Introduction

- Long slender members subjected to an axial compressive force are called ***columns***, and the lateral deflection that occurs is called ***buckling***
- Quite often the buckling of a column can lead to a sudden and dramatic failure of a structure
- As a result, special attention must be given to the design of columns so that they can safely support their intended loadings without buckling

- The maximum axial load that a column can support when it is on the *verge* of buckling is called the ***critical load*** (P_{cr})
- Any additional loading will cause the column to buckle and therefore deflect

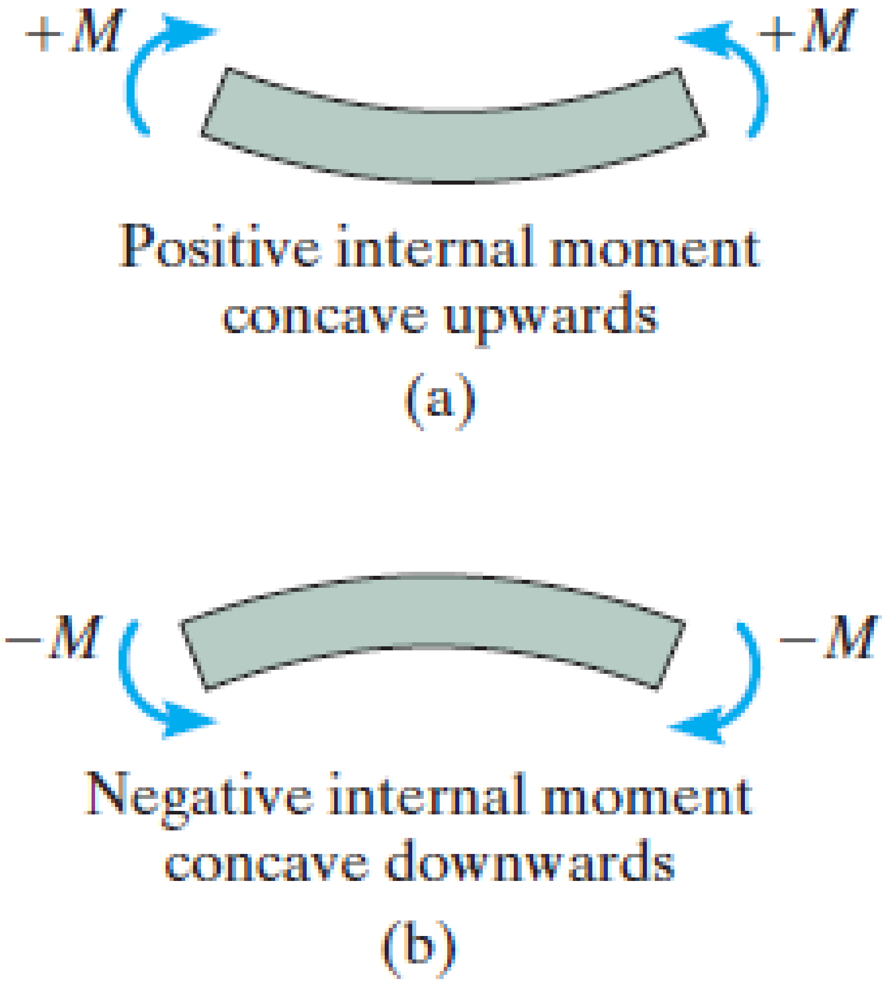
Ideal Column

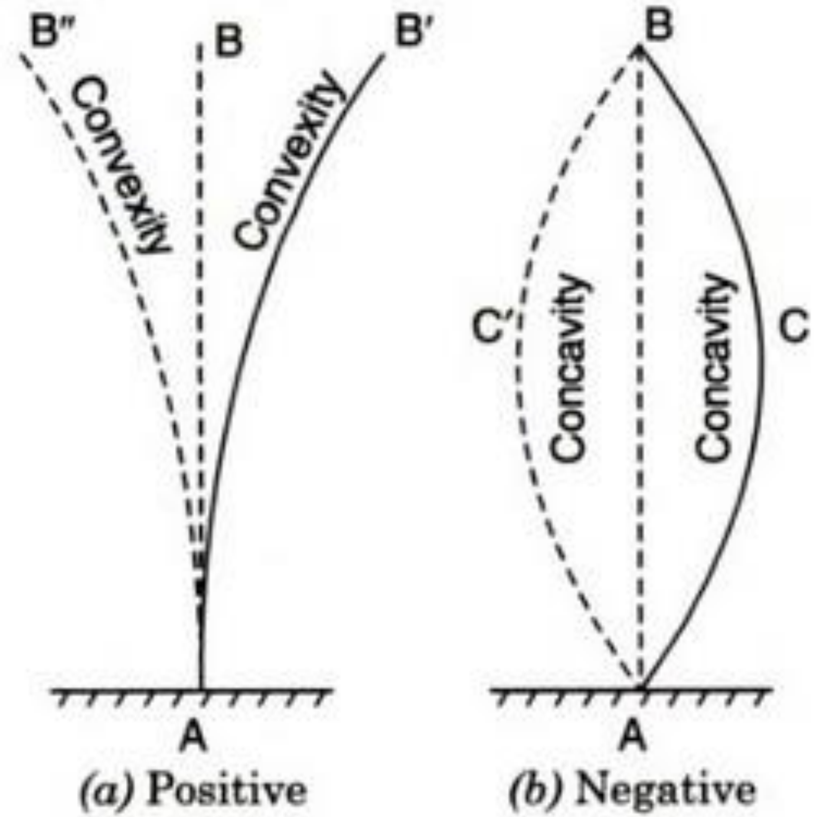
An ***ideal column***

- Perfectly straight before loading
- Made of homogeneous material
- The load is applied through the centroid of the cross section
- The material behaves in a linear-elastic manner
- The column buckles or bends in a single plane

Sign Convention

$$M = EI\frac{d^2y}{dx^2}$$





Euler's Formula for Pin-Ended Columns

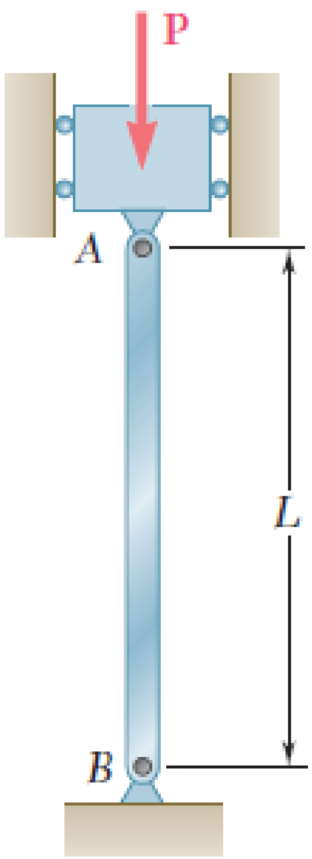


Fig. (a)

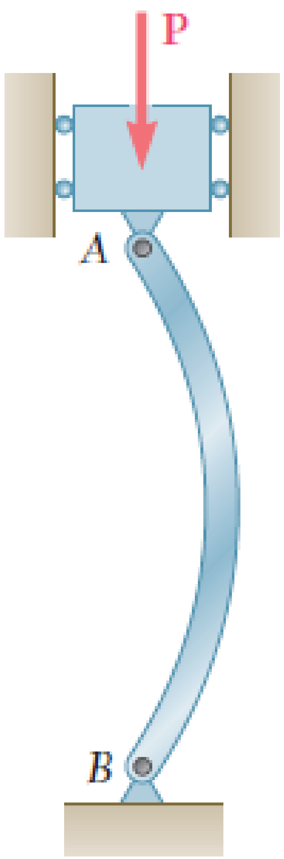


Fig. (b)

- we propose to determine the critical value of the load P, i.e., the value P_{cr} of the load for which the position shown in Fig. (a) ceases to be stable
- If $P > P_{cr}$, the slightest misalignment or disturbance will cause the column to buckle
- The approach will be to determine the conditions under which the configuration of Fig. (b) is possible

Considering the equilibrium of the free body *AQ*, we find that the bending moment at *Q* is *M = -Py*

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{P}{EI}y$$

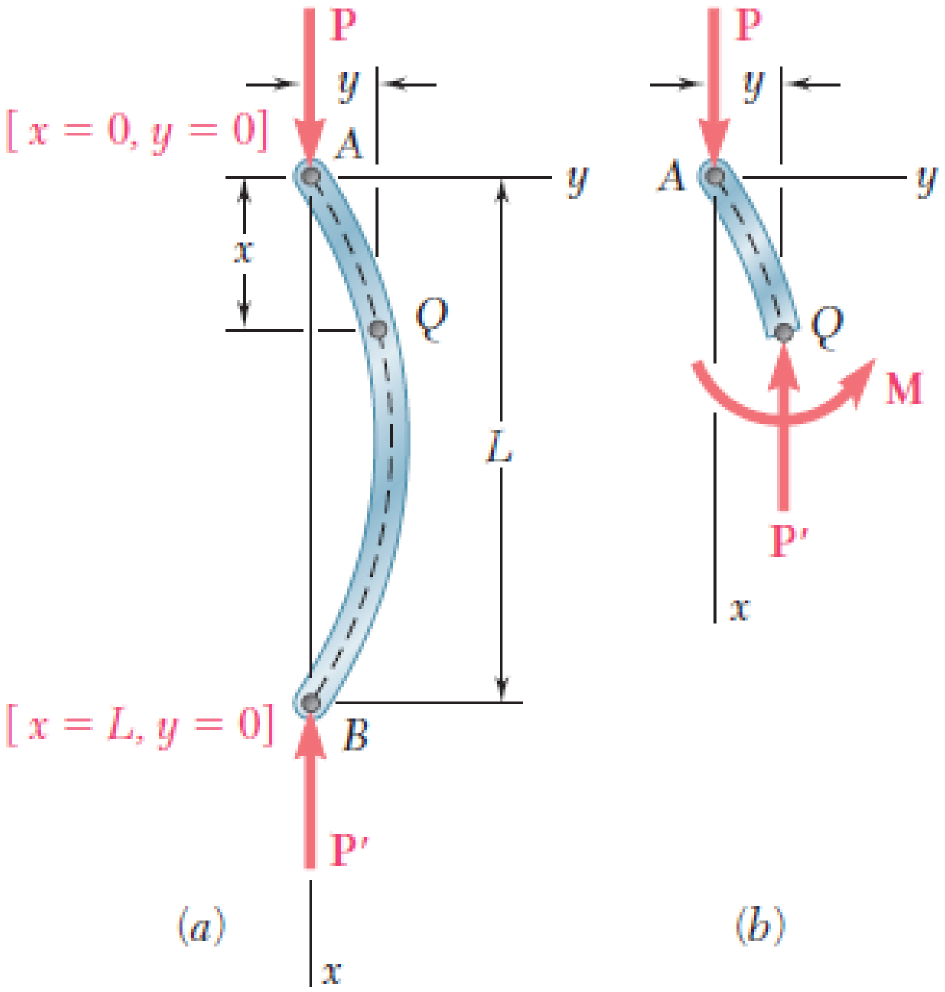
$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

Setting $$p^2 = \frac{P}{EI}$$

$$\frac{d^2y}{dx^2} + p^2y = 0$$

The general solution is

$$y = A\sin px + B\cos px$$



Column in buckled position

$$y = A \sin px + B \cos px$$

Applying boundary conditions

$x = 0$, $y = 0$ $\qquad\qquad$ $x = L$, $y = 0$

$$B = 0. \qquad\qquad A \sin pL = 0$$

if $A = 0$, then $y = 0$

$$\sin pL = 0$$

$$pL = n\pi$$

$$\boxed{P = \frac{n^2 \pi^2 EI}{L^2}}$$

The smallest of the values of P is corresponding to $n = 1$

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

Euler's formula

A 2-m-long pin-ended column of square cross section is to be made of wood. Assuming $E = 13$ GPa, $\sigma_{all} = 12$ MPa, and using a factor of safety of 2.5 in computing Euler's critical load for buckling, determine the size of the cross section if the column is to safely support (*a*) a 100-kN load, (*b*) a 200-kN load.

(a) For the 100-kN Load. Using the given factor of safety, we make

$$P_{cr} = 2.5(100\text{ kN}) = 250\text{ kN} \qquad L = 2\text{ m} \qquad E = 13\text{ GPa}$$

in Euler's formula (10.11) and solve for I. We have

$$I = \frac{P_{cr}L^2}{\pi^2 E} = \frac{(250 \times 10^3\text{ N})(2\text{ m})^2}{\pi^2(13 \times 10^9\text{ Pa})} = 7.794 \times 10^{-6}\text{ m}^4$$

Recalling that, for a square of side a, we have $I = a^4/12$, we write

$$\frac{a^4}{12} = 7.794 \times 10^{-6}\text{ m}^4 \qquad a = 98.3\text{ mm} \approx 100\text{ mm}$$

We check the value of the normal stress in the column:

We check the value of the normal stress in the column:

$$\sigma = \frac{P}{A} = \frac{100 \text{ kN}}{(0.100 \text{ m})^2} = 10 \text{ MPa}$$

Since σ is smaller than the allowable stress, a 100 × 100-mm cross section is acceptable.

(b) For the 200-kN Load. Solving again Eq. (10.11) for I, but making now $P_{cr} = 2.5(200) = 500$ kN, we have

$$I = 15.588 \times 10^{-6} \text{ m}^4$$

$$\frac{a^4}{12} = 15.588 \times 10^{-6} \qquad a = 116.95 \text{ mm}$$

The value of the normal stress is

$$\sigma = \frac{P}{A} = \frac{200 \text{ kN}}{(0.11695 \text{ m})^2} = 14.62 \text{ MPa}$$

Since this value is larger than the allowable stress, the dimension obtained is not acceptable, and we must select the cross section on the basis of its resistance to compression. We write

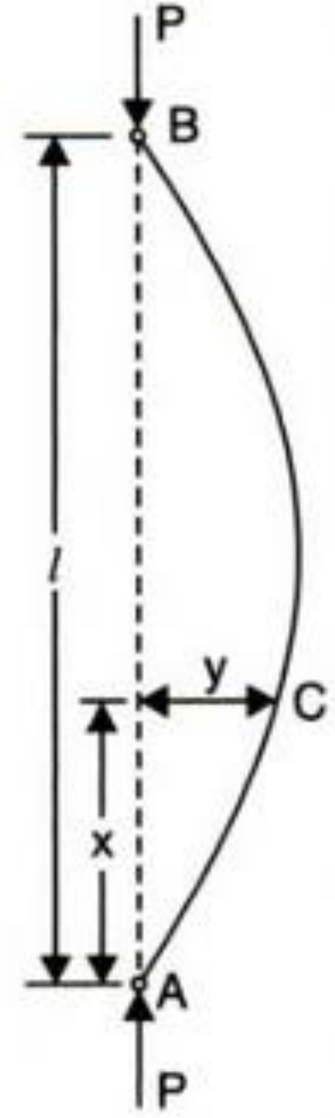
$$A = \frac{P}{\sigma_{all}} = \frac{200 \text{ kN}}{12 \text{ MPa}} = 16.67 \times 10^{-3} \text{ m}^2$$

$$a^2 = 16.67 \times 10^{-3} \text{ m}^2 \qquad a = 129.1 \text{ mm}$$

A 130 × 130-mm cross section is acceptable.

(i) Both Ends Hinged

$$\boxed{P_{\mathrm{cr}} = \frac{\pi^2 EI}{L^2}}$$



(ii) One End Fixed and Other End Free

$$M = EI\frac{d^2y}{dx^2}$$

At x,

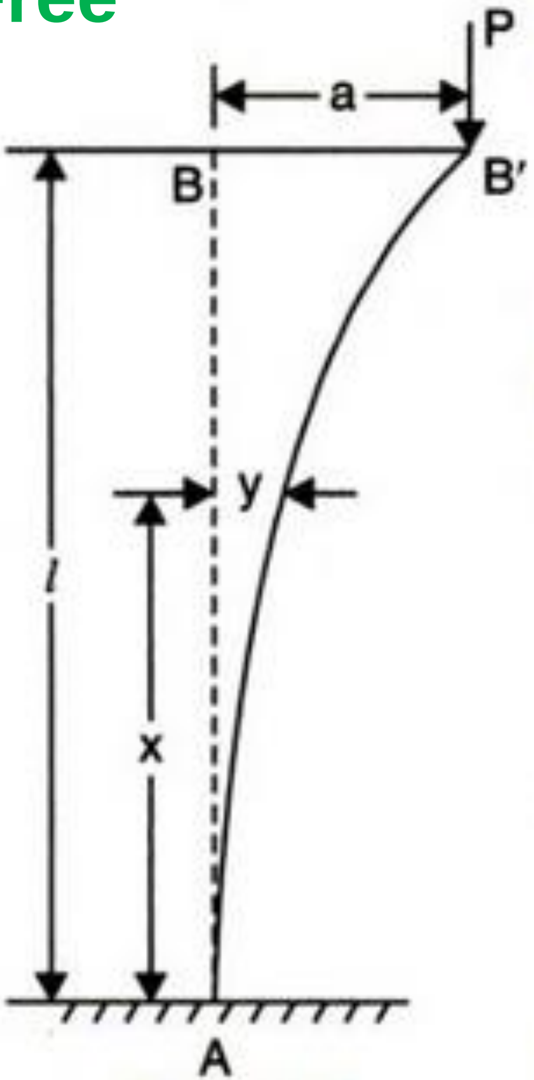
$$M = P(a - y)$$

$$EI\frac{d^2y}{dx^2} = P(a - y)$$

$$EI\frac{d^2y}{dx^2} + Py = Pa$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{P}{EI}a$$

$$y = A\cos\left(x\sqrt{\frac{P}{EI}}\right) + B\sin\left(x\sqrt{\frac{P}{EI}}\right) + a$$



At fixed end y=0 and $\frac{dy}{dx} = 0$,

$$\boxed{A = -a}$$

$$\frac{dy}{dx} = A\,(-1)\,sin\left(x\sqrt{\frac{P}{EI}}\right)\sqrt{\frac{P}{EI}} + B\,cos\left(x\sqrt{\frac{P}{EI}}\right)\sqrt{\frac{P}{EI}}$$

$$0 = A\,(-1)\,sin\left(0\sqrt{\frac{P}{EI}}\right)\sqrt{\frac{P}{EI}} + B\,cos\left(0\sqrt{\frac{P}{EI}}\right)\sqrt{\frac{P}{EI}}$$

$$\boxed{B = 0\ or\ \sqrt{\frac{P}{EI}} = 0}$$

$$\boxed{\therefore B = 0}$$

$$y = -a\,cos\left(x\sqrt{\frac{P}{EI}}\right) + a$$

$$y = -a\,cos\left(x\sqrt{\frac{P}{EI}}\right) + a$$

At free end x=L and y = a,

$$a = -a\,cos\left(L\sqrt{\frac{P}{EI}}\right) + a$$

$$a\,cos\left(L\sqrt{\frac{P}{EI}}\right) = 0$$

a cannot be zero

$$cos\left(L\sqrt{\frac{P}{EI}}\right) = 0$$

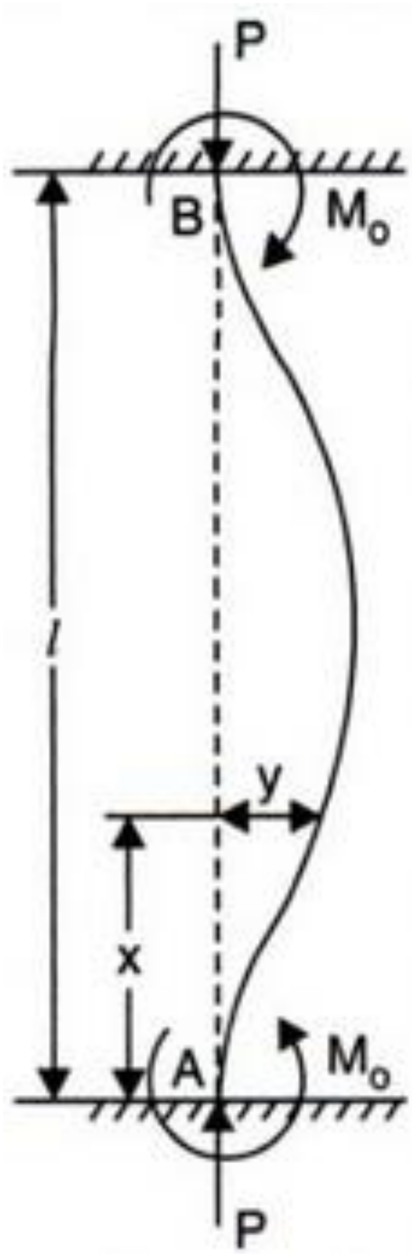
$$L\sqrt{\frac{P}{EI}} = \frac{\pi}{2}\ or\ \frac{3\pi}{2}\ or\ \frac{5\pi}{2}\ \ldots\ldots..$$

$$\sqrt{\frac{P}{EI}} = \frac{\pi}{2L}$$

$$\boxed{P_{\mathrm{cr}} = \frac{\pi^2 EI}{4L^2}}$$

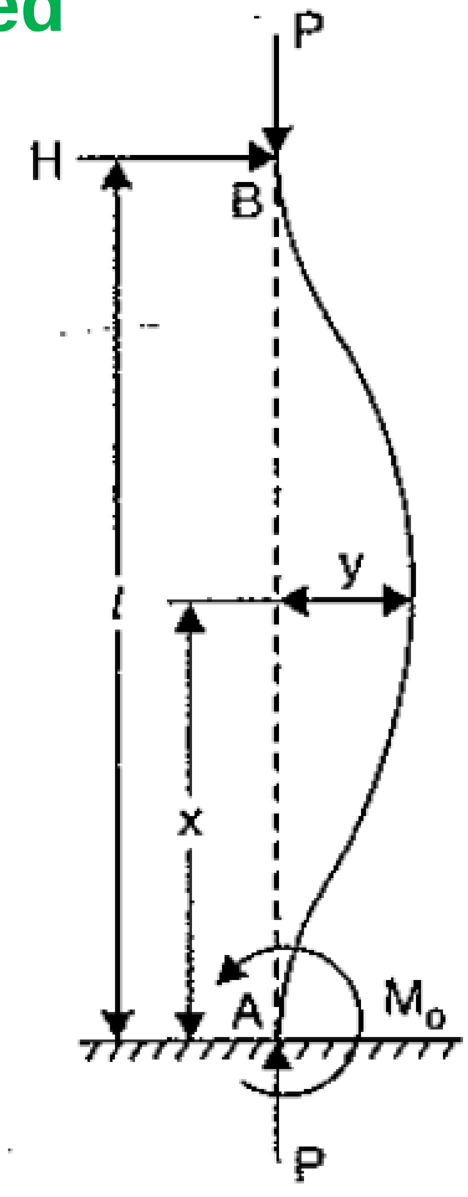
(iii) Both Ends Fixed

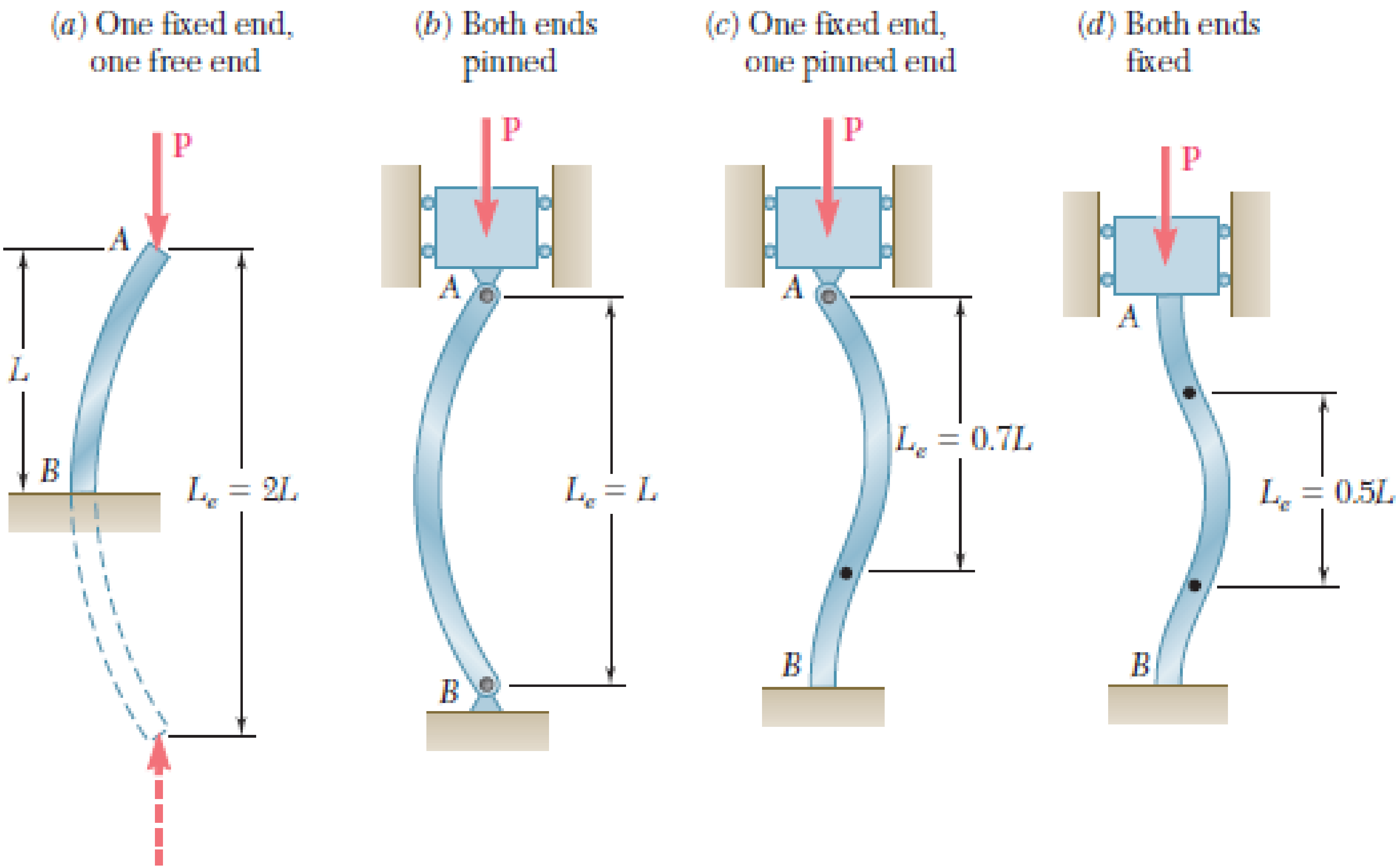
$$\boxed{P_{\text{cr}} = \frac{4\pi^2 EI}{L^2}}$$



(iv) One End Fixed Other End Hinged

$$\boxed{P_{\mathrm{cr}} = \frac{2\pi^2 EI}{L^2}}$$




(a) One fixed end, one free end
P
A
L
B
$L_e = 2L$
(b) Both ends pinned
P
A
B
$L_e = L$
(c) One fixed end, one pinned end
P
A
B
$L_e = 0.7L$
(d) Both ends fixed
P
A
B
$L_e = 0.5L$

The Crippling Load is given by $$\boxed{P_{\text{cr}} = \frac{\pi^2 EI}{{L_e}^2}}$$

(i) Both Ends Hinged

$$\boxed{P_{\text{cr}} = \frac{\pi^2 EI}{L^2}} \qquad \boxed{L_e = L}$$

(ii) One End Fixed and Other End Free

$$\boxed{P_{\text{cr}} = \frac{\pi^2 EI}{4L^2}} \qquad \boxed{L_e = 2L}$$

(iii) Both Ends Fixed

$$\boxed{P_{\text{cr}} = \frac{4\pi^2 EI}{L^2}} \qquad \boxed{L_e = L/2}$$

(iv) One End Fixed Other End Hinged

$$\boxed{P_{\text{cr}} = \frac{2\pi^2 EI}{L^2}} \qquad \boxed{L_e = L/\sqrt{2}}$$

Crippling Stress

$$\sigma_{cr} = \frac{Crippling\ Load\ (P_{cr})}{Area\ (A)}$$

$I = Ak^2$

k- Least radius of gyration of the column section

$$P_{cr} = \frac{\pi^2 EAk^2}{{L_e}^2} = \frac{\pi^2 EA}{\left(\frac{L_e}{k}\right)^2}$$

$$\sigma_{cr} = \frac{\pi^2 EA}{A\left(\frac{L_e}{k}\right)^2} = \frac{\pi^2 E}{\left(\frac{L_e}{k}\right)^2}$$

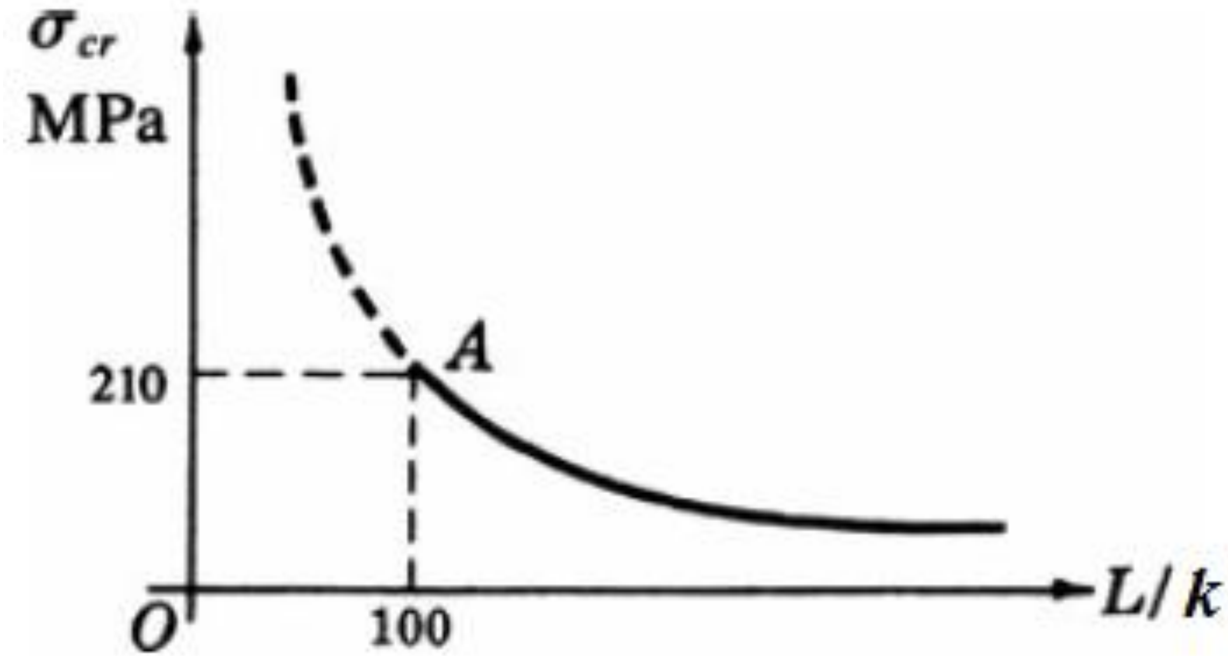
Slenderness ratio

Limitation of Euler's Formula

- Let us consider a steel column having a proportional limit of 210 MPa and E = 200 GPa.
- To find the value of L_e/k corresponding to these constants, we substitute in

$$\boxed{\sigma_{\mathrm{cr}} = \frac{\pi^2 E}{\left(\frac{L_e}{k}\right)^2}}$$

- Thus, for this material the buckling load and the axial stress are valid only for those columns having L_e/k 100.
- For those columns having L_e/k < 100, the compressive stress exceeds the proportional limit before elastic buckling takes place and the above equations are not valid.



- For the particular values of proportional limit and modulus of elasticity assumed above, the portion of the curve to the left of $L_e/k = 100$ is not valid. Thus for this material, point A marks the upper limit of applicability of the curve.

Rankine's Formula

- Euler's formula is applicable to long columns only in which the *l*/*k* ratio is larger than a certain value for a particular material
- It does not take in to account the direct compressive stress and therefore, it gives correct results for long columns only
- To take in to account this effect, Rankine forwarded an empirical relation which covers very short to very long columns

$$\boxed{\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e}}$$

P – Rankine's crippling load
P_c – Ultimate crushing load (σ_c x A), constant for a material
P_e – Euler's crippling load ($\pi^2 EI/l^2$)

- For short columns, P_e is very large and therefore $1/P_e$ is small in comparison to $1/P_c$. Thus the crippling load P is practically equal to P_c
- For long coumns, P_e is very small and therefore $1/P_e$ is quite large in comparison to $1/P_c$. Thus the crippling load P is practically equal to P_e

$$\boxed{\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e}} \qquad \frac{1}{P} = \frac{P_e + P_c}{P_c P_e}$$

$$P = \frac{P_c P_e}{P_e + P_c} = \frac{P_c}{1 + P_c/P_e} = \frac{P_c}{1 + \sigma_c A l^2/\pi^2 EI}$$

$$= \frac{\sigma_c A}{1 + \sigma_c A l^2/\pi^2 AEk^2} = \frac{\sigma_c A}{1 + (\sigma_c/\pi^2 E)\left(\frac{l}{k}\right)^2}$$

$$\boxed{P = \frac{\sigma_c A}{1 + a\left(\frac{l_e}{k}\right)^2}}$$

P – Rankine's crippling load
σ_c – Ultimate crushing stress
l_e – Effective length
a - Rankine's constant $(\sigma_c/\pi^2 E)$