Fourier series

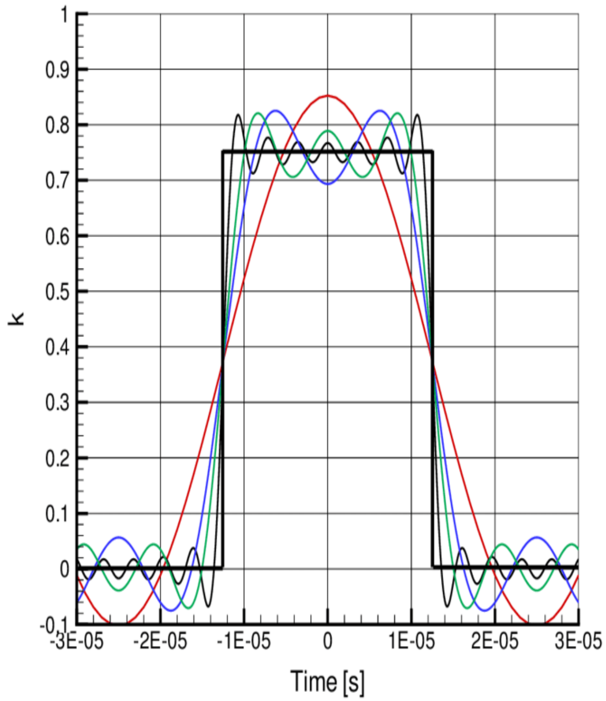
As we know that **TAYLOR SERIES** representation of functions are valid only for those functions which are continuous and differentiable. But there are many discontinuous periodic function which requires to express in terms of an infinite series containing **'sine'** and **'cosine'** terms

FOURIES SERIES, which is an infinite series representation of such functions in terms of 'sine' and 'cosine' terms, is useful here. Thus, FOURIER SERIES, are in certain sense, more UNIVERSAL than TAYLOR'S SERIES as it applies to all continuous, periodic functions and also to the functions which are discontinuous in their values and derivatives. FOURIER SERIES very powerful method to solve ordinary and partial differential equation, particularly with periodic functions appearing as non-homogenous terms.

Fourier series

- **Fourier series is an infinite series representation of periodic function in terms of the trigonometric sine and cosine functions.**
- **Most of the single valued functions which occur in applied mathematics can be expressed in the form of Fourier series, which is in the terms of sines and cosines.**
- **Fourier series is to be expressed in terms of periodic functions – sines and cosines.**
- **Fourier series is very powerful method to solve ordinary and partial differential equations, particularly with periodic functions appearing as non-homogeneous terms.**
- **We know that, TAYLOR SERIES EXPANSION is valid only for those functions which are continuous and differentiable. Fourier series is possible not only for continuous functions but also periodic functions, functions which are discontinuous in their values and derivatives. Further, because of the periodic nature, Fourier series constructed for one period is valid for all values.**

Fourier waves



Fourier series generally written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n sin(nx)$$

Where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)dx \text{..... (1.1)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx \text{ ..... (1.2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)\,dx \text{ ..... (1.3)}$$

Fourier series make use of the orthogonality relationships of the sine and cosine functions.

Basic formula of Fourier series

- **The Fourier series of the periodic function f(x) with period 2n is defined as the trigonometric series with the coefficient a_0, a_n, and b_n, known as FOURIER COEFFICIENTS, determined by formulae (1.1), (1.2), and (1.3).**
- **The individual terms of the series is known as HARMONICS.**
- **Every function f(x) of period 2n satisfying following conditions known as DIRICHLETS CONDITIONS, can be expressed in terms of Fourier series.**

Conditions

1. F(x) is bounded and single value(A function f(x) is called single valued if each point in the domain, it has unique value in the range)
2. F(x) has at most, a finite no. maxima and minima in the interval.
3. F(x) has at most, a finite no. of discontinuous in the interval.

EXAMPLE:

$\sin^{-1}x$, we can say that the function $\sin^{-1}x$ can't be expressed as Fourier series as it is not a single valued function

tanx, also in the interval (0, 2n) can't be expressed as a Fourier series because it is infinite at x = n/2.

Conditions for Fourier expansion (DIRCHLET CONDITIONS)

A function f(x) defined in [0,2π] has a valid Fourier series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n sin(nx)$$

Where a_0,a_n, b_n are constants provided

1) f(x) is well defined and single valued, except possibly at a finite number of point in the interval [0,2π].

2) f(x) has finite number of finite discontinuous in the interval [0,2π].

3) f(x) has finite number of finite maxima and minima.

Note: The above conditions are valid for functions defined in the intervals [π,π],[0,2L], [-l,+l]

- {1, cos1x, cos2x, cos3x, cos4x,…..cosnx,…….sin1x, sin2x, sin3x,……..sinnx……}
- {1, cos(πx/L), cos(2πx/L), cos(3πx/L),…… cos(nπx/L),……sin(πx/L), sin(2πx/L), sin(3πx/L), …..sin(nπx/L),…….}

All these have a common period 2L

These are called the complete set of orthogonal functions

Periodic functions

- A function f(x) is said to be periodic function with period T>0 if for all x, f(x+T) = f(x), and T is the least of such values.

Example:

1) sin x, cos x are periodic functions with period 2π.
2) tan x, cot x are periodic functions with period π.

Euler's formulae

The Fourier series of the function f(x) in the interval $(C \le x \le C + 2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n sin(nx)$$

Where

$$a_0 = \frac{1}{\pi}\int_C^{C+2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\sin(nx)\,dx$$

These values a_0,a_n, b_n are known as Euler's formulae.

Definition of Fourier series

- Let f(x) be the function defined in [0,2π]. Let $f(x+2\pi) = f(x)\ \forall\ x$, then the Fourier series of f(x) is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n sin(nx)$

Where,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$$

These values a_0,a_n, b_n are known as coefficients of f(x) in [0,2π].

- Let f(x) be a function defined in [-π,+ π]. Let $f(x+2\pi) = f(x)\ \forall\ x$, then the Fourier series of f(x) is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n sin(nx)$

Where,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)\,dx$$

These values a_0,a_n, b_n are known as coefficients of f(x) in [-π, +π].

- **Let f(x) be the function defined in [0, 2l]. Let $f(x+2\pi) = f(x)\ \forall\ x$, then the Fourier series of f(x) is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n sin(\frac{n\pi x}{l})$**

Where,

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{\pi}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

These values a_0,a_n, b_n are known as coefficients of f(x) in [0,2l].

- **Let f(x) be the function defined in [-l,+l]. Let $f(x+2\pi) = f(x)\ \forall\ x$, then the Fourier series of f(x) is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n sin(\frac{n\pi x}{l})$**

Where,

$$a_0 = \frac{1}{l}\int_{-l}^{+l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{+l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{\pi}\int_{-l}^{+l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

These values a_0,a_n, b_n are known as coefficients of f(x) in [-l,+l].

FOURIER SERIES FOR EVEN AND ODD FUNCTIONS

EVEN FUNCTIONS

If function f(x) is an even periodic function with period 2L $(-\mathrm{L} \leq x \leq L)$, then $f(x)\cos(\frac{n\pi x}{L})$ is even while $f(x)\, sin(\frac{n\pi x}{L})$ is odd

Thus the Fourier series expansion of an even periodic function with period 2L $(-\mathrm{L} \leq x \leq L)$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Where,

$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})dx$$

$$b_n = 0$$

ODD FUNCTIONS

If function f(x) is an even periodic function with period 2L $(-\mathrm{L} \leq x \leq L)$,

then $f(x)\cos(\frac{n\pi x}{L})$ is even while $f(x)\ sin(\frac{n\pi x}{L})$ is odd

Thus the Fourier series expansion of an ODD periodic function with period 2L $(-\mathrm{L} \leq x \leq L)$ is given by,

$$\boldsymbol{f(x) = \sum_{n=1}^{\infty} b_n\, sin\left(\frac{n\pi x}{L}\right)}$$

Where,

$$\boldsymbol{b_n = \frac{2}{L}\int_0^L f(x)\, sin(\frac{n\pi x}{L}) dx} \quad \text{n = 0,1,2,3....}$$