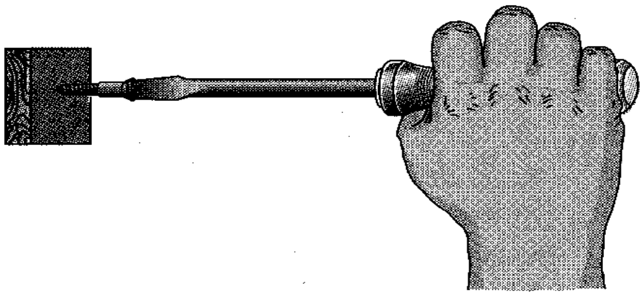
Module 6- Torsion

6	**Torsion:** Introduction to Torsion – derivation of shear strain – Torsion formula – stresses and deformations in circular and hollow shafts – Stepped shafts – shafts fixed at the both ends – Design of shafts according to theories of failure, Stresses in helical springs.	4

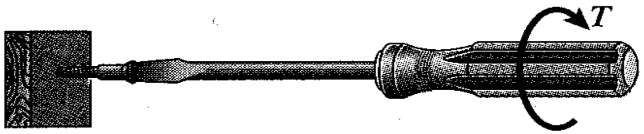
Torsion

Torsion is the twisting of a straight bar when it is loaded by twisting moments or **torques that tend to produce rotation** about the longitudinal axes of the bar.

For instance, when we turn a screw driver to produce torsion our hand applies torque ‘T’ to the handle and twists the shank of the screw driver.



(a)



(b)

TORSION OF CIRCULAR SHAFT

- <u>TORQUE OR TURNING MOMENT OR TWISTING MOMENT</u> :-

➢ In factories and workshops, shafts is used to transmit energy from one end to other end.

➢ To transmit the energy, a turning force is applied either to the rim of a pulley, keyed to the shafts, or to any other suitable point at some distance from the axis of the shaft.

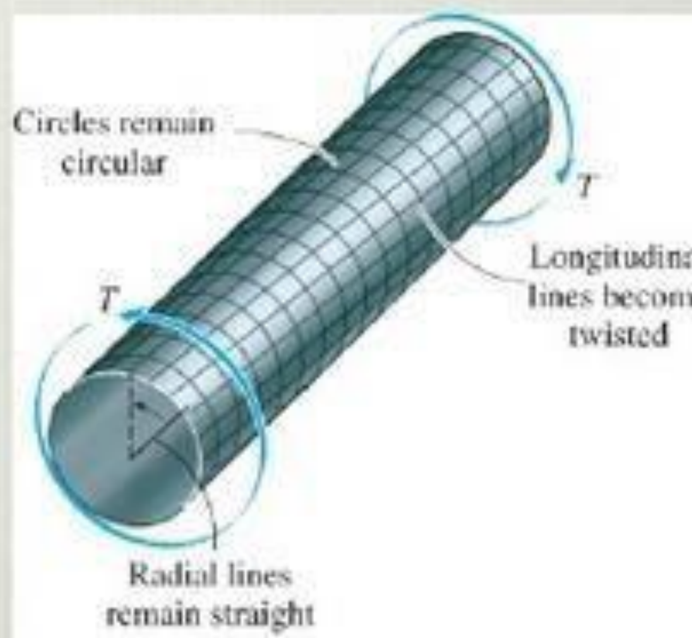
➢ The moment of couple acting on the shaft is called torque or turning moment or twisting moment.

Torsional deformation of a circular Shaft

- Torsion is a moment that twists/deforms a member about its longitudinal axis

- By observation, if angle of rotation is *small*, *length of shaft* and its *radius remain unchanged*

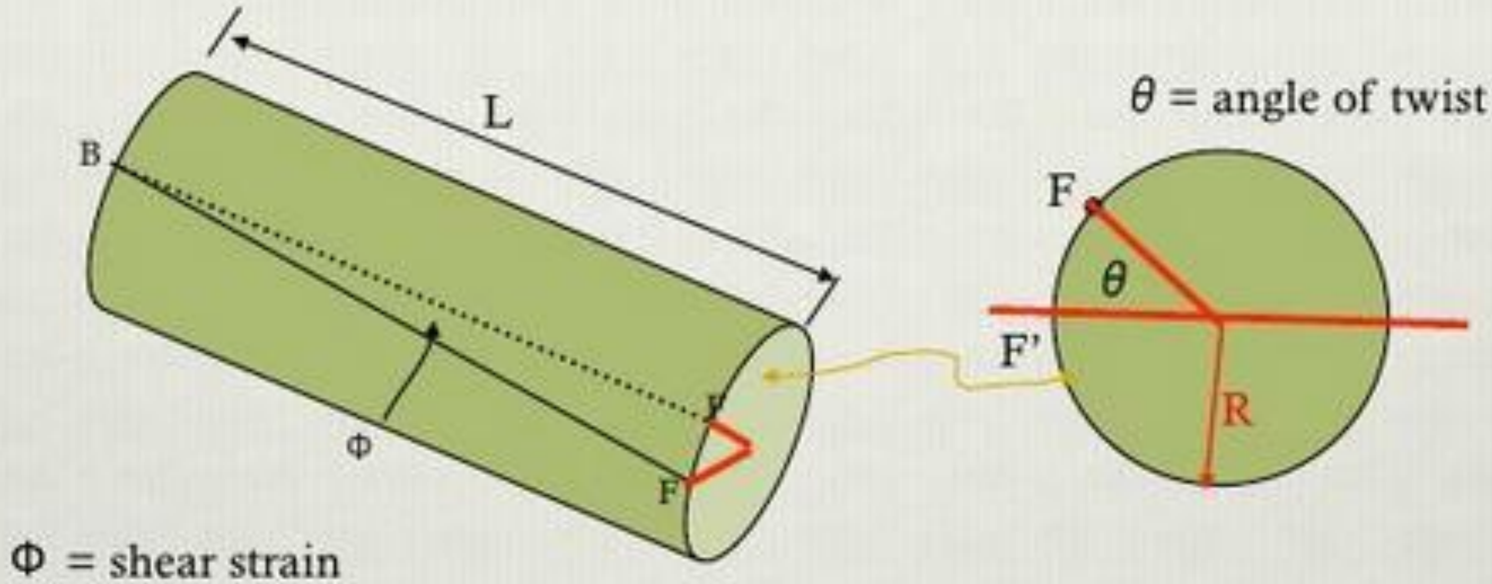
Before deformation
(a)



After deformation
(b)

SHEAR STRESS IN SHAFT:(τ)

- When a shaft is subjected to equals and opposite end couples, whose axes coincide with the axis of the shaft, the shaft is said to be in pure torsion and at any point in the section of the shaft stress will be induced.
- That stress is called shear stress in shaft.



ϕ is the shear strain, also remember that $\tan\phi = \phi$, thus :

$$\phi = \frac{F'F}{L} = \frac{R\theta}{L}$$

Note that shear strain does not only change with the amount of twist, but also, it varies along the radial direction such that it is zero at the center and increases linearly towards the outer periphery (see next slide)

STRENGTH OF SHAFTS

Maximum torque or power the shaft can transmit from one pulley to another, is called strength of shaft.

(a) For solid circular shafts:

Maximum torque (T)is given by :

$$T = \frac{\pi}{16} \times \tau \times D^3$$

where, D = dia. of the shaft

τ=shear stress in the shaft

(B) for hollow circular shaft

maximum torque (t) is given by.

$$T = \frac{\pi}{16} \times \tau \times \frac{D^4 - d^4}{D}$$

ASSUMPTION IN THE THEORY OF TORSION:

- The following assumptions are made while finding out shear stress in a circular shaft subjected to torsion.

1) The material of shaft is uniform throughout the length.
2) The twist along the shaft is uniform.
3) The shaft is of uniform circular section throughout the length.
4) Cross section of the shaft, which are plane before twist remain plain after twist.
5) All radii which are straight before twist remain straight after twist.

POLAR MOMENT OF INERTIA : (J)

- The moment of inertia of a plane area, with respect to an axis perpendicular to the plane of the figure is called polar moment of inertia.
- As per the perpendicular axis theorem.

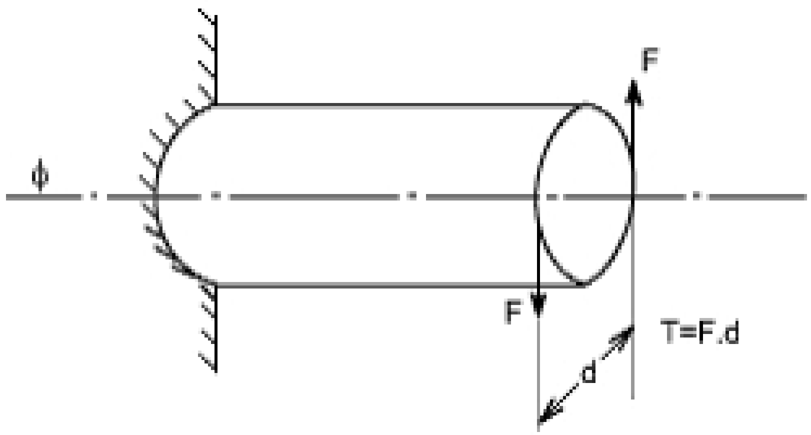
$$I_{ZZ} = I_{XX} + I_{YY} = J$$

$$= \frac{\pi}{64} \times D^4 + \frac{\pi}{64} \times D^4$$

$$J = \frac{\pi}{32} \times D^4$$

Torsion of circular shafts

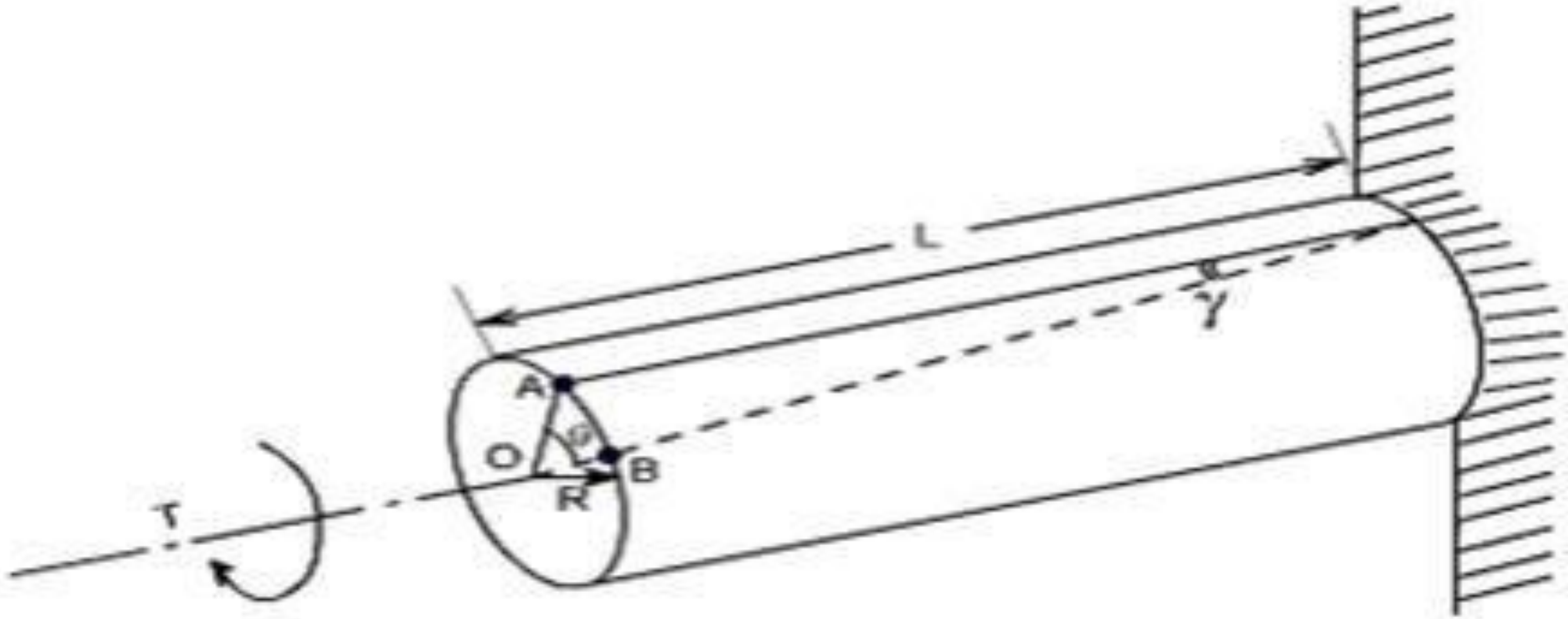
Definition of Torsion: Consider a shaft rigidly clamped at one end and twisted at the other end by a torque T = F.d applied in a plane perpendicular to the axis of the bar such a shaft is said to be in torsion.



Effects of Torsion: The effects of a torsional load applied to a bar are

(i) To impart an angular displacement of one end cross – section with respect to the other end.

(ii) To setup shear stresses on any cross section of the bar perpendicular to its axis.



L
A
O
B
R
T
γ

Consider now the solid circular shaft of radius R subjected to a torque T at one end, the other end being fixed Under the action of this torque a radial line at the free end of the shaft twists through an angle q , point A moves to B, and AB subtends an angle ' θ ' at the fixed end. This is then the angle of distortion of the shaft i.e the shear strain.

Since angle in radius = arc / Radius

arc AB = R θ

Thus, $\gamma = R\theta / L$ (1)

From the definition of Modulus of rigidity or Modulus of elasticity in shear

$$G = \frac{\text{shear stress}(\tau)}{\text{shear strain}(\gamma)}$$

where γ is the shear stress set up at radius R.

Then $\frac{\tau}{G} = \gamma$ (2)

Equating the equations (1) and (2) we get $\frac{R\theta}{L} = \frac{\tau}{G}$

$\frac{\tau}{R} = \frac{G\theta}{L}\left(= \frac{\tau'}{r}\right)$ where τ'is the shear stress at any radius r.

Consider an elementary ring of thickness dx at a radius x and let the shear stress at this radius be τ_x.

The turning force on the elementary ring

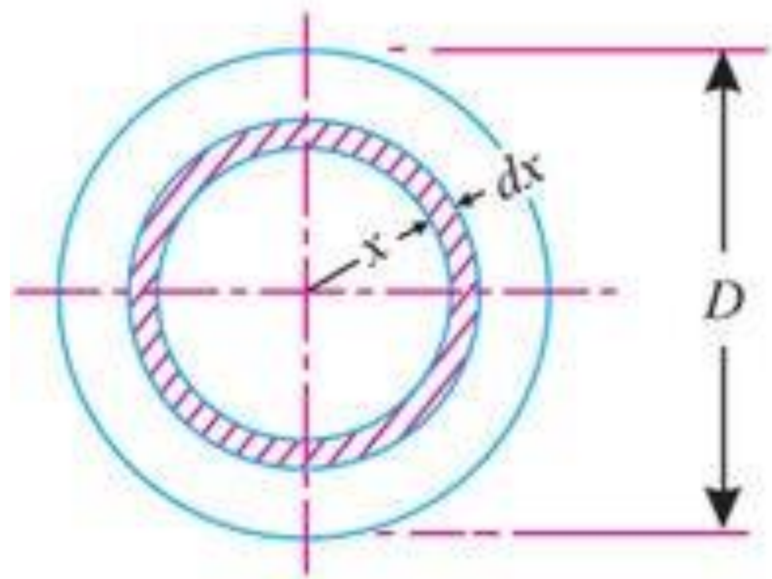
$$= \tau_x \,.\, 2\pi x \,.\, dx$$

Turning moment due to this turning force,

$$dT = \tau_x \,.\, 2\pi x \,.\, dx \times x$$

To get total turning moment integrating both sides, we get

$$\int dT = \int_0^R \tau_x \,.\, 2\pi x \; dx \times x$$



$$\int dT = 2\pi \int_0^R \tau_x \, . \, x^2 \, . \, dx = 2\pi \int_0^R \frac{\tau \, . \, x}{R} \, . \, x^2 \, . \, dx \qquad \left[\because \frac{\tau}{R} = \frac{\tau_x}{x} \quad \text{or } \tau_x = \frac{\tau . x}{R}\right]$$

$$= 2\pi \frac{\tau}{R} \int_0^R x^3 \, dx$$

$$T = 2\pi \frac{\tau}{R} \left|\frac{x^4}{4}\right|_0^R = 2\pi \frac{\tau}{R} \, . \, \frac{R^4}{4}$$

$$T = \tau \, . \, \frac{\pi R^3}{2} = \tau \cdot \frac{\pi}{16} D^3 \qquad \text{...(\textit{Strength of solid shaft})}$$

$$T = \frac{\tau}{R} \cdot \frac{\pi R^4}{2} = \frac{\tau}{R} I_p \qquad \left[\because I_p = \frac{\pi}{32} D^4 = \frac{\pi}{2} R^4\right]$$

$$\frac{T}{I_p} = \frac{\tau}{R} \qquad \text{...(13·2)}$$

From eqns. (13·1) and (13·2), we have

$$\frac{T}{I_p} = \frac{\tau}{R} = \frac{C\theta}{l} \qquad \text{...(13·3)}$$

$$\frac{T}{I_p} = \frac{\tau}{R} = \frac{C\theta}{l}$$

This is called **torsion equation.**

Note. From the relation, $\frac{T}{I_p} = \frac{\tau}{R}$,

We have $$T = \tau \times \frac{I_p}{R}$$

For a given shaft I_p and R are constants and $\frac{I_p}{R}$ is thus a constant and is known as **polar modulus** of the shaft section.

Thus $$T = \tau \times Z_p$$

For a shaft of given material τ, the maximum permissible shear stress is fixed and thus the greatest twisting moment that the shaft can withstand is proportional to the polar modulus of the shaft. *Polar modulus of the section is thus measure of strength of shaft in torsion.*

General Torsion Equation (Shafts of circular cross-section)

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\,\theta}{L}$$

1. For Solid Shaft

$$J = \frac{\pi}{2} r^4 = \frac{\pi d^4}{32}$$

2. For Hollow Shaft

$$J = \frac{\pi}{2} (r_1^4 - r_2^4)$$

$$= \frac{\pi}{32} (d_1^4 - d_2^4)$$

T = torque or twisting moment in newton metres
J = polar second moment of area of cross-section about shaft axis.
τ = shear stress at outer fibres in pascals
r = radius of shaft in metres
G = modulus of rigidity in pascals
θ = angle of twist in radians
L = length of shaft in metres
d = diameter of shaft in metres

TORSIONAL RIGIDITY/STIFFNESS OF SHAFTS

From the torsion equation, $$\frac{T}{J} = \frac{G\theta}{L}$$

we get, $$T = \frac{GJ\theta}{L}$$

Where (J=Ip)

- Hence the term *GJ* may be looked as torque required to introduce unit angle of twist in unit length, and is called torsional rigidity or stiffness of shaft.
- This term is analogous to the term flexural rigidity *(EI)* in theory of bending.

POWER TRANSMITTED BY THE SHAFT

Consider a force F newtons acting tangentially on the shaft of radius R. If the shaft due to this turning moment ($F \times R$) starts rotating at N r.p.m. then,

Work supplied to the shaft/sec.

$$= F \times \text{distance moved/sec.}$$

$$= F \times 2\pi RN/60 \text{ Nm/s}$$

or,
$$P = \frac{F \times 2\pi RN}{60} \text{ watts}$$

$$= \frac{T \times 2\pi N}{60 \times 1000} \text{ kW} \qquad (\because T = F \times R)$$

Hence,
$$P = \frac{2\pi NT}{60 \times 1000}$$

Where T is the mean/average torque in Nm.

PROBLEMS

What must be the length of a 5 mm diameter aluminum wire so that it can be twisted through one complete revolution without exceeding a shear stress of 42 MPa ? Take C= 27 GPa .

General Torsion Equation (Shafts of circular cross-section)

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$$

T = torque or twisting moment in newton metres
J = polar second moment of area of cross-section about shaft axis.
τ = shear stress at outer fibres in pascals
r = radius of shaft in metres
G = modulus of rigidity in pascals
θ = angle of twist in radians
L = length of shaft in metres
d = diameter of shaft in metres

(a) For solid circular shafts:
Maximum torque (T)is given by :

$$T = \frac{\pi}{16} \times \tau \times D^3$$

where, D = dia. of the shaft
τ=shear stress in the shaft

For Solid Shaft

$$J = \frac{\pi}{2} r^4 = \frac{\pi d^4}{32}$$

Solution. *Given*: $D = 5$ mm $= 0{\cdot}005$ m;

$\theta = 2\pi$ rad;

$\tau = 42$ MN/m^2;

$C = 27$ GN/m^2.

Let, l = Length of the aluminium wire.

Torque transmitted by the wire,

$$T = \tau \cdot \frac{\pi}{16} D^3 = 42 \times 10^6 \times \frac{\pi}{16} \times (0{\cdot}005)^3 \text{ Nm} = 1{\cdot}031 \text{ Nm}$$

Polar moment of inertia of a circular section,

$$I_p = \frac{\pi}{32} \times D^4 = \frac{\pi}{32}(0{\cdot}005)^4 = 6{\cdot}136 \times 10^{-11} \text{ m}^4$$

We know that,

$$\frac{T}{I_p} = \frac{C\theta}{l}$$

$$\frac{1{\cdot}031}{6{\cdot}136 \times 10^{-11}} = \frac{27 \times 10^9 \times 2\pi}{l}$$

or,

$$l = \frac{27 \times 10^9 \times 2\pi \times 6{\cdot}136 \times 10^{-11}}{1{\cdot}031} = \mathbf{10{\cdot}096 \text{ m}} \quad \textbf{(Ans.)}$$

A shaft 40 mm diameter is made from steel and the maximum allowable shear stress for the material is 50 MPa. Calculate the maximum torque that can be safely transmitted. Take G = 90 GPa.

General Torsion Equation (Shafts of circular cross-section)

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\,\theta}{L}$$

For Solid Shaft

$$J = \frac{\pi}{2} r^4 = \frac{\pi d^4}{32}$$

T	=	torque or twisting moment in newton metres
J	=	polar second moment of area of cross-section about shaft axis.
τ	=	shear stress at outer fibres in pascals
r	=	radius of shaft in metres
G	=	modulus of rigidity in pascals
θ	=	angle of twist in radians
L	=	length of shaft in metres
d	=	diameter of shaft in metres

SOLUTION

Important values to use are:

D = 0.04 m, R = 0.02 m, $\tau = 50 \times 10^6$ Pa and $G = 90 \times 10^9$ Pa

$$\frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{r}$$

$$J = \frac{\pi D^4}{32} = \frac{\pi \times 0.04^4}{32} = 251.32 \times 10^{-9}\ m^4$$

The complete torsion equation is $\frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{R}$ Rearrange and ignore the middle term.

$$T = \frac{\tau_{max} J}{R} = \frac{50 \times 10^6 \times 251.32 \times 10^{-9}}{0.02} = 628.3\ Nm$$

A hollow circular shaft 20mm thick transmits 294 kW at 200r.p.m.Determine the diameters of the shaft if shear strain due to torsion is not to exceed 8.6×10^{-4}

Take modulus of rigidity as 80GN/m^2

Solution

Let, d_1 = external diameter of the hollow shaft, in mm

d_2 = internal diameter of the hollow shaft, in mm

t = thickness of the shaft = 20 mm

Then,

$d_1 - d_2 = 2\mathrm{t} = 40\mathrm{mm}$

$[\mathrm{or}\ \ d_2 = (d_1 - 40)\mathrm{mm}]$

GIVEN:

Shear strain due to torsion,

$e_s = 8.6 \times 10^{-4}$

Modulus of rigidity, G $= 80\ \times 10^3 N/mm^2$

Power transmitted,

$P = 294kW$

= 294 000 Nm/sec

$= 294\ \times\ 10^6\ {}^{Nmm}/_{sec}$

Speed, $N = 200\ r.p.m$

Diameters of the shaft, d_1 and d_2:

We know that, $P = \frac{2\pi NT}{60}$

$$\frac{T}{J} = \frac{q_s}{R} \quad or \quad q_s = \frac{TR}{J}$$

From the definition of Modulus of rigidity or Modulus of elasticity in shear

$$G = \frac{\text{shear stress}(\tau)}{\text{shear strain}(\gamma)}$$

where γ is the shear stress set up at radius R.

Then $\frac{\tau}{G} = \gamma \qquad (2)$

Determine the diameter of solid shatt which will transmit 400 kW at 280 rpm. The angle of twist must not exceed one degree per metre length and maximum torsional shear stress is to be limited to 40 N/mm^2 . Assume G =84 kN/mm^2

$$P = \frac{2\pi NT}{60}$$

From the consideration of maximum shear stress, $\frac{T}{J} = \frac{q_s}{R}$

For Solid Shaft

$$J = \frac{\pi}{2} r^4 = \frac{\pi d^4}{32}$$

From the consideration of maximum angle of twist,

$$\frac{T}{J} = \frac{G\,\theta}{l}$$

COMPOUND SHAFTS

- Compound shafts are shafts made up of a number of small shafts of different cross-sections or of different materials connected to form a composite shaft to transmit or resist the torque applied.
- **<u>Types</u>**

  Shafts in series

  Shafts in parallel

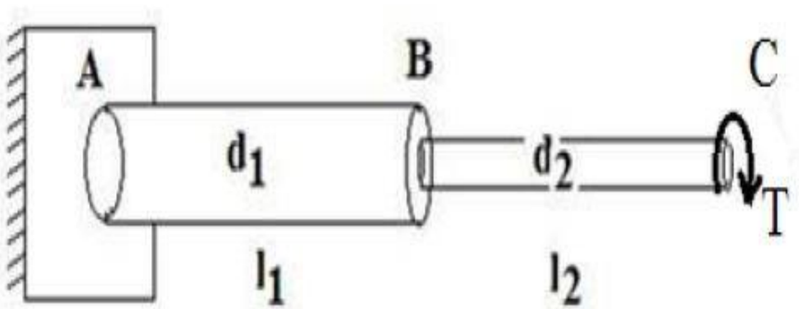
Shafts in series

In order to form a composite shaft sometimes number of small shafts of different cross-sections or of different materials shafts are connected in series.

To analyze these shafts, first torque resisted by each portion is found and then individual effects are clubbed.

Shafts in series

- At fixed end torque of required magnitude develops to keep the shaft in equilibrium.
- The torques developed at the ends of any portion are equal and opposite.
- The angle of twist is the sum of the angle of twist of the shafts connected in series.



$$T_1 = T_2$$

$$Total\ angle\ of\ twist,\ \theta = \theta_1 + \theta_2$$

$$\therefore \theta = \frac{Tl_1}{G_1J_1} + \frac{Tl_2}{G_2J_2} = T\left(\frac{l_1}{G_1J_1} + \frac{l_2}{G_2J_2}\right)$$

Shafts in Parallel

The shafts are said to be in parallel when the driving torque is applied at the junction of the shafts and the resisting torque is at the other ends of the shafts .

Here, the angle of twist is same for each shaft, but the applied torque is shared between the two shafts.

$$\text{i.e.}, \theta_1 = \theta_2$$

$$\frac{T_1 l_1}{G_1 J_1} = \frac{T_2 l_2}{G_2 J_2}$$

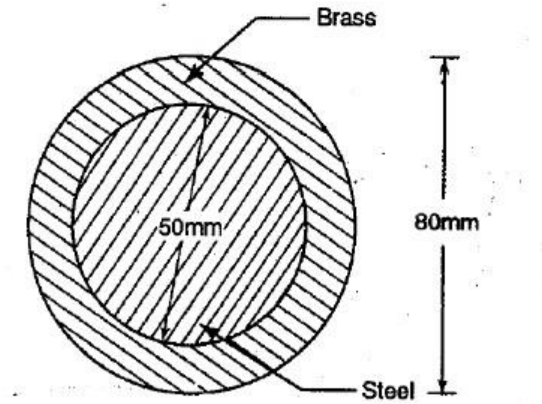
and

$$T = T_1 + T_2$$

A brass tube of external diameter 80 mm and internal diameter 50 mm is closely fitted to a steel rod of 50 mm diameter to form a composite shaft. If a torque of 10 kNm is to be resisted by this shaft, find the maximum stresses developed in each material and the angle of twist in 2 m length.

Take $G_b = 40 \times 10^3\ N/mm^2$

$G_s = 80 \times 10^3 N/mm^2$



$$J_s = \frac{\pi}{32} \times 50^4 = 613592.32\ mm^4$$

$$J_b = \frac{\pi}{32} \times (80^4 - 50^4) = 3407646.3\ mm^4$$

- Let T_s, be the torque resisted by steel and

  T_b the torque resisted by brass tube.

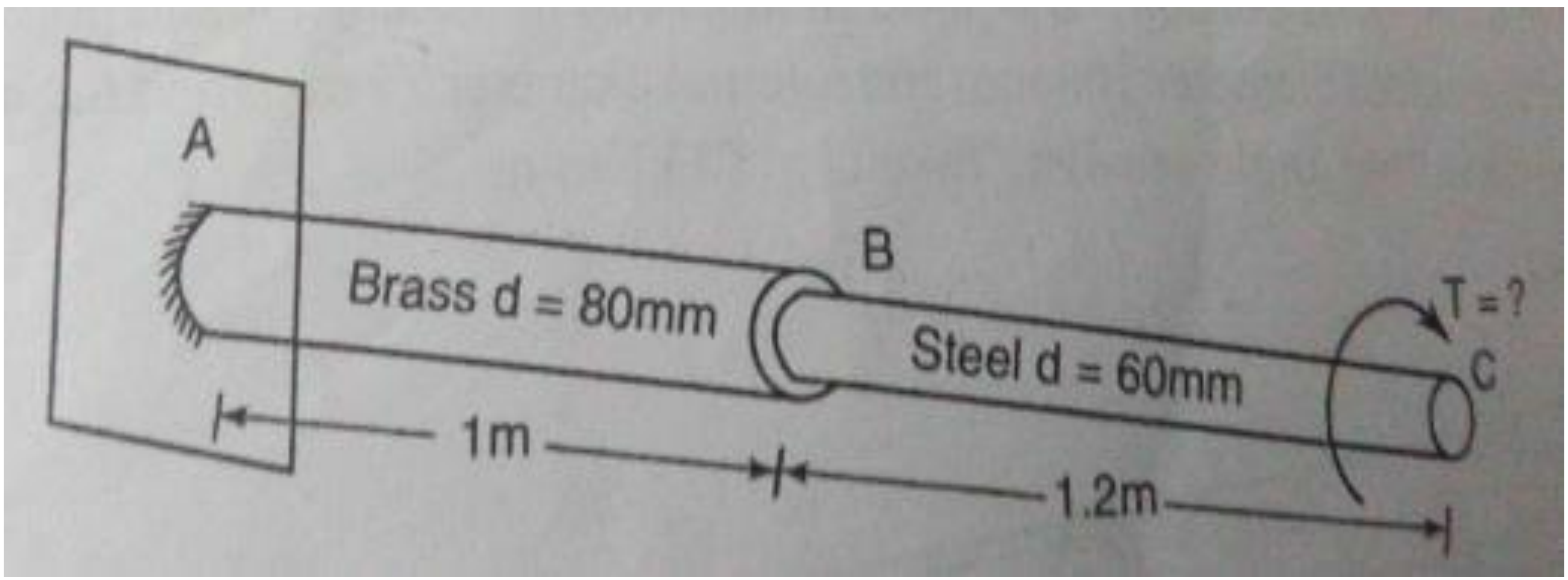
- Then, $T_s + T_b = T$ = 10 kNm

  = 10×10^6Nmm ----(1)

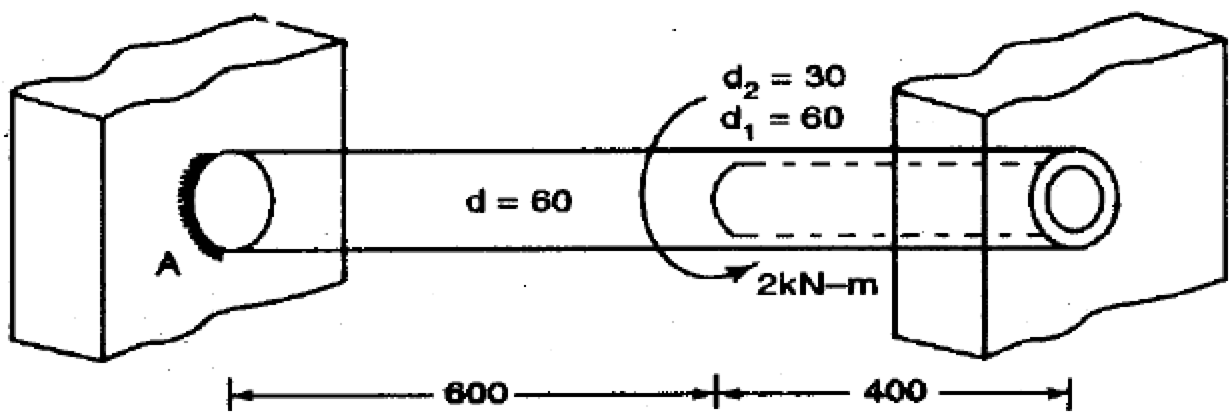
Since the angle of twist will be same in the two materials.

$$\theta_s = \theta_b$$

The allowable shear stress in brass is 80 MPa and in steel 100 Mpa. Find the maximum torque that can be applied in the stepped shaft as shown in figure. Find also total rotation of free end w.r.t. the fixed end if Gb=40kN/mm2 and Gs=80 kN/mm2.

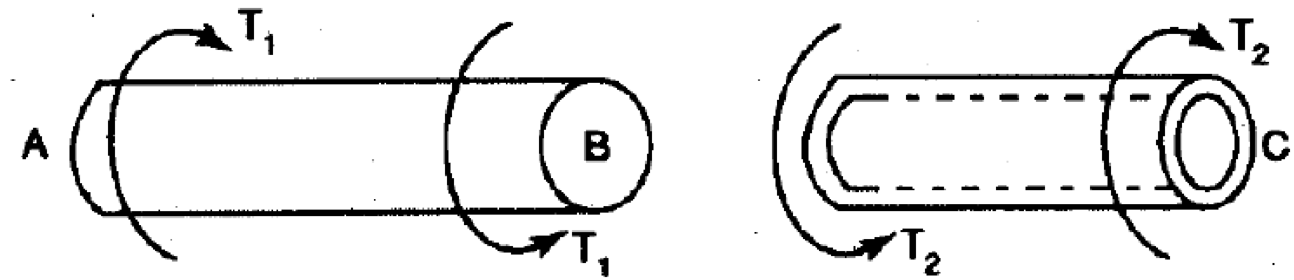


A bar of length 1000 mm and diameter 60 mm is centrally bored for 400 mm, the bore diameter being 30 mm as shown in Fig. (a). If the two ends are fixed and is subjected to a torque of 2 kNm as shown in figure, find the maximum stresses developed in the two portions.



(a)

Let T_1 be the torque resisted by portion AB and *T2 that* resisted by *BC.*



Rotation of shaft at *B* in portion *AB,*

$$\theta_1 = \frac{T_1 L_1}{G J_1} = \frac{T_1 \times 600}{G \times \frac{\pi}{32} \times 60^4} = \frac{32 \times 600}{\pi \times 60^4} \frac{T_1}{G}$$

Rotation of shaft at *B* in portion *BC*

$$\theta_2 = \frac{T_2 L_2}{G J_2} = \frac{T_2 \times 400}{G \times \frac{\pi}{32} \times (60^4 - 30^4)}$$

$$= \frac{32 \times 400}{\pi \times 60^4 (1 - 0.5^4)} \frac{T_2}{G}$$

Thank You