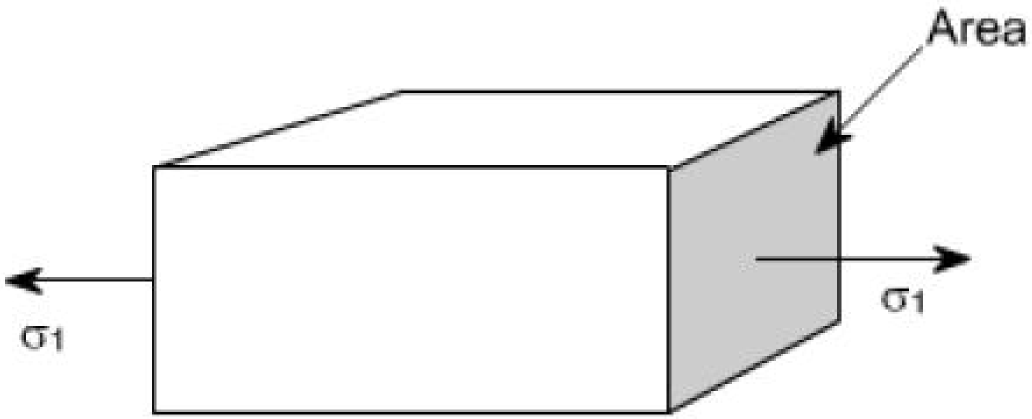
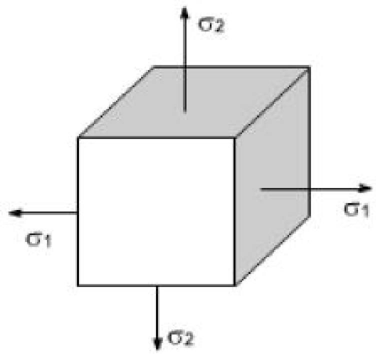
State of stress

- ***Uni-axial stress:*** only one non-zero principal stress, i.e. σ_1



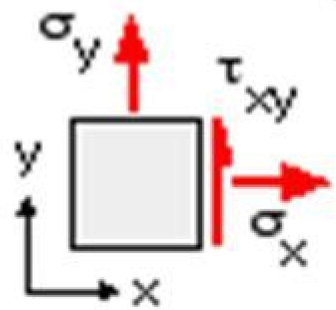
- ***Bi-axial stress:*** one principal stress equals zero, two do not, i.e. $\sigma_1 > \sigma_2$; $\sigma_3 = 0$



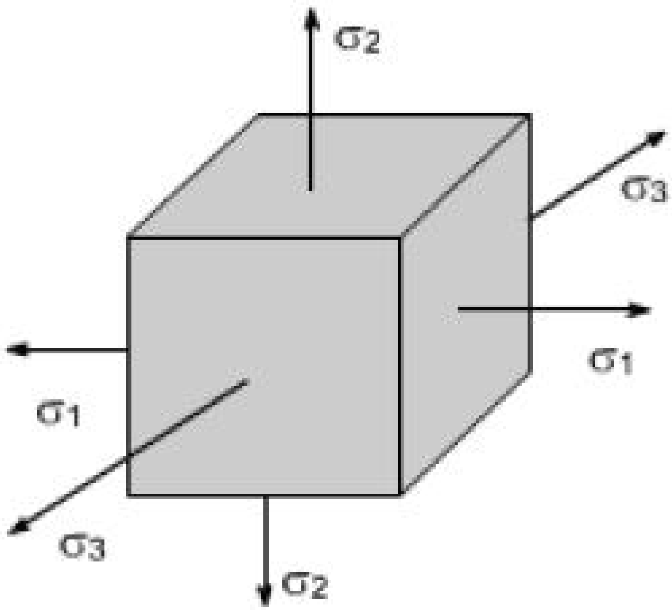
2 - dimensions

PLANE STRESS $\sigma_z = 0$

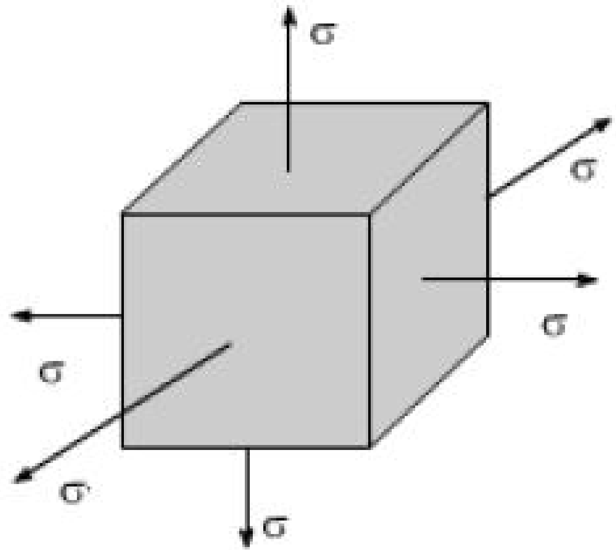
PLANE STRAIN $\varepsilon_z = 0$



- ***Tri-axial stress:*** three non-zero principal stresses, i.e. $\sigma_1 > \sigma_2 > \sigma_3$



- ***Isotropic stress:*** three principal stresses are equal, i.e. $\sigma_1 = \sigma_2 = \sigma_3$



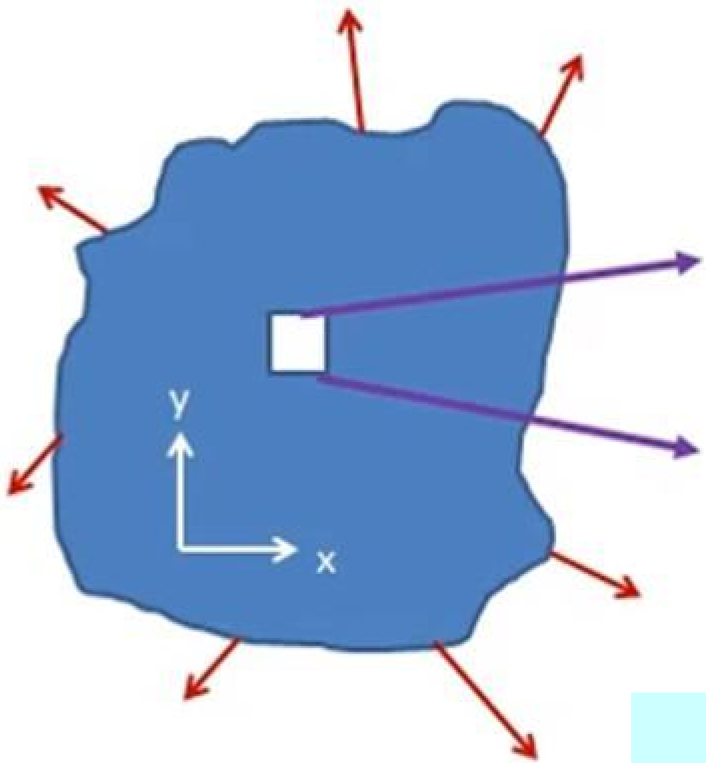
Stress

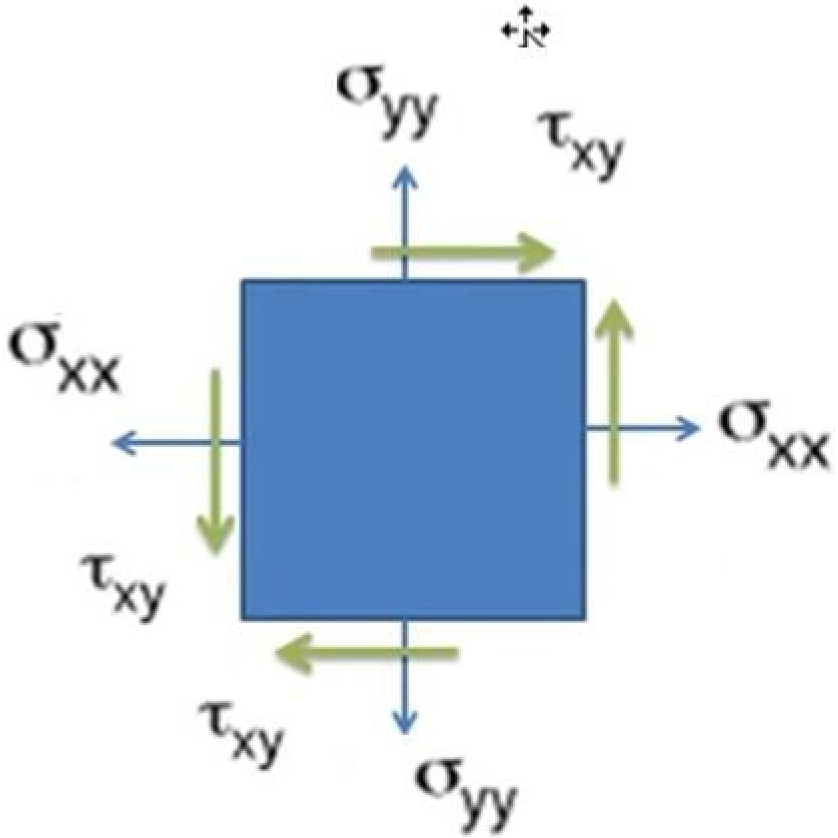
Failure Stress = Failure load/Area

$\sigma_u = P/A$

Factor of safety = Failure stress / Allowable stress

2D-Stresses

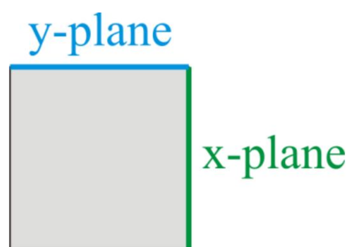




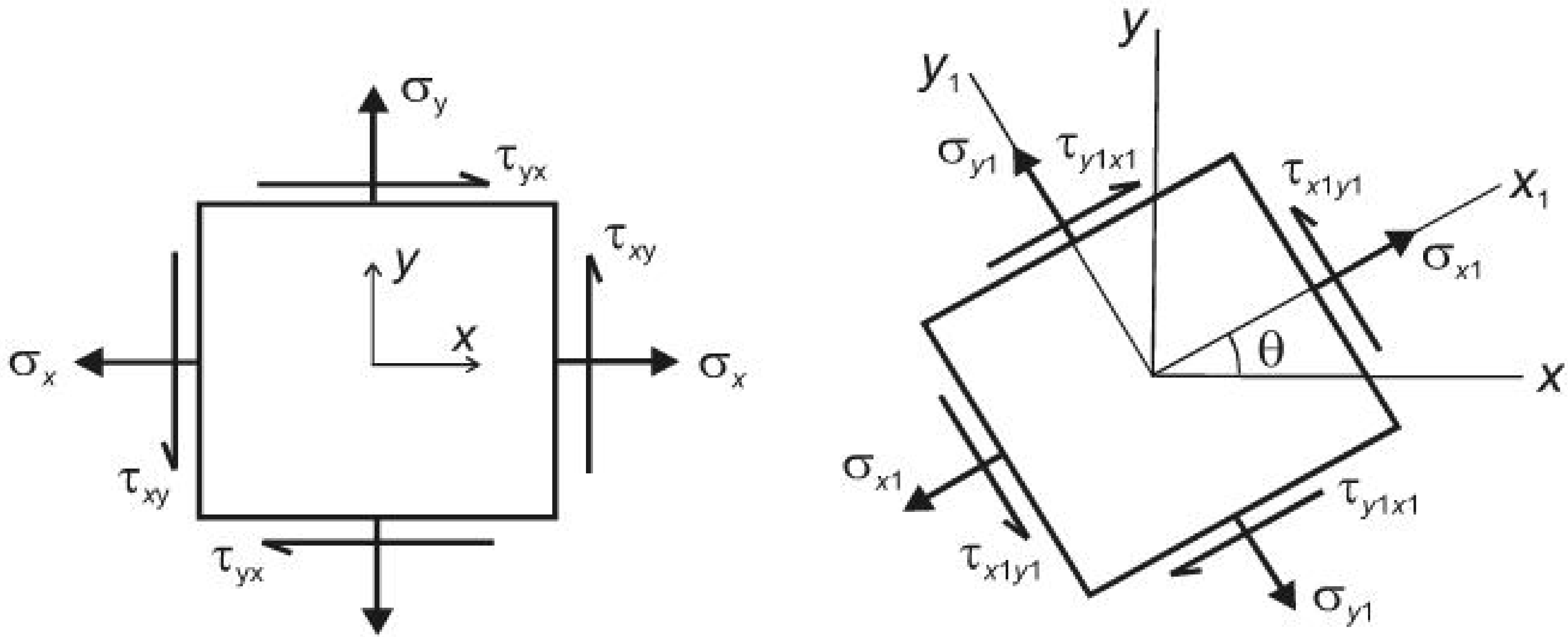
$$\sigma_{xx}$$

Direction

Plane



Stress Transformation



Compressive load

Failure in shear

Stresses are acting normal to the surface yet the material failed in a different plane

PRINCIPAL PLANES AND PRINCIPAL STRESSES

The planes, which have no shear stress, are known as principal planes. Hence principal planes are the planes of zero shear stress. These planes carry only normal stresses.

The normal stresses, acting on a principal plane, are known as principal stresses.

METHODS FOR DETERMINING STRESSES ON OBLIQUE SECTION

The stresses on oblique section are determined by the following methods :

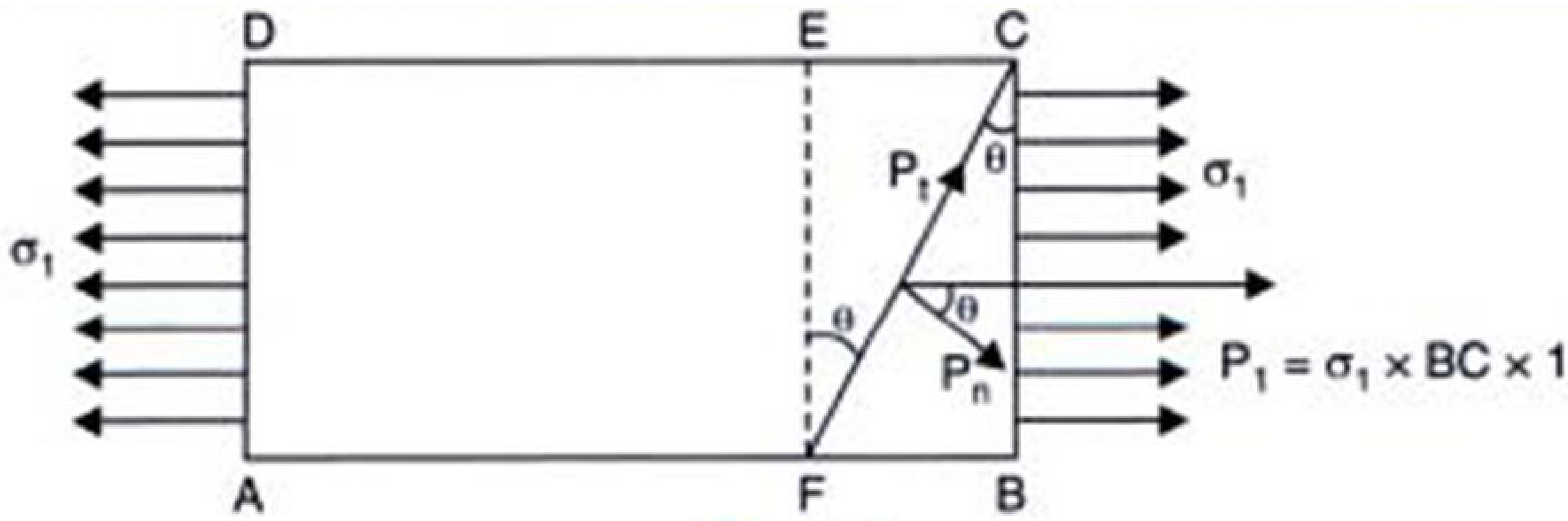
1. Analytical method, and
2. Graphical method.

ANALYTICAL METHOD FOR DETERMINING STRESSES ON OBLIQUE SECTION

The following two cases will be considered :

1. A member subjected to a direct stress in one plane.
2. The member is subjected to like direct stresses in two mutually perpendicular directions.

A Member Subjected to a Direct Stress in one Plane. Fig. 3.2 shows a rectangular member of uniform cross-sectional area *A* and of unit thickness. The bar is subjected to a principal tensile stress σ_1 on the faces *AD* and *BC*.



Area of cross-section $= BC \times$ Thickness of bar

$= BC \times 1$

Let the stresses on the oblique plane FC are to be calculated. The plane FC is inclined at an angle θ with the normal cross-section EF (or BC). This can be done by converting the stress σ_1 acting on face BC into equivalent force. Then this force will be resolved along the inclined planes FC and perpendicular to FC. (Please note that it is force and not the stress which is to be resolved).

Tensile stress on face $BC = \sigma_1$

Now, the tensile force on BC,

$$P_1 = \text{Stress } (\sigma_1) \times \text{Area of cross-section}$$

$$= \sigma_1 \times BC \times 1 \qquad (\because \text{ Area} = BC \times 1)$$

The above tensile force P_1 is also acting on the inclined section FC, in the axial direction as shown in Fig. 3.2. This force P_1 is resolved into two component, *i.e.*, one normal to the plane FC and other along the plane FC.

Let P_n = Component of the force P_1, normal to the section FC

$$= P_1 \cos\theta$$

$$= \sigma_1 \times BC \times 1 \times \cos\theta \qquad (\because P_1 = \sigma_1 \times BC \times 1)$$

P_t = Component of the force P_1, along the section FC

$$= P_1 \sin\theta$$

$$= \sigma_1 \times BC \times 1 \times \sin\theta$$

σ_n = Normal stress on the section FC

Then normal stress, $\sigma_n = \dfrac{\text{Force normal to section } FC}{\text{Area of section } FC}$

$$= \frac{P_n}{FC \times 1} \qquad (\because \text{ bar is of unit thickness})$$

$$= \frac{\sigma_1 \times BC \times \cos\theta}{FC} \qquad (\because P_n = \sigma_1 \times BC \times \cos\theta)$$

$$= \sigma_1 \times \cos\theta \times \cos\theta \qquad \left(\because \text{ In triangle } FBC, \frac{BC}{FC} = \cos\theta\right)$$

$$= \sigma_1 \times \cos^2\theta$$

Similarly, tangential (or shear) stress,

$$\sigma_t = \frac{\text{Force along section } FC}{\text{Area of section } FC} = \frac{P_t}{FC \times 1}$$

$$= \frac{\sigma_1 \times BC \times 1 \times \sin\theta}{FC} \qquad (\because P_t = \sigma_1 \times BC \times 1)$$

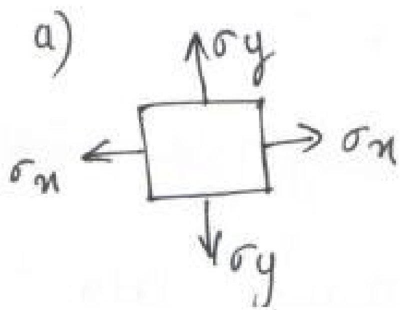
$$= \sigma_1 \times \cos\theta \times \sin\theta \qquad \left(\because \text{ In triangle } FBC, \frac{BC}{FC} = \cos\theta\right)$$

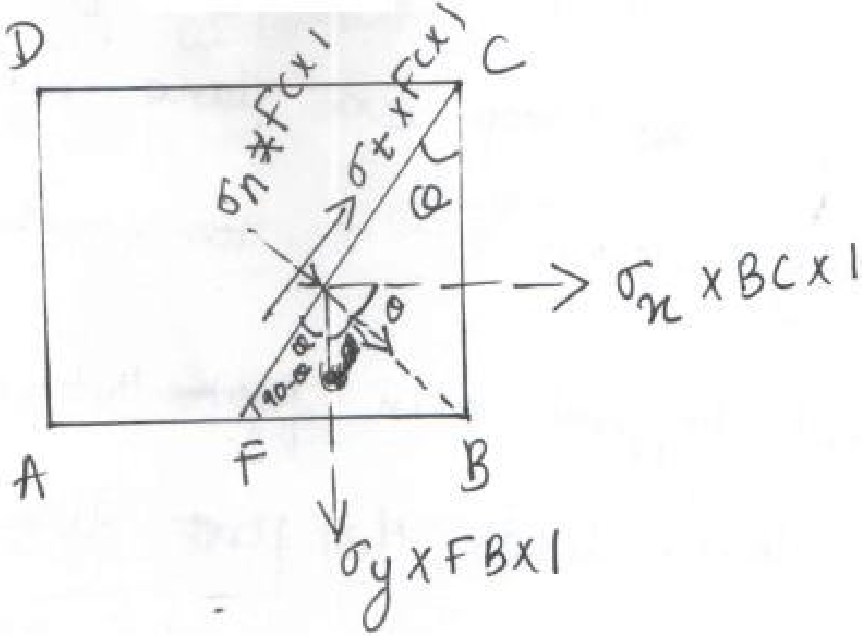
$$= \sigma_1 \times \cos\theta \times \sin\theta$$

$$= \frac{\sigma_1}{2} \times 2 \times \cos\theta \times \sin\theta \qquad (\text{Multiplying and dividing by two})$$

$$= \frac{\sigma_1}{2} \times \sin 2\theta \qquad (\because 2\sin\theta\cos\theta = \sin 2\theta)$$

A member subjected to like direct stresses in two mutually perpendicular directions





Let FC be the oblique section on which stresses are to be calculated. This can be done by converting the stresses σ_x and σ_y into equivalent forces. Then these forces will be resolved along the inclined plane FC and perpendicular to FC.

$$\sigma_n \times FC = \sigma_x \times BC \cos\theta + \sigma_y \times FB \times \sin\theta$$

$$\sigma_n = \sigma_x \left(\frac{BC}{FC}\right)^{\to \cos\theta} \cos\theta + \sigma_y \times \left(\frac{FB}{FC}\right)^{\to \sin\theta} \sin\theta .$$

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \times \sin^2\theta .$$

$$\sigma_n = \sigma_x \frac{(1+\cos 2\theta)}{2} + \sigma_y \left(\frac{1-\cos 2\theta)}{2}\right.$$

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \left(\frac{\sigma_x - \sigma_y}{2}\right) \cos 2\theta \quad \text{———} \quad (1)$$

(b) Resolving forces $\parallel^{el}$ to oblique plane

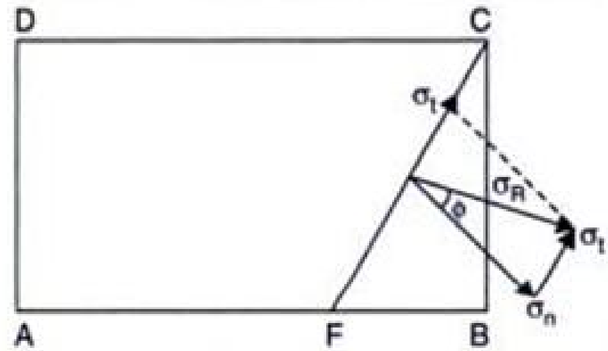
$$\sigma_t \times FC = \sigma_x \times BC \sin\theta - \sigma_y \times FB \cos\theta$$

$$\sigma_t = \sigma_x \left(\frac{BC}{FC}\right)^{-\cos\theta} \sin\theta - \sigma_y \left(\frac{FB}{FC}\right)^{-\sin\theta} \cos\theta$$

$$\sigma_t = (\sigma_x - \sigma_y) \sin\theta \cos\theta = \left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta \quad (2)$$

Obliquity The angle made by the resultant stress with the normal of the oblique plane, is known as obliquity. It is denoted by ϕ. Mathematically,

$$\tan \phi = \frac{\sigma_t}{\sigma_n}$$



The resultant stress on the section *FC* will be given as

$$\sigma_R = \sqrt{\sigma_n^2 + \sigma_t^2}$$

Maximum shear stress.

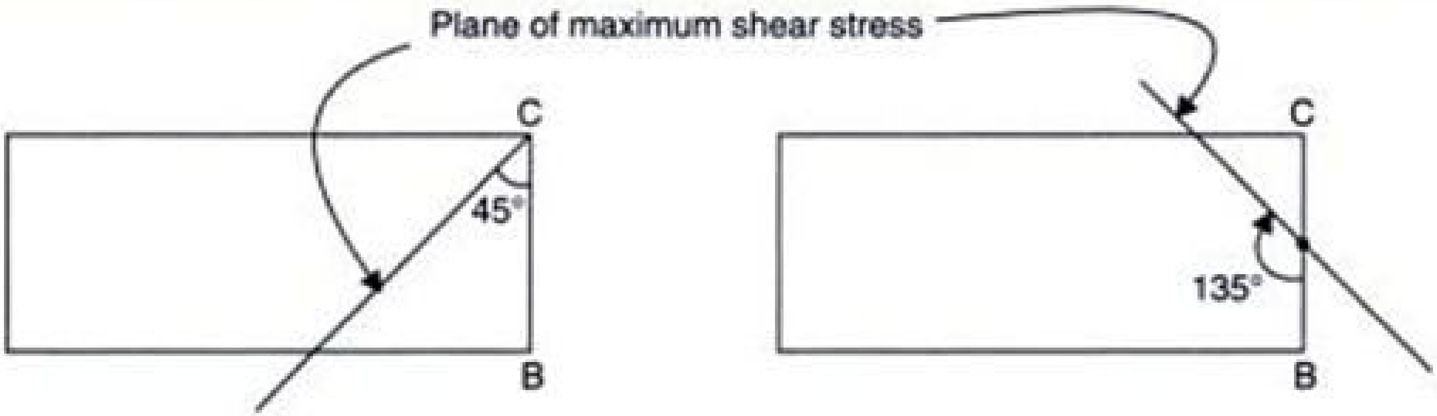
The shear stress (σ_t) will be maximum

when $\sin 2\theta = 1$ or $2\theta = 90°$ or $270°$ ($\because$ $\sin 90° = 1$ and also $\sin 270° = 1$)

or $\theta = 45°$ or $135°$

And maximum shear stress, $(\sigma_t)_{max} = \dfrac{\sigma_1 - \sigma_2}{2}$

The planes of maximum shear stress are obtained by making an angle of 45° and 135° with the plane *BC* (at any point on the plane *BC*) in such a wav that the planes of maximum shear stress lie within the material as shown in Fig.


Plane of maximum shear stress
C
45°
B
C
135°
B

Principal planes. Principal planes are the planes on which shear stress is zero. To locate the position of principal planes, the shear stress given by equation (3.7) should be equated to zero.

$\therefore$ For principal planes,

$$\frac{\sigma_1 - \sigma_2}{2} \sin 2\theta = 0$$

or $$\sin 2\theta = 0 \qquad [\because \ (\sigma_1 - \sigma_2) \text{ cannot be equal to zero}]$$

or $$2\theta = 0 \text{ or } 180^\circ$$

$\therefore$ $$\theta = 0 \text{ or } 90^\circ$$

when $\theta = 0$,
$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta$$
$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 0^\circ$$
$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \times 1 \qquad (\because \ \cos 0^\circ = 1)$$
$$= \sigma_1$$

when $\theta = 90^\circ$,
$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2 \times 90^\circ$$
$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 180^\circ$$
$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \times (-1) \qquad (\because \ \cos 180^\circ = -1)$$
$$= \sigma_2.$$

Problem . *The tensile stresses at a point across two mutually perpendicular planes are 120 N/mm² and 60 N/mm². Determine the normal, tangential and resultant stresses on a plane inclined at 30° to the axis of the minor stress.*

Sol. Given :

Major principal stress, $\sigma_1 = 120 \text{ N/mm}^2$

Minor principal, $\sigma_2 = 60 \text{ N/mm}^2$

Angle of oblique plane with the axis of minor principal stress,

$$\theta = 30°.$$

Normal stress

The normal stress (σ_n) is given by equation (3.6),

$$\therefore \quad \sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta$$

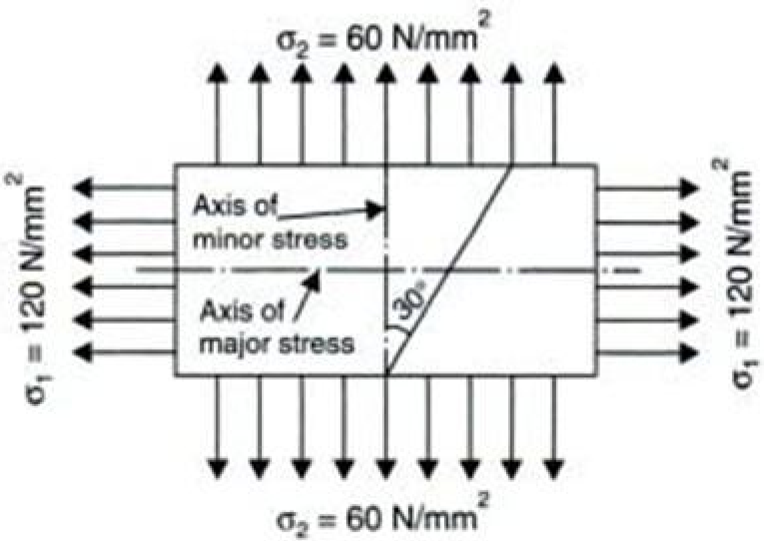
$$= \frac{120 + 60}{2} + \frac{120 - 60}{2} \cos 2 \times 30°$$

$$= 90 + 30 \cos 60° = 90 + 30 \times \frac{1}{2}$$

$$= \mathbf{105 \text{ N/mm}^2}. \quad \text{Ans.}$$

Tangential stress

The tangential (or shear stress) σ_t is given by equation (3.7).



$$\therefore \qquad \sigma_t = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta$$

$$= \frac{120 - 60}{2} \sin (2 \times 30°)$$

$$= 30 \times \sin 60° = 30 \times 0.866$$

$$= \mathbf{25.98\ N/mm^2.}$$

Resultant stress

The resultant stress (σ_R) is given by equation (3.8)

$$\therefore \qquad \sigma_R = \sqrt{{\sigma_n}^2 + {\sigma_t}^2} = \sqrt{105^2 + 25.98^2}$$

$$= \sqrt{11025 + 674.96} = \mathbf{108.16\ N/mm^2.}$$

Problem *At a point in a strained material the principal stresses are 100 N/mm² (tensile) and 60 N/mm² (compressive). Determine the normal stress, shear stress and resultant stress on a plane inclined at 50° to the axis of major principal stress. Also determine the maximum shear stress at the point.*

Major principal stress, $\sigma_1 = 100 \text{ N/mm}^2$

Minor principal stress, $\sigma_2 = -60 \text{ N/mm}^2$ (Negative sign due to compressive stre

Angle of the inclined plane with the axis of major principal stress = 50°

∴ Angle of the inclined plane with the axis of minor principal stress,

$$\theta = 90 - 50 = 40°.$$

Normal stress (σ_n)

Using equation (3.6),

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta$$

$$= \frac{100 + (-60)}{2} + \frac{100 - (-60)}{2} \cos (2 \times 40°)$$

$$= \frac{100 - 60}{2} + \frac{100 + 60}{2} \cos 80°$$

$$= 20 + 80 \times \cos 80° = 20 + 80 \times .1736$$

$$= 20 + 13.89 = \mathbf{33.89 \text{ N/mm}^2}.$$ **Ans.**

Shear stress (σ_t)

Using equation (3.7), $\sigma_t = \dfrac{\sigma_1 - \sigma_2}{2} \sin 2\theta$

$$= \frac{100 - (-60)}{2} \sin (2 \times 40°)$$

$$= \frac{100+60}{2} \sin 80° = 80 \times 0.9848 = \mathbf{78.785\ N/mm^2.}$$ Ans.

Resultant stress (σ_R)

Using equation on (3.8),

$$\sigma_R = \sqrt{\sigma_n^2 + \sigma_t^2} = \sqrt{33.89^2 + 78.785^2}$$

$$= \sqrt{1148.53 + 6207.07} = \mathbf{85.765\ N/mm^2.}$$ Ans.

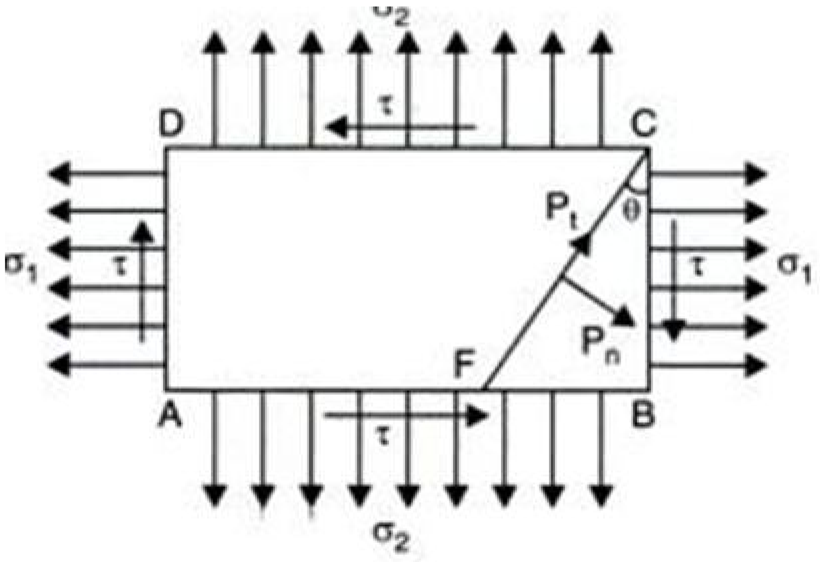
Maximum shear stress

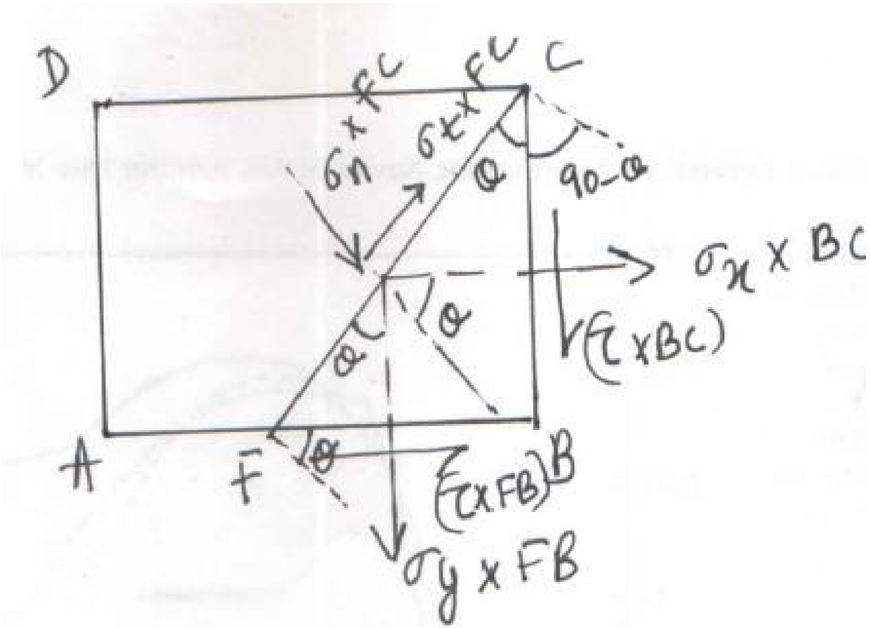
Using equation (3.9),

$$(\sigma_t)_{\max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{100-(-60)}{2}$$

$$= \frac{100+60}{2} = \mathbf{80\ N/mm^2.}$$ Ans.

A member subjected to direct stresses in two mutually perpendicular directions accompanied by a simple shear stress





Resolving forces perpendicular to oblique plane FC

$$\sigma_n \times FC = \sigma_x \times BC \cos\theta + \sigma_y \times FB \sin\theta + \tau \times BC \times \sin\theta + \tau \times FB \times \sin(90-\theta)$$

$$\sigma_n = \sigma_x \left(\frac{BC}{FC}\right) \cos\theta + \sigma_y \times \left(\frac{FB}{FC}\right) \sin\theta + \tau \sin\theta \left(\frac{BC}{FC}\right) + \tau \cos\theta \left(\frac{FB}{FC}\right)$$

$$\sigma_n = \sigma_x \times \cos^2\theta + \sigma_y \sin^2\theta + \tau \sin\theta \overset{\times \cos\theta}{} + \tau \sin\theta \cos\theta .$$

$$\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \cancel{\cos} \left(\frac{\sigma_x - \sigma_y}{2}\right) \cos 2\theta + 2\tau \sin\theta \cos\theta$$

$$\sigma_m = \frac{\sigma_x + \sigma_y}{2} + \left(\frac{\sigma_x - \sigma_y}{2}\right) \cos 2\theta + \tau \sin 2\theta \quad \text{——} \quad (1)$$

(b) Resolving forces ||el to the plane BC.

$$\sigma_t \times FC = \sigma_x \times BC \sin\theta - \sigma_y \times FB \sin\theta - \tau \times BC \times \cos\theta + \tau \times FB \times \sin\theta$$

$$\sigma_t = \sigma_x \sin\theta \cos\theta - \sigma_y \sin\theta \cos\theta - \tau \cos^2\theta + \tau \sin^2\theta ..$$

$$\sigma_t = \left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta - \tau \left(\cos^2\theta - \sin^2\theta \right)$$

$$\sigma_t = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta - \tau \cos 2\theta \quad \text{———} \quad (2)$$

eqn (1) & (2) are known as transformation equations of plane stress.

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau \sin 2\theta$$

$$\sigma_t = \frac{\sigma_1 - \sigma_2}{2} . \sin 2\theta - \tau \cos 2\theta$$

The above equations are known as transformation equations for plane stress.

One is interested in maximum and minimum normal and shear stresses acting on the inclined plane in order to design components against failure.

The maximum normal stress and shear stress can be found by differentiating the stress transformation equations with respect to θ and equate to zero.

The maximum and minimum stresses are called principal stresses and the plane on which they act are called principal planes.

On a plane on which the principal normal stress acts, the shear stress is zero. Similarly, on a plane on which the principal shear is acting, the normal stresses are zero.

Position of principal planes. The planes on which shear stress (*i.e.*, tangential stress) is zero, are known as principal planes. And the stresses acting on principal planes are known as principal stresses.

The position of principal planes are obtained by equating the tangential stress [given by equation (3.13)] to zero.

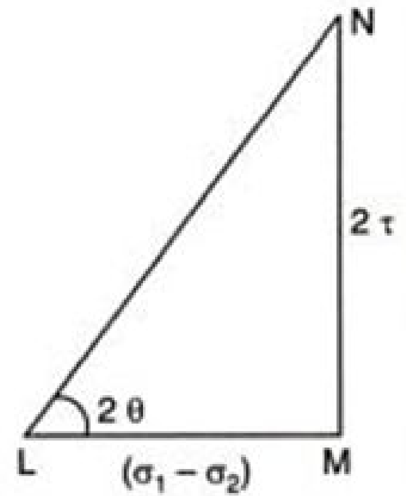
∴ For principal planes, $\sigma_t = 0$

or $$\frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta = 0$$

or $$\frac{\sigma_1 - \sigma_2}{2} \sin 2\theta = \tau \cos 2\theta$$

or $$\frac{\sin 2\theta}{\cos 2\theta} = \frac{\tau}{\frac{(\sigma_1 - \sigma_2)}{2}} = \frac{2\tau}{(\sigma_1 - \sigma_2)}$$

or $$\tan 2\theta = \frac{2\tau}{(\sigma_1 - \sigma_2)}$$



But the tangent of any angle in a right angled triangle

$$= \frac{\text{Height of right angled triangle}}{\text{Base of right angled triangle}}$$

∴ $$\frac{\text{Height of right angled triangle}}{\text{Base of right angled triangle}} = \frac{2\tau}{(\sigma_1 - \sigma_2)}$$

∴ Height of right angled triangle = 2τ

Base of right angled triangle = $(\sigma_1 - \sigma_2)$.

Now diagonal of the right angled triangle

$$= \pm\sqrt{(\sigma_1 - \sigma_2)^2 + (2\tau)^2} = \pm\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

$$= \sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2} \quad \text{and} \quad -\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

1st Case. $$\text{Diagonal} = \sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

Then $$\sin 2\theta = \frac{\text{Height}}{\text{Diagonal}} = \frac{2\tau}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

and $$\cos 2\theta = \frac{\text{Base}}{\text{Diagonal}} = \frac{(\sigma_1 - \sigma_2)}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}.$$

The value of major principal stress is obtained by substituting the values of $\sin 2\theta$ and $\cos 2\theta$ in equation (3.12).

$\therefore$ Major principal stress

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau \sin 2\theta$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \times \frac{(\sigma_1 - \sigma_2)}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}} + \tau \times \frac{2\tau}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{1}{2} \frac{(\sigma_1 - \sigma_2)^2}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}} + \frac{2\tau^2}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{(\sigma_1 - \sigma_2)^2 + 4\tau^2}{2\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

2nd Case. Diagonal $= -\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$

Then
$$\sin 2\theta = \frac{2\tau}{-\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

and
$$\cos 2\theta = \frac{(\sigma_1 - \sigma_2)}{-\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

Substituting these values in equation (3.12), we get minor principal stress.

$\therefore$ Minor principal stress

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau \sin 2\theta$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \times \frac{\sigma_1 - \sigma_2}{-\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}} + \tau \times \frac{2\tau}{-\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} - \frac{(\sigma_1 - \sigma_2)^2}{2\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}} - \frac{2\tau^2}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} - \frac{(\sigma_1 - \sigma_2)^2 + 4\tau^2}{2\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} - \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

$$= \frac{\sigma_1 + \sigma_2}{2} - \sqrt{\left(\frac{(\sigma_1 - \sigma_2)}{2}\right)^2 + \tau^2} \qquad \text{...(3.16)}$$

Principal Normal stress

Differentiating with θ and equating the sum to "0" we get

$$\sigma = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Back substituting the Two values of θ gives maximum and minimum values of normal stresses

$$\sigma_1 = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$$

$$\sigma_2 = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$$

Maximum shear stress. The shear stress is given by equation (3.13). The shear stress will be maximum or minimum when

$$\frac{d}{d\theta}(\sigma_t) = 0$$

or $$\frac{d}{d\theta}\left[\frac{\sigma_1 - \sigma_2}{2}\sin 2\theta - \tau\cos 2\theta\right] = 0$$

or $$\frac{\sigma_1 - \sigma_2}{2}(\cos 2\theta) \times 2 - \tau(-\sin 2\theta) \times 2 = 0$$

$$(\sigma_1 - \sigma_2) \,.\, \cos 2\theta + 2\tau \sin 2\theta = 0$$

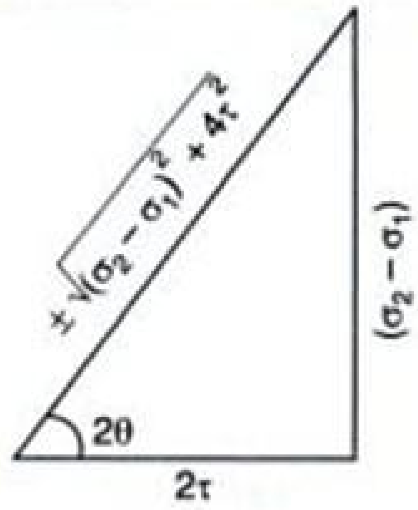
or $$2\tau \sin 2\theta = -(\sigma_1 - \sigma_2)\cos 2\theta$$

$$= (\sigma_2 - \sigma_1)\cos 2\theta$$

or $$\frac{\sin 2\theta}{\cos 2\theta} = \frac{\sigma_2 - \sigma_1}{2\tau}$$

or $$\tan 2\theta = \frac{\sigma_2 - \sigma_1}{2\tau}$$

Equation (3.17) gives condition for maximum or minimum shear stress.



If $\tan 2\theta = \dfrac{\sigma_2 - \sigma_1}{2\tau}$

Then $$\sin 2\theta = \pm\frac{\sigma_2 - \sigma_1}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$$

and $$\cos 2\theta = \pm\frac{2\tau}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$$

Substituting the values of sin 2θ and cos 2θ in equation (3.13), the maximum and minimum shear stresses are obtained.

∴ Maximum shear stress is given by

$$(\sigma_t)_{max} = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta$$

$$= \pm \frac{\sigma_1 - \sigma_2}{2} \times \frac{(\sigma_2 - \sigma_1)}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}} \pm \tau \times \frac{2\tau^2}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$$

$$= \pm \frac{(\sigma_1 - \sigma_2)^2}{2\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}} \pm \frac{2\tau^2}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$$

$$= \pm \frac{(\sigma_2 - \sigma_1)^2 + 4\tau^2}{2\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}} = \pm \frac{1}{2}\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}$$

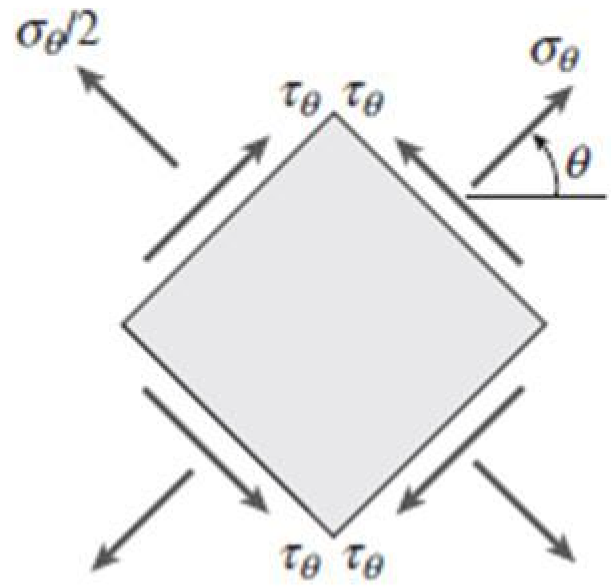
$$\therefore \quad (\sigma_t)_{max} = \frac{1}{2}\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}$$

$$= \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2} \qquad ...(3.18)$$

A plane stress element on a bar in uniaxial stress has tensile stress of $\sigma_\theta = 78$ MPa (see fig.). The maximum shear stress in the bar is approximately:

Solution

$\sigma_\theta = 78$ MPa



Plane stress transformation formulas for uniaxial stress:

$$\sigma_x = \frac{\sigma_\theta}{\cos(\theta)^2} \quad \text{and} \quad \sigma_x = \frac{\frac{\sigma_\theta}{2}}{\sin(\theta)^2}$$

$\wedge$ on element face at angle θ

$\wedge$ on element face at angle $\theta+90$

Equate above formulas and solve for σ_x

$$tan(\theta)^2 = \frac{1}{2}$$

so $\quad \theta = \text{atan}\left(\frac{1}{\sqrt{2}}\right) = 35.264°$

$$\sigma_x = \frac{\sigma_\theta}{\cos(\theta)^2} = 117.0\ \text{MPa} \qquad \text{also} \quad \tau_\theta = -\sigma_x \sin(\theta)\cos(\theta) = -55.154\ \text{MPa}$$

Max. shear stress is 1/2 of max. normal stress for bar in uniaxial stress and is on plane at 45 deg. to axis of bar:

$$\tau_{max} = \frac{\sigma_x}{2} = 58.5\ \text{MPa}$$

Problem *Direct stresses of 120 N/mm² tensile and 90 N/mm² compression exist on two perpendicular planes at a certain point in a body. They are also accompanied by shear stress on the planes. The greatest principal stress at the point due to these is 150 N/mm².*

(a) What must be the magnitude of the shearing stresses on the two planes ?

(b) What will be the maximum shearing stress at the point ?

Problem *Direct stresses of 120 N/mm² tensile and 90 N/mm² compression exist on two perpendicular planes at a certain point in a body. They are also accompanied by shear stress on the planes. The greatest principal stress at the point due to these is 150 N/mm².*

(a) What must be the magnitude of the shearing stresses on the two planes ?

(b) What will be the maximum shearing stress at the point ?

Sol. Given :

Major tensile stress, $\sigma_1 = 120 \text{ N/mm}^2$

Minor compressive stress, $\sigma_2 = -90 \text{ N/mm}^2$ (Minus sign due to compression)

Greatest principal stress $= 150 \text{ N/mm}^2$

(*a*) Let τ = Shear stress on the two planes.

Using equation (3.15) for greatest principal stress, we get

$$\text{Greatest principal stress} = \frac{\sigma_1 + \sigma_2}{2} + \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

or
$$150 = \frac{120 + (-90)}{2} + \sqrt{\left(\frac{120 - (-90)}{2}\right)^2 + \tau^2}$$

$$= \frac{120 - 90}{2} + \sqrt{\left(\frac{120 + 90}{2}\right)^2 + \tau^2}$$

$$= 15 + \sqrt{105^2 + \tau^2}$$

$$\text{or} \quad 150 - 15 = \sqrt{105^2 + \tau^2}$$

$$\text{or} \quad 135 = \sqrt{105^2 + \tau^2}$$

Squaring both sides, we get

$$135^2 = 105^2 + \tau^2$$

$$\text{or} \quad \tau^2 = 135^2 - 105^2 = 18225 - 11025 = 7200$$

$$\therefore \quad \tau = \sqrt{7200} = \mathbf{84.853\ N/mm^2}. \quad \textbf{Ans.}$$

(*b*) *Maximum shear stress at the point*

Using equation (3.18) for maximum shear stress,

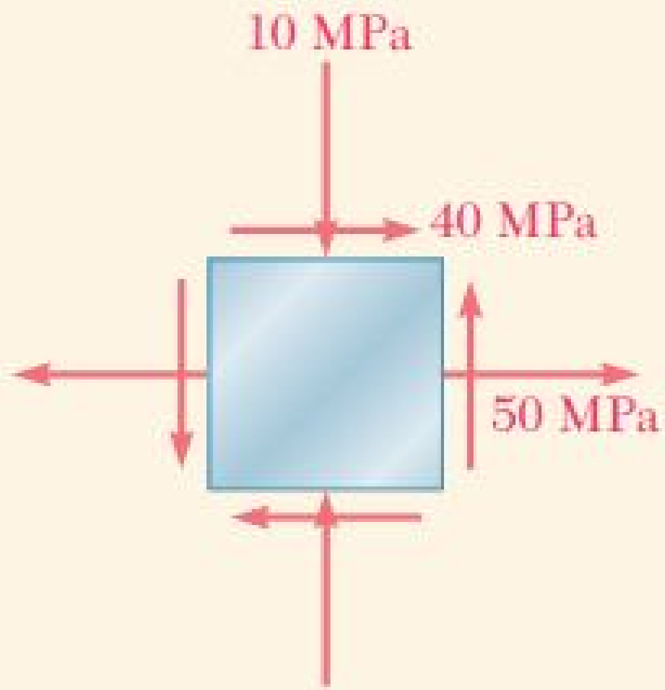
$$(\sigma_t)_{max} = \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

$$= \frac{1}{2}\sqrt{[120 - (-90)]^2 + 4 \times 7200} \qquad (\because \quad \tau^2 = 7200)$$

$$= \frac{1}{2}\sqrt{210^2 + 28800} = \frac{1}{2}\sqrt{44100 + 28800} = \frac{1}{2} \times 270$$

$$= \mathbf{135\ N/mm^2}. \quad \textbf{Ans.}$$

For the state of plane stress shown in Fig. determine (*a*) the principal planes, (*b*) the principal stresses, (*c*) the maximum shearing stress and the corresponding normal stress.



a. Principal Planes. Following the usual sign convention, the stress components are

$$\sigma_x = +50 \text{ MPa} \qquad \sigma_y = -10 \text{ MPa} \qquad \tau_{xy} = +40 \text{ MPa}$$

Substituting into Eq. (7.12),

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(+40)}{50 - (-10)} = \frac{80}{60}$$

$$2\theta_p = 53.1^\circ \quad \text{and} \quad 180^\circ + 53.1^\circ = 233.1^\circ$$

$$\theta_p = 26.6^\circ \quad \text{and} \quad 116.6^\circ$$

b. Principal Stresses. Equation (7.14) yields

$$\sigma_{\text{max, min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 20 \pm \sqrt{(30)^2 + (40)^2}$$

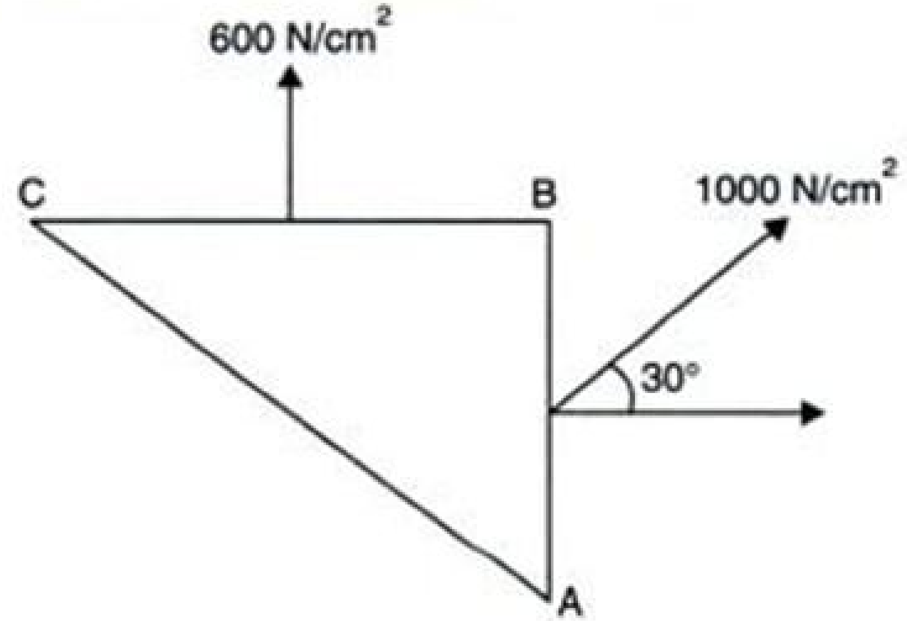
$$\sigma_{\text{max}} = 20 + 50 = 70 \text{ MPa}$$

$$\sigma_{\text{min}} = 20 - 50 = -30 \text{ MPa}$$

c. Maximum Shearing Stress.

$$\tau_{\text{max}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{(30)^2 + (40)^2} = 50 \text{ MPa}$$

Problem : *At a certain point in a material under stress the intensity of the resultant stress on a vertical plane is 1000 N/cm² inclined at 30° to the normal to that plane and the stress on a horizontal plane has a normal tensile component of intensity 600 N/cm² as shown in Fig. 3.16 (c). Find the magnitude and direction of the resultant stress on the horizontal plane and the principal stresses.*



Sol. Given :

Resultant stress on vertical plane AB = 1000 N/cm²

Inclination of the above stress = 30°

Normal stress on horizontal plane BC = 600 N/cm²

The resultant stress on plane AB is resolved into normal and tangential component.

The normal component

$= 1000 \times \cos 30° = 866\ N/cm^2$

Tangential component

$= 1000 \times \sin 30° = 500\ N/cm^2.$

Hence a shear stress of magnitude 500 N/cm² is acting on plane *AB*. To maintain the wedge in equilibrium, another shear stress of the same magnitude but opposite in direction must act on the plane *BC*. The free-body diagram of the element *ABCD* is shown in Fig. 3.16 (*d*), showing normal and shear stresses acting on different faces in which :

$$\sigma_1 = 866\ N/cm^2,$$

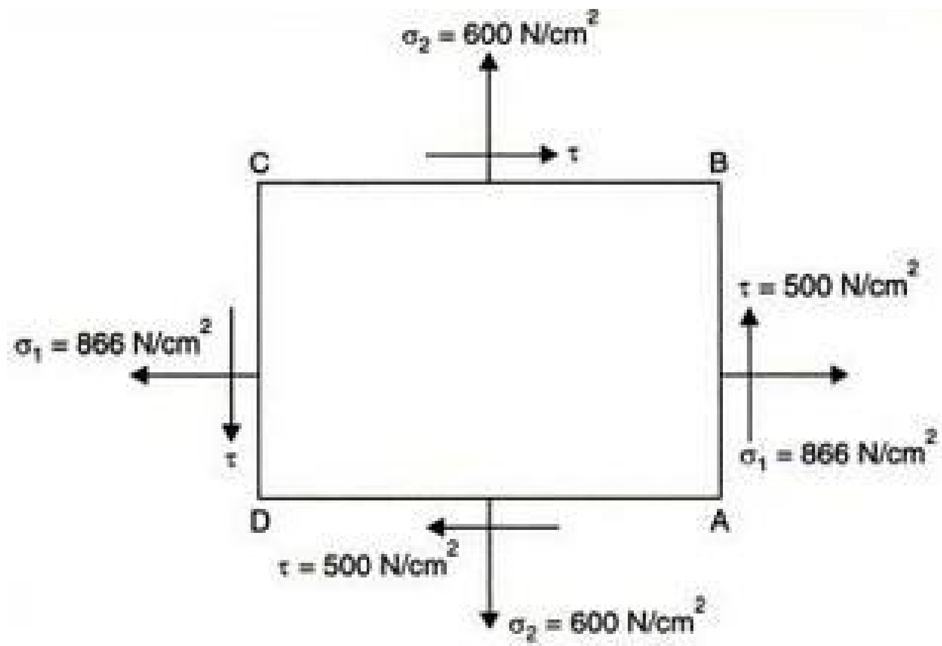
$$\sigma_2 = 600\ N/cm^2$$

and $$\tau = 500\ N/cm^2$$

(*i*) *Magnitude and direction of resultant stress on horizontal plane BC.*

Normal stress on plane *BC*, $\sigma_2 = 600\ N/cm^2$

Tangential stress on plane *BC*, $\tau = 500\ N/cm^2$



$\therefore$ Resultant stress $= \sqrt{\sigma_2^2 + \tau^2}$

$$= \sqrt{600^2 + 500^2} = \mathbf{781.02\ N/cm^2.} \quad \textbf{Ans.}$$

The direction of the resultant stress with the horizontal plane BC is given by,

$$\tan\theta = \frac{\sigma_2}{\tau} = \frac{600}{500} = 1.2$$

$$\theta = \tan^{-1} 1.2 = \mathbf{50.19°.} \quad \textbf{Ans.}$$

(*ii*) *Principal stresses*

The major and minor principal stresses are given by equations (3.15) and (3.16).

$$\therefore \text{ Principal stresses } = \frac{\sigma_1 + \sigma_2}{2} \pm \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

$$= \frac{866 + 600}{2} \pm \sqrt{\left(\frac{866 - 600}{2}\right)^2 + 500^2}$$

$$= 733 \pm 517.38$$

$$= (733 + 517.38) \text{ and } (733 - 517.38)$$

$$= 1250.38 \text{ and } 215.62\ \text{N/cm}^2.$$

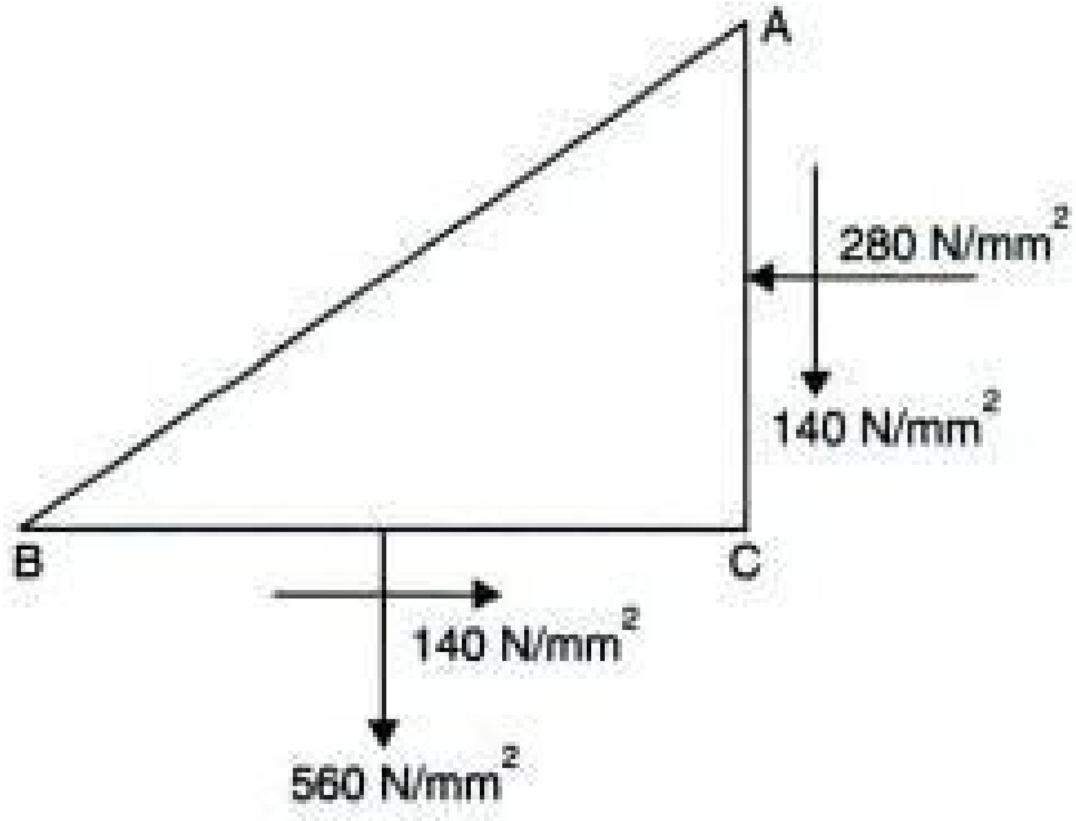
$\therefore$ Major principal stress $= \mathbf{1250.38\ N/cm^2.}$ **Ans.**

$\therefore$ Minor principal stress $= \mathbf{215.62\ N/cm^2.}$ **Ans.**

Problem *At a point in a strained material, on plane BC there are normal and shear stresses of 560 N/mm² and 140 N/mm² respectively. On plane AC, perpendicular to plane BC, there are normal and shear stresses of 280 N/mm² and 140 N/mm² respectively as shown in Fig. . Determine the following :*

(i) principal stresses and location of the planes on which they act,

(ii) maximum shear stress and the plane on which it acts.



On plane AC, $\sigma_1 = -280$ N/mm^2 (– ve sign due to compressive stress)

$\tau = 140$ N/mm^2

On plane BC, $\sigma_2 = 560$ N/mm^2

$\tau = 140$ N/mm^2

(*i*) *Principal stresses and location of the planes on which they act.*

Principal stress are given by equations (3.15) and (3.16)

$$\therefore \quad \text{Principal stresses} = \frac{\sigma_1 + \sigma_2}{2} \pm \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

$$= \frac{-280 + 560}{2} \pm \sqrt{\left(\frac{-280 - 560}{2}\right)^2 + 140^2}$$

$$= 140 \pm 442.7$$

$$= 582.7 \text{ and } (140 - 442.7) \text{ N/mm}^2$$

$$= 582.7 \text{ and } -302.7 \text{ N/mm}^2$$

$\therefore$ Major principal stress $=$ **582.7 N/mm^2 (Tensile). Ans.**

$\therefore$ Minor principal stress $=$ **– 302.7 N/mm^2. Ans.**

The planes on which principal stresses act, are given by equation (3.14) as

$$\tan 2\theta = \frac{2\tau}{\sigma_1 - \sigma_2} = \frac{2 \times 140}{-280 - 560} = \frac{280}{-840} = -0.33$$

$$\therefore \quad 2\theta = \tan^{-1} - 0.33 = -18.26^\circ$$

– ve sign shows that 2θ is lying in 2nd and 4th quadrant

$$\therefore \quad 2\theta = (180 - 18.26^\circ) \text{ or } (360 - 18.26^\circ)$$

$$= 161.34^\circ \text{ or } 341.34^\circ$$

$\therefore$ $\theta =$ **80.67° and 170.67°. Ans.**

(*ii*) *Maximum shear stress and the plane on which it acts.*

Maximum shear stress is given by equation (3.18).

$$\therefore \qquad (\sigma_t)_{max} = \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

$$= \sqrt{\left(\frac{-280 - 560}{2}\right)^2 + 140^2}$$

$$= \sqrt{420^2 + 140^2} = \mathbf{442.7\ N/mm^2.} \quad \text{Ans.}$$

The plane on which maximum shear stress acts is given by equation (3.17) as

$$\tan 2\theta = \frac{\sigma_2 - \sigma_1}{2\tau}$$

$$= \frac{560 - (-280)}{2 \times 140} = \frac{840}{280} = 3.0$$

$$\therefore \qquad 2\theta = \tan^{-1} 3.0 = 71.56° \text{ or } 251.56°$$

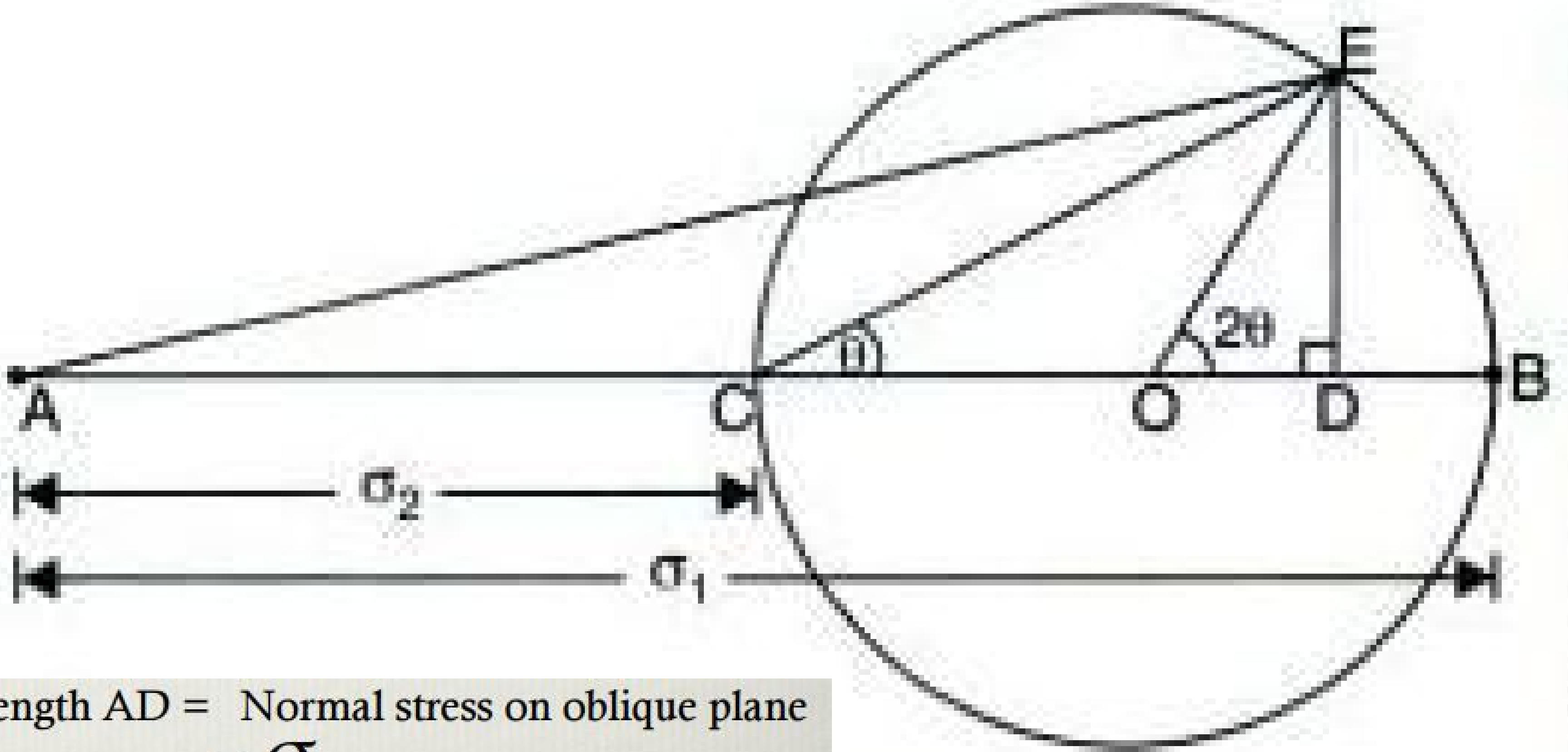
$$\therefore \qquad \theta = \mathbf{35.78° \text{ or } 125.78°.} \quad \text{Ans.}$$

Mohr Stress Circle

Graphical method to determine stresses.

- Body subjected to two mutually perpendicular principal stresses of unequal magnitude.
- Body subjected to two mutually perpendicular principal stresses of unequal magnitude and unlike (one tensile and other compressive).
- Body subjected to two mutually perpendicular principal stresses + simple shear stress.

Body subjected to two mutually perpendicular principal stresses of unequal magnitude



length AD = Normal stress on oblique plane

$$= \sigma_n$$

length ED = Tangential stress on Oblique plane

$$= \sigma_t$$

length AE = Resultant stress on Oblique plane $= \sqrt{\sigma_t^{\,2} + \sigma_n^{\,2}}$

Consider a rectangular body subjected to two mutually perpendicular principal tensile stresses of unequal intensities. It is required to find the resultant stress on an oblique plane.

Let σ_1 = Major tensile stress

σ_2 = Minor tensile stress, and

θ = Angle made by the oblique plane with the axis of minor tensile stress.

Mohr's circle is drawn as :

Take any point A and draw a horizontal line through A. Take $AB = \sigma_1$ and $AC = \sigma_2$ towards right from A to some suitable scale. With BC as diameter describe a circle. Let O is the centre of the circle. Now through O, draw a line OE marking an angle 2θ with OB.

From E, draw ED perpendicular on AB. Join AE. Then the normal and tangential stresses on the oblique plane are given by AD and ED respectively. The resultant stress on the oblique plane is given by AE.

From Fig. we have

Length AD = Normal stress on oblique plane

Length ED = Tangential stress on oblique plane

Length AE = Resultant stress on oblique plane.

Radius of Mohr's circle $= \dfrac{\sigma_1 - \sigma_2}{2}$

Angle ϕ = obliquity.

Proof.

$$CO = OB = OE = \text{Radius of Mohr's circle} = \frac{\sigma_1 - \sigma_2}{2}$$

$$\therefore \quad AO = AC + CO$$

$$= \sigma_2 + \frac{\sigma_1 - \sigma_2}{2} = \frac{2\sigma_2 + \sigma_1 - \sigma_2}{2} = \frac{\sigma_1 + \sigma_2}{2}$$

$$OD = OE \cos 2\theta$$

$$= \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta \qquad \left(\because \; OE = \frac{\sigma_1 - \sigma_2}{2}\right)$$

$$\therefore \quad AD = AO + OD$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta$$

$= \sigma_n$ or Normal stress

$$ED = OE \sin 2\theta$$

$$= \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta$$

$= \sigma_t$ or Tangential stress.

Important points.

(*i*) Normal stress is along the line *ACB*. Hence maximum normal stress will be when point *E* is at *B*. And minimum normal stress will be when point *E* is at *C*. Hence maximum normal stress = $AB = \sigma_1$ and minimum normal stress = $AB = \sigma_2$.

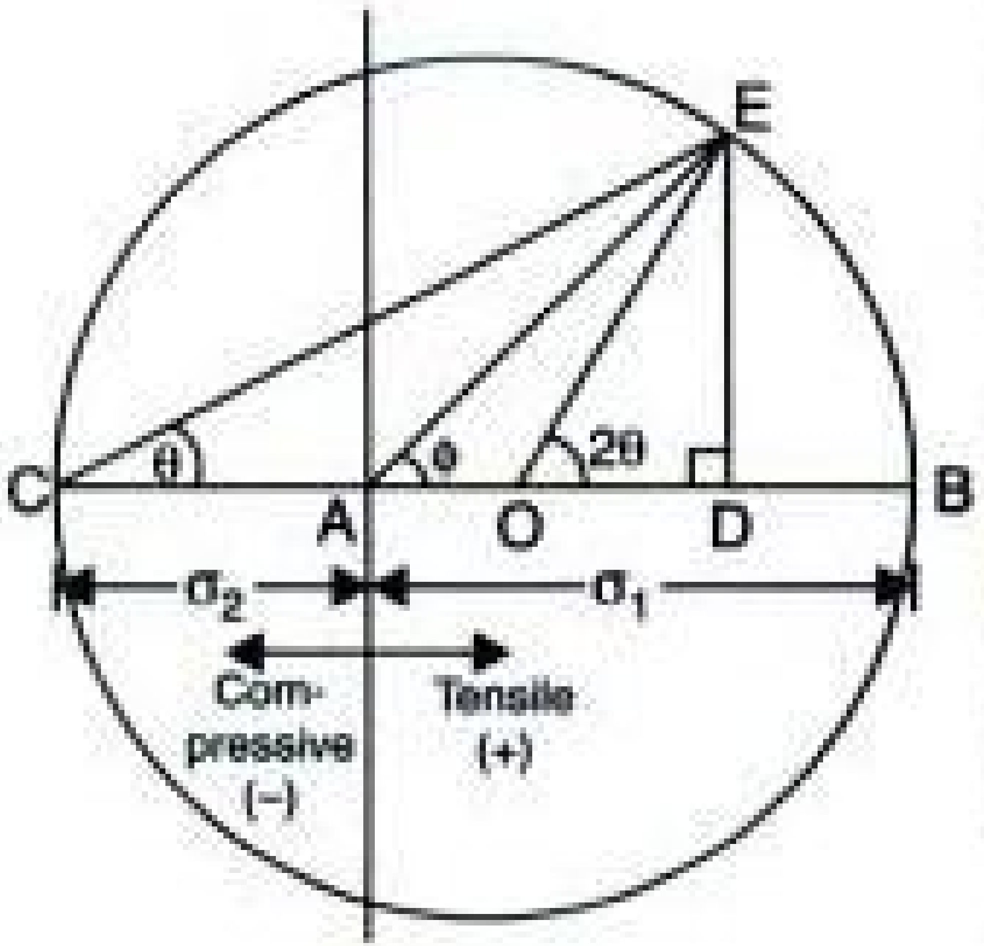
(*ii*) Tangential stress (or shear stress) is along a line which is perpendicular to line *CB*. Hence maximum shear stress will be when perpendicular to line *CB* is drawn from point *O*. Then maximum shear stress will be equal to the radius of the Mohr's circle.

$$\therefore \qquad (\sigma_t)_{max} = \frac{\sigma_1 - \sigma_2}{2}.$$

(*iii*) When the point *E* is at *B* or at *C*, the shear stress will be zero.

(*iv*) The angle ϕ (which is known as angle of obliquity) will be maximum, when the line *AE* is tangent to the Mohr's circle.

Body subjected to two mutually perpendicular principal stresses of unequal magnitude and unlike (one tensile and other compressive).



length AD = Normal stress on oblique plane

$$= \sigma_n$$

length ED = Tangential stress on Oblique plane

$$= \sigma_t$$

length AE = Resultant stress on Oblique plane

$$= \sqrt{\sigma_t^{\,2} + \sigma_n^{\,2}}$$

Consider a rectangular body subjected to two mutually perpendicular principal stresses which are unequal and one of them is tensile and the other is compressive. It is required to find the resultant stress on an oblique plane.

Let σ_1 = Major principal tensile stress,

σ_2 = Minor principal compressive stress, and

θ = Angle made by the oblique plane with the axis of minor principal stress.

Mohr's circle is drawn as :

Take any point A and draw a horizontal line through A on both sides of A as shown in Fig. 3.24. Take $AB = \sigma_1(+)$ towards right of A and $AC = \sigma_2(-)$ towards left of A to some suitable scale. Bisect BC at O. With O as centre and radius equal to CO or OB, draw a circle. Through O draw a line OE making an angle 2θ with OB.

From E, draw ED perpendicular to AB. Join AE and CE. Then normal and shear stress (*i.e.*, tangential stress) on the oblique plane are given by AD and ED. Length AE represents the resultant stress on the oblique plane.

$\therefore$ From Fig. 3.24, we have

Length AD = Normal stress on oblique plane,

Length ED = Shear stress on oblique plane,

Length AE = Resultant stress on oblique plane, and

Angle ϕ = Obliquity.

Radius of Mohr's circle = CO or $OB = \dfrac{\sigma_1 + \sigma_2}{2}$.

Proof.

$$CO = OB = OE = \text{Radius of Mohr's circle}$$

$$= \frac{\sigma_1 + \sigma_2}{2}$$

$$AO = OC - AC$$

$$= \frac{\sigma_1 + \sigma_2}{2} - \sigma_2 = \frac{\sigma_1 + \sigma_2 - 2\sigma_2}{2} = \frac{\sigma_1 - \sigma_2}{2}$$

$\therefore$

$$AD = AO + OD$$

$$= AO + OB \cos 2\theta \qquad (\because \quad OD = OE \cos 2\theta)$$

$$= \frac{\sigma_1 - \sigma_2}{2} + \frac{\sigma_1 + \sigma_2}{2} \cos 2\theta \qquad \left(\because \quad OE = \text{Radius} = \frac{\sigma_1 + \sigma_2}{2}\right)$$

$$= \sigma_n \text{ or Normal stress}$$

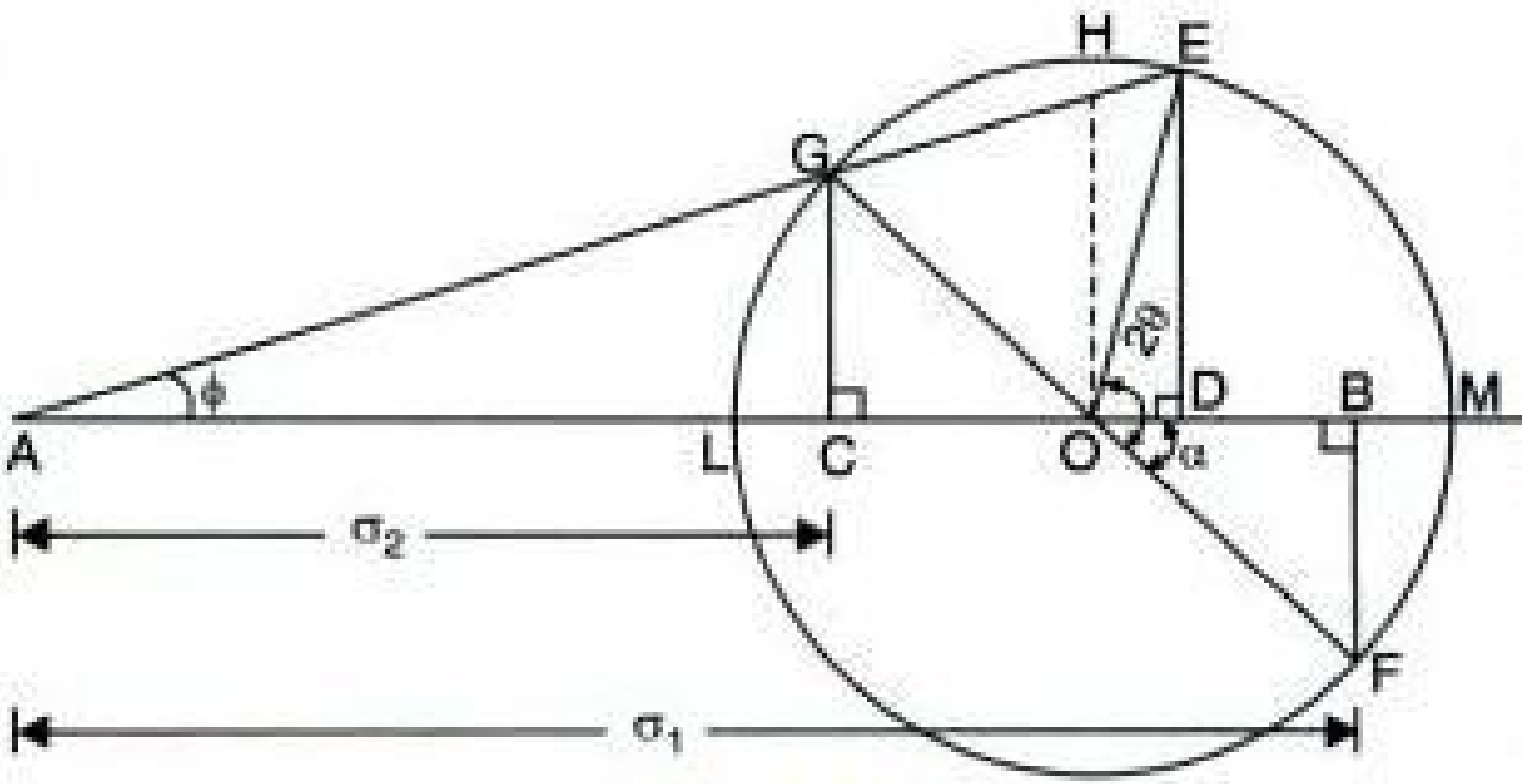
and

$$ED = OE \sin 2\theta$$

$$= \frac{\sigma_1 + \sigma_2}{2} \sin 2\theta \qquad \left(\because \quad OE = \frac{\sigma_1 + \sigma_2}{2}\right)$$

$$= \sigma_t \text{ or Tangential (or shear) stress.}$$

Body subjected to two mutually perpendicular principal stresses + simple shear stress.



Take any point A and draw a horizontal line through A. Take $AB = \sigma_1$ and $AC = \sigma_2$ towards right of A to some suitable scale. Draw perpendiculars at B and C and cut off BF and CG equal to shear stress τ to the same scale. Bisect BC at O. Now with O as centre and radius equal to OG or OF draw a circle. Through O, draw a line OE making an angle of 2θ with OF as shown in Fig. From E, draw ED perpendicular to CB. Join AE. Then length AE represents the resultant stress on the given oblique plane. And lengths AD and ED represents the normal stress and tangential stress respectively.

Hence from Fig. 3.27, we have

Length AE = Resultant stress on the oblique plane

Length AD = Normal stress on the oblique plane

Length ED = Shear stress on the oblique plane.

$$CO = \frac{1}{2} CB = \frac{1}{2} [\sigma_1 - \sigma_2]$$

$$AO = AC + CO = \sigma_2 + \frac{1}{2} [\sigma_1 - \sigma_2]$$

$$= \frac{2\sigma_2 + \sigma_1 - \sigma_2}{2} = \frac{\sigma_1 + \sigma_2}{2}$$

$$AD = AO + OD$$

$$= \frac{\sigma_1 + \sigma_2}{2} + OE \cos (2\theta - \alpha) \qquad [\because \quad OD = OE \cos (2\theta - \alpha)]$$

$$= \frac{\sigma_1 + \sigma_2}{2} + OE [\cos 2\theta \cos \alpha + \sin 2\theta \sin \alpha]$$

$$= \frac{\sigma_1 + \sigma_2}{2} + OE \cos 2\theta \cos \alpha + OE \sin 2\theta \sin \alpha$$

$$= \frac{\sigma_1 + \sigma_2}{2} + OE \cos \alpha \,.\, \cos 2\theta + OE \sin \alpha \,.\, \sin 2\theta$$

$$= \frac{\sigma_1 + \sigma_2}{2} + OF \cos \alpha \,.\, \cos 2\theta + OF \sin \alpha \,.\, \sin 2\theta$$

$$(\because \quad OE = OF = \text{Radius})$$

$$= \frac{\sigma_1 + \sigma_2}{2} + OB \cos 2\theta + BF \sin 2\theta$$

$$(\because \quad OF \cos \alpha = OB,\ OF \sin \alpha = BF)$$

$$= \frac{\sigma_1 + \sigma_2}{2} + CO \cos 2\theta + \tau \sin 2\theta \qquad (\because \quad OB = CO,\ BF = \tau)$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau \sin 2\theta \qquad \left(\because \quad CO = \frac{\sigma_1 - \sigma_2}{2}\right)$$

$$= \sigma_n \text{ or Normal stress}$$

Now $\quad ED = OE \sin(2\theta - \alpha) = OE(\sin 2\theta \cos\alpha - \cos 2\theta \sin\alpha)$

$$= OE \sin 2\theta \cos\alpha - OE \cos 2\theta \sin\alpha$$

$$= OE \cos\alpha \,.\, \sin 2\theta - OE \sin\alpha \,.\, \cos 2\theta$$

$$= OE \cos\alpha \,.\, \sin 2\theta - OE \sin\alpha \,.\, \cos 2\theta \qquad (\because \ OE = OF = \text{Radius})$$

$$= OB \,.\, \sin 2\theta - BF \cos 2\theta \qquad (\because \ OF\cos\alpha = OB,\ OF\sin\alpha = BF)$$

$$= CO \,.\, \sin 2\theta - \tau \cos 2\theta \qquad (\because \ OB = CO,\ BF = \tau)$$

$$= \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta \qquad \left(\because \ CO = \frac{\sigma_1 - \sigma_2}{2}\right)$$

$$= \sigma_t \text{ or Tangential stress.}$$

Maximum and minimum value of normal stress. In Fig. 3.27, the normal stress is given by AD. Hence the maximum value of AD will be when D coincides with M and minimum value of AD will be when D coincides with L.

$\therefore$ Maximum value of normal stress,

$$(\sigma_n)_{max} = AM = AO + OM$$

$$= \frac{\sigma_1 + \sigma_2}{2} + OF \qquad \left(\because \ AO = \frac{\sigma_1 + \sigma_2}{2}, OM = OF = \text{Radius}\right)$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \sqrt{OB^2 + BF^2} \qquad (\because \ \text{In triangle } OBF, OF = \sqrt{OB^2 + BF^2})$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2} \qquad \left(\because \ OB = \frac{\sigma_1 - \sigma_2}{2}, BF = \tau\right)$$

Minimum value of normal stress,

$$(\sigma_n)_{min} = AL = AO - LO$$

$$= \frac{\sigma_1 + \sigma_2}{2} - OF \qquad (\because \ LO = OF = \text{Radius})$$

$$= \frac{\sigma_1 + \sigma_2}{2} - \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

(*i*) For maximum normal stress, the point D coincides with M. But when the point D coincides with M, the point E also coincides with M. Hence for maximum value of normal stress,

$$\text{Angle} \quad 2\theta = \alpha \qquad (\because \text{ Line } OE \text{ coincides with line } OM)$$

$$\therefore \quad \theta = \frac{\alpha}{2} \qquad \ldots(i)$$

$$\text{Also} \quad \tan 2\theta = \tan \alpha = \frac{BF}{OB} = \frac{\tau}{\dfrac{\sigma_1 - \sigma_2}{2}} \qquad \left(\because \ BF = \tau, OB = \frac{\sigma_1 - \sigma_2}{2}\right)$$

$$= \frac{2\tau}{\sigma_1 - \sigma_2}.$$

(*ii*) For maximum and minimum normal stresses, the shear stress is zero and hence the planes, on which maximum and minimum normal stresses act, are known as *principal planes* and the stresses are known as *principal stresses*.

(*iii*) For minimum normal stress, the point D coincides with point L. But when the point D coincides with L, the point E also coincides with L. Then

$$\text{Angle} \quad 2\theta = \pi + \alpha \qquad (\because \text{ Line } OE \text{ coincides with line } OL)$$

$$\therefore \quad \theta = \frac{\pi}{2} + \frac{\alpha}{2} \qquad \ldots(ii)$$

From equations (*i*) and (*ii*), it is clear that the plane of minimum normal stress is inclined at an angle 90° to the plane of maximum normal stress.

Maximum value of shear stress. Shear stress is given by ED. Hence maximum value of ED will be when E coincides with G, and D coincides with O.

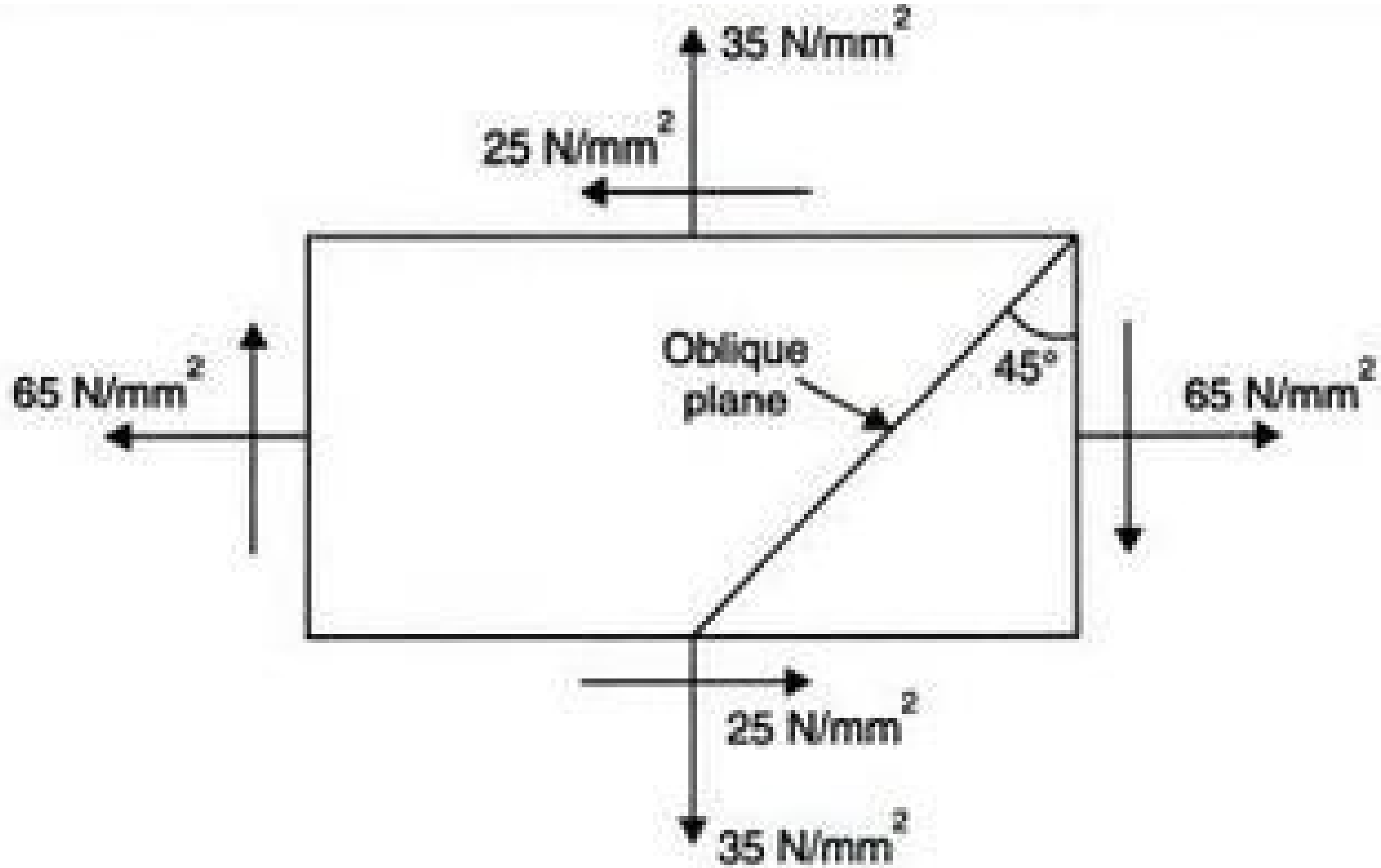
$\therefore$ Maximum shear stress,

$$(\sigma_t)_{max} = OH = OF \qquad (\because \ OH = OF = \text{radius})$$

$$= \sqrt{OB^2 + BF^2} \qquad (\because \text{ In triangle } OBF, OF = \sqrt{OB^2 + BF^2})$$

$$= \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2} \qquad \left(\because \ OB = \frac{\sigma_1 - \sigma_2}{2}, BF = \tau\right)$$

A point in a strained material is subjected to stresses shown in Fig. Using Mohr's circle method, determine the normal and tangential stresses across the oblique plane. Check the answer analytically.

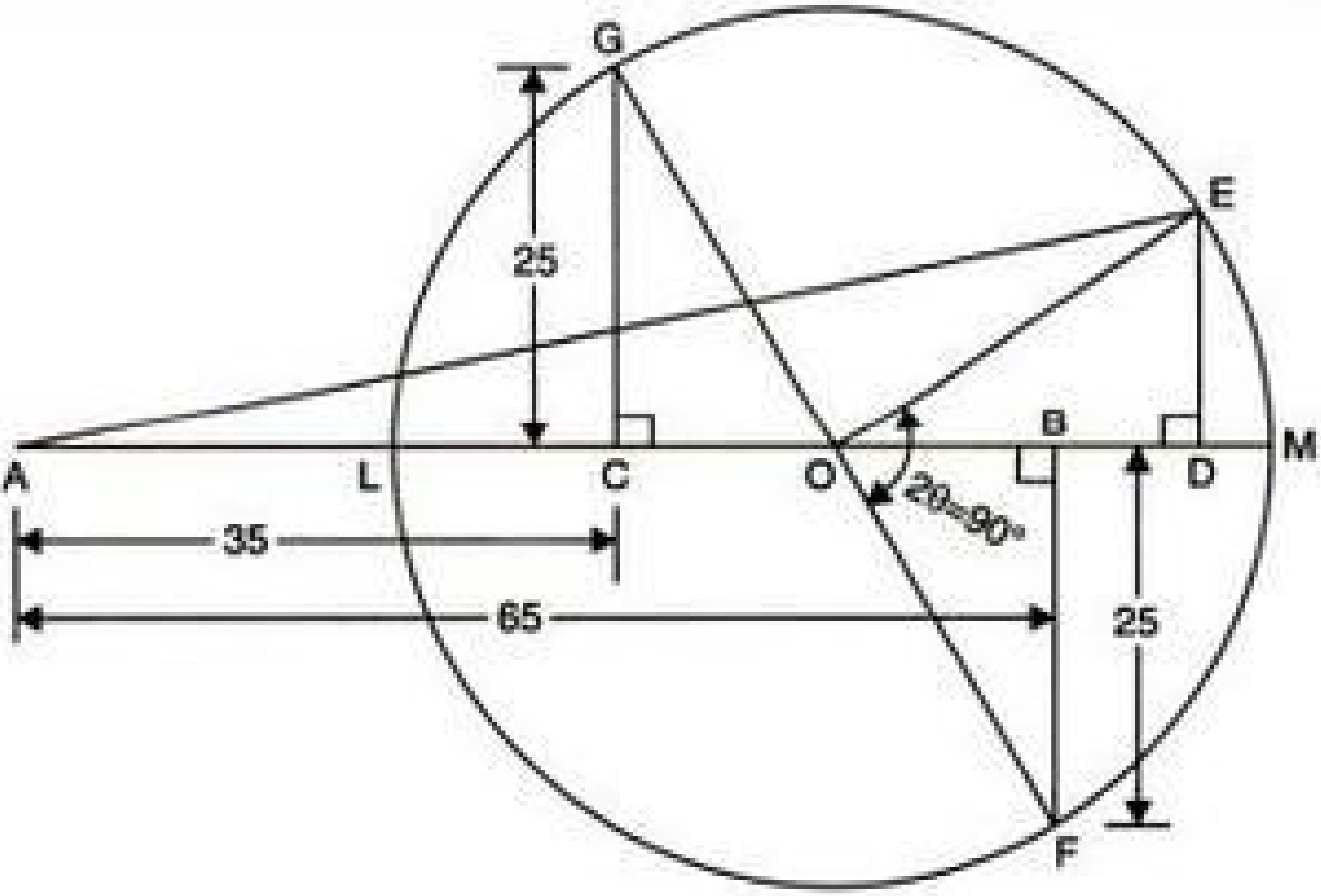


Mohr's circle method

Let $1 \text{ cm} = 10 \text{ N/mm}^2$

Then $\sigma_1 = \dfrac{65}{10} = 6.5 \text{ cm},$

$\sigma_2 = \dfrac{35}{10} = 3.5 \text{ cm and } \tau = \dfrac{25}{10} = 2.5 \text{ cm}$



Take any point A and draw a horizontal line through A. Take $AB = \sigma_1 = 6.5$ cm and $AC = \sigma_2 = 3.5$ cm towards right of A. Draw perpendicular at B and C and cut off BF and CG equal to shear stress $\tau = 2.5$ cm. Bisect BC at O. Now with O as centre and radius equal to OF (or OG) draw a circle. Through O, draw a line OE making an angle of 2θ (*i.e.*, $2 \times 45° = 90°$) with OF as shown in Fig. 3.29. From E, draw ED perpendicular to AB produced. Join AE. Then length AD represents the normal stress and length ED represents the shear stress.

By measurements, length $AD = 7.5$ cm and

length $ED = 1.5$ cm.

$\therefore$ Normal stress (σ_n) = Length AD × Scale = 7.5×10 = **75 N/mm²**. **Ans.**

$(\because \; 1 \text{ cm} = 10 \text{ N/mm}^2)$

And tangential stress (σ_t) = Length ED × Scale = 1.5×10 = **15 N/mm²**. **Ans.**

equal to shear stress $\tau = 2.5$ cm. Bisect *BC* at *O*. Now with *O* as centre and radius equal to *OF* (or *OG*) draw a circle. Through *O*, draw a line *OE* making an angle of 2θ (*i.e.*, $2 \times 45° = 90°$) with *OF* as shown in Fig. 3.29. From *E*, draw *ED* perpendicular to *AB* produced. Join *AE*. Then length *AD* represents the normal stress and length *ED* represents the shear stress.

By measurements, length $AD = 7.5$ cm and

length $ED = 1.5$ cm.

$\therefore$ Normal stress (σ_n) = Length $AD \times$ Scale $= 7.5 \times 10 =$ **75 N/mm²**. Ans.

($\because$ 1 cm = 10 N/mm²)

And tangential stress (σ_t) = Length $ED \times$ Scale $= 1.5 \times 10 =$ **15 N/mm²**. Ans.

Analytical Answers

Normal stress (σ_n) is given by equation (3.12).

$\therefore$ Using equation (3.12),

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau \sin 2\theta$$

$$= \frac{65 + 35}{2} + \frac{65 - 35}{2} \cos (2 \times 45°) + 25 \sin (2 \times 45°)$$

$$= 50 + 15 \cos 90° + 25 \sin 90°$$

$$= 50 + 15 \times 0 + 25 \times 1 \qquad (\because \cos 90° = 0, \sin 90° = 1)$$

$$= 50 + 0 + 25 = \mathbf{75\ N/mm^2}.$$ Ans.

Tangential stress is given by equation (3.13)

$\therefore$ Using equation (3.13),

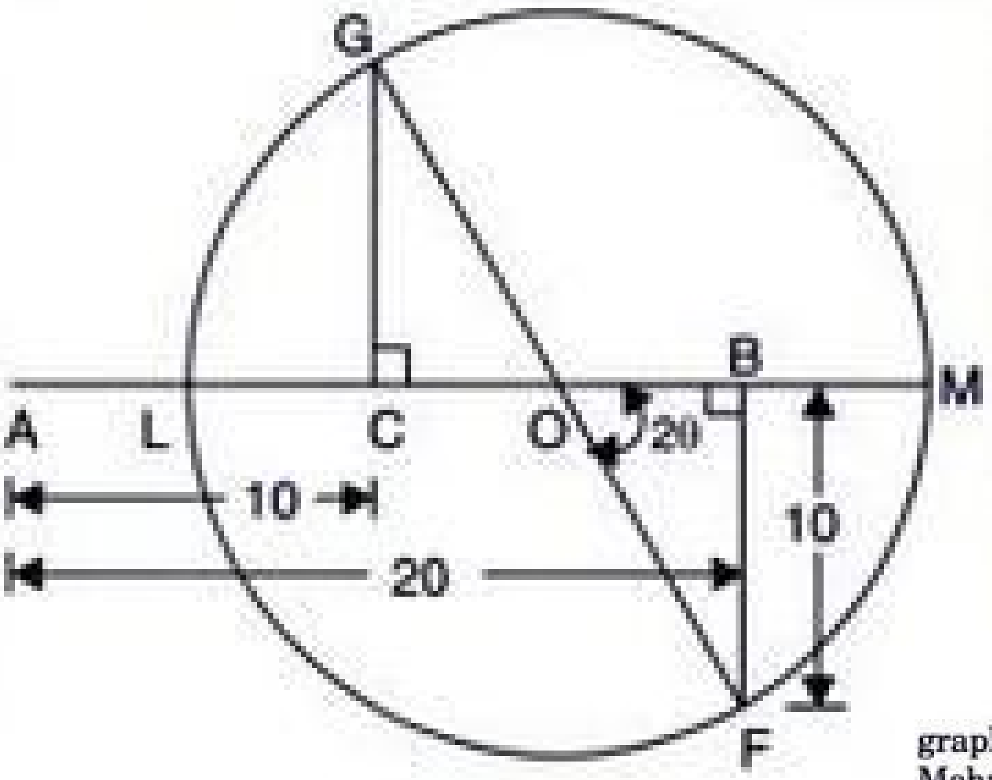
$$\sigma_t = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta$$

$$= \frac{65 - 35}{2} \sin (2 \times 45) - 25 \cos (2 \times 45)$$

$$= 15 \sin 90° - 25 \cos 90° = 15 \times 1 - 25 \times 0 = 15 - 0$$

$$= \mathbf{15\ N/mm^2}.$$ Ans.

Problem *At a certain point in a strained material, the intensities of stresses on two planes at right angles to each other are 20 N/mm² and 10 N/mm² both tensile. They are accompanied by a shear stress of magnitude 10 N/mm². Find graphically or otherwise, the location of principal planes and evaluate the principal stresses.*



Major tensile stress, $\sigma_1 = 20\ \text{N/mm}^2$

Minor tensile stress, $\sigma_2 = 10\ \text{N/mm}^2$

Shear stress, $\tau = 10\ \text{N/mm}^2$

This problem may be solved analytically or graphically. Here we shall solve it graphically (*i.e.*, by Mohr's circle method).

Scale. Take 1 cm = 2 N/mm²

Then $\sigma_1 = \dfrac{20}{2} = 10$ cm, $\sigma_2 = \dfrac{10}{2} = 5$ cm

and $\tau = \dfrac{10}{2} = 5$ cm.

Take any point A and draw a horizontal line through A. Take $AB = \sigma_1 = 10$ cm and $AC = \sigma_2 = 5$ cm towards right side of A. Draw perpendiculars at B and C and cut off $BF = CG = \tau = 5$ cm. Bisect BC at O. Now with O as centre and radius equal to OG (or OF), draw a circle cutting the horizontal line through A, at L and M as shown in Fig. 3.30. Then AM and AL represent the major principal and minor principal stresses.

By measurements, we have

Length $AM = 13.1$ cm and Length $AL = 1.91$ cm

$$\angle FOB \text{ (or } 2\theta) = 63.7°.$$

$\therefore$ Major principal stress $= \text{Length } AM \times \text{Scale}$

$= 13.1 \times 2 \text{ N/mm}^2$ $\qquad (\because \; 1 \text{ cm} = 2 \text{ N/mm}^2)$

$= \mathbf{26.2 \ N/mm^2}$. **Ans.**

Minor principal stress $= \text{Length } AL \times \text{Scale}$

$= 1.91 \times 2 = \mathbf{3.82 \ N/mm^2}$. **Ans.**

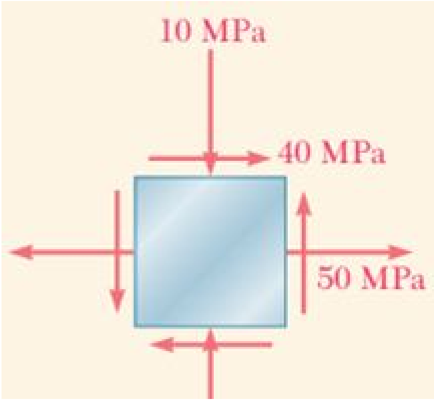
Location of principal planes

$$2\theta = 63.7°$$

$$\therefore \qquad \theta = \frac{63.7°}{2} = \mathbf{31.85°}. \quad \textbf{Ans.}$$

The second principal plane is given by

$$\theta + 90° \text{ or } 31.85° + 90° \text{ or } \mathbf{121.85°}. \quad \textbf{Ans.}$$



a. Construction of Mohr's Circle. Note from Fig. 7.16*a* that the normal stress exerted on the face oriented toward the *x* axis is tensile (positive) and the shearing stress tends to rotate the element counterclockwise. Therefore, point *X* of Mohr's circle is plotted to the right of the vertical axis and below the horizontal axis (Fig A similar inspection of the normal and shearing stresses exerted on the upper face of the element shows that point *Y* should be plotted to the left of the vertical axis and above the horizontal axis. Drawing the line *XY*, the center *C* of Mohr's circle is found. Its abscissa is

$$\sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} = \frac{50 + (-10)}{2} = 20 \text{ MPa}$$

Since the sides of the shaded triangle are

$$CF = 50 - 20 = 30 \text{ MPa} \qquad \text{and} \qquad FX = 40 \text{ MPa}$$

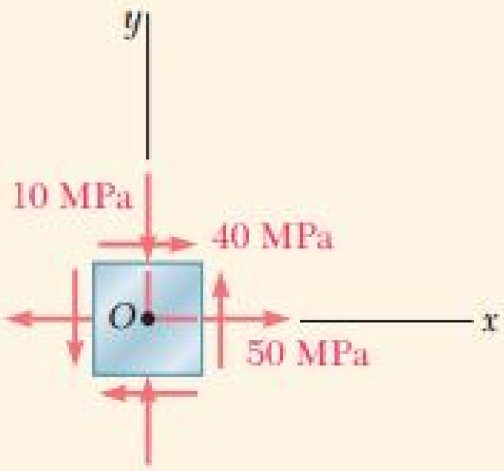
the radius of the circle is

$$R = CX = \sqrt{(30)^2 + (40)^2} = 50 \text{ MPa}$$

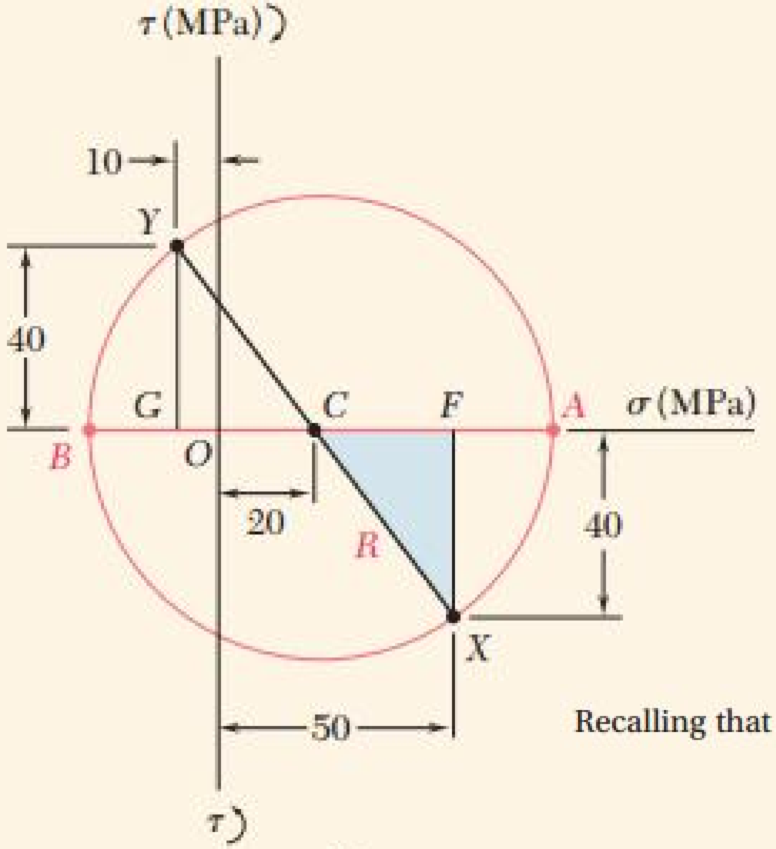
b. Principal Planes and Principal Stresses. The principal stresses are

$$\sigma_{\max} = OA = OC + CA = 20 + 50 = 70 \text{ MPa}$$

$$\sigma_{\min} = OB = OC - BC = 20 - 50 = -30 \text{ MPa}$$



(a)

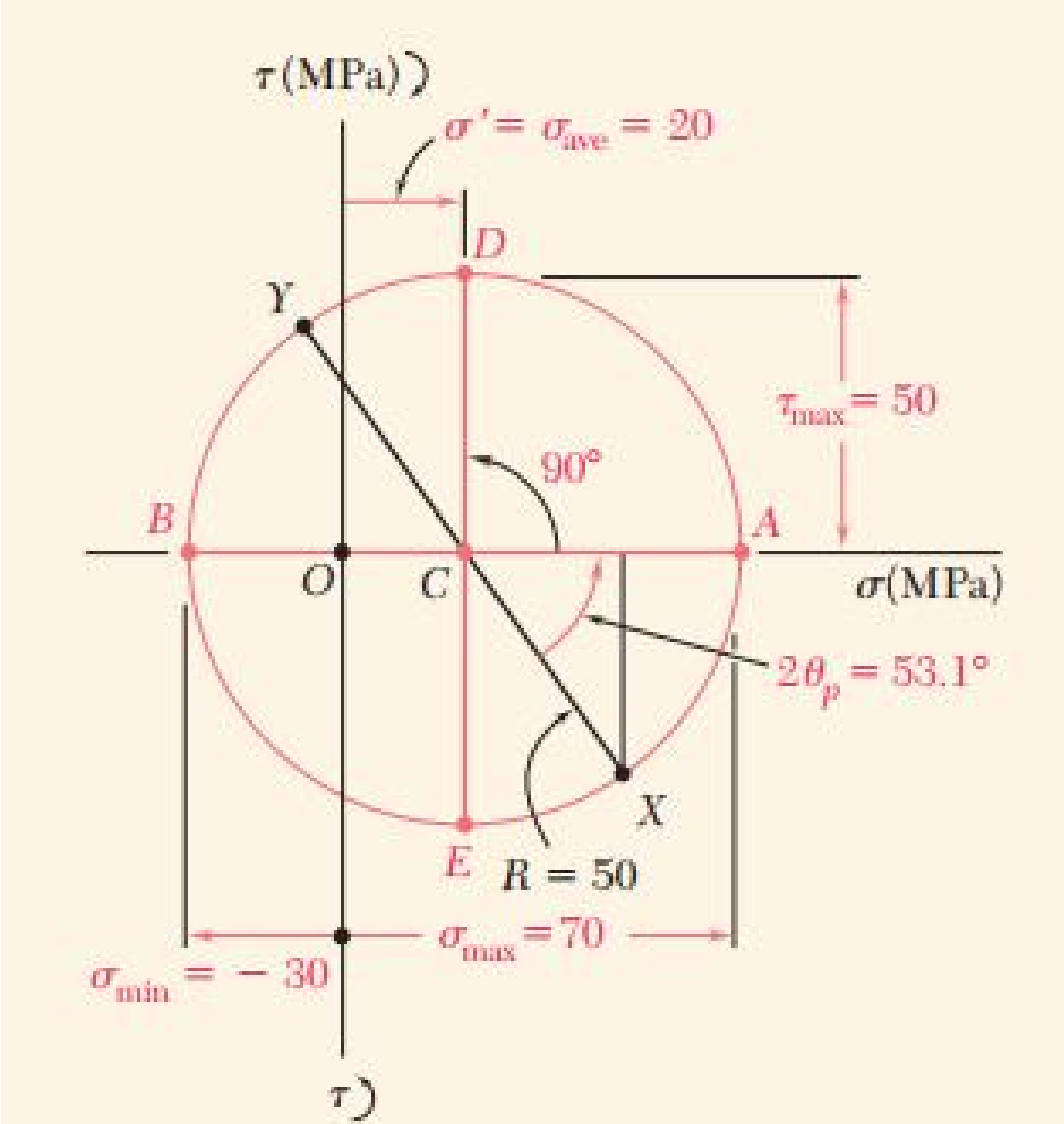


(b)

Recalling that the angle ACX represents $2\theta_p$

$$\tan 2\theta_p = \frac{FX}{CF} = \frac{40}{30}$$

$$2\theta_p = 53.1° \qquad \theta_p = 26.6°$$



τ (MPa)
σ′ = σave = 20
D
Y
τmax = 50
90°
B
A
O
C
σ(MPa)
2θp = 53.1°
X
E
R = 50
σmax = 70
σmin = − 30
τ

Theories of Failure

The material properties are usually determined by simple tension or compression tests.

The mechanical members are subjected to biaxial or triaxial stresses.

To determine whether a component will fail or not, some failure theories are proposed which are related to the properties of materials obtained from uniaxial tension or compression tests.

Initially we will consider failure of a mechanical member subjected to biaxial stresses

The Theories of Failures which are applicable for this situation are:

- Max principal or normal stress theory (Rankine’s theory)
- Maximum shear stress theory (Guest’s or Tresca’s theory)
- Max. Distortion energy theory (Von Mises & Hencky’s theory)
- Max. strain energy theory
- Max. principal strain theory

Ductile materials usually fail by yielding and hence the limiting strength is the yield strength of material as determined from simple tension test which is assumed the same in compression also.

For brittle materials limiting strength of material is ultimate tensile strength in tension or compression.

Max. Principal or Normal stress theory (Rankine's Theory):

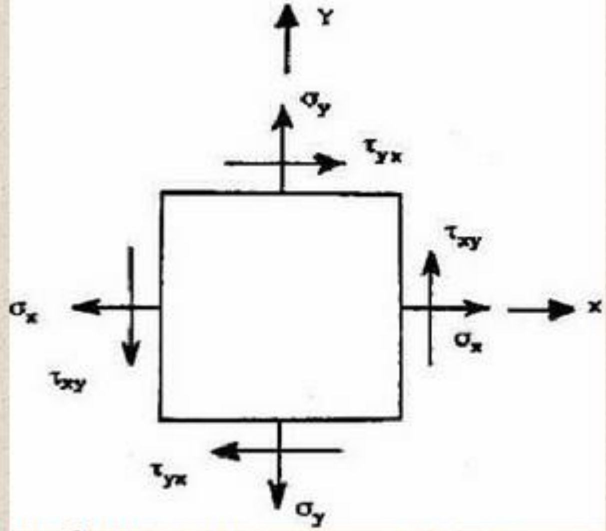
- It is assumed that the failure or yield occurs at a point in a member when the max. principal or normal stress in the biaxial stress system reaches the limiting strength of the material in a simple tension test.
- In this case max. principal stress is calculated in a biaxial stress case and is equated to limiting strength of the material.

Maximum principal stress

$$\sigma_1 = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Minimum principal stress

$$\sigma_2 = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$



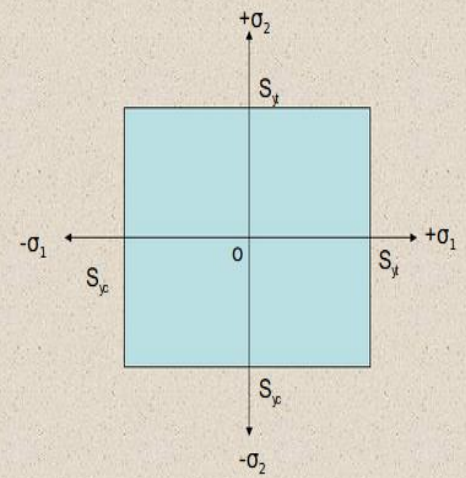
- For ductile materials

σ_1 should not exceed $\left(\frac{S_{yt}}{FOS}\right)$ in tension, FOS=Factor of safety

- For brittle materials

σ 1 should not exceed $\left(\frac{S_{ut}}{FOS}\right)$ in tension

This theory is basically applicable for brittle materials which are relatively stronger in shear and not applicable to ductile materials which are relatively weak in shear.



Boundary for maximum – normal – stress theory under bi – axial stresses

2. Maximum Shear Stress theory (Guest's or Tresca's theory):

- The failure or yielding is assumed to take place at a point in a member where the max shear stress in a biaxial stress system reaches a value equal to shear strength of the material obtained from simple tension test.

- In a biaxial stress case max shear stress developed is given by

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = \frac{\tau_{yt}}{FOS}$$

$$\text{where } \tau_{\max} = \frac{S_{yt}}{2 \times FOS}$$

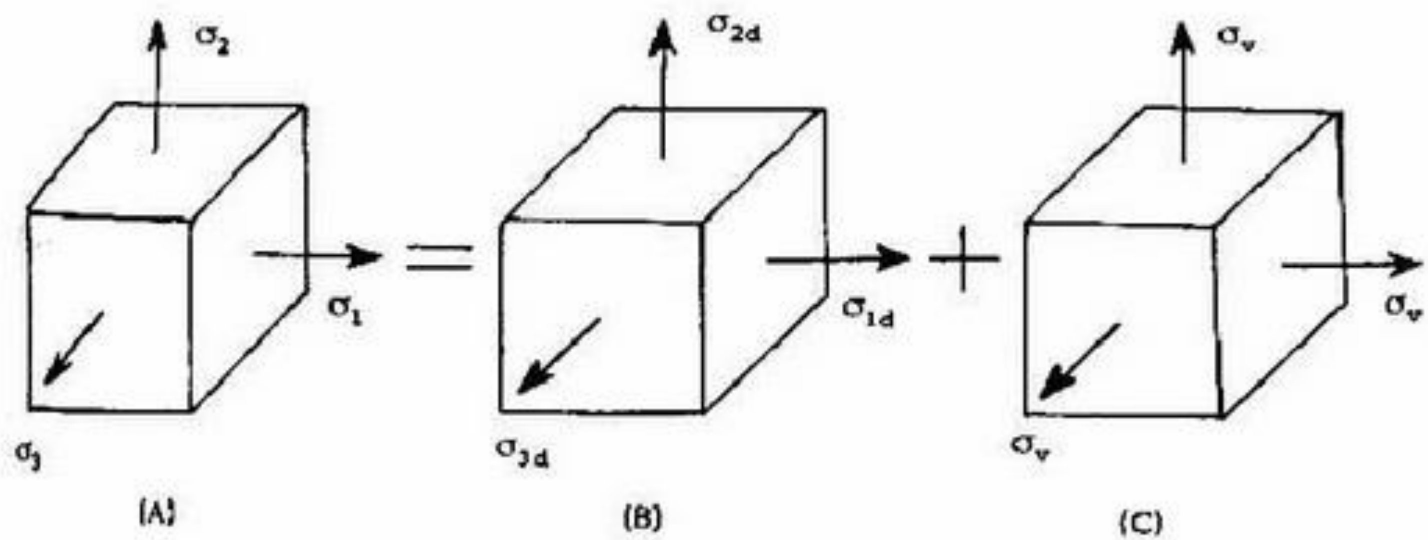
This theory is mostly used for ductile materials.

3.Max. Distortion energy theory (Von Mises & Hencky's theory):

- It is assumed that failure or yielding occurs at a point the member where the distortion strain energy (also called shear strain energy) per unit volume in a biaxial stress system reaches the limiting distortion energy (distortion energy at yield point) per unit volume as determined from a simple tension test.

- The maximum distortion energy is the difference between the total strain energy and the strain energy due to uniform stress.

3.Max. Distortion energy theory (Von Mises & Hencky's theory):



(a) Element with tri-axial stresses,
(b) Stress components due to distortion of element,
(c) Stress components due to change of volume.

3.Max. Distortion energy theory (Von Mises & Hencky's theory):

- The criteria of failure for the distortion – energy theory is expressed as

$$S_{yt} = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]}$$

- Considering the factor of safety

$$\frac{S_{yt}}{FOS} = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]}$$

- For bi – axial stresses (σ_3=0),

$$\frac{S_{yt}}{FOS} = \sqrt{\left(\sigma_1^{\,2} + \sigma_2^{\,2} - \sigma_1\,\sigma_2\right)}$$

4. Max. Strain energy theory (Heigh's Thoery):

- Failure is assumed to take place at a point in a member where strain energy per unit volume in a biaxial stress system reaches the limiting strain energy that is strain energy at yield point per unit volume as determined from a simple tension test.

- Strain energy per unit volume in a biaxial system is

$$U_1 = \frac{1}{2E}\left[\sigma_1^{\,2} + \sigma_2^{\,2} - \frac{2\sigma_1\sigma_2}{m}\right]$$

- The limiting strain energy per unit volume for yielding as determined from simple tension test is

$$U_2 = \frac{1}{2E}\left[\frac{S_{yt}}{FOS}\right]^2$$

Equating the above two equations then we get

$$\sigma_1^{\,2} + \sigma_2^{\,2} - \frac{2\sigma_1 \sigma_2}{m} = \left(\frac{S_{yt}}{FOS} \right)^2$$

In a biaxial case σ_1, σ_2 are calculated based as σ_x, σ_y & τ_{xy}

It will be checked whether the Left Hand Side of Equation is less than Right Hand Side of Equation or not. This theory is used for ductile materials.

5.Max. Principal Strain theory (Saint Venant's Theory):

- It is assumed that the failure or yielding occurs at a point in a member where the maximum principal (normal) strain in a biaxial stress exceeds limiting value of strain (strain at yield port) as obtained from simple tension test.

- In a biaxial stress case

$$E_{max} = \frac{\sigma_1}{E} - \frac{\sigma_2}{mE} = \frac{S_{yt}}{FOS \times E}$$

- One can calculate σ_1 & σ_2 given σ_x, σ_y & τ_{xy} and check whether the material fails or not, this theory is not used in general as reliable results could not be detained in variety of materials.