But B.M. at C is maximum

$\therefore$ Maximum B.M.,

$$M_{max} = \frac{8}{3} W \text{ Nm} = \frac{8}{3} W \times 1000 \text{ Nmm}$$

$$= \frac{8000}{3} W \text{ Nmm}$$

Now find the moment of inertia of the given I-section about the N.A.

$$\therefore \quad I = \frac{100 \times 225^3}{12} - \frac{(100 - 7.5) \times (225 - 2 \times 11.5)^3}{12}$$

$$= 94921875 - \frac{92.5 \times (202)^3}{12}$$

$$= 94921875 - 63535227.55 = 31386647.45 \text{ mm}^4$$

Now using the relation,

$$\frac{M}{I} = \frac{\sigma}{y}$$

or

$$\frac{M}{I} = \frac{\sigma_{max}}{y_{max}}$$

where $y_{max} = \dfrac{225}{2} = 112.5$ mm.

Now substituting the known values, we get

$$\frac{\left(\dfrac{8000}{3} W\right)}{31386647.45} = \frac{80}{112.5}$$

or

$$W = \frac{80}{112.5} \times 31386647.45 \times \frac{3}{8000} = \mathbf{8369.77\ N.} \quad \textbf{Ans.}$$

Problem 7.16. *A cast iron beam is of T-section as shown in Fig. 7.21. The beam is simply supported on a span of 8 m. The beam carries a uniformly distributed load of 1.5 kN/m length on the entire span. Determine the maximum tensile and maximum compressive stresses.*

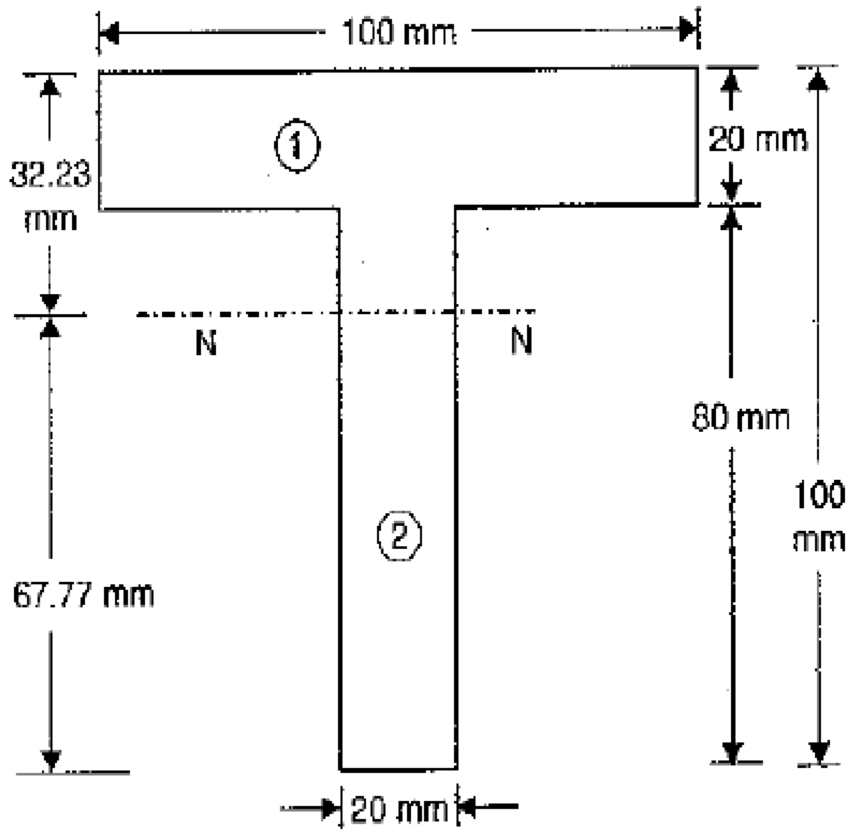


Fig. 7.21

$$\therefore \quad \bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2} = \frac{(100 \times 20) \times \left(80 + \frac{20}{2}\right) + 80 \times 20 \times \frac{80}{2}}{(100 \times 2) + (80 \times 20)}$$

$$= \frac{180000 + 64000}{2000 + 1600} = \frac{244000}{3600} = 67.77 \text{ mm}$$

$\therefore$ N.A. lies at a distance of 67.77 mm from the bottom face or 100 – 67.77 = 32.23 mm from the top face.

Now moment of inertia of the section about N.A. is given by,

$$I = I_1 + I_2$$

where I_1 = M.O.I. of top flange about N.A.

= M.O.I. of top flange about its C.G. + A_1 × (Distance of its C.G. from N.A.)^2

$$= \frac{100 \times 20^3}{12} + (100 \times 20) \times (32.23 - 10)^2$$

$$= 66666.7 + 988345.8 = 1055012.5 \text{ mm}^4$$

I_2 = M.O.I. of web about N.A.

= M.O.I. of web about its C.G. + A_2 × (Distance of its C.G. from N.A.)^2

$$= \frac{20 \times 80^3}{12} + (80 \times 20) \times (67.77 - 40)^2$$

$$= 853333.3 + 1233876.6 = 2087209.9 \text{ mm}^4$$

$$I = I_1 + I_2 = 1055012.5 + 2087209.9 = 3142222.4 \text{ mm}^4.$$

For a simply supported beam, the maximum tensile stress will be at the extreme bottom fibre and maximum compressive stress will be at the extreme top fibre.

Maximum B.M. is given by,

$$M = \frac{w \times L^2}{8} = \frac{1500 \times 8^2}{8} = 12000 \text{ Nm}$$

$$= 12000 \times 1000 = 12000000 \text{ Nmm}$$

Now using the relation

$$\frac{M}{I} = \frac{\sigma}{y} \quad \text{or} \quad \sigma = \frac{M}{I} \times y$$

(*i*) For maximum tensile stress,

y = Distance of extreme bottom fibre from N.A. = 67.77 mm

$$\therefore \quad \sigma = \frac{12000000}{3142222.4} \times 67.77 = \mathbf{258.81 \ N/mm^2}. \quad \textbf{Ans.}$$

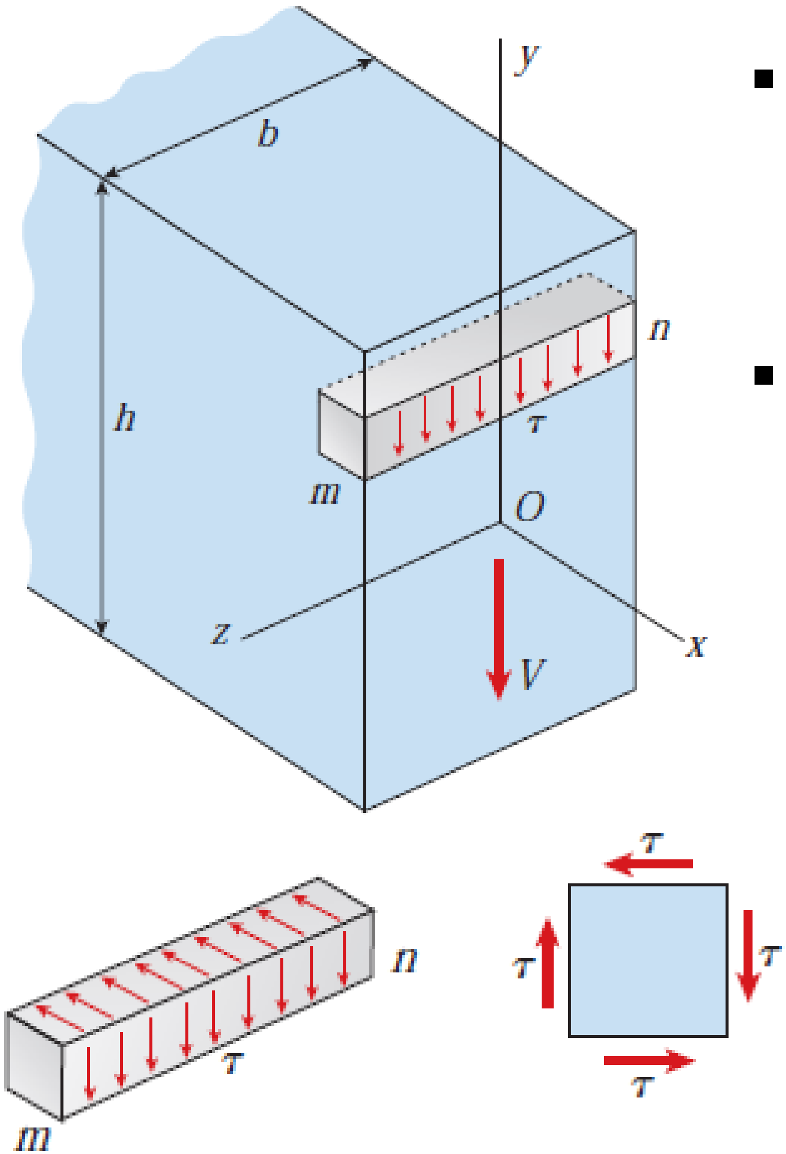
(*ii*) For maximum compressive stress,

y = Distance of extreme top fibre from N.A. = 32.23 mm

$$\therefore \quad \sigma = \frac{M}{I} \times y = \frac{12000000}{3142222.4} \times 32.23 = \mathbf{123.08 \ N/mm^2}. \quad \textbf{Ans.}$$

Shear Stresses in Beams

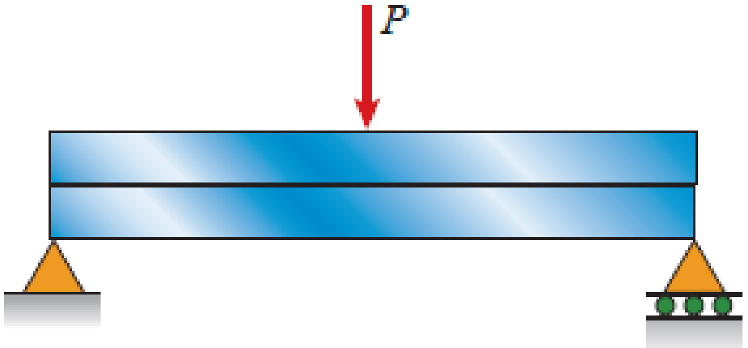
Assumptions



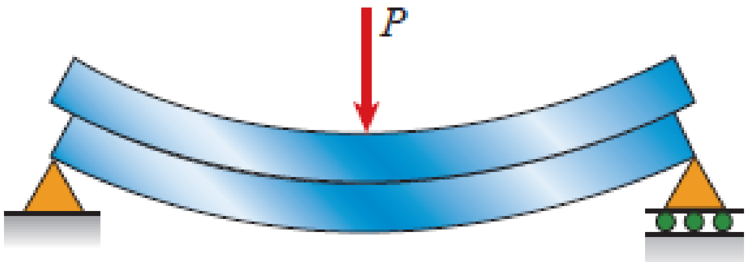
- The shear stresses $\boldsymbol{\tau}$ acting on the cross section are parallel to the shear force

- The shear stresses are uniformly distributed across the width of the beam, although they may vary over the height

$$\boldsymbol{\tau = 0 \; at \; y = \pm \frac{h}{2}}$$

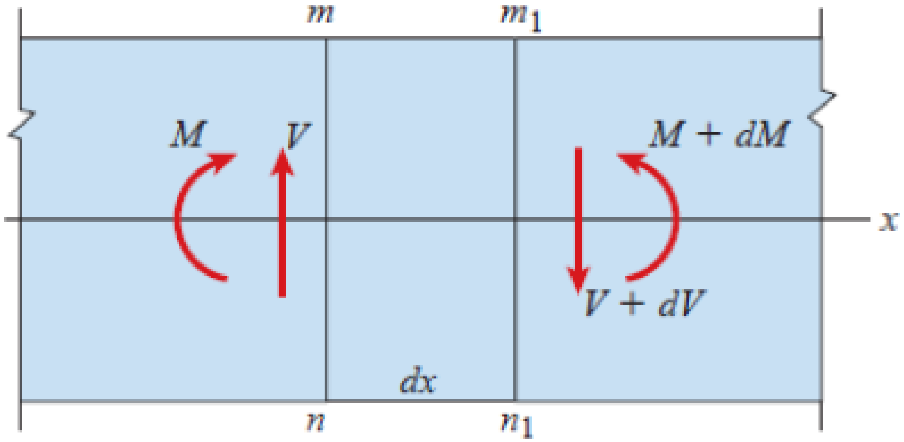


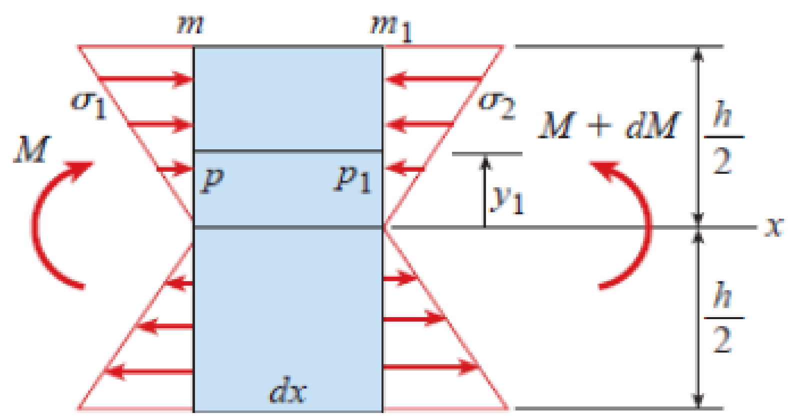
(a)



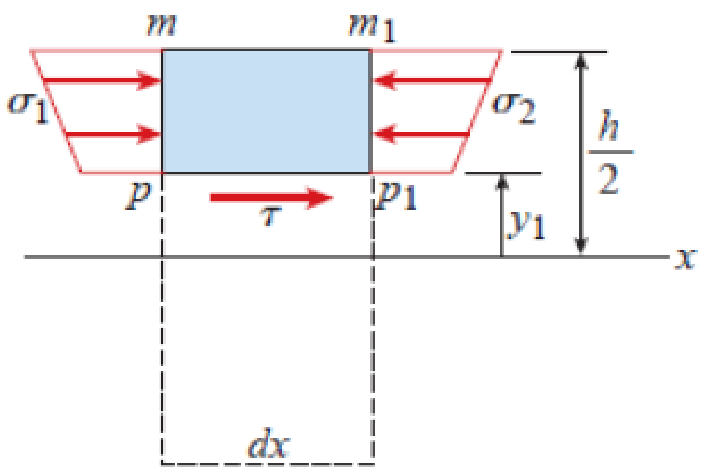
- Each beam will be in compression above its own neutral axis and in tension below its neutral axis
- Therefore the bottom surface of the upper beam will slide with respect to the top surface of the lower beam
- When this beam is glued and loaded, horizontal shear stresses must develop along the glued surface in order to prevent the sliding
- Because of the presence of these shear stresses, the single solid beam is much stiffer and stronger than the two separate beams

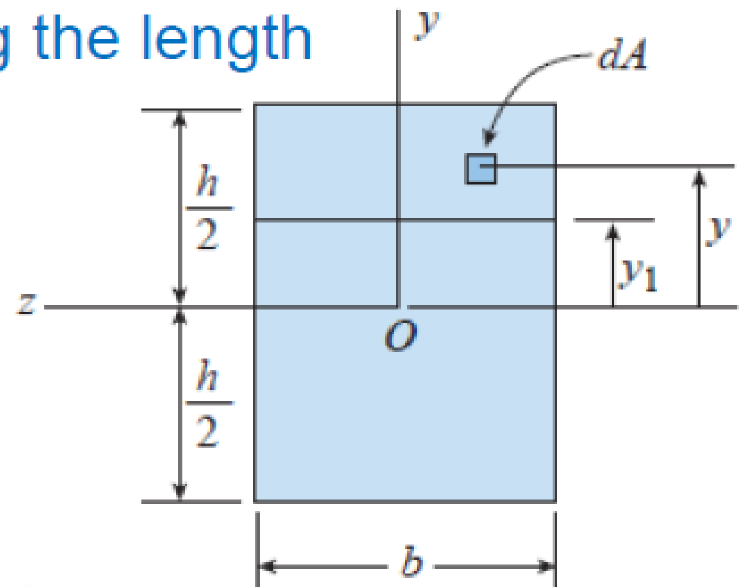
Derivation of Shear Formula





Portion of loaded beam along the length



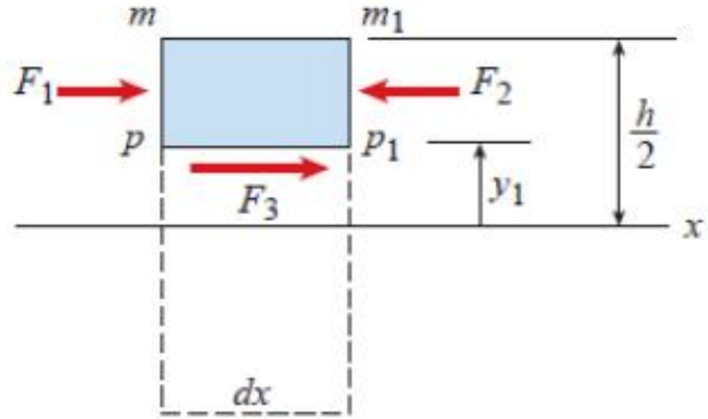


$$\sigma_1 = -\frac{My}{I} \text{ and } \sigma_2 = -\frac{(M + dM)y}{I}$$

Beam cross section

$$\sigma_1 dA = \frac{My}{I} dA$$

$$F_1 = \int \sigma_1 \, dA = \int \frac{My}{I} \, dA$$



$$F_2 = \int \sigma_2 dA = \int \frac{(M+dM)y}{I} \, dA$$

$$\overset{+}{\rightarrow} \Sigma F_x = 0:$$

$$F_3 = F_2 - F_1$$

$$F_3 = \int \frac{(M + dM)y}{I} \, dA - \int \frac{My}{I} dA = \int \frac{(dM)y}{I} dA$$

$$F_3 = \tau b \, dx$$

$$\tau = \frac{dM}{dx}\left(\frac{1}{Ib}\right)\int y \, dA$$

$$\tau = \frac{V}{Ib}\int y \, dA$$

$$Q = \int y \, dA$$

$$\boxed{\tau = \frac{VQ}{Ib}}$$

$$\tau = \frac{VQ}{Ib}$$

$\tau =$ shear stress in a member at any distance from neutral axis

$V =$ internal resultant shear force

$Q =$ $\bar{y}A'$, first moment of area of the cross section above or below the distance y_1 from the neutral axis

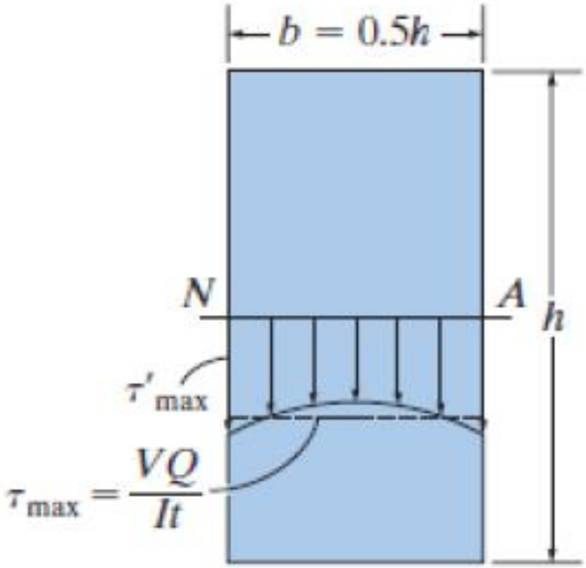
$\bar{y}$ – distance of centroid of the cross section above (or below) y_1 from the neutral axis

A' – Area of the cross section above or below y_1 from the neutral axis

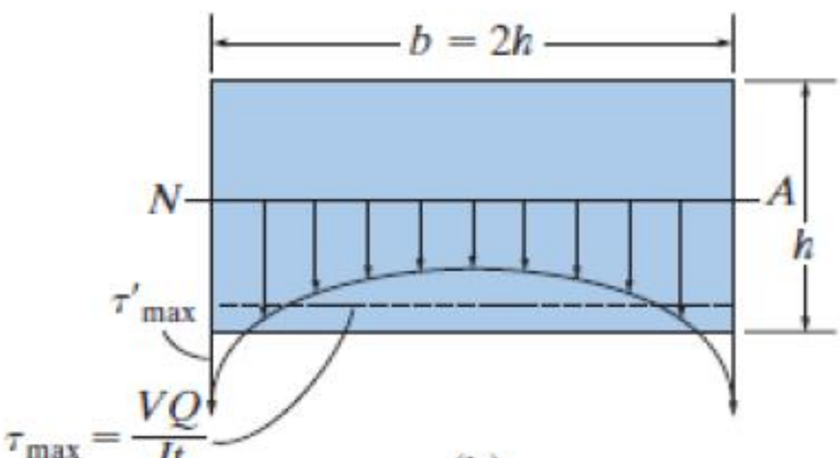
$I =$ moment of inertia of the entire cross section

$b =$ width of the section at y_1 from the neutral axis

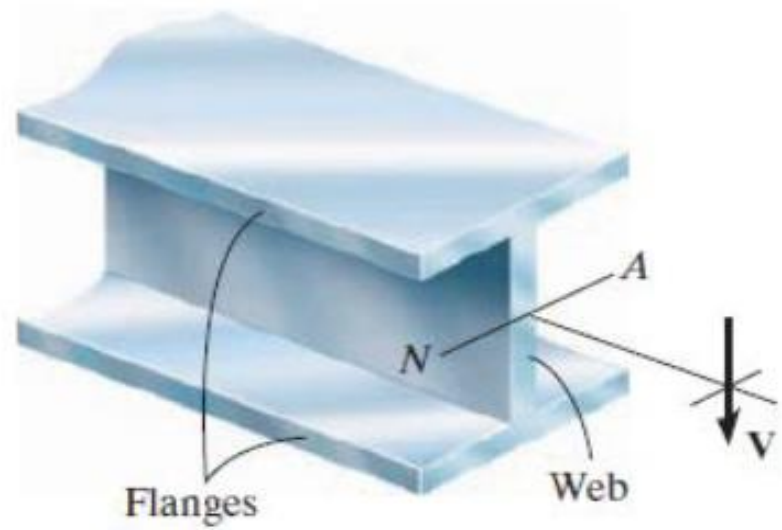
Limitations on the Use of the Shear Formula



The shear formula **does not** give accurate results when applied to members having cross sections that are *short or flat*, or at points where the cross section suddenly changes.



(b)



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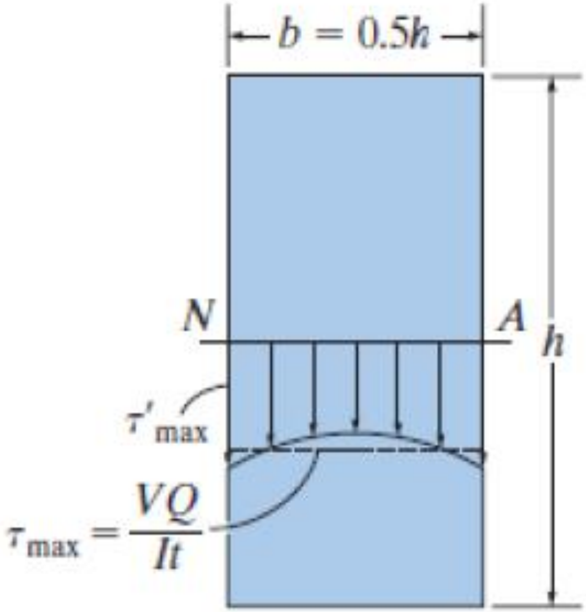
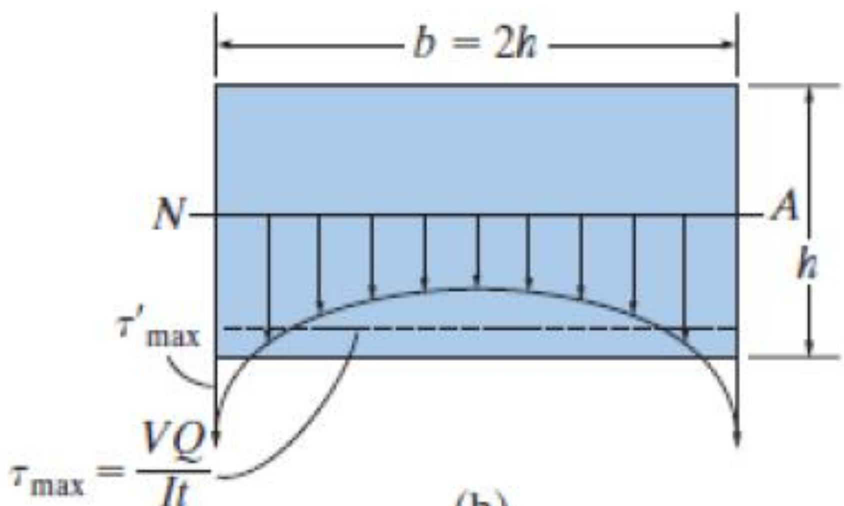
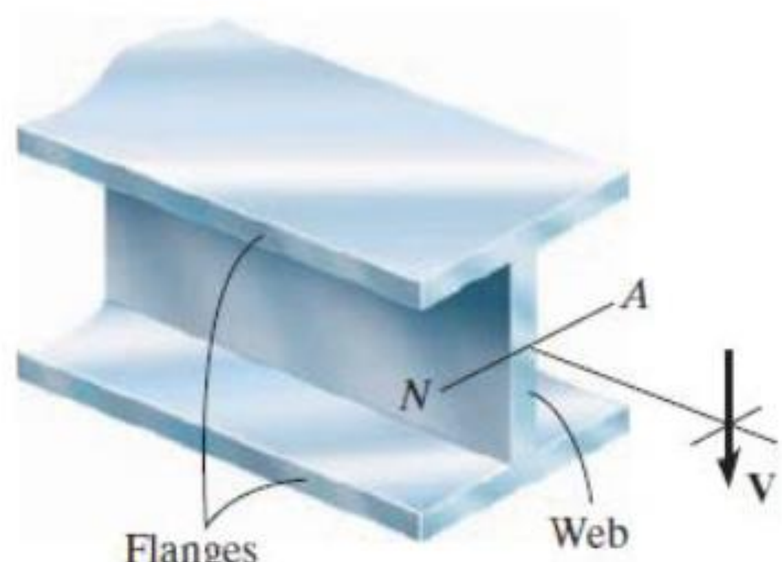
Limitations on the Use of the Shear Formula



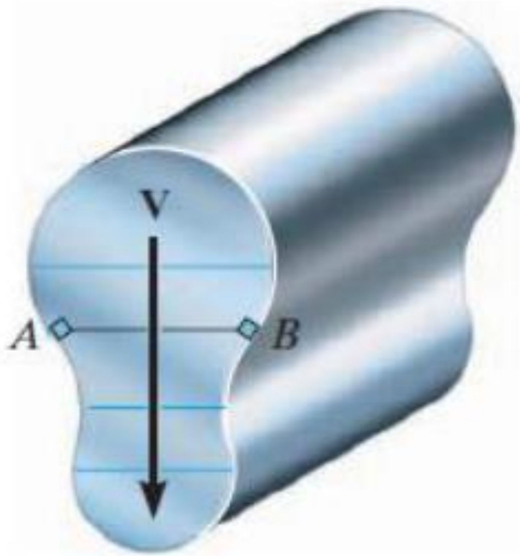
The shear formula **does not** give accurate results when applied to members having cross sections that are *short or flat*, or at points where the cross section suddenly changes.



(b)

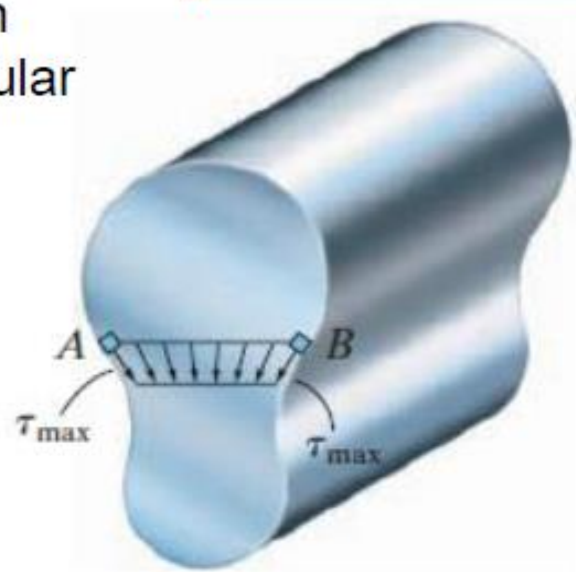


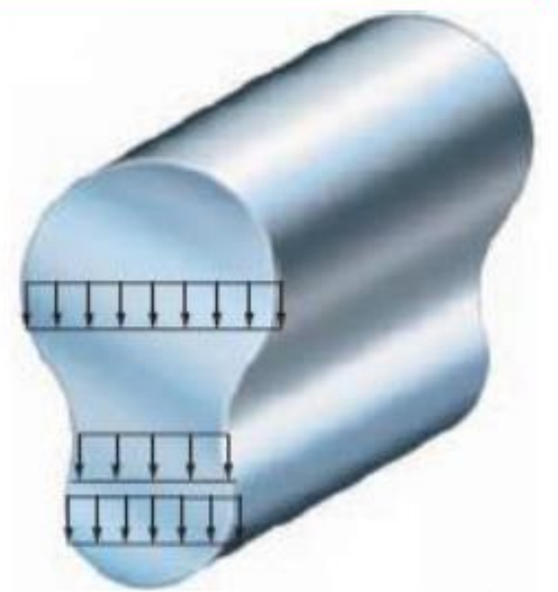
Limitations on the Use of the Shear Formula



cross section with an irregular or nonrectangular boundary

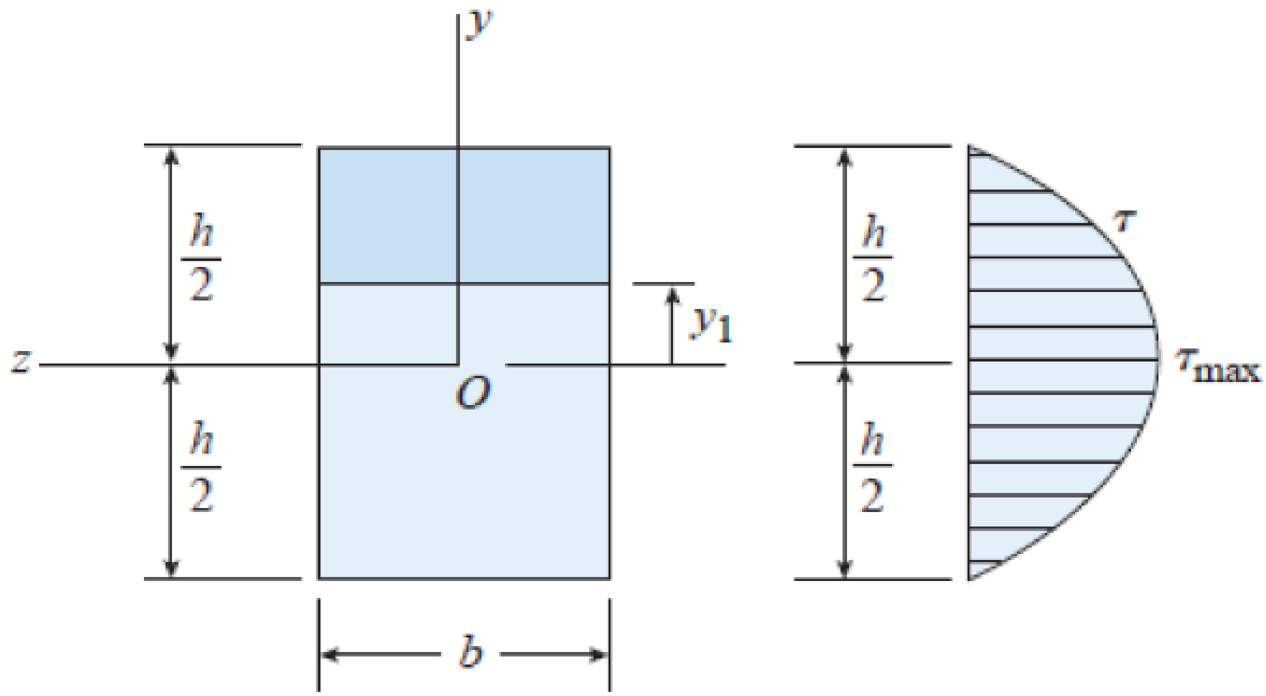
The shear formula does not give accurate results when applied across a section that intersects the boundary of the member at an angle other than 90°.





Exercise 1

Distribution of Shear Stresses in a Rectangular Beam



$$Q = b\left(\frac{h}{2} - y_1\right)\left(y_1 + \frac{h/2 - y_1}{2}\right) = \frac{b}{2}\left(\frac{h^2}{4} - y_1^2\right)$$

$$Q = \int y\,dA = \int_{y_1}^{h/2} yb\,dy = \frac{b}{2}\left(\frac{h^2}{4} - y_1^2\right)$$

$$\boxed{\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y_1^2\right)}$$

$$\boxed{\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y_1^2\right)}$$

The shear stress is zero when $y_1 = \pm\, h/2$.

When $y_1 = 0$

$$\tau_{\max} = \frac{Vh^2}{8I} = \frac{3V}{2A}$$

Where, $I = bh^3/12$ and $A = bh$

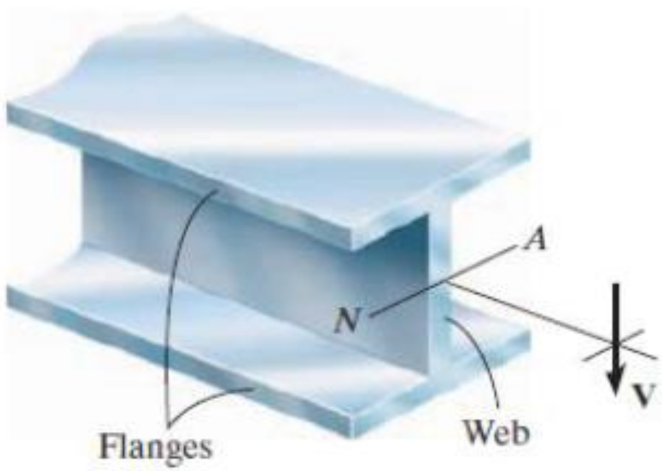
$$\tau_{ave} = \frac{V}{A}$$

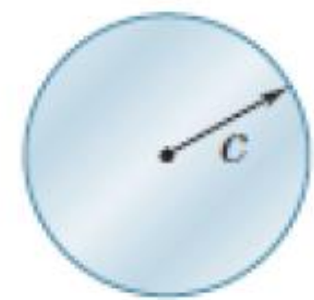
$$\boxed{\therefore\ \tau_{max} = 1.5\ \tau_{ave}}$$

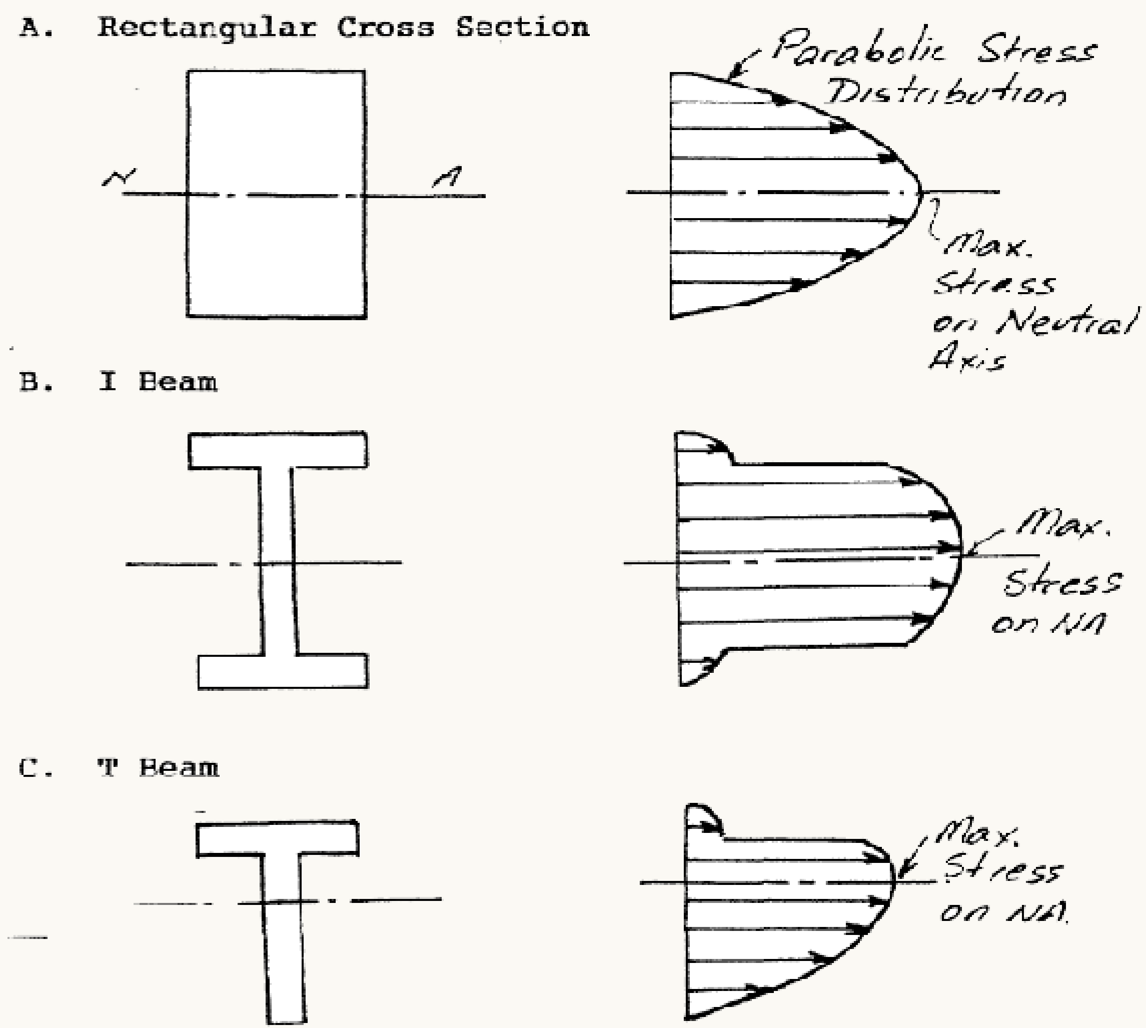
Exercises

Find the distribution of Shear Stresses in the following cross sections

- I section (video lecture available in youtube)
- T section
- Circular cross-section



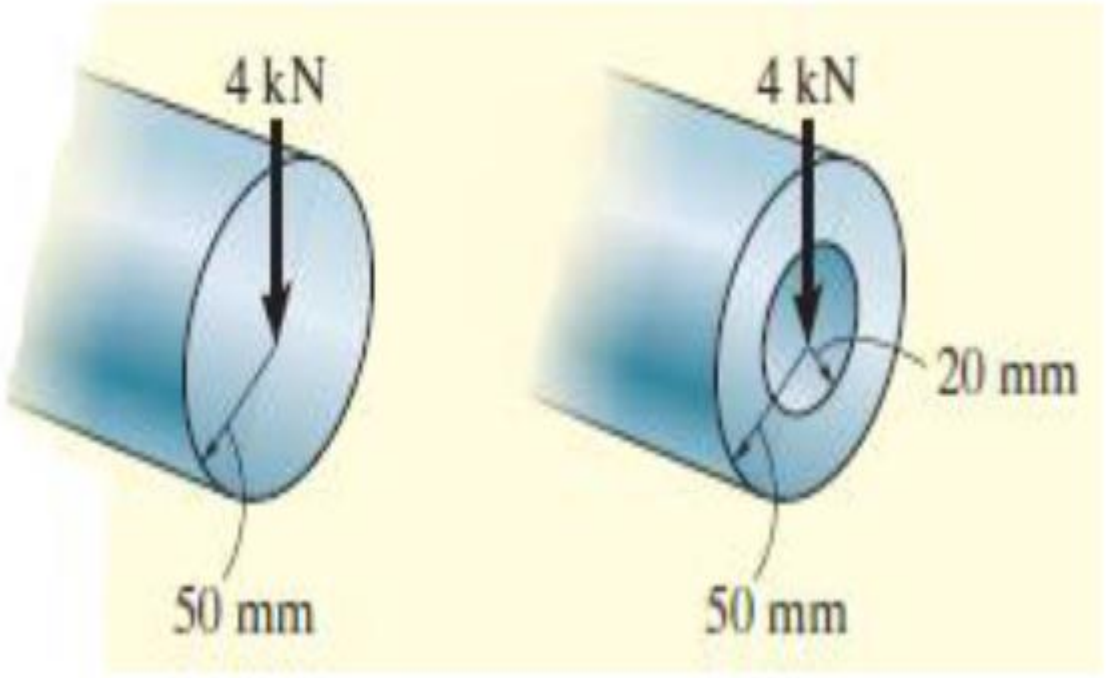




Distribution of shear stress across a beam section.

Problem 1

The solid shaft and tube shown in Figure are subjected to the shear force of 4 kN. Determine the shear stress acting over the diameter of each cross section.



Solution

$$I_{\text{solid}} = \frac{1}{4}\pi c^4 = \frac{1}{4}\pi(0.05\text{ m})^4 = 4.909(10^{-6})\text{ m}^4$$

$$I_{\text{tube}} = \frac{1}{4}\pi(c_o^4 - c_i^4) = \frac{1}{4}\pi[(0.05\text{ m})^4 - (0.02\text{ m})^4] = 4.783(10^{-6})\text{ m}^4$$

$$Q_{\text{solid}} = \overline{y}'A' = \frac{4c}{3\pi}\left(\frac{\pi c^2}{2}\right) = \frac{4(0.05\text{ m})}{3\pi}\left(\frac{\pi(0.05\text{ m})^2}{2}\right) = 83.33\,(10^{-6})\text{ m}^3$$

$$Q_{\text{tube}} = \Sigma\overline{y}'A' = \frac{4c_o}{3\pi}\left(\frac{\pi c_o^2}{2}\right) - \frac{4c_i}{3\pi}\left(\frac{\pi c_i^2}{2}\right)$$

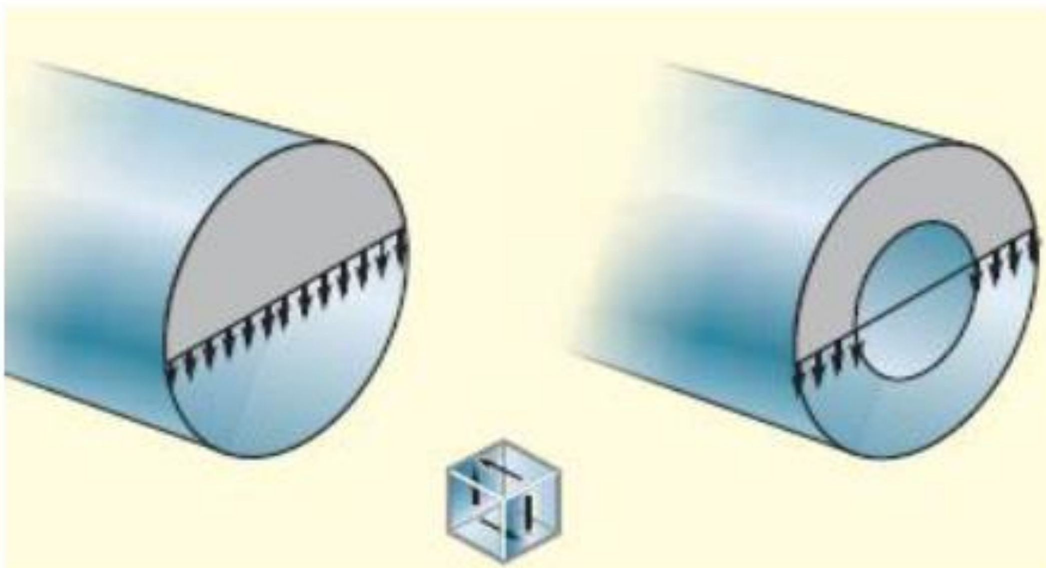
$$= \frac{4(0.05\text{ m})}{3\pi}\left(\frac{\pi(0.05\text{ m})^2}{2}\right) - \frac{4(0.02\text{ m})}{3\pi}\left(\frac{\pi(0.02\text{ m})^2}{2}\right)$$

$$= 78.0(10^{-6})\text{ m}^3$$

Shear Stress. Applying the shear formula where $t = 0.1$ m for t
solid section, and $t = 2(0.03 \text{ m}) = 0.06$ m for the tube, we have

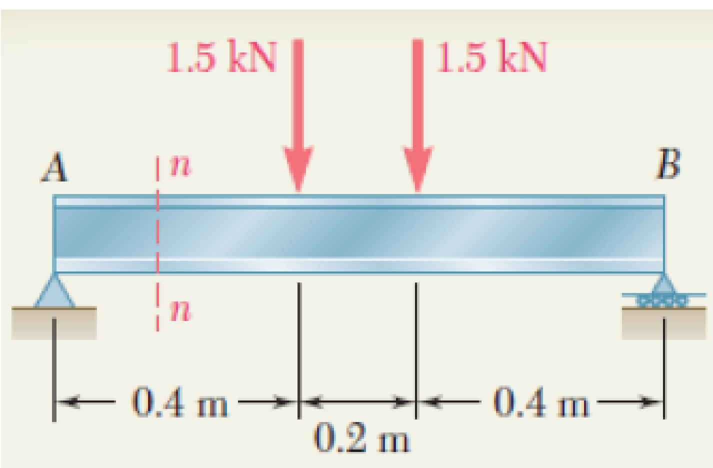
$$\tau_{\text{solid}} = \frac{VQ}{It} = \frac{4(10^3)\text{ N}(83.33(10^{-6})\text{ m}^3)}{4.909(10^{-6})\text{ m}^4(0.1\text{ m})} = 679\text{ kPa}$$

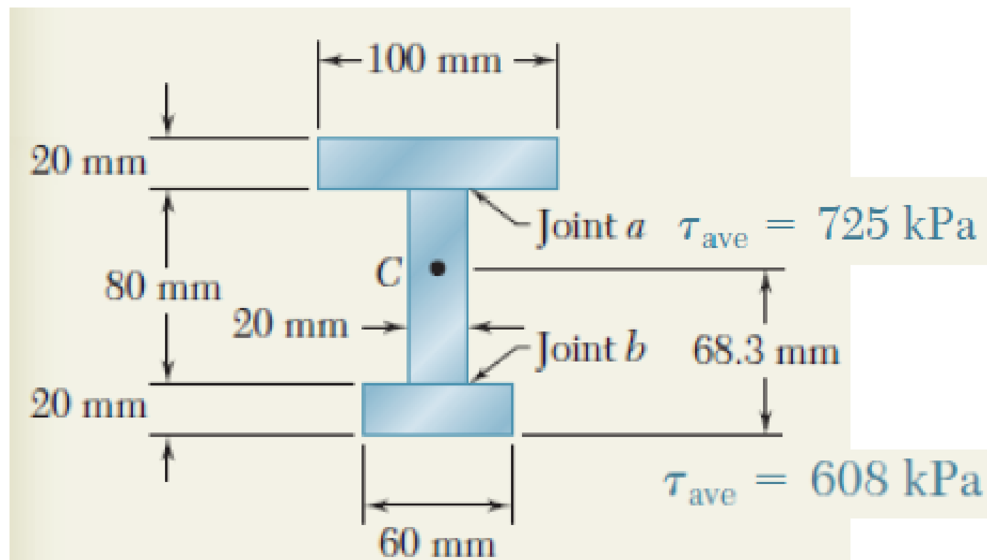
$$\tau_{\text{tube}} = \frac{VQ}{It} = \frac{4(10^3)\text{ N}(78.0(10^{-6})\text{ m}^3)}{4.783(10^{-6})\text{ m}^4(0.06\text{ m})} = 1.09\text{ MPa}$$

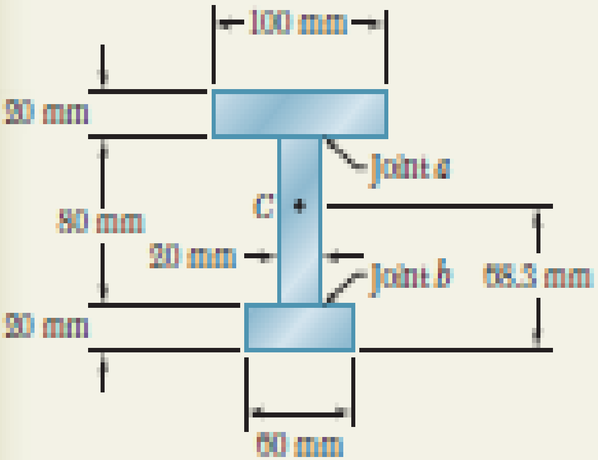


Problem 2

Beam *AB* is made of three planks glued together and is subjected, in its plane of symmetry, to the loading shown. Knowing that the width of each glued joint is 20 mm, determine the average shearing stress in each joint at section *n-n* of the beam. The location of the centroid of the section is given in the sketch and the centroidal moment of inertia is known to be $I = 8.63 \times 10^{-6}\ m^4$.

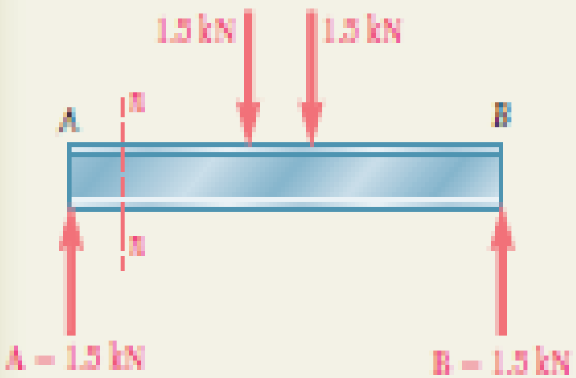


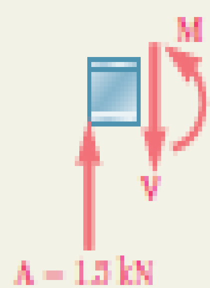




SOLUTION

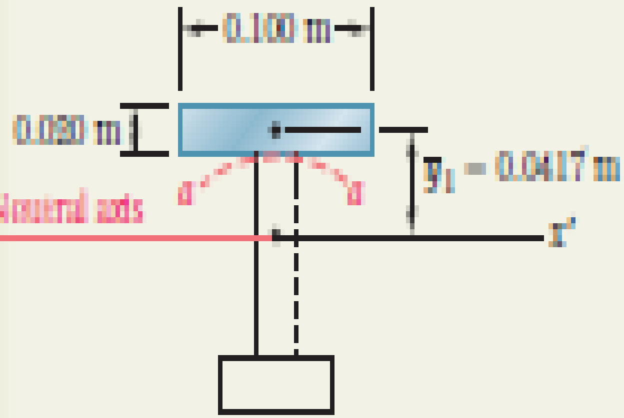
Vertical Shear at Section n-n. Since the beam and loading are both symmetric with respect to the center of the beam, we have $\mathbf{A} = \mathbf{B} = 1.5 \text{ kN} \uparrow$.





Considering the portion of the beam to the left of section n-n as a free body, we write

$$+\uparrow \sum F_y = 0: \qquad 1.5 \text{ kN} - V = 0 \qquad V = 1.5 \text{ kN}$$

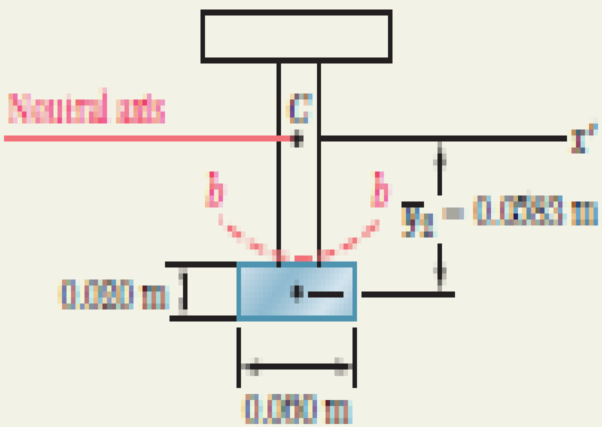


Shearing Stress in Joint *a*. We pass the section *a-a* through the glued joint and separate the cross-sectional area into two parts. We choose to determine Q by computing the first moment with respect to the neutral axis of the area above section *a-a*.

$$Q = A\bar{y}_1 = [(0.100\text{ m})(0.020\text{ m})](0.0417\text{ m}) = 83.4 \times 10^{-6}\text{ m}^3$$

Recalling that the width of the glued joint is $t = 0.020$ m, we use Eq. (6.7) to determine the average shearing stress in the joint.

$$\tau_{ave} = \frac{VQ}{It} = \frac{(1500\text{ N})(83.4 \times 10^{-6}\text{ m}^3)}{(8.63 \times 10^{-6}\text{ m}^4)(0.020\text{ m})} \qquad \tau_{ave} = 725\text{ kPa}$$



Shearing Stress in Joint *b*. We now pass section *b-b* and compute Q by using the area below the section.

$$Q = A\bar{y}_2 = [(0.060\text{ m})(0.020\text{ m})](0.0583\text{ m}) = 70.0 \times 10^{-6}\text{ m}^3$$

$$\tau_{ave} = \frac{VQ}{It} = \frac{(1500\text{ N})(70.0 \times 10^{-6}\text{ m}^3)}{(8.63 \times 10^{-6}\text{ m}^4)(0.020\text{ m})} \qquad \tau_{ave} = 608\text{ kPa}$$