VIT®

Vellore Institute of Technology

(Deemed to be University under section 3 of UGC Act, 1956)

Winter 2022-23

School of Mechanical Engineering

BMEE207L KINEMATICS AND DYNAMICS OF MACHINES

MODULE 6

Balancing

By

Dr. Tapan Kumar Mahanta

Outline

- Why balancing?
- Balancing of rotating masses
  - Static
  - Dynamic
- Problems
- Basics of Balancing of Reciprocating Masses

Balancing of Rotating Masses

- Static balancing
- Dynamic balancing

1. Kinematics and Dynamics of Machinery by R. L. Norton
2. Theory of Machines and Mechanisms by John J, Uicker Jr, Gordon R. Pennock and Joseph E. Shigley
3. Theory of Machines by S. S. Rattan

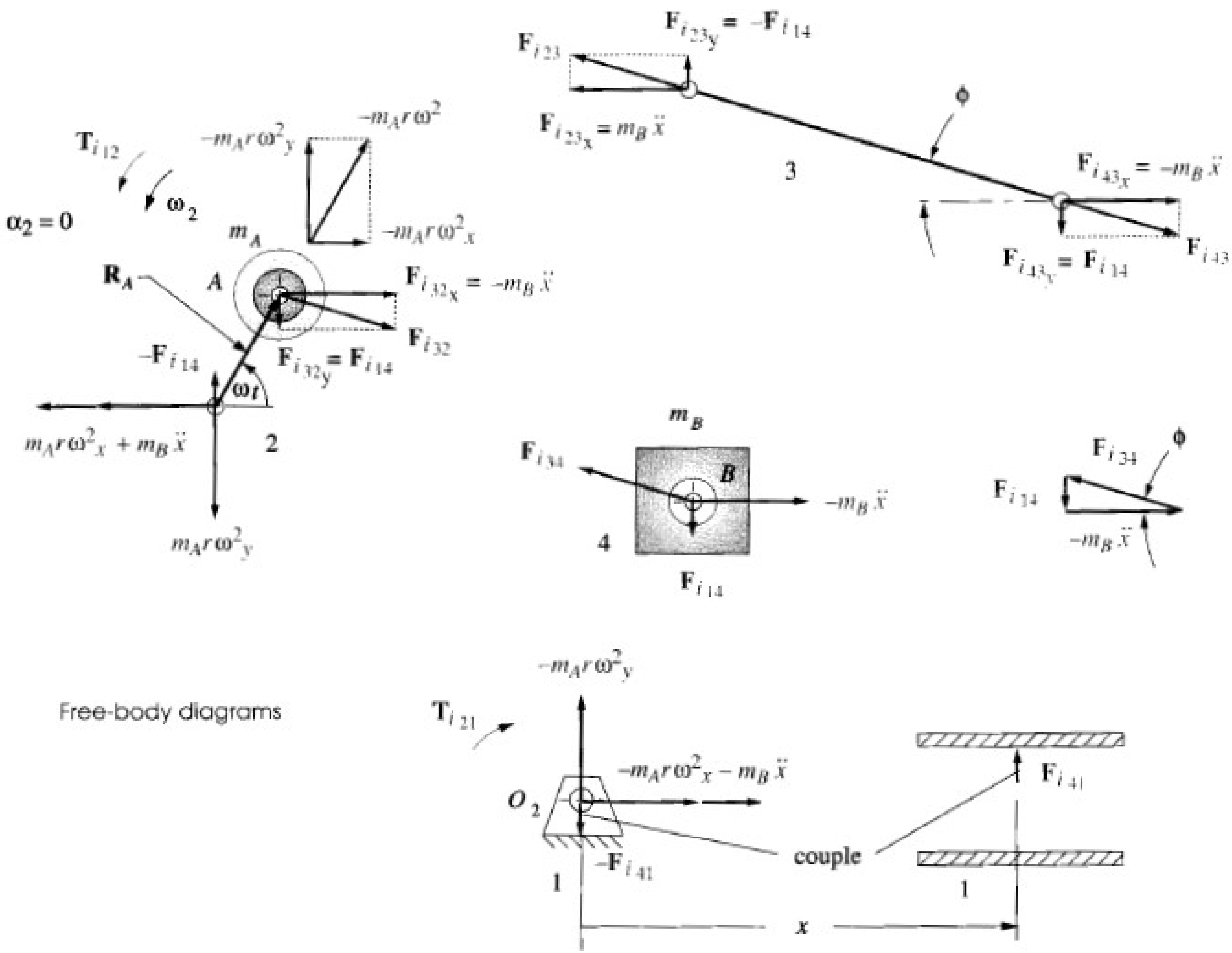
Why Balancing?

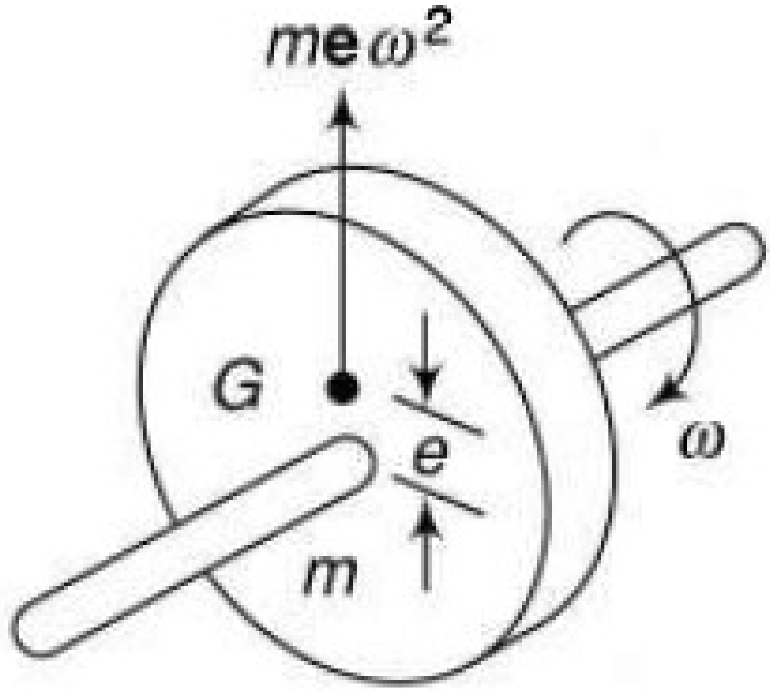
- Inertia forces associated with the moving masses causes unbalance
- The inertia forces transmitted to the frame or foundation of a machine tend to shake or vibrate the machine
- Balancing is the technique of correcting or eliminating unwanted inertia forces and moments in rotating machinery
- Members in pure rotation can (theoretically) be balanced to eliminate all shaking forces and shaking moments

Why Balancing?

- Rotating parts can be designed to be inherently balanced by their geometry
- The unexpected changes in the production tolerances guarantee that there will still be some small unbalance in each part
- Thus a balancing procedure will have to be applied to each part after manufacture
- The imbalance can be measured quite accurately and the material is added or removed in the correct locations

Engine Forces




meω2
G
e
m
ω

Static Balance (Single-plane Balance)

- The requirement for static balance is that the sum of all forces on the moving system must be zero

According to D' Alembert's principle,

$$\sum \boldsymbol{F} - \boldsymbol{ma} = 0$$

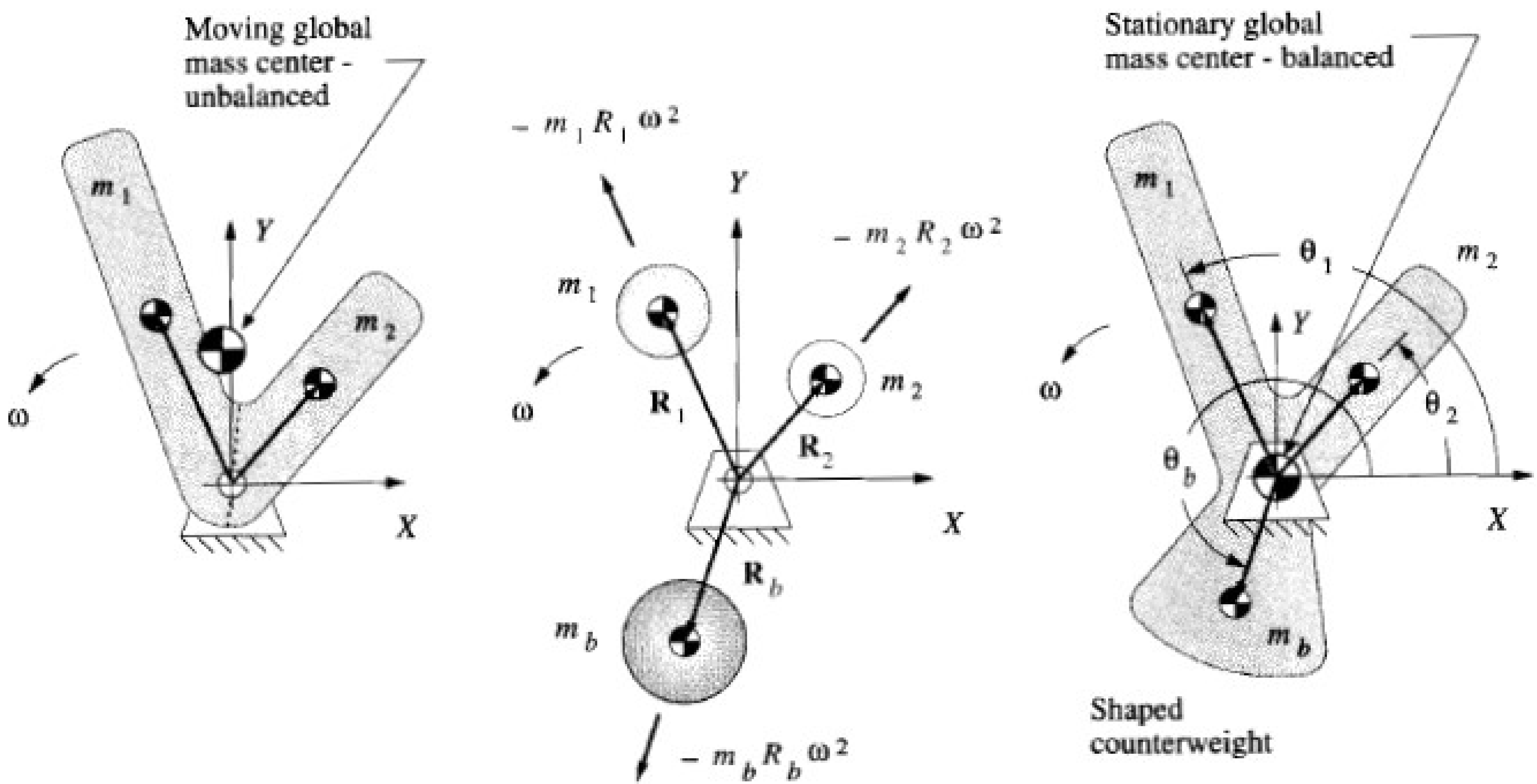
- External forces cannot be balanced by making any changes to the system's internal geometry
- Hence, only the inertia forces are equated to zero
- Single-plane balance means that the masses which are generating the inertia forces are in (nearly in) the same plane

Some examples are,

- A single gear or pulley on a shaft
- A bicycle or motorcycle tire and wheel
- A thin flywheel
- An airplane propeller
- An individual turbine blade-wheel (but not the entire turbine)

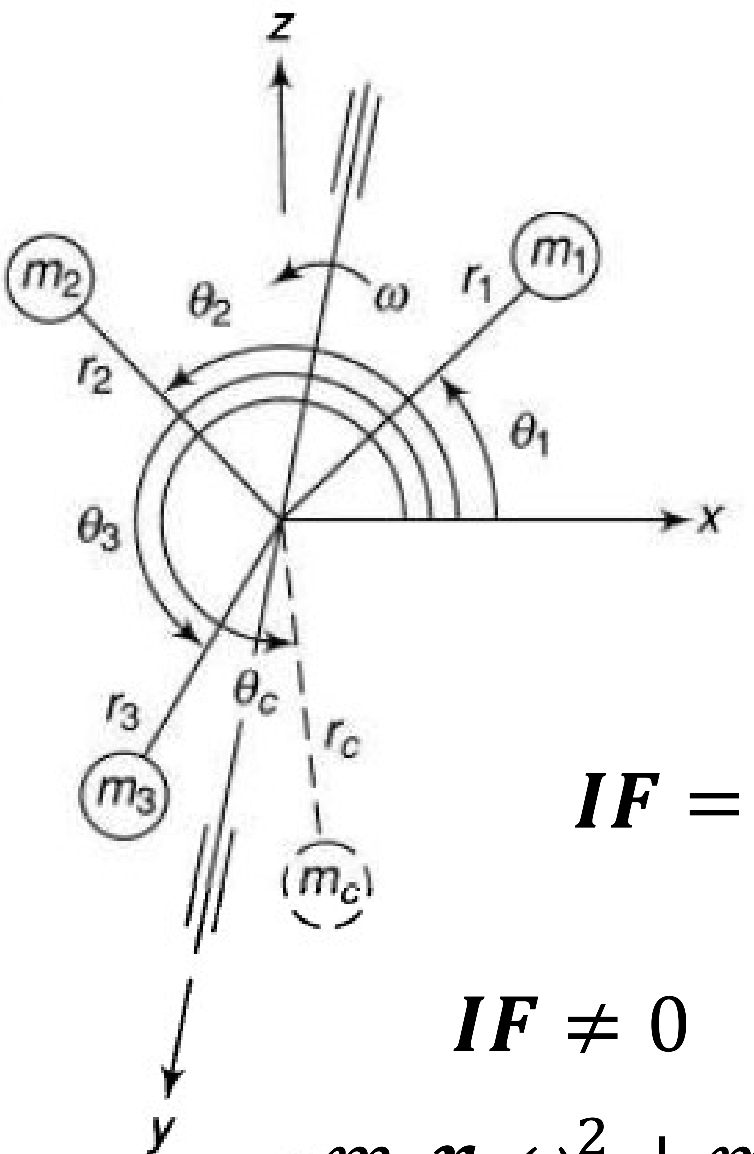
The common denominator among these devices is that they are all short in the axial direction compared to the radial direction, and thus can be considered to exist in a single plane.

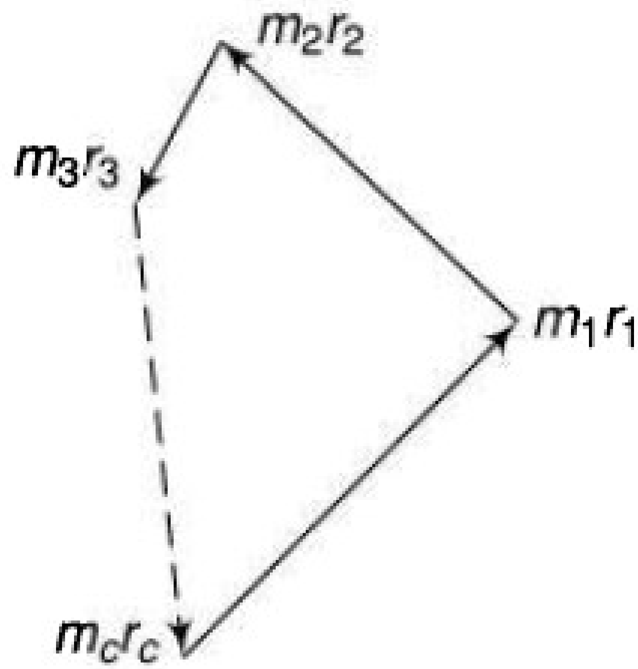
A Link in Pure Rotation - Static Balancing



(a) Unbalanced link (b) Dynamic model (c) Statically balanced link

Static Balance-Analytical Approach





$$\boldsymbol{IF} = m_1\boldsymbol{r}_1\omega^2 + m_2\boldsymbol{r}_2\omega^2 + m_3\boldsymbol{r}_3\omega^2 = 0$$

Statically balanced

$$\boldsymbol{IF} \neq 0$$

$$m_1\boldsymbol{r}_1\omega^2 + m_2\boldsymbol{r}_2\omega^2 + m_3\boldsymbol{r}_3\omega^2 + m_c\boldsymbol{r}_c\omega^2 = 0$$

In general,

$$\sum m\boldsymbol{r}\omega^2 + m_c\boldsymbol{r}_c\omega^2 = 0$$

To solve mathematically, resolve the each force in to two components

$$\sum mr\ \cos\theta + m_c r_c \cos\theta_c = 0$$

$$\sum mr\ \sin\theta + m_c r_c \sin\theta_c = 0$$

$$m_c r_c \cos\theta_c = -\sum mr\ \cos\theta$$

$$m_c r_c \sin\theta_c = -\sum mr\ \sin\theta$$

Squaring and adding,

$$m_c r_c = \sqrt{(\sum mr\ \cos\theta)^2 + (\sum mr\ \sin\theta)^2}$$

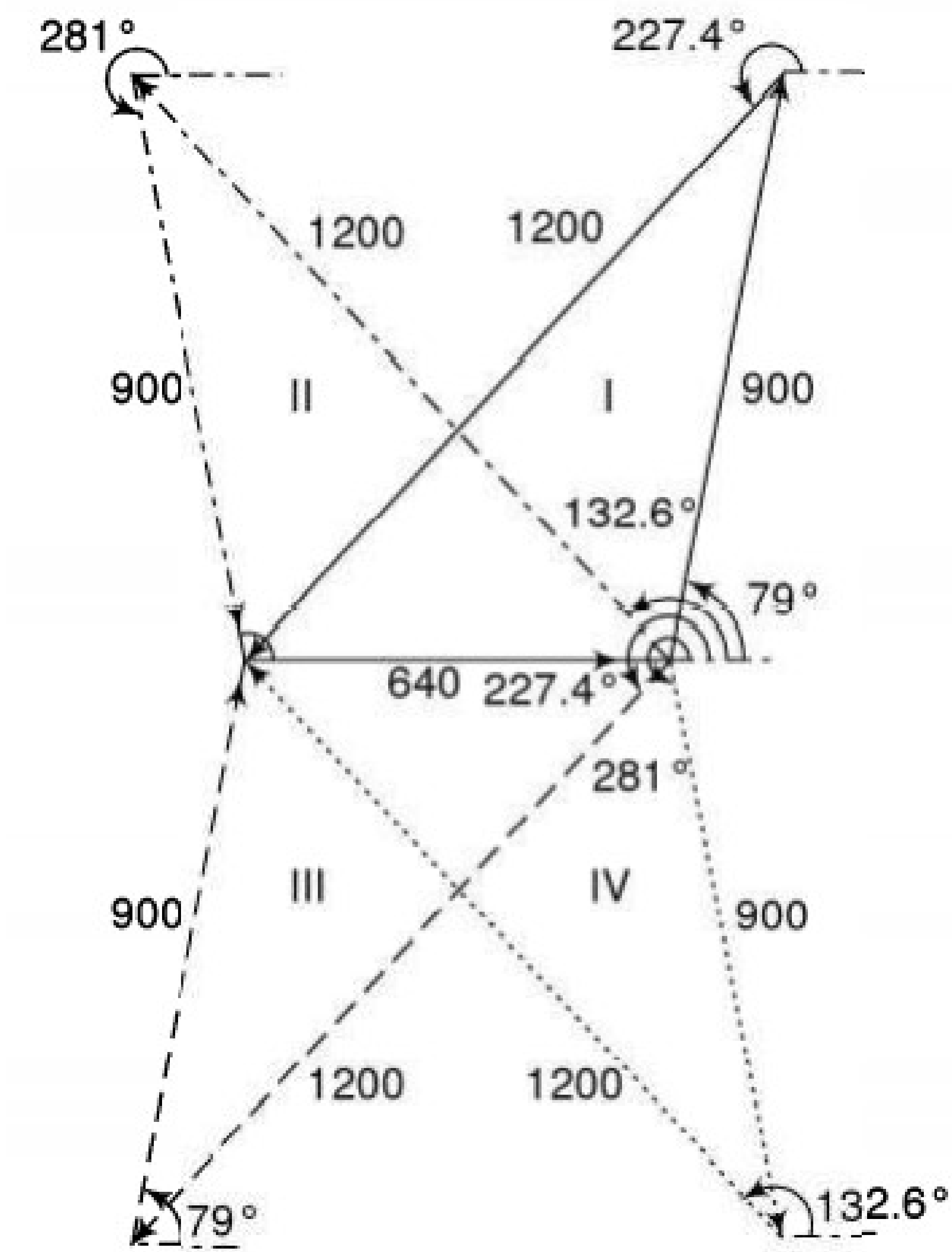
$$\tan\theta_c = \frac{-\sum mr\ \sin\theta}{-\sum mr\ \cos\theta}$$

Problem 1

Three masses of 8 kg, 12 kg and 15 kg attached at radial distances of 80 mm, 100 mm and 60 mm respectively to a disc on a shaft are in complete balance. Determine the angular positions of the masses of 12 kg and 15 kg relative to the 8 kg mass.

$$m_1 r_1 = 8 \times 80 = 640$$
$$m_2 r_2 = 12 \times 100 = 1200$$
$$m_3 r_3 = 15 \times 60 = 900$$

Graphical Solution



Analytical Solution

$$m_1 r_1 = 8 \times 80 = 640$$
$$m_2 r_2 = 12 \times 100 = 1200$$
$$m_3 r_3 = 15 \times 60 = 900$$

$$\sum m\mathbf{r} = 0$$

$$640 \cos 0° + 1200 \cos\theta_2 + 900 \cos\theta_3 = 0$$
$$1200 \cos\theta_2 = -(640 + 900 \cos\theta_3) \quad \text{(i)}$$
$$640 \sin 0° + 1200 \sin\theta_2 + 900 \sin\theta_3 = 0$$
$$1200 \sin\theta_2 = -900 \sin\theta_3 \quad \text{(ii)}$$

Squaring and adding (i) and (ii),

$$1200^2 = 640^2 + 900^2 \cos^2\theta_3 + 2 \times 640 \times 900 \times \cos\theta_3 + 900^2 \sin^2\theta_3$$
$$= 640^2 + 900^2 + 2 \times 640 \times 900 \times \cos\theta_3$$
$$\cos\theta_3 = 0.1913$$

$$\cos\theta_2 = -0.677$$

$$\theta_3 = 79° \text{ or } 281°$$

When $\theta_3 = 79°$, $1200 \sin\theta_2 = -900 \sin 79°$

$$\sin\theta_2 = -0.736$$
$$\theta_2 = -47.4° \text{ or } 227.4°$$

When $\theta_3 = \mathbf{281}°$

$$1200 \sin\theta_2 = -900 \sin \mathbf{281}°$$
$$\sin\theta_2 = 0.736$$
$$\theta_2 = 47.4° \text{ or } 132.6°$$

Problem 2

A circular disc mounted on a shaft carries three attached masses of 4 kg, 3 kg and 2.5 kg at radial distances of 75 mm, 85 mm and 50 mm and at the angular positions of 45°, 135° and 240° respectively. The angular positions are measured counter clockwise from the reference line along the x-axis. Determine the amount of the counter mass at a radial distance of 75 mm required for the static balance.

Graphical Solution

$$m_1 r_1 = 4 \times 75 = 300,$$

$$m_2 r_2 = 3 \times 85 = 255,$$

$$m_3 r_3 = 2.5 \times 50 = 125$$

$$m_c r_c = 282.75$$

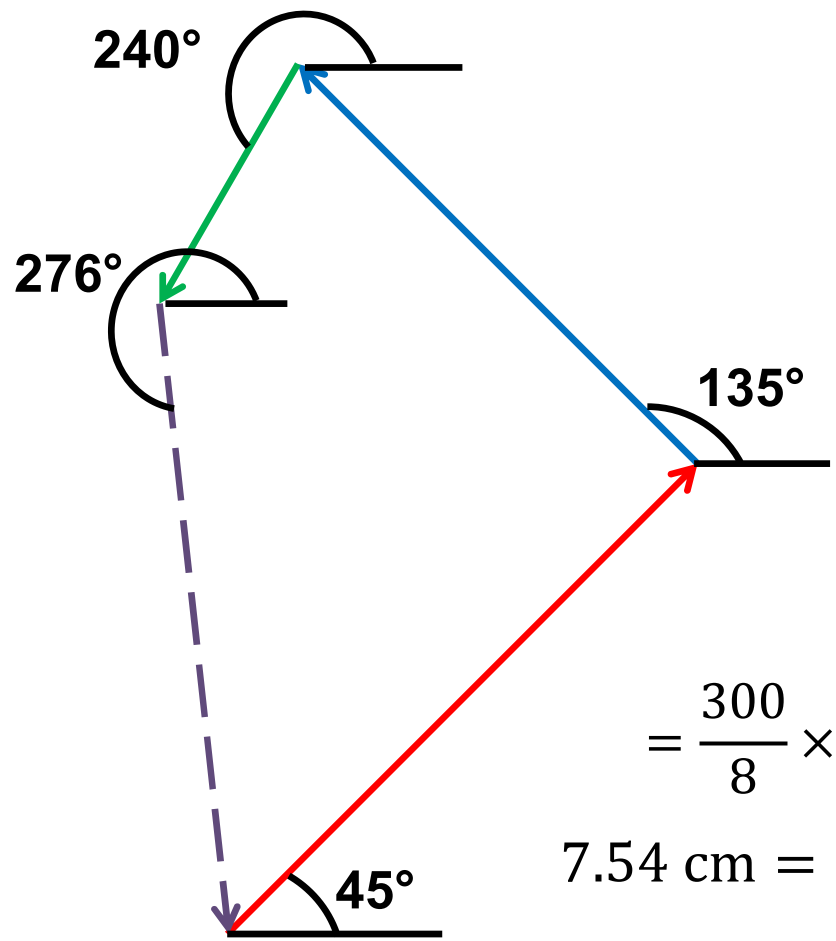
$$r_c = 75 \text{ mm}$$

$$\therefore m_c = \frac{282.75}{75} = 3.77 \text{ kg}$$

$$300 = 8 \text{ cm}$$

$$255 = 6.8 \text{ cm} \qquad = \frac{8}{300} \times 255 \text{ cm}$$

$$125 = 3.33 \text{ cm} \qquad = \frac{8}{300} \times 125 \text{ cm}$$



$$= \frac{300}{8} \times 7.54 \text{ cm}$$

$$7.54 \text{ cm} = 282.75$$

Analytical Solution

$$m_1 r_1 = 4 \times 75 = 300,$$
$$m_2 r_2 = 3 \times 85 = 255,$$
$$m_3 r_3 = 2.5 \times 50 = 125$$

$$\sum m\mathbf{r} + m_c \mathbf{r}_c = 0$$

$$300 \cos 45° + 255 \cos 135° + 125 \cos 240° + m_c r_c \cos \theta_c = 0$$

$$300 \cos 45° + 255 \sin 135° + 125 \cos 240° + m_c r_c \sin \theta_c = 0$$

Squaring, adding and then solving,

$$m_c r_c = \left[\begin{pmatrix} 300 \cos 45° + 255 \cos 135° \\ +125 \cos 240° \end{pmatrix}^2 + \begin{pmatrix} 300 \sin 45° + 255 \sin 135° \\ +125 \cos 240° \end{pmatrix}^2 \right]^{1/2}$$

$$m_c \times 75 = [(-30.68)^2 + (284.2)^2]^{1/2}$$
$$= 285.8 \text{ kg.mm}$$
$$m_c = 3.81 \text{ kg}$$

$$\tan \theta_c = \frac{-284.2}{-(-30.68)}$$

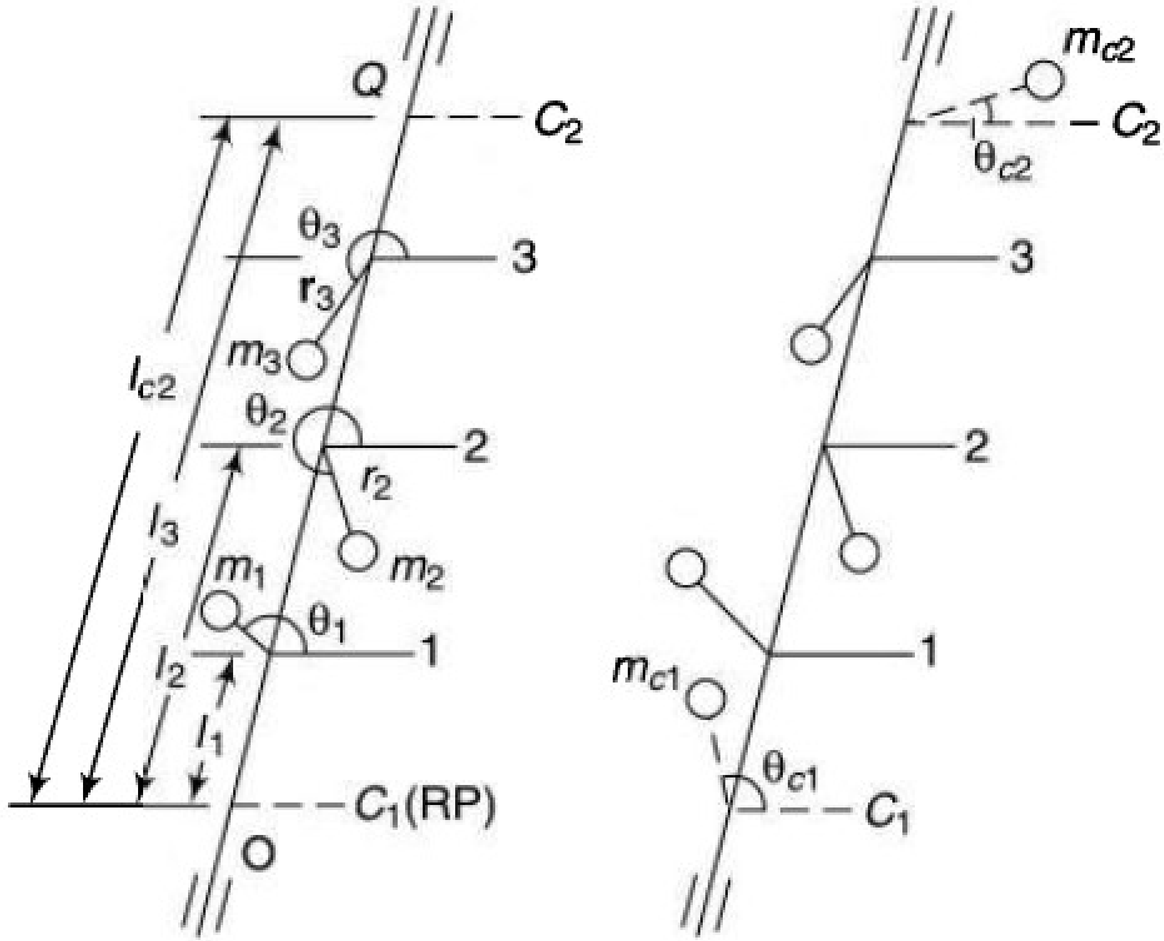
$$= \frac{-284.2}{+30.68} = -9.26$$

$$\therefore \quad \theta_c = 276°12'$$

θ_c lies in the fourth quadrant

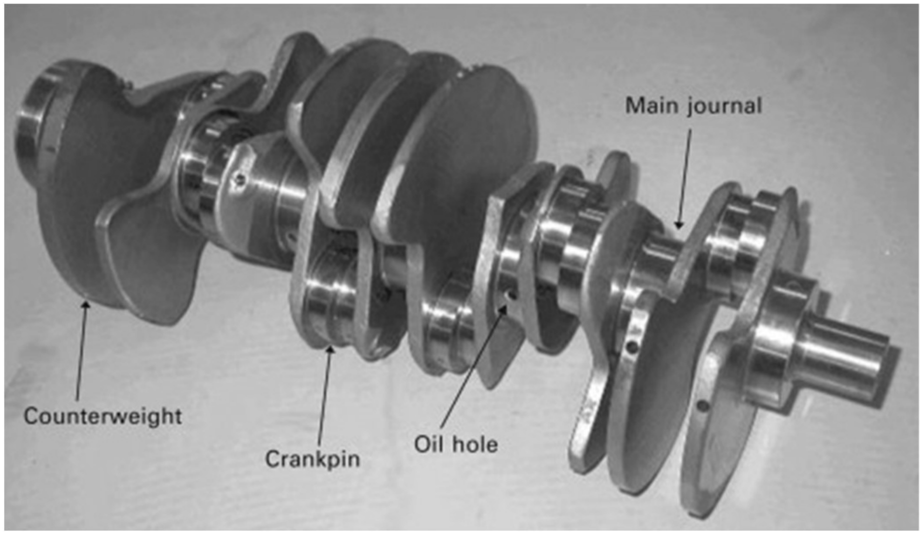
Dynamic Balancing

- When several masses rotate in different planes, the centrifugal force, in addition to being out of balance, also form couples
- A system of rotating masses is in dynamic balance when there does not exist any resultant centrifugal force as well as resultant couple



Some examples are,

- Rollers
- Crank shafts
- Cam shafts
- Motor rotors
- Clusters of multiple gears
- Aircraft turbines
- Propeller shafts



The common denominator among these devices is their mass may be unevenly distributed both rotationally around their axis and also longitudinally along their axis.

Balancing of Several Masses in Different Planes

For complete balancing of rotating masses in different planes, the resultant forces and resultant couples both should be zero.

$$m_1\mathbf{r}_1\omega^2 + m_2\mathbf{r}_2\omega^2 + m_3\mathbf{r}_3\omega^2 = 0$$

$$m_1\mathbf{r}_1 l_1\omega^2 + m_2\mathbf{r}_2\, l_2\omega^2 + m_3\mathbf{r}_3\, l_3\omega^2 = 0$$

- If not satisfied the above equations, then there are unbalanced forces and couples
- A mass placed in the reference plane may satisfy the force equation but the couple equation is satisfied only by two forces in different transverse planes
- Thus, two planes are needed to balance a system of rotating masses

The force equation becomes,

$$m_1\mathbf{r}_1\omega^2 + m_2\mathbf{r}_2\omega^2 + m_3\mathbf{r}_3\omega^2 + m_{c1}\mathbf{r}_{c1}\omega^2 + m_{c2}\mathbf{r}_{c2}\omega^2 = 0$$

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 + m_{c1}\mathbf{r}_{c1} + m_{c2}\mathbf{r}_{c2} = 0$$

$$\boxed{\Sigma m\mathbf{r} + m_{c1}\mathbf{r}_{c1} + m_{c2}\mathbf{r}_{c2} = 0}$$

Taking moment about reference plane O,

$$m_1\mathbf{r}_1 l_1\omega^2 + m_2\mathbf{r}_2\, l_2\omega^2 + m_3\mathbf{r}_3\, l_3\omega^2 + m_{c2}\mathbf{r}_{c2}\, l_{c2}\omega^2 = 0$$

$$m_1\mathbf{r}_1\, l_1 + m_2\mathbf{r}_2\, l_2 + m_3\mathbf{r}_3\, l_3 + m_{c2}\mathbf{r}_{c2}\, l_{c2} = 0$$

$$\boxed{\Sigma m\mathbf{r}l + m_{c2}\mathbf{r}_{c2}\, l_{c2} = 0}$$

To solve mathematically, resolve the each force in to two components to find m_{c2} *or* r_{c2} and θ_{c2}

$$\sum mrl\ \cos\theta + m_{c2}r_{c2}\,l_{c2}\cos\theta_{c2} = 0$$

$$\sum mrl\ \sin\theta + m_{c2}r_{c2}\,l_{c2}\,sin\,\theta_{c2} = 0$$

$$m_{c2}r_{c2}\,l_{c2}\cos\theta_{c2} = -\sum mrl\ \cos\theta$$

$$m_{c2}r_{c2}\,l_{c2}\,sin\,\theta_{c2} = -\sum mrl\ \sin\theta$$

Squaring and adding,

$$\boxed{m_{c2}r_{c2}l_{c2} = \sqrt{(\sum mrl\ \cos\theta)^2+(\sum mrl\ \sin\theta)^2}}$$

$$\boxed{\tan\theta_{c2} = \frac{-\sum mrl\ \sin\theta}{-\sum mrl\ \cos\theta}}$$

After finding m_{c2} $(or\ r_{c2})$ and θ_{c2}, using force equations, find m_{c1} $(or\ r_{c1})$ and θ_{c1}

$$\sum mr\ \cos\theta + m_{c1}r_{c1}\cos\theta_{c1} + m_{c2}r_{c2}\cos\theta_{c2} = 0$$

$$\sum mr\ \sin\theta + m_{c1}r_{c1}\sin\theta_{c1} + m_{c2}r_{c2}\sin\theta_{c2} = 0$$

$$m_{c1}r_{c1}\cos\theta_{c1} = -(\sum mr\ \cos\theta + m_{c2}r_{c2}\cos\theta_{c2})$$

$$m_{c1}r_{c1}\sin\theta_{c1} = -(\sum mr\ \sin\theta + m_{c2}r_{c2}\sin\theta_{c2})$$

Squaring and adding,

$$\boxed{m_{c1}r_{c1} = \sqrt{(\sum mr\ \cos\theta + m_{c2}r_{c2}\cos\theta_{c2})^2 + (\sum mr\ \sin\theta + m_{c2}r_{c2}\sin\theta_{c2})^2}}$$

$$\boxed{\tan\theta_{c1} = \frac{-(\sum mr\ \sin\theta + m_{c2}r_{c2}\sin\theta_{c2})}{-(\sum mr\ \cos\theta + m_{c2}r_{c2}\cos\theta_{c2})}}$$

Problem 3

A rotating shaft carries three unbalanced masses of 4 kg, 3 kg and 2.5 kg at radial distances 75 mm, 85 mm and 50 mm and at the angular positions of 45°, 135° and 240° respectively. The second and the third masses are in the planes at 200 mm and 375 mm from the plane of the first mass. The angular positions are measured counter-clockwise from the reference line along x-axis and viewing the shaft from the first mass end.

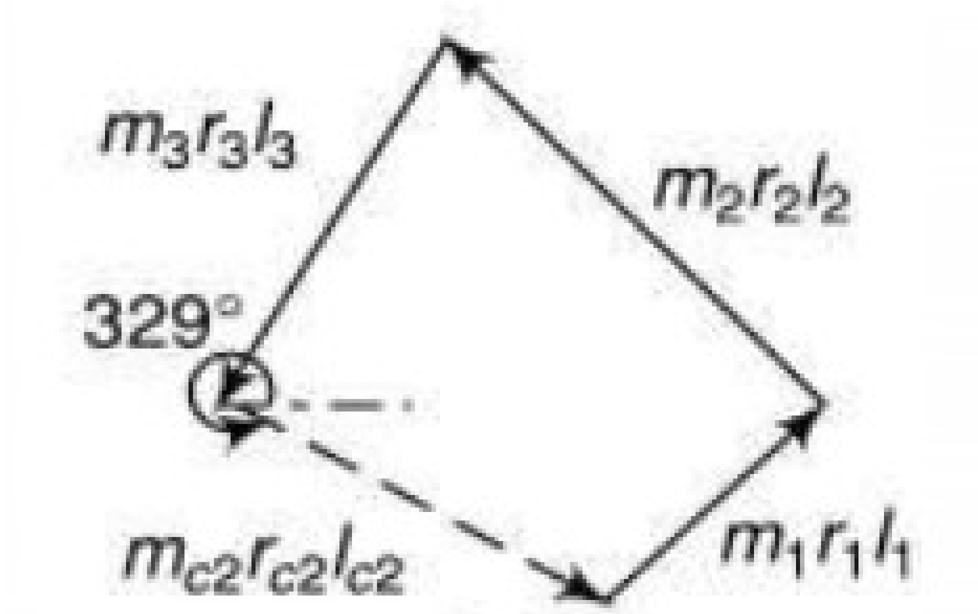
The shaft length is 800 mm between bearings and the distance between the plane of the first mass and the bearing at that end is 225 mm. Determine the amount of the counter masses in planes at 75 mm from the bearings for the complete balancing of the shaft. The first counter mass is to be in a plane between the first mass and the bearing and the second mass in a plane between the third mass and the bearing at that end.

Graphical Solution

Plane	Mass (m), kg	Radius (r), mm	Length (l), mm	Force mr	Couple mrl	
C_1 (RP)	m_{c1}	r_{c1}=75	0	$75m_{c1}$	0	
1	4	75	150	300	45000	**= 5 cm**
2	3	85	350	255	89250	**= 9.92 cm**
3	2.5	50	525	125	65625	**= 7.29 cm**
C_2	m_{c2}	r_{c2}=40	675	$40m_{c2}$	$27000m_{c2}$	**= 8.55 cm**

= 76950

m_{c2} = 2.85 kg



Couple Polygon

Graphical Solution

m_{c2} = 2.85 kg

Plane	Mass (m), kg	Radius (r), mm	Length (l), mm	Force mr
C_1 (RP)	m_{c1}	r_{c1}=75	0	$75m_{c1}$
1	4	75	150	300
2	3	85	350	255
3	2.5	50	525	125
C_2	2.85	r_{c2}=40	675	114

= 3.9 cm **= 234**

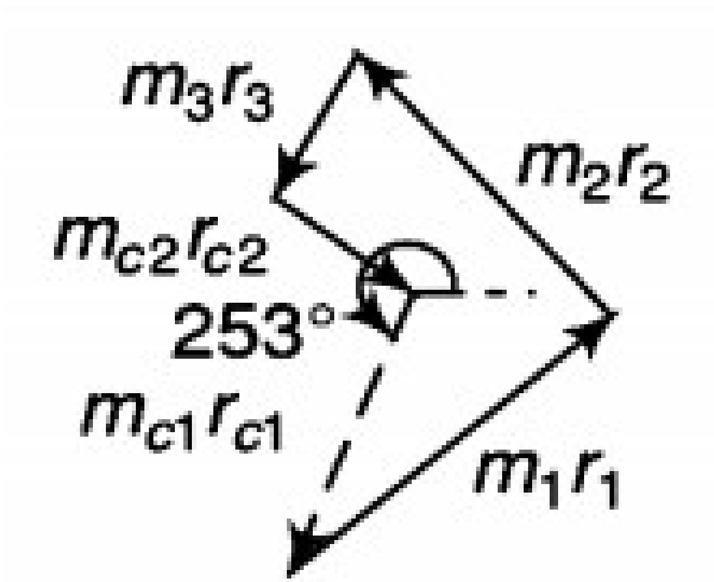
= 5 cm

= 4.25 cm

= 2.1 cm

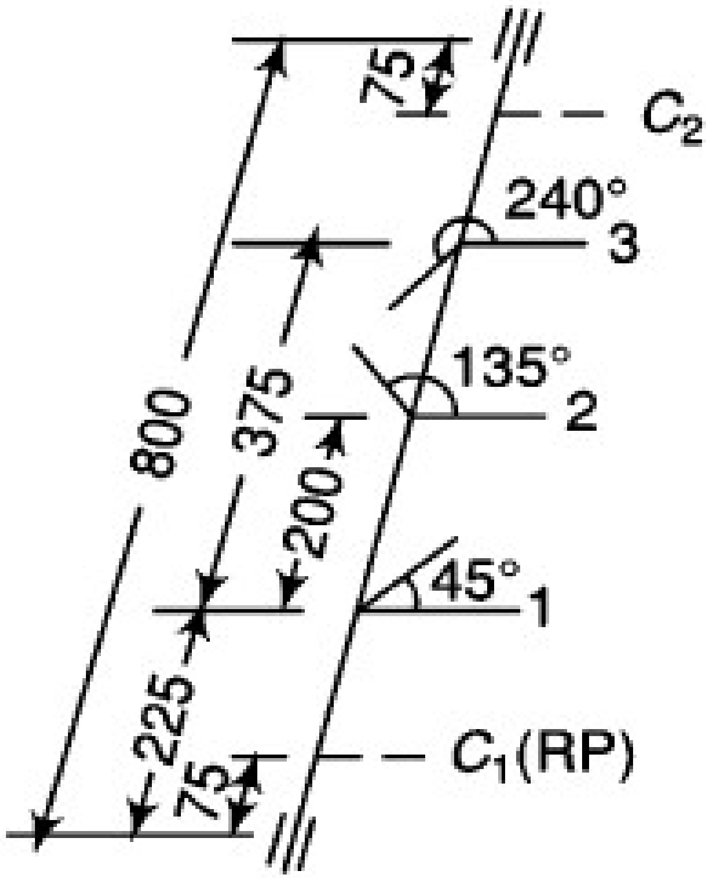
= 1.9 cm

m_{c1} = 3.12 kg



Force Polygon

Solution



$l_{c2} = (800 - 75 \times 2) = 650$ mm

$l_1 = 225 - 75 = 150$ mm

$l_2 = 150 + 200 = 350$ mm

$l_3 = 150 + 375 = 525$ mm

$$m_1 r_1 l_1 = 4 \times 75 \times 150 = 45\,000 \qquad m_1 r_1 = 4 \times 75 = 300$$

$$m_2 r_2 l_2 = 3 \times 85 \times 350 = 89\,250 \qquad m_2 r_2 = 3 \times 85 = 255$$

$$m_3 r_3 l_3 = 2.5 \times 50 \times 525 = 65\,625 \qquad m_3 r_3 = 2.5 \times 50 = 125$$

$$\Sigma m\mathbf{r}l + m_{c2}\mathbf{r}_{c2}\, l_{c2} = 0$$

or $4500 \cos 45° + 89\,250 \cos 135° + 65\,625 \cos 240° + m_{c2}\mathbf{r}_{c2} l_{c2} \cos\theta_{c2} = 0$

and $45000 \sin 45° + 89\,250 \sin 135° + 65\,625 \sin 240° + m_{c2}\mathbf{r}_{c2} l_{c2} \sin\theta_{c2} = 0$

Squaring, adding and then solving,

$$m_{c2} r_{c2} l_{c2} = \left[\begin{pmatrix} 45\,000 \cos 45° + 89\,250 \\ \cos 135° + 65\,625 \cos 240° \end{pmatrix}^2 + \begin{pmatrix} 45\,000 \sin 45° + 89\,250 \\ \sin 135° + 65\,625 \sin 240° \end{pmatrix}^2 \right]^{1/2}$$

$$= [(-64\,102)^2 + (38\,096)^2]^{1/2}$$

or $$m_{c2} \times 40 \times 650 = 74\,568$$

$$m_{c2} = \underline{2.868 \text{ kg}}$$

$$\tan \theta_{c2} = \frac{-38\,096}{-(-64\,102)} = -0.594$$

$$\theta_{c2} = 329.3° \text{ or } \underline{329°18'}$$

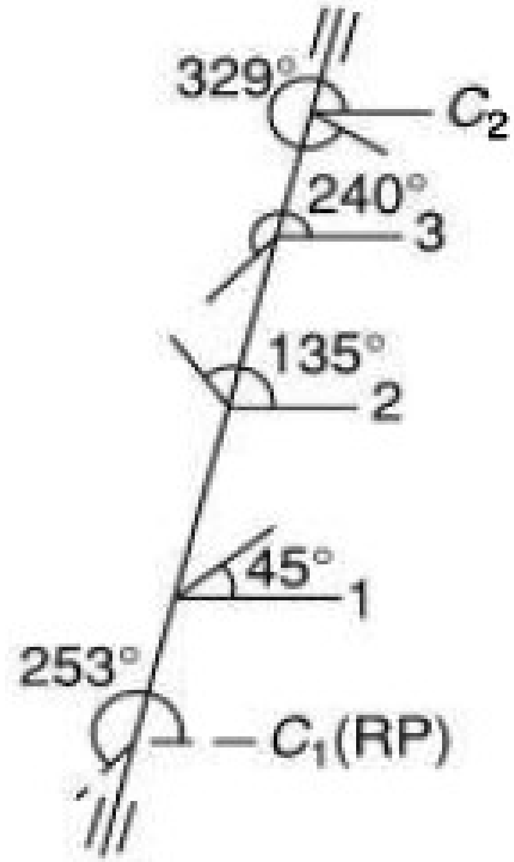
$$\Sigma m\mathbf{r} + m_{c1}\mathbf{r}_{c1} + m_{c2}\mathbf{r}_{c2} = 0$$

or $$300\cos 45° + 255\cos 135° + 125\cos 240° + m_{c1}\mathbf{r}_{c1}\cos\theta_1 + 2.868 \times 40\cos 329.3 = 0$$

and $$300\sin 45° + 255\sin 135° + 125\sin 240° + m_{c1}\mathbf{r}_{c1}\sin\theta_1 + 2.868 \times 40\sin 329.3 = 0$$

Squaring, adding and then solving,

$$m_{c1}r_{c1} = \begin{bmatrix} (300\cos 45° + 255\cos 135° \\ + 125\cos 240° + 2.868 \\ \times 40\cos 329.3°)^2 + \\ (300\sin 45° + 255\sin 135° \\ + 125\sin 240° + 2.868 \\ \times 40\sin 329.3°)^2 \end{bmatrix}^{1/2}$$



$$m_{c1} \times 75 = [(67.96)^2 + (225.62)^2]^{1/2} = 235.63$$

$$m_{c1} = \underline{3.14\text{ kg}}$$

$$\tan\theta_{c1} = \frac{-225.62}{-67.96} = 3.32;\ \theta_{c1} = 253.2° \text{ or } \underline{253°12'}$$

Balancing of Reciprocating Masses

Single Slider Crank Mechanism

Acceleration of the reciprocating mass of a slider-crank mechanism is given by

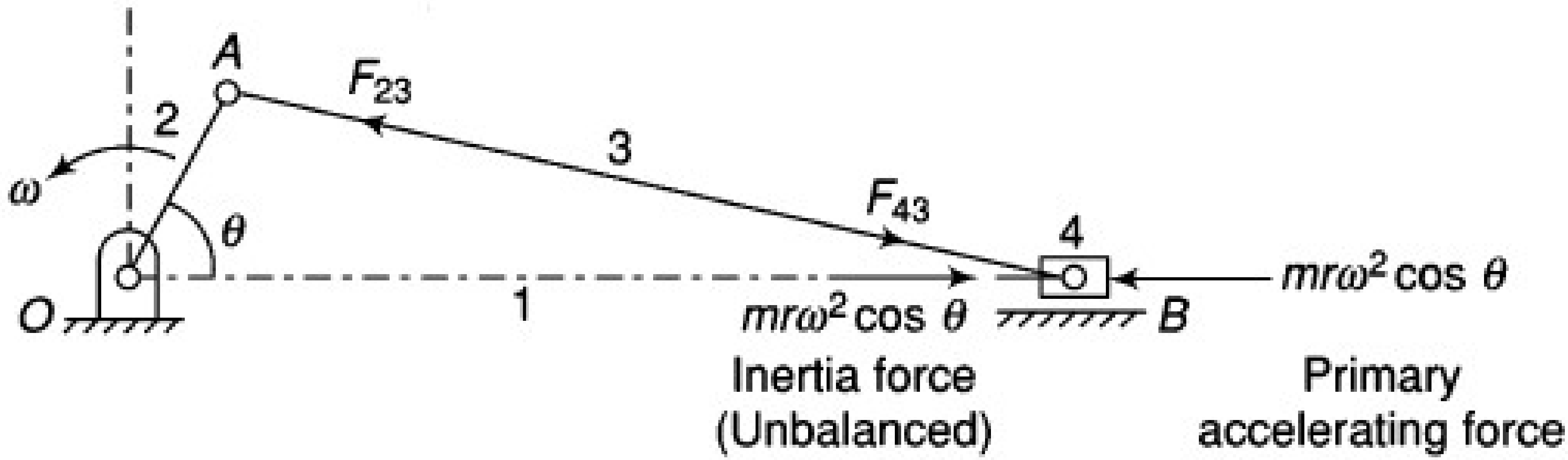
$$a = r\omega^2\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

$$\boxed{n = \frac{l}{r}}$$

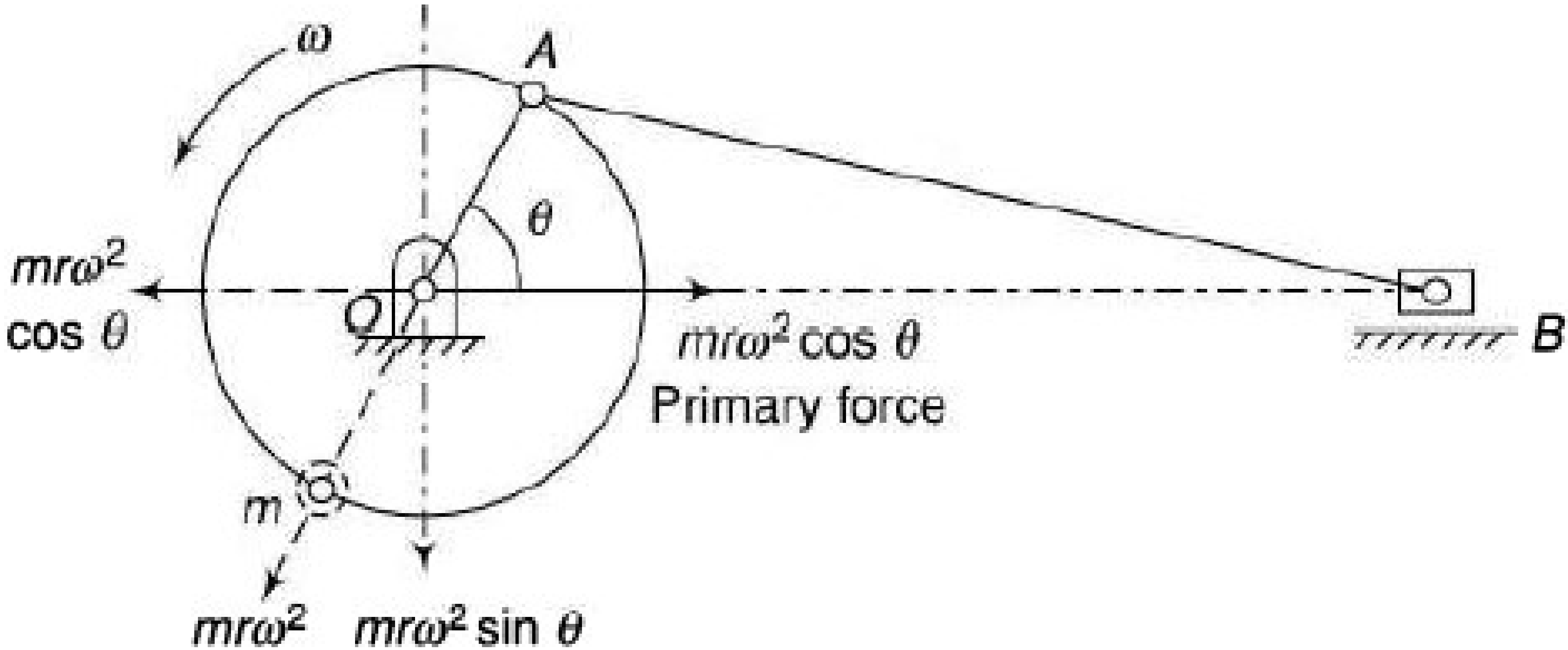
The force required to accelerate the mass *m* is

$$F = mr\omega^2\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

$$F = \underbrace{mr\omega^2\cos\theta}_{\text{Primary accelerating force}} + \underbrace{mr\omega^2\frac{\cos 2\theta}{n}}_{\text{Secondary accelerating force}}$$

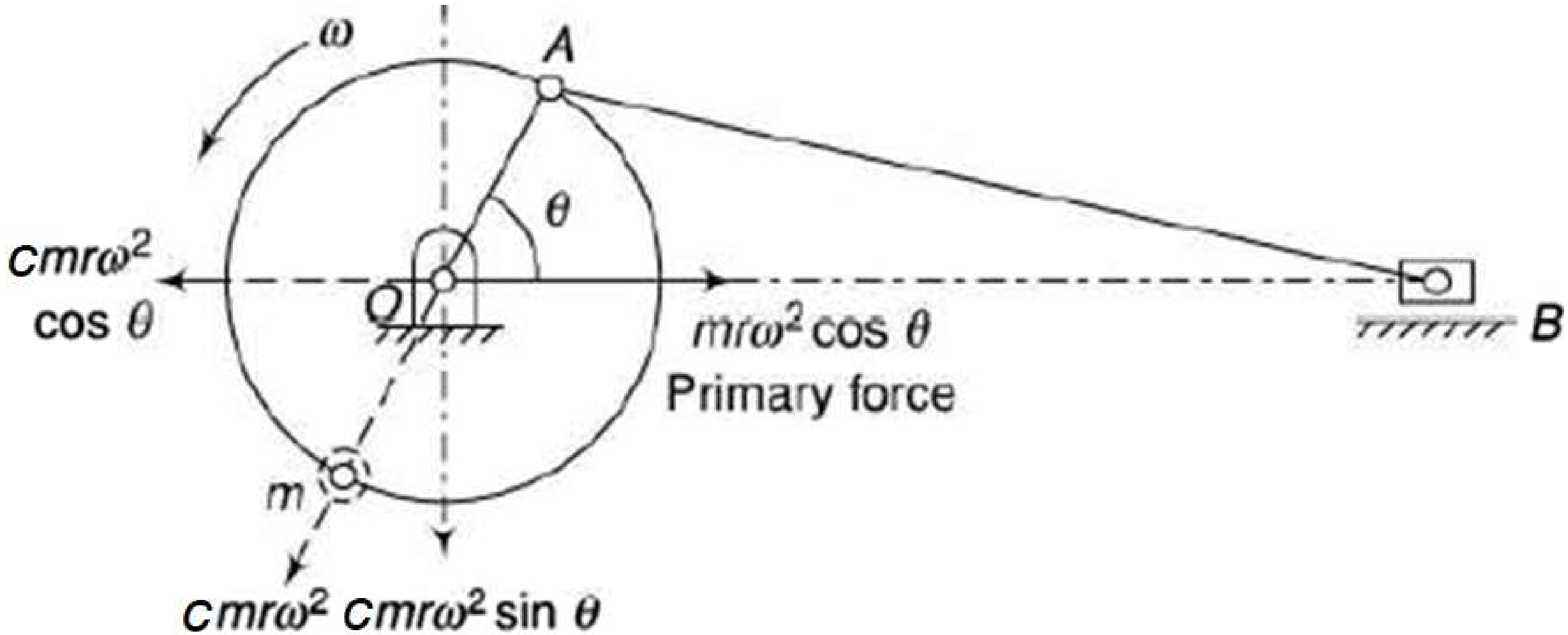


- The mass of the connecting rod is lumped at A and B by assuming a dynamically equivalent connecting rod (massless)
- Thus the complex motion (translation and rotation or general plane motion) of connecting rod is not to be considered in balancing the mechanism
- The unbalance primary inertia force is shown in Figure



- The unbalance primary inertia force can be balanced by adding a counter mass as shown in Figure and thus the horizontal component of the counter mass ($\boldsymbol{mr\omega^2 \cos\theta}$) balances the reciprocating unbalance.
- But the vertical component ($\boldsymbol{mr\omega^2 \sin\theta}$) is now introduced
- To minimize the effect of the vertical component, partial balancing preferred

Partial Balancing



If ***C*** is the fraction of the reciprocating mass, then

primary force balanced by the mass $= cmr\omega^2 \cos\theta$

primary force unbalanced by the mass $= (1 - c)cmr\omega^2 \cos\theta$

vertical component of centrifugal force which remains unbalanced

$$= cmr\omega^2 \sin\theta$$

Resultant unbalanced force at any instant

$$= \sqrt{[(1-c)\,mr\omega^2 \cos\theta]^2 + [cmr\omega^2 \sin\theta]^2}$$

The resultant unbalanced force is minimum when $c = 1/2$.

If m_p is the mass at the crank pin and C is the fraction of the reciprocating mass m, the mass at the crank pin may be considered as $(cm + m_p)$ which is to be completely balanced