School of Mechanical and Building Sciences

MEE 2002 Strength of Materials

Unit 2

Biaxial State of Stress

By

Dr. S.JEYANTHI

Associate Professor

Outline of the presentation

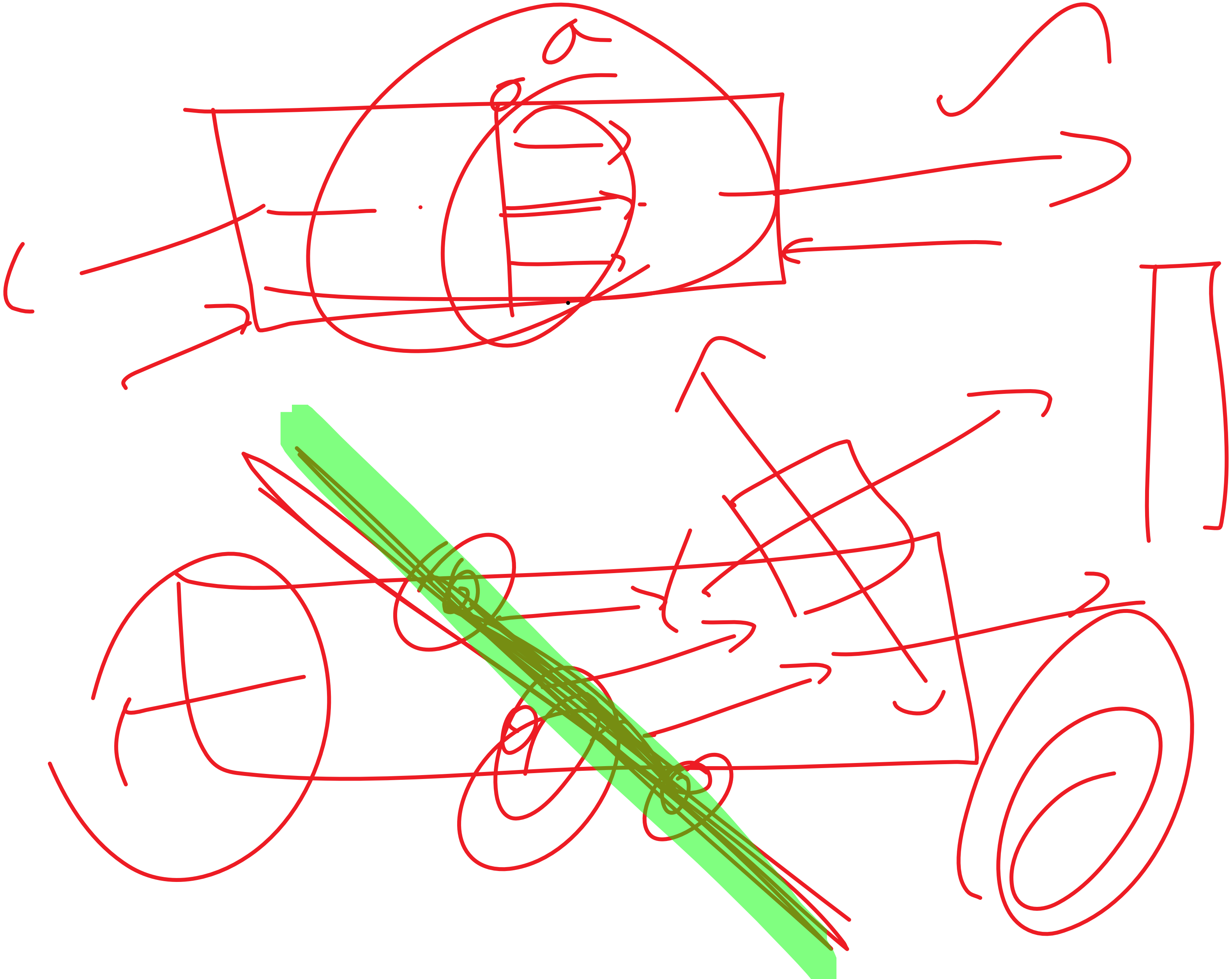
- Introduction
- Stresses on an inclined section of a bar under axial loading
- Compound stresses
- Normal and tangential stresses on an inclined plane for biaxial stresses
- Two perpendicular normal stresses accompanied by a state of simple shear
- Mohr's circle of stresses –
- Principal stresses and strains – Analytical and graphical solutions,
- Theories of Failure.

Introduction

- So far we have considered normal stresses arising in bars subject to axial loading as well as several cases involving shearing stresses etc.

- It is to be noted that we have considered a bar, for example, to be subject to only one loading at a time.

- But frequently such bars are simultaneously subject to several kinds of loadings, and it is required to determine the state of stress under these conditions.



INTRODUCTION

the concept and definition of stress, strain, types of stresses (*i.e.*, tensile, compressive and simple shear) and types of strain (*i.e.*, tensile, compressive, shear and volumetric strains etc.) are discussed. These stresses were acting in a plane, which was at right angles to the line of action of the force. In many engineering problems both direct (tensile or compressive stress) and shear stresses are acting at the same time. In such situation the resultant stress across any section will be neither normal nor tangential to the plane. In this chapter the stresses, acting on an inclined plane (or oblique section) will be analysed.

PRINCIPAL PLANES AND PRINCIPAL STRESSES

The planes, which have no shear stress, are known as principal planes. Hence principal planes are the planes of zero shear stress. These planes carry only normal stresses.

The normal stresses, acting on a principal plane, are known as principal stresses.

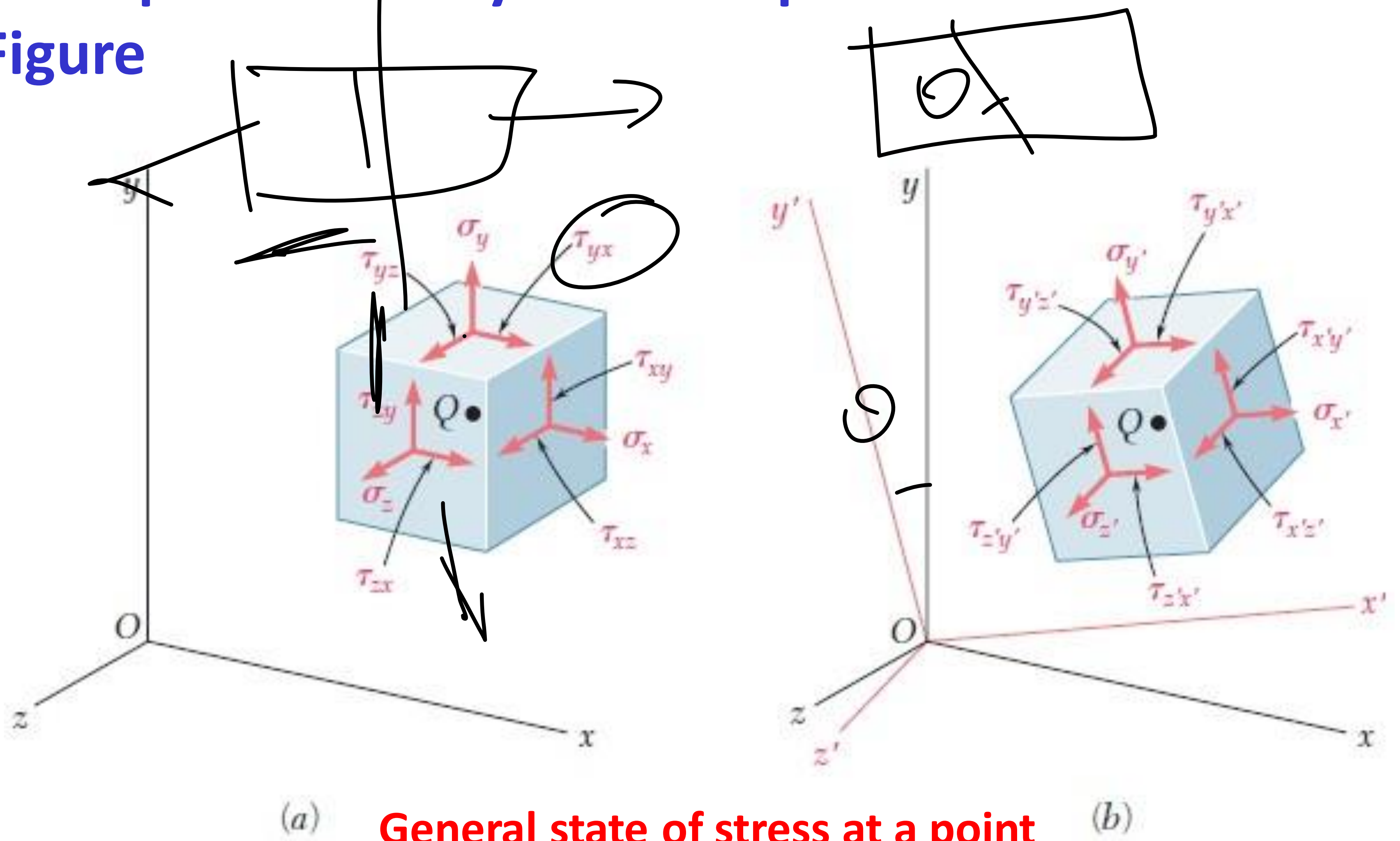
Bi-axial Stress system

- Since normal and shearing stress are tensor quantities, considerable care must be exercised in combining the stresses given by the expressions for single loadings as derived earlier.

- It is the purpose of this module is to investigate the state of stress on an arbitrary plane through an element in a body subject to both normal and shearing stresses.

Bi-axial Stress system

•Most general state of stress at a given point Q may be represented by six components as shown in Figure



General state of stress at a point

- Engineers most often wants to determine maximum normal stress induced at a given point for their design purpose. But there can be infinite number of planes passing through a point, and normal stress on each plane will be different from one another.
- There will be one plane on which normal stress value is maximum, this plane is known as Principal plane (more precisely maximum principal plane) and normal stress on this plane is known as principal stress (more precisely maximum principal stress).
- Similarly there will be one more plane on which normal stress value is minimum, this is also a principal plane (minimum principal plane) and normal stress on this plane is known as Principal stress (minimum principal stress).
- 2 Dimensional Stress Analysis - Stress acting on a 2D element is shown in figure below

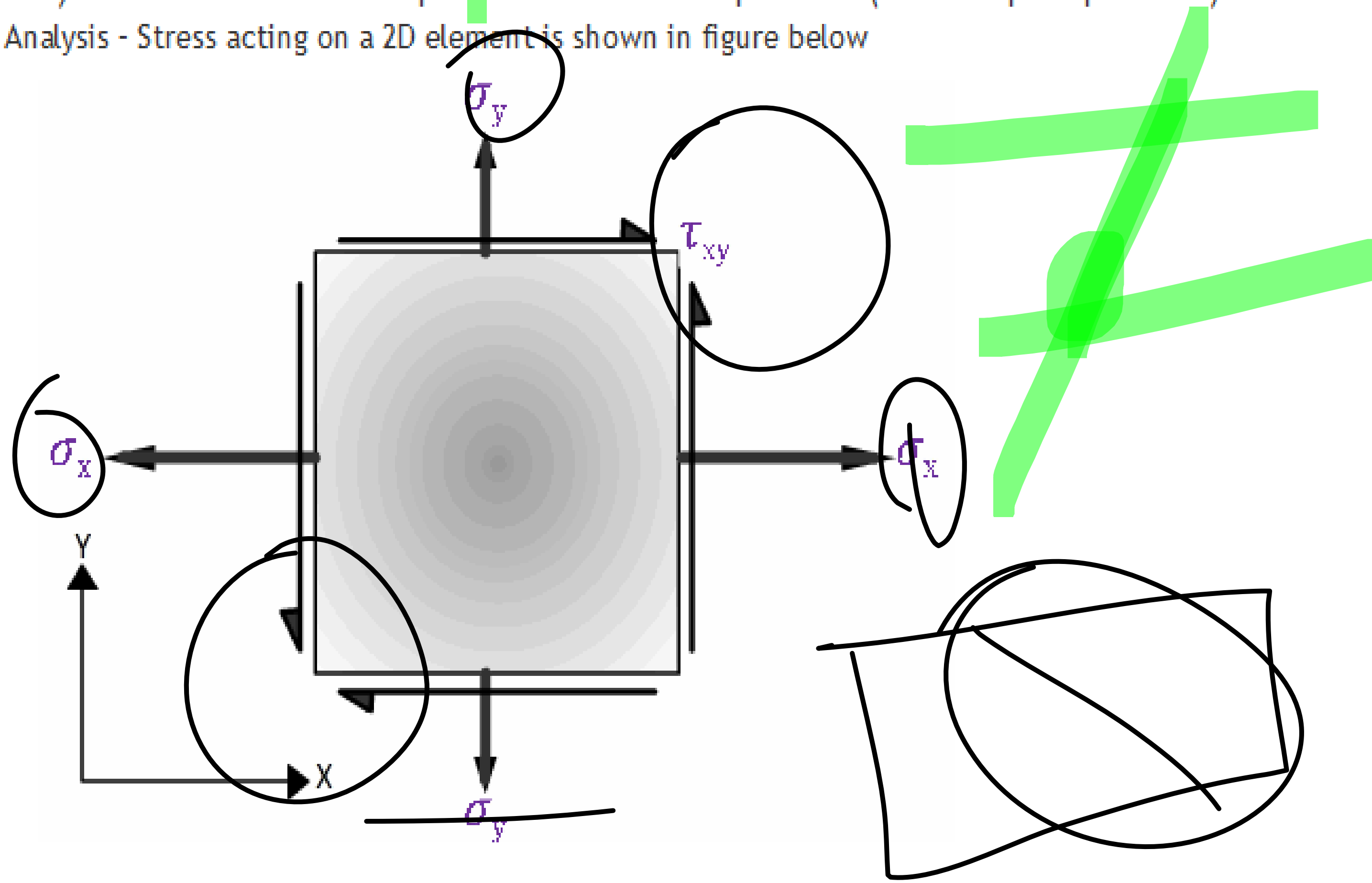
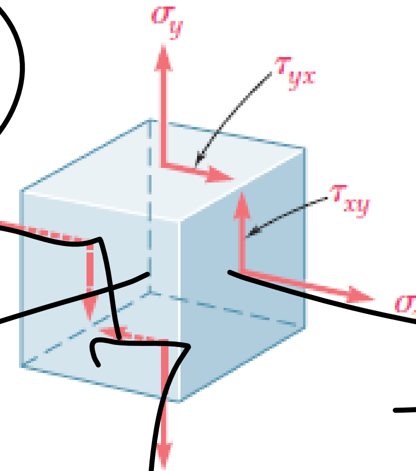
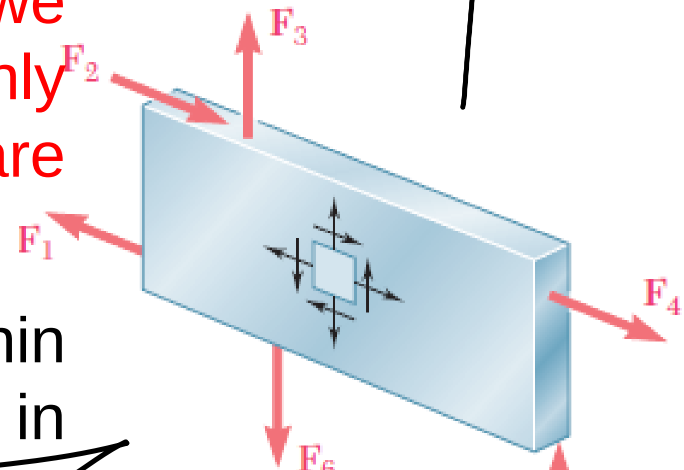


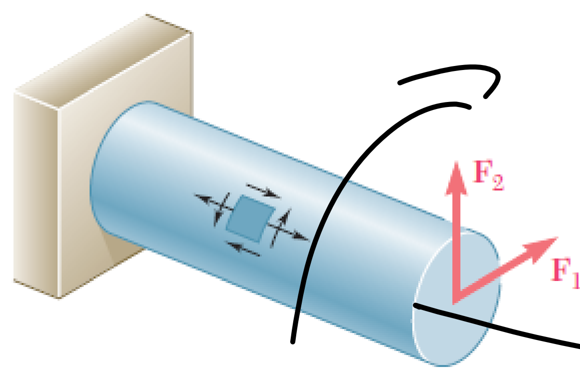
Fig.1 Stress boundary conditions on a 2 dimensional element

Plane Stress

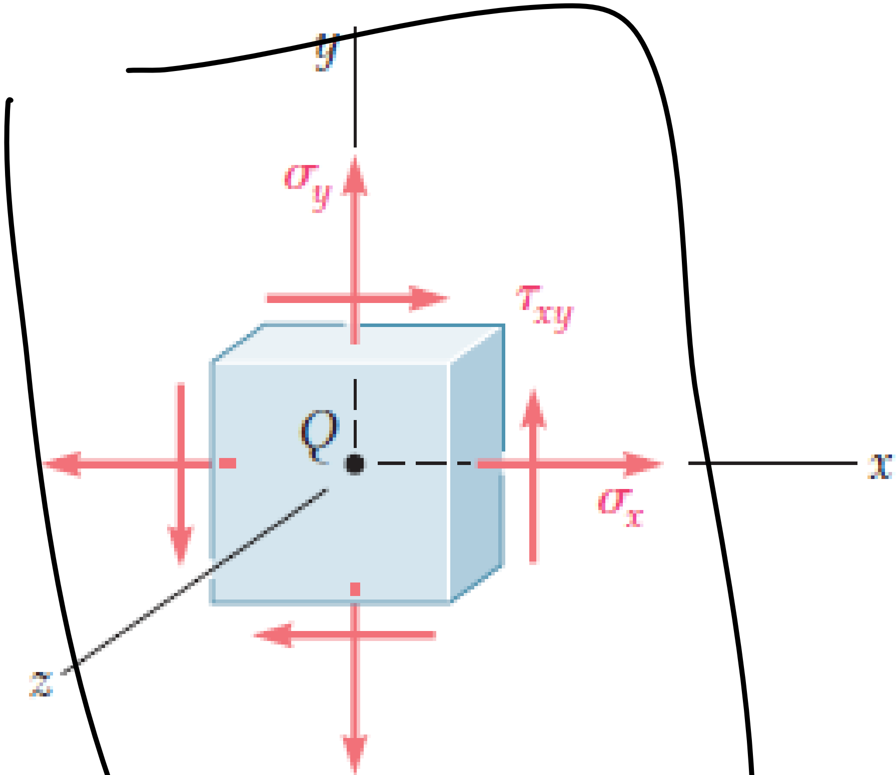
- Plane stress means, two of the faces of the cubic element are free of any stress
- If the z axis is chosen perpendicular to these faces, we have $\sigma_z = \tau_{zx} = \tau_{zy} = 0$, and the only remaining stress components are σ_x, σ_y, and τ_{xy}
- Such a situation occurs in a thin plate subjected to forces acting in the mid plane of the plate
- It also occurs on the free surface of a structural element or machine component



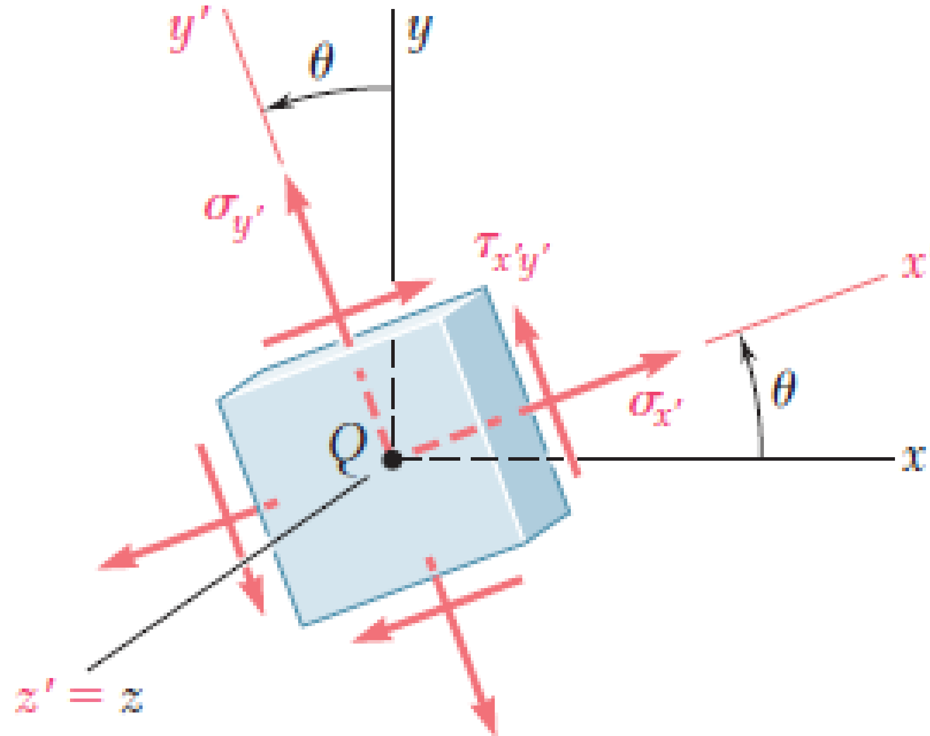




Transformation of Plane Stress

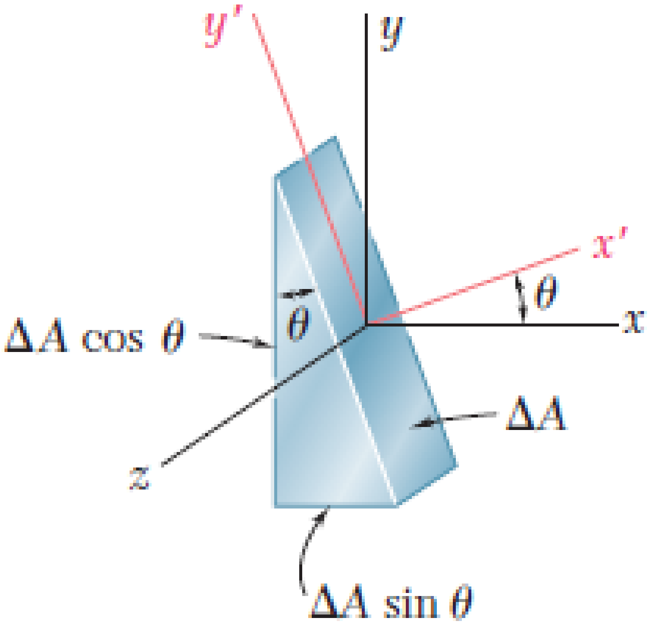


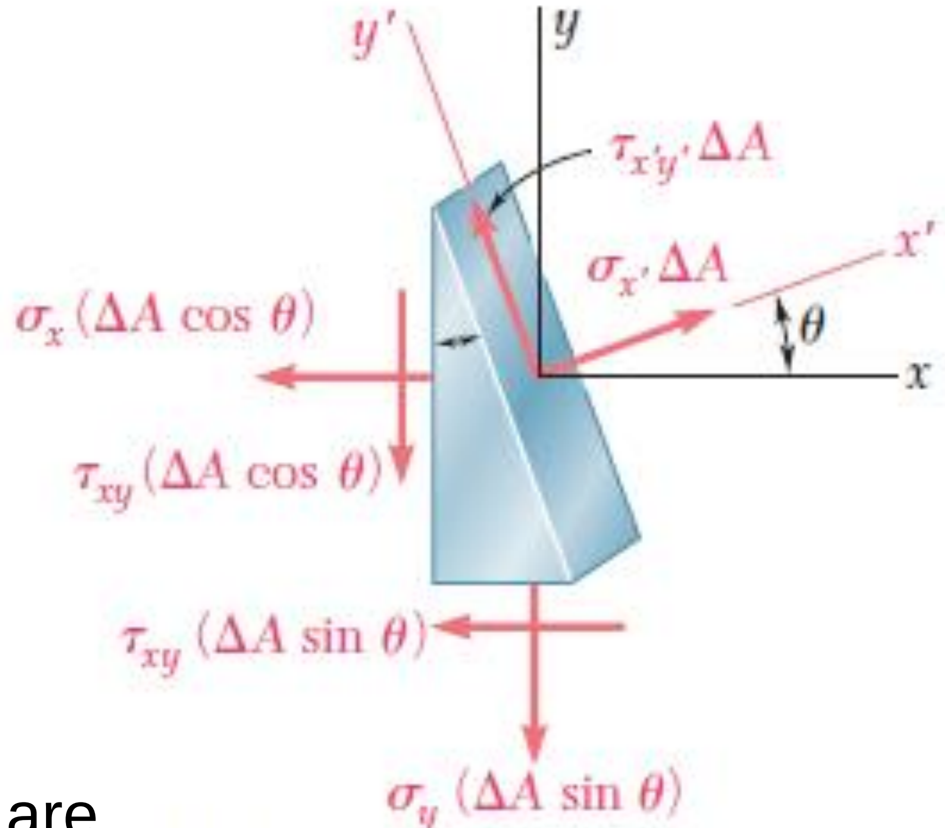
$$\boxed{\sigma_z = \tau_{zx} = \tau_{zy} = 0}$$



To determine: Stress components of the element after it has been rotated through an angle θ about z-axis

$$\boxed{\sigma_{x'}, \sigma_{y'}, \tau_{x'y'}}$$





The equilibrium equations are

$$\sum F_{x'} = 0: \quad \sigma_{x'}\,\Delta A - \sigma_x(\Delta A\cos\theta)\cos\theta - \tau_{xy}(\Delta A\cos\theta)\sin\theta$$
$$-\sigma_y(\Delta A\sin\theta)\sin\theta - \tau_{xy}(\Delta A\sin\theta)\cos\theta = 0$$

$$\sum F_{y'} = 0: \quad \tau_{x'y'}\,\Delta A + \sigma_x(\Delta A\cos\theta)\sin\theta - \tau_{xy}(\Delta A\cos\theta)\cos\theta$$
$$-\sigma_y(\Delta A\sin\theta)\cos\theta + \tau_{xy}(\Delta A\sin\theta)\sin\theta = 0$$

Solving the first equation for $\sigma_{x'}$ and the second for $\tau_{x'y'}$, we have

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$

$$\tau_{x'y'} = -(\sigma_x - \sigma_y) \sin\theta \cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\boxed{\sin 2\theta = 2 \sin\theta \cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta}$$

$$\boxed{\cos^2\theta = \frac{1 + \cos 2\theta}{2} \qquad \sin^2\theta = \frac{1 - \cos 2\theta}{2}}$$

$$\sigma_{x'} = \sigma_x \frac{1 + \cos 2\theta}{2} + \sigma_y \frac{1 - \cos 2\theta}{2} + \tau_{xy} \sin 2\theta$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

$\sigma_{y'}$ is obtained by replacing θ in $\sigma_{x'}$ by the angle θ+90° that the y' axis forms with the *x* axis. Since
cos (2θ + 180°) = -cos 2θ and
sin (2θ + 180°) = -sin 2θ

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y$$

We thus verify in the case of plane stress that the sum of the normal stresses exerted on a cubic element of material is independent of the orientation of that element

Principal Stresses and Maximum shear stress

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 = \left(\frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\right)^2$$

$$\left(\tau_{x'y'}\right)^2 = \left(-\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\,cos\,2\theta\right)^2$$

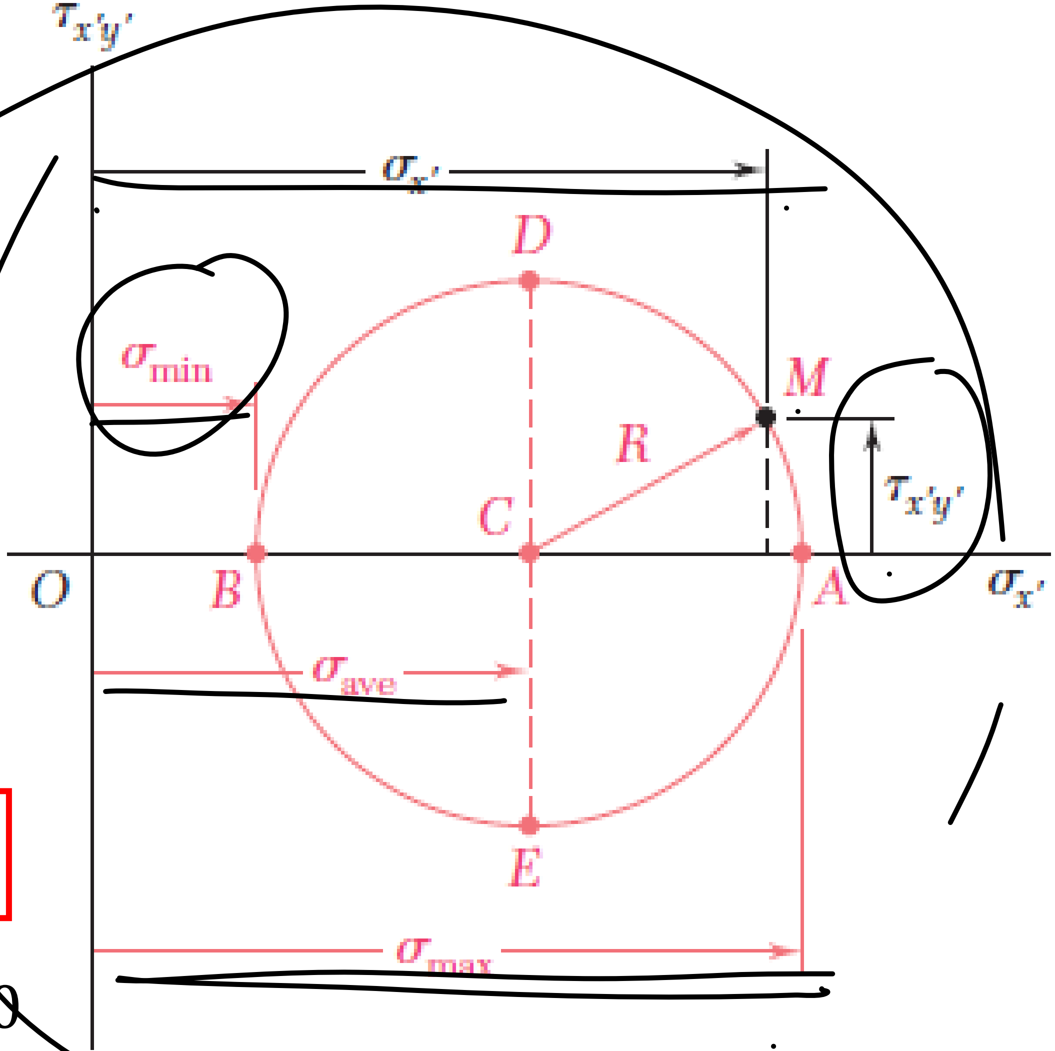
$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \left(\tau_{x'y'}\right)^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2$$

Setting

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\boxed{(\sigma_{x'} - \sigma_{ave})^2 + \tau_{x'y'}^2 = R^2}$$



$$\theta = \theta_p \quad setting\ \tau_{x'y'} = 0$$

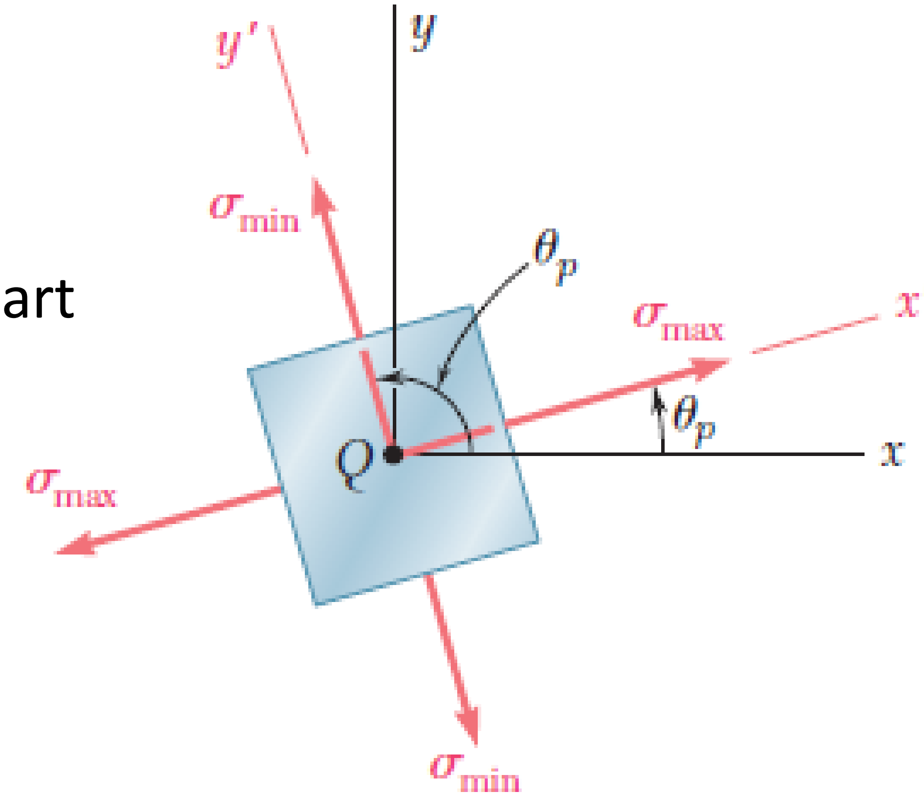
$$0 = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta_p + \tau_{xy}\cos 2\theta_p$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Two values of $2\theta_p$ that are 180° apart

Therefore, two values of θ_p are 90° apart



- The planes containing the faces of the element obtained in this way are called the principal planes of stress at point Q
- The corresponding values σ_{max} and σ_{min} of the normal stress exerted on these planes are called the principal stresses at Q
- No shearing stress is exerted on the principal planes

$$\sigma_{\max} = \sigma_{\text{ave}} + R \qquad \text{and} \qquad \sigma_{\min} = \sigma_{\text{ave}} - R$$

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Maximum Shear Stress

- The points *D* and *E* located on the vertical diameter of the circle correspond to the largest numerical value of the shearing stress

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

At points *D* and *E*

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} \qquad \theta = \theta_s$$

Therefore,

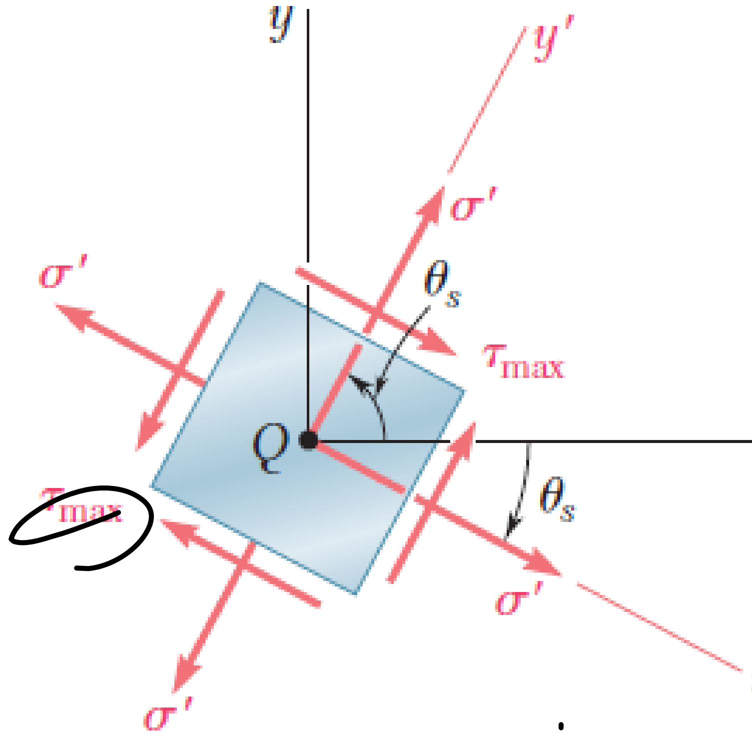
$$\frac{\sigma_x - \sigma_y}{2}\cos 2\theta_s + \tau_{xy}\sin 2\theta_s = 0$$

$$\boxed{\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}}$$

Two values of $2\theta_s$ that are 180° apart

Therefore, two values of θ_s are 90° apart

The maximum value of the shearing stress is equal to the radius *R* of the circle,



$$\boxed{\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

The normal stress corresponding to the condition of maximum shearing stress is,

$$\sigma' = \sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2}$$

Comparing the equations

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{\tan 2\theta_p(\sigma_x - \sigma_y)}$$

$$\boxed{\tan 2\theta_s = -\frac{1}{\tan 2\theta_p}}$$

This means that the angles $2\theta_s$ and $2\theta_p$ are 90° apart and, therefore, that the angles θ_s and θ_p are 45° apart

We thus conclude that the planes of maximum shearing stress are at 45° to the principal planes

Problem 3.1. *A rectangular bar of cross-sectional area 10000 mm² is subjected to an axial load of 20 kN. Determine the normal and shear stresses on a section which is inclined at an angle of 30° with normal cross-section of the bar.*

Sol. Given :

Cross-sectional area of the rectangular bar,

$$A = 10000 \text{ mm}^2$$

Axial load, $P = 20 \text{ kN} = 20{,}000 \text{ N}$

Angle of oblique plane with the normal cross-section of the bar,

$$\theta = 30°$$

Now direct stress, $\sigma = \dfrac{P}{A} = \dfrac{20000}{10000} = 2 \text{ N/mm}^2$

Let σ_n = Normal stress on the oblique plane

σ_t = Shear stress on the oblique plane.

Using equation (3.2) for normal stress, we get

$$\sigma_n = \sigma \cos^2 \theta$$

$$= 2 \times \cos^2 30° \qquad (\because \sigma = 2 \text{ N/mm}^2)$$

$$= 2 \times 0.866^2 \qquad (\because \cos 30° = 0.866)$$

$$= \mathbf{1.5 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

Using equation (3.3) for shear stress, we get

$$\sigma_t = \frac{\sigma}{2} \sin 2\theta = \frac{2}{2} \times \sin(2 \times 30°)$$

$$= 1 \times \sin 60° = \mathbf{0.866 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

Problem 3.2. *Find the diameter of a circular bar which is subjected to an axial pull of 160 kN, if the maximum allowable shear stress on any section is 65 N/mm².*

Sol. Given :

Axial pull, $P = 160 \text{ kN} = 160000 \text{ N}$

Maximum shear stress $= 65 \text{ N/mm}^2$

Let D = Diameter of the bar

$\therefore$ Area of the bar $= \dfrac{\pi}{4} D^2$

$\therefore$ Direct stress, $\sigma = \dfrac{P}{A} = \dfrac{160000}{\frac{\pi}{4} D^2} = \dfrac{640000}{\pi D^2} \text{ N/mm}^2$

Maximum shear stress is given by equation (3.5).

$\therefore$ Maximum shear stress $= \dfrac{\sigma}{2} = \dfrac{640000}{2 \times \pi D^2}$.

$$D^2 = \frac{640000}{2 \times \pi \times 65} = 1567$$

$D = 39.58$ **mm. Ans.**

Problem 3.3. *A rectangular bar of cross-sectional area of 11000 mm^2 is subjected to a tensile load P as shown in Fig. 3.3. The permissible normal and shear stresses on the oblique plane BC are given as 7 N/mm^2 and 3.5 N/mm^2 respectively. Determine the safe value of P.*

Sol. Given :

Area of cross-section, $A = 11000\ mm^2$

Normal stress, $\sigma_n = 7\ N/mm^2$

Shear stress, $\sigma_t = 3.5\ N/mm^2$

Angle of oblique plane with the axis of bar = 60°.

$\therefore$ Angle of oblique plane BC with the normal cross-section of the bar,

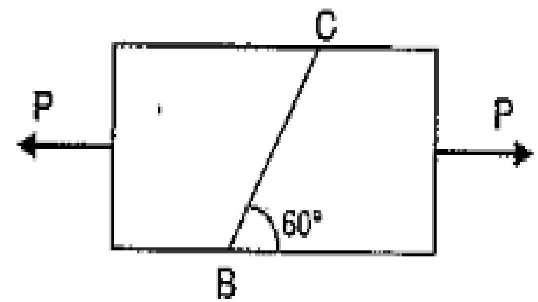


Fig. 3.3

$$\theta = 90° - 60° = 30°$$

Let P = Safe value of axial pull

σ = Safe stress in the member.

Using equation (3.2),

$$\sigma_n = \sigma \cos^2 \theta \quad \text{or} \quad 7 = \sigma \cos^2 30°$$

$$= \sigma (0.866)^2. \qquad (\because \cos 30° = 0.866)$$

$$\therefore \qquad \sigma = \frac{7}{0.866 \times 0.866} = 9.334\ N/mm^2$$

$$\therefore \qquad \sigma = \frac{7}{0.866 \times 0.866} = 9.334 \text{ N/mm}^2$$

Using equation (3.3),

$$\sigma_t = \frac{\sigma}{2} \sin 2\theta$$

or

$$3.5 = \frac{\sigma}{2} \sin 2 \times 30° = \frac{\sigma}{2} \sin 60° = \frac{\sigma}{2} \times 0.866$$

$$\therefore \qquad \sigma = \frac{3.5 \times 2}{0.866} = 8.083 \text{ N/mm}^2.$$

The safe stress is the least of the two, *i.e.*, 8.083 N/mm^2.

$\therefore$ Safe value of axial pull,

$$P = \text{Safe stress} \times \text{Area of cross-section}$$

$$= 8.083 \times 11000 = 88913 \text{ N} = \mathbf{88.913\ kN.} \quad \textbf{Ans.}$$

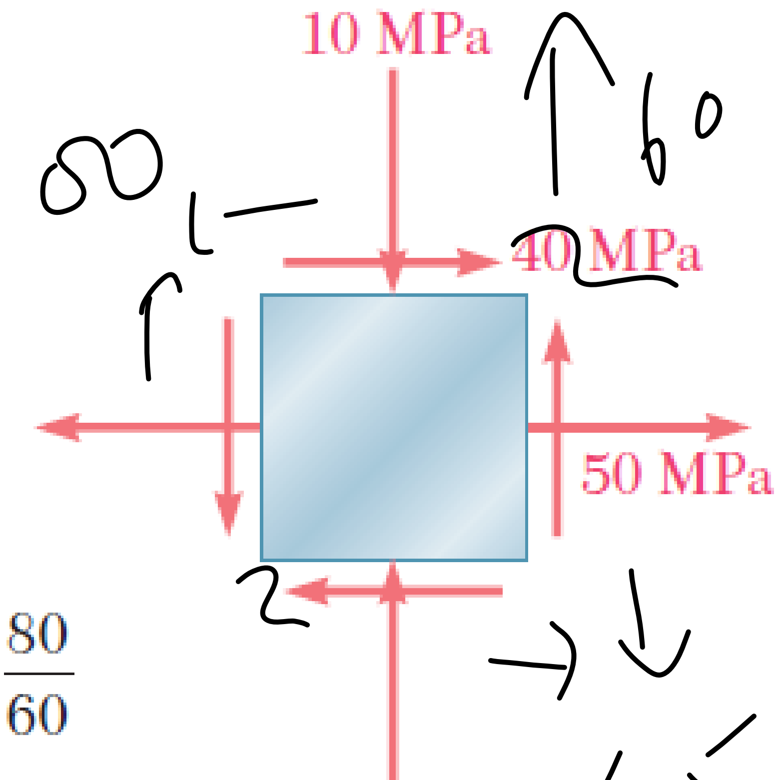
$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau \sin 2\theta$$

$$\sigma_t = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta$$

$$\sigma_{\text{max, min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Example 1

For the state of plane stress shown in figure, determine (*a*) the principal planes, (*b*) the principal stresses, (*c*) the maximum shearing stress and the corresponding normal stress.



Solution

$$\sigma_x = +50 \text{ MPa}$$
$$\sigma_y = -10 \text{ MPa}$$
$$\tau_{xy} = +40 \text{ MPa}$$

180 60 45

Principal Planes

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(+40)}{50-(-10)} = \frac{80}{60}$$

$$2\theta_p = 53.1^\circ \quad \text{and} \quad 180^\circ + 53.1^\circ = 233.1^\circ$$
$$\theta_p = 26.6^\circ \quad \text{and} \quad 116.6^\circ$$

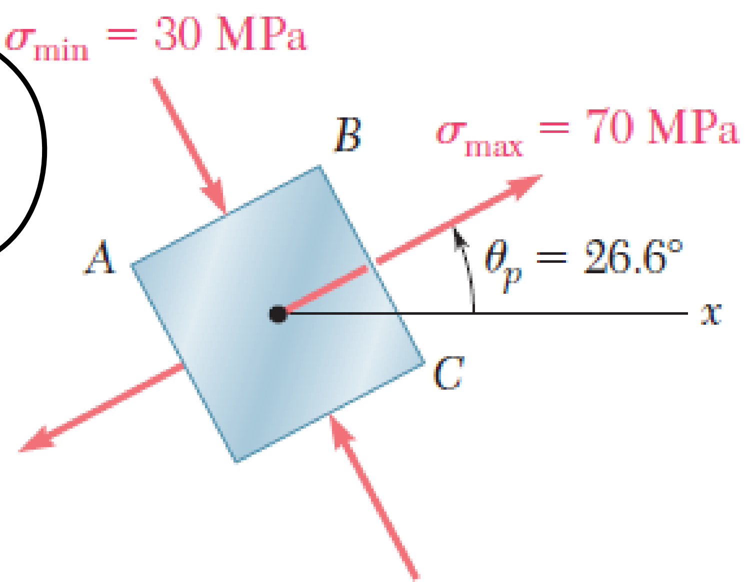
Principal Stresses

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 20 \pm \sqrt{(30)^2 + (40)^2}$$

$$\sigma_{\max} = 20 + 50 = 70 \text{ MPa}$$

$$\sigma_{\min} = 20 - 50 = -30 \text{ MPa}$$



we check that the normal stress exerted on face *BC* of the element is the maximum stress:

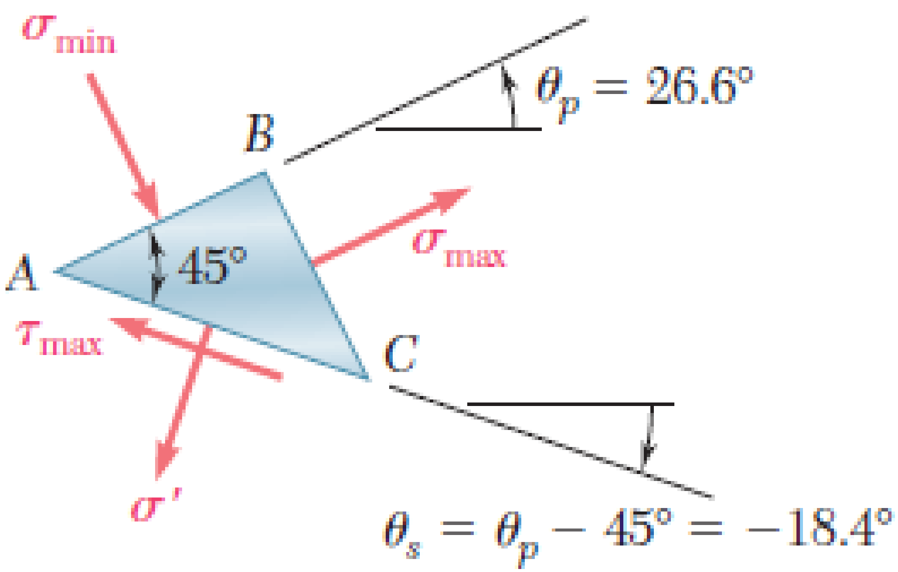
$$\sigma_{x'} = \frac{50 - 10}{2} + \frac{50 + 10}{2} \cos 53.1° + 40 \sin 53.1°$$

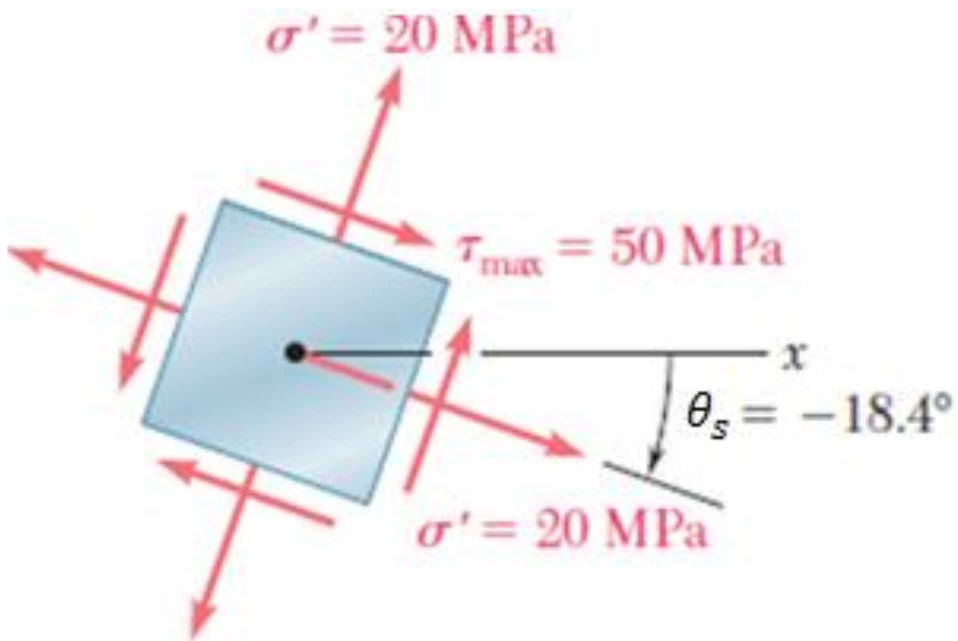
$$= 20 + 30 \cos 53.1° + 40 \sin 53.1° = 70 \text{ MPa} = \sigma_{\max}$$

Maximum shearing Stress

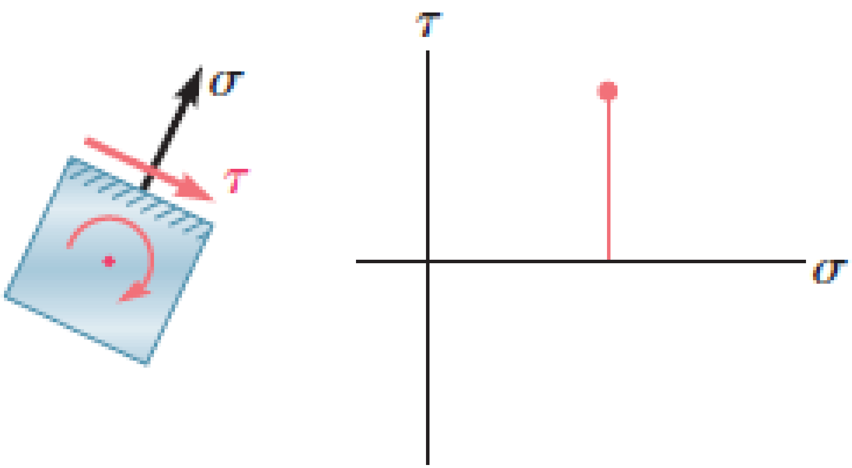
$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{(30)^2 + (40)^2} = 50 \text{ MPa}$$

$$\sigma' = \sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} = \frac{50 - 10}{2} = 20 \text{ MPa}$$

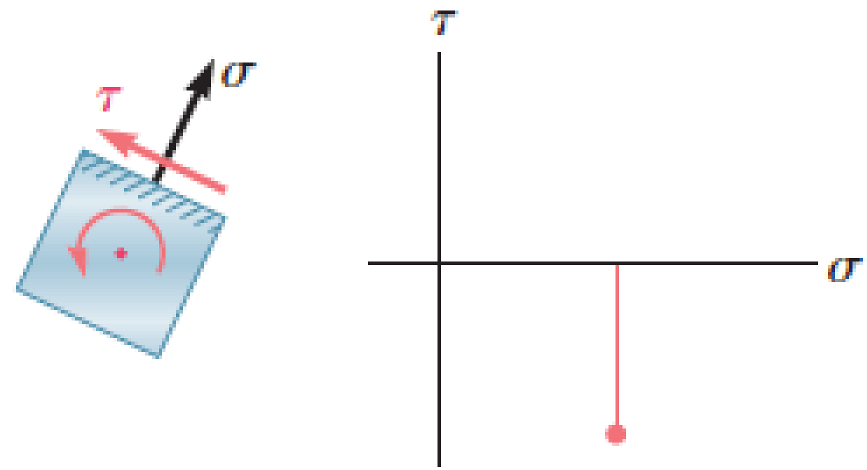




Mohr’s Circle for Plane Stress

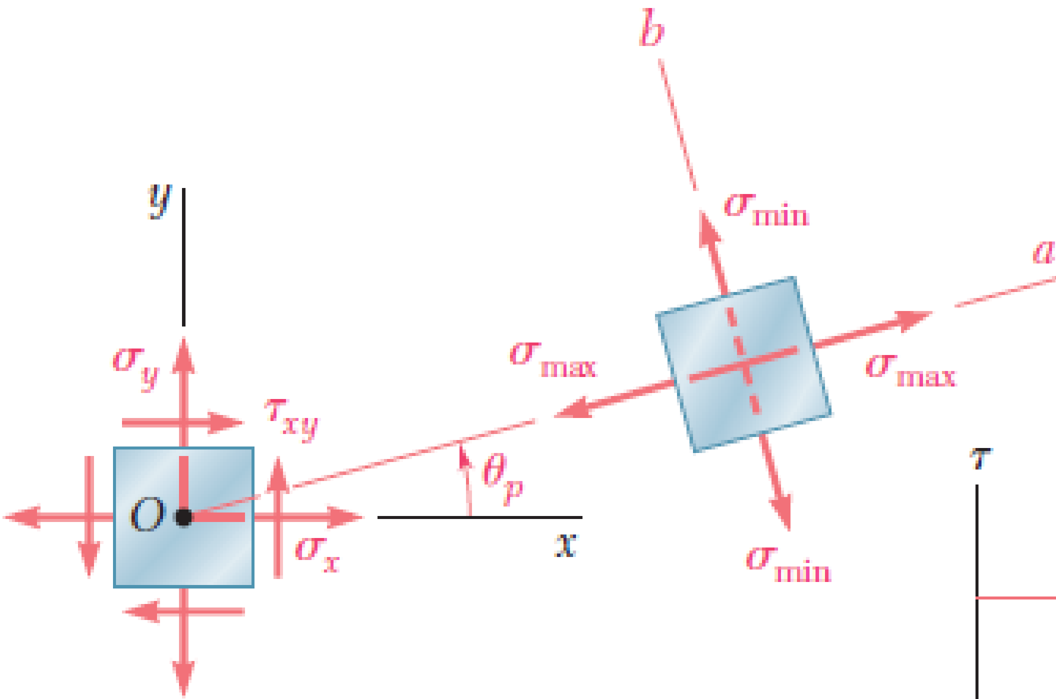


(*a*) Clockwise ⟶ Above

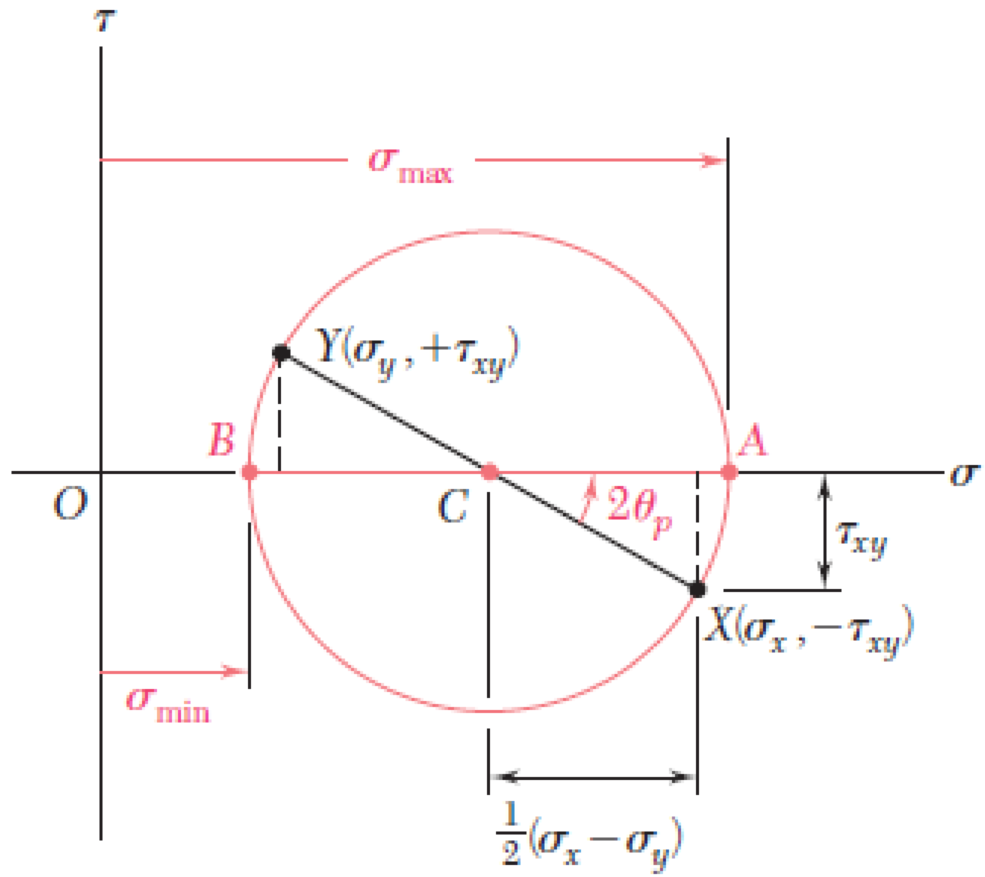


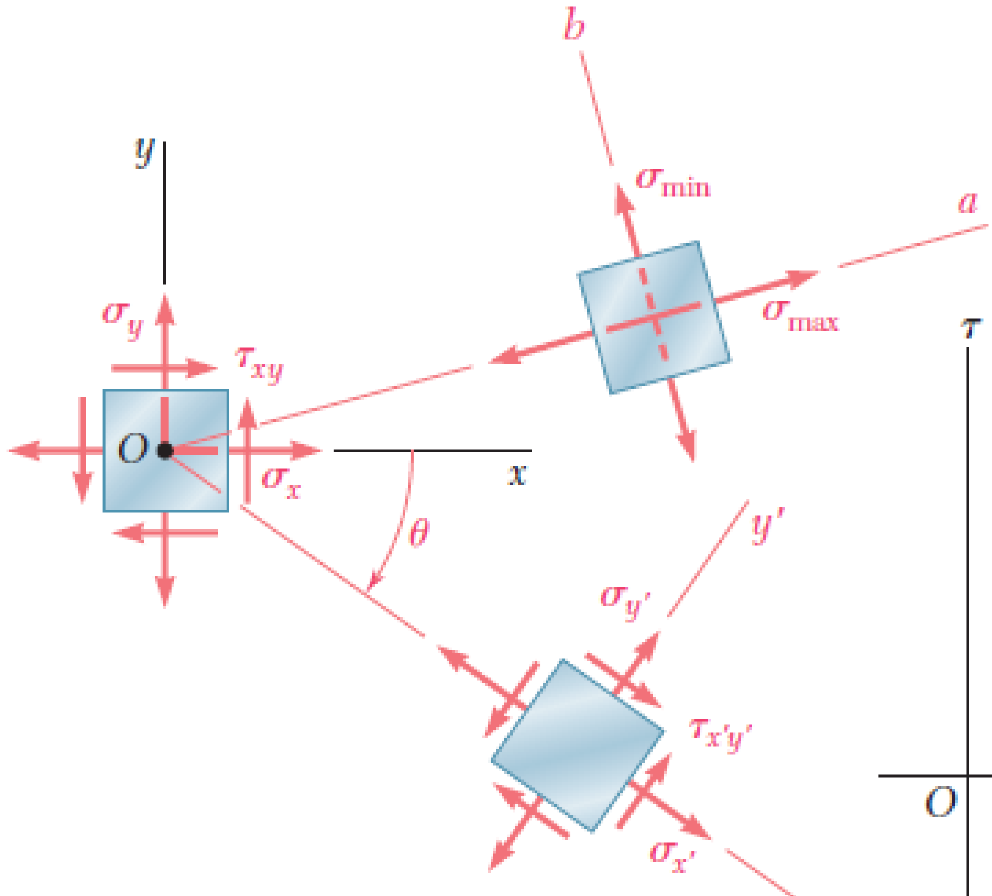
(*b*) Counterclockwise ⟶ Below

$\sigma_x, \sigma_y, \tau_{xy}$



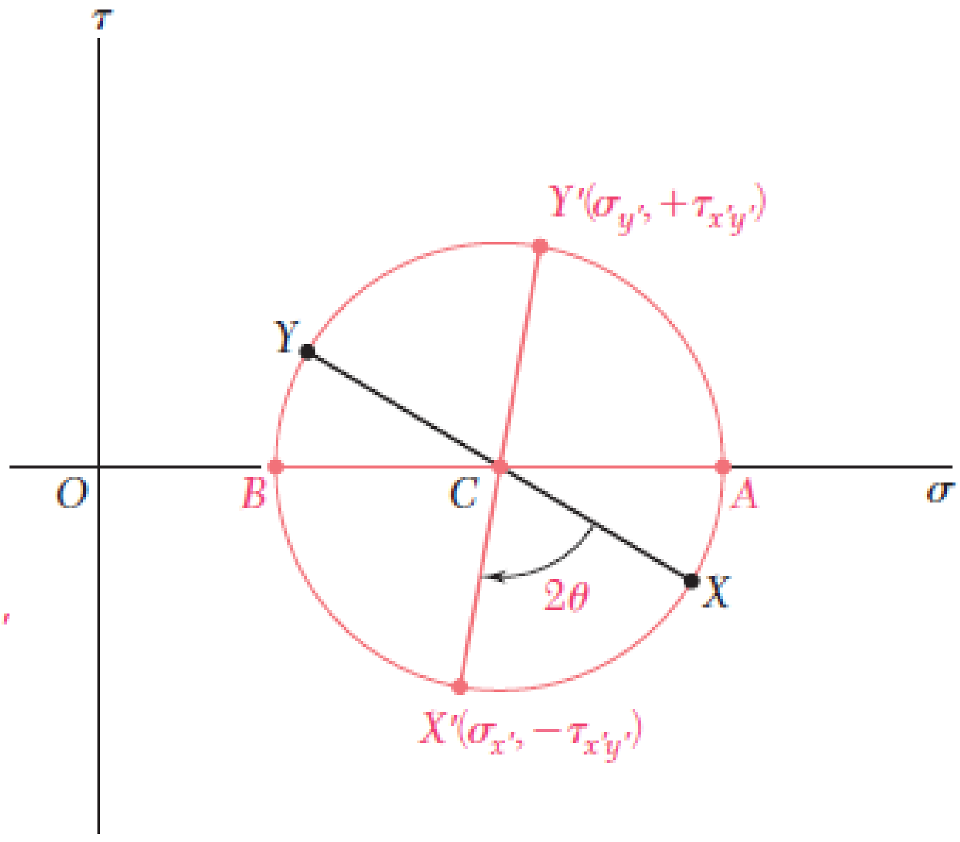
$$(\sigma_{x'} - \sigma_{ave})^2 + \tau_{x'y'}^2 = R^2$$



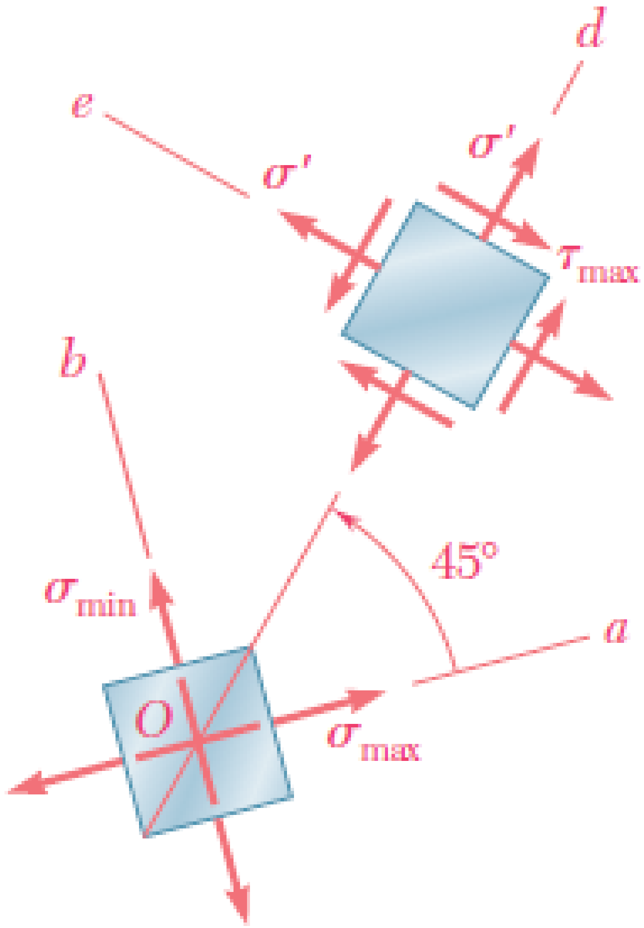


b
a
y
σ_min
σ_max
σ_y
τ_xy
O
σ_x
x
θ
y′
σ_y′
τ_x′y′
σ_x′
x′

(a)

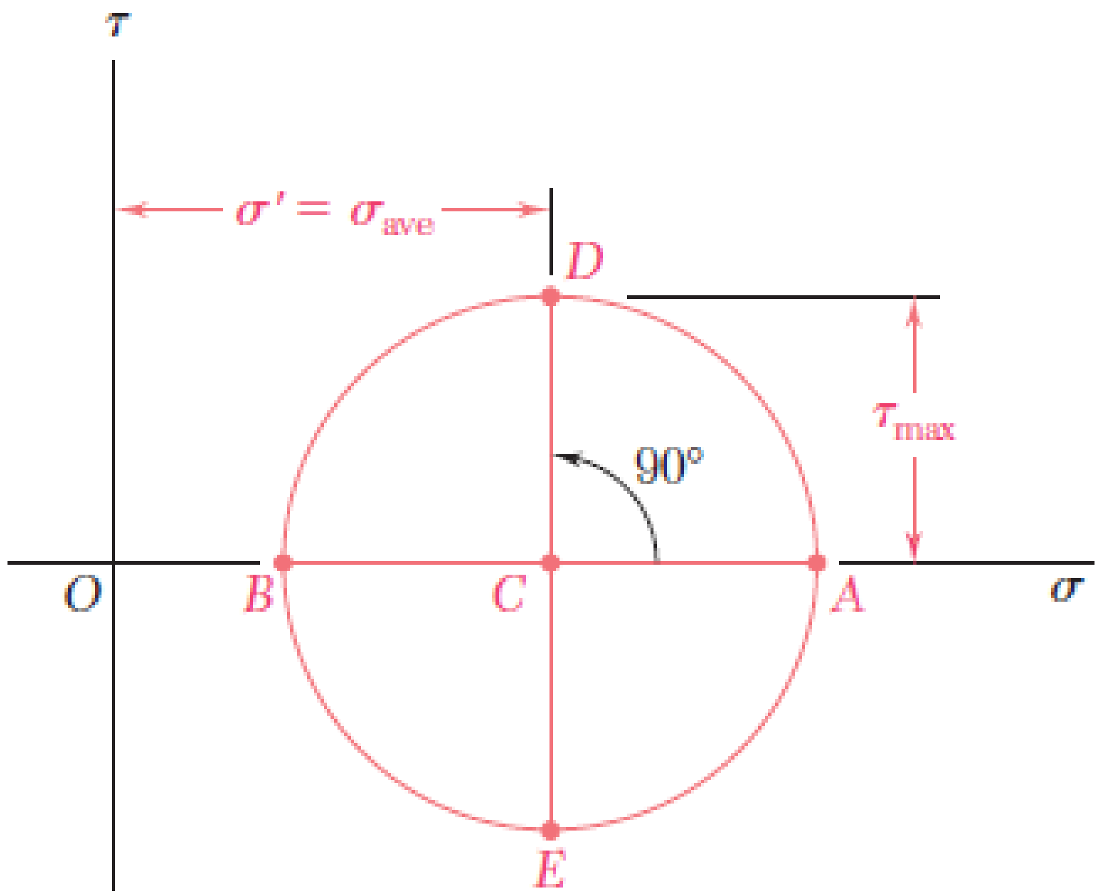


τ
Y′(σ_y′, +τ_x′y′)
Y
O
B
C
A
σ
2θ
X
X′(σ_x′, −τ_x′y′)



d
e
σ'
σ'
τ_{max}
b
45°
σ_{min}
a
O
σ_{max}

(a)

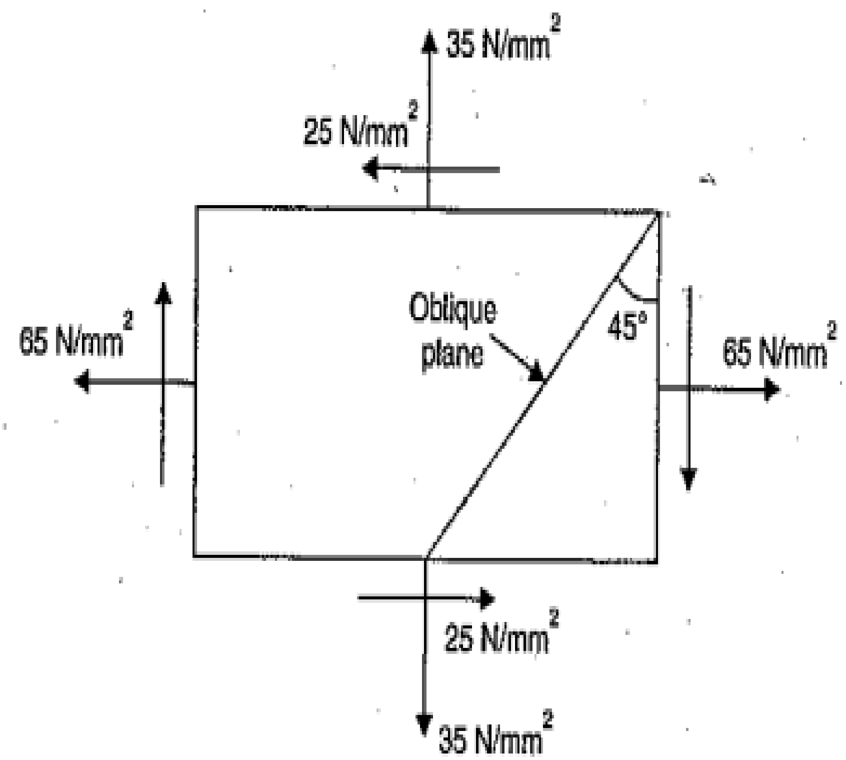


τ
$\sigma' = \sigma_{ave}$
D
τ_{max}
90°
O
B
C
A
σ
E

(b)

Example 2

For the state of plane stress of problem 1, (*a*) construct Mohr's circle, (*b*) determine the principal stresses, (*c*) determine the maximum shearing stress and the corresponding normal stress.



Sol. Given :

Major principal stress,	$\sigma_1 = 65\ N/mm^2$
Minor principal stress,	$\sigma_2 = 35\ N/mm^2$
Shear stress,	$\tau = 25\ N/mm^2$
Angle of oblique plane,	$\theta = 45°$.

Major principal stress, $\sigma_1 = 65\ \text{N/mm}^2$
Minor principal stress, $\sigma_2 = 35\ \text{N/mm}^2$
Shear stress, $\tau = 25\ \text{N/mm}^2$
Angle of oblique plane, $\theta = 45°$.

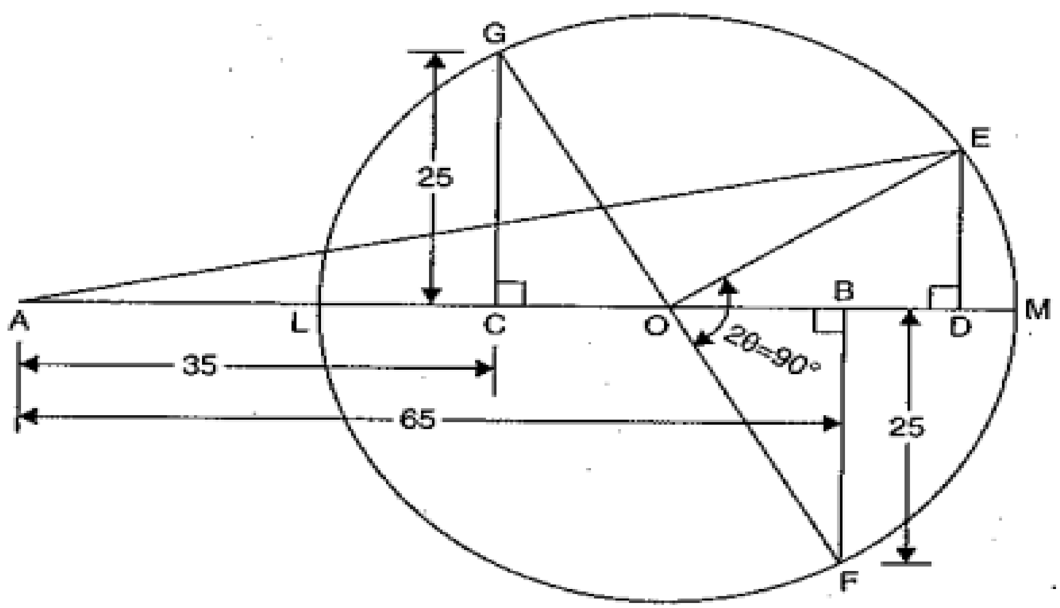
Mohr's circle method

Let $1\ \text{cm} = 10\ \text{N/mm}^2$

Then $$\sigma_1 = \frac{65}{10} = 6.5\ \text{cm},$$

$$\sigma_2 = \frac{35}{10} = 3.5\ \text{cm and } \tau = \frac{25}{10} = 2.5\ \text{cm}$$

Mohr's circle is drawn as given in Fig. 3.29.



equal to shear stress $\tau = 2.5$ cm. Bisect BC at O. Now with O as centre and radius equal to OF (or OG) draw a circle. Through O, draw a line OE making an angle of 2θ (*i.e.*, $2 \times 45° = 90°$) with OF as shown in Fig. 3.29. From E, draw ED perpendicular to AB produced. Join AE. Then length AD represents the normal stress and length ED represents the shear stress.

By measurements, length $AD = 7.5$ cm and

length $ED = 1.5$ cm.

$\therefore$ Normal stress (σ_n) = Length $AD \times$ Scale $= 7.5 \times 10 =$ **75 N/mm²**. **Ans.**

$(\because \quad 1 \text{ cm} = 10 \text{ N/mm}^2)$

And tangential stress (σ_t) = Length $ED \times$ Scale $= 1.5 \times 10 =$ **15 N/mm²**. **Ans.**

Analytical Answers

Normal stress (σ_n) is given by equation (3.12).

$\therefore$ Using equation (3.12),

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau \sin 2\theta$$

$$= \frac{65 + 35}{2} + \frac{65 - 35}{2} \cos (2 \times 45°) + 25 \sin (2 \times 45°)$$

$$= 50 + 15 \cos 90° + 25 \sin 90°$$

$$= 50 + 15 \times 0 + 25 \times 1 \qquad (\because \quad \cos 90° = 0, \sin 90° = 1)$$

$$= 50 + 0 + 25 = \mathbf{75\ N/mm^2}. \quad \textbf{Ans.}$$

Tangential stress is given by equation (3.13)

$\therefore$ Using equation (3.13),

$$\sigma_t = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta$$

$$= \frac{65 - 35}{2} \sin (2 \times 45) - 25 \cos (2 \times 45)$$

$$= 15 \sin 90° - 25 \cos 90° = 15 \times 1 - 25 \times 0 = 15 - 0$$

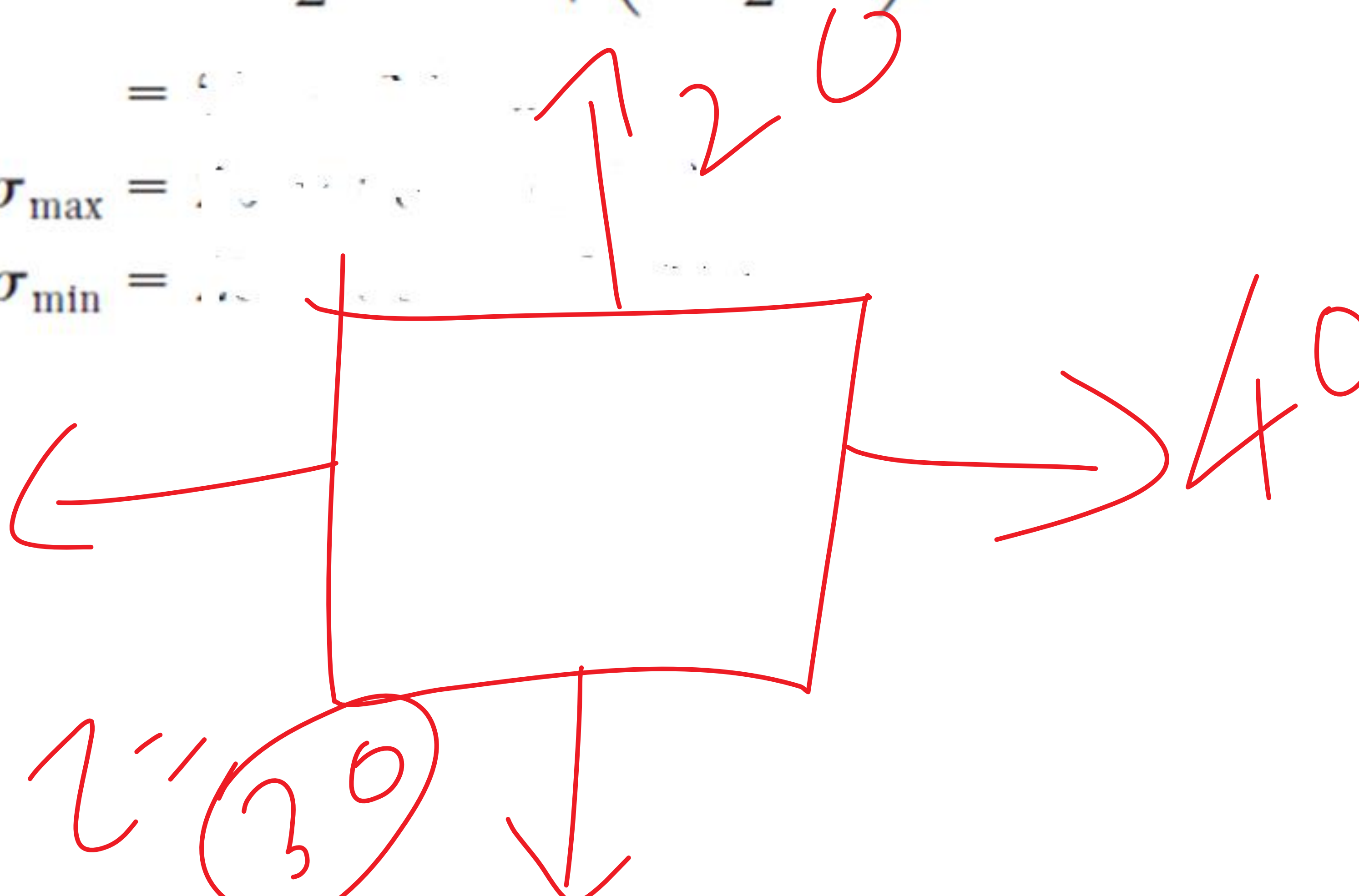
$$= \mathbf{15\ N/mm^2}. \quad \textbf{Ans.}$$

$$\sigma\,\sigma_{\text{max, min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$=$$

$$\sigma_{\text{max}} =$$

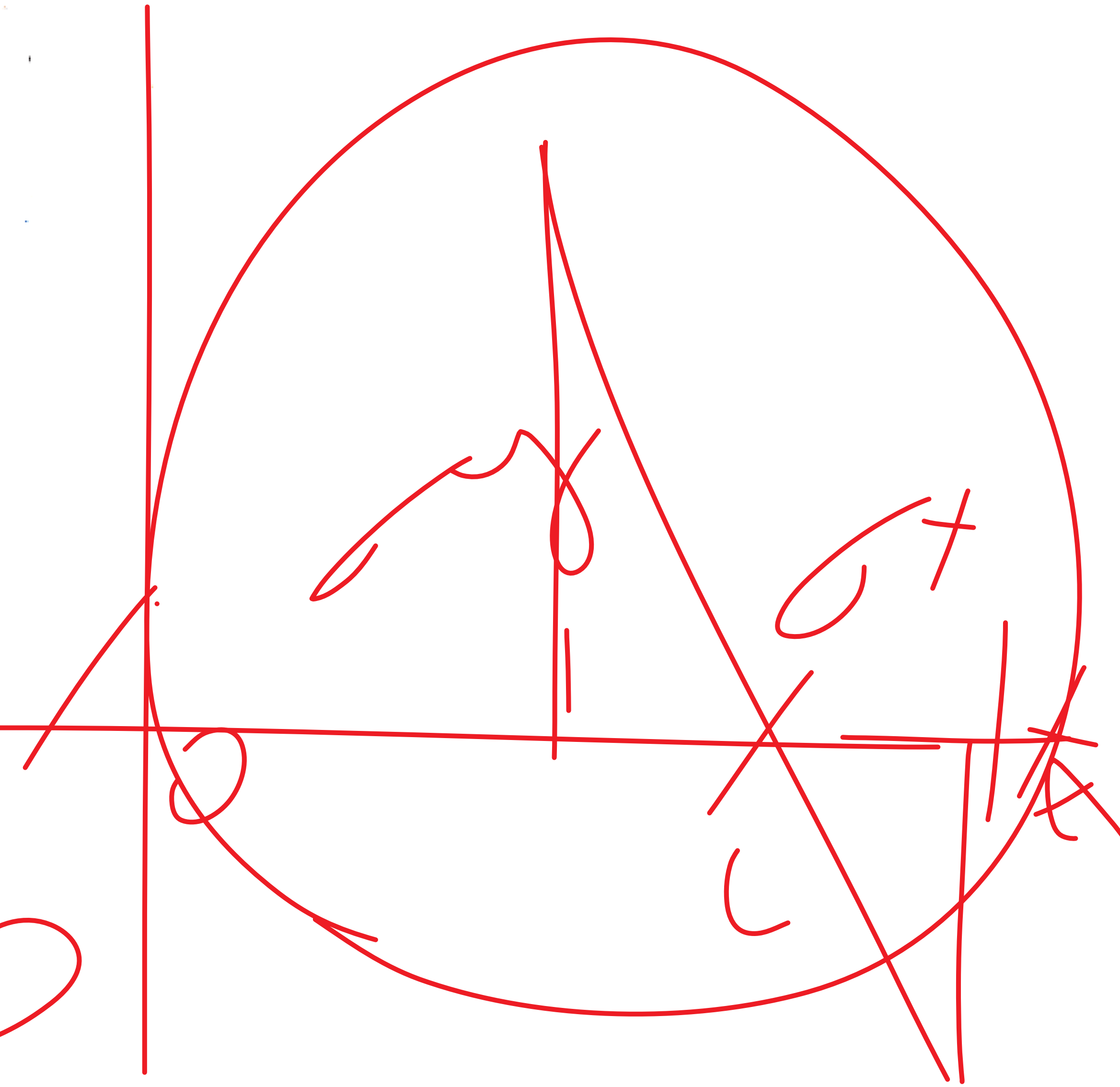
$$\sigma_{\text{min}} =$$



$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} =$$

CF=

FX=



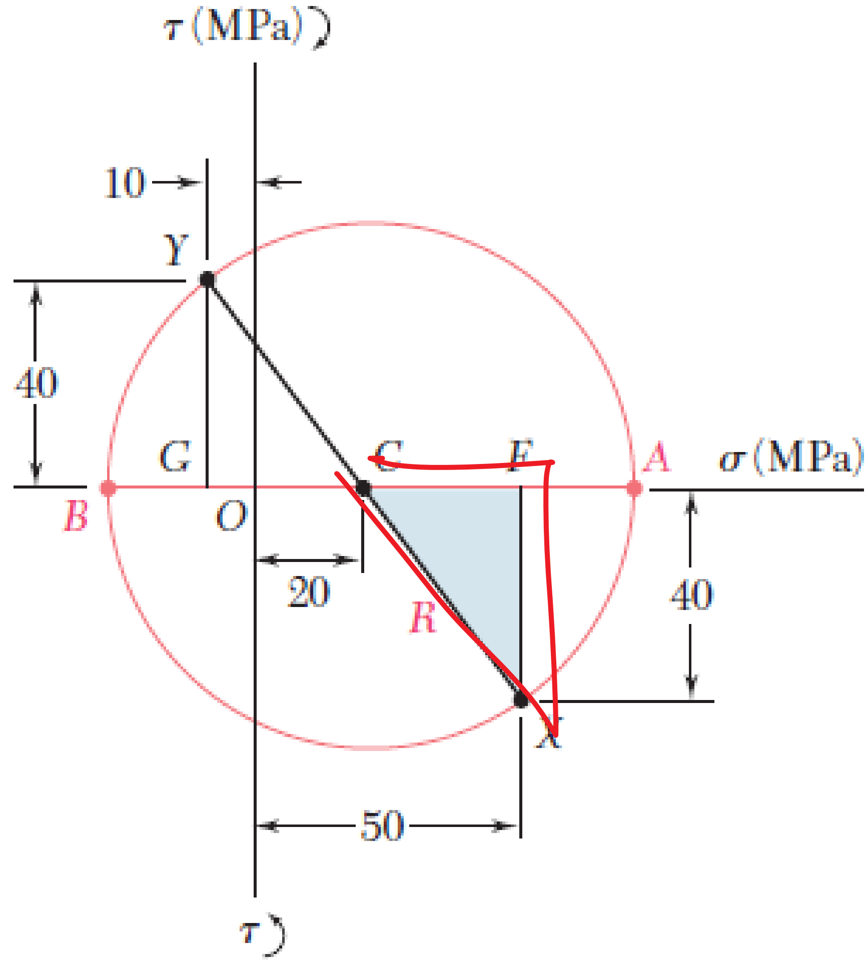
Construction of Mohr's Circle

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} =$$

$$CF = 50 - 20 = 30 \text{ MPa}$$

$$FX = 40 \text{ MPa}$$

$$R = CX = \sqrt{(30)^2 + (40)^2} = 50 \text{ MPa}$$



Principal Planes and Principal Stresses

$$\sigma_{\max} = OA = OC + CA = 20 + 50 = 70 \text{ MPa}$$

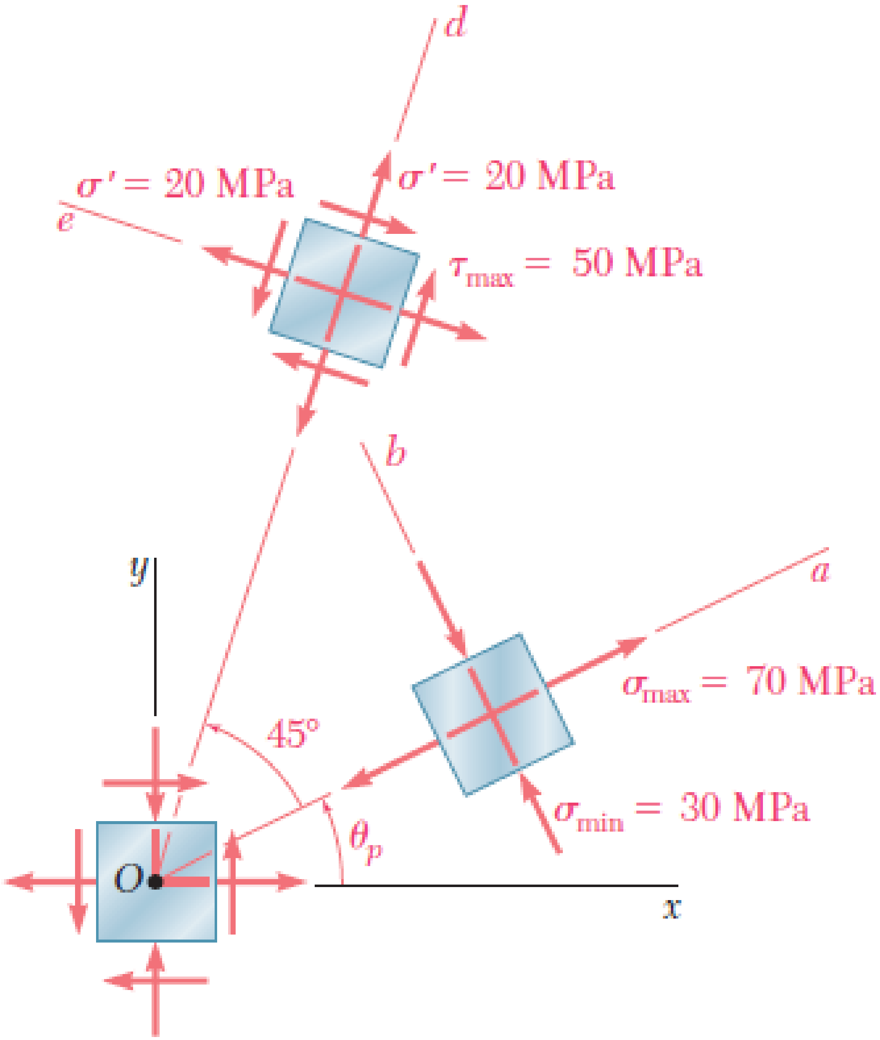
$$\sigma_{\min} = OB = OC - BC = 20 - 50 = -30 \text{ MPa}$$

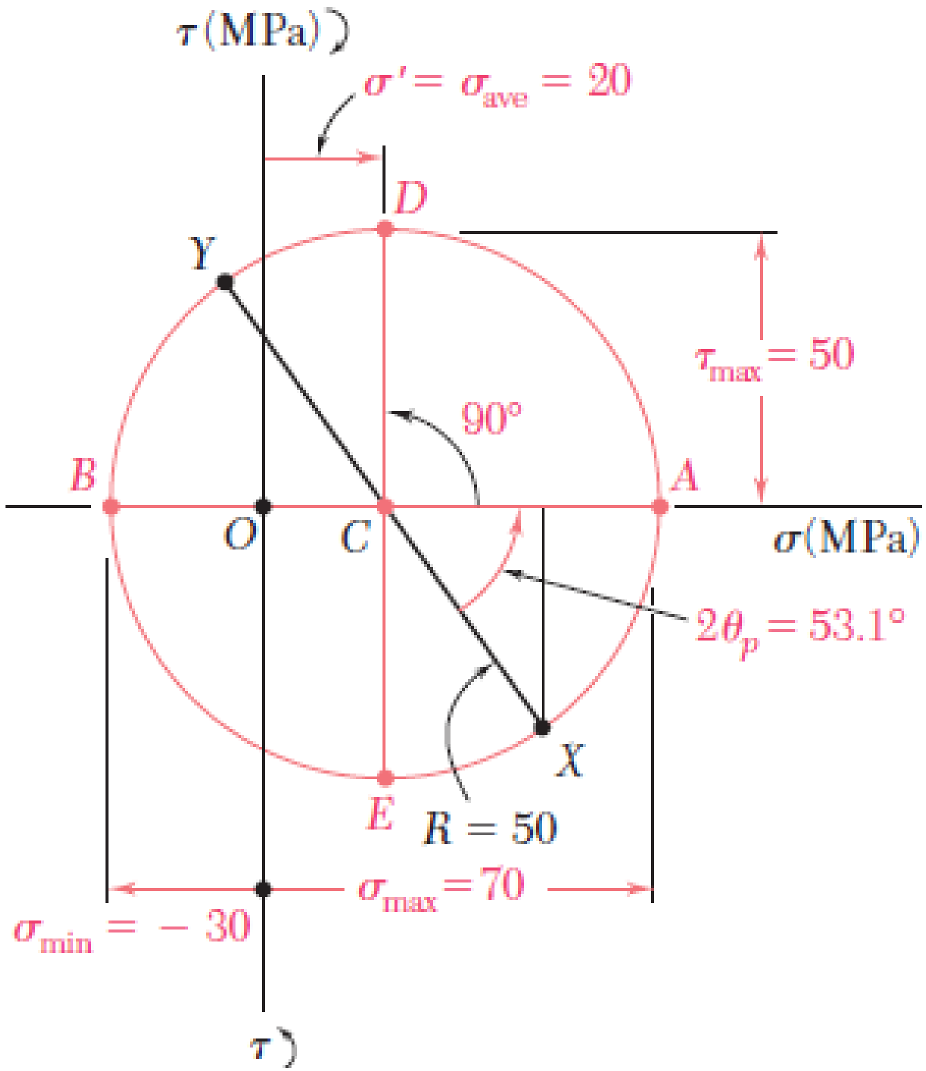
Recalling that the angle ACX represents $2\theta_p$

$$\tan 2\theta_p = \frac{FX}{CF} = \frac{40}{30}$$

$$2\theta_p = 53.1^\circ \qquad \theta_p = 26.6^\circ$$

Maximum Shearing Stress





Problem 3.5. *The tensile stresses at a point across two-mutually perpendicular planes are 120 N/mm² and 60 N/mm². Determine the normal, tangential and resultant stresses on a plane inclined at 30° to the axis of the minor stress.*

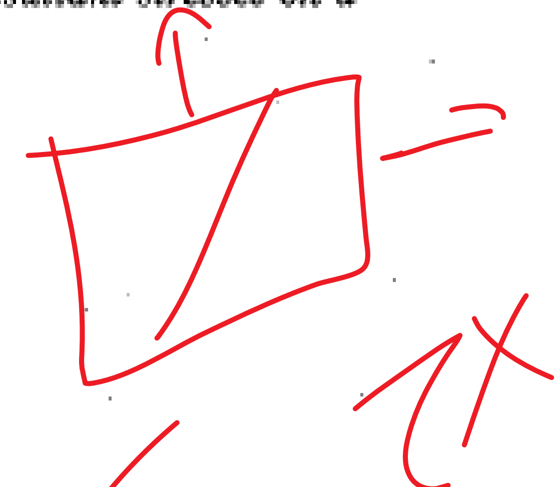
Sol. Given :

Major principal stress, $\sigma_1 = 120$ N/mm²

~~Minor principal, $\sigma_2 = 60$ N/mm²~~

~~Angle of oblique plane with the axis of minor~~ principal stress,

$\theta = 30°$.



Normal stress

The normal stress (σ_n) is given by equation (3.6),

$$\therefore \quad \sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta$$

$$= \frac{120 + 60}{2} + \frac{120 - 60}{2} \cos 2 \times 30°$$

$$= 90 + 30 \cos 60° = 90 + 30 \times \frac{1}{2}$$

$$= \mathbf{105\ N/mm^2}. \quad \textbf{Ans.}$$

σ_n

σ_t

σ_r

Tangential stress

The tangential (or shear stress) σ_t is given by equation (3.7).

$$\therefore \quad \sigma_t = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta$$

$$= \frac{120 - 60}{2} \sin (2 \times 30°)$$

$$= 30 \times \sin 60° = 30 \times 0.866$$

$$= \mathbf{25.98\ N/mm^2.} \quad \textbf{Ans.}$$

Resultant stress

The resultant stress (σ_R) is given by equation (3.8)

$$\therefore \quad \sigma_R = \sqrt{\sigma_n{}^2 + \sigma_t{}^2} = \sqrt{105^2 + 25.98^2}$$

$$= \sqrt{11025 + 674.96} = \mathbf{108.16\ N/mm^2.} \quad \textbf{Ans.}$$

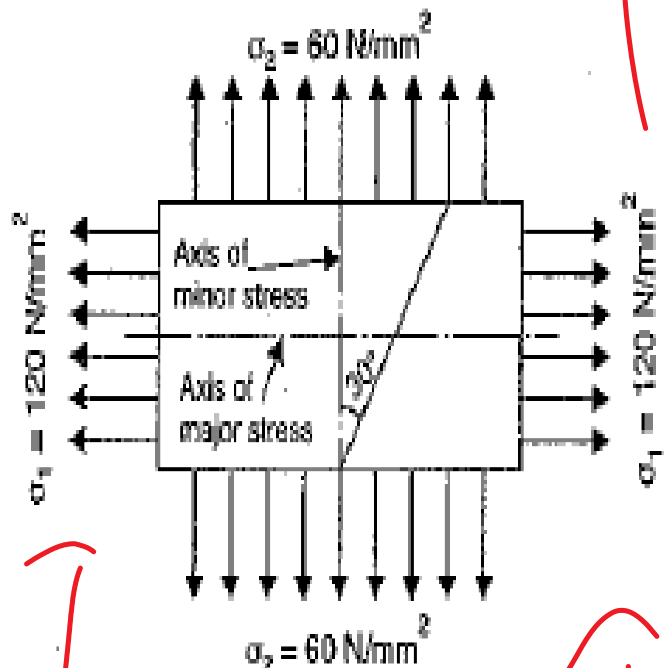
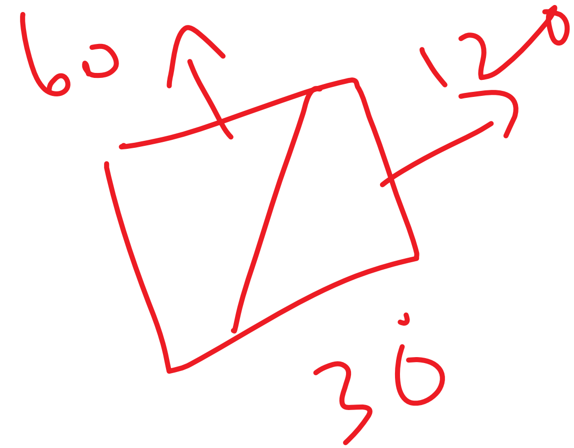


Fig. 3.5

Problem 3.21. *Solve the problem 3.5 by graphical method.*



Sol. The data given in problem 3.5, is

$$\sigma_1 = 120 \text{ N/mm}^2, \sigma_2 = 60 \text{ N/mm}^2, \theta = 30°.$$

Scale

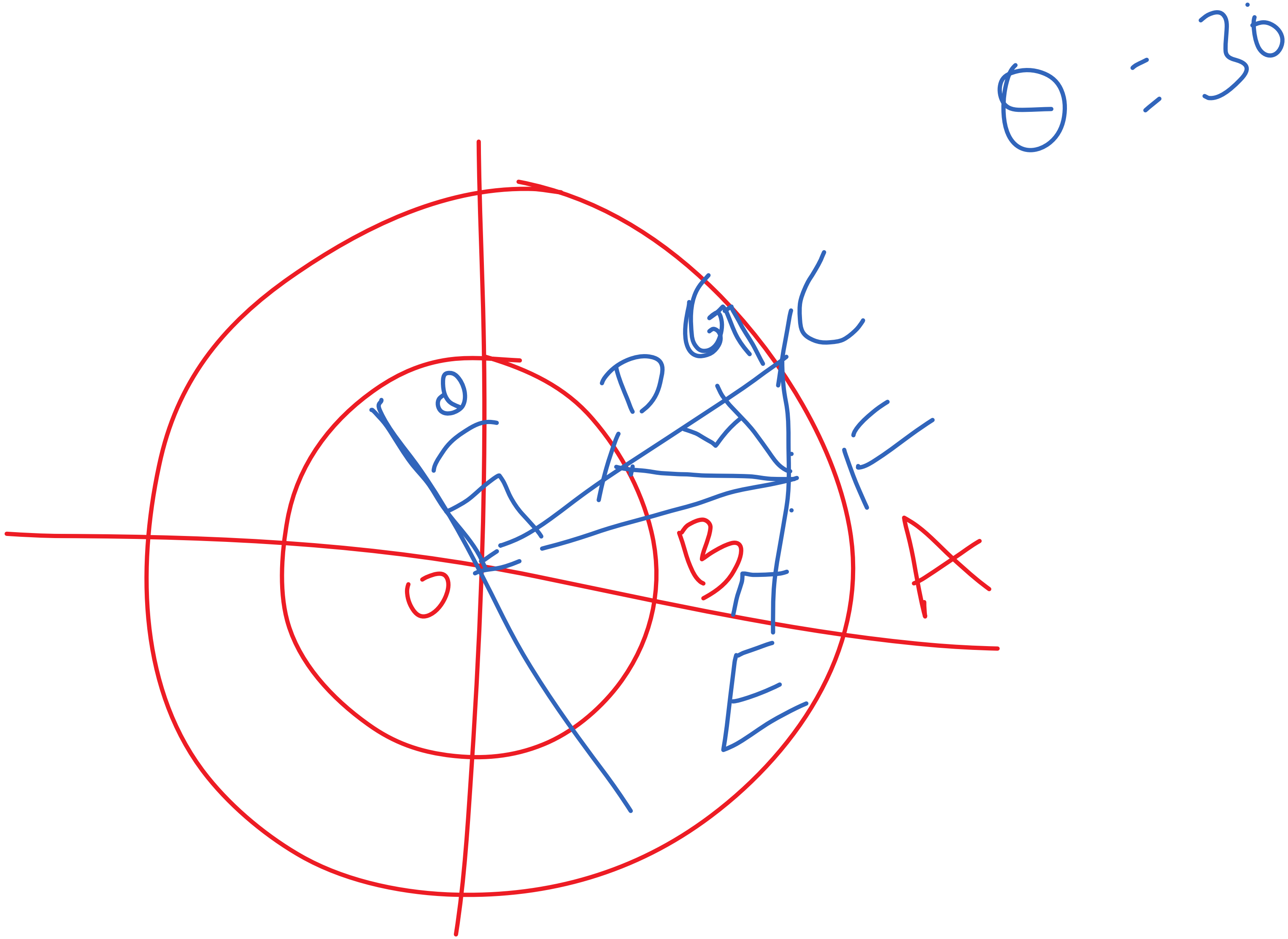
Take 1 cm = 20 N/mm^2

Then $\sigma_1 = \frac{120}{20} = 6$ cm and $\sigma_2 = \frac{60}{20} = 3$ cm.

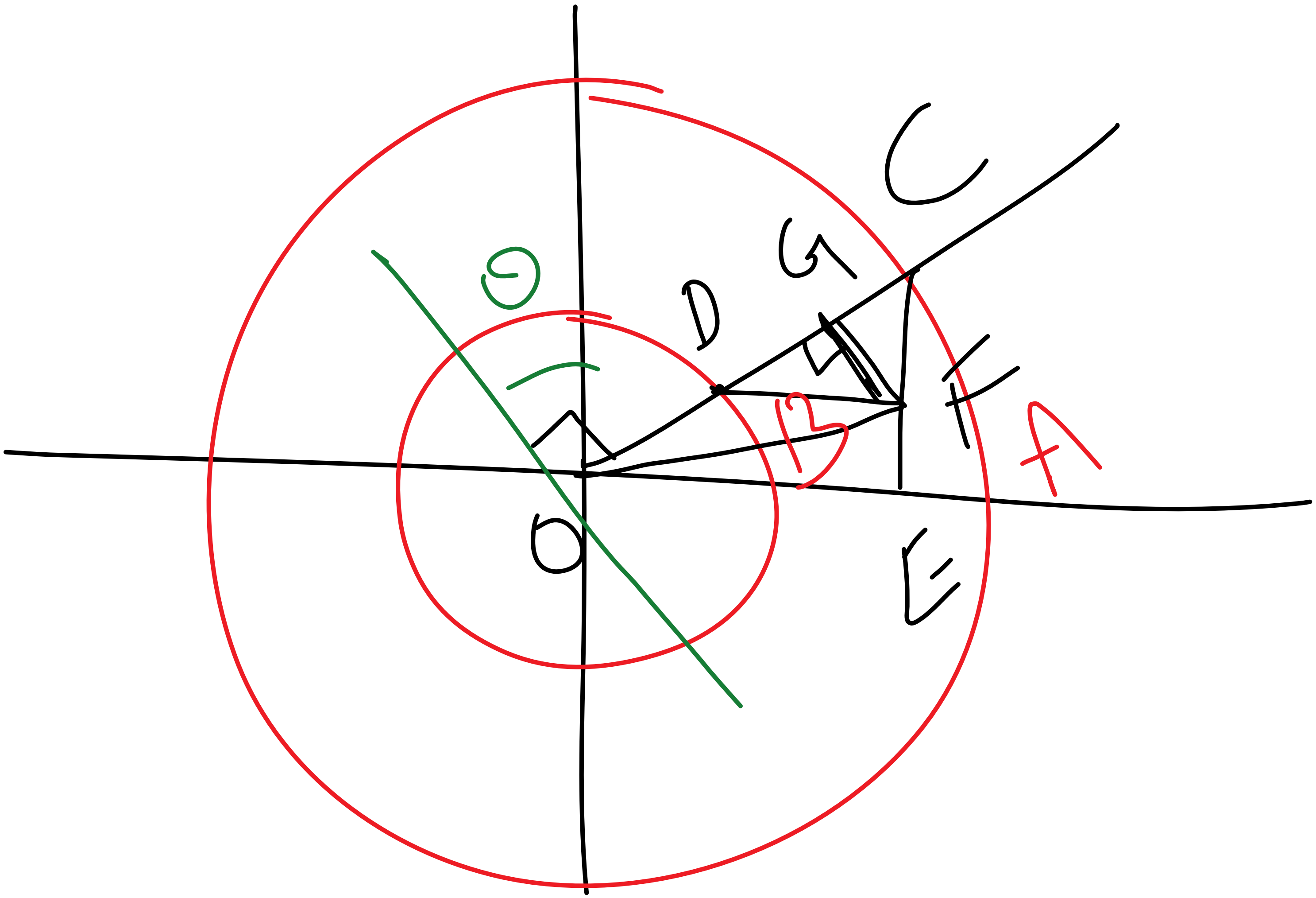
(*i*) Draw two mutually perpendicular lines meeting at *O* as shown in Fig. 3.20.

(*ii*) Take *OA* = 6 cm and *OB* = 3 cm.

(*iii*) Draw two concentric circles with centre *O* and radii equal to *OA* = 6 cm and *OB* = 3 cm.

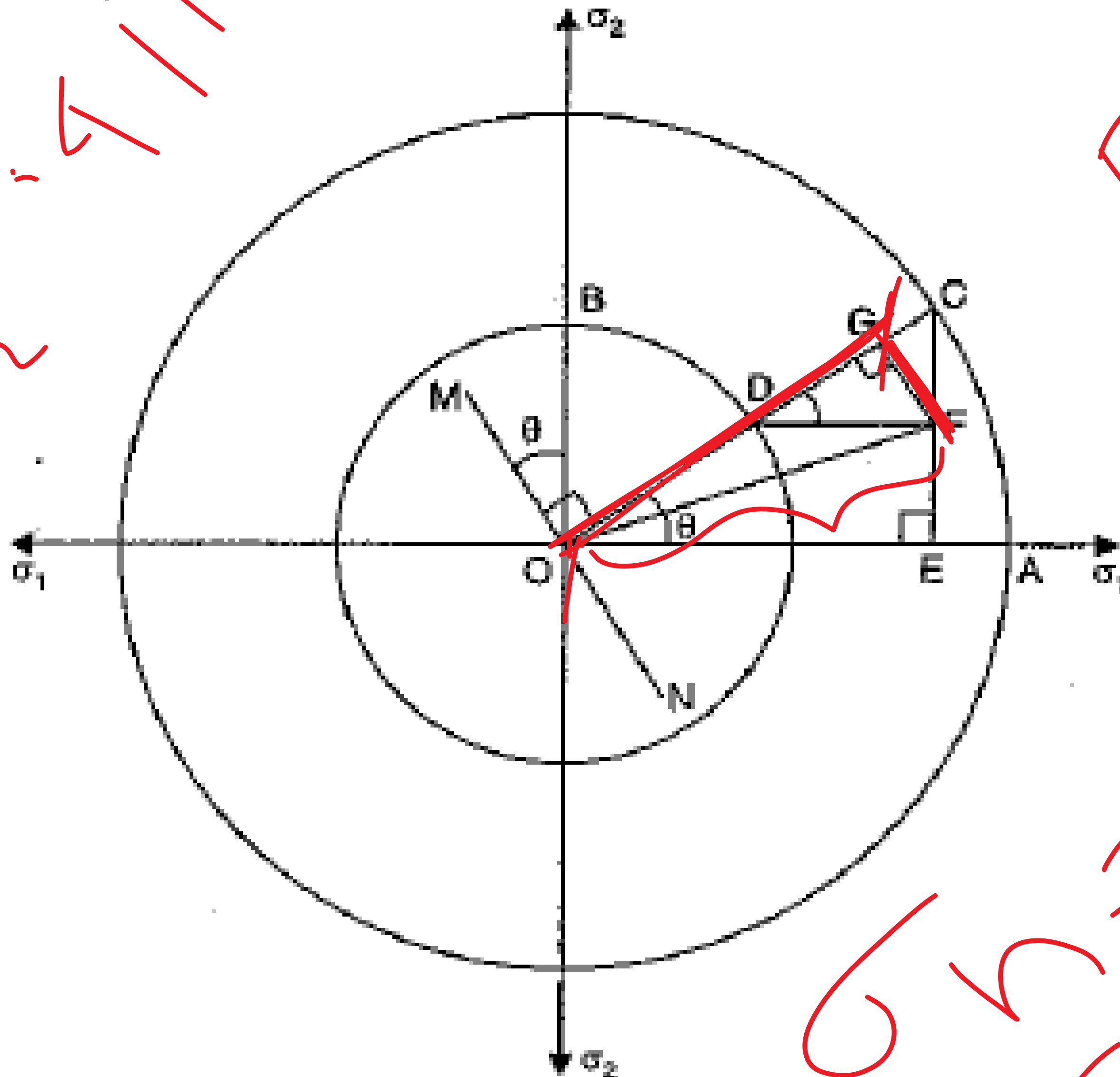


θ = 30
θ
O
D
G
C
H
F
A
B
E


C
G
D
H
A
B
O
E
θ

OF = 5.411 × 20 = 108.2

σ_f =



FG × 8 = OG × 50

$\sigma_n = \sigma_f : FG$

(*iv*) Draw a line *MN* through *O*, making an angle 30° with *OB*.

(*v*) Through *O*, draw a line *OC* at right angles to *MN*, cutting the two circles at *D* and *C*. From *C*, draw a line *CE* perpendicular to *OA*.

(*vi*) From *D*, draw a line *DF* parallel to *OA*, meeting the line *GE* at *F*.

(*vii*) Join *OF*. Then *OF* represents the resultant stress on the oblique plane.

(*viii*) From *F*, draw a line *FG* perpendicular to line *OC*. Then *FG* represents the tangential stress and *OG* represents the normal stress.

(*ix*) Measure the lengths *OF*, *FG* and *OG*.

By measurements, we get

Length $OF = 5.411$ cm

Length $FG = 1.30$ cm

Length $OG = 5.25$ cm.

$\therefore$ Resultant stress, σ_R = Length $OF \times$ Scale

$= 5.41 \times 20$ $\quad$ ($\because$ 1 cm = 20 N/mm^2)

$=$ **108.2 N/mm^2. Ans.**

Normal stress, σ_n = Length $OG \times 20$ N/mm^2 $= 5.25 \times 20 =$ **105 N/mm^2. Ans.**

Tangential stress, σ_t = Length $FG \times 20$ N/mm^2 $= 1.30 \times 20 =$ **26 N/mm^2. Ans.**

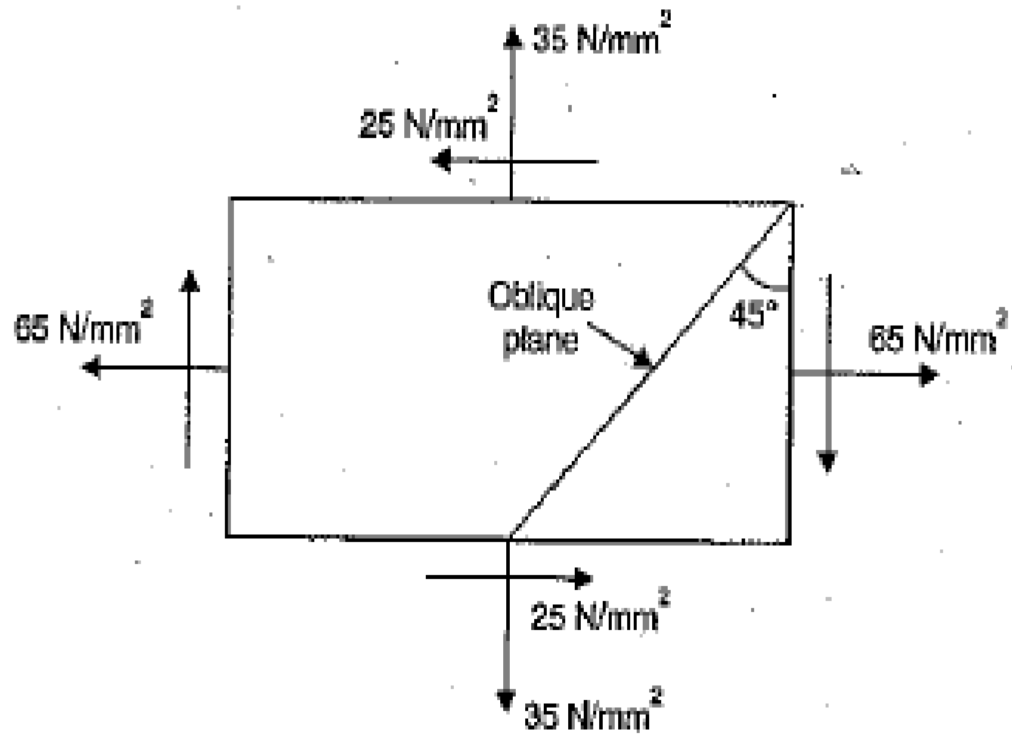


Fig. 3.28

Sol. Given :

Major principal stress, $\sigma_1 = 65\ N/mm^2$

Minor principal stress, $\sigma_2 = 35\ N/mm^2$

Shear stress, $\tau = 25\ N/mm^2$

Angle of oblique plane, $\theta = 45°$.

Mohr's circle method

Let $1 \text{ cm} = 10 \text{ N/mm}^2$

Then $\sigma_1 = \frac{65}{10} = 6.5 \text{ cm},$

$$\sigma_2 = \frac{35}{10} = 3.5 \text{ cm and } \tau = \frac{25}{10} = 2.5 \text{ cm}$$

Mohr's circle is drawn as given in Fig. 3.29.

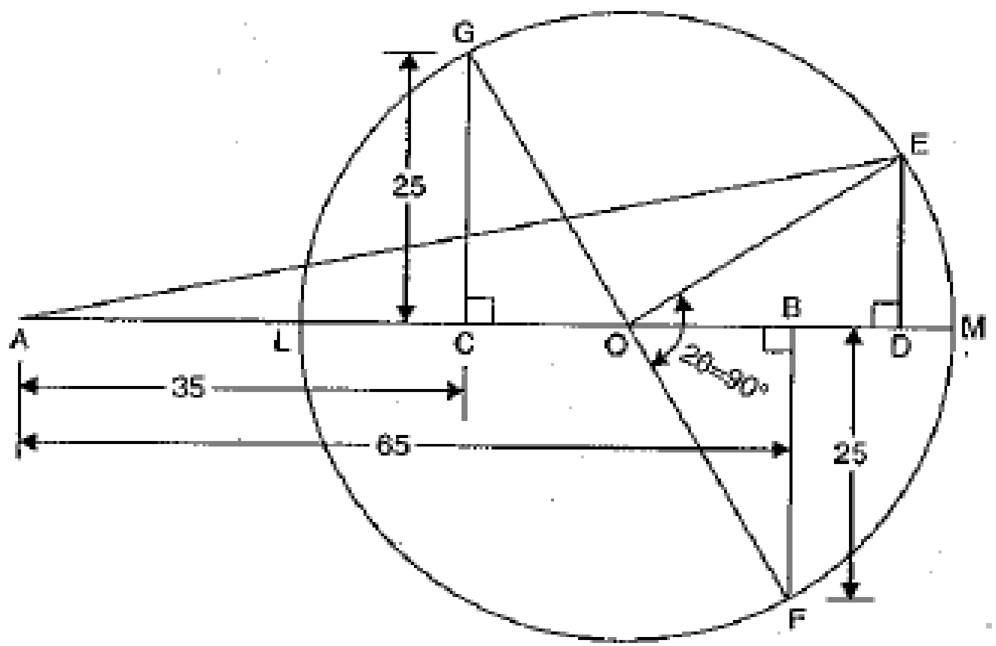
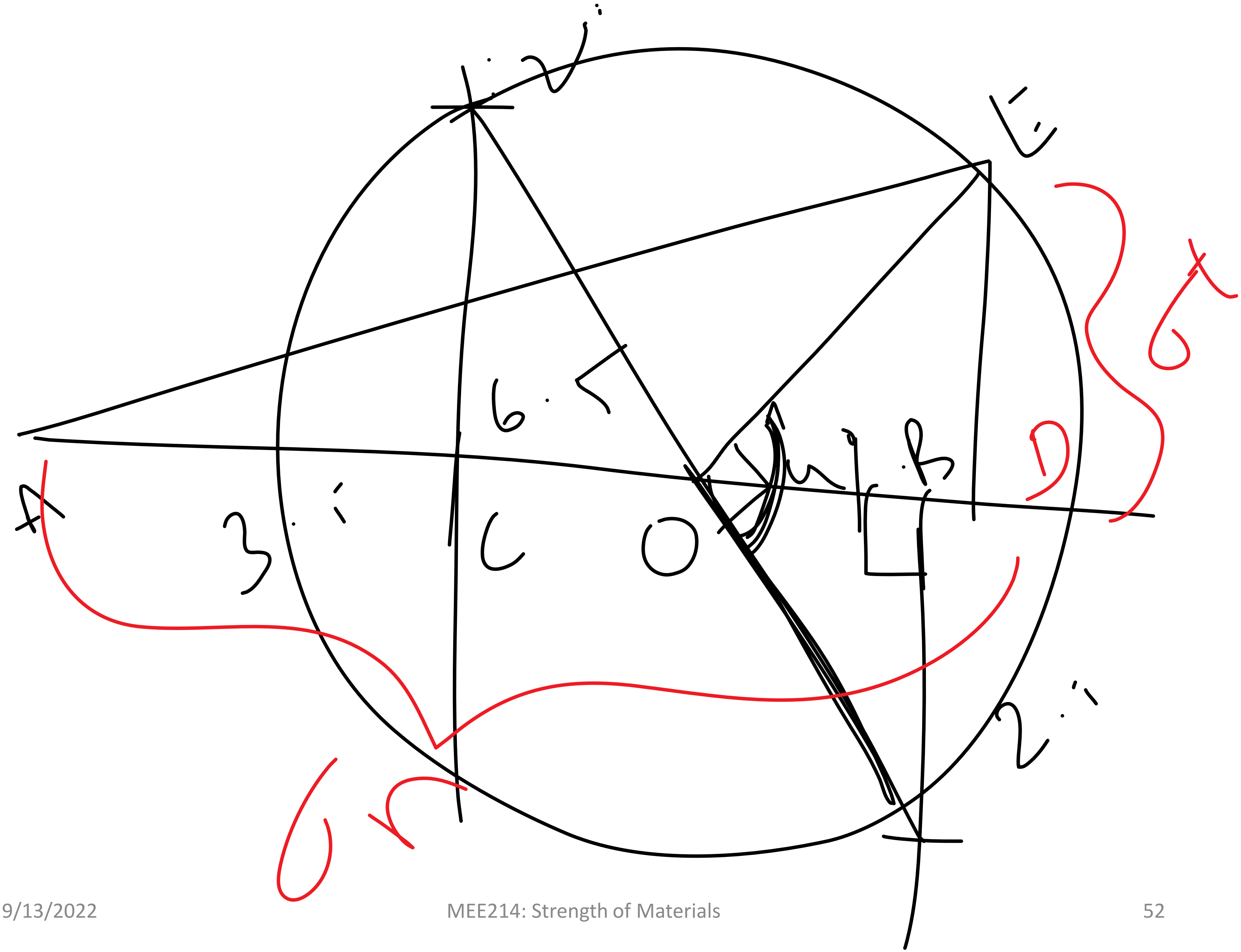


Fig. 3.29

Take any point A and draw a horizontal line through A. Take $AB = \sigma_1 = 6.5$ cm and $AC = \sigma_2 = 3.5$ cm towards right of A. Draw perpendicular at B and C and cut off BF and CG



A
C
O
E
D

equal to shear stress τ = 2.5 cm. Bisect BC at O. Now with O as centre and radius equal to OF (or OG) draw a circle. Through O, draw a line OE making an angle of 2θ (i.e., $2 \times 45° = 90°$) with OF as shown in Fig. 3.29. From E, draw ED perpendicular to AB produced. Join AE. Then length AD represents the normal stress and length ED represents the shear stress.

By measurements, length AD = 7.5 cm and

length ED = 1.5 cm.

$\therefore$ Normal stress (σ_n) = Length AD x Scale = 7.5 x 10 = **75 N/mm²**. **Ans.**

($\because$ 1 cm = 10 N/mm²)

And tangential stress (σ_t) = Length ED x Scale = 1.5 x 10 = **15 N/mm²**. **Ans.**

Analytical Answers

Normal stress (σ_n) is given by equation (3.12).

$\therefore$ Using equation (3.12),

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau \sin 2\theta$$

$$= \frac{65 + 35}{2} + \frac{65 - 35}{2} \cos (2 \times 45°) + 25 \sin (2 \times 45°)$$

$$= 50 + 15 \cos 90° + 25 \sin 90°$$

$$= 50 + 15 \times 0 + 25 \times 1 \qquad (\because \cos 90° = 0, \sin 90° = 1)$$

$$= 50 + 0 + 25 = \mathbf{75\ N/mm^2}. \quad \textbf{Ans.}$$

Tangential stress is given by equation (3.13)

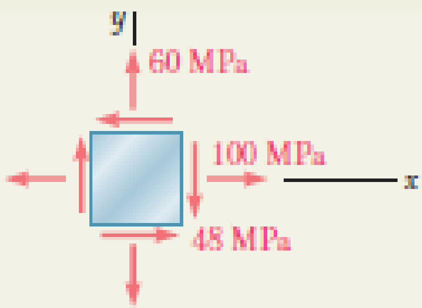
$\therefore$ Using equation (3.13),

$$\sigma_t = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta$$

$$= \frac{65 - 35}{2} \sin (2 \times 45) - 25 \cos (2 \times 45)$$

$$= 15 \sin 90° - 25 \cos 90° = 15 \times 1 - 25 \times 0 = 15 - 0$$

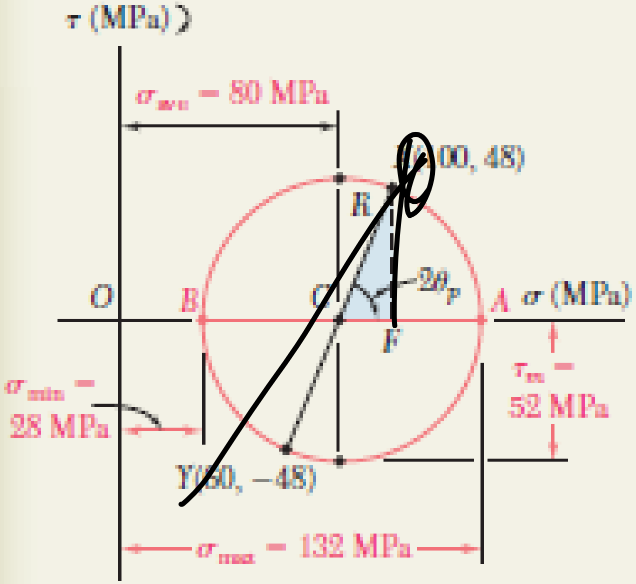
$$= \mathbf{15\ N/mm^2}. \quad \textbf{Ans.}$$



SAMPLE PROBLEM 7.2

For the state of plane stress shown, determine (*a*) the principal planes and the principal stresses, (*b*) the stress components exerted on the element obtained by rotating the given element counterclockwise through 30°.

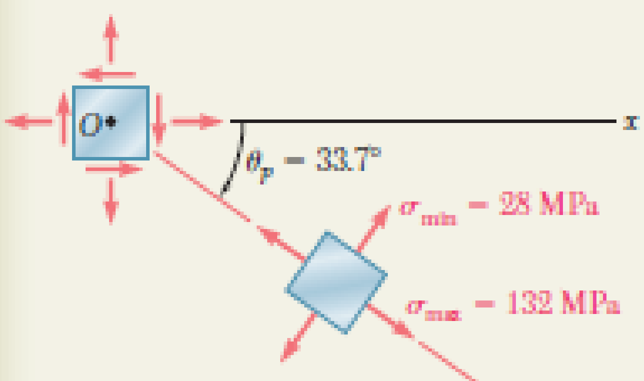
σ_{max} σ_{min} θ



SOLUTION

Construction of Mohr's Circle. We note that on a face perpendicular to the x axis, the normal stress is tensile and the shearing stress tends to rotate the element clockwise; thus, we plot X at a point 100 units to the right of the vertical axis and 48 units above the horizontal axis. In a similar fashion, we examine the stress components on the upper face and plot point $Y(60, -48)$. Joining points X and Y by a straight line, we define the center C of Mohr's circle. The abscissa of C, which represents σ_{ave}, and the radius R of the circle can be measured directly or calculated as follows:

$$\sigma_{ave} = OC = \tfrac{1}{2}(\sigma_x + \sigma_y) = \tfrac{1}{2}(100 + 60) = 80 \text{ MPa}$$
$$R = \sqrt{(CF)^2 + (FX)^2} = \sqrt{(20)^2 + (48)^2} = 52 \text{ MPa}$$

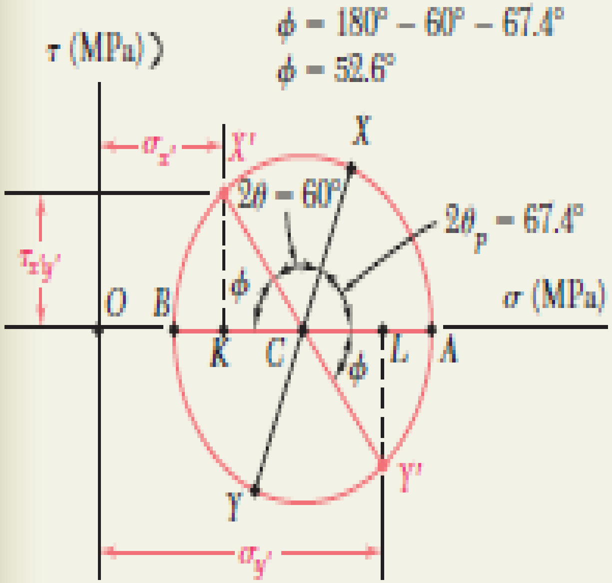


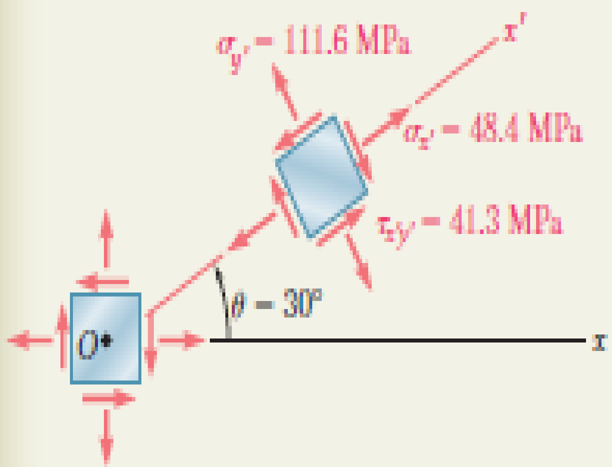
a. Principal Planes and Principal Stresses. We rotate the diameter XY clockwise through $2\theta_p$ until it coincides with the diameter AB. We have

$$\tan 2\theta_p = \frac{XF}{CF} = \frac{48}{20} = 2.4 \quad 2\theta_p = 67.4° \curvearrowright \quad \theta_p = 33.7° \curvearrowright \blacktriangleleft$$

The principal stresses are represented by the abscissas of points A and B:

$$\sigma_{max} = OA = OC + CA = 80 + 52 \qquad \sigma_{max} = +132 \text{ MPa} \blacktriangleleft$$





Since the rotation that brings XY into AB is clockwise, the rotation that brings Ox into the axis Oa corresponding to σ_{max} is also clockwise; we obtain the orientation shown for the principal planes.

b. Stress Components on Element Rotated 30°↺. Points X' and Y' on Mohr's circle that correspond to the stress components on the rotated element are obtained by rotating XY counterclockwise through $2\theta = 60°$. We find

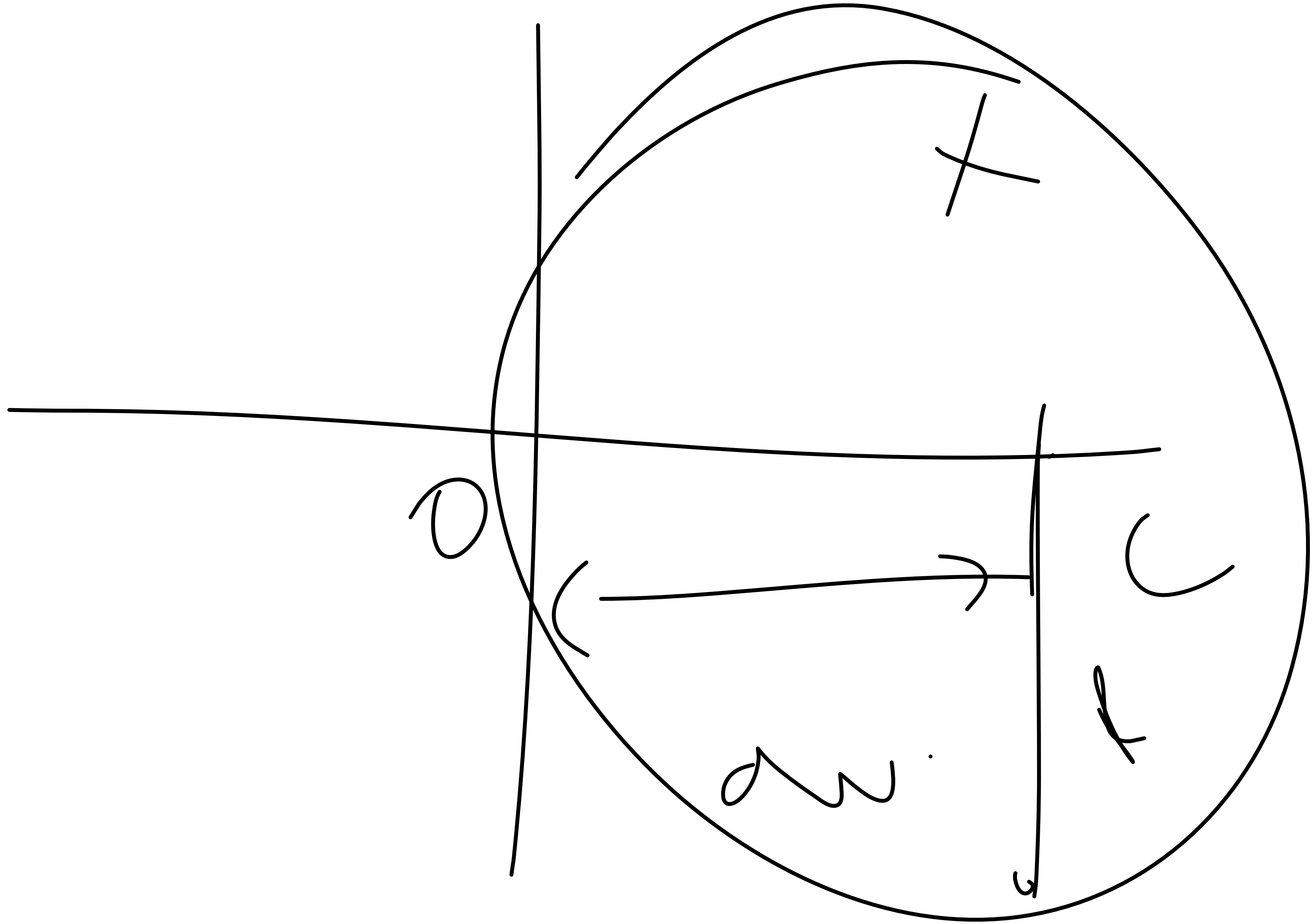
$$\phi = 180° - 60° - 67.4° \qquad \phi = 52.6° \blacktriangleleft$$

$$\sigma_{x'} = OK = OC - KC = 80 - 52 \cos 52.6° \qquad \sigma_{x'} = +\ 48.4 \text{ MPa} \blacktriangleleft$$

$$\sigma_{y'} = OL = OC + CL = 80 + 52 \cos 52.6° \qquad \sigma_{y'} = +111.6 \text{ MPa} \blacktriangleleft$$

$$\tau_{x'y'} = KX' = 52 \sin 52.6° \qquad \tau_{x'y'} = \quad 41.3 \text{ MPa} \blacktriangleleft$$

Since X' is located above the horizontal axis, the shearing stress on the face perpendicular to Ox' tends to rotate the element clockwise.


x
O
c
dw
y

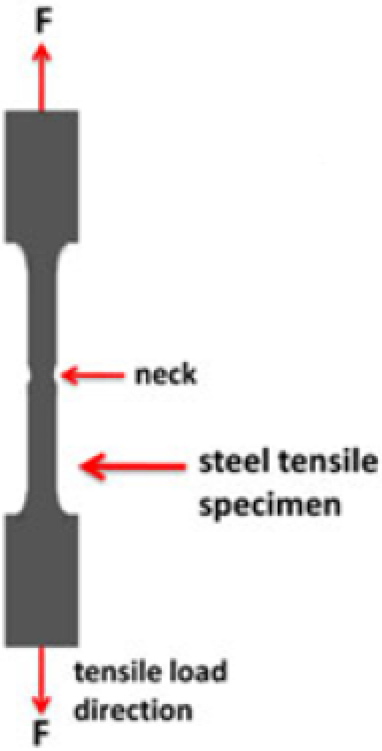
THEORY OF FAILURES

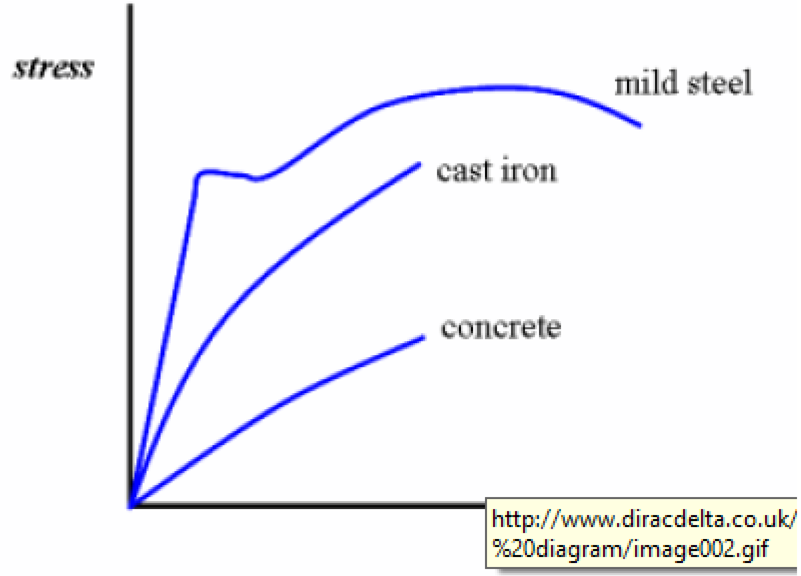
Failure Theories-Importance

- Theories of elastic failure provide a relationship between the strength of the machine component subjected to complex state of stresses with the mechanical properties obtained in tension test.

- With the help of these theories, the data obtained in the tension test can be used to determine the dimensions of the component, irrespective of the nature of stresses induced in the component due to complex loads.

Simple Tensile Test





- The loading is in one direction, the stress induced is tensile
- The maximum principal stress is σ_1
- Failure point for ductile material is σ_{yt}

$$\therefore \sigma_1 = \sigma_{yt}$$

- Failure point for brittle material is σ_{ut}

$$\therefore \sigma_1 = \sigma_{ut}$$

Maximum principal stress theory (Rankine's theory)

Failure occurs whenever one of the maximum principal stresses equals or exceeds the strength.

For the actual 3D situation, $\sigma_1 > \sigma_2 > \sigma_3$

σ_1 = maximum principal stress

In simple tensile test, failure point for brittle material is σ_{ut}

Therefore, to avoid failure,

$$\sigma_1 = \sigma_{ut}$$

When factor of safety is considered,

$$\sigma_1 = \frac{\sigma_{ut}}{fos}$$

It is suitable for brittle material

Maximum shear stress theory (Coulomb, Tresca and Guest's theory)

This theory predicts that yielding begins whenever the maximum shear stress in any element equals or exceeds the maximum shear stress in a tension-test specimen of the same material when that specimen begins to yield.

For the actual 3D situation, if $\sigma_1 > \sigma_2 > \sigma_3$

$$\text{Maximum shear stress}, \tau_{max} = \frac{\sigma_1 - \sigma_3}{2}$$

In simple tensile test,

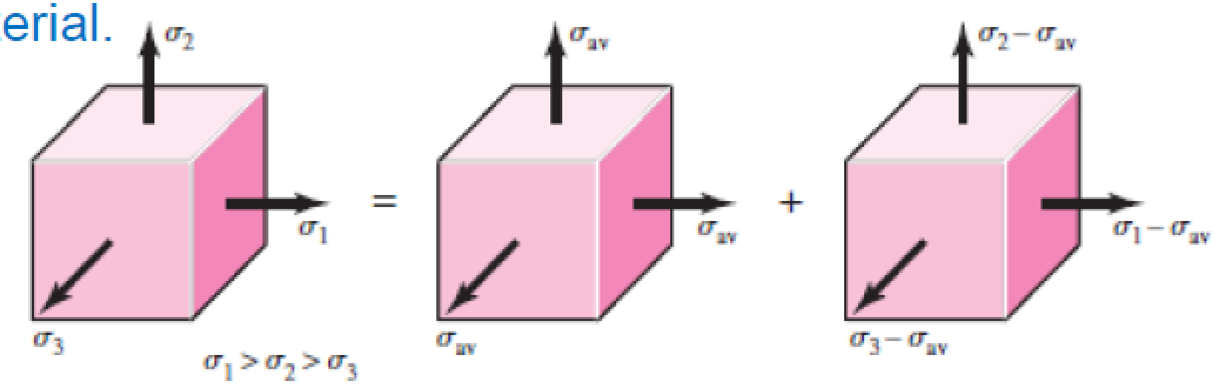
$$\text{Maximum shear stress}, \tau_{max} = \frac{\sigma_{yt}}{2}$$

$$\therefore \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_{yt}}{2}$$

$$\therefore \sigma_1 - \sigma_3 = \sigma_{yt} \ or \ \ \sigma_1 - \sigma_3 = \frac{\sigma_{yt}}{fos}$$

Distortion energy theory (Huber von Mises and Henky's theory)

This theory predicts that yielding occurs when the distortion strain energy per unit volume reaches or exceeds the distortion strain energy per unit volume for yield in simple tension or compression of the same material.



(a) Triaxial stresses (b) Hydrostatic component (c) Distortional component

- Element with triaxial stresses will undergo both volume change and angular distortion.
- Strain energy/unit volume,

$$u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

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- Strain energy for producing volume change,

$$u_v = \frac{3\sigma_{av}^2}{2E}(1-2\nu)$$

$$\boxed{\sigma_{av} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}}$$

$$u_v = \frac{1-2\nu}{6E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_2\sigma_3 + 2\sigma_3\sigma_1\right)$$

- Distortion energy obtained by,

$$u_d = u - u_v = \frac{1+\nu}{3E}\left[\frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}{2}\right]$$

- For the simple tensile test, $\sigma_1 = \sigma_{yt}$ and $\sigma_2 = \sigma_3 = 0$

$$\therefore u_d = \frac{1+\nu}{3E}{\sigma_{yt}}^2$$

$$\left[\frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}{2}\right]^{1/2} = \frac{\sigma_{yt}}{FOS}$$

4.12 Working Stress

When designing machine parts, it is desirable to keep the stress lower than the maximum or ultimate stress at which failure of the material takes place. This stress is known as the ***working stress*** or ***design stress.*** It is also known as ***safe*** or ***allowable stress.***

Note : By failure it is not meant actual breaking of the material. Some machine parts are said to fail when they have plastic deformation set in them, and they no more perform their function satisfactory.

4.13 Factor of Safety

It is defined, in general, as the **ratio of the maximum stress to the working stress.** Mathematically,

$$\text{Factor of safety} = \frac{\text{Maximum stress}}{\text{Working or design stress}}$$

In case of ductile materials *e.g.* mild steel, where the yield point is clearly defined, the factor of safety is based upon the yield point stress. In such cases,

$$\text{Factor of safety} = \frac{\text{Yield point stress}}{\text{Working or design stress}}$$

In case of brittle materials *e.g.* cast iron, the yield point is not well defined as for ductile materials. Therefore, the factor of safety for brittle materials is based on ultimate stress.

$$\therefore \qquad \text{Factor of safety} = \frac{\text{Ultimate stress}}{\text{Working or design stress}}$$

This relation may also be used for ductile materials.

4.14 Selection of Factor of Safety

The selection of a proper factor of safety to be used in designing any machine component depends upon a number of considerations, such as the material, mode of manufacture, type of stress, general service conditions and shape of the parts. Before selecting a proper factor of safety, a design engineer should consider the following points :

1. The reliability of the properties of the material and change of these properties during service ;
2. The reliability of test results and accuracy of application of these results to actual machine parts ;
3. The reliability of applied load ;
4. The certainty as to exact mode of failure ;
5. The extent of simplifying assumptions ;
6. The extent of localised stresses ;
7. The extent of initial stresses set up during manufacture ;
8. The extent of loss of life if failure occurs ; and
9. The extent of loss of property if failure occurs.

Each of the above factors must be carefully considered and evaluated. The high factor of safety results in unnecessary risk of failure. The values of factor of safety based on ultimate strength for different materials and type of load are given in the following table:

Table 4.3. Values of factor of safety.

Material	*Steady load*	*Live load*	*Shock load*
Cast iron	5 to 6	8 to 12	16 to 20
Wrought iron	4	7	10 to 15
Steel	4	8	12 to 16
Soft materials and alloys	6	9	15
Leather	9	12	15
Timber	7	10 to 15	20