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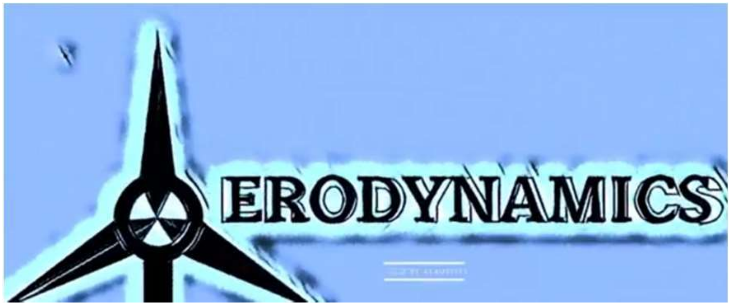

MEE1004-FLUID MECHANICS

Dimensional Analysis
Lecture 1 & 2

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Dimension and Units

Example

Dimension	***Units***
Mass, [M]	**Kilogram or gram or Milligram**
Length, [L]	**Kilometer, meter**
Time, [T]	**Seconds, Milliseconds**
Temperature, [t]	**Kelvin, Celcius**
Electric current, [I]	**Ampere**

Primary / Fundamental Dimensions

Secondary Dimensions

All non-primary dimensions can be formed by some combination of the seven primary dimensions.

Example: Velocity, Acceleration, Force, Pressure…

Dimension and Units

Example

Dimension	**Units**	**Secondary Dimensions**
Velocity	meter / second	L/T (or) LT^{-1}
Acceleration	meter / second^2	LT^{-2}
Force	mass*acceleration	MLT^{-2}
Pressure	force / Area	$\frac{MLT^{-2}}{L^2}$
		$MT^{-2}L^{-1}$
Energy	mass* velocity^2	
	density* acceleration*height	

Application of an Dimensional Analysis

Example

Dimension	***Units***	***Secondary Dimensions***
Velocity	**meter / second**	**L/T (or) LT^{-1}**
Acceleration	**meter / $second^2$**	**LT^{-2}**
Force	**mass*acceleration**	**MLT^{-2}**
Pressure	**force / Area**	$\frac{MLT^{-2}}{L^2}$ $MT^{-2}L^{-1}$
Energy	**mass* $velocity^2$**	
	density* acceleration*height	

Dimensional Homogeneity

- The **law of dimensional homogeneity,** stated as

  Every additive term in an equation must have the same dimensions. For example,

Change of total energy of a system:

$$\Delta E = \Delta U + \Delta \mathrm{KE} + \Delta \mathrm{PE}$$

$$\Delta U = m(u_2 - u_1)$$ ← Internal Energy

$$\Delta \mathrm{KE} = \frac{1}{2} m(V_2^2 - V_1^2)$$

$$\Delta \mathrm{PE} = mg(z_2 - z_1)$$

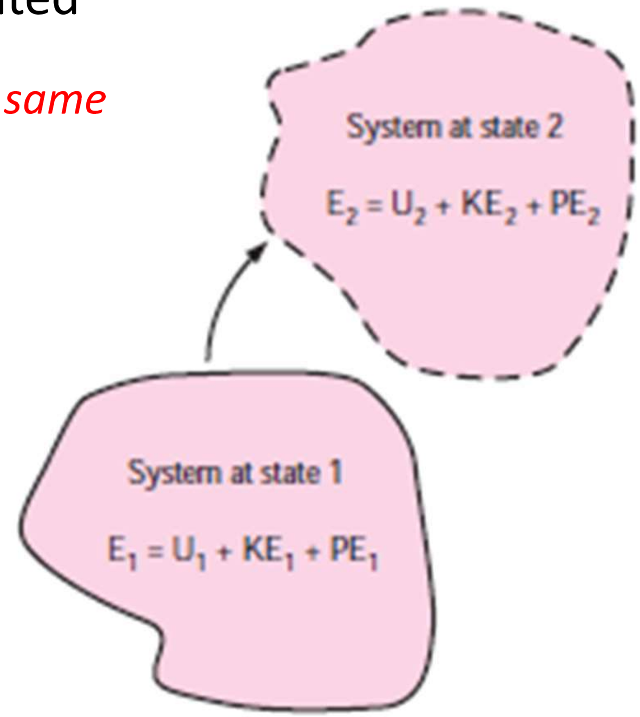


FIGURE 7–4
Total energy of a system at state 1 and at state 2.

Ref. : ÇENGEL, Y. A., & CIMBALA, J. M. (2006). Fluid mechanics: fundamentals and applications. Boston, McGraw-HillHigher Education.

Dimensional Homogeneity

$$\Delta E = \Delta U + \Delta \mathrm{KE} + \Delta \mathrm{PE}$$

$$\{\Delta E\} = \{\mathrm{Energy}\} = \{\mathrm{Force} \cdot \mathrm{Length}\} \quad \rightarrow \quad \{\Delta E\} = \{\mathrm{mL^2/t^2}\}$$

$$\{\Delta U\} = \left\{\mathrm{Mass}\,\frac{\mathrm{Energy}}{\mathrm{Mass}}\right\} = \{\mathrm{Energy}\} \quad \rightarrow \quad \{\Delta U\} = \{\mathrm{mL^2/t^2}\}$$

$$\{\Delta \mathrm{KE}\} = \left\{\mathrm{Mass}\,\frac{\mathrm{Length}^2}{\mathrm{Time}^2}\right\} \quad \rightarrow \quad \{\Delta \mathrm{KE}\} = \{\mathrm{mL^2/t^2}\}$$

$$\{\Delta \mathrm{PE}\} = \left\{\mathrm{Mass}\,\frac{\mathrm{Length}}{\mathrm{Time}^2}\,\mathrm{Length}\right\} \quad \rightarrow \quad \{\Delta \mathrm{PE}\} = \{\mathrm{mL^2/t^2}\}$$

An equation that is not dimensionally homogeneous is sign of an error.

Ref. : ÇENGEL, Y. A., & CIMBALA, J. M. (2006). Fluid mechanics: fundamentals and applications. Boston, McGraw-HillHigher Education.

Dimensional homogeneity of the Bernoulli equation

Bernoulli equation: $P + \frac{1}{2}\rho V^2 + \rho g z = C$

(*a*) Verify that each additive term in the Bernoulli equation has the same dimensions. (*b*) What are the dimensions of the constant *C*?

$$\{P\} = \{\text{Pressure}\} = \left\{\frac{\text{Force}}{\text{Area}}\right\} = \left\{\text{Mass}\frac{\text{Length}}{\text{Time}^2}\frac{1}{\text{Length}^2}\right\} = \left\{\frac{m}{t^2 L}\right\}$$

$$\left\{\frac{1}{2}\rho V^2\right\} = \left\{\frac{\text{Mass}}{\text{Volume}}\left(\frac{\text{Length}}{\text{Time}}\right)^2\right\} = \left\{\frac{\text{Mass}\times\text{Length}^2}{\text{Length}^3\times\text{Time}^2}\right\} = \left\{\frac{m}{t^2 L}\right\}$$

$$\{\rho g z\} = \left\{\frac{\text{Mass}}{\text{Volume}}\frac{\text{Length}}{\text{Time}^2}\text{Length}\right\} = \left\{\frac{\text{Mass}\times\text{Length}^2}{\text{Length}^3\times\text{Time}^2}\right\} = \left\{\frac{m}{t^2 L}\right\}$$

Indeed, **all three additive terms have the same dimensions.**

(*b*) From the law of dimensional homogeneity, the constant must have the same dimensions as the other additive terms in the equation. Thus,

Primary dimensions of the Bernoulli constant: $\{C\} = \left\{\frac{m}{t^2 L}\right\}$

Discussion If the dimensions of any of the terms were different from the others, it would indicate that an error was made somewhere in the analysis.

Ref. : ÇENGEL, Y. A., & CIMBALA, J. M. (2006). Fluid mechanics: fundamentals and applications. Boston, McGraw-HillHigher Education.

Non-dimensionalization of Equations

- Dimensional homogeneity $\Rightarrow$ every term in an equation has the same dimensions.
- **Non-dimensional** $\Rightarrow$ divide each term in the equation by a collection of variables and constants whose product has those same dimensions.
- If the non-dimensional terms in the equation are of order unity $\Rightarrow$ called **normalized.**
- Normalization is thus more restrictive than non-dimensionalization. (often used interchangeably).
- **Non-dimensional parameters** are named after a notable scientist or engineer (e.g., the Reynolds number and the Froude number). This process is referred to by some authors as **inspectional analysis.**

Rayleigh's Method of Dimensional analysis

- This method is used for determining the expression for a variable which depends up on maximum three or four variable only.
- If the number of independent variables becomes more than four, then it is very difficult to find the expression for the dependent variable.

Let X is a variable, which depends on X_1, X_2 and X_3 variables.

Rayleigh's Method of Dimensional analysis

Let X is a variable, which depends on X_1, X_2 and X_3 variables.

Then according to Rayleigh's method, X is function of X_1, X_2 and X_3 a

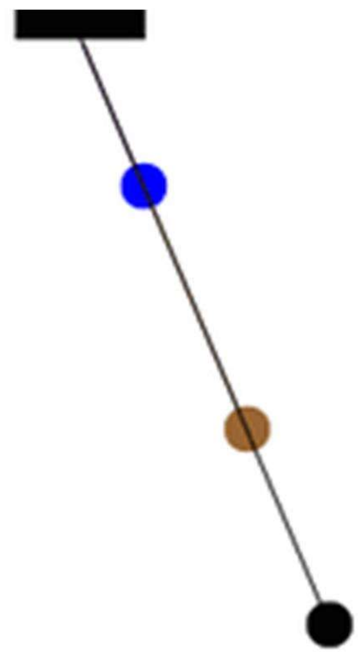
This can also be written as $X = KX_1^a \cdot X_2^b \cdot X_3^c$

where K is constant and a, b and c are arbitrary powers.

The values of a, b and c are obtained by comparing the powers of the fundamental dimension on both sides. Thus the expression is obtained for dependent variable.

Rayleigh's Method of Dimensional analysis

The time period (t) of a pendulum depends upon the length (L) of the pendulum and acceleration due to gravity (g). Derive an expression for the time period.



Rayleigh's Method of Dimensional analysis

The time period (t) of a pendulum depends upon the length (L) of the pendulum and acceleration due to gravity (g). Derive an expression for the time period.

Solution. Time period t is a function of (i) L and (ii) g

$\therefore$ $$t = KL^a \, . \, g^b, \text{ where } K \text{ is a constant}$$

Substituting the dimensions on both sides $T^1 = KL^a \, . \, (LT^{-2})^b$

Equating the powers of M, L and T on both sides, we have

Power of T, $\quad 1 = -2b \quad \therefore \quad b = -\frac{1}{2}$

Power of L, $\quad 0 = a + b \quad \therefore \quad a = -b = -\left(-\frac{1}{2}\right) = \frac{1}{2}$

Substituting the values of a and b in equation (i),

$$t = KL^{1/2} \, . \, g^{-1/2} = K\sqrt{\frac{L}{g}}$$

The value of K is determined from experiments which is given as

$$K = 2\pi$$

$\therefore$ $$t = 2\pi\sqrt{\frac{\mathbf{L}}{\mathbf{g}}}. \textbf{ Ans.}$$

Rayleigh's Method of Dimensional analysis

Find an expression for the drag force on smooth sphere of diameter D, moving with a uniform velocity V in a fluid of density ρ and dynamic viscosity μ.

Solution. Drag force F is a function of

(*i*) Diameter, D (*ii*) Velocity, V (*iii*) Density, ρ

(*iv*) Viscocity, μ

$$\therefore \qquad F = KD^a \cdot V^b \cdot \rho^c \cdot \mu^d$$

$D^a \quad V^b \quad \rho^c \quad \mu^d$

$a \quad b \quad c \quad d$

where K is non-dimensional factor.

Substituting the dimensions on both sides,

$$MLT^{-2} = KL^a \cdot (LT^{-1})^b \cdot (ML^{-3})^c \cdot (ML^{-1}T^{-1})^d$$

Equating the powers of M, L and T on both sides,

Power of M, $\qquad 1 = c + d$

Power of L, $\qquad 1 = a + b - 3c - d$

Power of T, $\qquad -2 = -b - d.$

Rayleigh's Method of Dimensional analysis

Equating the powers of M, L and T on both sides,

Power of M, $\quad 1 = c + d$

Power of L, $\quad 1 = a + b - 3c - d$

Power of T, $\quad -2 = -b - d.$

3 eq. — a, b, c & d

There are four unknowns (a, b, c, d) but equations are three. Hence it is not possible to find the values of a, b, c and d. But three of them can be expressed in terms of fourth variable which is most important. Here viscosity is having a vital role and hence a, b, c are expressed in terms of d which is the power to viscosity.

$$\therefore \quad c = 1 - d$$

$$b = 2 - d$$

$$a = 1 - b + 3c + d = 1 - 2 + d + 3(1 - d) + d$$

$$= 1 - 2 + d + 3 - 3d + d = 2 - d$$

Substituting these values of a, b and c in (i), we get

$$F = KD^{2-d} \, . \, V^{2-d} \, . \, \rho^{1-d} \, . \, \mu^{d}$$

$$= KD^2V^2\rho \, (D^{-d} \, . \, V^{-d} \, . \, \rho^{-d} \, . \, \mu^{d}) = K\rho D^2V^2 \left(\frac{\mu}{\rho VD}\right)^{d}$$

$$= \mathbf{K\rho D^2V^2\phi\left(\frac{\mu}{\rho VD}\right)}. \textbf{ Ans.}$$

$F = K\rho V^2 D^2 \phi\left(\frac{1}{Re}\right)$

$1/Re$

D, ρ, V, μ

12.4.3 Method of Selecting Repeating Variables. The number of repeating variables are equal to the number of fundamental dimensions of the problem. The choice of repeating variables is governed by the following considerations :

1. As far as possible, the dependent variable should not be selected as repeating variable.
2. The repeating variables should be choosen in such a way that one variable contains geometric property, other variable contains flow property and third variable contains fluid property.

Variables with Geometric Property are

(*i*) Length, l (*ii*) d (*iii*) Height, H etc.

Variables with flow property are

(*i*) Velocity, V (*ii*) Acceleration etc.

Variables with fluid property : (*i*) μ, (*ii*) ρ, (*iii*) ω etc.

3. The repeating variables selected should not form a dimensionless group.
4. The repeating variables together must have the same number of fundamental dimensions.
5. No two repeating variables should have the same dimensions.

In most of fluid mechanics problems, the choice of repeating variables may be (*i*) d, v, ρ (*ii*) l, v, ρ or (*iii*) l, v, μ or (*iv*) d, v, μ.

Buckingham Pi Theorem Application

12.4.2 Buckingham's π-Theorem. The Rayleigh's method of dimensional analysis becomes more laborious if the variables are more than the number of fundamental dimensions (M, L, T). This difficulty is overcame by using Buckingham's π-theorem, which states, "If there are n variables (independent and dependent variables) in a physical phenomenon and if these variables contain m fundamental dimensions (M, L, T), then the variables are arranged into $(n - m)$ dimensionless terms. Each term is called π-term".

Let X_1, X_2, X_3, ..., X_n are the variables involved in a physical problem. Let X_1 be the dependent variable and X_2, X_3, ..., X_n are the independent variables on which X_1 depends. Then X_1 is a function of X_2, X_3, ..., X_n and mathematically it is expressed as

$$X_1 = f(X_2, X_3, \ldots, X_n) \qquad \ldots(12.1)$$

Equation (12.1) can also be written as

$$f_1(X_1, X_2, X_3, \ldots, X_n) = 0. \qquad \ldots(12.2)$$

Equation (12.2) is a dimensionally homogeneous equation. It contains n variables. If there are m fundamental dimensions then according to Buckingham's π-theorem, equation (12.2) can be written in terms of number of dimensionless groups or π-terms in which number of π-terms is equal to $(n - m)$. Hence equation (12.2) becomes as

$$f(\pi_1, \pi_2, \ldots, \pi_{n-m}) = 0. \qquad \ldots(12.3)$$

Each of π-terms is dimensionless and is independent of the system. Division or multiplication by a constant does not change the character of the π-term. Each π-term contains $m + 1$ variables, where m is the number of fundamental dimensions and is also called repeating variables. Let in the above case X_2, X_3 and X_4 are repeating variables if the fundamental dimension m (M, L, T) = 3. Then each π-term is written as

The resisting force R of a supersonic plane during flight can be considered as dependent upon the length of the aircraft l, velocity V, air viscosity μ, air density ρ and bulk modulus of air K. Express the functional relationship between these variables and the resisting force.

Solution. Step 1. The resisting force R depends upon (*i*) l, (*ii*) V, (*iii*) μ, (*iv*) ρ and (*v*) K. Hence R is a function of l, V, μ, ρ and K. Mathematically,

$$R = f(l, V, \mu, \rho, K) \qquad ...(i)$$

or it can be written as $f_1\ (R, l, V, \mu, \rho, K) = 0$ ...(*ii*)

$\therefore$ Total number of variables, $n = 6$.

Number of fundamental dimensions, $m = 3$.

[m is obtained by writing dimensions of each variables as $R = MLT^{-2}$, $V = LT^{-1}$, $\mu = ML^{-1}T^{-1}$, $\rho = ML^{-3}$, $K = ML^{-1}T^{-2}$. Thus as fundamental dimensions in the problem are M, L, T and hence $m = 3$.]

Number of dimensionless π-terms $= n - m = 6 - 3 = 3$.

Thus three π-terms say π_1, π_2 and π_3 are formed. Hence equation (*ii*) is written as

$$f_1\ (\pi_1, \pi_2, \pi_3) = 0. \qquad ...(iii)$$

Step 2. Each π term $= m + 1$ variables, where m is equal to 3 and also called repeating variables.

Step 3. Each π-term is written as according to equation (12.4)

$$\left.\begin{aligned} \pi_1 &= l^{a_1} \,.\, V^{b_1} \,.\, \rho^{c_1} \,.\, R \\ \pi_2 &= l^{a_2} \,.\, V^{b_2} \,.\, \rho^{c_2} \,.\, \mu \\ \pi_3 &= l^{a_3} \,.\, V^{b_3} \,.\, \rho^{c_3} \,.\, K \end{aligned}\right\} \qquad \text{...(iv)}$$

Step 4. Each π-term is solved by the principle of dimensional homogeneity. For the first π-term, we have

$$\pi_1 = M^0L^0T^0 = L^{a_1} \,.\, (LT^{-1})^{b_1} .\, (ML^{-3})^{c_1} \,.\, MLT^{-2}.$$

Equating the powers of M, L, T on both sides, we get

Power of M, $\quad 0 = c_1 + 1 \quad \therefore \quad c_1 = -1$

Power of L, $\quad 0 = a_1 + b_1 - 3c_1 + 1,$

$\therefore \quad a_1 = -b_1 + 3c_1 - 1 = 2 - 3 - 1 = -2$

Power of T, $\quad 0 = -b_1 - 2 \quad \therefore \quad b_1 = -2$

Substituting the values of a_1, b_1 and c_1 in equation (*iv*),

$$\pi_1 = l^{-2} \,.\, V^{-2} \,.\, \rho^{-1} \,.\, R$$

or

$$\pi_1 = \frac{R}{l^2V^2\rho} = \frac{R}{\rho l^2 V^2} \qquad \text{...(v)}$$

Similarly for the 2nd π-term, we get $\pi_2 = M^0L^0T^0 = L^{a_2} \,.\, (LT^{-1})^{b_2} \,.\, (ML^{-3})^{c_2} \,.\, ML^{-1}T^{-1}$.

Equating the powers of M, L, T on both sides

Power of M, $\quad 0 = c_2 + 1, \quad \therefore \quad c_2 = -1$

Power of L, $\quad 0 = a_2 + b_2 - 3c_2 - 1,$

$a_2 = -b_2 + 3c_2 + 1 = 1 - 3 + 1 = -1$

Power of T, $\quad 0 = -b_2 - 1, \quad \therefore \quad b_2 = -1$

Substituting the values of a_2, b_2 and c_2 in π_2 of (*iv*)

$$\pi_2 = l^{-1} \,.\, V^{-1} \,.\, \rho^{-1} \,.\, \mu = \frac{\mu}{lV\rho}.$$

3rd π-term

$$\pi_3 = l^{a_3} \,.\, V^{b_3} \,.\, \rho^{c_3} \,.\, K$$

or
$$M^0L^0T^0 = L^{a_3} \,.\, (LT^{-3})^{b_3} \,.\, (ML^{-3})^{a_3} \,.\, ML^{-1}T^{-2}$$

Equating the powers of M, L, T on both sides, we have

Power of M, $\quad 0 = c_3 + 1, \quad \therefore \quad c_3 = -1$

Power of L, $\quad 0 = a_3 + b_3 - 3c_3 - 1, \quad \therefore \quad a_3 = -b_3 + 3c_3 + 1 = 2 - 3 + 1 = 0$

Power of T, $\quad 0 = -b_3 - 2, \quad \therefore \quad b_3 = -2$

Substituting the values of a_3, b_3 and c_3 in π_3 term

$$\pi_3 = l^0 \cdot V^{-2} \,.\, \rho^{-1} \,.\, K = \frac{K}{V^2\rho}.$$

Step 5. Substituting the values of π_1, π_2 and π_3 in equation (*iii*), we get

$$f_1\left(\frac{R}{\rho l^2V^2}, \frac{\mu}{lV\rho}, \frac{K}{V^2\rho}\right) = 0 \quad \text{or} \quad \frac{R}{\rho l^2V^2} = \phi\left[\frac{\mu}{lV\rho}, \frac{K}{V^2\rho}\right]$$

$$R = \rho\mathbf{l}^2\mathbf{V}^2\phi\left[\frac{\mu}{\mathbf{lV}\rho}, \frac{\mathbf{K}}{\mathbf{V}^2\rho}\right].$$ **Ans.**

Credits to the members of Aerodynamics laboratory, School of Mechanical Engineering, VIT Chennai.

Thank you very much..☺

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https://www.youtube.com/channel/UCPHV628SolZbTxH9Y2tMIjQ