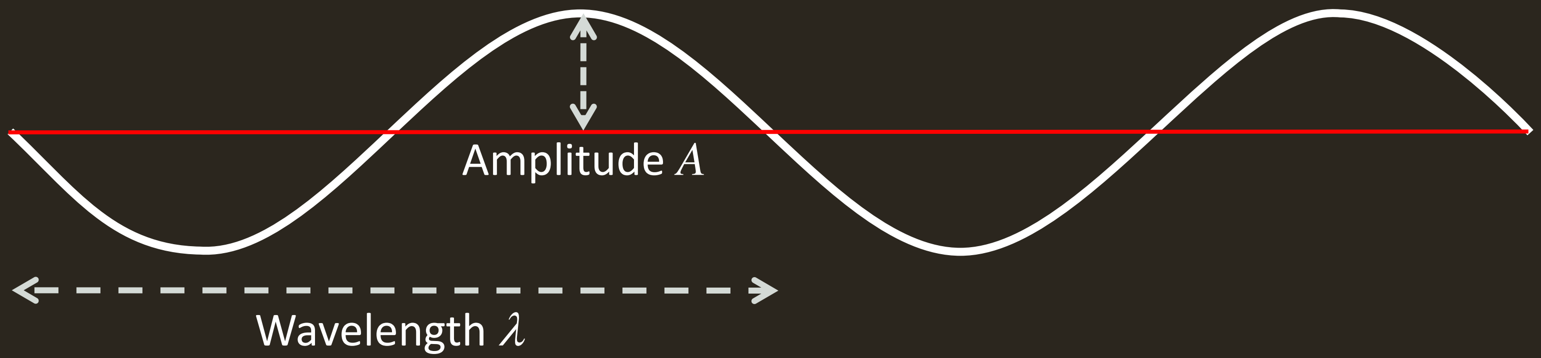
Overview of Module - 1

Topics

- The wave equation
- Energy and power of waves
- Superposition
- Standing waves as sums of traveling waves
- Fourier series

Harmonic Waves

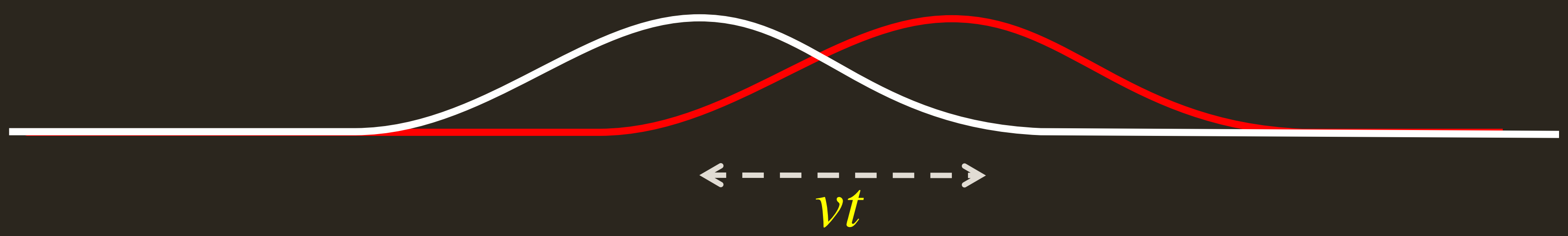
- A simple harmonic wave has sinusoidal form:



- For a string along the *x*-axis, this is local displacement in *y*-direction at some instant.
- For a sound wave traveling in the *x*-direction, this is local *x*-displacement at some instant.

Traveling Wave

- Experimentally, a pulse traveling down a string under tension maintains its shape:



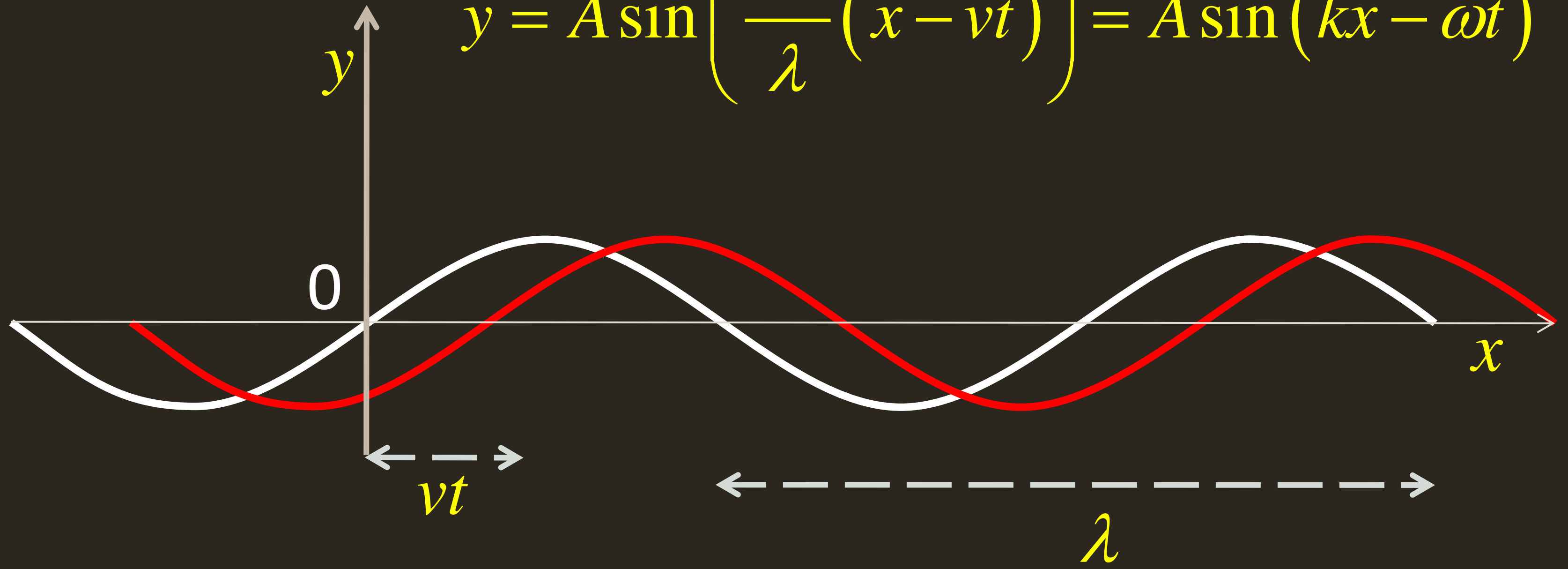
- Mathematically, this means the perpendicular displacement y stays the same function of x, but with an origin moving at velocity v:

$$y = f\left(x,t\right) = f\left(x - vt\right)$$

Traveling Harmonic Wave

- A sine wave of wavelength λ, amplitude A, traveling at velocity v has displacement

$$y = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right) = A\sin(kx - \omega t)$$



Harmonic Wave Notation

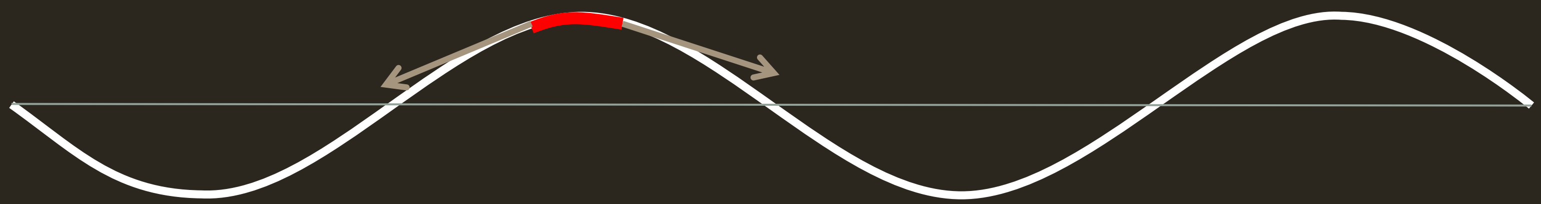
- A sine wave of wavelength λ, amplitude A, traveling at velocity v has displacement

$$y = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

- This is usually written $y = A\sin(kx - \omega t)$, where the “wave number” $k = 2\pi/\lambda$ and $\omega = vk$.
- As the wave is passing, a single particle of string has simple harmonic motion with frequency ω radians/sec, or $f = \omega/2\pi$ Hz. Note that $\boxed{v = \lambda f}$

The Wave Equation

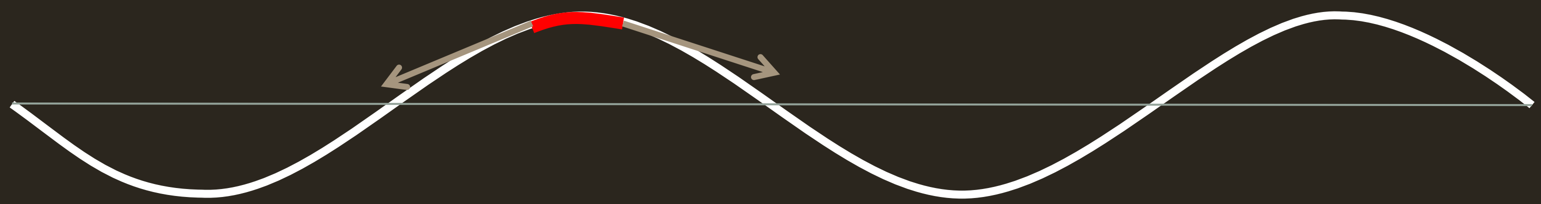
- The wave equation is just Newton's law $F = ma$ applied to a little bit of the vibrating string:



- The tiny length of string shown in red has length $m = \mu dx$, is accelerating in the y-direction with acceleration $a = \partial^2 f\left(x,t\right)/\partial t^2$, and the force F is the sum of the tensions at the two ends of the bit of string, which don't cancel because they're not parallel. Animation!

The Wave Equation

- The y-direction component of the tension T at the front end of the string is just T multiplied by the slope (for small amplitudes), $T\partial f\left(x+dx,t\right)/\partial x$.
- At the back end, T points backwards, so the downward component is $-T\partial f\left(x,t\right)/\partial x$.



- The total y-direction force is therefore

$$F = T\partial f\left(x+dx,t\right)/\partial x - T\partial f\left(x,t\right)/\partial x = T\left(\partial^2 f\left(x,t\right)/\partial x^2\right)dx$$

Wave Equation

- We're ready to write $F = ma$ for that bit of string:

$$F = T\partial f\left(x+dx,t\right)/\partial x - T\partial f\left(x,t\right)/\partial x = T\left(\partial^2 f\left(x,t\right)/\partial x^2\right)dx$$

- $m = \mu dx,\ \ a = \partial^2 f\left(x,t\right)/\partial t^2$.
- Putting it all together:

$$\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2}}$$

- Since this is nothing but Newton's second law, it must be true for *any* wave on a string.

Traveling Wave Equation

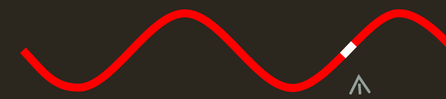
- Recall that from observation a traveling wave has the form $y = f\left(x - vt\right)$.
- From the chain rule, for that function

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial\left(x - vt\right)} \frac{\partial\left(x - vt\right)}{\partial t} = -v\frac{\partial f}{\partial x}, \quad \frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}$$

- Comparing this with the wave equation, we see that $$\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}}$$

This <u>proves</u> that $v = \sqrt{T / \mu}$.

Harmonic Wave Energy



- Writing the wave $y = A\sin\left(kx - \omega t\right)$ where remember $k = 2\pi / \lambda, \quad \omega = vk$ it's clear that at any fixed point x a bit of string dx is oscillating up and down in simple harmonic motion with amplitude A and frequency $f = \omega/2\pi$ Hz.
- The energy of that bit dx is <u>all</u> kinetic when $y = 0$, ($kx = \omega t$), the y-velocity at that instant is

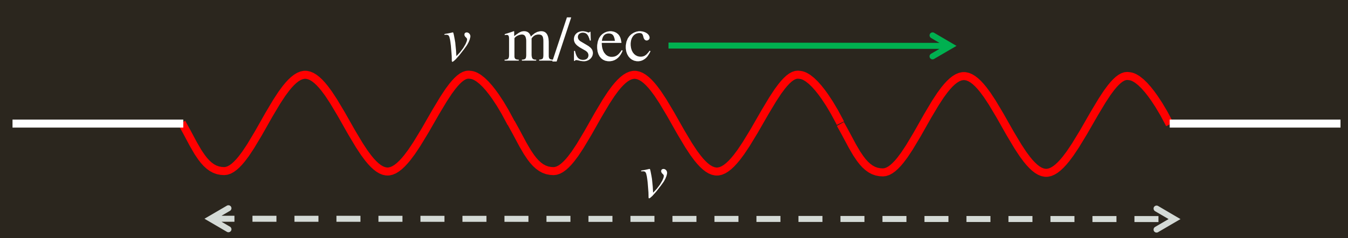
$$v = \partial y / \partial t = -\omega A\cos\left(kx - \omega t\right) = -\omega A$$

so the <u>total</u> energy in dx is $\frac{1}{2}mv^2 = \frac{1}{2}\left(\mu dx\right)A^2\omega^2$.

Harmonic Wave Energy

- The <u>total</u> energy in dx is $\frac{1}{2}mv^2 = \frac{1}{2}(\mu dx)A^2\omega^2$,

  so in length L the wave energy is $\frac{1}{2}\mu L A^2\omega^2$.
- Imagine now a group of waves, choose length *v*, moving to the right at speed *v* (passes you in just one second!):



- The power delivered by the waves is the energy passing a fixed point per second—that is

$$\boxed{\bar{P} = \frac{1}{2}\mu v A^2\omega^2 = 2\pi^2\mu v A^2 f^2}$$

The Wave Equation and Superposition

- If you have two solutions to the wave equation, $y = f(x,t)$ and $y = g(x,t)$, then $y = f + g$ is <u>also</u> a solution to the wave equation!
- This can be checked with the actual equation:

$$\frac{\partial^2 (f+g)}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} + \frac{1}{v^2}\frac{\partial^2 g}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 (f+g)}{\partial t^2}$$

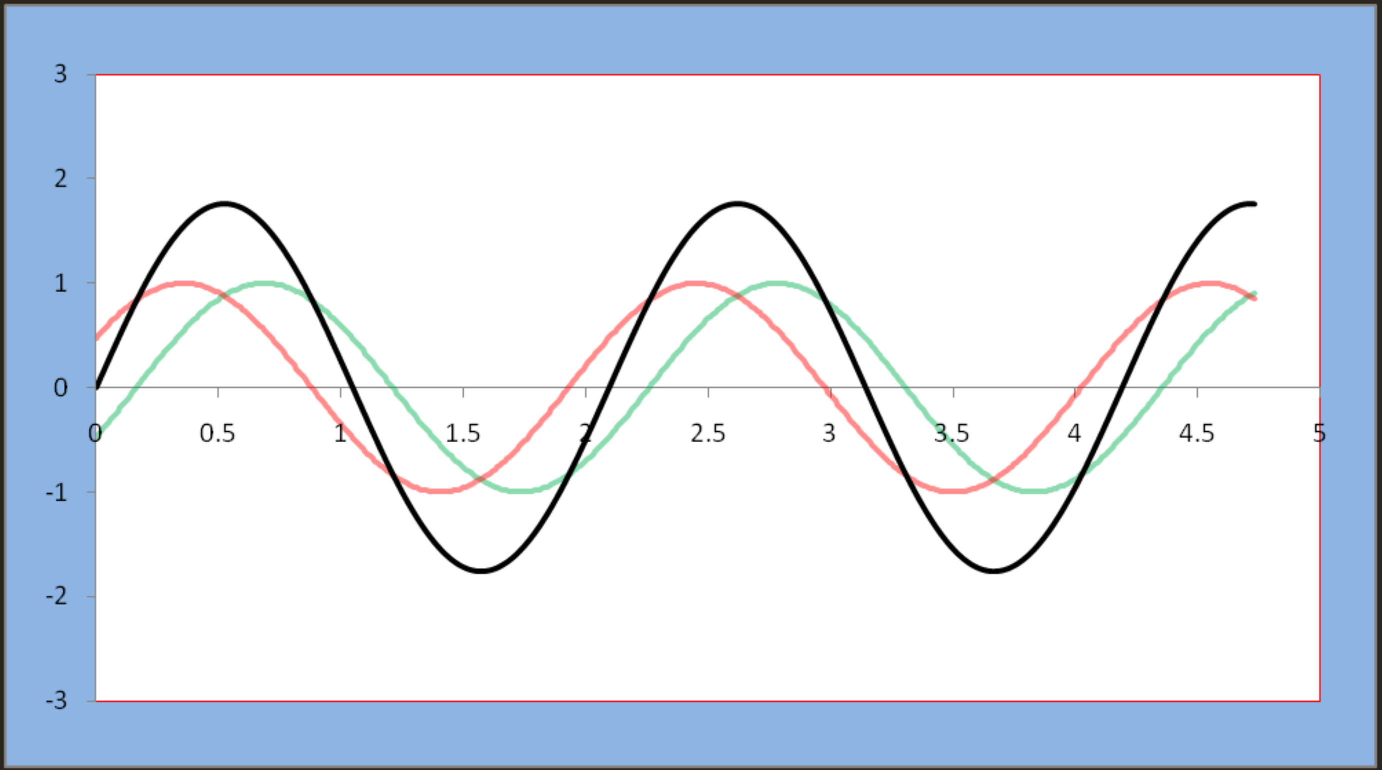
- Differential equations with this property are called "linear". It means you can build up any shape wave from harmonic waves.

A Wave Hits a Wall…

- When a wave hits a wall, the energy and wave form are reflected.
- What does this look like? Let's take the case of a wave on a string, the string fixed at one end.
- Now think about a harmonic wave hitting a wall!

Harmonic Wave Addition

Two harmonic waves of the same wavelength and amplitude, but moving in opposite directions, add to give a standing wave.



Notice the standing wave also satisfies $\lambda f = v$, even though it's not traveling!

Standing Wave Formula

- To add two traveling waves of equal amplitude and wavelength moving in opposite directions, we use the trig formula for addition of sines:

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

- Applying this,

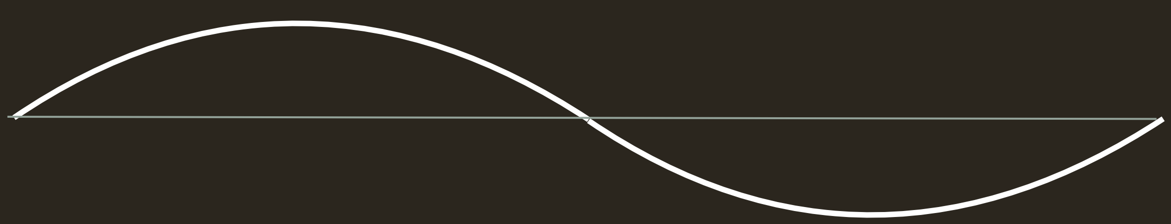
$$A\sin(kx-\omega t) + A\sin(kx+\omega t) = 2A\sin kx\cos\omega t$$

- Allowed values of k are given by $k\ell = \pi, 2\pi, 3\pi \ldots$ where ℓ is the string length.

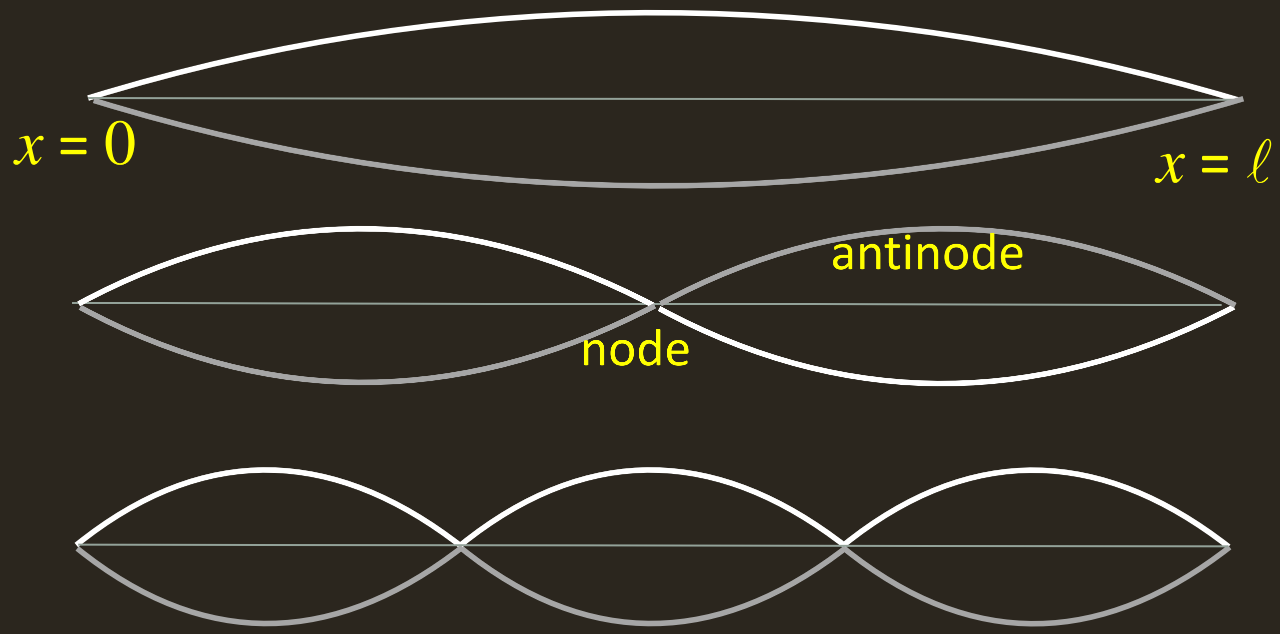
Harmonic Wave on String

- The amplitude must always be zero at the ends of the string. From $\lambda v = f,$ the lowest frequency note (the fundamental, or first harmonic) has the longest allowed wavelength: $\lambda = 2\ell$.

- The second harmonic has $\lambda = \ell$:



Nodes and Antinodes



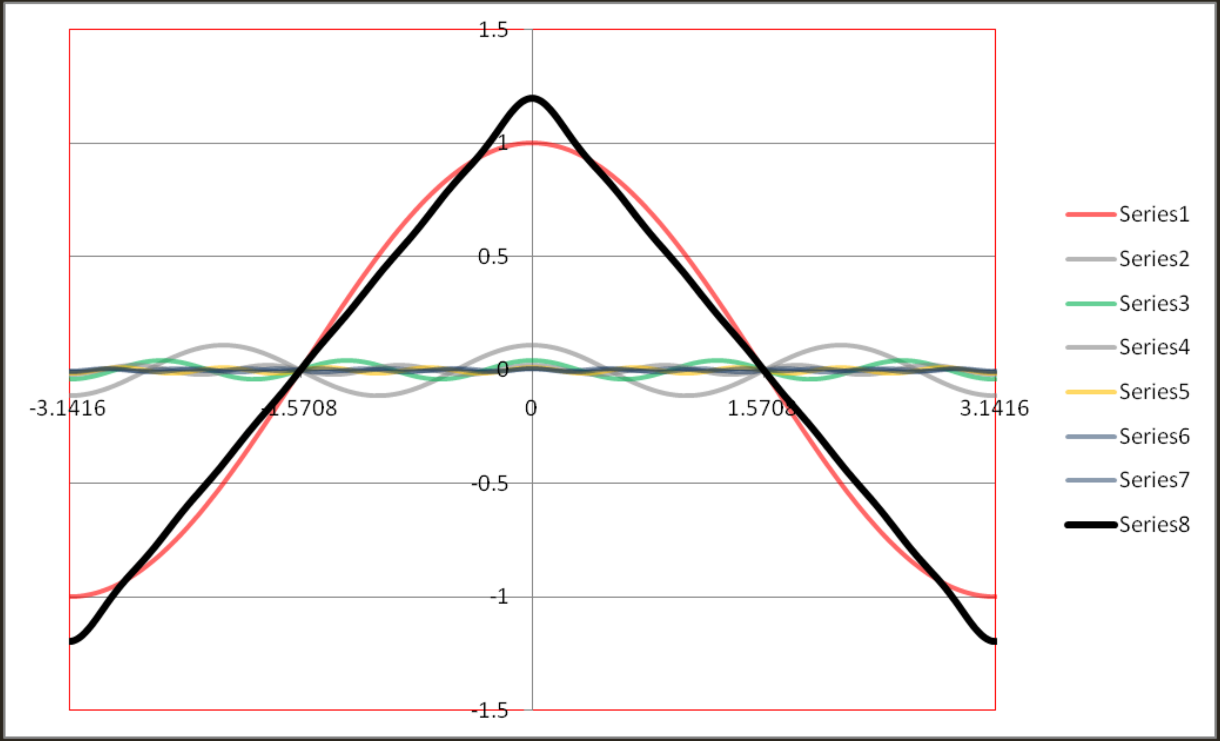
The standing wave has form $y(x,t) = A\sin kx\cos\omega t = A\sin\dfrac{2\pi x}{\lambda}\cos 2\pi ft$

For a pure note on a string with fixed ends, $\lambda = 2\ell, \ell, \tfrac{2}{3}\ell, \ldots$

<u>At a node, the string never moves</u>: $\sin\dfrac{2\pi x}{\lambda} = 0, \quad x = 0, \tfrac{1}{2}\lambda, \lambda, \tfrac{3}{2}\lambda, \ldots$

Fourier Series

We can also build up any type of periodic wave by adding harmonic waves with the right amplitudes—this is called "Fourier analysis": in music, it's building up a complex note from its harmonics: here's a triangle (formed by pulling an instrument string up at the midpoint then letting go?).



Pulse Encounter

It's worth seeing how two pulses traveling in opposite directions pass each other: