Mechanics of Machines

Mechanical Vibration

Module 6:

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(Deemed to be University under section 3 of UGC Act, 1956)

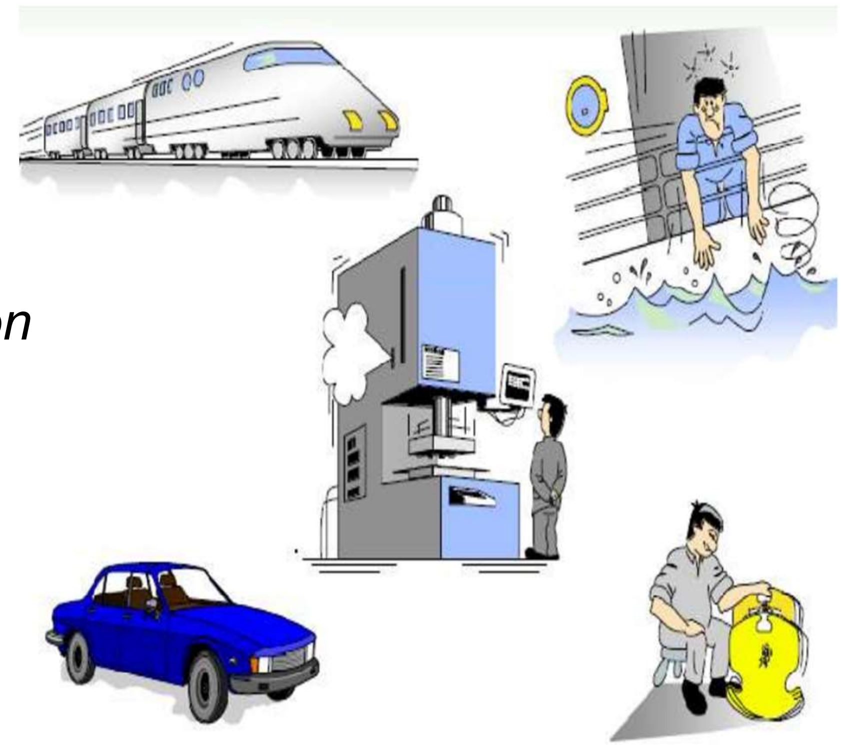
Mechanical vibration:

- Class Objective1: What is vibration ?
- Class Objective2: Why vibration?
- Class objective3: Free vibration

Vibration:

- Vibrations are the oscillations of a structural system about the equilibrium position.
- Vibration system involves transfer of PE to KE, and KE to PE.

It is also an everyday phenomenon we meet on everyday life

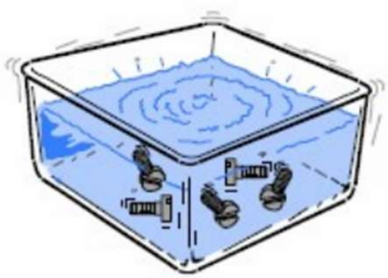


Vibration:

Useful Vibration

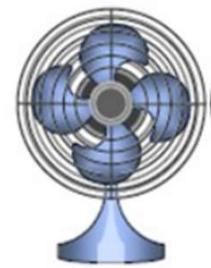
Compressor

Testing



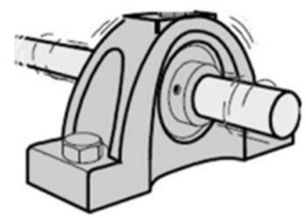
Ultrasonic cleaning

Harmful vibration

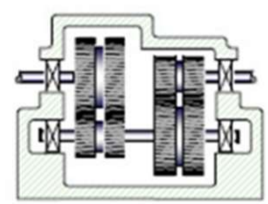


Noise

Destruction



Wear



Fatigue

Vibration definition:

- Mechanical vibration is the motion of a particle or body which oscillates about a position of equilibrium. Most vibrations in machines and structures are undesirable due to increased stresses and energy losses.

- Time interval required for a system to complete a full cycle of the motion is the period of the vibration.

- Number of cycles per unit time defines the frequency of the vibration.

- Maximum displacement of the system from the equilibrium position is the amplitude of the vibration.

Vibration definition:

- **Free (or natural) vibrations:** A vibration in which after the initial displacement, no external forces act and the motion is maintained by the internal elastic forces, is termed as free or natural vibration

- **Forced vibration:** These type of vibrations are caused when a periodic disturbing force is continuously applied to the body. The vibrations then has the same frequency as the applied force.

- **Damping:** It is the resistance to the motion of a vibrating body.

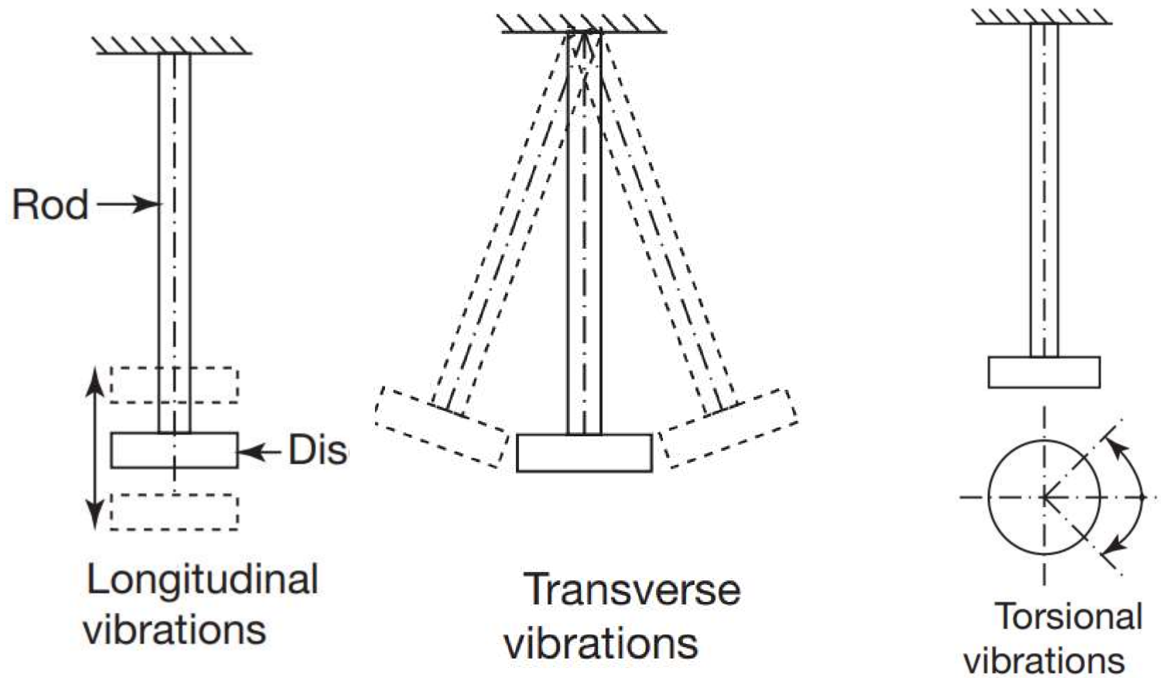
- **Natural frequency:** It is the frequency of free vibrations of a body vibrating of its own without the help of an external agency.

Vibration definition:

- **Resonance:** When the frequency of external excitation is equal to the natural frequency of a vibrating body.
- **Degrees of freedom:** The minimum number of independent coordinates required to specify the motion of a system.

Types of vibrations:

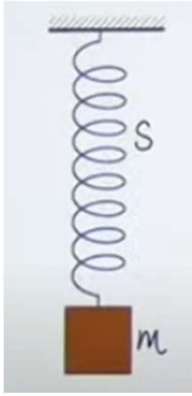
- Longitudinal vibrations
- Transverse vibrations
- Torsional vibrations



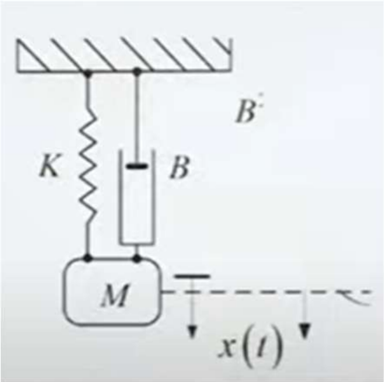
A vibrating system :

1. KE storing device (mass) or *Inertial elements*
2. PE storing device (spring) or *Restoring elements*
3. Friction (damper) or *Damping elements*
4. Unbalance force

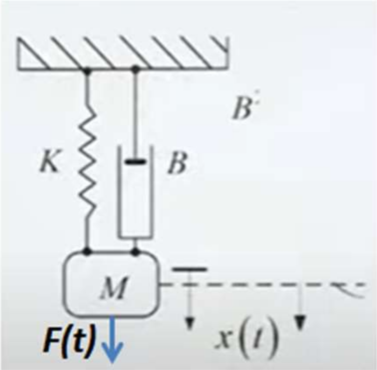
Free vibration system



Damped vibration system



Forced damped vibration system



Free vibration system:

Now the forces acting on the system are, applying D'Alembert's principle, we have

Upward force $= k(x + \delta_{st})$

Downward force $= -m\ddot{x} + mg$

For the equilibrium of the system, $-m\ddot{x} + mg = k(x + \delta_{st})$

$\therefore$ $m\ddot{x} + kx = 0$

$$x(t) = A \sin \omega_n t + B \cos \omega_n t$$

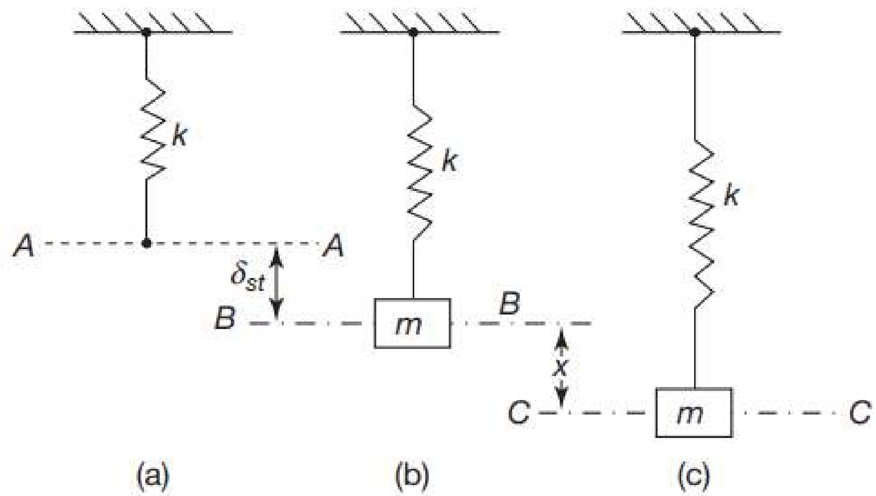
$$\dot{x}(t) = \omega_n \left[A \cos \omega_n t - B \sin \omega_n t \right]$$

$$x(0) = x_o \quad \text{and} \quad \dot{x}(0) = v_o$$

$$x_o = B$$

$$v_o = \omega_n A$$

$$A = \frac{v_o}{\omega_n}$$



Stages in the extension of a spring

Free vibration system:

$$x(t) = A \sin \omega_n t + B \cos \omega_n t$$

$$\dot{x}(t) = \omega_n \left[A \cos \omega_n t - B \sin \omega_n t \right]$$

$$x(0) = x_o \quad \text{and} \quad \dot{x}(0) = v_o$$

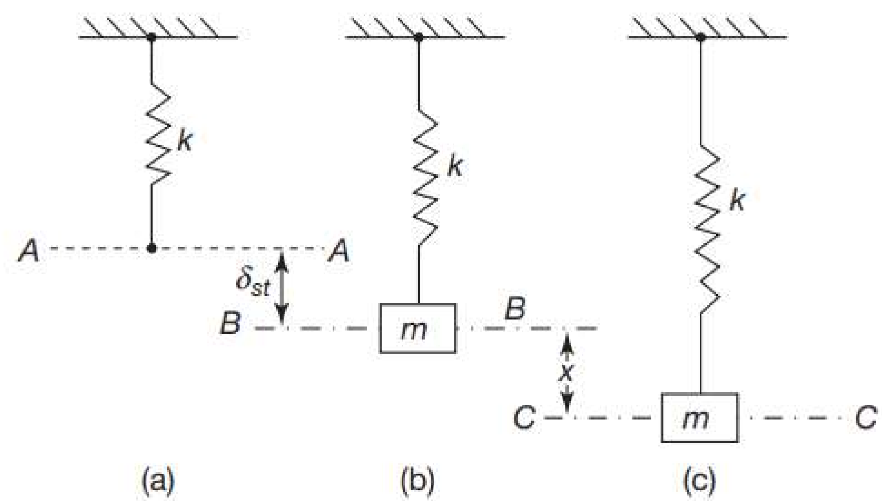
$$x_o = B$$

$$v_o = \omega_n A$$

$$A = \frac{v_o}{\omega_n}$$

$$x(t) = \left(\frac{v_o}{\omega_n} \right) \sin \omega_n t + x_o \cos \omega_n t$$

$$x(t) = X \sin (\omega_n t + \phi)$$



Stages in the extension of a spring

Free vibration system:

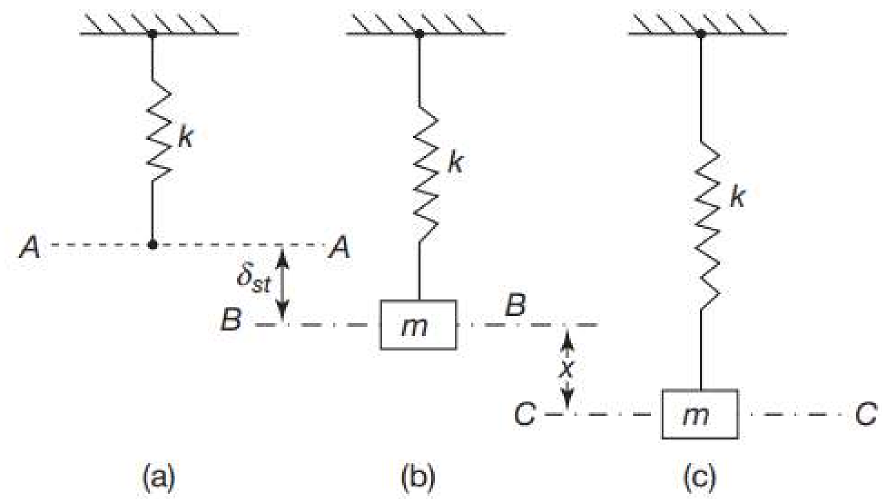
$$x(t) = X \sin(\omega_n t + \phi)$$

Velocity, $\dot{x}(t) = \mathrm{x}\,\omega_n \cos(\omega_n t + \phi)$

$$= \mathrm{x}\,\omega_n \sin\left[\frac{\pi}{2} + (\omega_n t + \phi)\right]$$

Acceleration, $\ddot{x}(t) = -\mathrm{x}\,\omega_n^2 \sin(\omega_n t + \phi)$

$$= \mathrm{x}\,\omega_n^2 \sin\left[\pi + (\omega_n t + \phi)\right]$$

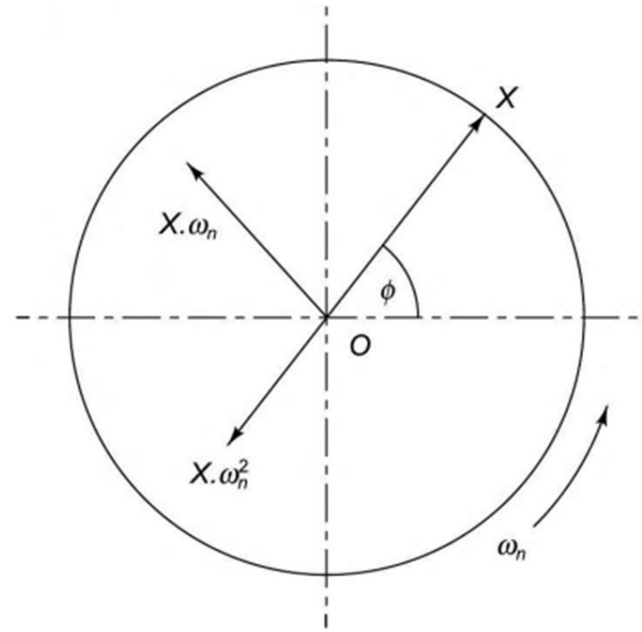


Stages in the extension of a spring

Free vibration system:

Acceleration, $\ddot{x}(t) = -\mathrm{x}\ \omega_n^2 \sin(\omega_n t + \phi)$

$$= \mathrm{x}\ \omega_n^2 \sin\left[\pi + (\omega_n t + \phi)\right]$$



$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta_{st}}} \text{ rad/s}$$

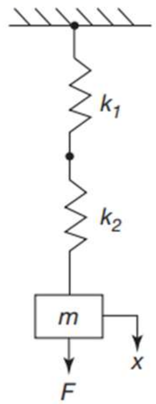
$$\text{Time period, } T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}} \text{ s}$$

$$\text{Natural frequency, } f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ Hz (Cycles/s)}$$

Equivalent Stiffness of Springs:

Springs in series:

$$F = k_1 x_1 = k_2 x_2$$



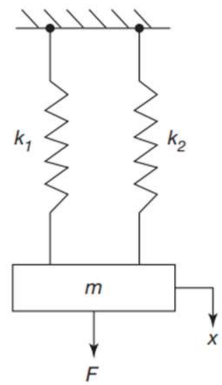
Total extension, $x = x_1 + x_2$

$$= F\left(\frac{1}{k_1} + \frac{1}{k_2}\right)$$

Equivalent stiffness, $k_e = \dfrac{F}{x} = \dfrac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}} = \dfrac{k_1 k_2}{k_1 + k_2}$

$$\boxed{\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2}}$$

Springs in parallel:



$F_1 = k_1 x$ and $F_2 = k_2 x$

$$F = F_1 + F_2$$

$$= (k_1 + k_2)\, x$$

$$\boxed{k_e = \frac{F}{x} = k_1 + k_2}$$

Problem1:

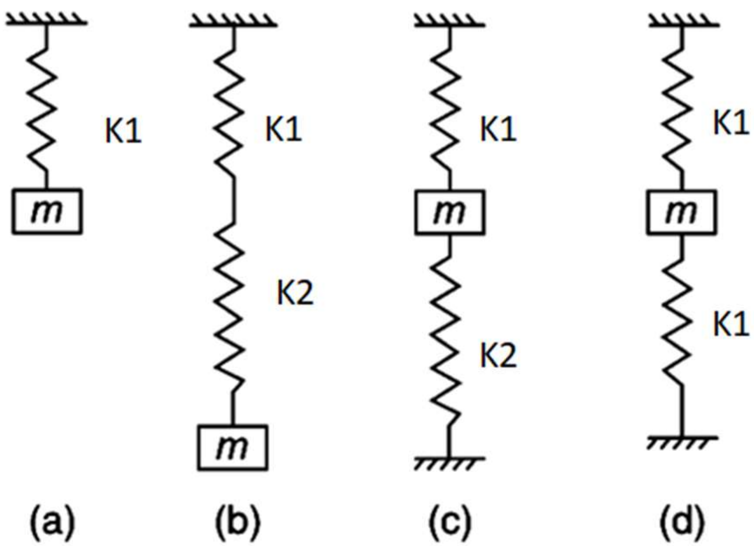
Determine the equivalent spring stiffness and natural frequency of the following vibrating systems when

(a) Mass is suspended to a spring

(b) Mass is suspended at the bottom of the two springs in series.

(c) Mass fixed in between two springs

(d) Mass is fixed to the midpoint of a spring



Take

$K_1 = 5$ N/mm, $K_2 = 8$ N/mm

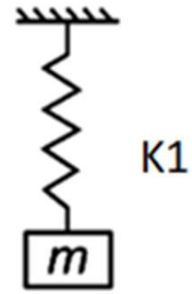
$m = 10$ kg,

Continued:

(a) **Mass is suspended to a spring**

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{5\times 10^3}{10}} = 3.56 \text{ Hz}$$

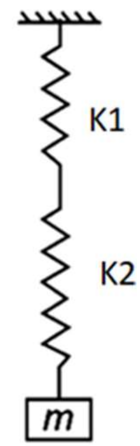


(b) Mass is suspended at the bottom of the two springs in series.

$$k_e = \frac{k_1 k_2}{k_1 + k_2}$$

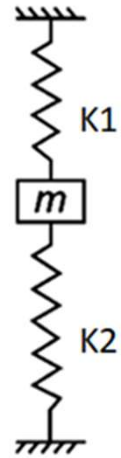
$$f_n = \frac{1}{2\pi}\sqrt{\frac{\text{Keq}}{m}} = \frac{1}{2\pi}\sqrt{\frac{\mathbf{K1\ K2}}{(\mathbf{K1+\ K2})m}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{(5\times 10^3)\times(8\times 10^3)}{(5+8)\times 10^3\times 10}} = \underline{2.79 \text{ Hz}}$$



Continued:

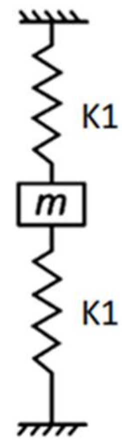
(c) Mass fixed in between two springs



$$k_e = k_1 + k_2$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{\text{Keq}}{m}} = \frac{1}{2\pi}\sqrt{\frac{13\times10^3}{10}} = \underline{5.74\ \text{Hz}}$$

(d) Mass is fixed to the midpoint of a spring



$$\text{Stiffness of spring on each side} = \frac{\text{K1}}{1/2} = 2\,\text{K1}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{4\text{K1}}{m}} = \frac{1}{2\pi}\sqrt{\frac{4\times5\times10^3}{10}} = \underline{7.12\ \text{Hz}}$$

Damped vibration system:

$$m\ddot{x} + c\dot{x} + kx = 0$$

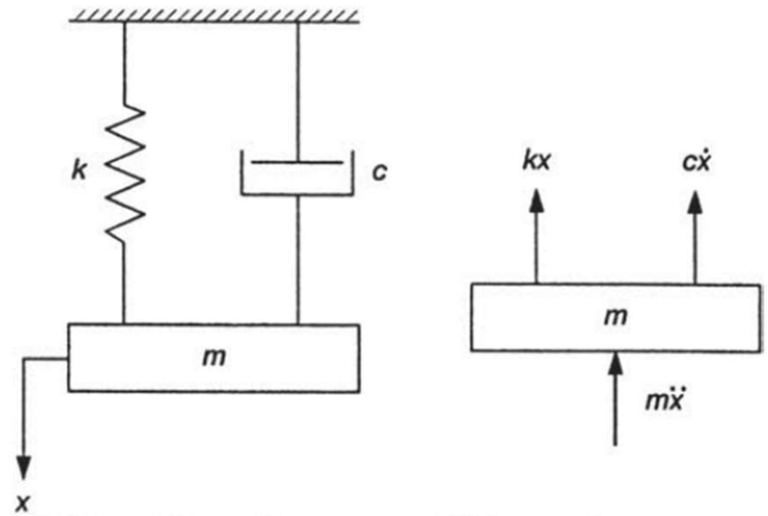
$$\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = 0$$

$$x = e^{\alpha t}$$

$$\alpha^2 + \frac{c}{m}\alpha + \frac{K}{m} = 0$$

$$\alpha_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{K}{m}\right)}$$

$$x = A e^{\alpha_1 t} + B e^{\alpha_2 t}$$



(a) Damped free system (b) Forces acting on mass

For critical damping, the term under the square root is zero

Continued:

$$\alpha_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{K}{m}\right)}$$

For critical damping, the term under the square root is zero, the damping coefficient is called the critical damping coefficient, *C_c.*

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$$

$$\frac{c_c}{2m} = \left(\frac{k}{m}\right)^{1/2} = \omega_n$$

$$c_c = 2m\omega_n = 2(km)^{1/2}$$

$$\boxed{c_c = 2m\omega_n = 2(km)^{1/2}}$$

$$\boxed{\text{damping ratio,} \quad \zeta = \frac{c}{c_c} = \frac{\text{damping coefficient}}{\text{critical damping coefficient}}}$$

Continued:

$$\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = 0$$

$$\frac{c}{2m} = \left(\frac{c}{c_c}\right)\left(\frac{c_c}{2m}\right) = \zeta\omega_n$$

$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0}$$

$$\alpha_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{K}{m}\right)}$$

$$\alpha_{1.2} = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2} = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_n$$

Over damped system ($\zeta > 1$)

Criticality damped system ($\zeta = 1$)

Under damped system ($\zeta < 1$)

Over damped system ($\zeta > 1$)

The roots of the auxiliary equation are real.

$$\alpha_{1,2} = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_n$$

Therefore, the solution is $x = A\,e^{\alpha_1 t} + B\,e^{\alpha_2 t}$

$$\boxed{x = Ae^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + Be^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t}}$$

Constants A and B can be determined from the initial conditions. This is the equation of an aperiodic motion, i.e., the system cannot vibrate due to over-damping. The magnitude of the resultant displacement approaches zero with time.

Under damped system ($\zeta < 1$)

The roots of the auxiliary equation are imaginary. $\alpha_{1,2} = (-\zeta \pm i\sqrt{1-\zeta^2})\omega_n$

$$x = Ae^{(-\zeta + i\sqrt{1-\zeta^2})\omega_n t} + Be^{(-\zeta - i\sqrt{1-\zeta^2})\omega_n t}$$

$$= e^{-\zeta\omega_n t}\left[Ae^{(i\sqrt{1-\zeta^2})\omega_n t} + Be^{(-i\sqrt{1-\zeta^2})\omega_n t} \right]$$

Put $\sqrt{1-\zeta^2}\,\omega_n = \omega_d$

$$x = e^{-\zeta\omega_n t}[Ae^{i\omega_d t} + Be^{-i\omega_d t}]$$

$$= e^{-\zeta\omega_n t}[A(\cos\omega_d t + i\sin\omega_d t) + B(\cos\omega_d t - i\sin\omega_d t)]$$

$$= e^{-\zeta\omega_n t}[(A+B)\cos\omega_d t + i(A-B)\sin\omega_d t]$$

$$= e^{-\zeta\omega_n t}[C\cos\omega_d t + D\sin\omega_d t]$$

where

$C = A + B \qquad D = i\,(A - B)$

$A + B = X\sin\varphi$

$i\,(A - B) = X\cos\varphi$

$$x = e^{-\zeta\omega_n t}(X\sin\varphi\cos\omega_d t + X\cos\varphi\sin\omega_d t)$$

$$= Xe^{-\zeta\omega_n t}\sin(\omega_d t + \varphi)$$

$$\boxed{x = Xe^{-\zeta\omega_n t}\sin(\omega_d t + \varphi)}$$

Continued:

$$x = Xe^{-\zeta\omega_n t}\sin(\omega_d t + \varphi)$$

Constants X and φ are to be determined from initial conditions. This equation indicates that the system oscillates with frequency $\omega_d (= \sqrt{1-\zeta^2}\,\omega_n)$. As ζ is less than 1, ω_d is always less than ω_n.

- X, which is constant
- $e^{-\zeta\omega_n t}$, which decreases with time and finally $e^{-\infty} = 0$
- sin $(\omega_d t + \varphi)$ which represents a repetition of motion

  Thus, the resultant motion is oscillatory with decreasing amplitudes having a frequency of ω_d. Ultimately, the motion dies down with time.

  Also,

$$\text{linear frequency, } f_d = \frac{\omega_d}{2\pi}$$

$$\text{time period, } T_d = \frac{\omega_d}{2\pi}$$

Continued:

let X_0 = displacement at the start of motion when $t = 0$

X_1 = displacement at the end of first oscillation when $t = T_d$

$$= Xe^{-\zeta\omega_n T_d} \sin(\omega_d T_d + \varphi)$$

$$= Xe^{-\zeta\omega_n T_d} \sin\left(\omega_d \frac{2\pi}{\omega_d} + \varphi\right)$$

$$= Xe^{-\zeta\omega_n T_d} \sin\varphi$$

X_2 = displacement at the end of second oscillation $= Xe^{-\zeta\omega_n \times 2T_d} \sin\varphi$

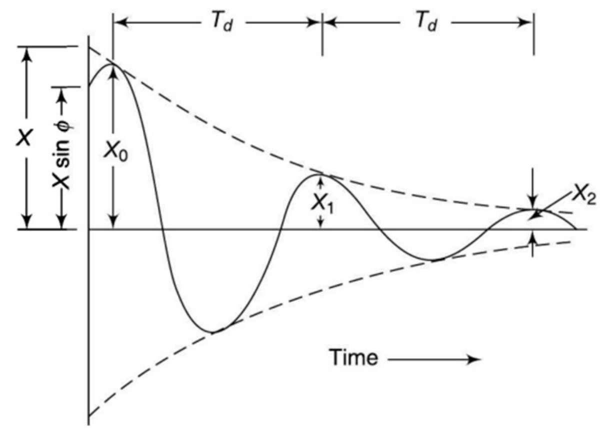
Similarly, $X_3 = Xe^{-\zeta\omega_n \times 3T_d} \sin\varphi$

....

$$X_n = Xe^{-\zeta\omega_n \times nT_d} \sin\varphi$$

$$X_{n+1} = Xe^{-\zeta\omega_n \times (n+1)T_d} \sin\varphi$$

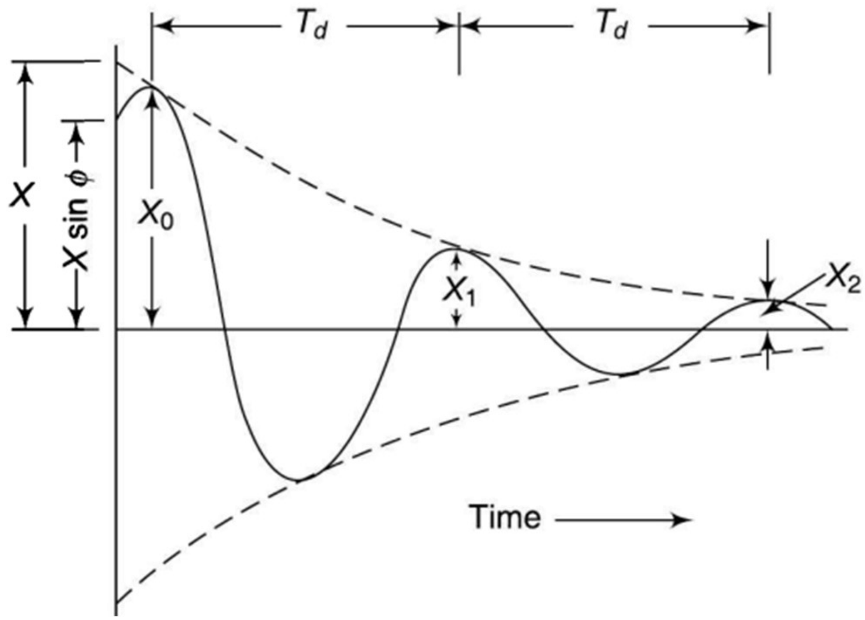
$$\frac{X_n}{X_{n+1}} = e^{\zeta\omega_n T_d} = \frac{X_0}{X_1} = \frac{X_1}{X_2} = \frac{X_2}{X_3} = \ldots$$



Continued:

$$\frac{X_n}{X_{n+1}} = e^{\zeta\omega_n T_d} = \frac{X_0}{X_1} = \frac{X_1}{X_2} = \frac{X_2}{X_3} = \ldots$$

ratio of amplitudes of two successive oscillations is constant



Criticality damped system ($\zeta = 1$)

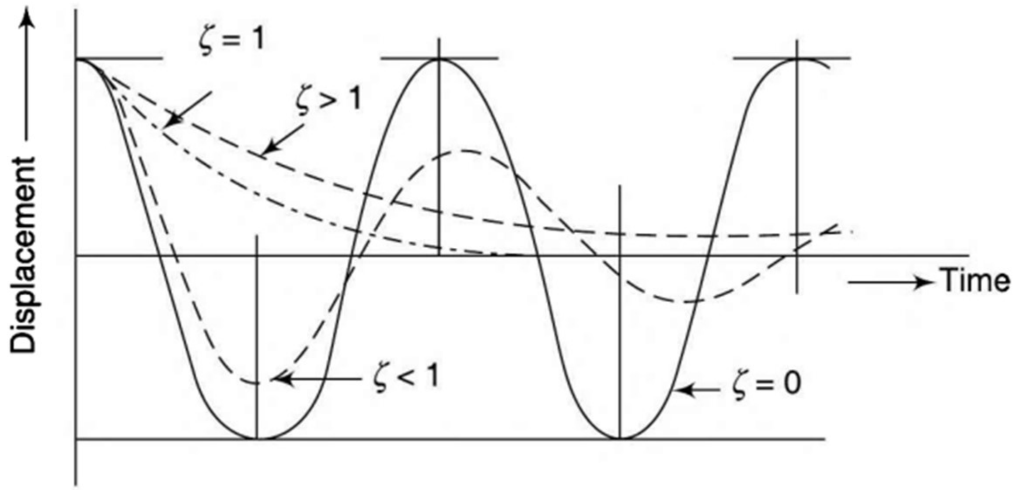
$$\alpha_{1.2} = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_n$$

$$x = A e^{\alpha_1 t} + B e^{\alpha_2 t}$$

The roots of the auxiliary equation are equal, each being equal to $-\omega_n$ and the solution is $x = (A + Bt)\ e^{-\omega_n t}$

Since $e^{-}\omega_n^{\,t}$ approaches zero as $t \to \infty$, the motion is aperiodic. The displacement will be approaching to zero with time.

Comparison:



(i) An undamped system ($\zeta = 0$) vibrates at its frequency which depends upon the static deflection under the weight of its mass ($\omega_n = \sqrt{g / \Delta}$).

(ii) When the system is underdamped ($\zeta < 1$), the frequency of the system decreases to $\omega_d (= \sqrt{1 - \zeta^2}\, \omega_n)$ and the time period increases to $T_d = 2\pi/\omega_d$. The amplitudes of the vibrations decrease with time, the ratio of successive amplitudes being constant. The vibrations die down with time.

(iii) At critical damping, $\zeta = 1$, $\omega_d = 0$ and $T_d = \infty$. The system does not vibrate and the mass m moves back slowly to the equilibrium position.

(iv) For an overdamped system, $\zeta > 1$, the system behaves in the same manner as for critical damping.

(v) ζ is the ratio of the existing damping in a system to that required for critical damping, i.e., $\zeta = c/c_c$.

Logarithmic Decrement

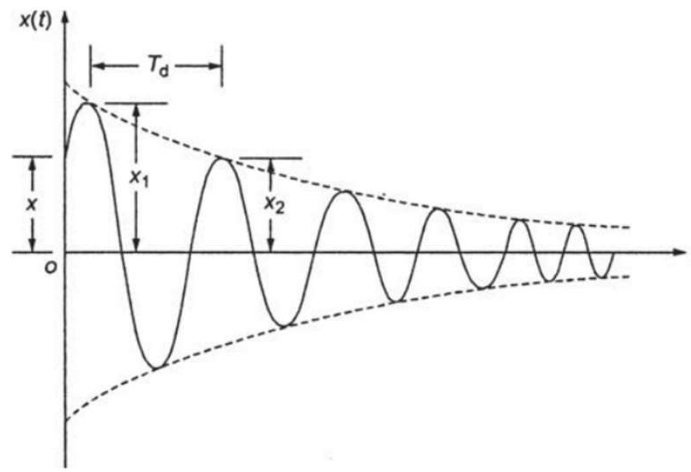
ratio of two successive oscillations is constant in an underdamped system.

Natural logarithm of this ratio is called logarithmic decrement and is denoted by $\delta = \ln\left(\frac{X_n}{X_{n+1}}\right)$

$$\delta = \ln e^{(\zeta\omega_n T_d)} = \zeta\omega_n T_d$$

$$\delta = \zeta\omega_n \frac{2\pi}{\omega_d} = \zeta\omega_n \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\boxed{\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}$$



$$\frac{X_n}{X_{n+1}} = e^{\zeta\omega_n T_d} = \frac{X_0}{X_1} = \frac{X_1}{X_2} = \frac{X_2}{X_3} = \ldots$$

Problem 1:

A vibrating system consists of a mass of 50 kg, a spring with a stiffness of 30 kN/m and a damper. The damping provided is only 20% of the critical value.

Determine the

(i) *damping factor*
(ii) *critical damping coefficient*
(iii) *natural frequency of damped vibrations*
(iv) *logarithmic decrement*
(v) *ratio of two consecutive amplitudes*

$m = 50$ kg $\quad s = 30\ 000$ N/m $\quad c = 0.2\ c_c$

(i) $\zeta = \dfrac{c}{c_c} = \underline{0.2}$

(ii) $c_c = 2\sqrt{sm} = 2\sqrt{30\ 000 \times 50} = 2450$ N/m/s

$= \underline{2.45 \text{ N/mm/s}}$

(iii) $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$

where

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{30\ 000}{50}} = 24.5 \text{ rad/s}$$

$$\omega_d = \sqrt{1-(0.2)^2} \times 24.5 = \underline{24 \text{ rad/s}}$$

(iv) $\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \dfrac{2\pi \times 0.2}{\sqrt{1-(0.2)^2}} = \underline{1.28}$

(v) $\dfrac{X_n}{X_{n+1}} = e^{\delta} = e^{1.28} = \underline{3.6}$

Problem 2:

Determine the time in which the mass in a damped vibrating system would settle down to 1/50 th of its initial deflection for the following data: m = 200 kg $\qquad \zeta = 0.22$

Also, find the number of oscillations completed to reach this value of deflection.

$$\frac{X_0}{X_N} = e^{\zeta \omega_n N T_d}$$

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{40 \times 10^3}{200}} = 14.14 \text{ rad/s}$$

$$\therefore \qquad 50 = e^{0.22 \times 14.14 N T_d}$$

or

Total time $NT_d = \underline{1.26 \text{ s}}$

$$T_d = \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n}$$

$$= \frac{2\pi}{(\sqrt{1-(0.22)^2}) \times 14.14} = 0.455 \text{ s}$$

$$\text{Number of oscillations completed} = \frac{1.26}{0.455} = \underline{2.76}$$

Problem 3:

In a single-degree damped vibrating system, a suspended mass of 8 kg makes 30 oscillations in 18 seconds. The amplitude decreases to 0.25 of the initial value after 5 oscillations. Determine the

(i) *stiffness of the spring*
(ii) *logarithmic decrement*
(iii) *damping factor, and*
(iv) *damping coefficient*

$$m = 8 \text{ kg}, N = 30, t = 18\text{s}$$

$$f_n = \frac{30}{18} = 1.67 \text{ Hz}$$

$$\omega_n = 2\pi f_n = 2\pi \times 1.67 = 10.47 \text{ rad/s}$$

(i) $\omega_n = \sqrt{\dfrac{s}{m}}$

$$10.47 = \sqrt{\frac{s}{8}}$$

$\therefore$ $s = 877$ N/m or <u>0.877 N/mm</u>

(ii) $$\frac{X_0}{X_5} = \frac{X_0}{X_1} \times \frac{X_1}{X_2} \times \frac{X_2}{X_3} \times \frac{X_3}{X_4} \times \frac{X_4}{X_5}$$

$$= \left(\frac{X_0}{X_1}\right)^5 \cdots \left(\frac{X_0}{X_1} = \frac{X_1}{X_2} = \frac{X_2}{X_3} = \frac{X_3}{X_4} = \frac{X_4}{X_5}\right)$$

$$\therefore \left(\frac{X_0}{X_1}\right) = \left(\frac{X_0}{X_5}\right)^{1/5} = \left(\frac{1}{0.25}\right)^{1/5} = 1.32$$

$$\delta = \ln\left(\frac{X_0}{X_5}\right) = \ln 1.32 = 0.278$$

(iii) $\dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 0.278$

or

$$\sqrt{1-\zeta^2} = 22.6\zeta$$

$$1-\zeta^2 = 510.82\zeta^2$$

$$\zeta^2 = 0.00195$$

$$\zeta = \underline{0.0442}$$

(iv) $c = 2\, m\, \omega_n\, \zeta$

$$= 2 \times 8 \times 10.47 \times 0.0442$$

$$= \underline{7.4 \text{ N/m/s}}$$

Problem 4:

The measurements on a mechanical vibrating system show that it has a mass of 8 kg and that the springs can be combined to give an equivalent spring of stiffness 5.4 N/mm. If the vibrating system have a dashpot attached which exerts a force of 40 N when the mass has a velocity of 1 m/s, find : 1. critical damping coefficient, 2. damping factor, 3. logarithmic decrement, and 4. ratio of two consecutive amplitudes.

1. critical damping coefficient :

$$c_c = 2m.\omega_n \;\; = 2\times 8\sqrt{\frac{5400}{8}} \;\; = 416 \text{ N/m/s}$$

2. damping factor :

$$\zeta = \frac{c}{c_c} = \frac{40}{416} \;\; = 0.096$$

3. logarithmic decrement :

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 0.6$$

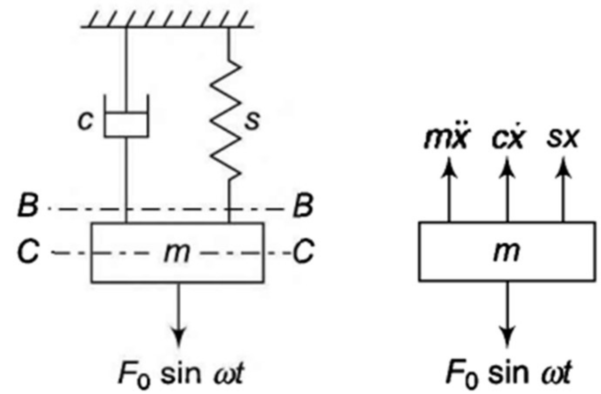
4. ratio of two consecutive amplitudes.

$$\delta = \log_e\left[\frac{x_n}{x_{n+1}}\right]$$

$$\frac{x_n}{x_{n+1}} = e^{\delta} = (2.7)^{0.6} \;\; = 1.82$$

Forced damped Vibrations:

$$m\ddot{x} + c\dot{x} + sx - F_0 \sin\omega t = 0$$
$$m\ddot{x} + c\dot{x} + sx = F_0 \sin\omega t$$



Complete solution of this equation consists of two parts, the complementary function (CF) and the particular integral (PI).

$$CF = X e^{-\zeta\omega_n t} \sin(\omega_d t + \varphi_1)$$

To obtain the PI, let

$$\frac{c}{m} = a, \frac{s}{m} = b, \text{ and } \frac{F_0}{m} = d$$

Then, using the operator D, the equation becomes

$$(D^2 + aD + b)\,x = d \sin\omega t$$

Continued:

$$PI = \frac{d \sin \omega t}{D^2 + aD + b}$$

$$= \frac{d \sin \omega t}{-\omega^2 + aD + b}$$

$$= \frac{1}{(b-\omega^2) + aD} \times \frac{(b-\omega^2) - aD}{(b-\omega^2) - aD} d \sin \omega t$$

$$= d\left[\frac{\sin \omega t (b-\omega^2) - aD \sin \omega t}{(b-\omega^2)^2 - a^2 D^2}\right]$$

$$= d\left[\frac{\sin \omega t (b-\omega^2) - a\omega \cos \omega t}{(b-\omega^2)^2 + (a\omega)^2}\right]$$

Take $(b-\omega^2) = R\cos\varphi$ and $a\,\omega = R\sin\varphi$
Constants R and φ are given by

$$R = \sqrt{(b-\omega^2)^2 + (a\omega)^2} \quad \text{and} \quad \varphi = \tan^{-1}\frac{a}{b-\omega^2}$$

$$PI = \frac{dR(\sin \omega t \cos\varphi - \cos \omega t \sin \varphi)}{(b-\omega^2)^2 + (a\omega)^2}$$

$$= \frac{d\sqrt{(b-\omega^2)^2 + (a\omega)^2}}{(b-\omega^2) + (a\omega)^2} \sin(\omega t - \varphi)$$

$$= \frac{d}{\sqrt{(b-\omega^2)^2 + (a\omega)^2}} \sin(\omega t - \varphi)$$

$$= \frac{F_0 / m}{\sqrt{\left(\frac{s}{m} - \omega^2\right)^2 + \left(\frac{c}{m}\omega\right)^2}} \sin(\omega t - \varphi)$$

$$= \frac{F_0}{\sqrt{(s - m\omega^2)^2 + (c\omega)^2}} \sin(\omega t - \varphi)$$

Continued:

$$x = CF + PI$$

$$Xe^{-\zeta\omega_n t}\sin(\omega_d t - \varphi_1) + \frac{F_0}{\sqrt{(s - m\omega^2)^2 + (c\omega)^2}}\sin(\omega t - \varphi)$$