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Vellore Institute of Technology
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School of Mechanical Engineering

B.Tech. – Mechatronics and Automation

BMEE207L Kinematics & Dynamics of Machines

MODULE 4

Synthesis of mechanisms

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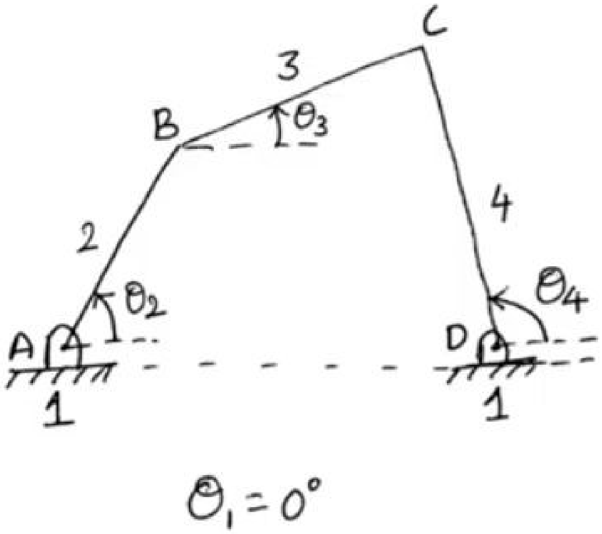
Freudenstein’s Equation

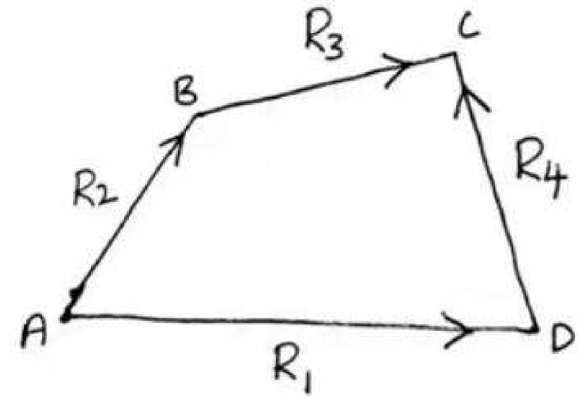
- Freudenstein’s Equation tells us how to design four bar and crank-slider mechanisms for function generation.

- Freudenstein’s Equation is named for its discoverer, *Ferdinand Freudenstein*, the originator of mechanism design in the United States.

- His equation is one of the most useful and easy to use tools for designing function-generating four bar mechanisms and crank-slider mechanisms.

Freudenstein's Equation for Four Bar Mechanism





Vectorial Representation

Fig: The four bar mechanism and Vectorial Representation necessary for true analysis

The Vector loop closure equation can be written as:

$$R_2 + R_3 - R_4 - R_1 = 0. \quad \text{—— (1)}$$

Freudenstein's Equation for Four Bar Mechanism

Equation -(1) Can be written in Complex polar form as,

$$r_2 e^{j\theta_2} + r_3 e^{j\theta_3} - r_4 e^{j\theta_4} - r_1 e^{j\theta_1} = 0$$

The above equation Can be re-written using Euler's formula as,

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$r_2(\cos\theta_2 + j\sin\theta_2) + r_3(\cos\theta_3 + j\sin\theta_3) - r_4(\cos\theta_4 + j\sin\theta_4) - r_1 = 0$$

Freudenstein's Equation for Four Bar Mechanism

Comparing Real and Imaginary parts of the above,

Real part: $r_2 \cos\theta_2 + r_3 \cos\theta_3 - r_4 \cos\theta_4 - r_1 = 0$ — (2)

Imaginary Part: $r_2 \sin\theta_2 + r_3 \sin\theta_3 - r_4 \sin\theta_4 = 0$ — (3)

Our aim is to establish the relationship between θ_2 and θ_4 as determined by link lengths r_1, r_2, r_3 and r_4.

Therefore, to eliminate θ_3 from eqn -(2) and -(3), re-write the equations as below,

$$r_3 \cdot \cos\theta_3 = -r_2 \cos\theta_2 + r_4 \cos\theta_4 + r_1 \quad \text{— (4)}$$

Freudenstein's Equation for Four Bar Mechanism

There fore, to eliminate θ_3 from equⁿ -② and -③, re-write the equations as below,

$$r_3 . \cos\theta_3 = -r_2 \cos\theta_2 + r_4 \cos\theta_4 + r_1 \quad — (4)$$

$$r_3 . \sin\theta_3 = -r_2 \sin\theta_2 + r_4 \sin\theta_4 \quad — (5)$$

Squaring both Sides of the above equations and adding them as below,

$$(r_3 . \cos\theta_3)^2 = (\underbrace{-r_2 \cos\theta_2}_{a} + \underbrace{r_4 . \cos\theta_4}_{b} + \underbrace{r_1}_{c})^2 \quad — (6)$$

$$(r_3 . \sin\theta_3)^2 = (\underbrace{-r_2 \sin\theta_2}_{a} + \underbrace{r_4 . \sin\theta_4}_{b})^2 \quad — (7)$$

Freudenstein's Equation for Four Bar Mechanism

We Know that,

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$$

$$(a+b)^2 = a^2+b^2+2ab$$

Expanding -(6) and -(7) using above formulae,

$$\left.\begin{aligned} r_3^2 \cdot \cos^2\theta_3 = {} & r_2^2\cos^2\theta_2 + r_4^2\cos^2\theta_4 + r_1^2 - 2r_2r_4\cos\theta_2 \cdot \cos\theta_4 \\ & + 2r_4r_1\cos\theta_4 \\ & - 2r_1r_2\cos\theta_2 \end{aligned}\right\} \quad -(8)$$

$$r_3^2 \cdot \sin^2\theta_3 = r_2^2\sin^2\theta_2 + r_4^2\sin^2\theta_4 - 2r_2r_4\sin\theta_2 \cdot \sin\theta_4 \quad -(9)$$

$$\begin{aligned} r_3^2 = {} & r_2^2+r_4^2+r_1^2-2r_2r_4\cos\theta_2\cos\theta_4 + 2r_4r_1\cos\theta_4 - 2r_1r_2\cos\theta_2 \\ & - 2r_2r_4\sin\theta_2 \cdot \sin\theta_4 \quad -(10) \end{aligned}$$

Freudenstein's Equation for Four Bar Mechanism

$$r_3^2 = r_2^2 + r_4^2 + r_1^2 - 2 r_2 r_4 \cos\theta_2 \cos\theta_4 + 2 r_4 r_1 \cos\theta_4 - 2 r_1 r_2 \cos\theta_2$$
$$- 2 r_2 r_4 \sin\theta_2 . \sin\theta_4 \quad — (10)$$

Re-arranging Eqn - (10) as below,

$$r_2^2 + r_4^2 + r_1^2 - r_3^2 + 2 r_4 r_1 \cos\theta_4 - 2 r_1 r_2 \cos\theta_2$$
$$= 2 r_2 r_4 \cos\theta_2 . \cos\theta_4 + 2 r_2 r_4 \sin\theta_2 . \sin\theta_4$$
$$— (11)$$

Divide both Sides of Equation - (11) by "$2r_2r_4$",

$$\frac{r_2^2 + r_4^2 + r_1^2 - r_3^2}{2 r_2 r_4} + \frac{r_1}{r_2} . \cos\theta_4 - \frac{r_1}{r_4} . \cos\theta_2 = \cos\theta_2 . \cos\theta_4 + \sin\theta_2 . \sin\theta_4$$
$$- (12)$$

Freudenstein's Equation for Four Bar Mechanism

$$\frac{r_2^2 + r_4^2 + r_1^2 - r_3^2}{2 r_2 r_4} + \frac{r_1}{r_2} \cdot \cos\theta_4 - \frac{r_1}{r_4} \cdot \cos\theta_2 = \cos\theta_2 \cdot \cos\theta_4 + \sin\theta_2 \cdot \sin\theta_4 \quad -(12)$$

W.K.T. $\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$

and let $K_1 = \frac{r_1}{r_4}$, $K_2 = \frac{r_1}{r_2}$ and $K_3 = \frac{r_2^2 + r_4^2 + r_1^2 - r_3^2}{2 r_2 r_4}$

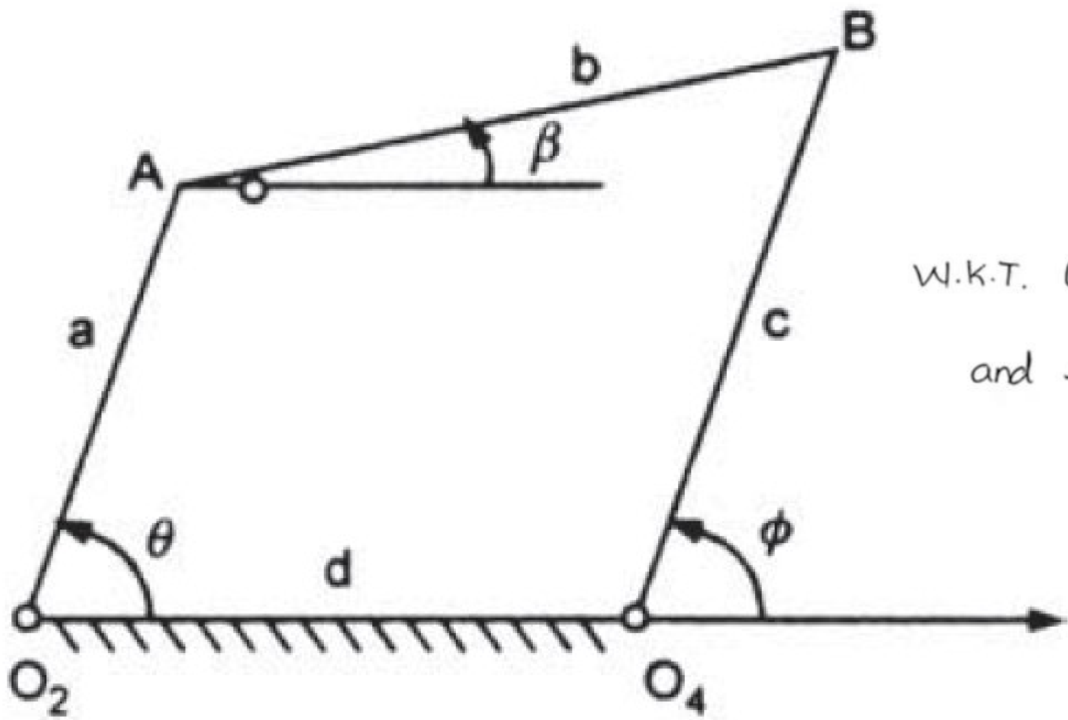
Now, Equation $-(2)$ becomes.

$$\boxed{K_3 + K_2 \cdot \cos\theta_4 - K_1 \cdot \cos\theta_2 = \cos(\theta_2 - \theta_4)}$$

The above is the **Freudenstein's Equation** for four bar mechanism

Freudenstein's Equation for the Precision Points

Freudenstein's equation helps to determine the length of links of a four-bar mechanism. The displacement equation of a four-bar mechanism is given by:



$$K_3 + K_2 \cdot \cos\theta_4 - K_1 \cdot \cos\theta_2 = \cos(\theta_2 - \theta_4)$$

W.K.T. $\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$

and let $K_1 = \frac{r_1}{r_4}$, $K_2 = \frac{r_1}{r_2}$ and $K_3 = \frac{r_2^2 + r_4^2 + r_1^2 - r_3^2}{2 r_2 r_4}$

Figure: Freudenstein's equation for the precision points

$$\frac{d}{a}\cos\phi - \frac{d}{c}\cos\theta + \frac{a^2 - b^2 + c^2 + d^2}{2ac} = \cos(\theta - \phi)$$

Freudenstein's Equation for the Precision Points

$$\frac{d}{a}\cos\phi - \frac{d}{c}\cos\theta + \frac{a^2 - b^2 + c^2 + d^2}{2ac} = \cos(\theta - \phi)$$

Where, $$k_1 = \frac{d}{a}, k_2 = -\frac{d}{c} \quad \text{and} \quad k_3 = \frac{a^2 - b^2 + c^2 + d^2}{2ac}$$

$$k_1 \cos\phi + k_2 \cos\theta + k_3 = \cos(\theta - \phi)$$

Let the input and output are related by some function, such as, $y = f(x)$.

For three specified positions, let

$$\theta_1, \theta_2, \theta_3 = \text{three positions of input link}$$

$$\phi_1, \phi_2, \phi_3 = \text{three positions of output link}$$

Then substituting these values

$$k_1 \cos\phi_1 + k_2 \cos\theta_1 + k_3 = \cos(\theta_1 - \phi_1)$$

$$k_1 \cos\phi_2 + k_2 \cos\theta_2 + k_3 = \cos(\theta_2 - \phi_2)$$

$$k_1 \cos\phi_3 + k_2 \cos\theta_3 + k_3 = \cos(\theta_3 - \phi_3)$$

Freudenstein's Equation for the Precision Points

$$k_1 \cos\phi_1 + k_2 \cos\theta_1 + k_3 = \cos(\theta_1 - \phi_1)$$
$$k_1 \cos\phi_2 + k_2 \cos\theta_2 + k_3 = \cos(\theta_2 - \phi_2)$$
$$k_1 \cos\phi_3 + k_2 \cos\theta_3 + k_3 = \cos(\theta_3 - \phi_3)$$

These equations can be written in the matrix form as:

$$\begin{bmatrix} \cos\phi_1 & \cos\theta_1 & 1 \\ \cos\phi_2 & \cos\theta_2 & 1 \\ \cos\phi_3 & \cos\theta_3 & 1 \end{bmatrix} \begin{Bmatrix} k_1 \\ k_2 \\ k_3 \end{Bmatrix} = \begin{Bmatrix} \cos(\theta_1 - \phi_1) \\ \cos(\theta_2 - \phi_2) \\ \cos(\theta_3 - \phi_3) \end{Bmatrix}$$

These equations can be solved by any numerical technique.
Using Cramer's rule, let

$$A = \begin{vmatrix} \cos\phi_1 & \cos\theta_1 & 1 \\ \cos\phi_2 & \cos\theta_2 & 1 \\ \cos\phi_3 & \cos\theta_3 & 1 \end{vmatrix}, \quad A_1 = \begin{vmatrix} \cos(\theta_1 - \phi_1) & \cos\theta_1 & 1 \\ \cos(\theta_2 - \phi_2) & \cos\theta_2 & 1 \\ \cos(\theta_3 - \phi_3) & \cos\theta_3 & 1 \end{vmatrix}$$

Freudenstein's Equation for the Precision Points

$$A = \begin{vmatrix} \cos\phi_1 & \cos\theta_1 & 1 \\ \cos\phi_2 & \cos\theta_2 & 1 \\ \cos\phi_3 & \cos\theta_3 & 1 \end{vmatrix}, A_1 = \begin{vmatrix} \cos(\theta_1 - \phi_1) & \cos\theta_1 & 1 \\ \cos(\theta_2 - \phi_2) & \cos\theta_2 & 1 \\ \cos(\theta_3 - \phi_3) & \cos\theta_3 & 1 \end{vmatrix}$$

$$A_2 = \begin{vmatrix} \cos\phi_1 & \cos(\theta_1 - \phi_1) & 1 \\ \cos\theta_2 & \cos(\theta_2 - \phi_2) & 1 \\ \cos\phi_3 & \cos(\theta_3 - \phi_2) & 1 \end{vmatrix}, A_3 = \begin{vmatrix} \cos\phi_1 & \cos\theta_1 & \cos(\theta_1 - \phi_1) \\ \cos\phi_2 & \cos\theta_2 & \cos(\theta_2 - \phi_2) \\ \cos\phi_3 & \cos\theta_3 & \cos(\theta_3 - \phi_3) \end{vmatrix}$$

$$\text{Then } k_1 = \frac{A_1}{A}, k_2 = \frac{A_2}{A}, k_3 = \frac{A_3}{A}$$

Knowing k_1, k_2 and k_3, the values of a, b, c, and d can be calculated. Value of either 'a' or 'd' can be assumed to be unity to obtain the proportionate values of other parameters.

Freudenstein's Equation for the Precision Points

Numerical

Design a four-bar mechanism to coordinate three positions of the input and output links given by:
$\theta_1 = 25$, $\phi_1 = 30°$; $\theta_2 = 35°$, $\phi_2 = 40°$; $\theta_3 = 50°$, $\phi_3 = 60°$

Solution

$$\cos\,\theta_1 = \cos\,25° = 0.9063,\ \cos\,\theta_2 = \cos\,35° = 0.8191,\ \cos\,\theta_3 = \cos\,50° = 0.6428$$

$$\cos\,\phi_1 = \cos\,30° = 0.8660,\ \cos\,\phi_2 = \cos\,40° = 0.7660,\ \cos\,\phi_3 = \cos\,60° = 0.5000$$

$$\cos\,(\theta_1 - \phi_1) = \cos\,(25° - 30°) = 0.9962$$

$$\cos\,(\theta_2 - \phi_2) = \cos\,(35° - 40°) = 0.9962$$

$$\cos\,(\theta_2 - \phi_2) = \cos\,(50° - 60°) = 0.9848$$

$$A = \begin{vmatrix} 0.8660 & 0.9063 & 1 \\ 0.7660 & 0.8191 & 1 \\ 0.5000 & 0.6428 & 1 \end{vmatrix}$$

$$A = \begin{vmatrix} \cos\phi_1 & \cos\theta_1 & 1 \\ \cos\phi_2 & \cos\theta_2 & 1 \\ \cos\phi_3 & \cos\theta_3 & 1 \end{vmatrix},\ A_1 = \begin{vmatrix} \cos(\theta_1 - \phi_1) & \cos\theta_1 & 1 \\ \cos(\theta_2 - \phi_2) & \cos\theta_2 & 1 \\ \cos(\theta_3 - \phi_3) & \cos\theta_3 & 1 \end{vmatrix}$$

$$A_2 = \begin{vmatrix} \cos\phi_1 & \cos(\theta_1 - \phi_1) & 1 \\ \cos\theta_2 & \cos(\theta_2 - \phi_2) & 1 \\ \cos\phi_3 & \cos(\theta_3 - \phi_2) & 1 \end{vmatrix},\ A_3 = \begin{vmatrix} \cos\phi_1 & \cos\theta_1 & \cos(\theta_1 - \phi_1) \\ \cos\phi_2 & \cos\theta_2 & \cos(\theta_2 - \phi_2) \\ \cos\phi_3 & \cos\theta_3 & \cos(\theta_3 - \phi_3) \end{vmatrix}$$

Freudenstein's Equation for the Precision Points

$$A = \begin{vmatrix} 0.8660 & 0.9063 & 1 \\ 0.7660 & 0.8191 & 1 \\ 0.5000 & 0.6428 & 1 \end{vmatrix}$$

$$A = \begin{vmatrix} \cos\phi_1 & \cos\theta_1 & 1 \\ \cos\phi_2 & \cos\theta_2 & 1 \\ \cos\phi_3 & \cos\theta_3 & 1 \end{vmatrix}, A_1 = \begin{vmatrix} \cos(\theta_1 - \phi_1) & \cos\theta_1 & 1 \\ \cos(\theta_2 - \phi_2) & \cos\theta_2 & 1 \\ \cos(\theta_3 - \phi_3) & \cos\theta_3 & 1 \end{vmatrix}$$

$$A_2 = \begin{vmatrix} \cos\phi_1 & \cos(\theta_1 - \phi_1) & 1 \\ \cos\theta_2 & \cos(\theta_2 - \phi_2) & 1 \\ \cos\phi_3 & \cos(\theta_3 - \phi_2) & 1 \end{vmatrix}, A_3 = \begin{vmatrix} \cos\phi_1 & \cos\theta_1 & \cos(\theta_1 - \phi_1) \\ \cos\phi_2 & \cos\theta_2 & \cos(\theta_2 - \phi_2) \\ \cos\phi_3 & \cos\theta_3 & \cos(\theta_3 - \phi_3) \end{vmatrix}$$

$$= 0.8660\,(0.8191 - 0.6428) - 0.9063\,(0.7660 - 0.5000) + 1(0.7666 \times 0.6428 - 0.8191 \times 0.5000)$$

$$= 5.5652 \times 10^{-3}$$

$$A_1 = \begin{vmatrix} 0.9962 & 0.9063 & 1 \\ 0.9962 & 0.8191 & 1 \\ 0.9848 & 0.6428 & 1 \end{vmatrix}$$

$$= 0.9962\,(0.8191 - 0.6428) - 0.9063\,(0.9962 - 0.9848) + 1(0.9962 \times 0.6428 - 0.8191 \times 0.9848)$$

$$= -9.9408 \times 10^{-4}$$

Freudenstein's Equation for the Precision Points

$$A = \begin{vmatrix} \cos\phi_1 & \cos\theta_1 & 1 \\ \cos\phi_2 & \cos\theta_2 & 1 \\ \cos\phi_3 & \cos\theta_3 & 1 \end{vmatrix}, \; A_1 = \begin{vmatrix} \cos(\theta_1 - \phi_1) & \cos\theta_1 & 1 \\ \cos(\theta_2 - \phi_2) & \cos\theta_2 & 1 \\ \cos(\theta_3 - \phi_3) & \cos\theta_3 & 1 \end{vmatrix}$$

$$A_2 = \begin{vmatrix} \cos\phi_1 & \cos(\theta_1 - \phi_1) & 1 \\ \cos\theta_2 & \cos(\theta_2 - \phi_2) & 1 \\ \cos\phi_3 & \cos(\theta_3 - \phi_2) & 1 \end{vmatrix}, \; A_3 = \begin{vmatrix} \cos\phi_1 & \cos\theta_1 & \cos(\theta_1 - \phi_1) \\ \cos\phi_2 & \cos\theta_2 & \cos(\theta_2 - \phi_2) \\ \cos\phi_3 & \cos\theta_3 & \cos(\theta_3 - \phi_3) \end{vmatrix}$$

$$A_2 = \begin{vmatrix} 0.8660 & 0.9063 & 1 \\ 0.7660 & 0.9962 & 1 \\ 0.5000 & 0.9848 & 1 \end{vmatrix}$$

$$= 0.8660\,(0.9962 - 0.9848) - 0.9962\,(0.7660 - 0.5000) + 1(0.7660 \times 0.9848 - 0.9962) \times 0.5000)$$

$$= -1.14 \times 10^2$$

$$A_3 = \begin{vmatrix} 0.8660 & 0.9962 & 0.9962 \\ 0.7660 & 0.8191 & 0.9962 \\ 0.5000 & 0.6428 & 0.9848 \end{vmatrix}$$

$$= 0.8660\,(0.8191 \times 0.9848 - 0.9962 \times 0.6428) - 0.9063\,(0.7660 \times 0.9848 - 0.9962 \times 0.5000)$$

$$= 5.746 \times 10^{-3}$$

Freudenstein's Equation for the Precision Points

$$k_1 = \frac{A_1}{A} = \frac{-9.9408 \times 10^{-4}}{-5.5652 \times 10^{-3}} = 0.1786 = \frac{d}{a}$$

Let $d = 1$ unit, then $a = 5.598$ units

$$k_2 = \frac{A_2}{A} = \frac{1.14 \times 10^{-3}}{-5.5652 \times 10^{-3}} = -0.2048$$

$$= -\frac{d}{c}, \quad c = 4.88 \text{ units}$$

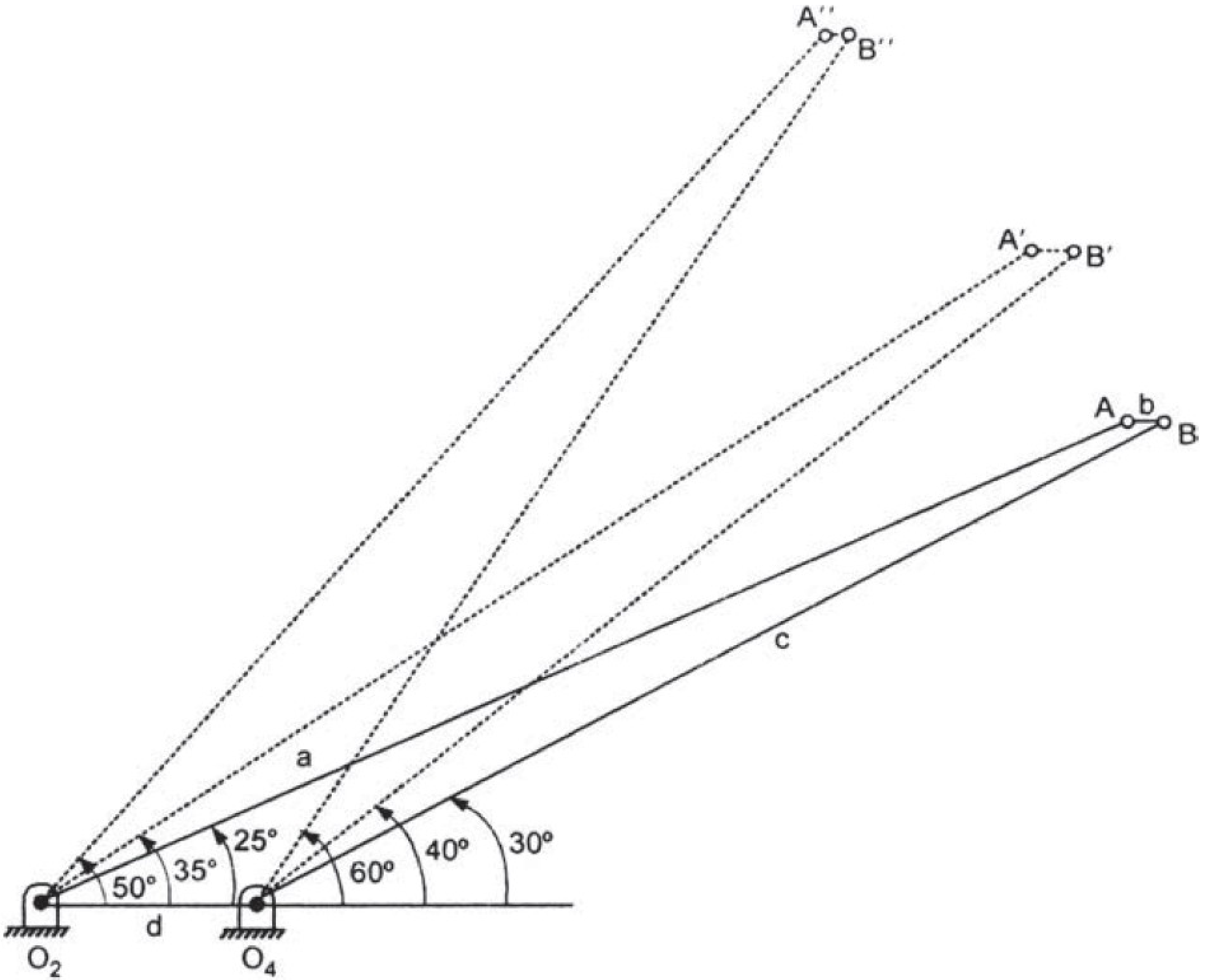
$$k_3 = \frac{A_3}{A} = \frac{-5.764 \times 10^{-3}}{-5.5652 \times 10^{-3}} = 1.0272$$

$$= \frac{a^2 - b^2 + c^2 + d^2}{2ac}$$

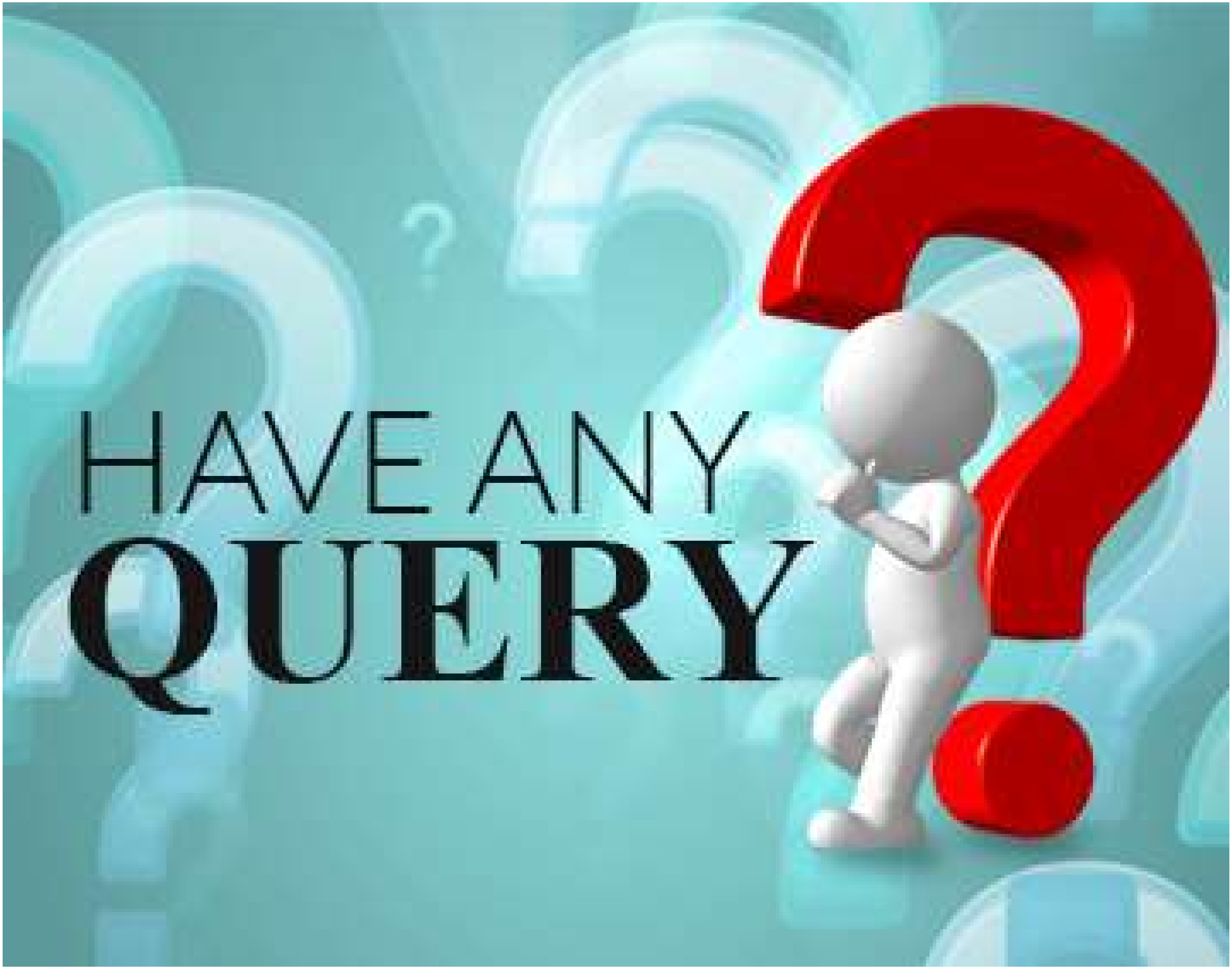
$$= \frac{(5.598)^2 - b^2 + (4.88)^2}{2 \times 5.598 \times 4.88}, \qquad b = 0.176$$

Freudenstein's Equation for the Precision Points

The mechanism is



Four-bar mechanism developed by three position precision points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