Solution of the Wave Equation of String

The solution of the wave equation $$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Here the variables are x and t. we know that any arbitrary functions either $y = f_1(ct - x)$ or $y = f_2(ct + x)$ will be a solution of the wave equation. Hence, generally, their liner combination $y = f_1(ct - x) + f_2(ct + x)$ is the complete solution.

$$\frac{\partial y}{\partial x} = -f_1^{'}(ct - x)$$

Here f1′ represents the differentiation of the function with respect to the bracket (ct – x),

and $$\frac{\partial^2 y}{\partial x^2} = f_1^{''}(ct - x)$$

also $$\frac{\partial y}{\partial t} = cf_1^{'}(ct - x)$$

$$\frac{\partial^2 y}{\partial t^2} = c^2 f_1^{''}(ct - x)$$

$$\because \frac{\partial^2 y}{\partial x^2} = f_1^{''}(ct - x)$$

so that $$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

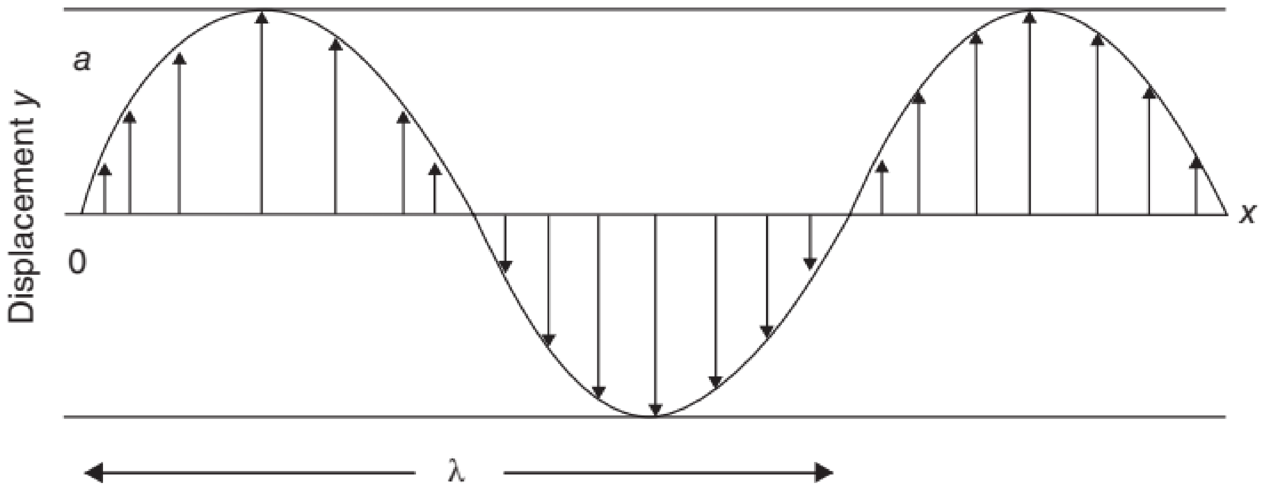
for y = f_1(ct – x) and when y = f_2(ct + x) a similar result we can get.

Solution of The Wave Equation: sinusoidal function

Now consider $y = a\sin(\omega t - \varphi) = a\sin\frac{2\pi}{\lambda}(ct - x)$

As a solution to the wave equation if $2\pi c/\lambda = \omega = 2\pi\nu$, where ν is the oscillation frequency and $\phi = 2\pi x/\lambda$.

- This means that if a wave, moving to the right, passes over the oscillators in a medium and a photograph is taken at time $t = 0$, the locus of the oscillator displacements will be given by the expression $y = a\sin(\omega t - \phi) = a\sin 2\pi(ct - x)/\lambda$. If we now observe the motion of the oscillator at the position $x = 0$ it will be given by $y = a\sin\omega t$.



- Any oscillator to its right at some position x will be set in motion at some later time by the wave moving to the right; this motion will be given by

$$y = a\sin(\omega t - \varphi) = a\sin\frac{2\pi}{\lambda}(ct - x)$$

having a phase lag of ϕ with respect to the oscillator at $x = 0$. This phase lag $\phi = 2\pi x/\lambda$, so that if $x = \lambda$ the phase lag is 2π rad that is, equivalent to exactly one complete vibration of an oscillator.

- If the wave is moving to the left the sign of ϕ is changed because the oscillation at x begins before that at $x = 0$. Thus, the bracket

(ct – x) denotes a wave moving to the right and

(ct + x) gives a wave moving in the direction of negative x.

Consider $$y = a\sin(\omega t - kx)$$

where $k = 2\pi/\lambda = \omega/c$ is called the wave number; also $y = ae^{i(\omega t - kx)}$, the exponential representation of both sine and cosine.

This wave equation giving the displacement of an oscillator and its phase with respect to some reference oscillator.

we have
$$\frac{\partial y}{\partial t} = \omega \, a\cos(\omega t - kx)$$

$$\frac{\partial y}{\partial x} = -ka\cos(\omega t - kx)$$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 a\sin(\omega t - kx)$$

$$\frac{\partial^2 y}{\partial x^2} = -k^2 a\sin(\omega t - kx)$$

$$\because -\frac{\partial^2 y}{\partial t^2}\frac{1}{\omega^2} = a\sin(\omega t - kx)$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

- The wave or phase velocity is, of course, ∂x/∂t, the rate at which the disturbance moves across the oscillators; the oscillator or particle velocity is the simple harmonic velocity ∂y/∂t.

So that $$\frac{\partial y}{\partial t} = -\frac{\omega}{k}\frac{\partial y}{\partial x} = -c\frac{\partial y}{\partial x}\left(= \frac{\partial x}{\partial t}\frac{\partial y}{\partial x}\right)$$

$$\text{Particle velocity} = \text{Wave velocity} \times \text{slope of the displacement curve}$$

- The particle velocity ∂y/∂t is therefore given as the product of the wave velocity

$$c = \frac{\partial x}{\partial t} \quad (\text{wave velocity"v"})$$

The gradient of the wave profile preceded by a negative sign for a right-going wave,

$$y = f(ct - x)$$