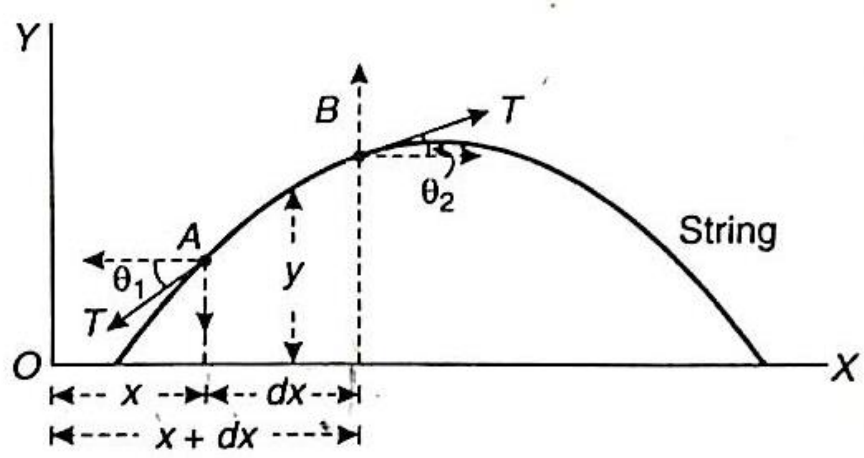
# Wave equation on a string (Derivation)

- Consider an ideal string is perfectly elastic, uniform, and flexible cord having very large length in comparison to its diameter. When a string stretched between two points is plucked in a direction perpendicular to its length, transverse vibrations are setup in the string.
- Now derive the wave equation for transverse vibrations of string fixed between two rigid supports and stretched under a tension T along X-axis.
- In displaced position, consider an infinitesimal string element AB of length dx between the coordinates x and x+dx.
- Let y be its displacement at time t. Let $\Theta_1$ and $\Theta_2$ be the angles which the tension makes with X-axis.
- Components of tension T:

| Component | Tension at A | Tension at B |
|---|---|---|
| Horizontal | $T \cos \Theta_1$ | $T \cos \Theta_2$ |
| Vertical | $T \sin \Theta_1$ | $T \sin \Theta_2$ |



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**The resultant vertical force in the upward direction is given by**

$$F_Y = T\sin\theta_2 - T\sin\theta_1 = T[\sin\theta_2 - \sin\theta_1]$$

$$\sin\theta_1 \approx \tan\theta_1 \approx \left(\frac{\partial y}{\partial x}\right)_x$$

(Since the displacement is small hence $\Theta_1$ and $\Theta_2$ are small)

$$\sin\theta_2 \approx \tan\theta_2 \approx \left(\frac{\partial y}{\partial x}\right)_{x+dx}$$

$$F_Y = T\left[\left(\frac{\partial y}{\partial x}\right)_{x+dx} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

Using Taylor's series we can expand

$$(\frac{\partial y}{\partial x})_{x+dx}, ie.,$$

$$(\frac{\partial y}{\partial x})_{x+dx} = (\frac{\partial y}{\partial x})_x + (\frac{\partial^2 y}{\partial x^2})dx + (\frac{\partial^3 y}{\partial x^3})\frac{(dx)^2}{2}$$

$$(\frac{\partial y}{\partial x})_{x+dx} = (\frac{\partial y}{\partial x})_x + (\frac{\partial^2 y}{\partial x^2})dx$$

Since, Taylor's series expansion of f(x + dx)) is given by

$$f(x+dx) = f(x) + f^{'}(x)\triangle x + f^{''}(x)\frac{(\Delta x^2)}{1\times 2}$$

$$F_Y = T\left[\{(\frac{\partial y}{\partial x})_x + (\frac{\partial^2 y}{\partial x^2})dx\} - (\frac{\partial y}{\partial x})_x\right]$$

$$F_Y = T\left[\frac{\partial^2 y}{\partial x^2}\right]dx$$

$$F_Y = mass \times acceleration = (mdx)(\frac{\partial^2 y}{\partial t^2})$$ **(m – mass per unit length of the wire)**

$$m(\frac{\partial^2 y}{\partial t^2})dx = T\left[\frac{\partial^2 y}{\partial x^2}\right]dx$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2}$$

**This equation is called wave equation of motion of the string.**

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2}$$

The differential equation of a wave motion is given by,

$$\frac{\partial^2 y}{\partial t^2} = v^2\left[\frac{\partial^2 y}{\partial x^2}\right]$$

Comparing the above two equations, we get

$$v^2 = \frac{T}{m}$$

$$v = \sqrt{(\frac{T}{m})}$$

This gives the velocity of transverse waves along the string