UNIT-5

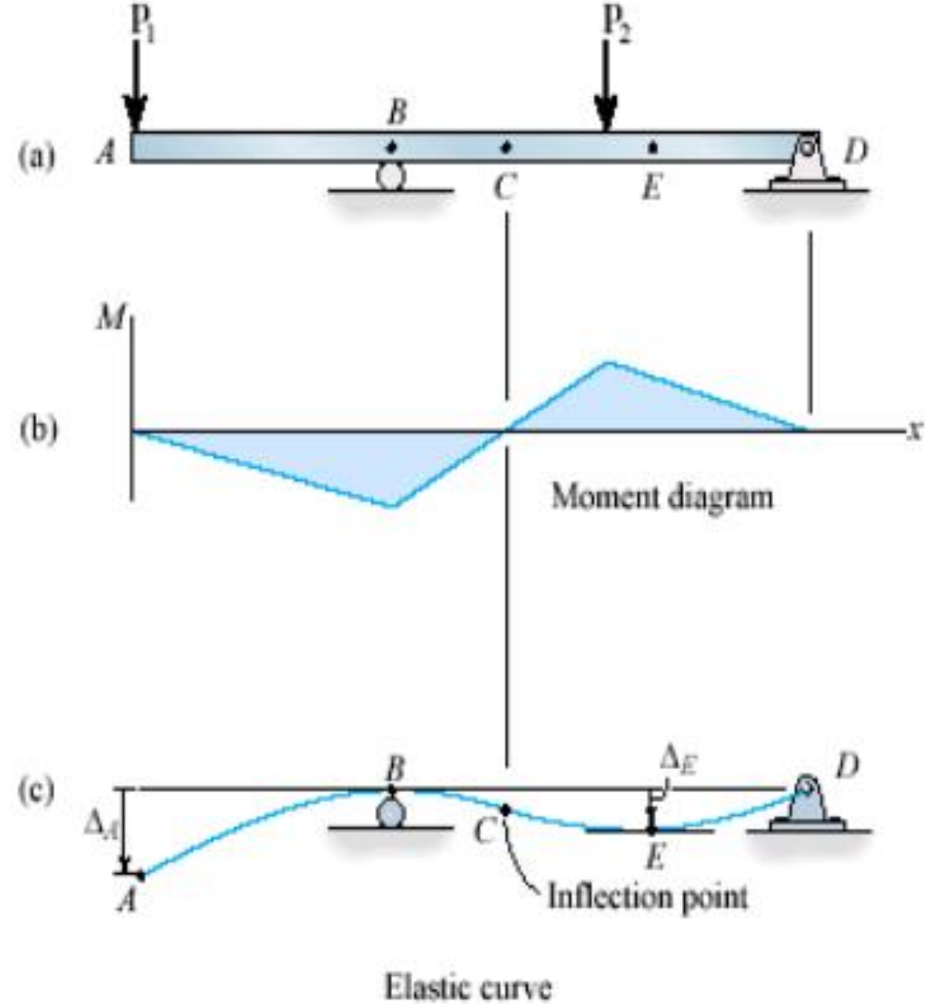
Deflection Of Beams

Deflection of beams by Double integration method, Macaulay's method, Area moment theorems for computation of slopes and deflections in beams, Conjugate beam method.

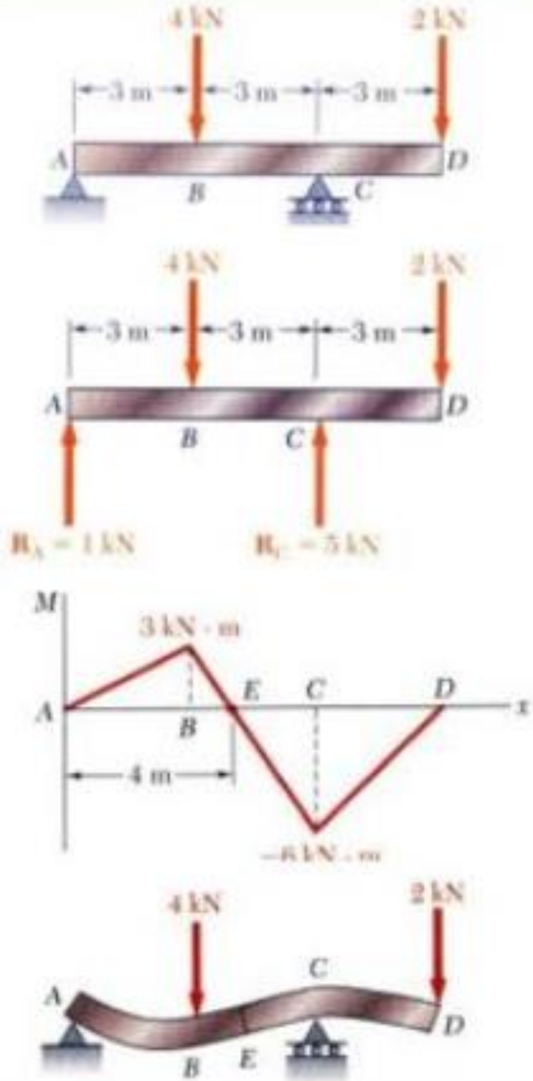
Introduction

- Calculation of deflections is an important part of structural analysis
- Excessive beam deflection can be seen as a mode of failure.
  - Extensive glass breakage in tall buildings can be attributed to excessive deflections
  - Large deflections in buildings are unsightly (and unnerving) and can cause cracks in ceilings and walls.
  - Deflections are limited to prevent undesirable vibrations

- Bending changes the initially straight longitudinal axis of the beam into a curve that is called the **Deflection Curve** or **Elastic Curve**



Deformation of a Beam Under Transverse Loading



- Overhanging beam
- Reactions at *A* and *C*
- Bending moment diagram
- Curvature is zero at points where the bending moment is zero, i.e., at each end and at *E*.

$$\frac{1}{\rho}=\frac{M(x)}{EI}$$

- Beam is concave upwards where the bending moment is positive and concave downwards where it is negative.
- Maximum curvature occurs where the moment magnitude is a maximum.
- An equation for the beam shape or *elastic curve* is required to determine maximum deflection and slope.

Sign Convention

Positive Bending

Negative Bending

Assumptions and Limitations

- ❖Deflections caused by shearing action negligibly small compared to bending
- ❖Deflections are small compared to the cross-sectional dimensions of the beam
- ❖All portions of the beam are acting in the elastic range
- ❖Beam is straight prior to the application of loads

Factors

The deflection of a beam depends on four general factors:

1. Stiffness of the materials that the beam is made of,
2. Dimensions of the beam,
3. Applied loads, and
4. Supports

METHODS OF DETERMINATION OF SLOPE AND DEFLECTION

Following are the important methods which are used for finding out the slope and deflection at a section in a loaded beam:

1. Double integration method
2. Moment–area method
3. Macaulay's method
4. Conjugate beam method.

Double Integration Method

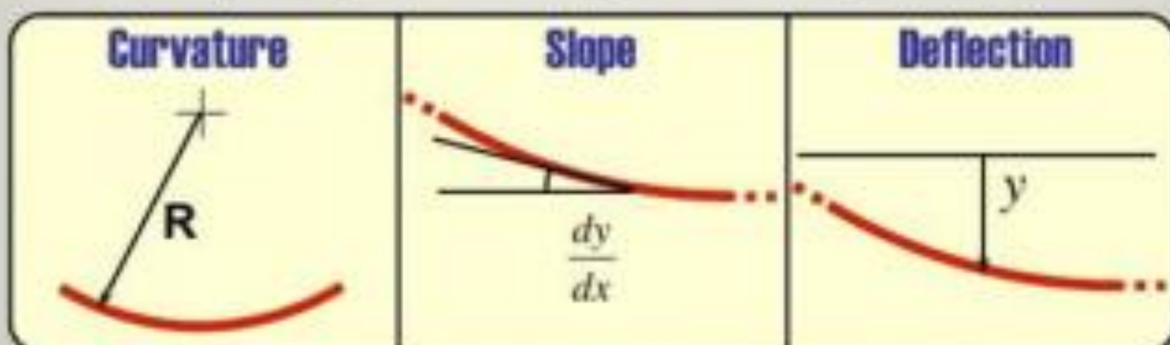
Method of double integration The primary advantage of the double- integration method is that it produces the equation for the deflection everywhere along the beams.

$$\frac{d^2y}{dx^2} = \left(\frac{1}{EI}\right)M \quad \Leftarrow \textbf{Curvature}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1}{EI}\right)\int M \cdot dx + C_1 \quad \Leftarrow \textbf{Slope}$$

$$\Rightarrow y = \left(\frac{1}{EI}\right)\int\int M \cdot dx \cdot dx + \int C_1 \cdot dx + C_2 \quad \Leftarrow \textbf{Deflection}$$

Where C_1 and C_2 are found using the boundary conditions.



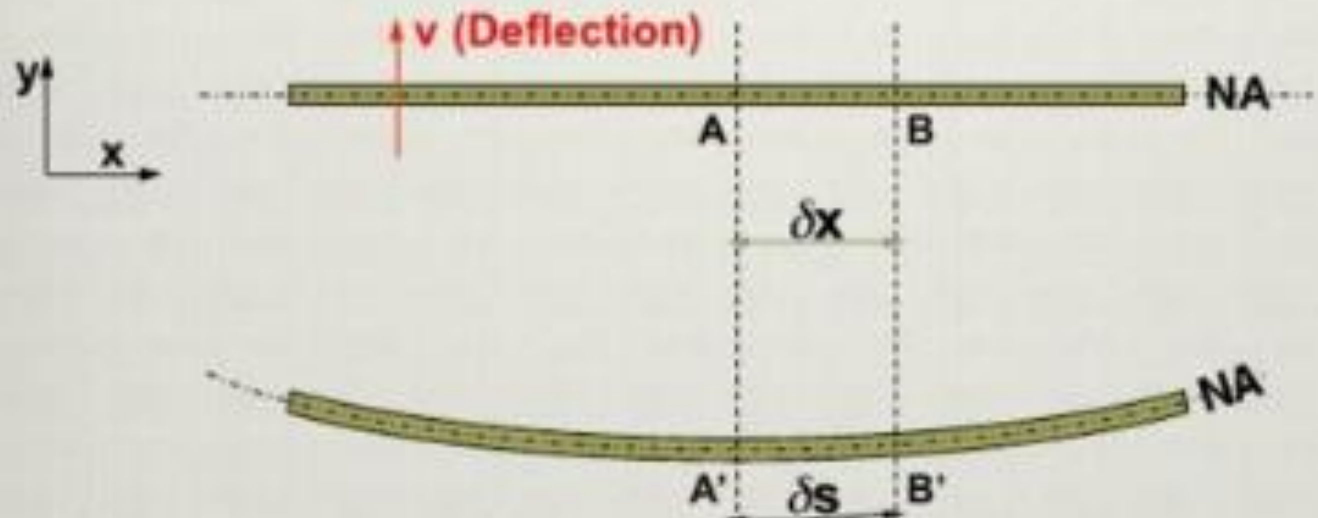
Proof

Beam Deflection

Recall: THE ENGINEERING BEAM THEORY

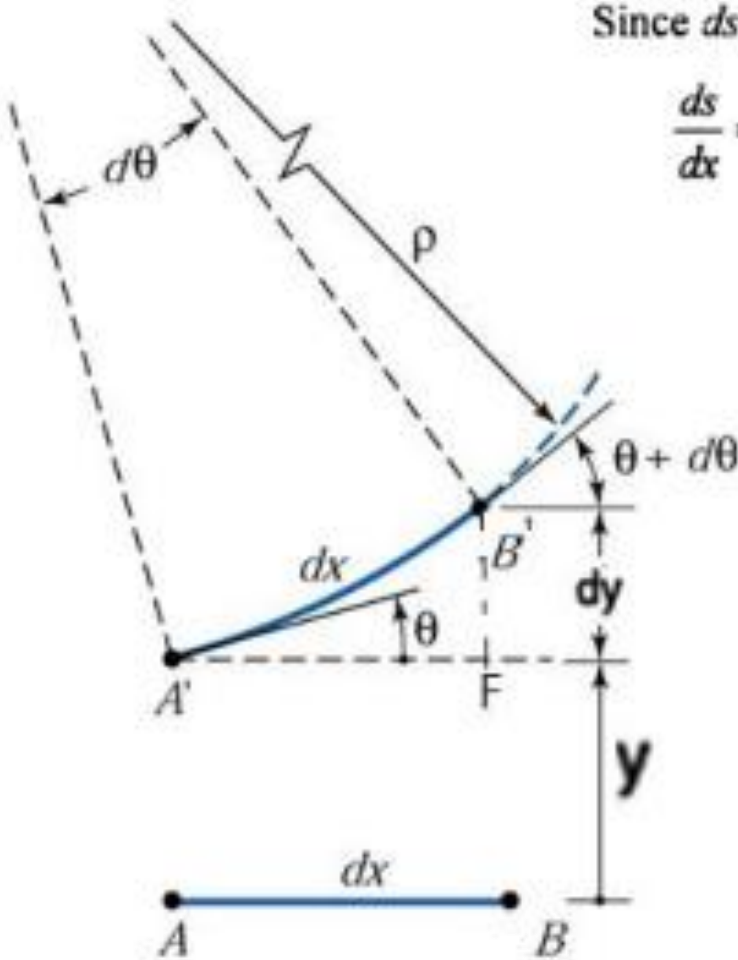
$$\frac{\sigma}{y} = \boxed{\frac{M}{I} = \frac{E}{R}}$$

Moment-Curvature Equation



If deformation is small (i.e. slope is "flat"):

$$\delta s \approx \delta x$$



Since ds is an elemental length, treating ABF as a triangle

$$\frac{ds}{dx} = \sec\theta \qquad \text{.............(1)}$$

$$ds = R d\theta$$

$$\frac{1}{R} = \frac{d\theta}{ds} \qquad \text{..............(2)}$$

$$\frac{dy}{dx} = \tan\theta$$

Differentiating equation w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \sec^2\theta \frac{d\theta}{dx}$$

Put 1 & 2 in 3

$$= \sec^2\theta \frac{d\theta}{ds}\frac{ds}{dx} \qquad \text{.............(3)}$$

$$= \sec^2\theta \times \frac{1}{R}\sec\theta$$

$$= \sec^3\theta \times \frac{1}{R}$$

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left(1+\tan^2\theta\right)^{3/2}} \qquad \text{since} \sec^2\theta = 1+\tan^2\theta$$

$$= \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

In beams, deflections are small and, hence, slope dy/dx is small. Therefore, in this theory, which may be called small deflection theory, $(dy/dx)^2$ is neglected compared to unity and hence,

$$\frac{1}{R} = \frac{d^2y}{dx^2}$$

Hence, $$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

or $$EI\frac{d^2y}{dx^2} = M$$

This equation is called differential equation for deflection.

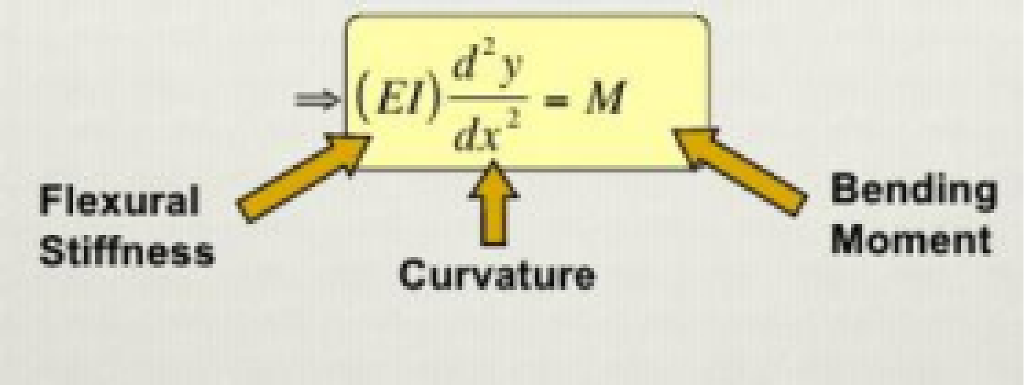
Note that the following sign conventions are used in deriving equation

(*a*) the *y*-axis is upward.

(*b*) Curvature is concave towards the positive y-axis.

(*c*) This type of curvature occurs in the beam due to the sagging moment. Hence, the sagging moment is to be considered as the +ve moment.

In some text books, $EI\frac{d^2y}{dx^2} = -M$ is taken to get downward deflection positive



In this method, the first moment, M at any distance x from one of the supports, is written with the sagging moment as positive.

Then $$EI\frac{d^2y}{dx^2} = M$$

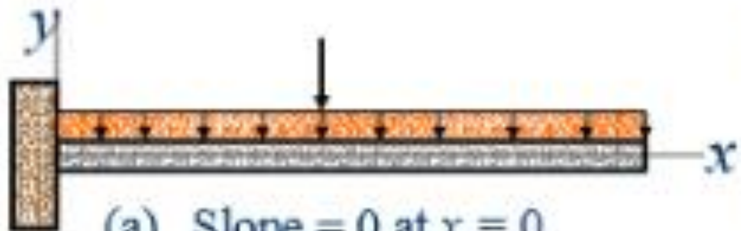
Integrating, $$EI\frac{dy}{dx} = \int_0^x M dx + C_1$$

and $$EIy = \int_0^x \int_0^x M dx + C_1 x + C_2$$

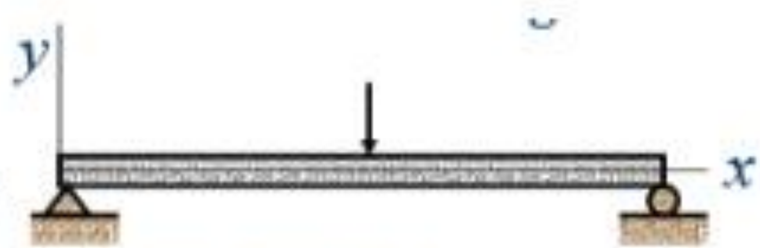
where C_1 and C_2 are constants.

Constants C_1 and C_2 are found by making use of boundary conditions.

- Example Boundary Conditions



(a) Slope = 0 at $x = 0$
Deflection = 0 at $x = 0$



(b) Slope at $L/2 = 0$
Deflection = 0 at $x = 0$, and L

(c) Slope at rollers ?
Deflection at rollers = 0

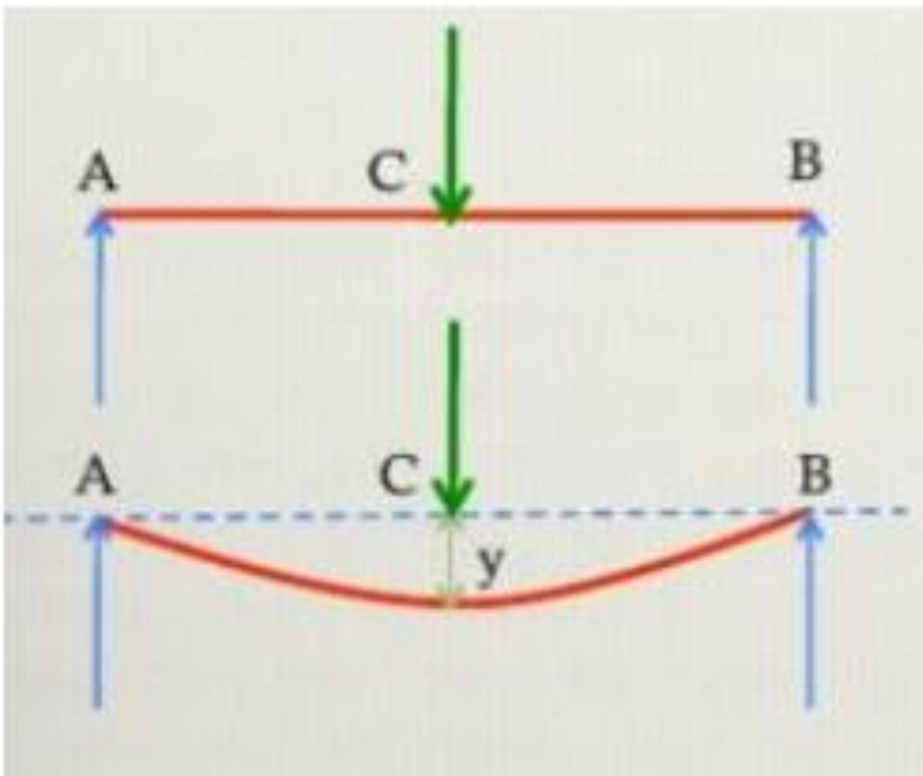
(d) Slope = 0 at $x = 0$
Deflection = 0 at $x = 0$ and $x = L$

(*a*) **At simply supported/roller ends:**
deflection, $y = 0$

(*b*) **At fixed ends:**
deflection, $y = 0$
and slope $\frac{dy}{dx} = 0$

(*c*) At point of symmetry $\frac{dy}{dx} = 0$

Relationship



Deflection $= y$

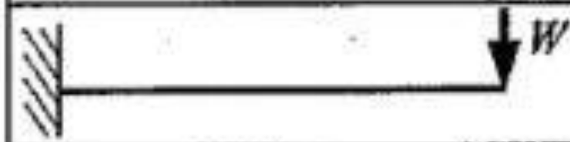
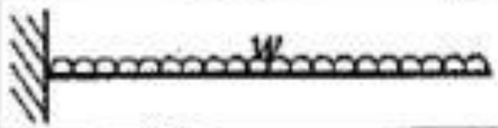
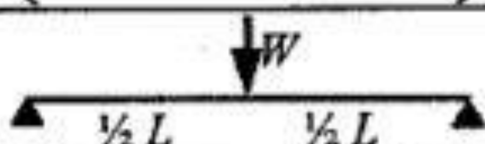
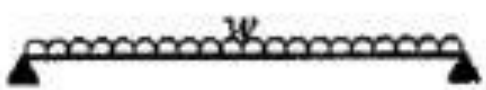
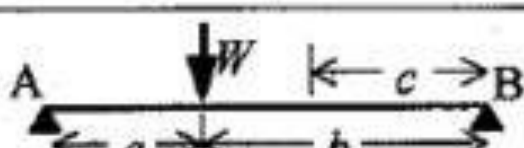
Slope $= \dfrac{dy}{dx}$

Bending moment $= EI\dfrac{d^2y}{dx^2}$

Shearing force $= EI\dfrac{d^3y}{dx^3}$

Rate of loading $= EI\dfrac{d^4y}{dx^4}$

BEAM BENDING

L = overall length W = point load, M = moment w = load per unit length	End Slope	Max Deflection	Max bending moment
	$\frac{ML}{EI}$	$\frac{ML^2}{2EI}$	M
	$\frac{WL^2}{2EI}$	$\frac{WL^3}{3EI}$	WL
	$\frac{wL^3}{6EI}$	$\frac{wL^4}{8EI}$	$\frac{wL^2}{2}$
	$\frac{ML}{2EI}$	$\frac{ML^2}{8EI}$	M
	$\frac{WL^2}{16EI}$	$\frac{WL^3}{48EI}$	$\frac{WL}{4}$
	$\frac{wL^3}{24EI}$	$\frac{5wL^4}{384EI}$	$\frac{wL^2}{8}$
 $a \le b, \quad c = \sqrt{\frac{1}{3} b(L+a)}$	$\theta_B = \frac{Wac^2}{2LEI}$ $\theta_A = \frac{L+b}{L+a}\theta_B$	$\frac{Wac^3}{3LEI}$ (at position c)	$\frac{Wab}{L}$ (under load)

Concept Application 9.1

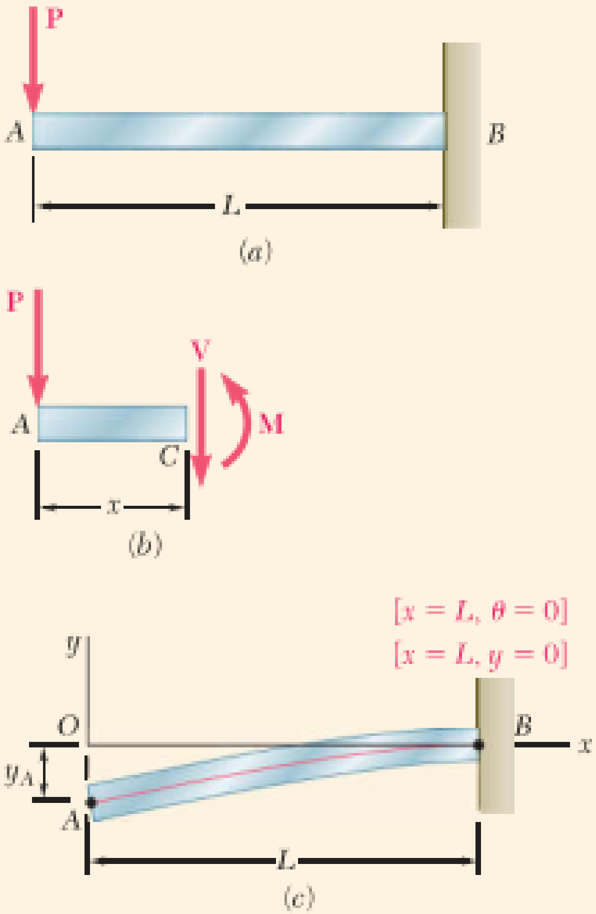


Fig. 9.9 *(a)* Cantilever beam with end load. *(b)* Free-body diagram of section *AC*. *(c)* Deformed shape and boundary conditions.

The cantilever beam *AB* is of uniform cross section and carries a load **P** at its free end A (Fig. 9.9a). Determine the equation of the elastic curve and the deflection and slope at *A*.

Using the free-body diagram of the portion *AC* of the beam (Fig. 9.9*b*), where *C* is located at a distance *x* from end *A*,

$$M = -Px \tag{1}$$

Substituting for *M* into Eq. (9.4) and multiplying both members by the constant *EI* gives

$$EI\frac{d^2y}{dx^2} = -Px$$

Integrating in *x*,

$$EI\frac{dy}{dx} = -\tfrac{1}{2}Px^2 + C_1 \tag{2}$$

Now observe the fixed end *B* where $x = L$ and $\theta = dy/dx = 0$ (Fig. 9.9c). Substituting these values into Eq. (2) and solving for C_1 gives

$$C_1 = \tfrac{1}{2}PL^2$$

which we carry back into Eq. (2):

$$EI\frac{dy}{dx} = -\tfrac{1}{2}Px^2 + \tfrac{1}{2}PL^2 \tag{3}$$

Integrating both members of Eq. (3),

$$EI\,y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x + C_2 \tag{4}$$

But at B, $x = L$, $y = 0$. Substituting into Eq. (4),

$$0 = -\tfrac{1}{6}PL^3 + \tfrac{1}{2}PL^3 + C_2$$
$$C_2 = -\tfrac{1}{3}PL^3$$

Carrying the value of C_2 back into Eq. (4), the equation of the elastic curve is

$$EI\,y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x - \tfrac{1}{3}PL^3$$

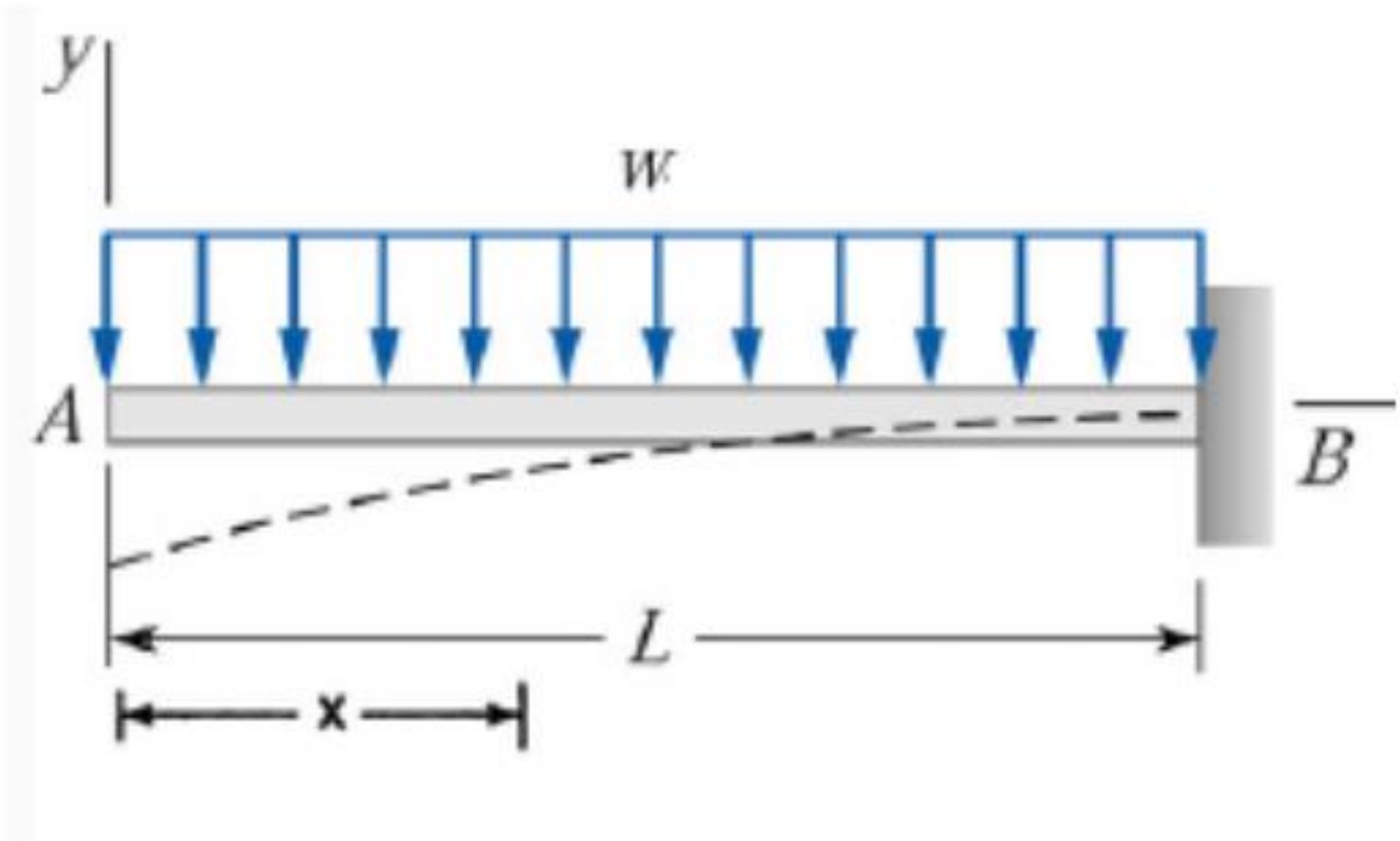
or

$$y = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3) \tag{5}$$

The deflection and slope at A are obtained by letting $x = 0$ in Eqs. (3) and (5).

$$y_A = -\frac{PL^3}{3EI} \quad \text{and} \quad \theta_A = \left(\frac{dy}{dx}\right)_A = \frac{PL^2}{2EI}$$

A Cantilever Subjected to Uniformly Distributed Load



$$M_x = \frac{-wx^2}{2}$$

i.e. $$EI\frac{d^2y}{dx^2} = \frac{-wx^2}{2}$$

$\therefore$ $$EI\frac{dy}{dx} = \frac{-wx^3}{6} + C_1$$

At $x = L$, $$\frac{dy}{dx} = 0$$

$\therefore$ $$0 = \frac{-wL^3}{6} + C_1$$

or $$C_1 = \frac{wL^3}{6}$$

i.e. $$EI\frac{dy}{dx} = \frac{-wx^3}{6} + \frac{wL^3}{6}$$

Integrating again, we get $$EIy = \frac{-wx^4}{24} + \frac{wL^3x}{6} + C_2$$

Integrating again, we get $EIy = \dfrac{-wx^4}{24} + \dfrac{wL^3x}{6} + C_2$

At $x = L$, $\quad y = 0$

$$\therefore \quad 0 = \frac{-wL^4}{24} + \frac{wL^4}{6} + C_2$$

or

$$C_2 = -\frac{wL^4}{6} + \frac{wL^4}{24}$$

$$= \frac{-wL^4}{8}$$

$$\therefore \quad EIy = \frac{-wx^4}{24} + \frac{wL^3}{6}x - \frac{wL^4}{8}$$

At free end where $x = 0$ we get

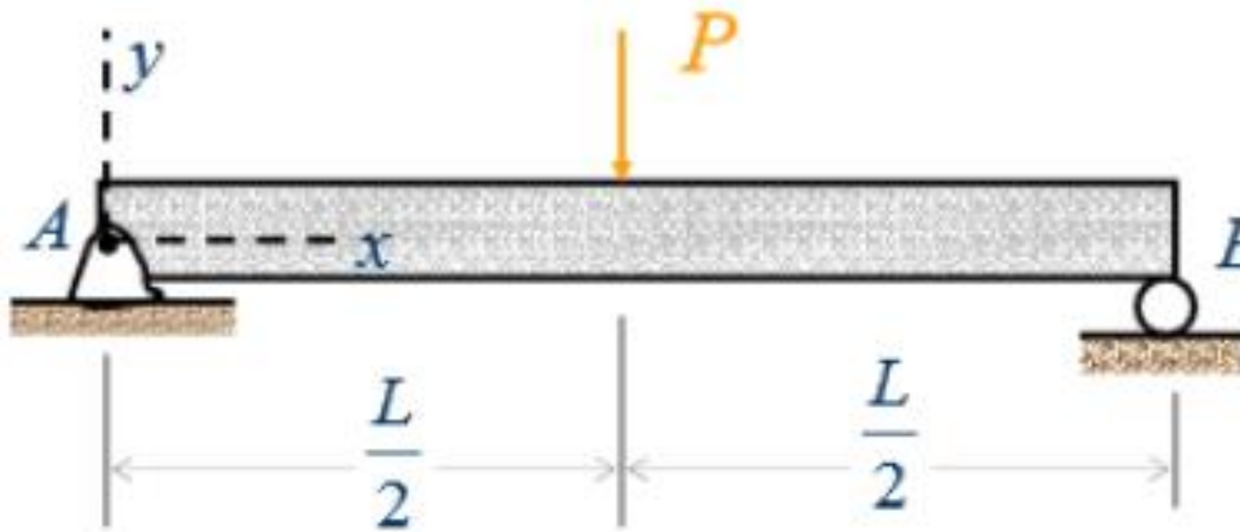
$$\frac{dy}{dx} = \frac{wL^3}{6EI}$$

and

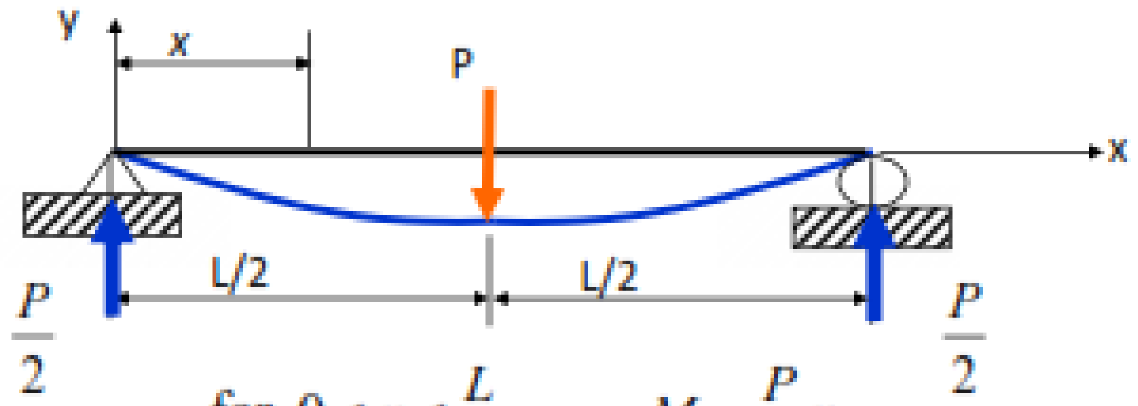
$$y = \frac{1}{EI}\left(\frac{-wL^4}{8}\right) = \frac{-wL^4}{8EI}$$

$$= \frac{wL^4}{8EI} \text{ downward}$$

Simply Supported Beam Subjected to a Central Concentrated Load



Example



$$\text{for } 0 < x < \frac{L}{2} \qquad M = \frac{P}{2}x$$

$$EI\frac{d^2y}{dx^2} = \frac{P}{2}x \qquad \text{for } 0 < x < \frac{L}{2}$$

Integrating

$$EI\frac{dy}{dx} = \frac{P}{2}\frac{x^2}{2} + c_1$$

Since the beam is symmetric

$$@\ x = \frac{L}{2} \qquad \frac{dy}{dx} = 0$$

$$@\ x = \frac{L}{2} \qquad EI(0) = \frac{P}{2}\frac{\left(\frac{L}{2}\right)^2}{2} + c_1 \Rightarrow \boxed{c_1 = -\frac{PL^2}{16}}$$

$$\boxed{\therefore \quad EI\frac{dy}{dx} = \frac{P}{4}x^2 - \frac{PL^2}{16}}$$

Integrating $$EIy = \frac{P}{4}\frac{x^3}{3} - \frac{PL^2}{16}x + c_2$$

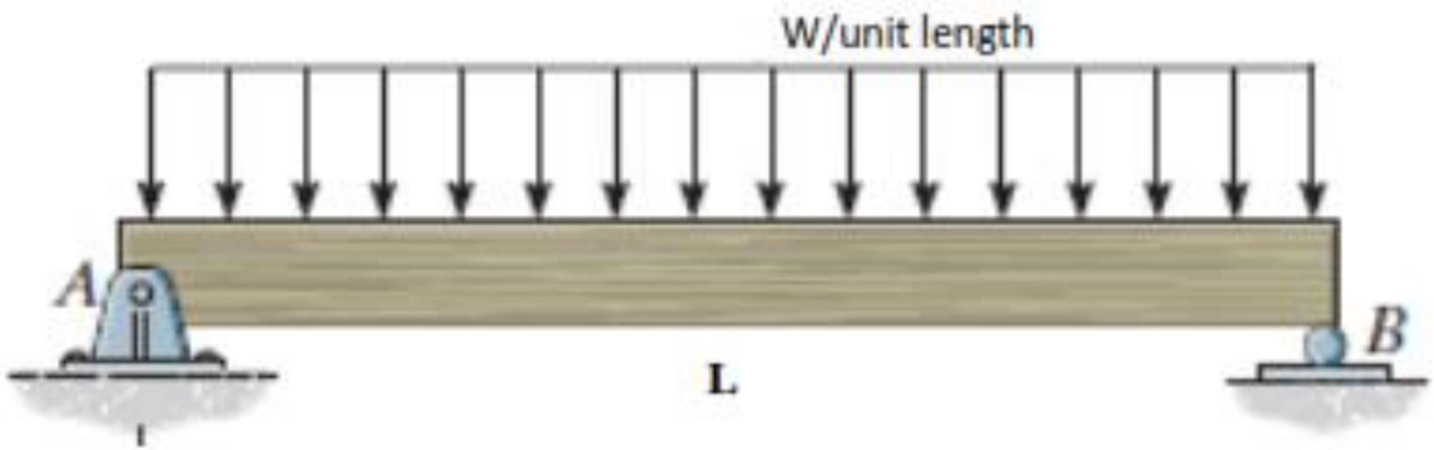
@ x = 0 y = 0 $$\Rightarrow EI(0) = \frac{P}{4}\frac{(0)^3}{3} - \frac{PL^2}{16}(0) + c_2 \quad \Rightarrow \boxed{c_2 = 0}$$

$$\therefore \boxed{EIy = \frac{P}{12}x^3 - \frac{PL^2}{16}x}$$

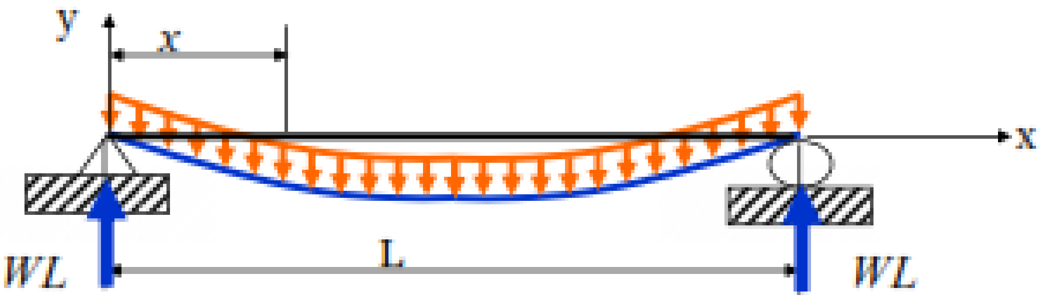
Max. occurs @ x = L /2 $$\boxed{EIy_{max} = -\frac{PL^3}{48}}$$

$$\boxed{\Delta_{max} = \frac{PL^3}{48EI}}$$

Simply Supported Beam Subject to Uniformly Distributed Load



Example



$$\frac{WL}{2} \qquad\qquad \frac{WL}{2}$$

$$M = \frac{WL}{2}x - Wx\frac{x}{2}$$

$$EI\frac{d^2y}{dx^2} = \frac{WL}{2}x - W\frac{x^2}{2}$$

Integrating $$EI\frac{dy}{dx} = \frac{WL}{2}\frac{x^2}{2} - \frac{W}{2}\frac{x^3}{3} + c_1$$

Since the beam is symmetric $\quad @\ x = \frac{L}{2} \quad \frac{dy}{dx} = 0$

$$@\ x = \frac{L}{2} \qquad EI(0) = \frac{WL}{2}\frac{\left(\frac{L}{2}\right)^2}{2} - \frac{W}{2}\frac{\left(\frac{L}{2}\right)^3}{3} + c_1 \Rightarrow \boxed{c_1 = -\frac{WL^3}{24}}$$

$$\therefore \quad \boxed{EI\frac{dy}{dx} = \frac{WL}{4}x^2 - \frac{W}{6}x^3 - \frac{WL^3}{24}}$$

Integrating $$EIy = \frac{WL}{4}\frac{x^3}{3} - \frac{W}{6}\frac{x^4}{4} - \frac{WL^3}{24}x + c_2$$

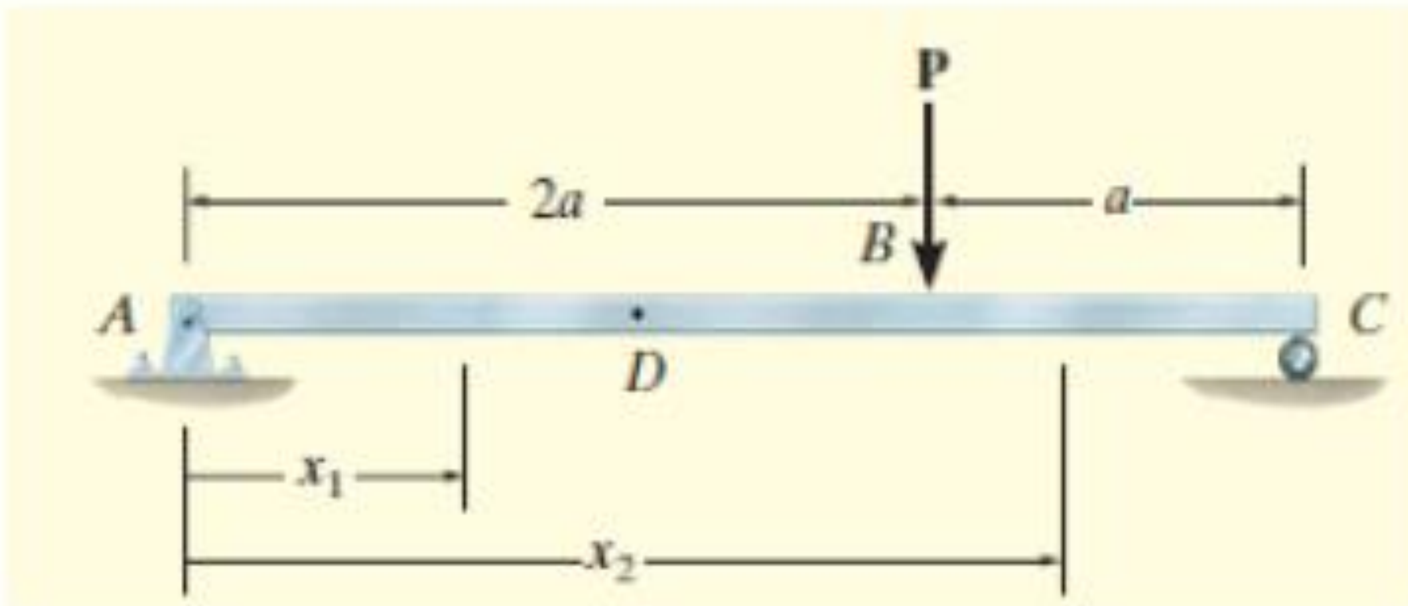
@ x = o y = o $$\Rightarrow EI(0) = \frac{WL}{4}\frac{(0)^3}{3} - \frac{W}{6}\frac{(0)^4}{4} - \frac{WL^3}{24}(0) + c_2 \quad \Rightarrow \boxed{c_2 = 0}$$

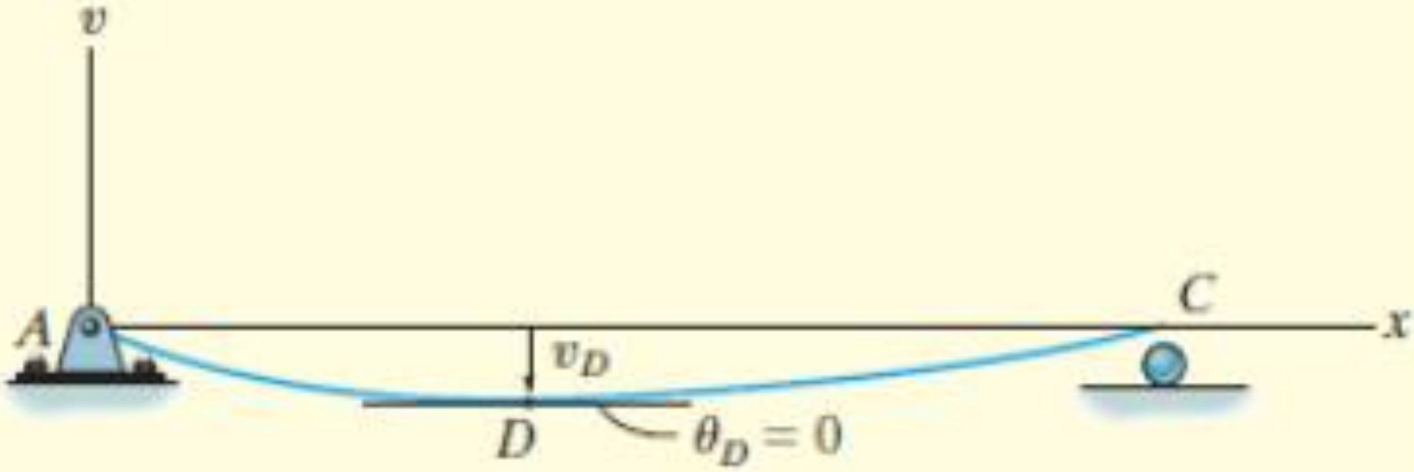
$$\therefore \boxed{EIy = \frac{WL}{12}x^3 - \frac{W}{24}x^4 - \frac{WL^3}{24}x}$$

Max. occurs @ x = L /2 $$\boxed{EIy_{\max} = -\frac{5WL^4}{384}}$$

$$\boxed{\Delta_{\max} = \frac{5WL^4}{384EI}}$$

The simply supported beam shown in Fig. is subjected to the concentrated force **P**. Determine the maximum deflection of the beam. EI is constant.



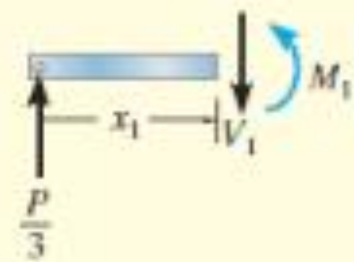


Elastic Curve. The beam deflects as shown in Fig. 12–12*b*. Two coordinates must be used, since the moment function will change at *P*. Here we will take x_1 and x_2, having the *same origin* at *A*.

Moment Function. From the free-body diagrams shown in Fig. 12–12*c*,

$$M_1 = \frac{P}{3}x_1$$

$$M_2 = \frac{P}{3}x_2 - P(x_2 - 2a) = \frac{2P}{3}(3a - x_2)$$



Slope and Elastic Curve. Applying Eq. 12–10 for M_1, for $0 \leq x_1 < 2a$, and integrating twice yields

$$EI\frac{d^2v_1}{dx_1^{\,2}} = \frac{P}{3}x_1$$

$$EI\frac{dv_1}{dx_1} = \frac{P}{6}x_1^{\,2} + C_1 \tag{1}$$

$$EIv_1 = \frac{P}{18}x_1^{\,3} + C_1x_1 + C_2 \tag{2}$$

Likewise for M_2, for $2a < x_2 \leq 3a$,

$$EI\frac{d^2v_2}{dx_2^{\,2}} = \frac{2P}{3}(3a - x_2)$$

$$EI\frac{dv_2}{dx_2} = \frac{2P}{3}\left(3ax_2 - \frac{x_2^{\,2}}{2}\right) + C_3 \tag{3}$$

$$EIv_2 = \frac{2P}{3}\left(\frac{3}{2}ax_2^{\,2} - \frac{x_2^{\,3}}{6}\right) + C_3x_2 + C_4 \tag{4}$$

The four constants are evaluated using *two* boundary conditions, namely, $x_1 = 0$, $v_1 = 0$ and $x_2 = 3a$, $v_2 = 0$. Also, *two* continuity conditions must be applied at B, that is, $dv_1/dx_1 = dv_2/dx_2$ at $x_1 = x_2 = 2a$ and $v_1 = v_2$ at $x_1 = x_2 = 2a$. Substitution as specified results in the following four equations:

$v_1 = 0$ at $x_1 = 0$; $\qquad 0 = 0 + 0 + C_2$

$v_2 = 0$ at $x_2 = 3a$; $\qquad 0 = \dfrac{2P}{3}\left(\dfrac{3}{2}a(3a)^2 - \dfrac{(3a)^3}{6}\right) + C_3(3a) + C_4$

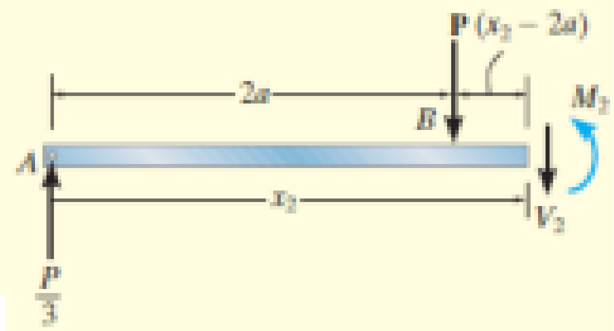
$\dfrac{dv_1(2a)}{dx_1} = \dfrac{dv_2(2a)}{dx_2}$; $\qquad \dfrac{P}{6}(2a)^2 + C_1 = \dfrac{2P}{3}\left(3a(2a) - \dfrac{(2a)^2}{2}\right) + C_3$

$v_1(2a) = v_2(2a)$; $\qquad \dfrac{P}{18}(2a)^3 + C_1(2a) + C_2 = \dfrac{2P}{3}\left(\dfrac{3}{2}a(2a)^2 - \dfrac{(2a)^3}{6}\right) + C_3(2a) + C_4$

Solving, we get

$$C_1 = -\frac{4}{9}Pa^2 \qquad C_2 = 0$$

$$C_3 = -\frac{22}{9}Pa^2 \qquad C_4 = \frac{4}{3}Pa^3$$



Thus Eqs. 1–4 become

$$\frac{dv_1}{dx_1} = \frac{P}{6EI}x_1^2 - \frac{4Pa^2}{9EI} \tag{5}$$

$$v_1 = \frac{P}{18EI}x_1^3 - \frac{4Pa^2}{9EI}x_1 \tag{6}$$

$$\frac{dv_2}{dx_2} = \frac{2Pa}{EI}x_2 - \frac{P}{3EI}x_2^2 - \frac{22Pa^2}{9EI} \tag{7}$$

$$v_2 = \frac{Pa}{EI}x_2^2 - \frac{P}{9EI}x_2^3 - \frac{22Pa^2}{9EI}x_2 + \frac{4Pa^3}{3EI} \tag{8}$$

By inspection of the elastic curve, Fig. 12–12*b*, the maximum deflection occurs at *D*, somewhere within region *AB*. Here the slope must be zero. From Eq. 5,

$$\frac{1}{6}x_1^2 - \frac{4}{9}a^2 = 0$$

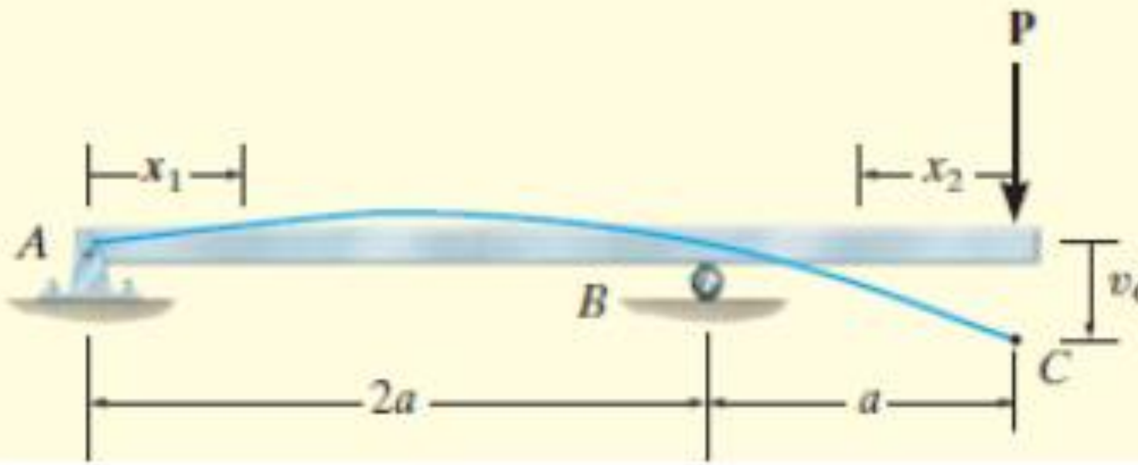
$$x_1 = 1.633a$$

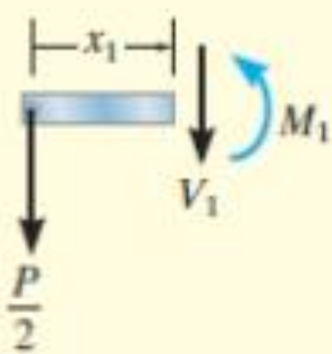
Substituting into Eq. 6,

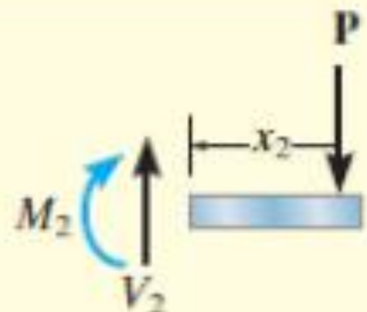
$$v_{max} = -0.484\frac{Pa^3}{EI} \qquad \textit{Ans.}$$

The negative sign indicates that the deflection is downward.

The beam in Fig. is subjected to a load **P** at its end. Determine the displacement at C. EI is constant.

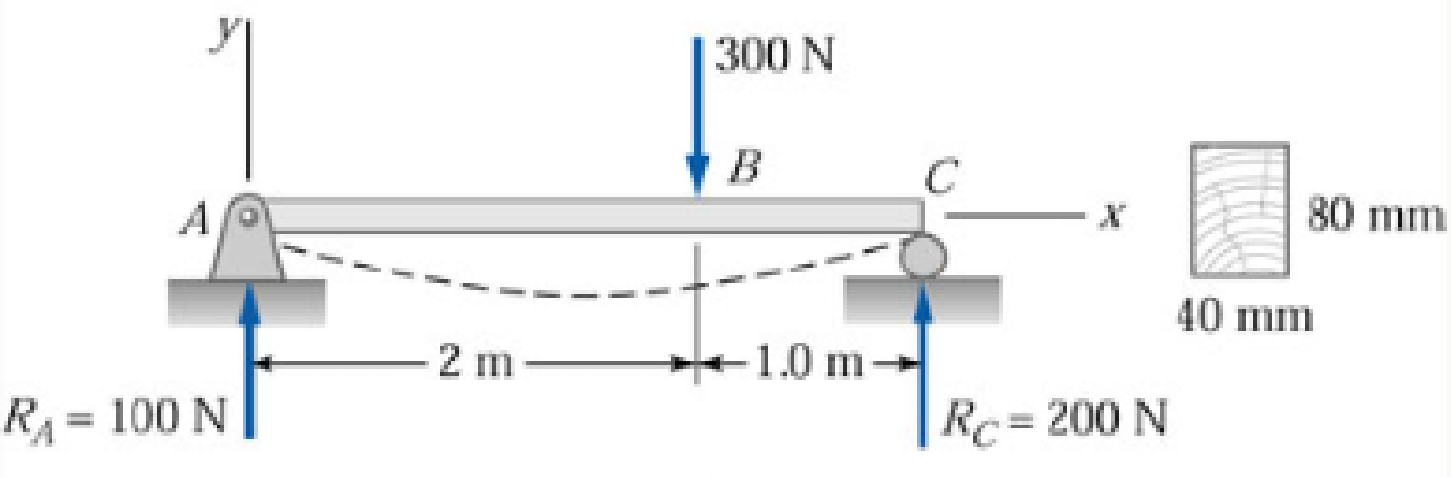






Sample Problem

The simply supported wood beam ABC in Fig. (a) has the rectangular cross section shown. The beam supports a concentrated load of 300 N located 2 m from the left support. Determine the maximum displacement and maximum slope angle of the beam. Use E = 12 Gpa for the modulus of elasticity. Neglect the weight of the beam.



(a)

The negative sign indicates that the deflection is downward, as expected. Thus, the maximum displacement is

$$\delta_{max} = |\upsilon|_{x=1.633m} = 7.09mm \downarrow \qquad Answer$$

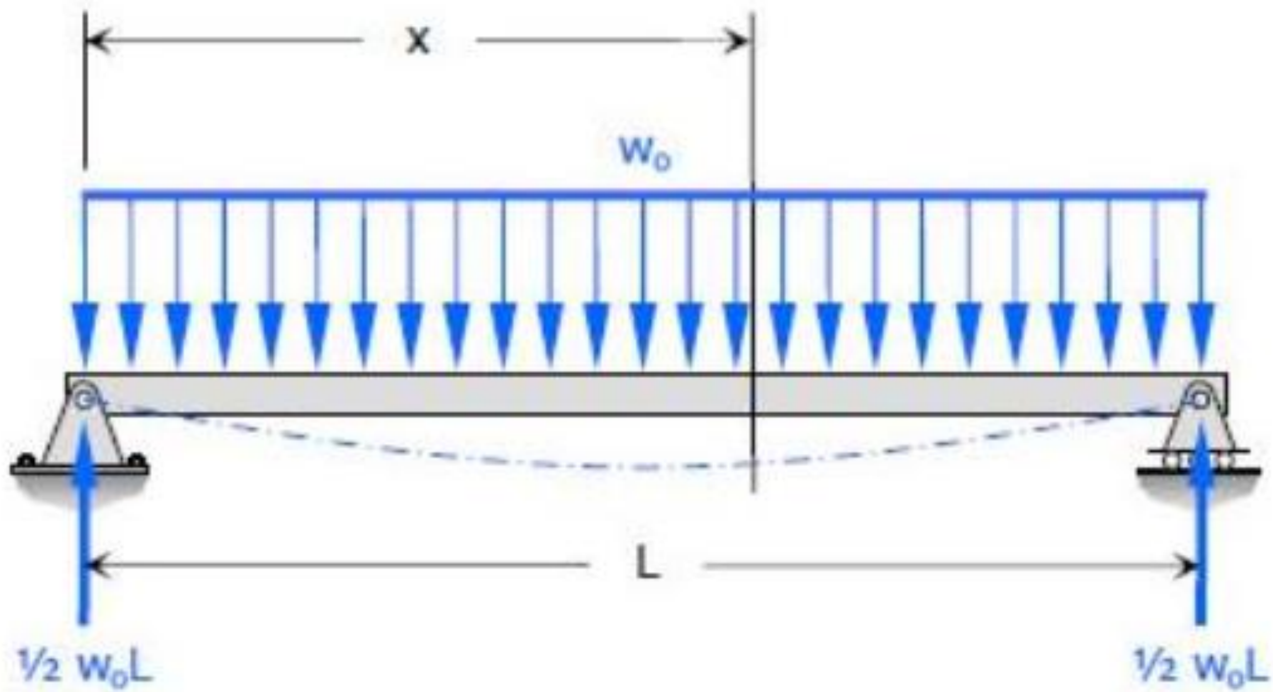
$$\theta_{max} = |\upsilon'|_{x=3m} = 8.50\times10^{-3}\,rad = 0.487^{0} \quad \circlearrowleft \qquad Answer$$

Macaulay's method

For discontinuous uniformly distributed load and point loads at various points in a beam, a single equation of M_x will not be sufficient to determine the slope and deflection everywhere in the beam.

In such situation's Mecaulay's method may be used. In Mecaulay's method, a single equation of moment is formed for all loads on the beam in such a way that the constants of integration apply equally to all positions of the beam.

Determine the maximum deflection δ in a simply supported beam of length L carrying a uniformly distributed load of intensity w_o applied over its entire length.



$$EI\,y'' = \frac{1}{2}w_oLx - w_ox(\frac{1}{2}x)$$

$$EI\,y'' = \frac{1}{2}w_oLx - \frac{1}{2}w_ox^2$$

$$EI\,y' = \frac{1}{4}w_oLx^2 - \frac{1}{6}w_ox^3 + C_1$$

$$EI\,y = \frac{1}{12}w_oLx^3 - \frac{1}{24}w_ox^4 + C_1x + C_2$$

At x = 0, y = 0, therefore $C_2 = 0$

At x = L, y = 0

$$0 = \frac{1}{12}w_oL^4 - \frac{1}{24}w_oL^4 + C_1L$$

$$C_1 = -\frac{1}{24}w_oL^3$$

Therefore,

$$EI\,y = \frac{1}{12}w_oLx^3 - \frac{1}{24}w_ox^4 - \frac{1}{24}w_oL^3x$$

Maximum deflection will occur at x = ½ L (midspan)

$$EI\,y_{max} = \frac{1}{12}w_oL(\frac{1}{2}L)^3 - \frac{1}{24}w_o(\frac{1}{2}L)^4 - \frac{1}{24}w_oL^3(\frac{1}{2}L)$$

$$EI\,y_{max} = \frac{1}{96}w_oL^4 - \frac{1}{384}w_oL^4 - \frac{1}{48}w_oL^4$$

$$EI\,y_{max} = -\frac{5}{384}w_oL^4$$

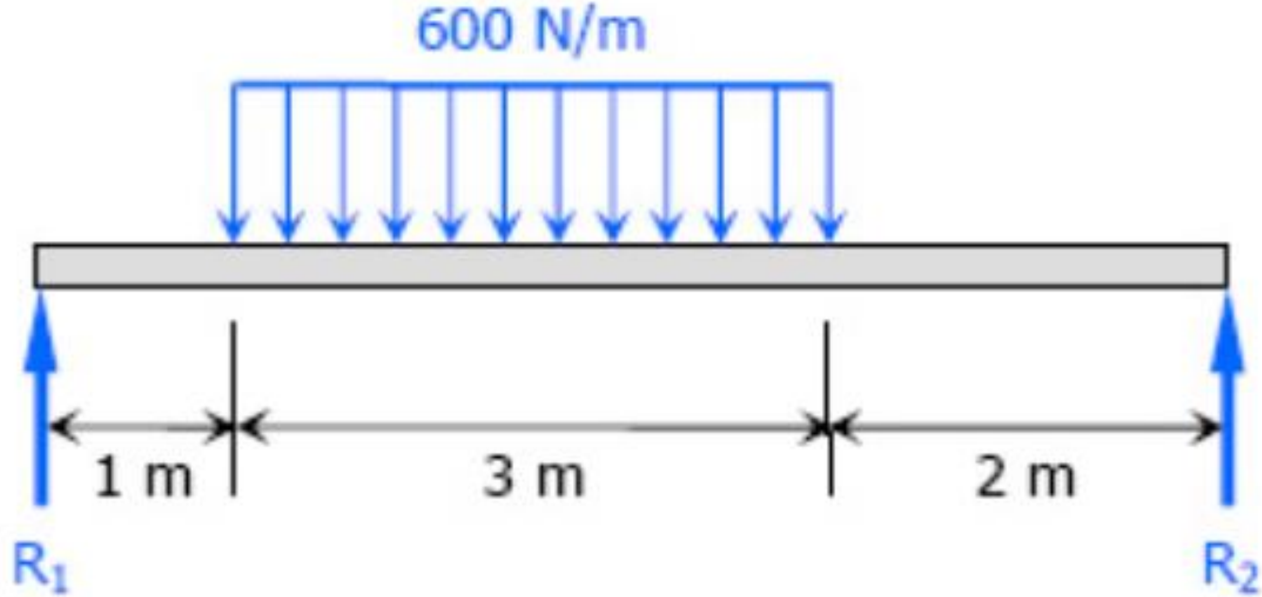
$$\delta_{max} = \frac{5w_oL^4}{384EI}$$ *answer*

Taking W = w_oL:

$$\delta_{max} = \frac{5(w_oL)(L^3)}{384EI}$$

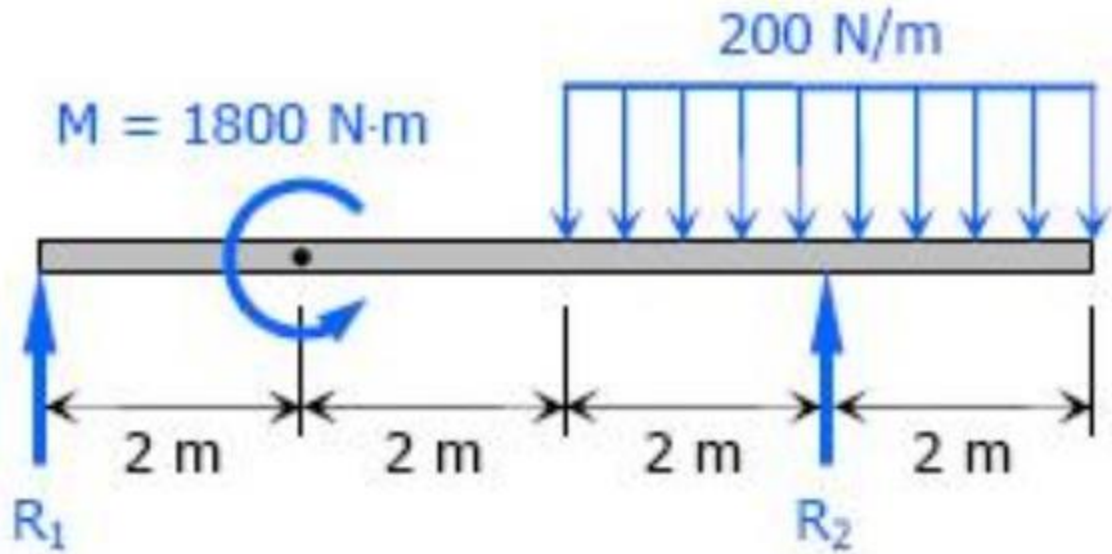
$$\delta_{max} = \frac{5WL^3}{384EI}$$ *answer*

Compute the midspan value of EI δ for the beam loaded as shown in Fig.



$$EI\,\delta_{midspan} = 6962.5\,\mathrm{N \cdot m^3}$$ *answer*

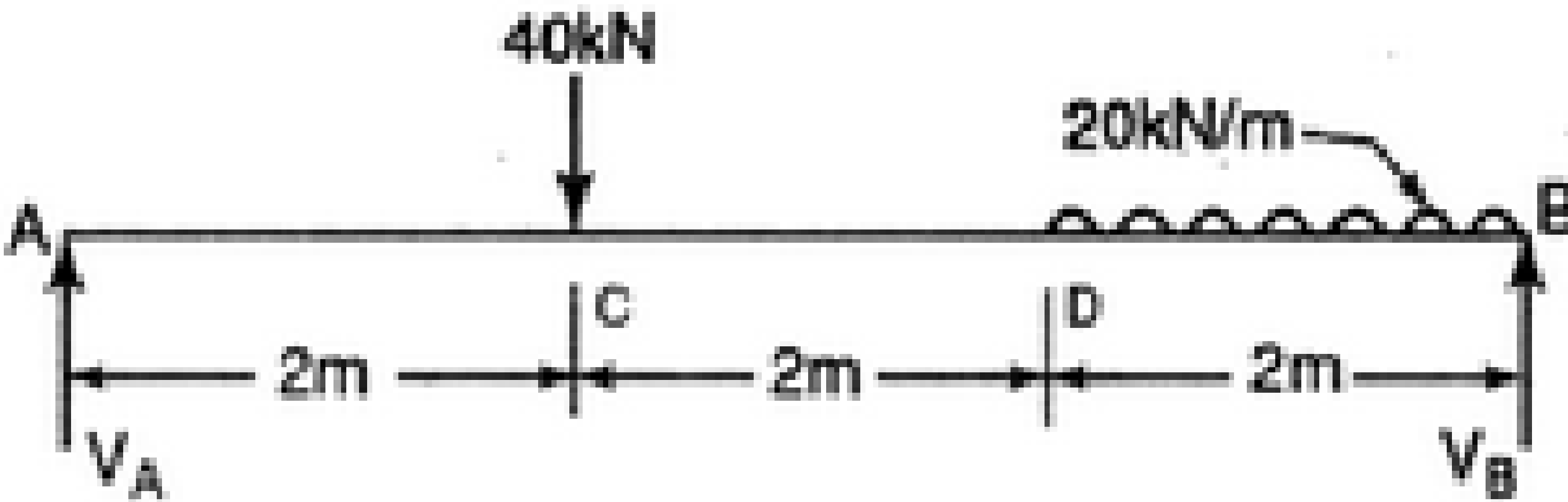
Determine the value of EIy midway between the supports for the beam loaded as shown in Fig. I



$$EI\,y = \frac{6950}{3}\ \mathrm{N \cdot m^3}$$

answer

Find the maximum deflection and the maximum slope for the beam loaded as shown in Fig. Take flexural rigidity $EI = 15 \times 10^9$ kN–mm^2 = 15000 kN–m^2



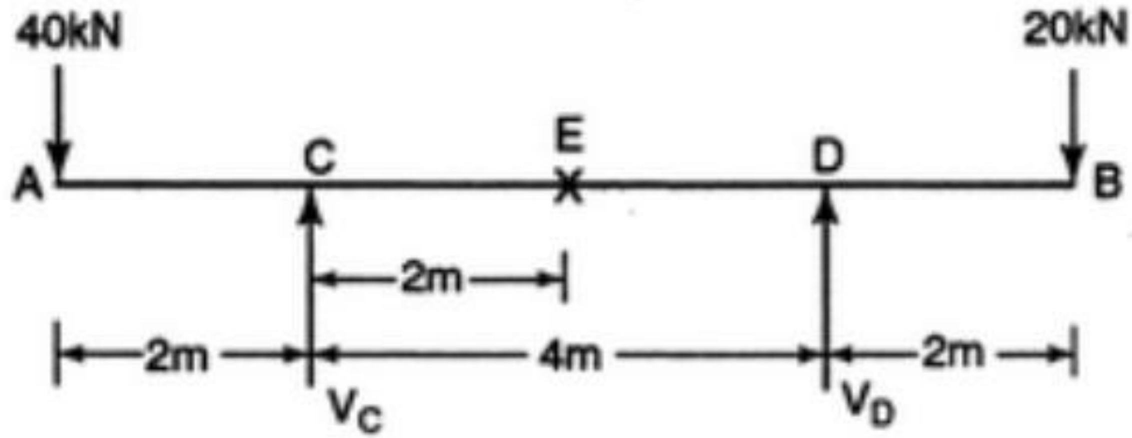
$$y_{max} = -15.8 \times 10^{-3}\ \text{m}$$
$$= -15.8\ \text{mm} \qquad \textbf{(Ans)}$$

$$\left(\frac{dy}{dx}\right)_A = -8.4444 \times 10^{-3}\ \text{radians} \qquad \textbf{(Ans)}$$

Example A double overhanging beam AB of 8 m length rests symmetrically on supports C and D, 4 metres apart. A load of 40 kN acts at free end A and a load of 20 kN at the free end B. Neglecting the self weight of the beam, calculate the deflection relative to the level of the supports

a) at the ends A and B *b*) at the centre of CD

Take $E = 200$ GPa, $I = 50 \times 10^6$ mm^4



$y_A = -373.333 \times 10^{-3}$ m

$= 37.333$ mm (Downward) **(Ans)**

$y_B = -26.667$mm

$= 26.667$mm (downward) **(Ans)**

Moment–area method

Moment-area method The moment- area method is a semigraphical procedure that utilizes the properties of the area under the bending moment diagram. It is the quickest way to compute the deflection at a **specific location** if the bending moment diagram has a simple shape.

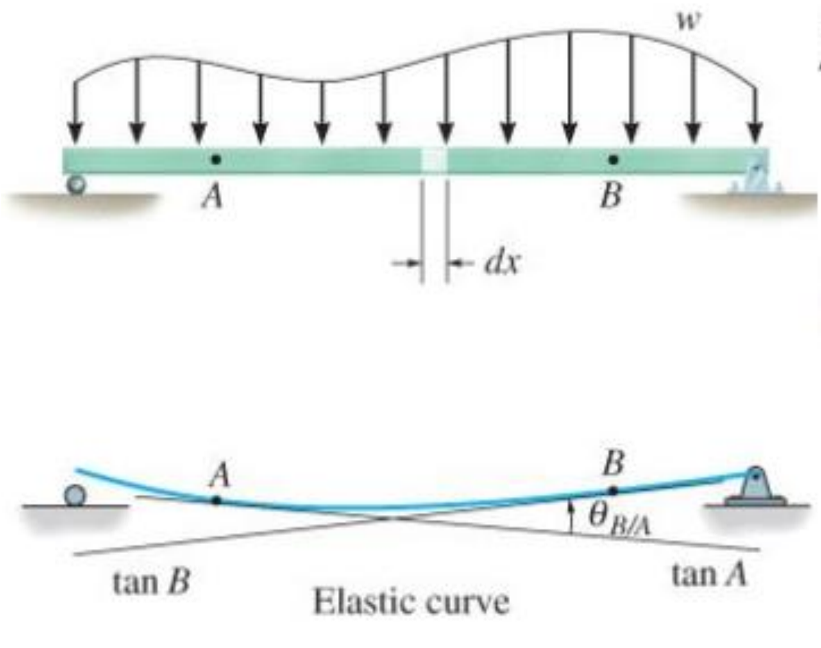
Assumptions:

- beam is initially straight,
- is elastically deformed by the loads, such that the slope and deflection of the elastic curve are very small, and
- deformations are caused by bending.

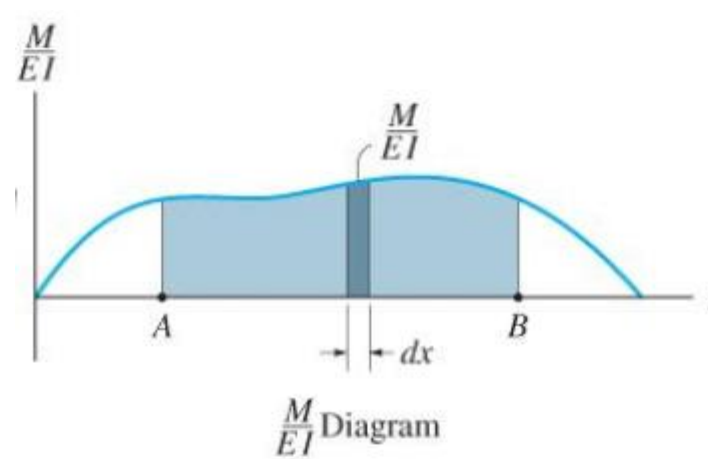
First Moment –Area Theorem

- The angle between the tangents at any two pts on the elastic curve equals the area under the M/EI diagram between these two pts.

$$\theta_{B/A} = \int_A^B \frac{M}{EI} dx \qquad \theta_B - \theta_A = \int_A^B \frac{M}{EI} dx$$



(a)



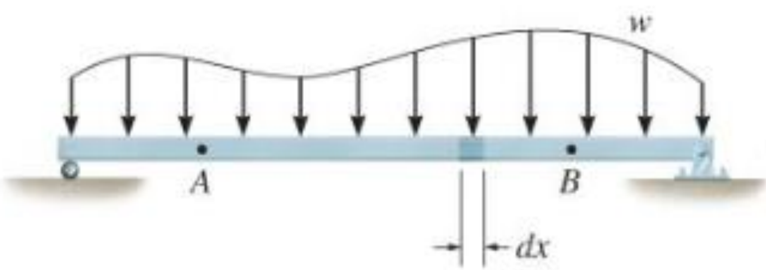
(c)

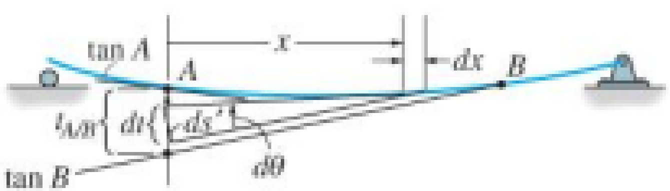
Second moment area theorem :

The vertical distance of point A on a deflection curve from the tangent drawn to the curve at B is equal to the moment with respect to the vertical through A of the area of the bending diagram between A and B, divided by the product EI.

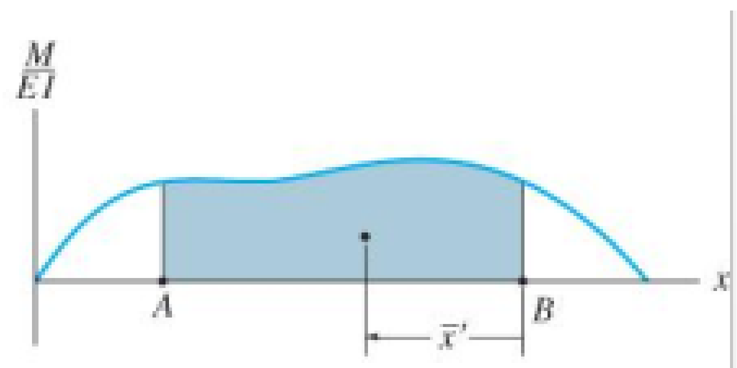
This moment is computed about pt (A) where the vertical deviation ($t_{A/B}$) is to be determined.

$$t_{A/B} = \int_A^B \frac{M}{EI}\,\bar{x}\,dx$$

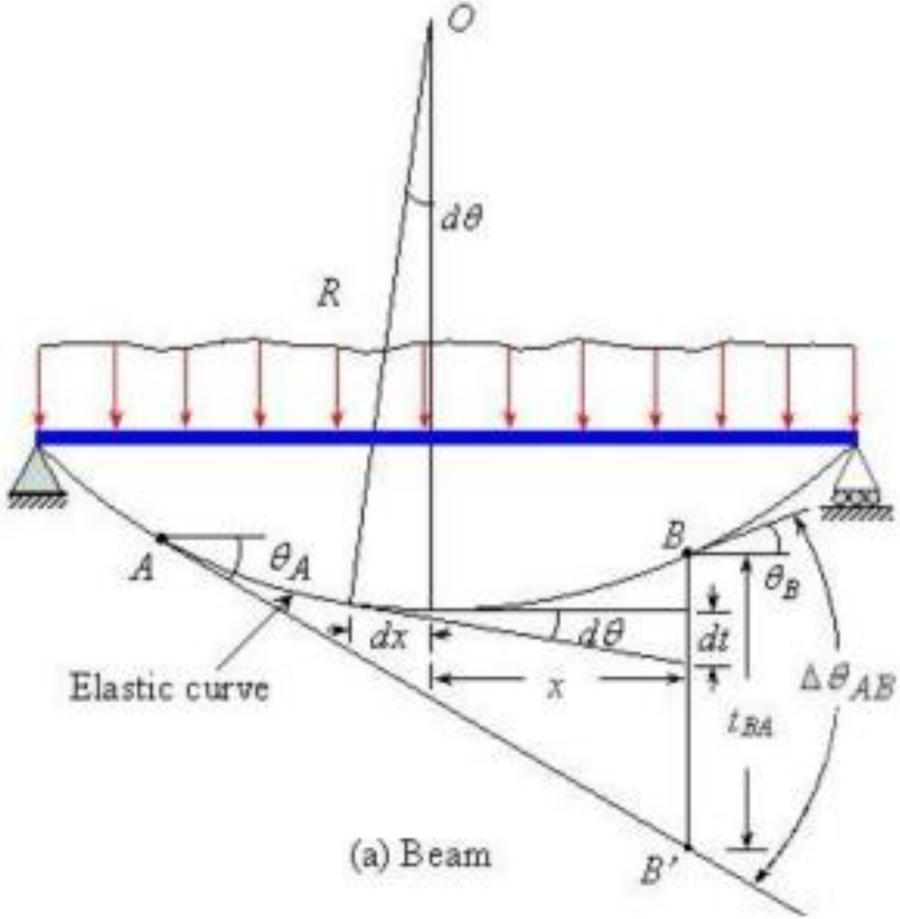




(a)



(c)

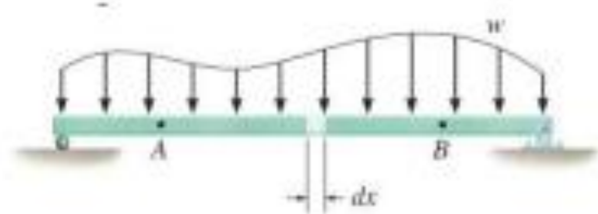


(a) Beam

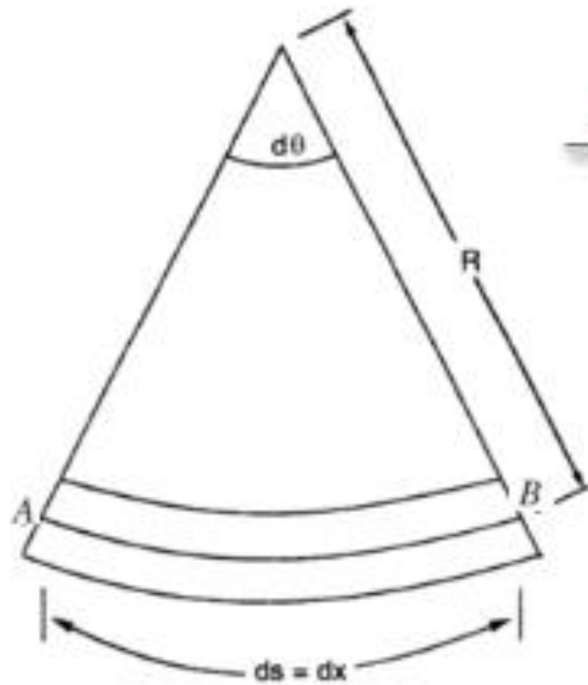
DERIVATION OF MOMENT AREA THEOREMS

Fig shows the elemental length 'dx' Let R be the radius of curvature. Then, from flexure formula

$$\frac{M}{I} = \frac{E}{R}$$



(a)



$R d\theta = ds = dx$, since axial deformations are considered negligible

$$\therefore \qquad R = \frac{dx}{d\theta}$$

Substituting this value of R in equation 2.3, we get

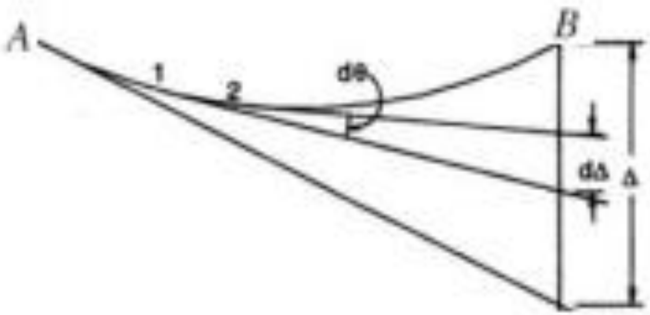
$$\frac{M}{I} = \frac{E}{(dx/d\theta)}$$

or

$$d\theta = \frac{M}{EI} dx$$

$$\theta_{B/A} = \int_A^B \frac{M}{EI} dx \qquad \theta_B - \theta_A = \int_A^B \frac{M}{EI} dx$$

Let the change of slope in elemental length dx be '$d\theta$.' Distance of elemental length from B is x
Hence deflection.



$$d\Delta = x d\theta = x \frac{M}{EI} dx \quad = \quad \Delta = \int_A^B \frac{M}{EI} x \, dx \quad = \quad \boxed{t_{B/A} = \int_A^B \frac{M}{EI} \bar{x} dx}$$

The Moment Area Procedure

1. The reactions of the beam are determined
2. An approximate deflection curve is drawn. This curve must be consistent with the known conditions at the supports, such as zero slope or zero deflection
3. The bending moment diagram is drawn for the beam. Construct M/EI diagram
4. Convenient points A and B are selected and a tangent is drawn to the assumed deflection curve at one of these points, say A
5. The deflection of point B from the tangent at A is then calculated by the second moment area theorem

Comparison of Moment Area and Double Integration Methods

If the deflection of only a single point of a beam is desired, the moment-area method is usually more convenient than the double integration method.

If the equation of the deflection curve of the entire beam is desired the double integration method is preferable.

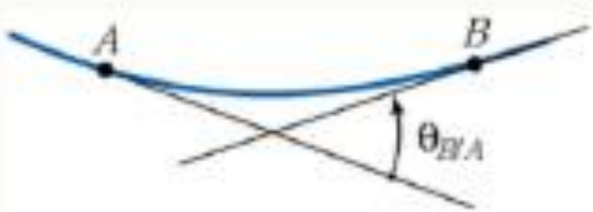
SignConventions for tangential deviation and change of slope.

- The tangential deviation $t_{B/A}$ is positive if B lies above the tangent line drawn to the elastic curve at A, and negative if B lies above the tangent line.

(a) Positive deviation; B located above reference tangent

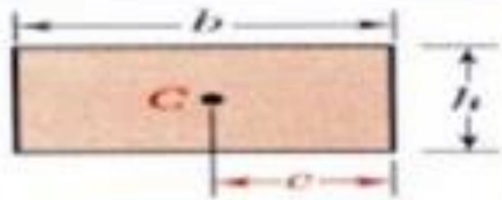
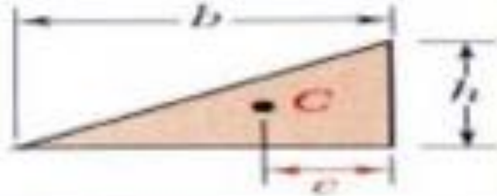
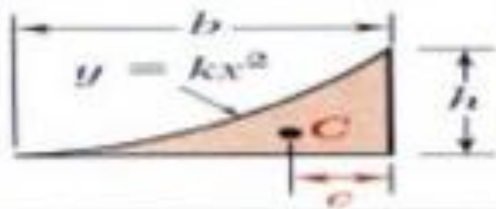
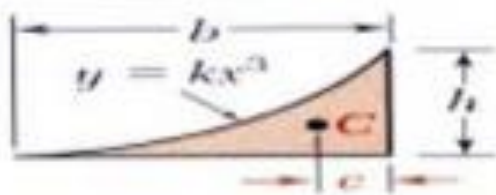
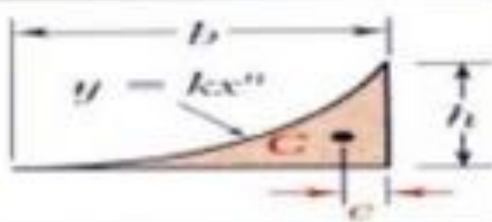
(b) Negative deviation; B located below reference tangent

- Positive $\theta_{B/A}$ has a counterclockwise direction, whereas negative $\theta_{B/A}$ has a clockwise direction.

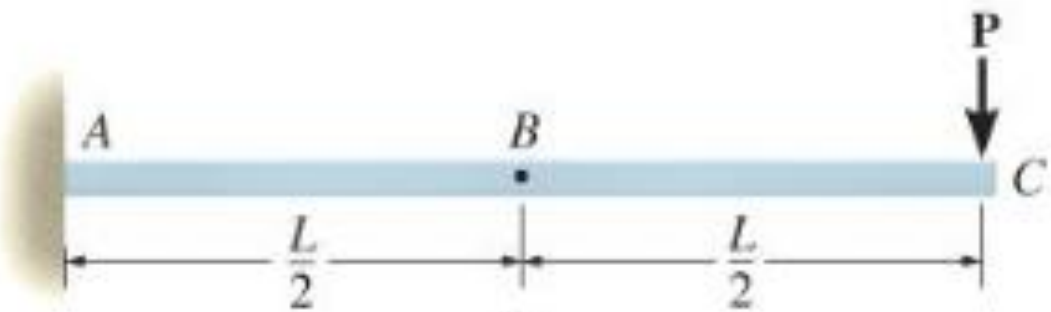


(c) Positive change of slope is counterclockwise from left tangent

(d) Negative change of slope is clockwise from left tangent

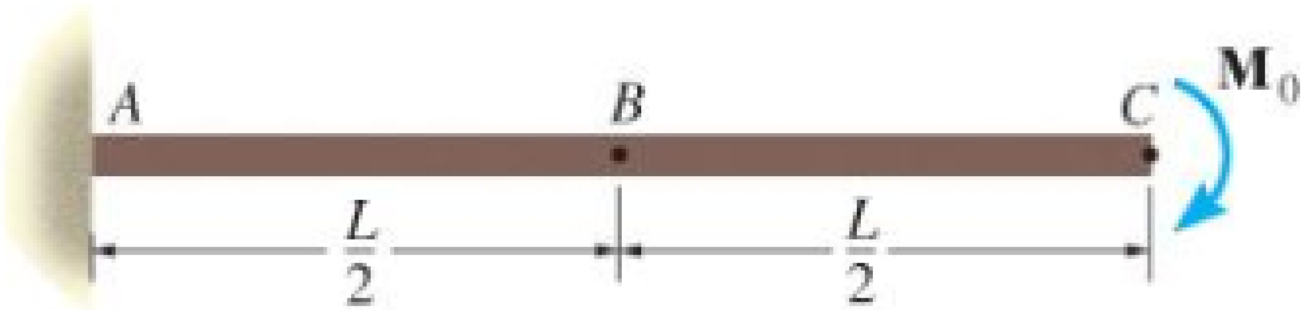
Shape		Area	c
Rectangle		bh	$\frac{b}{2}$
Triangle		$\frac{bh}{2}$	$\frac{b}{3}$
Parabolic spandrel		$\frac{bh}{3}$	$\frac{b}{4}$
Cubic spandrel		$\frac{bh}{4}$	$\frac{b}{5}$
General spandrel		$\frac{bh}{n+1}$	$\frac{b}{n+2}$

Determine the slope of the beam shown at pts B and C. EI is constant.



(a)

Determine the displacement of pts B and C of beam shown. EI is constant.



(a)

A Cantilever Subjected to Concentrated Load at Free End

Slope at $A = \frac{1}{EI}$ [area of B.M. diagram between A and B

$$= \frac{1}{EI}\left[\frac{L}{2}WL\right] = \frac{WL^2}{2EI}$$

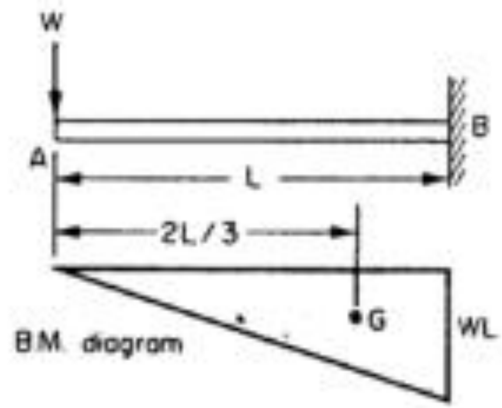


Fig. 5.20.

Deflection at A (relative to B)

$= \frac{1}{EI}$ [first moment of area of B.M. diagram between A and B about A]

$$= \frac{1}{EI}\left[\left(\frac{L}{2}WL\right)\frac{2L}{3}\right] = \frac{WL^3}{3EI}$$

A Cantilever Subjected to Uniformly Distributed Load

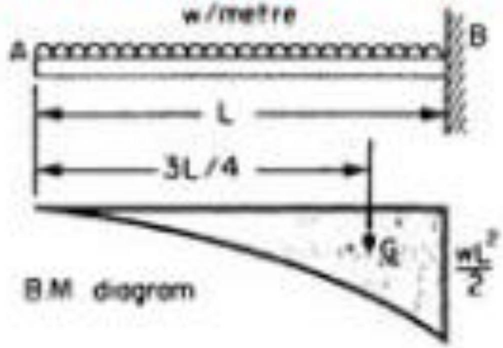


Fig. 5.21.

Again B is a point of zero slope.

$$\therefore \quad \text{slope at } A = \frac{1}{EI}[\text{area of B.M. diagram}$$

$$= \frac{1}{EI}\left[\frac{1}{3}L\frac{wL^2}{2}\right]$$

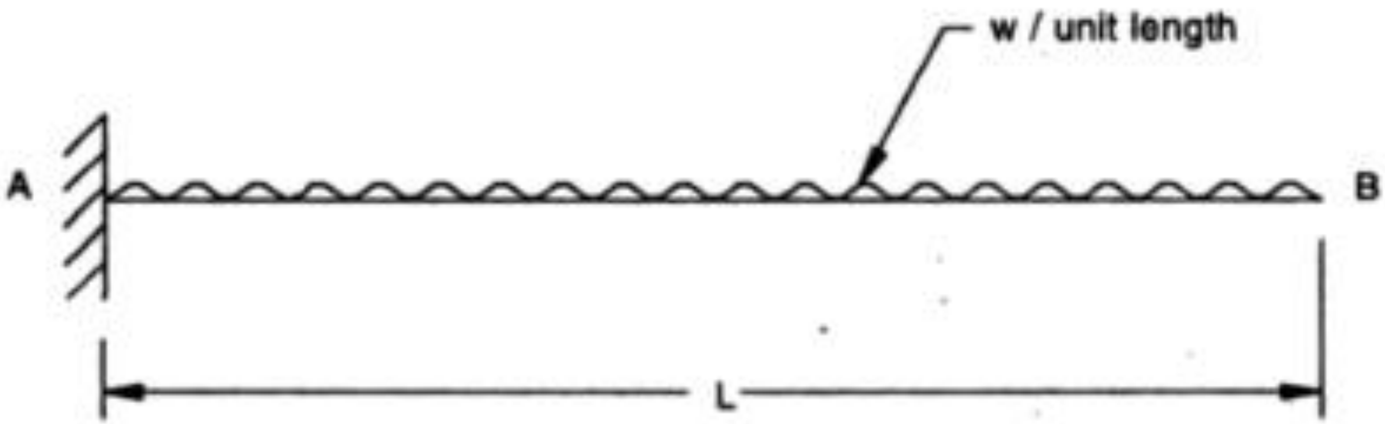
$$= \frac{wL^3}{6EI}$$

$$\text{Deflection at } A = \frac{1}{EI}[\text{moment of B.M. diagram about } A]$$

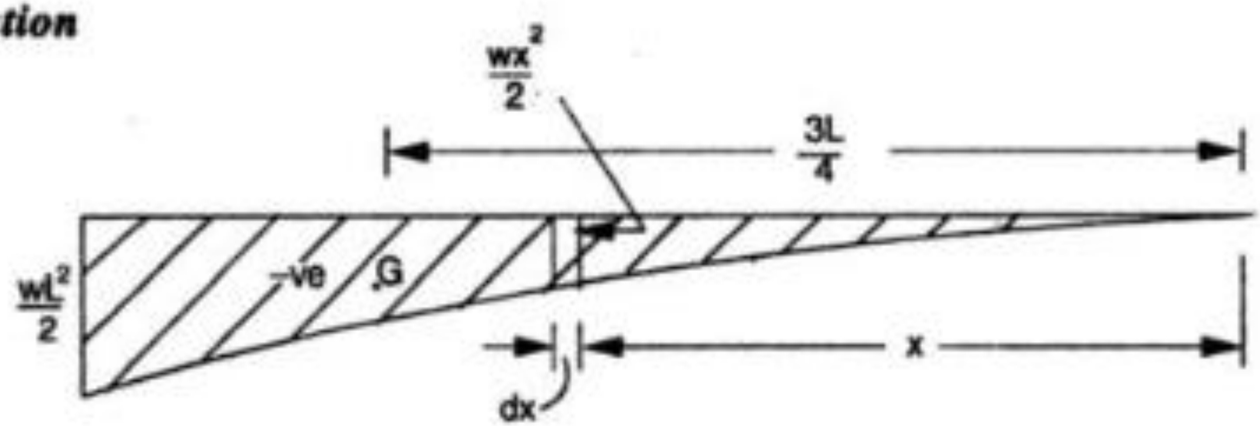
$$= \frac{1}{EI}\left[\left(\frac{1}{3}L\frac{wL^2}{2}\right)\frac{3L}{4}\right] = \frac{wL^4}{8EI}$$

OR

Example Determine the rotation and deflection at the free end of the cantilever beam subjected to uniformly distributed load over an entire span as shown in Fig.



Solution



The bending moment diagram is shown in Fig.2.4(b). At any distance x from free end, bending moment is $-\frac{wx^2}{2}$.

Now,

$$\theta_{BA} = \theta_B - \theta_A = \theta_B \qquad \because \quad \theta_A = 0$$

∴ From the moment area theorem,

$$\theta_B = \int_0^L \frac{M}{EI} dx$$

$$= \int_0^L -\frac{wx^2}{2EI} dx$$

$$= -\frac{w}{2EI}\left[\frac{x^3}{3}\right]_0^L$$

$$= \frac{-wL^3}{6EI}$$

$$= \frac{wL^3}{6EI}, \text{ clockwise with tangent at A}$$

Δ_B = deflection of B with respect to tangent at A
= vertical deflection, since tangent at A is horizontal.

From the second moment area theorem,

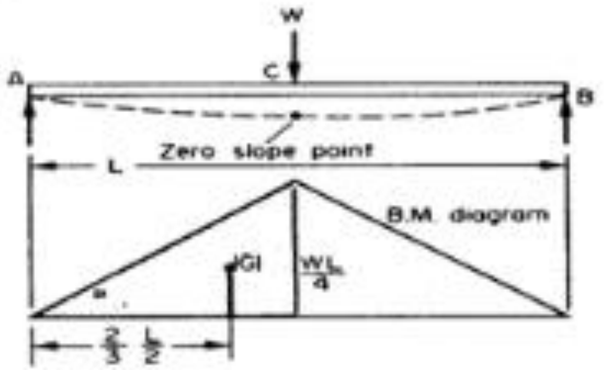
$$\Delta_B = \int_0^L \frac{M}{EI} x \, dx = \int_0^L -\frac{wx^3}{2EI} dx$$

$$= -\frac{w}{2EI}\left[\frac{x^4}{4}\right]_0^L = \frac{-wL^4}{8EI}$$

$$= \frac{wL^4}{8EI}, \text{ downward}$$

Note: Area of such a parabolic curve is = $\frac{1}{3} \times L \times$ Ordinate at the end and its centre of gravity is at a distance $\frac{3L}{4}$, from the end where the value is zero.

Determine the end slope and deflection of the mid-point C in the beam shown below using moment area method .



Again working relative to the zero slope point at the centre C.

$$\text{slope at } A = \frac{1}{EI}[\text{area of B.M. diagram between } A \text{ and } C \text{ (Fig. 5.23)}]$$

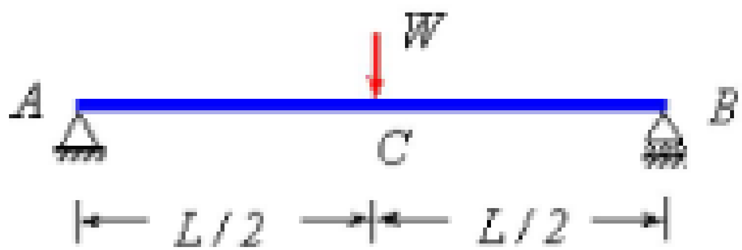
$$= \frac{1}{EI}\left[\frac{1}{2}\frac{L}{2}\frac{WL}{4}\right] = \frac{WL^2}{16EI}$$

Deflection of A relative to C (= central deflection of C)

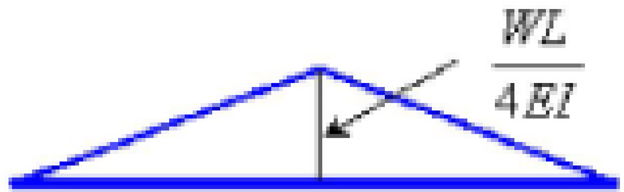
$$= \frac{1}{EI}[\text{moment of B.M. diagram between } A \text{ and } C \text{ about } A]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\frac{L}{2}\frac{WL}{4}\right)\left(\frac{2}{3}\frac{L}{2}\right)\right] = \frac{WL^3}{48EI}$$

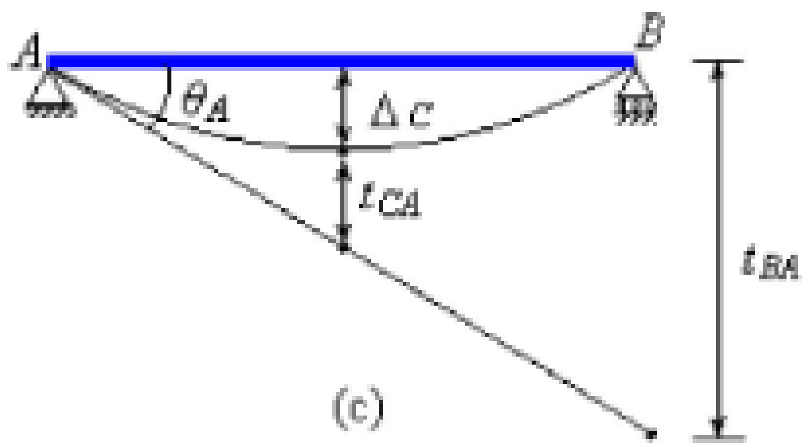
OR



(a)



(b)



(c)

Solution: The M / EI diagram of the beam is shown in Figure 4.2(a). The slope at A , θ_A can be obtained by computing the t_{BA} using the second moment area theorem i.e.

$$\theta_A \times L = t_{BA}$$

$$\theta_A = \frac{1}{L} \times \left(\frac{1}{2} \times \frac{WL}{4EI} \times L \times \frac{L}{2} \right) = \frac{WL^2}{16EI} \text{ (clockwise direction)}$$

The slope at B can be obtained by using the first moment area theorem between points A and B i.e.

$$\theta_B - \theta_A = \Delta\theta_{AB}$$

$$\theta_B - \theta_A = \frac{1}{2} \times \frac{WL}{4EI} \times L = \frac{WL^2}{8EI}$$

$$\theta_B = \frac{WL^2}{8EI} - \frac{WL^2}{16EI} = \frac{WL^2}{16EI} \text{ (anti-clockwise)}$$

(It is to be noted that the $\theta_A = -\frac{WL^2}{16EI}$. The negative sign is because of the slope being in the clockwise direction. As per sign convention a positive slope is in the anti-clockwise direction)

The deflection at the centre of the beam can be obtained with the help of the second moment area theorem between points A and C i.e.

$$\theta_A \times \frac{L}{2} = \Delta_C + t_{CA}$$

$$\frac{WL^2}{16EI} \times \frac{L}{2} = \Delta_C + \left(\frac{1}{2} \times \frac{WL}{4EI} \times \frac{L}{2} \times \frac{L}{6} \right)$$

$$\Delta_C = \frac{WL^3}{48EI} \text{ (downward direction)}$$

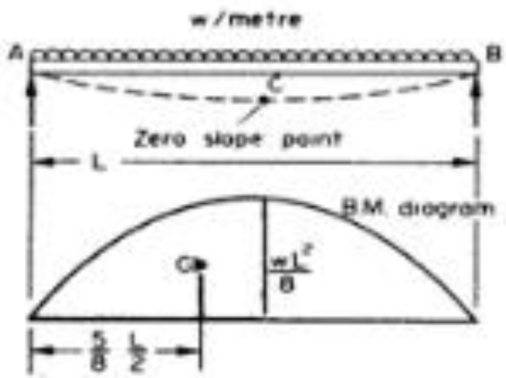


Fig. 5.22.

lere the point of zero slope is at the centre of the beam C. Working relative to C,

$$\text{slope at } A = \frac{1}{EI}\,[\text{area of B.M. diagram between } A \text{ and } C \text{ (Fig. 5.22)}]$$

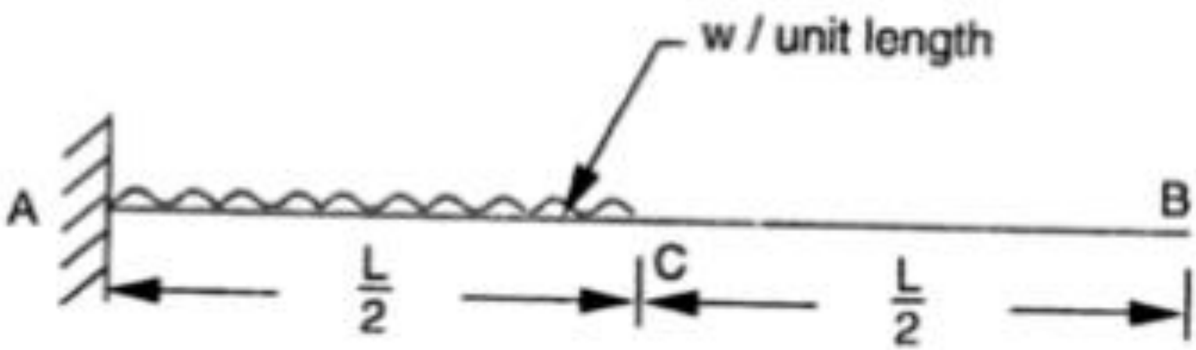
$$= \frac{1}{EI}\left[\frac{2}{3}\frac{wL^2}{8}\frac{L}{2}\right] = \frac{wL^3}{24EI}$$

lection of A relative to C (= central deflection relative to A)

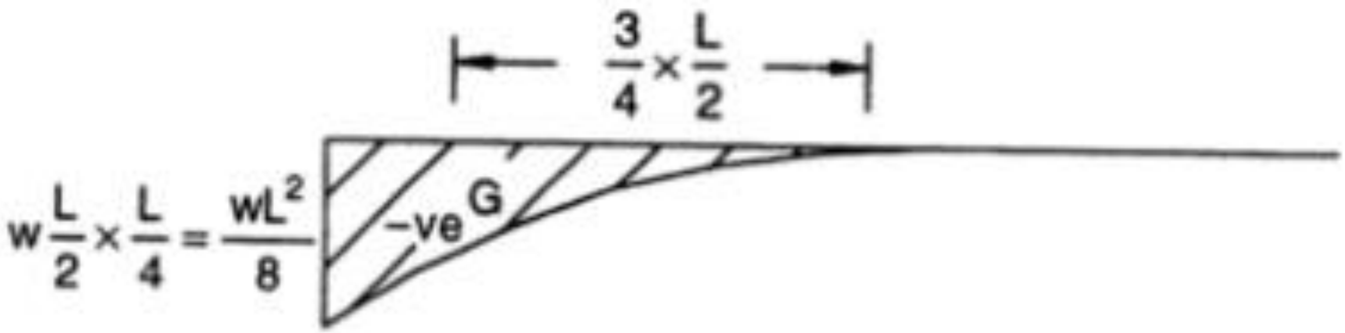
$$= \frac{1}{EI}\,[\text{moment of B.M. diagram between } A \text{ and } C \text{ about } A]$$

$$= \frac{1}{EI}\left[\left(\frac{2}{3}\frac{wL^2}{8}\frac{L}{2}\right)\left(\frac{5L}{16}\right)\right] = \frac{5wL^4}{384EI}$$

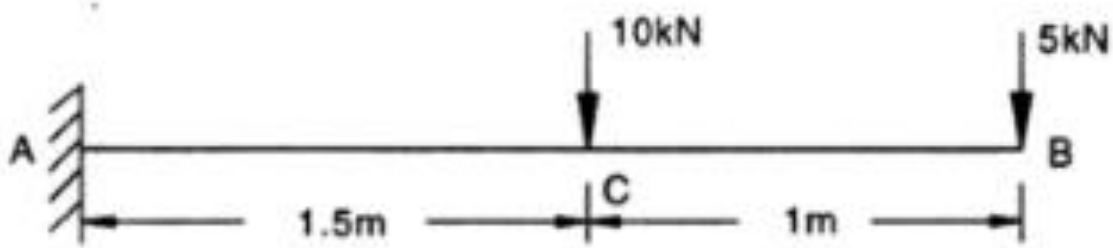
Find the rotation and deflection at the free end in the cantilever beam shown in Fig.

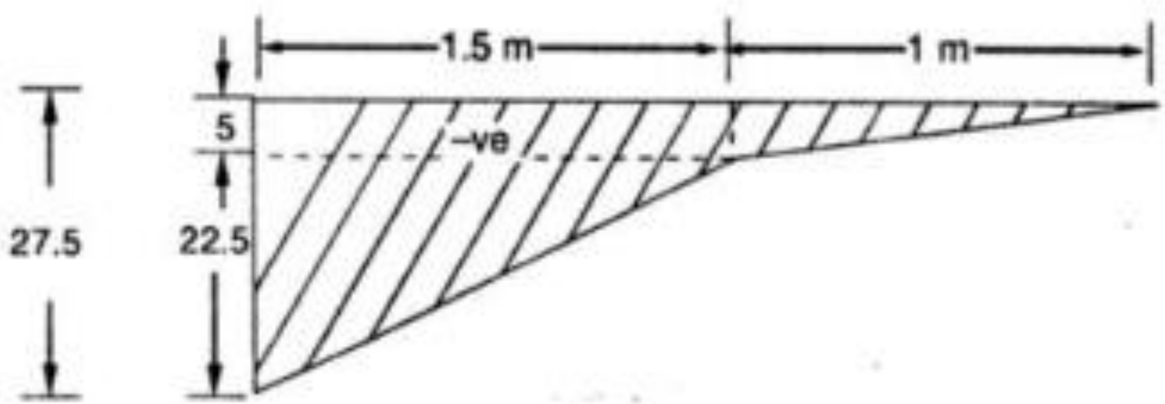


Solution

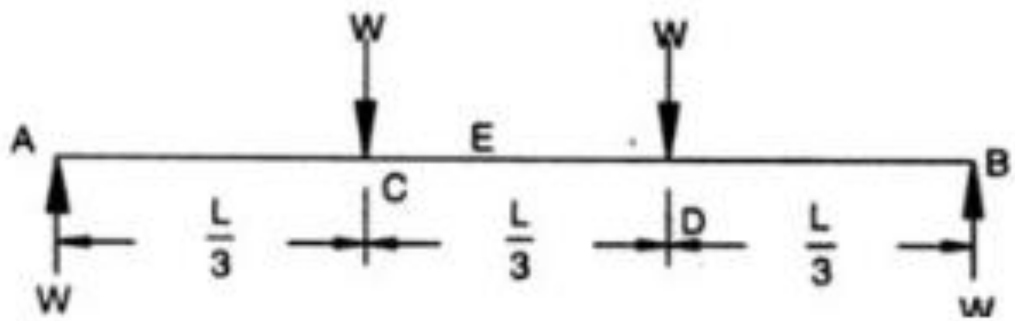


Example Determine the slope and deflection at the free end of a cantilever beam as shown in Fig. (a) by moment area method. (Take EI = 4000 kNm^2).





Example Determine the rotation at supports and deflection at mid-span and under the loads in the simply supported beam as shown in Fig.



Solution

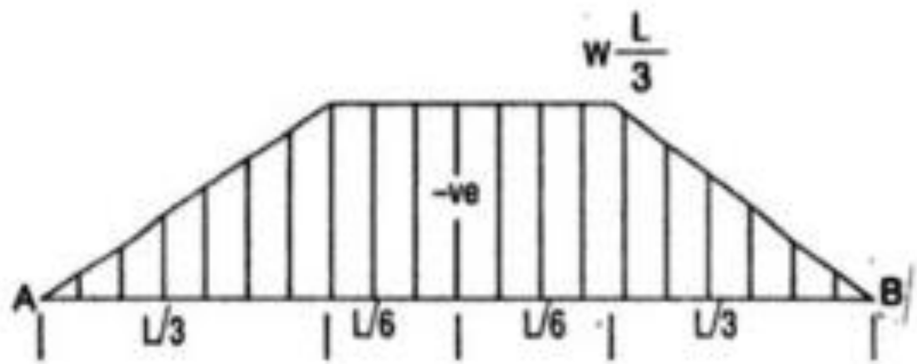


Figure 2.7 (b)

In this case, the bending moment diagram is as shown in Fig.2.7(b). Due to symmetry, the slope at E is zero. In other words, the tangent at E is horizontal. Hence, it is convenient to workout rotations and deflection with respect to the tangent at E.