# MEE 2002: Strength of Materials

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# Syllabus

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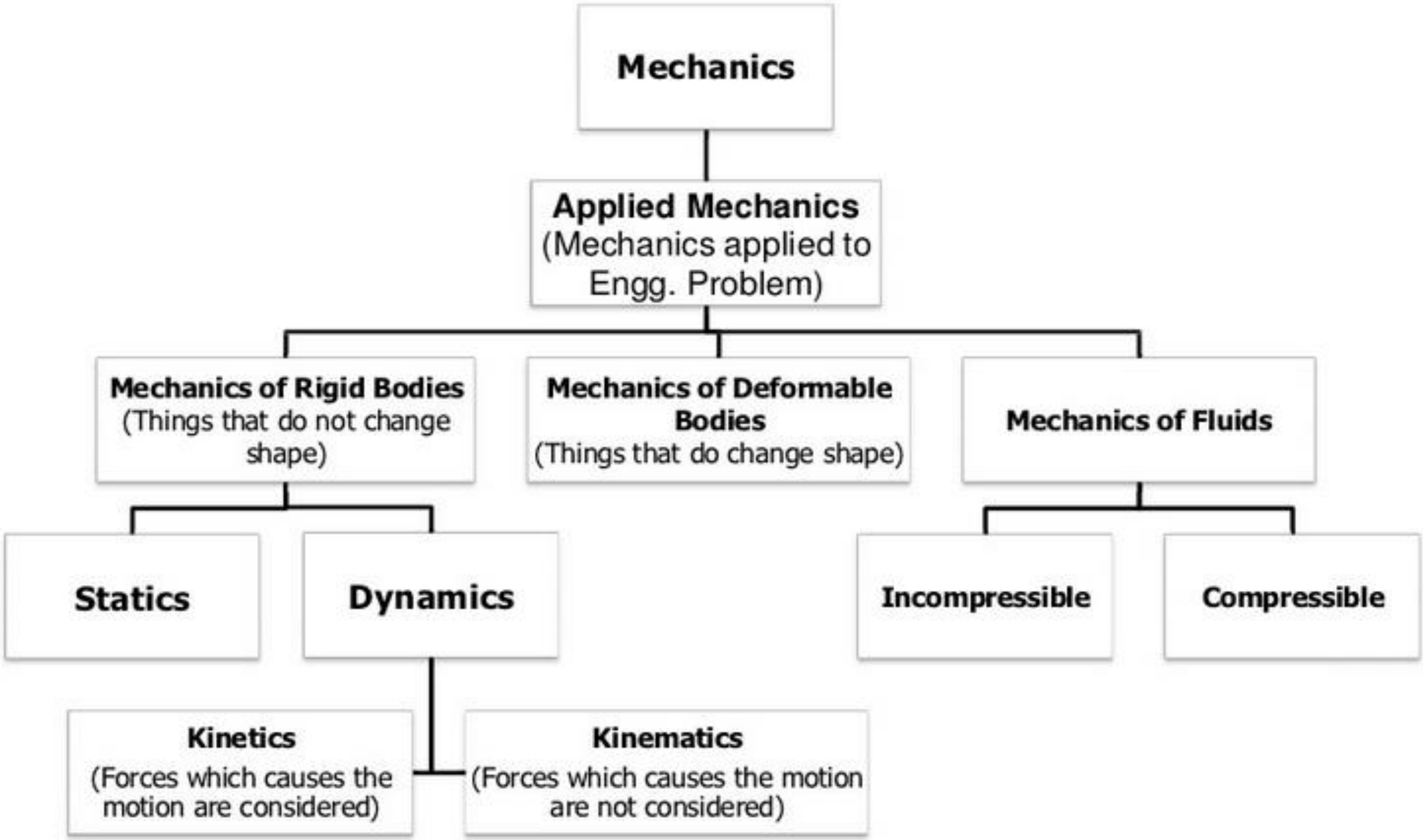
| **Simple Stresses and strains:** Definition/derivation of normal stress, shear stress and normal strain and shear strain – Stre ss-strain diagram for brittle and ductile materials - Poisson's ratio & volumetric strain – Elastic constants – relationship between elastic constants and Poisson's ratio – Generalised Hook's law – Deformation of simple and compound bars – Strain energy – Resilience – Gradual, sudden, impact and shock loadings – thermal stresses. |
| **Bi-axial Stress system:** Introduction – Stresses on an inclined section of a bar under axial loading – compound stresses – Normal and tangential stresses on an inclined plane for biaxial stresses – Two perpendicular normal stresses accompanied by a state of simple shear – Mohr's circle of stresses – Principal stresses and strains – Analytical and graphical solutions, Theories of Failure. |
| **Shear Force and Bending Moment:** Definition of beam – Types of beams – Concept of shear force and bending moment – S.F and B.M diagrams for cantilever, simply supported and overhanging beams subjected to point loads, uniformly distributed loads, uniformly varying loads and combination of these loads – Point of contra flexure – Relation between S.F., B.M and rate of loading at a section of a beam. |
| **Stresses in beams:** Theory of simple bending – Assumptions – Derivation of bending equation - Neutral axis – Determination of bending stresses – section modulus of rectangular and circular sections (Solid and Hollow), I, T, Angle and Channel sections – Design of simple beam sections, Shear Stresses: Derivation of formula – Shear stress distribution across various beams sections like rectangular, circular, triangular, I, T angle sections. |
| **Deflection of beams:** Deflection of beams by Double integration method – Macaulay's method – Area moment theorems for computation of slopes and deflections in beams – Conjugate beam method. |
| **Torsion:** Introduction to Torsion – derivation of shear strain – Torsion formula – stresses and deformations in circular and hollow shafts – Stepped shafts – shafts fixed at the both ends – Design of shafts according to theories of failure, Stresses in helical springs. |
| **Columns:** Theory of columns – Long column and short column - Euler's formula – Rankine's formula.<br>**Thin and thick cylinders:** Thin cylinders and shells – deformation of thin cylinders and shells – thick cylindrical shell – Lame's equation. |

# Unit-1
# Stresses and Strains

Definition/derivation of normal stress, shear stress, and normal strain and shear strain –Stress-strain diagram- Elastic constants – Poisson's ratio – relationship between elastic constants and Poisson's ratio – Generalised Hook's law–Strain energy– Deformation of simple and compound bars – thermal stresses.

# Structure of Mechanics



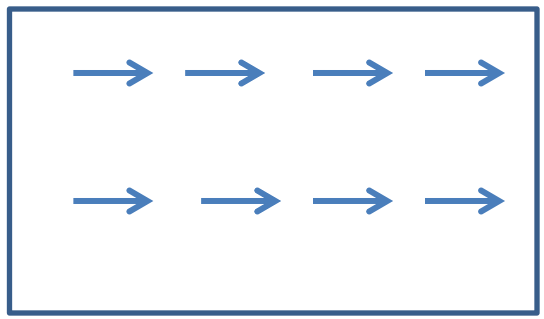
# <u>Solid mechanics</u>

**Mechanics of rigid bodies**

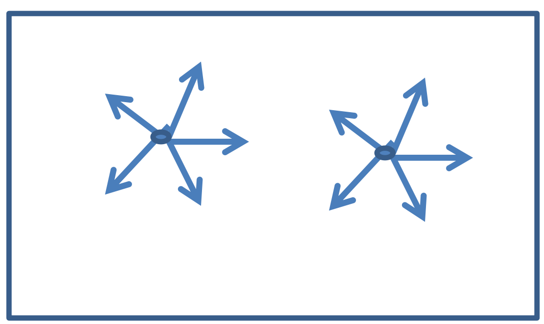
**Mechanics of deformable bodies**
**or**
**Strength of Materials**

# Assumptions

- **Material composition is continuous and homogenous**

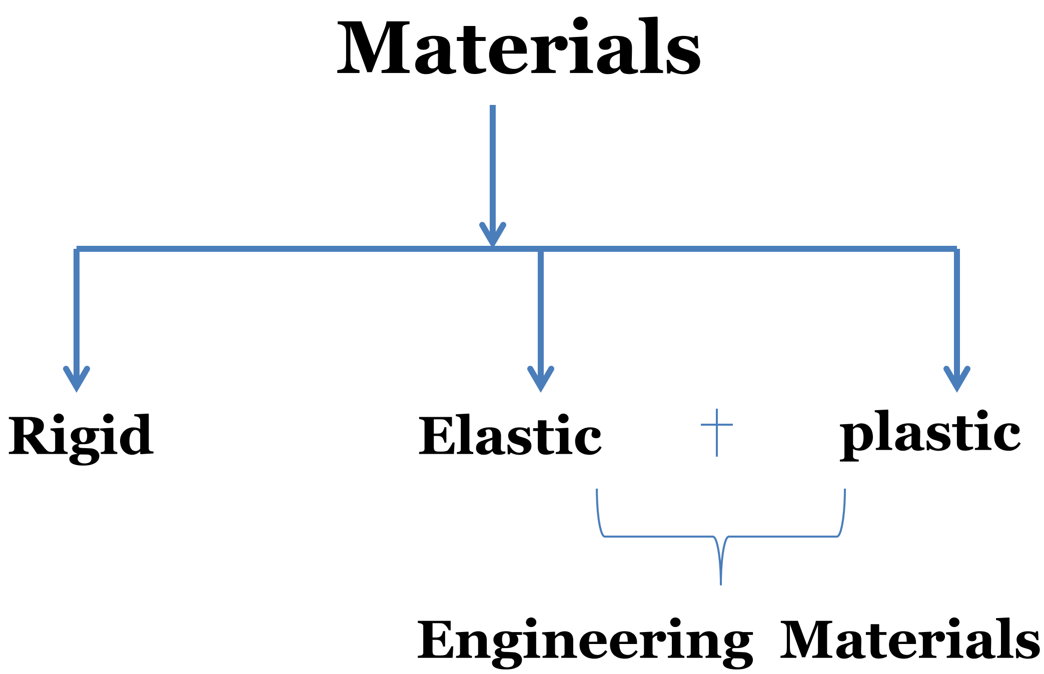


- **Material is isotropic**



"**Isotropic**" means having identical values of a property in all directions.

# Classifications Of Materials



# INTRODUCTION TO STRENGTH OF MATERIALS

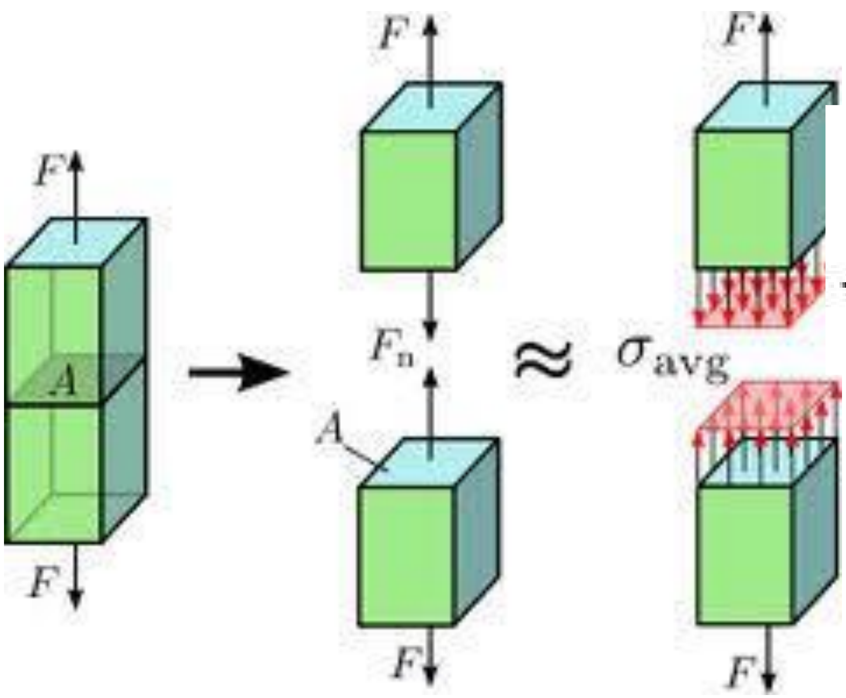
Study of internal effects (stresses and strains) caused by external loads (forces and moments) acting on a deformable body/structure.

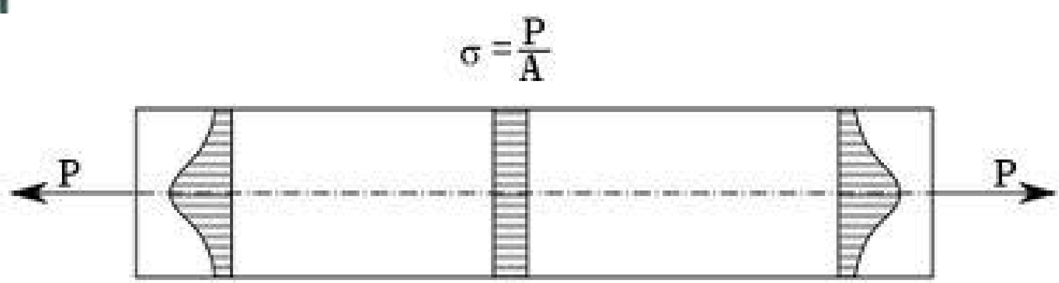
Determines:

- **Strength** (determine by stress at failure)
- **Deformation** (determined by strain)
- **Stiffness** (ability to resist deformation; load needed to cause a specific deformation; determined by the stress- strain relationship)
- **Stability** (ability to avoid rapidly growing deformations caused by an initial disturbance; e.g., buckling)

## NORMAL STRESS

It is the internal resistance offered by a unit area of the material, which is subjected to external loading




**Direct Stress** = $\frac{\textbf{Applied Force (P)}}{\textbf{Cross Sectional Area (A)}}$

**Units:** Usually N/m² (Pa), N/mm², MN/m², GN/m² or N/cm²

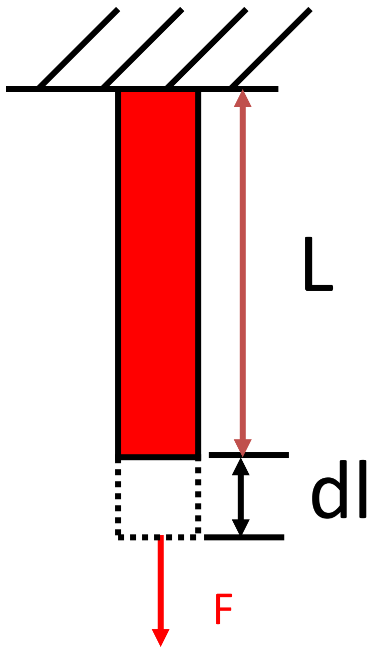
**Note:** 1 N/mm² = 1 MN/m² = 1 MPa

# Normal Stress Contd.

Direct stress may be tensile, $\sigma_t$ or compressive, $\sigma_c$ and result from forces acting perpendicular to the plane of the cross-section


# Normal Strain

Strain is defined as deformation per unit length or Change in dimension of an object under application of external force




Consider a bar, subjected to axial tensile loading force, F. If the bar extension is dl and its original length (before loading) is L, then tensile strain is:

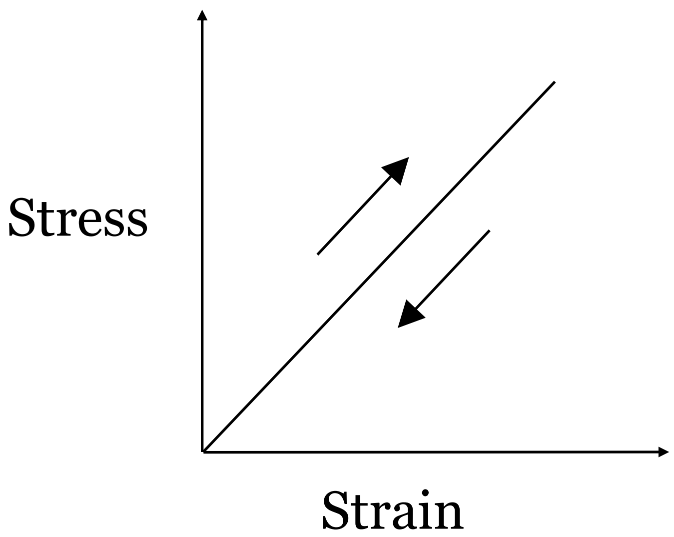
$$\textbf{Normal Strain } (\varepsilon) = \frac{\textbf{Change in Length}}{\textbf{Original Length}}$$

**i.e.** $\varepsilon = dl/L$

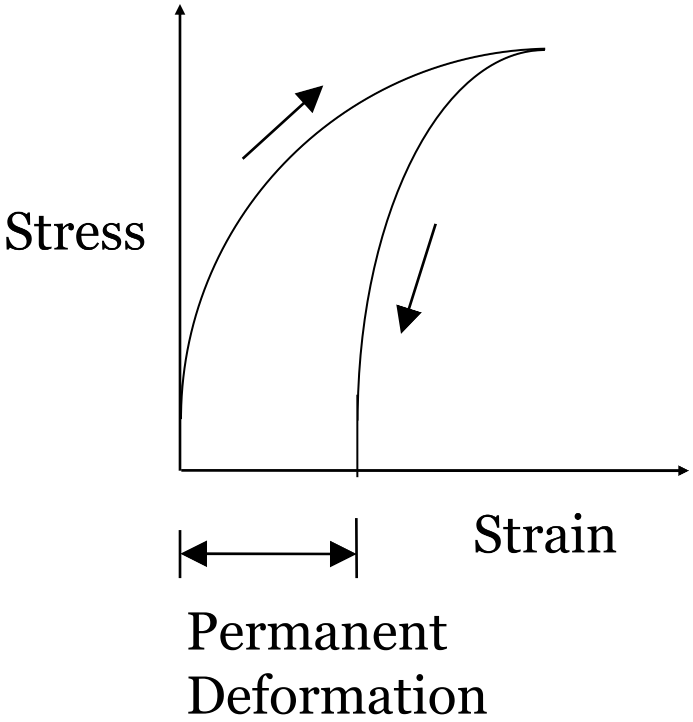
- As strain is a ratio of lengths, it is dimensionless.
- Similarly, for compression by amount, dl:

  Compressive strain = - dl/L
- **Note:** Strain is positive for an increase in dimension and negative for a reduction in dimension.

# Elastic and Plastic deformation




Elastic deformation




Plastic deformation

# Hooke's Law

- States that providing the limit of proportionality of a material is not exceeded, the stress is directly proportional to the strain produced.

  By Hooke's Law

  $\sigma \propto \in$

  $\sigma = E \in$; where E=Young's modulus

- If a graph of stress and strain is plotted as load is gradually applied, the first portion of the graph will be a straight line.

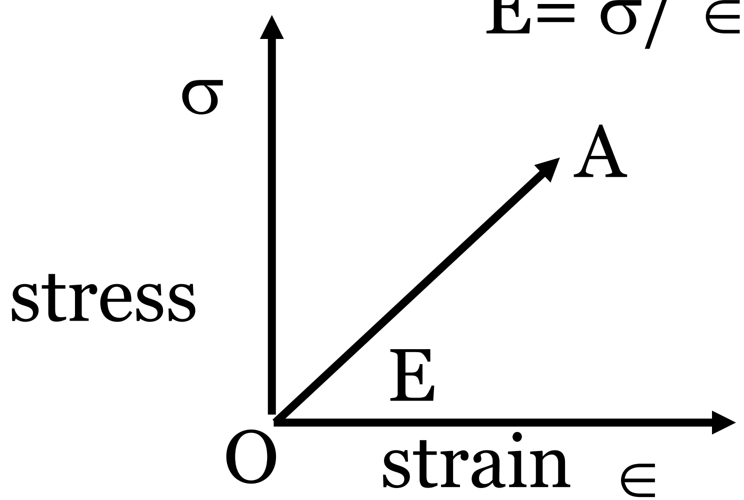
- The slope of this line is the constant of proportionality called modulus of Elasticity, E or Young's Modulus.

# Modulus of Elasticity

Young's modulus is also called the modulus of elasticity or stiffness and is a measure of how much strain occurs due to a given stress. Because strain is dimensionless Young's modulus has the units of stress or pressure

$$\text{Youngs Modulus} \quad E = \frac{\text{Stress}}{\text{Strain}} = \frac{W}{x} \times \frac{L}{A}$$

$$E = \sigma / \in$$




Value of E is same in Tension & Compression.

Example 1:A short hollow, cast iron cylinder with wall thickness of 10 mm is to carry compressive load of 100 KN. Compute the required outside diameter `D',if the working stress in compression is 80 N/mm².

## Solution:

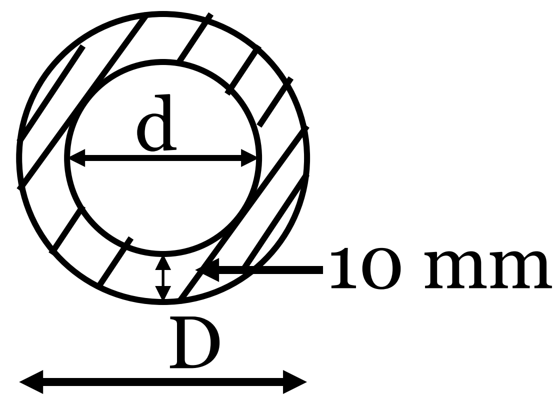
$\sigma = 80 N/mm^2$;

$P = 100\ kN = 100*10^3\ N$

$A = (\pi/4) * \{D^2 - (D-20)^2\}$

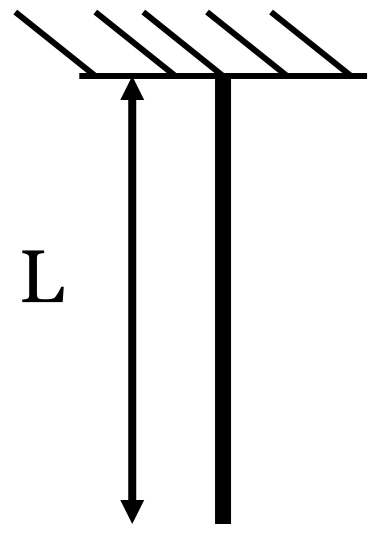
as $\sigma = P/A$

substituting in above eq. and solving. D = 49.8 mm



Example 2: A Steel wire hangs vertically under its weight. What is the greatest length it can have if the allowable tensile stress $\sigma_t$ =200MPa? Weight Density of steel $\gamma$=80 kN/m$^3$.

## Solution:




$\sigma_t$ =200 MPa= 200*10$^3$ kN/m$^2$ ;

$\gamma$=80 kN/m$^3$.

Wt. of wire $P=(\pi/4)*D^2*L*\gamma$

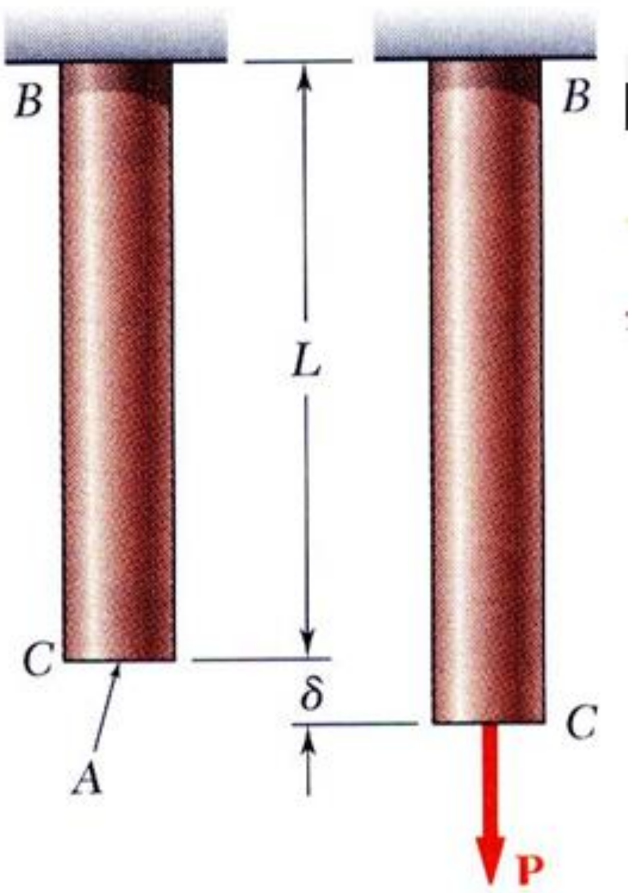
c/s area of wire $A=(\pi/4)*D^2$

$\sigma_t$ = P/A

solving above eqn. We get

L =2500m

# Deformation Under Axial Loading




- From Hooke's Law:

$$\sigma = E\varepsilon \qquad \varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

- From the definition of strain:

$$\varepsilon = \frac{\delta}{L}$$

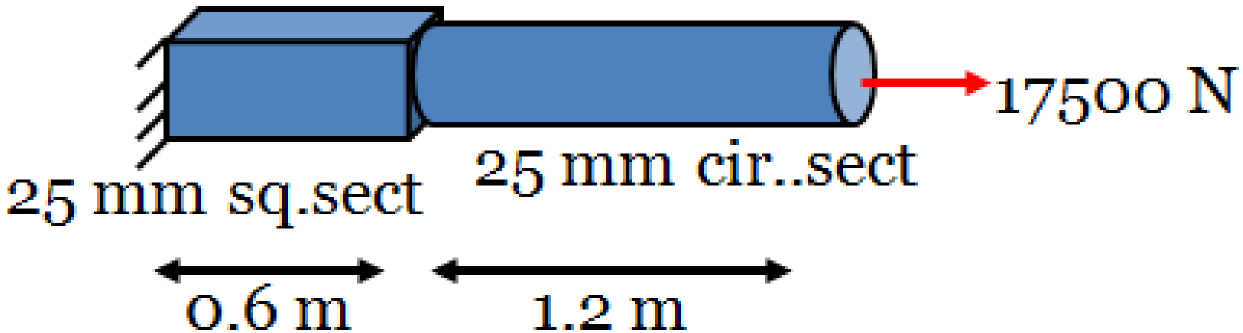
- Equating and solving for the deformation,

$$\delta = \frac{PL}{AE}$$

- With variations in loading, cross-section or material properties,

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i}$$

Example: An aluminium bar 1.8 meters long has a 25 mm square c/s over 0.6 meters of its length and 25 mm circular c/s over other 1.2 meters . How much will the bar elongate under a tensile load P=17500 N, if E = 75000 Mpa.




Solution :- $\delta = \sum PL/AE$

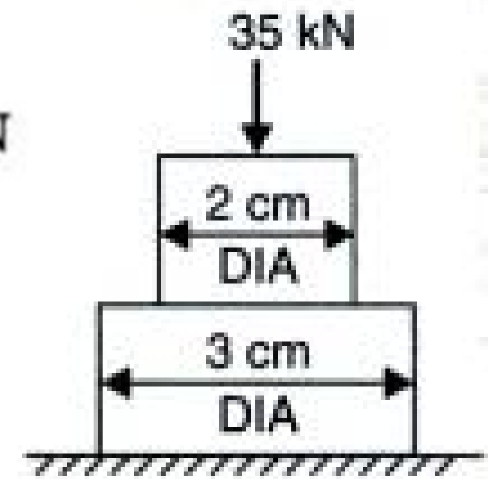
$=17500*600 / (25^2*75000) +$
$17500*1200/(0.785*25^2*75000) = 0.794$ mm

*A stepped bar shown in Fig. is subjected to an axially applied compressive load of 35 kN. Find the maximum and minimum stresses produced.*

**Sol.** Given :

Axial load, $P = 35 \text{ kN} = 35 \times 10^3 \text{ N}$

Dia. of upper part, $D_1 = 2 \text{ cm} = 20 \text{ mm}$




$\therefore$ Area of upper part, $A_1 = \frac{\pi}{4}(20^2) = 100\,\pi \text{ mm}^2$

Area of lower part, $A_2 = \frac{\pi}{4} D_2^2 = \frac{\pi}{4}(30^2) = 225\,\pi \text{ mm}^2$

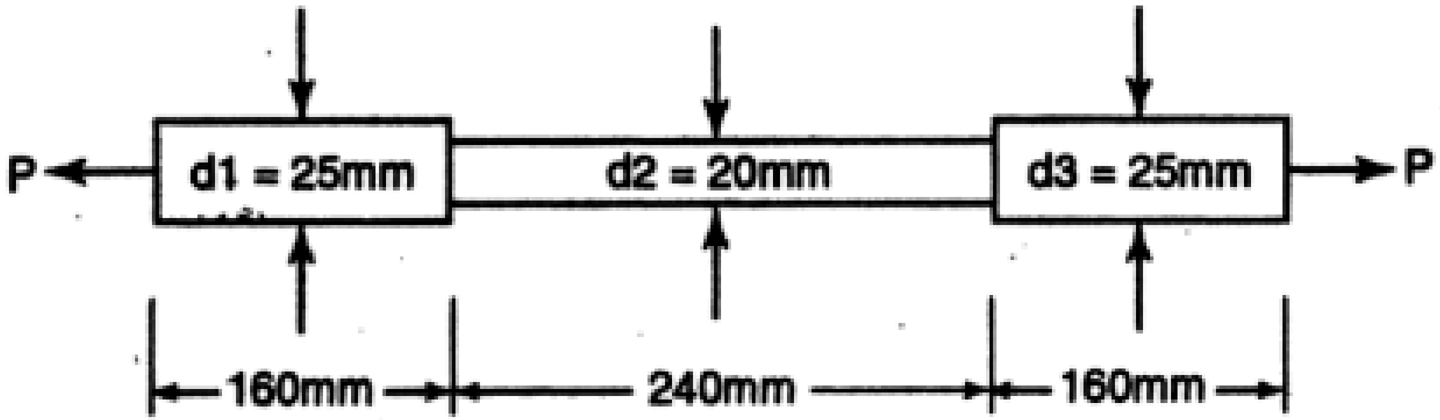
The stress is equal to load divided by area. Hence stress will be maximum where area is minimum. Hence stress will be maximum in upper part and minimum in lower part.

$\therefore$ Maximum stress $= \frac{\text{Load}}{A_1} = \frac{35 \times 10^3}{100 \times \pi} = \mathbf{111.408\ N/mm^2}$. **Ans.**

Minimum stress $= \frac{\text{Load}}{A_2} = \frac{35 \times 10^3}{225 \times \pi} = \mathbf{49.5146\ N/mm^2}$. **Ans.**

The bar shown in Fig. is tested in universal testing machine. It is observed that at a load of 40 kN the total extension of the bar is 0.285 mm.

Determine the Young's Modulus of the material.




Extension of portion 1, $\frac{PL_1}{A_1E} = \frac{40\times10^3\times160}{\frac{\pi}{4}\times25^2\,E}$

Extension of portion 2, $\frac{PL_2}{A_2E} = \frac{40\times10^3\times240}{\frac{\pi}{4}\times20^2\,E}$

Extension of portion 3, $\frac{PL_3}{A_3E} = \frac{40\times10^3\times160}{\frac{\pi}{4}\times25^2\,E}$

$$\text{Total extension} = \frac{40\times10^3}{E}\times\frac{4}{\pi}\left\{\frac{160}{625}+\frac{240}{400}+\frac{160}{625}\right\}$$

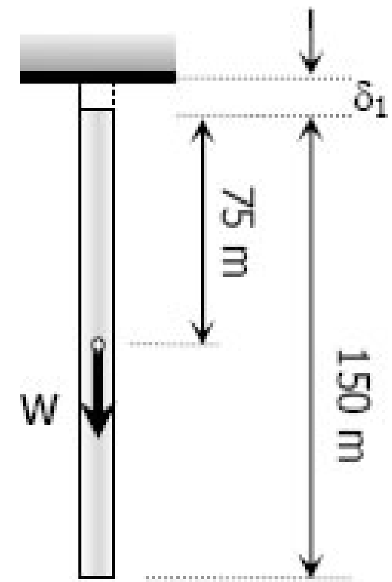
$$0.285 = \frac{40\times10^3}{E}\times\frac{4}{\pi}\times\frac{1.112}{E}$$

$$E = 198714.72\ \text{N/mm}^2 \quad \textbf{(Ans)}$$

A steel rod having a cross-sectional area of 300 mm$^2$ and a length of 150 m is suspended vertically from one end. It supports a tensile load of 20 kN at the lower end. If the unit mass of steel is 7850 kg/m$^3$ and E = 200 × 10$^3$ MN/m$^2$, find the total elongation of the rod.

**Elongation due to its own weight:**

$$\delta_1 = \frac{PL}{AE}$$




Where:

P = W = $7850(1/1000)^3(9.81)[300(150)(1000)]$

P = 3465.3825 N

L = 75(1000) = 75 000 mm

A = 300 mm$^2$

E = 200 000 MPa

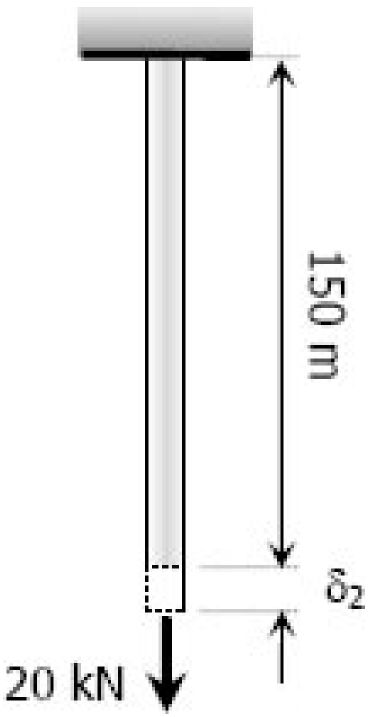
Thus,

$$\delta_1 = \frac{3\,465.3825(75\,000)}{300(200\,000)}$$

$$\delta_1 = 4.33 \text{ mm}$$

**Elongation due to applied load:**

$$\delta_2 = \frac{PL}{AE}$$




Where:

P = 20 kN = 20 000 N

L = 150 m = 150 000 mm

A = 300 $mm^2$

E = 200 000 MPa

Thus,

$$\delta_2 = \frac{20\,000(150\,000)}{300(200\,000)}$$

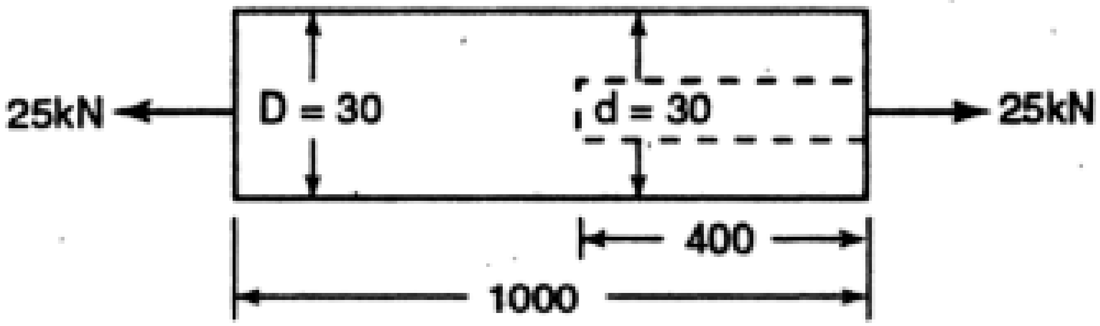
$$\delta_2 = 50 \text{ mm}$$

**Total elongation:**

$$\delta = \delta_1 + \delta_2$$

$$\delta = 4.33 + 50 = 54.33 \text{ mm}$$

**Example** A bar of length 1000 mm and diameter 30 mm is centrally bored for 400 mm, the bore diameter being 10 mm as shown in Fig.. Under a load of 25 kN, if the extension of the bar is 0.185 mm, what is the modulus of elasticity of the bar?




$$L_1 = 1000 - 400 = 600\,\text{mm}$$

$$L_2 = 400\,\text{mm}$$

$$A_1 = \frac{\pi}{4} \times 30^2 = 225\pi \qquad \Delta_1 = \frac{PL_1}{A_1 E}$$

$$A_2 = \frac{\pi}{4} \times (30^2 - 10^2) = 200\pi \qquad \Delta_2 = \frac{PL_2}{A_2 E}$$

$$\therefore \quad \Delta = \Delta_1 + \Delta_2 = \frac{P}{E}\left(\frac{L_1}{A_1} + \frac{L_2}{A_2}\right)$$

i.e.
$$0.185 = \frac{25 \times 10^3}{E}\left(\frac{600}{225\pi} + \frac{400}{200\pi}\right)$$

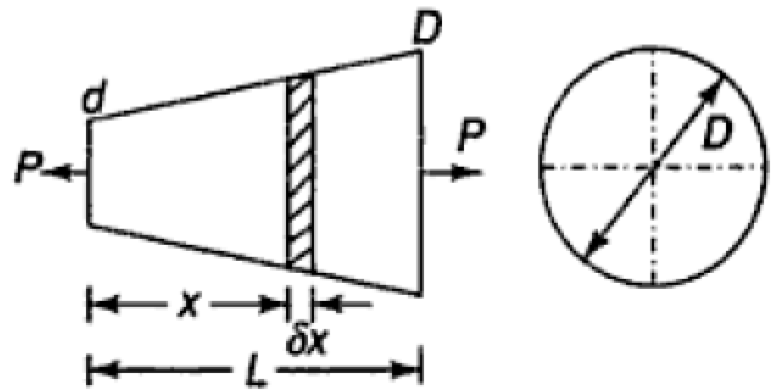
$$\therefore \quad E = 200736 \ \text{N/mm}^2 \quad \textbf{(Ans)}$$

# Analysis of Uniformly Tapering Circular Rod

Bars of tapering section can be of conical section or of trapezoidal section with uniform thickness.

## Conical Section

Consider a bar of conical section under the action of axial force $P$ as shown in Fig. 1.8.




Let $D$ = diameter at the larger end

$d$ = diameter at the smaller end

$L$ = length of the bar

$E$ = Young's modulus of the bar material

Consider a very small length $\delta x$ at a distance $x$ from the small end.

The diameter at a distance $x$ from the small end $= d + \dfrac{D-d}{L}.x$

The extension of a small length

$$= \frac{P.\delta x}{\frac{\pi}{4}\left(d + \frac{D-d}{L}x\right)^2 .E} \qquad \cdots\cdots \left(\Delta = \frac{PL}{AE}\right)$$

Extension of the whole rod $= \int_0^L \frac{4P}{\pi\left(d \;+ (D-d)x/L\right)^2 .E}.dx$

$$= \frac{4P}{\pi E}\int_0^L \left(d + \frac{D-d}{L}.x\right)^{-2}.dx = -\frac{4P}{\pi E}.\frac{L}{(D-d)}\left(\frac{1}{\left(d \;+(D-d)x/L\right)}\right)_0^L$$

$$= \frac{4PL}{\pi E(D-d)}\left(\frac{1}{d} - \frac{1}{D}\right) = \frac{4PL}{\pi E(D-d)}\left(\frac{D-d}{dD}\right) = \frac{4PL}{\pi EdD}$$

**Problem 1.** *Find the modulus of elasticity for a rod, which tapers uniformly from 30 mm to 15 mm diameter in a length of 350 mm. The rod is subjected to an axial load of 5.5 kN and extension of the rod is 0.025 mm.*

**Sol.** Given :

| | |
|---|---|
| Larger diameter, | $D_1 = 30$ mm |
| Smaller diameter, | $D_2 = 15$ mm |
| Length of rod, | $L = 350$ mm |
| Axial load, | $P = 5.5$ kN $= 5500$ N |
| Extension, | $dL = 0.025$ mm |

Using equation (1.10), we get

$$dL = \frac{4PL}{\pi E\, D_1 D_2}$$

or

$$E = \frac{4PL}{\pi\, D_1 D_2\, dL} = \frac{4 \times 5500 \times 350}{\pi \times 30 \times 15 \times 0.025}$$

$$= \mathbf{217865\ N/mm^2 \text{ or } 2.17865 \times 10^5\ N/mm^2}. \quad \text{Ans.}$$

**Example** A brass bar uniformly tapered from diameter 20 mm at one end to diameter 10 mm at the other end over an axial length 300 mm is subjected to an axial compressive load of 7.5 kN. If $E = 100$ kN/mm$^2$ for brass, determine
(a) the maximum and minimum axial stresses in bar and
(b) the total change in length of the bar.

**Solution**

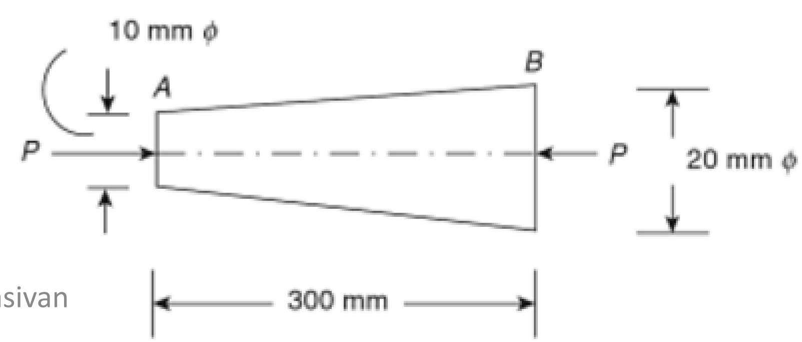
Axial load, $P = 7.5$ kN. Maximum stress occurs at end $A$ with minimum cross-sectional area:

$$\text{(a)}\quad \sigma_{max} = -\frac{4P}{\pi(10)^2} = -\frac{4\times 7.5\times 1{,}000}{\pi\times 100} = -95.94\,\text{N/mm}^2 \quad \text{at end } A$$

$$\text{(b)}\quad \sigma_{min} = -\frac{4P}{\pi(20)^2} = -\frac{4\times 7.5\times 1{,}000}{\pi\times 20} = -23.87\,\text{N/mm}^2 \quad \text{at end } B$$

$$\delta L = -\frac{4PL}{\pi E d_1 d_2}, \text{ as the load is compressive}$$

$$= -\frac{4\times 7.5\times 1{,}000\times 300}{\pi\times 100\times 1{,}000\times 10\times 20} = -0.143\text{ mm}$$

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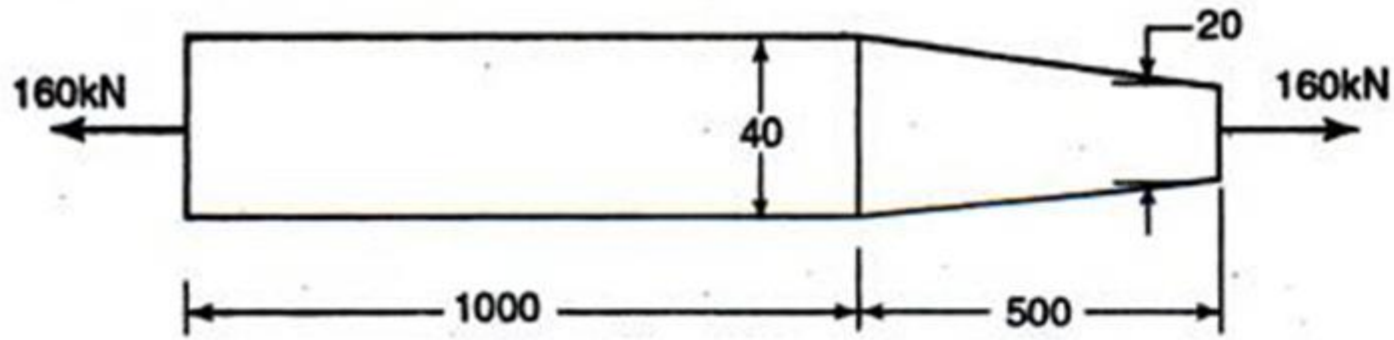
**Example** A 1.5 m long steel bar is having uniform diameter of 40 mm for a length of 1 m and in the next 0.5 m its diameter gradually reduces from 40 mm to 20 mm as shown in Fig. Determine the elongation of this bar when subjected to an axial tensile load of 160 kN. Given $E = 200$ GN/m$^2$.




***Solution***

Now,

$$P = 160 \times 10^3 \text{ N}$$

$$E = 200\, GN/m^2 = \frac{200 \times 10^9}{(1000)^2} \text{ N/mm}^2$$

$$= 200 \times 1000 \text{ N/mm}^2$$

$$= 2 \times 10^5 \text{ N/mm}^2$$

Extensions of uniform portion and tapering portion are worked out separately and then added to get extension of the given bar.

$$\text{Extension of uniform portion } \Delta_1 = \frac{PL}{AE}$$

$$= \frac{160 \times 10^3 \times 1000}{\frac{\pi}{4} \times 40^2 \times 2 \times 10^5} = 0.6366\,\text{mm.}$$

$$\text{Extension of tapering portion } \Delta_2 = \frac{4PL}{E\pi d_1 d_2}$$

$$= \frac{4 \times 160 \times 10^3 \times 500}{2 \times 10^5 \times \pi \times 20 \times 40}$$

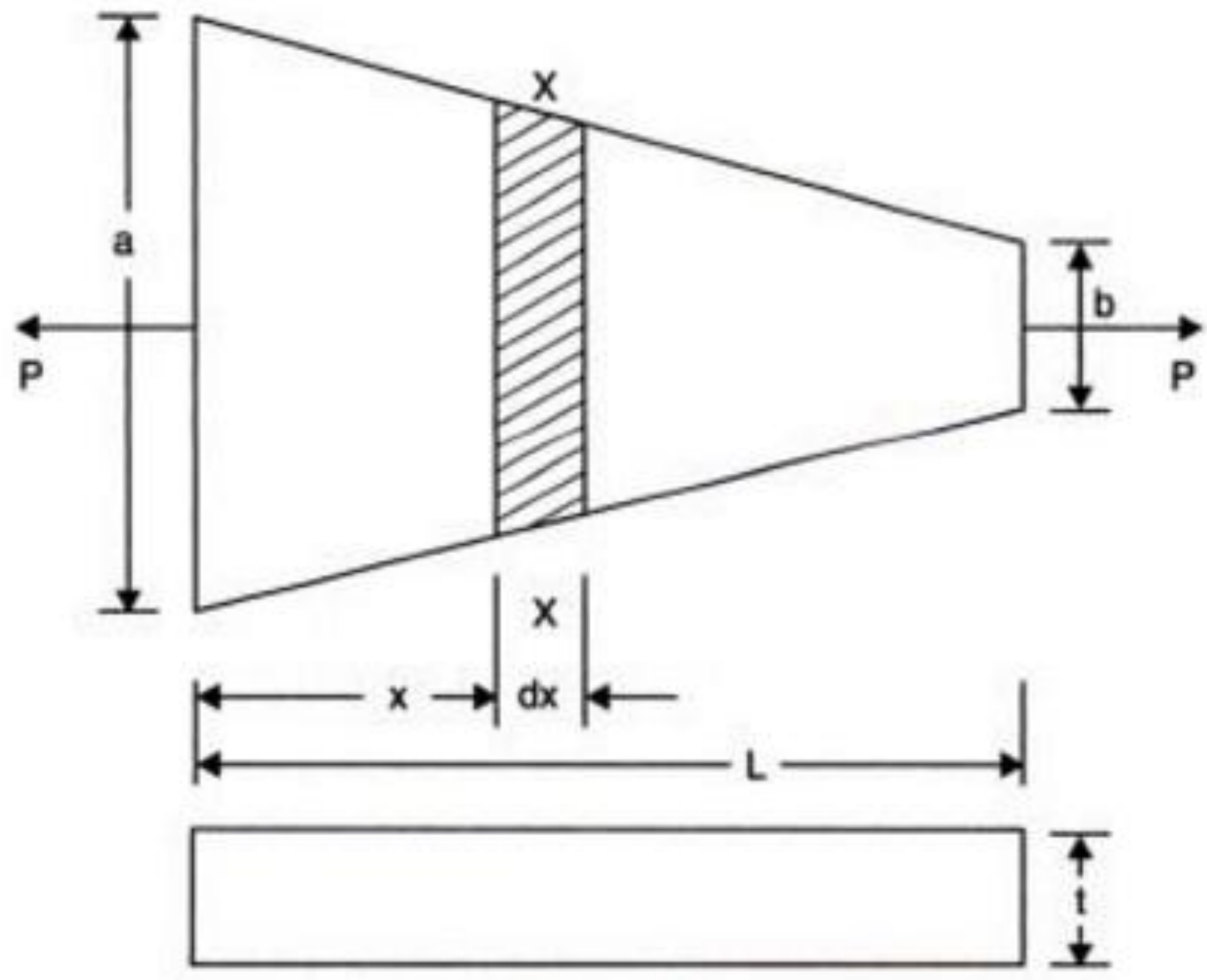
$$= 0.6366\,\text{mm}$$

$$\text{Total Extension } = \Delta_1 + \Delta_2 = 0.6366 + 0.6366$$

$$= 1.2732\,\text{mm} \qquad \textbf{(Ans)}$$

# Analysis of Uniformly Tapering Rectangular Bar

A bar of constant thickness and uniformly tapering in width from one end to the other end is shown in Fig.



Let $P$ = Axial load on the bar
$L$ = Length of bar
$a$ = Width at bigger end
$b$ = Width at smaller end
$E$ = Young's modulus
$t$ = Thickness of bar

Consider any section $X$-$X$ at a distance $x$ from the bigger end.

Width of the bar at the section $X$-$X$

$$= a - \frac{(a-b)x}{L}$$

$$= a - kx \qquad \text{where } k = \frac{a-b}{L}$$

Thickness of bar at section $X$-$X$ = $t$

$\therefore$ Area of the section $X$-$X$

$$= \text{Width} \times \text{thickness}$$

$$= (a - kx)t$$

$\therefore$ Stress on the section $X$-$X$

$$= \frac{\text{Load}}{\text{Area}} = \frac{P}{(a-kx)t}$$

Thickness of bar at section $X$-$X = t$

$\therefore$ Area of the section $X$-$X$

$$= \text{Width} \times \text{thickness}$$

$$= (a - kx)t$$

$\therefore$ Stress on the section $X$-$X$

$$= \frac{\text{Load}}{\text{Area}} = \frac{P}{(a - kx)t}$$

Extension of the small elemental length $dx$

$$= \text{Strain} \times \text{Length } dx$$

$$= \frac{\text{Stress}}{E} \times dx \qquad \left(\because \ \text{Strain} = \frac{\text{Stress}}{E}\right)$$

$$= \frac{\left(\dfrac{P}{(a - kx)t}\right)}{E} \times dx \qquad \left(\because \ \text{Stress} = \frac{P}{(a - kx)t}\right)$$

$$= \frac{P}{E(a - kx)t}\, ax$$

Total extension of the bar is obtained by integrating the above equation between the limits 0 and $L$.

$\therefore$ Total extension,

$$dL = \int_0^L \frac{P}{E(a-kx)t}\,dx = \frac{P}{Et}\int_0^L \frac{dx}{(a-kx)}$$

$$= \frac{P}{Et}\,.\,\log_e\Big[(a-kx)\Big]_0^L \times \left(-\frac{1}{k}\right) = -\frac{P}{Etk}\,[\log_e(a-kL)-\log_e a]$$

$$= \frac{P}{Etk}\,[\log_e a - \log_e(a-kL)] = \frac{P}{Etk}\left[\log_e\left(\frac{a}{a-kL}\right)\right]$$

$$= \frac{P}{Et\left(\dfrac{a-b}{L}\right)}\left[\log_e\left(\frac{a}{a-\left(\dfrac{a-b}{L}\right)L}\right)\right] \qquad \left(\because \;\; k = \frac{a-b}{L}\right)$$

$$= \frac{PL}{Et(a-b)}\log_e\frac{a}{b}\,.$$

*The extension in a rectangular steel bar of length 400 mm and thickness 10 mm, is found to be 0.21 mm. The bar tapers uniformly in width from 100 mm to 50 mm. If E for the bar is $2 \times 10^5$ N/mm$^2$, determine the axial load on the bar.*

**Sol.** Given :

Extension, $dL = 0.21$ mm

Length, $L = 400$ mm

Thickness, $t = 10$ mm

Width at bigger end, $a = 100$ mm

Width at smaller end, $b = 50$ mm

Value of $E = 2 \times 10^5$ N/mm$^2$

Let $P$ = axial load.

Using equation (1.12), we get

$$dL = \frac{PL}{Et(a-b)} \log_e \left(\frac{a}{b}\right)$$

or

$$0.21 = \frac{P \times 400}{2 \times 10^5 \times 10(100-50)} \log_e \left(\frac{100}{50}\right)$$

$$= 0.000004\, P \times 0.6931$$

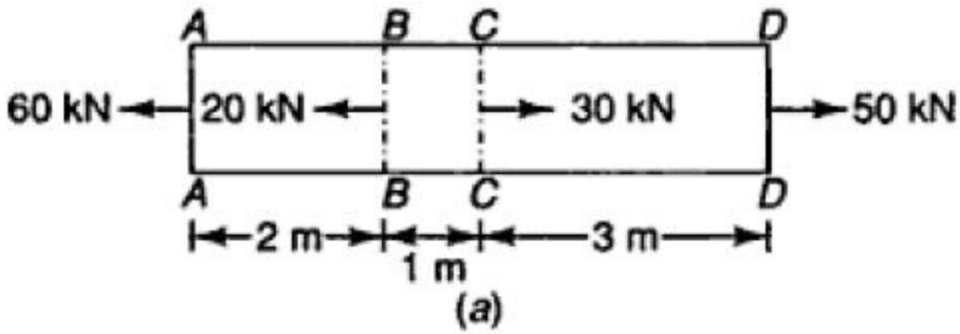
$$\therefore \quad P = \frac{0.21}{0.000004 \times 0.6931} = 75746 \text{ N}$$

$$= \mathbf{75.746\ kN.} \quad \textbf{Ans.}$$

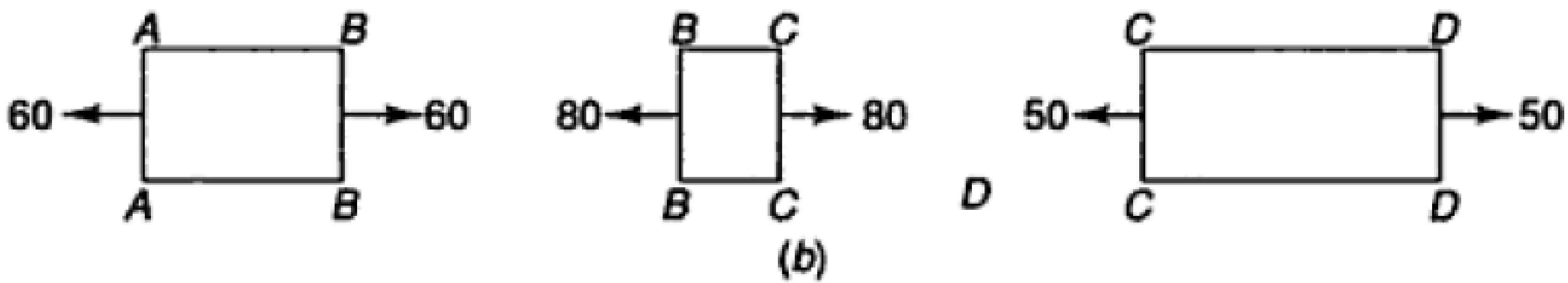
# Principle of Superposition

- It states that the effects of several actions taking place simultaneously can be reproduced exactly by adding the effect of each action separately.
- The principle is general and has wide applications and holds true if:
- (i) The structure is elastic
- (ii) The stress-strain relationship is linear
- (iii) The deformations are small.

**Example 1** *A steel bar of 25-mm diameter is acted upon by forces as shown in Fig. 1. What is the total elongation of the bar? Take E = 190 GPa.*



(a)



(b)

*Solution* Area of the section = $\frac{\pi}{4}(25)^2 = 490.88\ \text{mm}^2$, $E = 190$ GPa

$$= 190\,000\ \text{N/mm}^2$$

Forces in various segments are considered by taking free-body diagram of each segment as follows (Fig. 1.6*b*):

*Segment AB*: At section *AA*, it is 60 kN tensile and for force equilibrium of this segment, it is to be 60 kN tensile at *BB* also.

*Segment BC*:

Force at section *BB* = 60 kN (as above) + 20 kN (tensile force at section *BB*)

= 80 kN (tensile) = Force at section *CC*

*Segment CD*:

Force at section *CC* = 80 kN (as above) – 30 kN (compressive force at section *CC*)

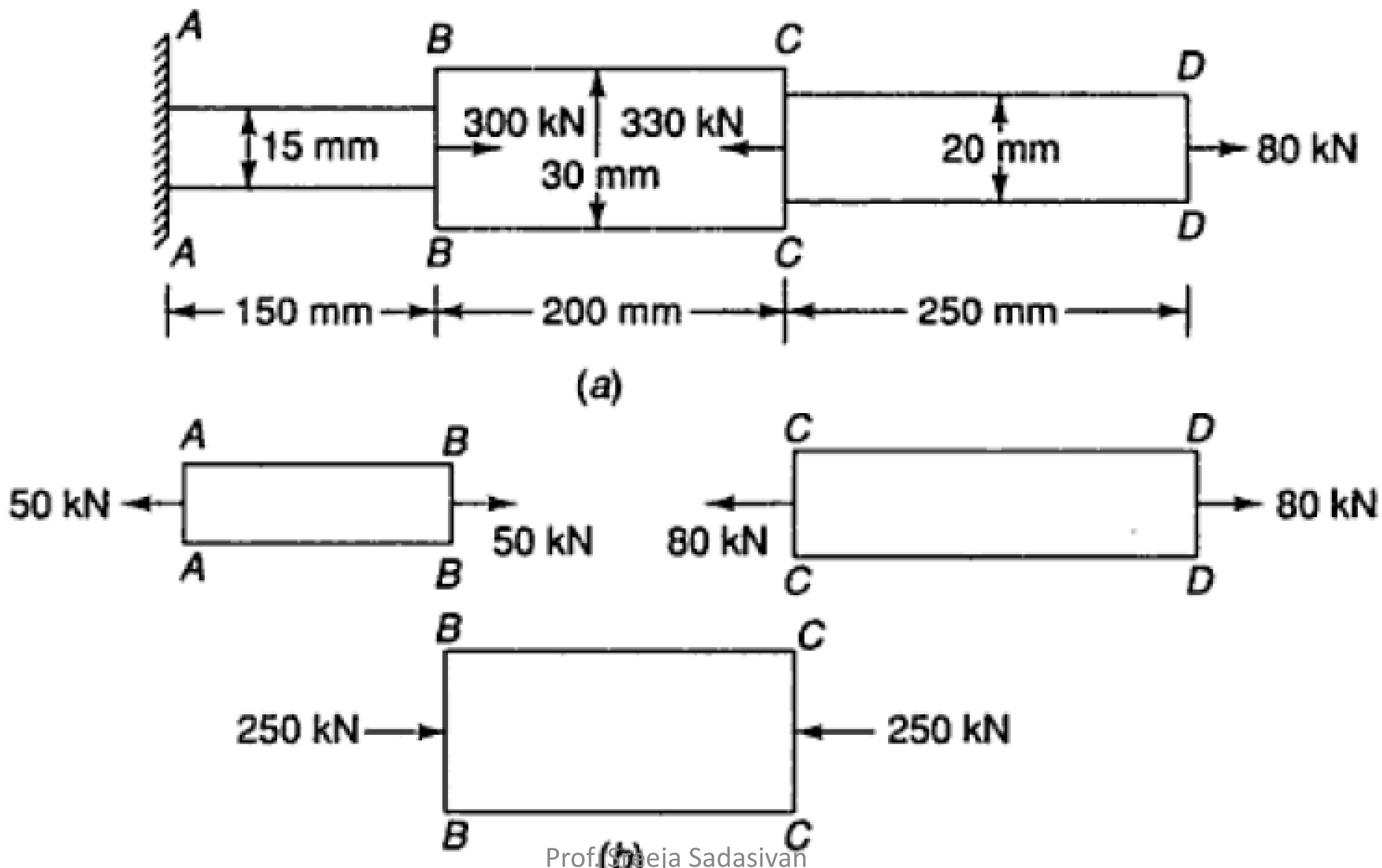
= 50 kN (tensile) = Force at section *DD*

Elongation is given by, $\Delta = \frac{PL}{AE}$

$$= \frac{1}{490.88 \times 190\,000}(60\,000 \times 2000 + 80\,000 \times 1000 + 50\,000 \times 3000) = 3.75\ \text{mm}$$

**Example 1.2** *A steel circular bar has three segments as shown in Fig. 1.7a. Determine*

*(i) the total elongation of the bar*
*(ii) the length of the middle segment to have zero elongation of the bar*
*(iii) the diameter of the last segment to have zero elongation of the bar*
*Take E = 205 GPa.*



(i) *Segment CD*: At section *DD*, it is 80 kN tensiles and for force equilibrium of this segment, at *CC* also it is to be 80 kN tensile.

*Segment BC*:

Force at section *CC* = 80 kN (as above) – 330 kN (compressive force at section *CC*)

= – 250 kN (compressive) = Force at section *BB*

*Segment AB*:

Force at section *BB* = – 250 kN (as above) + 300 kN (tensile force at section *BB*)

= 50 kN (tensile) = Force at section *AA*

Total elongation,

$$\Delta = \frac{1}{(\pi/4) \times 205\,000}\left(\frac{50\,000 \times 150}{15^2} - \frac{250\,000 \times 200}{30^2} + \frac{80\,000 \times 250}{20^2}\right)$$

$$= \frac{1}{161\,007}(33\,333.3 - 55\,555.5 + 50\,000) = 0.173 \text{ mm}$$

(ii) Let the length of the middle segment be $L$ to have zero elongation of the bar.

Then $$\Delta = \frac{1}{161\,007}\left(33\,333.3 - \frac{250\,000 \times L}{30^2} + 50\,000\right) = 0$$

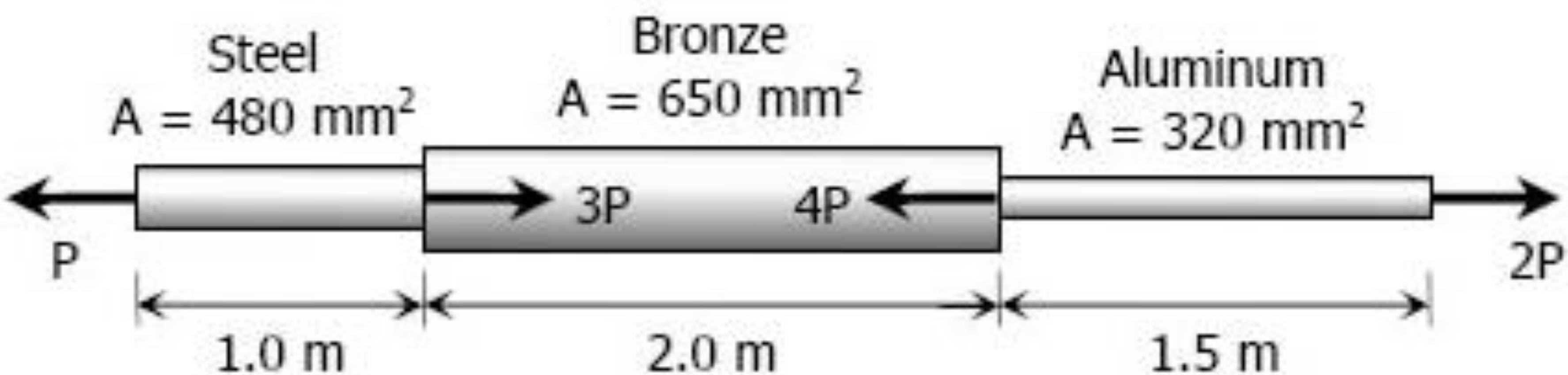
or $$L = \frac{30^2}{250\,000} \times 83\,333.3 = 300 \text{ mm}$$

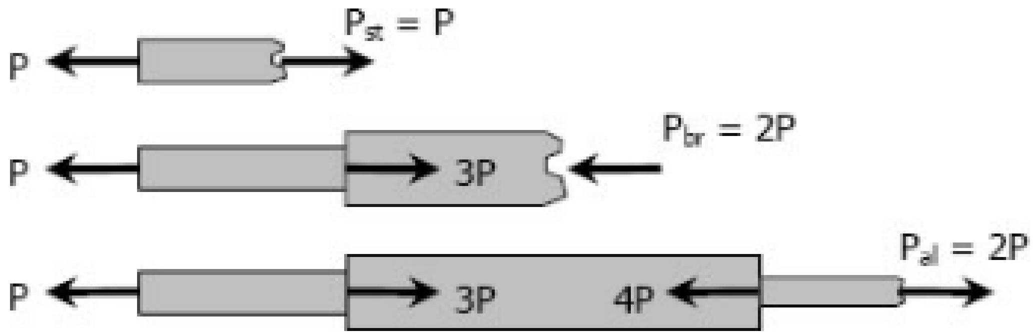
(iii) Let the diameter of the last segment be $d$ to have zero elongation of the bar.

$\therefore$ $$\Delta = \frac{1}{161\,007}\left(33\,333.3 - 55\,555.5 + \frac{80\,000 \times 250}{d^2}\right) = 0$$

$$d^2 = \frac{80\,000 \times 250}{22\,222.2} = 900 \quad \text{or} \quad d = 30 \text{ mm}$$

A bronze bar is fastened between a steel bar and an aluminum bar as shown in Fig. Axial loads are applied at the positions indicated. Find the largest value of P that will not exceed an overall deformation of 3.0 mm, or the following stresses: 140 MPa in the steel, 120 MPa in the bronze, and 80 MPa in the aluminum. Assume that the assembly is suitably braced to prevent buckling. Use $E_{st}$ = 200 GPa, $E_{al}$ = 70 GPa, and $E_{br}$ = 83 GPa.






**Based on allowable stresses:**

**Steel:**

$$P_{st} = \sigma_{st}\, A_{st}$$

$$P = 140(480) = 67\,200\ \text{N}$$

$$P = 67.2\ \text{kN}$$

**Bronze:**

$$P_{br} = \sigma_{br}\, A_{br}$$

$$2P = 120(650) = 78\,000\ \text{N}$$

$$P = 39\,000\ \text{N} = 39\ \text{kN}$$

**Aluminum:**

$$P_{al} = \sigma_{al}\, A_{al}$$

$$2P = 80(320) = 25\,600\ \text{N}$$

$$P = 12\,800\ \text{N} = 12.8\ \text{kN}$$

**Based on allowable deformation:**

*(steel and aluminum lengthens, bronze shortens)*

$$\delta = \delta_{st} - \delta_{br} + \delta_{al}$$

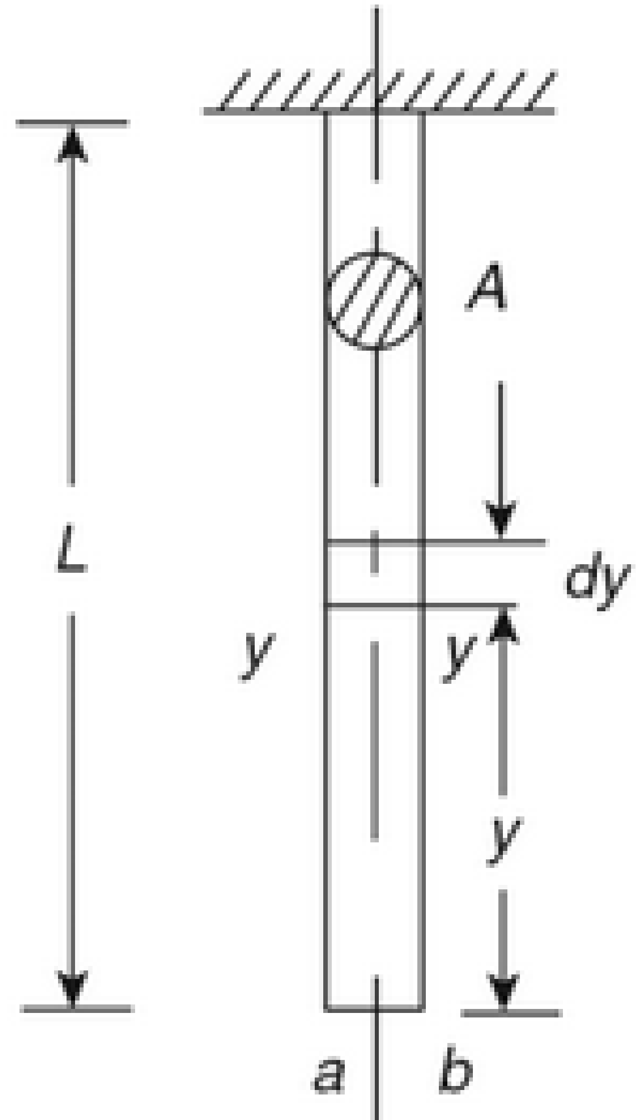
$$3 = \frac{P(1000)}{480(200\,000)} - \frac{2P(2000)}{650(83\,000)} + \frac{2P(1500)}{320(70\,000)}$$

$$3 = \left(\frac{1}{96\,000} - \frac{2}{26\,975} + \frac{3}{22\,400}\right) P$$

$$P = 42\,733.52\,\mathrm{N} = 42.73\,\mathrm{kN}$$

Use the smallest value of P, **P = 12.8 kN**

# Elongation of a bar due to it's own weight



Problem of stresses or strains exists in long hanging cables where the length of the cable becomes large; self-weight of the cable causes tensile stress in cable with increasing magnitude from top to bottom.

A bar of a cross-sectional area of $A$ and a length of $L$ is suspended vertically with its upper end rigidly fixed in ceiling as shown in Fig . Say the weight density of the bar is $w$. Consider a section $yy$ at a distance y from the lower end.

Weight of the lower portion $abyy = yAw$

Cross-sectional area at $yy = A$

Normal stress of $yy$, $\sigma_y = \dfrac{Ayw}{A} = wy$ (tensile)

(weight is acting downwards on section $yy$)

Strain in section $yy$, $\varepsilon_y = wy/E$, where $E$ is the Young's modulus of material.

Change in length over small length $dy$ of element

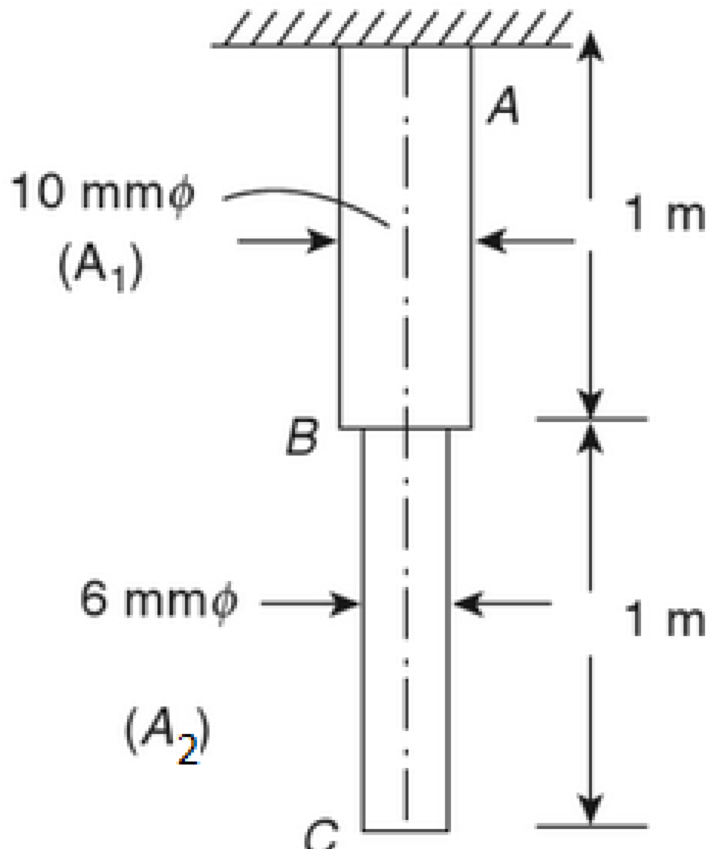
$$\delta dy = \frac{wydy}{E}$$

Total change in length, $\delta L = \int_0^L \dfrac{wydy}{E}$

$$\delta L = \frac{wL^2}{2E} = \frac{AwL^2}{2AE} = \frac{(wLA)L}{2AE} = \frac{WL}{2AE}$$

where $W$ is the total weight of the bar. Stress in the bar or cable due to self-weight gradually increases from bottom to top as shown in Fig.

**Example 1** A stepped steel bar is suspended vertically. The diameter in the upper half portion is 10 mm, while the diameter in the lower half portion is 6 mm. What are the stresses due to self-weight in sections $B$ and $A$ as shown in Fig. 1.24. $E = 200\ kN/mm^2$. Weight density, $w : 0.7644 \times 10^{-3}\ N/mm^3$. What is the change in its length?



**Solution**

$$w = 0.7644 \times 10^{-3}\ \text{N/mm}^3$$

Cross-sectional area, $A_1 = \dfrac{\pi}{4} \times 10^2 = 78.54\ \text{mm}^2$

Cross-sectional area, $A_2 = \dfrac{\pi}{4} \times 6^2 = 28.27\ \text{mm}^2$

Weight of upper portion, $W_1 = A \times l \times w$

$$= 78.54 \times 1{,}000 \times 0.7644 \times 10^{-3} = 60.03\ \text{N}$$

Weight of lower portion, $W_2 = 28.27 \times 1{,}000 \times 0.7644 \times 10^{-3} = 21.61\ \text{N}$

$$\sigma \text{ at } A = \frac{W_1 + W_2}{A_1} = \frac{60.03 + 21.61}{78.54} = 1.0395\ \text{N/mm}^2$$

$$\sigma \text{ at } B = \frac{W_2}{A_2} = \frac{21.61}{28.27} = 0.7644\ \text{N/mm}^2$$

Change in length

at $B$, $$\delta l = \frac{W_2 L}{2A_2 E} = \frac{21.01 \times 1000}{2 \times 28.27 \times 200 \times 10^3} = 0.191 \times 10^{-2} \text{ mm}$$

at $A$, $$\delta l = 0.191 \times 10^{-2} + \frac{W_1}{2A_1 E}$$

$$= 0.191 \times 10^{-2} + \frac{60.03 \times 1000}{2 \times 78.54 \times 200 \times 10^3}$$

$$= 0.191 \times 10^{-2} + 0.191 \times 10^{-2} \text{ mm}$$

$$= 0.382 \times 10^{-2} \text{ mm due to self-weight}$$

$\delta l'$ = change in length of upper part due to weight of lower part.

$$= \frac{W_2}{A_1} \times \frac{1,000}{E}$$

$$= \frac{21.61 \times 1,000}{78.54 \times 200 \times 10^3}$$

$$= 137.57 \times 10^{-3} \text{ mm}$$

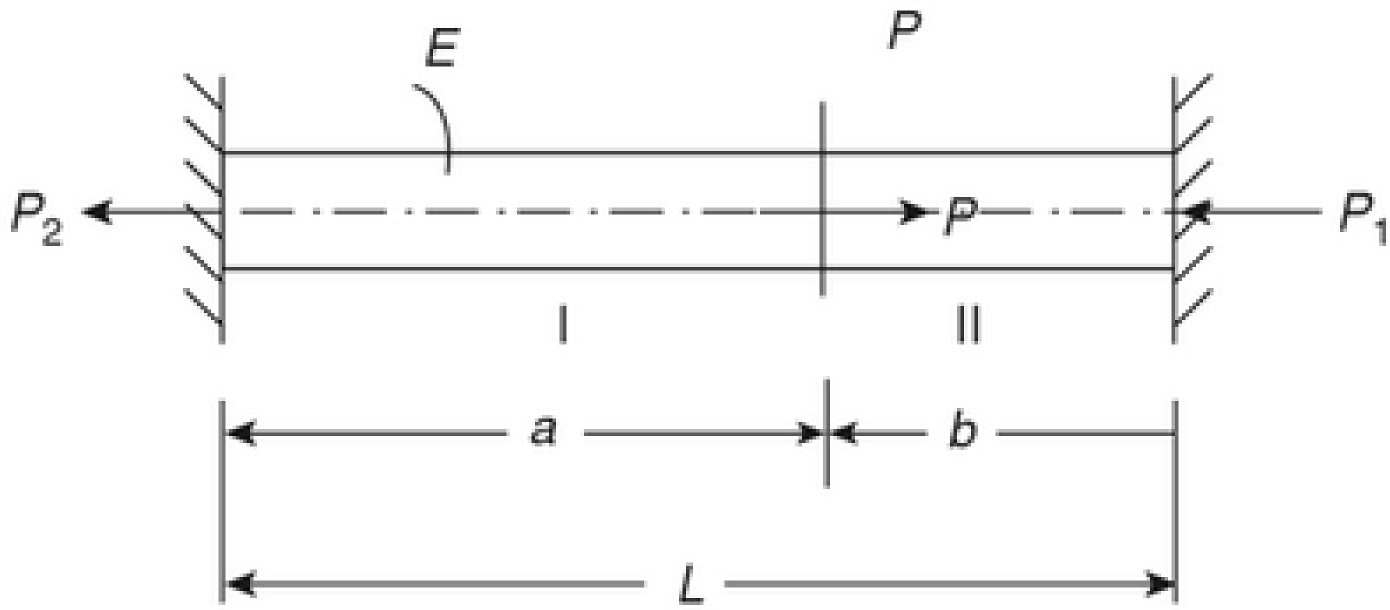
$$= 0.13757 \times 10^{-2} \text{ mm}$$

Over all change in length $= (0.382 + 0.13757) \times 10^{-2}$

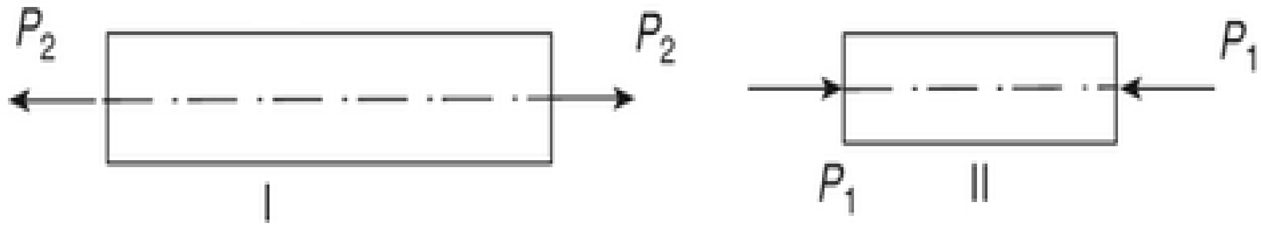
$$= 0.5196 \times 10^{-2} \text{ mm}$$

# Statically Indeterminate problems

In a statically determinate problem, the equations of equilibrium of forces are sufficient to determine the unknown forces as reactions in a structure. However, in certain problems, the equations of equilibrium of forces are not sufficient to determine the unknown forces. These are called statically indeterminate problems and an *additional* equation involving the geometry of deformed structure is necessary to determine unknown forces. This type of problem is a *statically indeterminate problem*. This equation provides condition for structural compatibility.




Bar field between two rigid supports




Two portions under axial loads

In Fig. a bar held between two rigid supports is shown. A force $P$ is applied to the bar. Reactions at ends are $P_1$ and $P_2$. Then,

$$P = P_1 + P_2 \qquad (1)$$

This equation is not sufficient to determine the values of $P_1$ and $P_2$ independently.

## Additional Equation

A bar is held between two rigid supports; as shown in Fig. therefore, net deformation in the bar along its axis is zero.

Due to reactions $P_1$ and $P_2$, portion I comes under tension and portion II comes under compression, as shown in Fig. If $E$ is the Young's modulus of this material, then

$$\delta l_1 = \frac{P_2 a}{AE} \quad \text{(extension in portion I)}$$

$$\delta l_{11} = \frac{P_1 b}{AE} \quad \text{(contraction in portion II)}$$

where $A$ is the cross-sectional area of the bar.

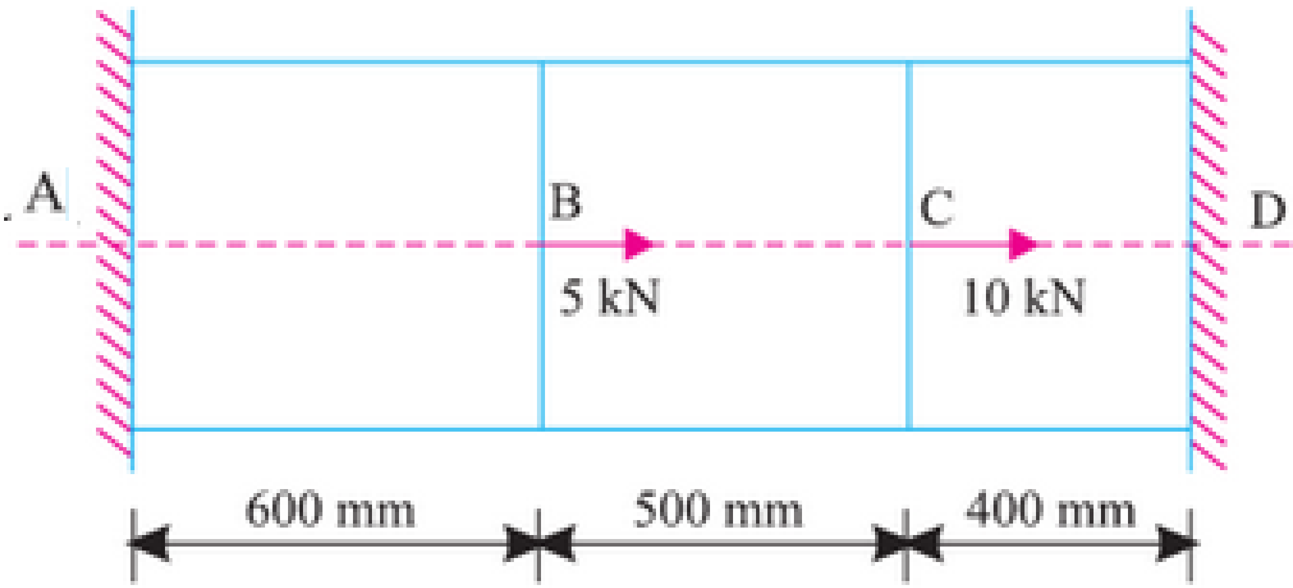
Then, $$\frac{P_2 a}{AE} - \frac{P_1 b}{AE} = 0 \qquad (2)$$

or $$P_2\, a = +P_1\, b$$

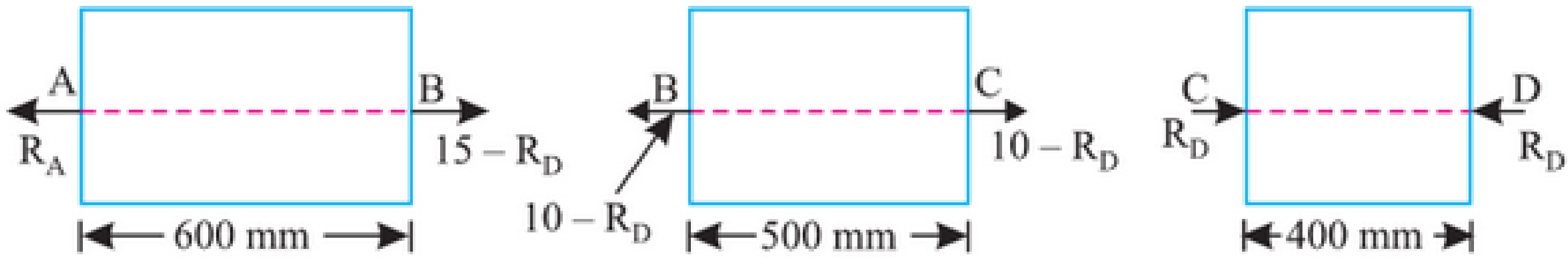
$$P_2 = \frac{b}{a}\, P_1 \qquad (3)$$

Now, Eqs ( 1 ) and ( 2 ) can determine the reactions, $P_1$ and $P_2$. The second equation is known as deformation compatibility equation.

*A straight uniform bar AD is clamped at both ends and loaded as shown in Fig. Initially the bar is stress free. Determine the stresses in all the three parts (AB, BC, CD) of the bar if the cross-sectional area of bar is 1000 mm².*



Let $R_A$ and $R_D$ be the reactions at the supports $A$ and $D$ respectively. For equilibrium of the bar $AD$, these reactions must act towards the left.




Hence $$R_A + R_D = 5 + 10 = 15 \text{ kN} \qquad \ldots(i)$$

Fig. 1.17 shows the free-body diagram for the parts $AB$, $BC$ and $CD$.

Let $\delta l_{AB}$, $\delta l_{BC}$ and $\delta l_{CD}$ be the extensions in the parts $AB$, $BC$ and $CD$ respectively.

Then $$\delta l_{AB} = \frac{(15 - R_D) \times l_{AB}}{A_{AB} \times E} \qquad \ldots\ldots \text{(extension)}$$

$$\delta l_{BC} = \frac{(10 - R_D) \times l_{BC}}{A_{BC} \times E} \qquad \ldots\ldots \text{(extension)}$$

$$\delta l_{CD} = \frac{R_D \times l_{CD}}{A_{CD} \times E} \qquad \ldots\ldots \text{(compression)}$$

Since the *supports are rigid*, therefore elongation of $AB$ and $BC$ shall be *equal to* the compressions of $CD$.

Hence, $\delta l_{AB} + \delta l_{BC} = \delta l_{CD}$

Substituting the values, we have

$$\frac{(15 - R_D) \times 600}{1000 \times E} + \frac{(10 - R_D) \times 500}{1000 \times E} = \frac{R_D \times 400}{1000 \times E}$$

$$(\because A_{AB} = A_{BC} = A_{CD} = 1000 \text{ mm}^2 \text{ ........ given})$$

or, $6\,(15 - R_D) + 5\,(10 - R_D) = 4\,R_D$

or, $90 - 6\,R_D + 50 - 5\,R_D = 4\,R_D$

or, $15\,R_D = 140$

$\therefore$ $R_D = \dfrac{140}{15} = 9.33 \text{ kN}$

From eqn. (*ii*), we have

$$R_A + 9.33 = 15$$

or, $R_A = 15 - 9.33 = 5.67 \text{ kN}$

Stress in part AB, $\sigma_{AB} = \dfrac{\text{Load in part } AB}{\text{Cross-sectional area of part } AB}$

$$= \frac{(15 - R_D) \times 1000}{1000} \text{ N/mm}^2 = \frac{15 - 9.33}{1000} \times 1000$$

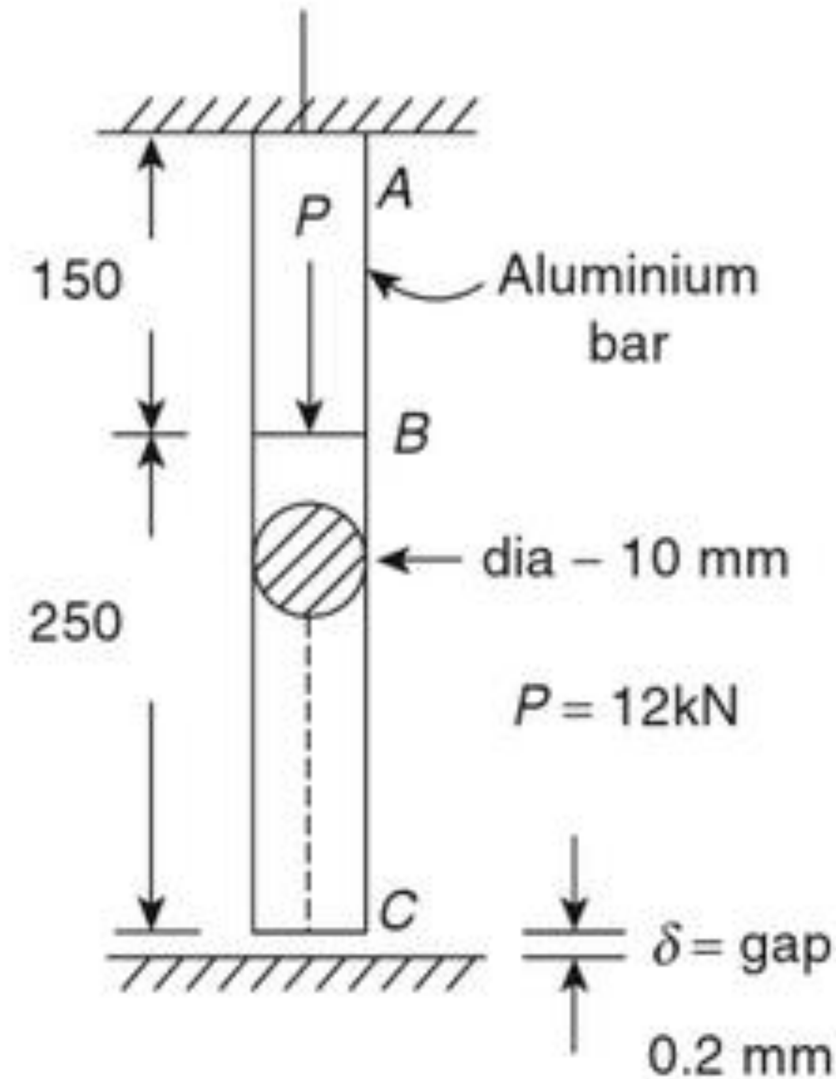
$$= \mathbf{5.67\ N/mm^2\ (tensile)} \quad \mathbf{(Ans.)}$$

Stress in part $BC$, $$\sigma_{BC} = \frac{(10 - R_D) \times 1000}{1000} \text{ N/mm}^2 = \frac{10 - 9.33}{1000} \times 1000$$

$$= \mathbf{0.67\ N/m^2\ (tensile)} \quad \mathbf{(Ans.)}$$

Stress in part $CD$, $$\sigma_{CD} = \frac{R_D \times 1000}{1000} \text{ N/mm}^2 = \mathbf{9.33\ N/mm^2} \quad \mathbf{(Ans.)}$$

**Example** An aluminium bar of a diameter of 10 mm and a length of 400 mm is rigidly held at its upper end. It is subjected to an axial force $P = 12$ kN at section $B$ as shown in Fig. 1. . $E$ for aluminium = 67 kN/mm$^2$. Gap between the lower end and the floor is 0.2 mm. Determine the stresses in portions $AB$ and $BC$ of the bar.



A
P
150
Aluminium
bar
B
dia – 10 mm
250
P = 12kN
C
δ = gap
0.2 mm

**Solution**

Under the action of the force $P$, the portion $AB$ of the bar extends and the end $C$ of the bar touches the floor. Then, reaction offered from floor provides a compressive force for portion $BC$ of bar.

Say $P_1$ = tensile force in portion $AB$

$P_2$ = compressive force in portion $BC$

$$P = P_1 + P_2$$

Extension in $AB$, $\delta l_1 = \dfrac{P_1}{AE} \times 150$

Contraction in $BC$, $\delta l_{11} = \dfrac{P_2}{AE} \times 250$

where $A$ is the cross-sectional area of the bar and $E$ is the Young's modulus of the bar.

Then, $\delta l_1 - \delta l_{11} = 0.2$ mm gap

Area, $A = \dfrac{\pi}{4} \times 10^2 = 78.54 \text{ mm}^2$

$$\frac{P_1 \times 150}{78.54 \times E} - \frac{P_2 \times 250}{78.54 \times E} = 0.2$$

$$150\,P_1 - 250\,P_2 = 0.2 \times 78.54 \times 67{,}000$$

$$150\,P_1 - 250\,P_2 = 1{,}052{,}436$$

$$P_1 = 1.6667\,P_2 + 7016.24$$

$$P_1 + P_2 = 12\text{ kN} = 12{,}000$$

or $$1.6667\,P_2 + 7016.24 = 12{,}000 - P_2$$

or $$2.6667\,P_2 = 4983.76$$

$$P_2 = 1868.9\text{ N}$$

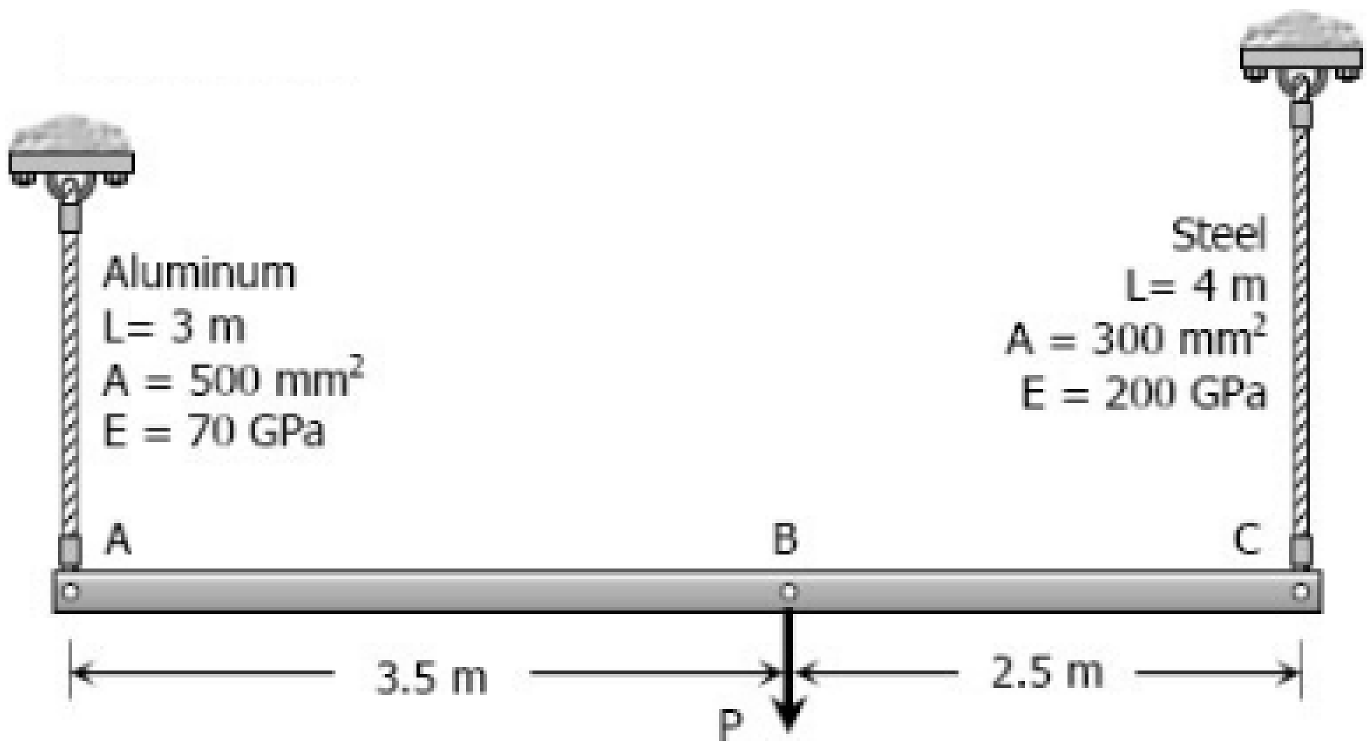
$$P_1 = 12{,}000 - 1888.9 = 10131.1\text{ N}$$

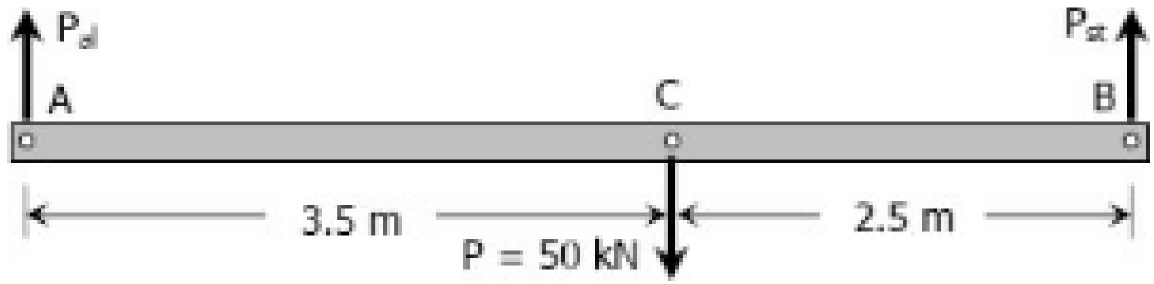
Stresses in portions *AB* and *BC*:

$$\sigma_1 = +\frac{10131.1}{78.54} = +129.0\text{ N/mm}^2$$

$$\sigma_{II} = -\frac{1868.9}{78.54} = -23.8\text{ N/mm}^2$$

The rigid bar AC, attached to two vertical rods as shown in Fig. is horizontal before the load P is applied. Determine the vertical movement of P if its magnitude is 50 kN.






**For aluminum:**

$\Sigma M_B = 0$

$6P_{al} = 2.5(50)$

$P_{al} = 20.83 \text{ kN}$

$$\delta = \frac{PL}{AE}$$

$$\delta_{al} = \frac{20.83(3)1000^2}{500(70\,000)}$$

$\delta_{al} = 1.78 \text{ mm}$

**For steel:**

$\Sigma M_A = 0$

$6P_{st} = 3.5(50)$

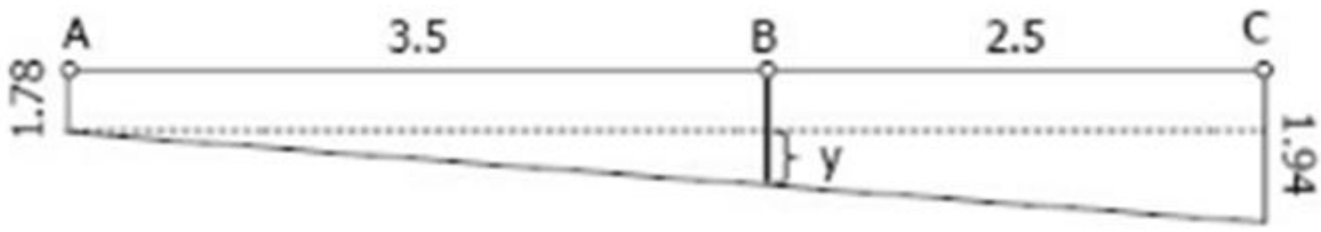
$P_{st} = 29.17 \text{ kN}$

$$\delta = \frac{PL}{AE}$$

$$\delta_{st} = \frac{29.17(4)1000^2}{300(200\,000)}$$

$\delta_{st} = 1.94 \text{ mm}$

**Movement diagram:**




$$\frac{y}{3.5} = \frac{1.94 - 1.78}{6}$$

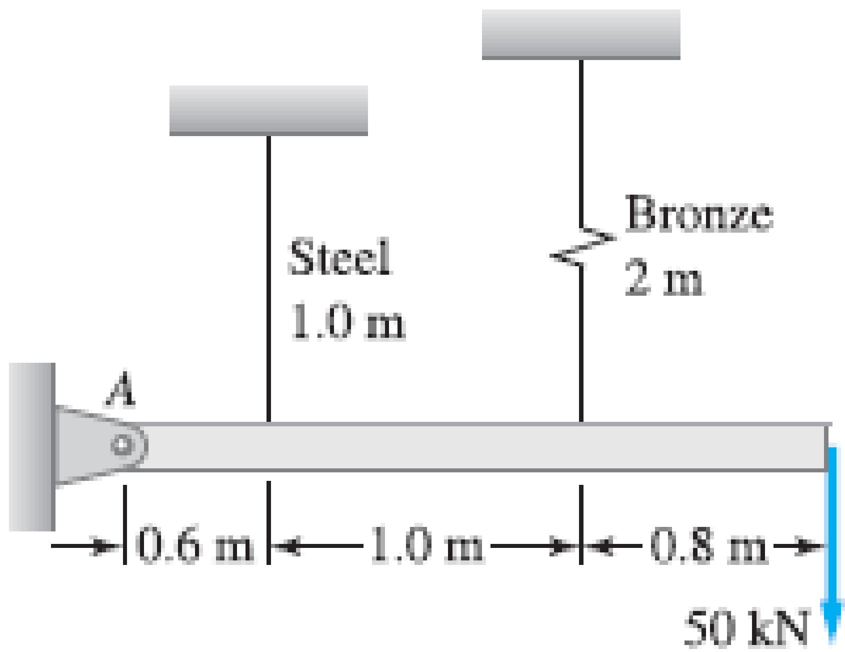
$y = 0.09\,\text{mm}$

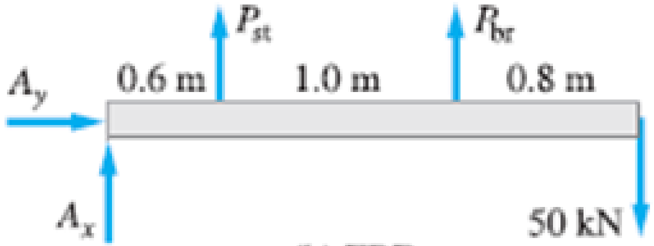
$\delta_B =$ vertical movement of $P$

$\delta_B = 1.78 + y = 1.78 + 0.09$

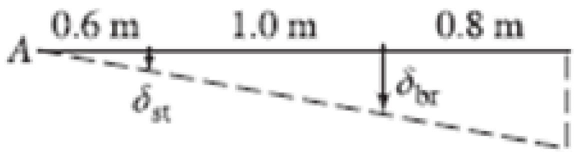
$\delta_B = 1.87\,\text{mm}$ *answer*

A rigid bar shown in fig .is supported by a pin at A and two rods, one made of steel and the other of bronze. Neglecting the weight of the bar, compute the stress in each rod caused by the 50kN load. The area of steel and bronze rods are 600mm² and 300mm² respectively. $E_{steel}$ = 200GPa and $E_{bronze}$ = 83Gpa.






(b) FBD




$$\Sigma M_A = 0 \quad +\circlearrowleft \quad 0.6P_{st} + 1.6P_{br} - 2.4(50 \times 10^3) = 0$$

$$\frac{\delta_{st}}{0.6} = \frac{\delta_{br}}{1.6}$$

(b)

$$\frac{1}{0.6}\left(\frac{PL}{EA}\right)_{st} = \frac{1}{1.6}\left(\frac{PL}{EA}\right)_{br}$$

$$P_{st} = 3.614P_{br}$$

$$P_{st} = 115.08 \times 10^3 \text{ N} \qquad P_{br} = 31.84 \times 10^3 \text{ N}$$

$$\sigma_{st} = \frac{P_{st}}{A_{st}} = \frac{115.08 \times 10^3}{600 \times 10^{-6}} = 191.8 \times 10^6 \text{ Pa} = 191.8 \text{ MPa}$$

$$\sigma_{br} = \frac{P_{br}}{A_{br}} = \frac{31.84 \times 10^3}{300 \times 10^{-6}} = 106.1 \times 10^6 \text{ Pa} = 106.1 \text{ MPa}$$