Laws of transverse vibrations of strings

The followings are the laws of transverse vibrations of strings:

- **Law of length:** The frequency of the fundamental note emitted is inversely proportional to the length of the string when the tension and mass per unit length are constant

$$\nu \propto \frac{1}{l} \text{ when T and m are constant}$$

$$\nu_1 = \frac{1}{2l}\sqrt{\left(\frac{T}{m}\right)}$$

- **Law of Tension:** *The frequency of the fundamental note emitted is directly proportional to the square root of the tension when the length and linear density are constants*

$$\nu \propto \sqrt{T} \text{ when l and m are constant}$$

- **Law of linear mass of the string:** *The frequency of the fundamental note emitted is inversely proportional to the square root of mass per unit length of the wire when length and tension are constant.*

$$\nu \propto \frac{1}{\sqrt{m}} \text{ when l and T are constant}$$

Modes of vibration of stretched string clamped at both the ends

- Consider the most general simple harmonic solution of the wave equation in case of a uniform string of length l having mass per unit length m and stretched by a tension T.
- The general solution of the wave equation is given by

$$y = a_1 \sin(wt - k\,x) + a_2 \sin(wt + k\,x) + b_1 \cos(wt - k\,x) + b_2 \cos(wt + k\,x) \ldots (1)$$

where a_1, a_2, b_1, and b_2 are arbitrary constants.

- As the string is rigidity supported at the two ends, we have the following boundary conditions

$$y = 0 \text{ at } x = 0 \quad \text{at all time } t \ldots\ldots\ldots (2a)$$

$$y = 0 \text{ at } x = l \quad \text{at all time } t \ldots\ldots\ldots (2b)$$

- Applying boundary conditions (2a) in eq. (1), we get

$$0 = a_1 \sin wt + a_2 \sin wt + b_1 \cos wt + b_2 \cos wt$$

Modes of vibration of stretched string clamped at both the ends

$$Thus, we\ have \quad a_1 = -a_2 \quad and \quad b_1 = -b_2$$

$$Now\ eq.1\ becomes\ \ y = (-2a_1 \cos wt + 2b_2 \sin wt) \sin kx$$

$$As,\ \cos wt \neq 0\ and\ \sin wt \neq 0,\ \ hence,$$

$$\sin kl = 0$$

$$which\ gives\ the\ general\ solution\ for\ angle\ kl\ to\ be : kl = n\pi \quad where\ n = 1, 2, 3....$$

As l is constant, k is limited to discrete set of values known as eigenvalues.

$$k_n = \frac{n\pi}{l} \quad where\ n = 1, 2, 3...$$

$$\nu_n = n\left(\frac{V}{2l}\right) \quad where\ n = 1, 2, 3...$$

$$\because\ k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{\lambda\nu} = \frac{2\pi\nu}{V}$$

$$\nu = \frac{kV}{2\pi}$$

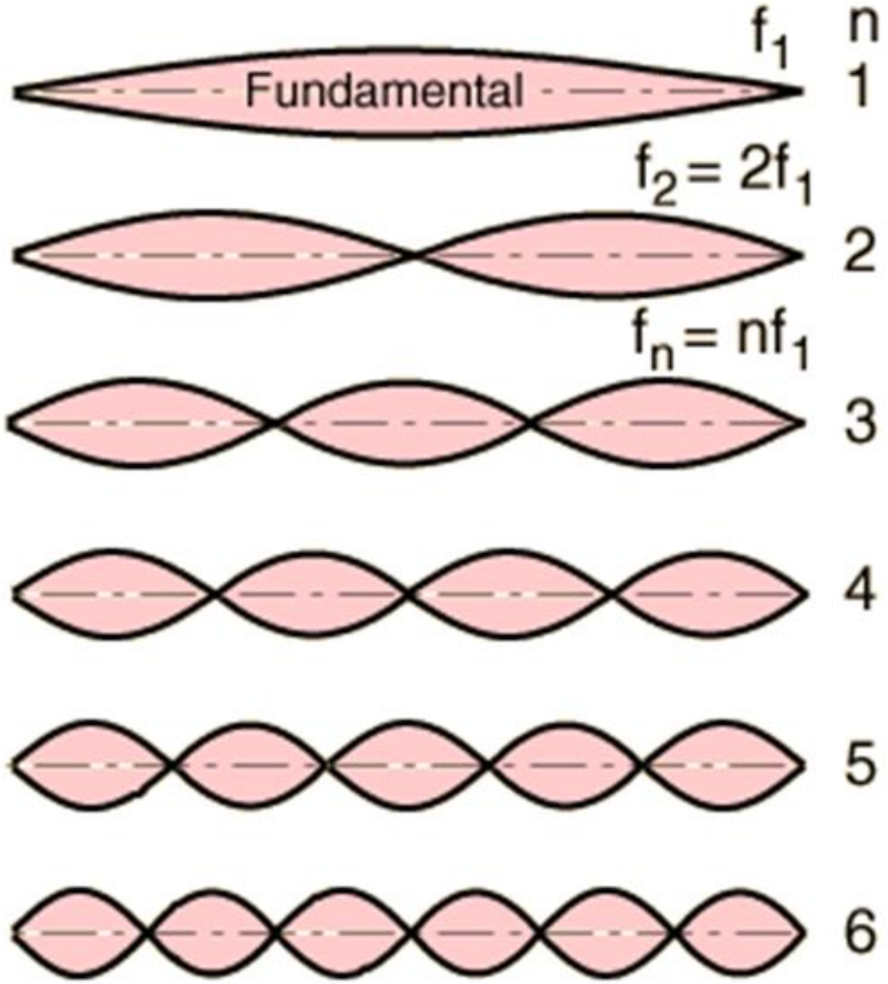
$$\nu = \frac{n\pi V}{2\pi l} = n(\frac{V}{2l})$$

$$\nu_1 = n(\frac{V}{2l}) = \frac{1}{2l}\sqrt{(\frac{T}{m})}$$

$$\nu_n = \frac{n}{2l}\sqrt{(\frac{T}{m})}$$

- When the longest wavelength (2L) and lowest frequency are achieved, it is defined as the fundamental frequency (first harmonic)
- when the string is plucked at the middle, it vibrates with nodes at the ends and antinodes at the middle. The tone emitted under this condition is known as fundamental frequency.

Overtones and harmonics



- These standing waves are referred as normal modes of vibration of a String
- Higher frequencies are often multiples of fundamental frequency: $\nu_n = n\nu_1$ $_{n=1,\ 2,\ 3,4\ldots}$
- These frequencies are known as harmonics or resonant frequencies
- Fundamental harmonic: $\nu_1 = \frac{1}{2l}\sqrt{\left(\frac{T}{m}\right)}$
- Second harmonic: $2\nu_1 = \nu_2 = \frac{2}{2l}\sqrt{\left(\frac{T}{m}\right)}$ First Overtone
- Third harmonic: $3\nu_1 = \nu_3 = \frac{3}{2l}\sqrt{\left(\frac{T}{m}\right)}$ Second Overtone
- Fourth harmonic: $4\nu_1 = \nu_4 = \frac{4}{2l}\sqrt{\left(\frac{T}{m}\right)}$ Third Overtone

$$\nu_1 : \nu_2 : \nu_3 .. = 1:2:3..$$

- The string fixed at both ends has all possible harmonics, frequencies of the harmonics, are integral multiples of the fundamental frequency.