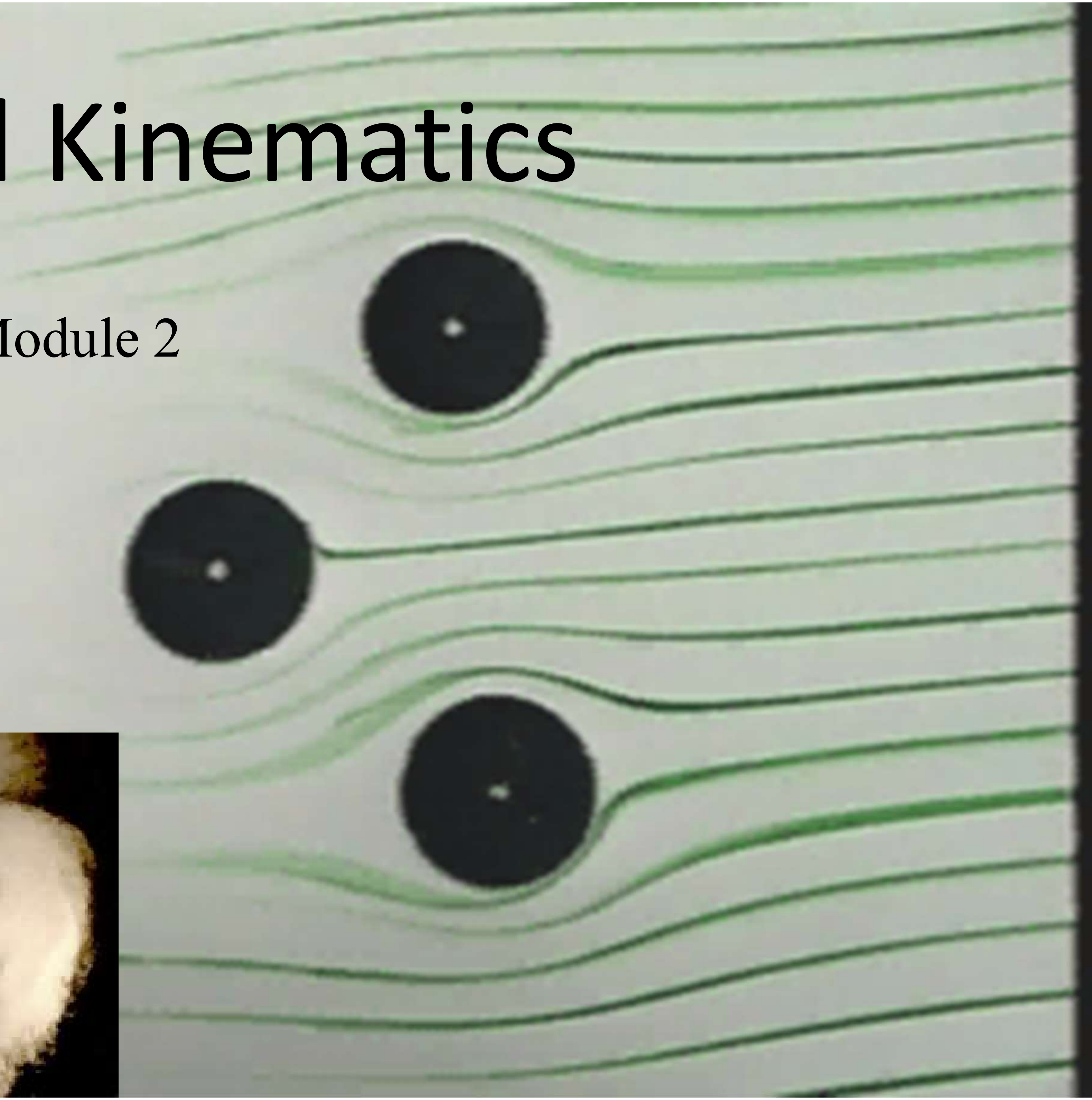
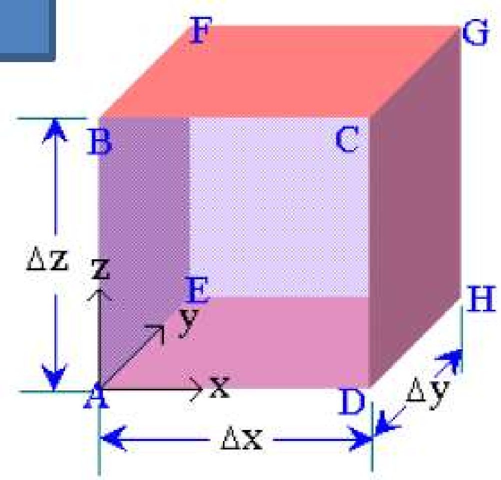
# Fluid Kinematics

Lecture 4 of Module 2



# Continuity Equation in Cartesian

Let us make the mass balance for a fluid element as shown below: (an open-faced cube)




Let us denote the sides by with the following corresponding numbers:

| x -direction | | y -direction | | z -direction | |
|---|---|---|---|---|---|
| ABFE | 1 | ABCD | 3 | AEHD | 5 |
| DCGH | 2 | EFGH | 4 | BFGC | 6 |

<u>Mass balance:</u>

Accumulation rate of mass in the system = all mass flow rates in - all mass flow rates out --> 1

The mass in the system at any instant is r Dx Dy Dz.

The flow into the system through face 1 is

$$\dot{m}_1 = \rho_1 v_{x_1} \Delta y \Delta z$$

and the flow out of the system through face 2 is

$$\dot{m}_2 = \rho_2 v_{x_2} \Delta y \Delta z$$

Similarly for the fcaes 3, 4, 5, and 6 are written as follows:

$$\dot{m}_3 = \rho_3 v_{y_3} \Delta x \Delta z$$

$$\dot{m}_4 = \rho_4 v_{y_4} \Delta x \Delta z$$

$$\dot{m}_5 = \rho_5 v_{z_5} \Delta x \Delta y$$

$$\dot{m}_6 = \rho_6 v_{z_6} \Delta x \Delta y$$

Substituting these quantities in equn.1, we get

$$\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} = \Delta y \Delta z (\rho_1 v_{x_1} - \rho_2 v_{x_2}) + \Delta x \Delta z (\rho_3 v_{y_3} - \rho_4 v_{y_4}) + \Delta x \Delta y (\rho_5 v_{z_5} - \rho_6 v_{z_6})$$

Dividing the above equation by $\Delta x\ \Delta y\ \Delta z$:

$$-\frac{\partial \rho}{\partial t} = \frac{\rho_2 v_{x_2} - \rho_1 v_{x_1}}{\Delta x} + \frac{\rho_4 v_{y_4} - \rho_3 v_{y_3}}{\Delta y} + \frac{\rho_6 v_{z_6} - \rho_5 v_{z_5}}{\Delta z}$$

Now we let D$x$, D$y$, and D$z$ each approach zero simulaneously, so that the cube shrinks to a point. Taking the limit of the three ratios on the right-hand side of this equation, we get the partial derivatives.

$$\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = -\frac{\partial \rho}{\partial t}$$

This is the continuity equation for every point in a fluid flow whether steady or unsteady, compressible or incompressible.

$$\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = -\frac{\partial \rho}{\partial t}$$

In most books the authors specified that u is the velocity in the x-direction, v in the y-direction and w in the z-direction. So purposefully lets modify the same and say the

$$u = v_x \qquad v = v_y \qquad w = v_z$$

$$0 = \left[\left(\rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x}\right) + \left(\rho\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y}\right) + \left(\rho\frac{\partial w}{\partial z} + w\frac{\partial \rho}{\partial z}\right)\right] dxdydz + \left(\frac{\partial \rho}{\partial t}\right) d\forall$$

$$\Rightarrow \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] d\forall = 0$$

This is the **Equation of Continuity** for a compressible fluid in a rectangular cartesian coordinate system.

# Continuity Equation in Cartesian

The continuity equation can be written in a vector form as

$$\frac{\partial \rho}{\partial t}+\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right).\left[\rho u\hat{i}+\rho v\hat{j}+\rho w\hat{k}\right]=0$$

$$\frac{\partial \rho}{\partial t}+\nabla.(\rho\vec{V})=0$$

Where $\vec{V}=u\hat{i}+v\hat{j}+w\hat{k}$ is the velocity of the point

In case of a **steady flow**, $\frac{\partial \rho}{\partial t}=0$ $\qquad \nabla.(\rho\vec{V})=0$

In a rectangular cartesian coordinate system $\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$

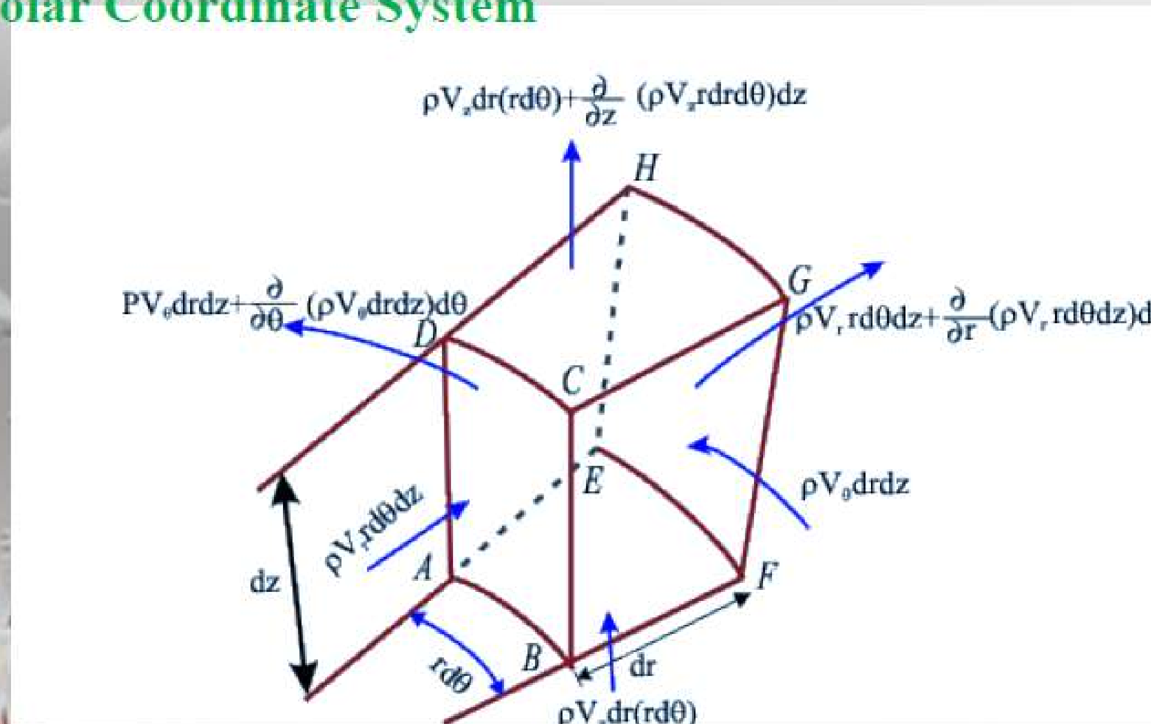
In case of an incompressible flow, $\rho=\text{constant}$ $\frac{\partial \rho}{\partial t}=0$

$$\nabla.(\rho\vec{V})=\rho\nabla.(\vec{V})$$

$$\nabla.(\vec{V})=0 \qquad \text{or,} \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

# Continuity Equation in Polar co-ordinates

**Continuity Equation- Cylindrical Polar Coordinate System**




$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho V_r r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta)+\frac{\partial}{\partial z}(\rho V_z)=0$$

For an incompressible flow,

$$\frac{\partial V_r}{\partial r}+\frac{V_r}{r}+\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{\partial V_z}{\partial z}=0$$

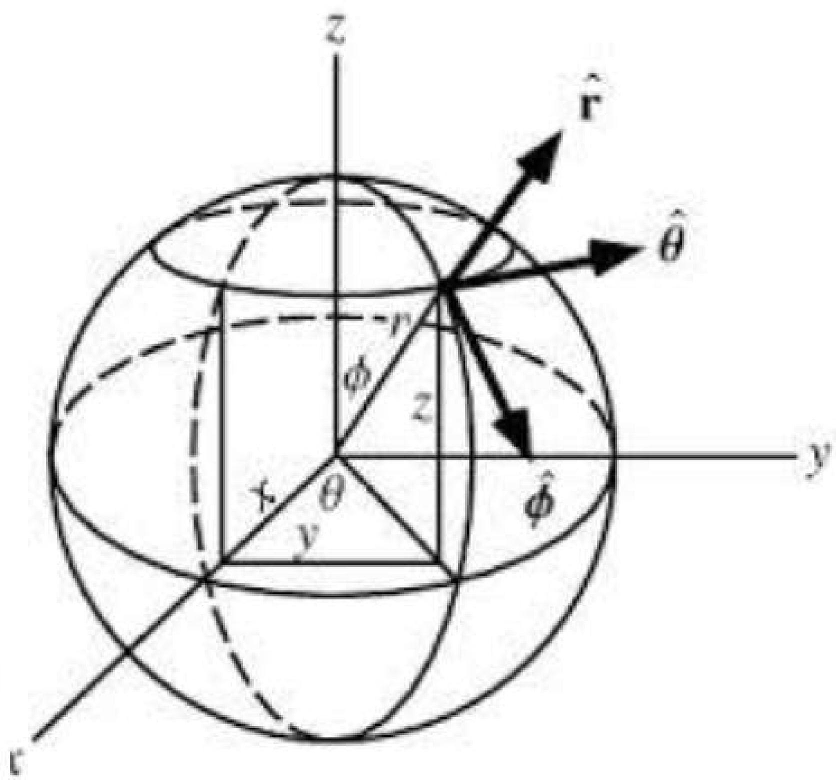
# Continuity Equation in Spherical co-ordinates

Spherical polar coordinates are a system of curvilinear coordinates that are natural for describing atmospheric flows.

Define $\theta$ to be the azimuthal angle in the $x$-$y$ -plane from the $x$-axis with $0 \leq \theta < 2\pi$.

$\phi$ to be the zenith angle and colatitude, with $0 \leq \phi < \pi$

$r$ to be distance (radius) from a point to the origin.




The spherical coordinates $(r, \theta, \phi)$ are related to the Cartesian coordinates $(x, y, z)$ by

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \phi = \cos^{-1}\left(\frac{z}{r}\right)$$

$$x = r\cos\theta\sin\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = \cos\phi$$

# Continuity Equation in Spherical co-ordinates

The gradient is $$\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{1}{r\sin\phi}\hat{\theta}\frac{\partial}{\partial\theta} + \frac{1}{r}\hat{\phi}\frac{\partial}{\partial\phi}$$

The equation of continuity in a spherical polar coordinate system can be written by expanding the term of Eq. as

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial\left(\rho r^2 v_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\rho v_\theta \sin\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\left(\rho v_\phi\right)}{\partial\phi} = 0$$

End of Lecture 4 (Module 2)