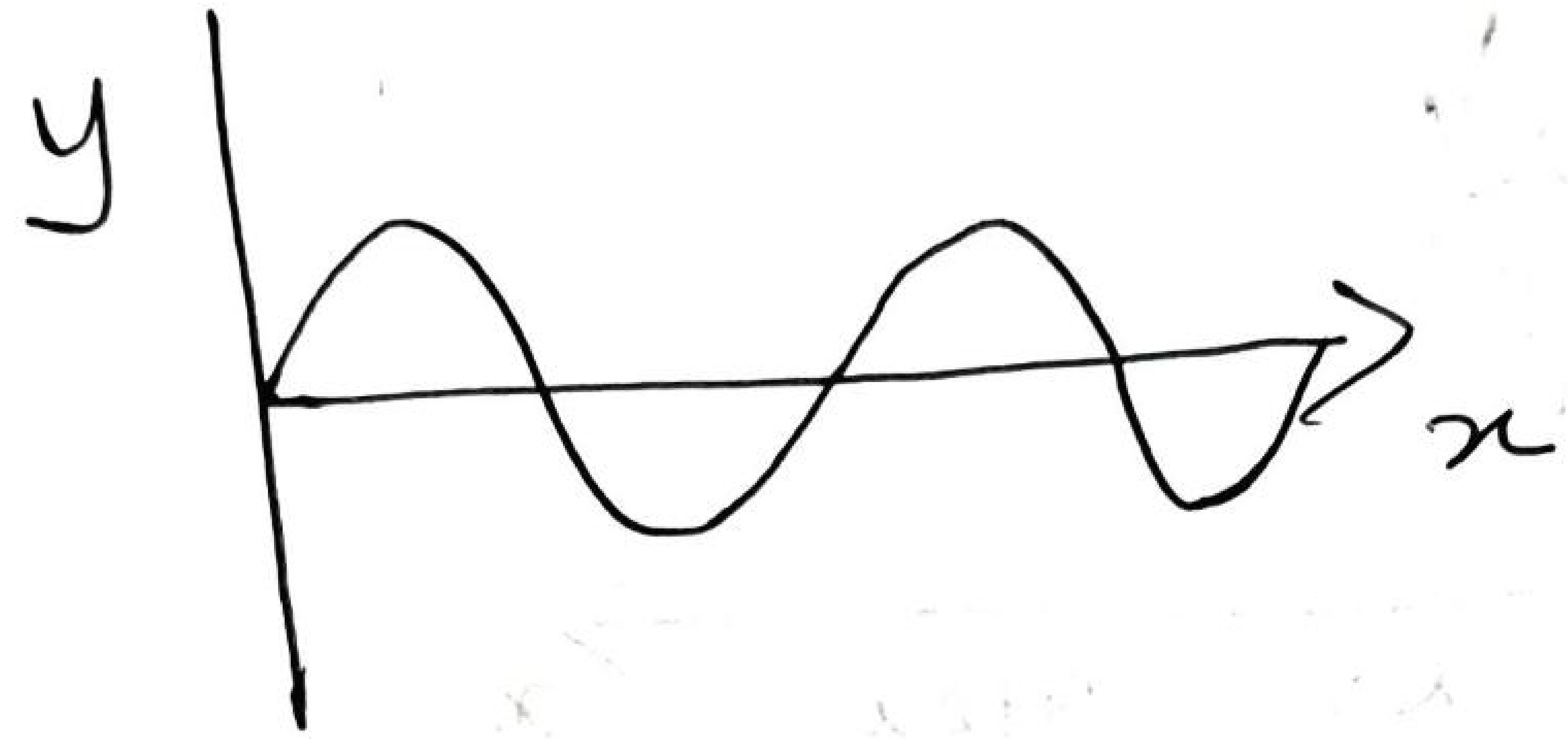
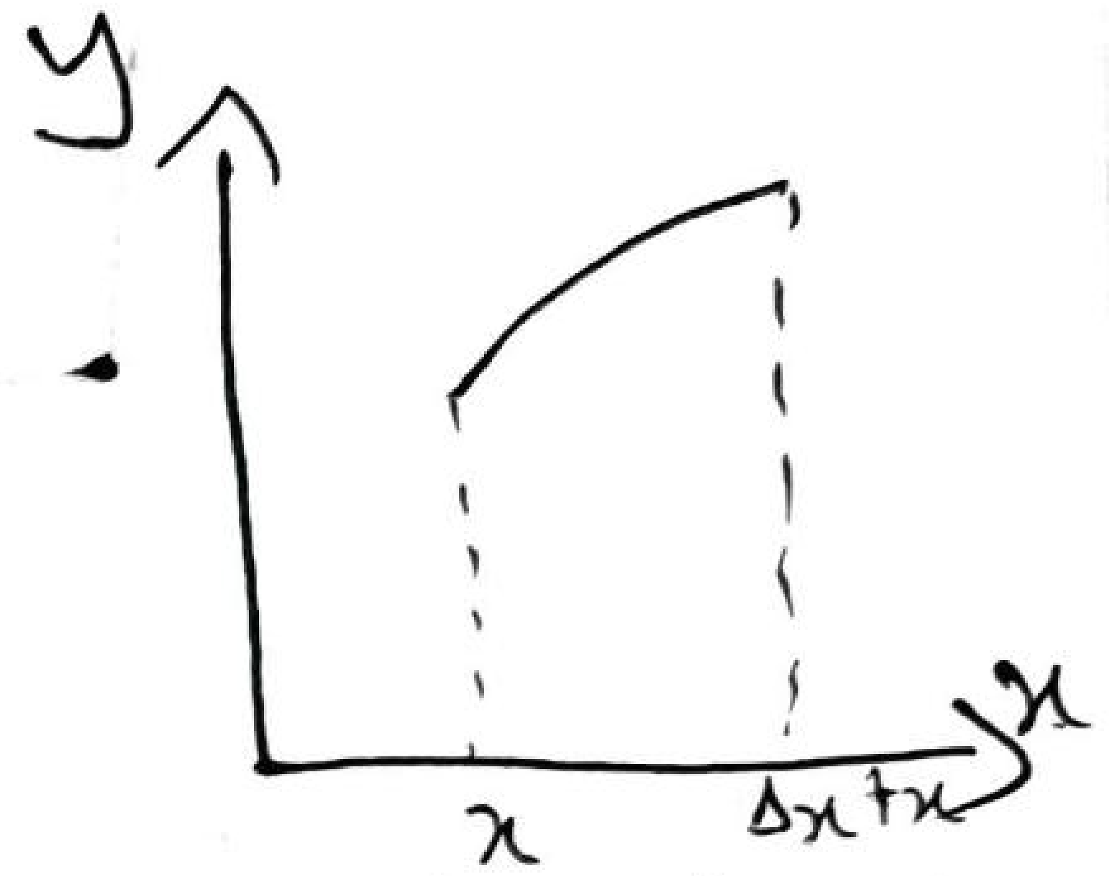
1-D Wave equation

Wave propogation in a string.





Assumptions:

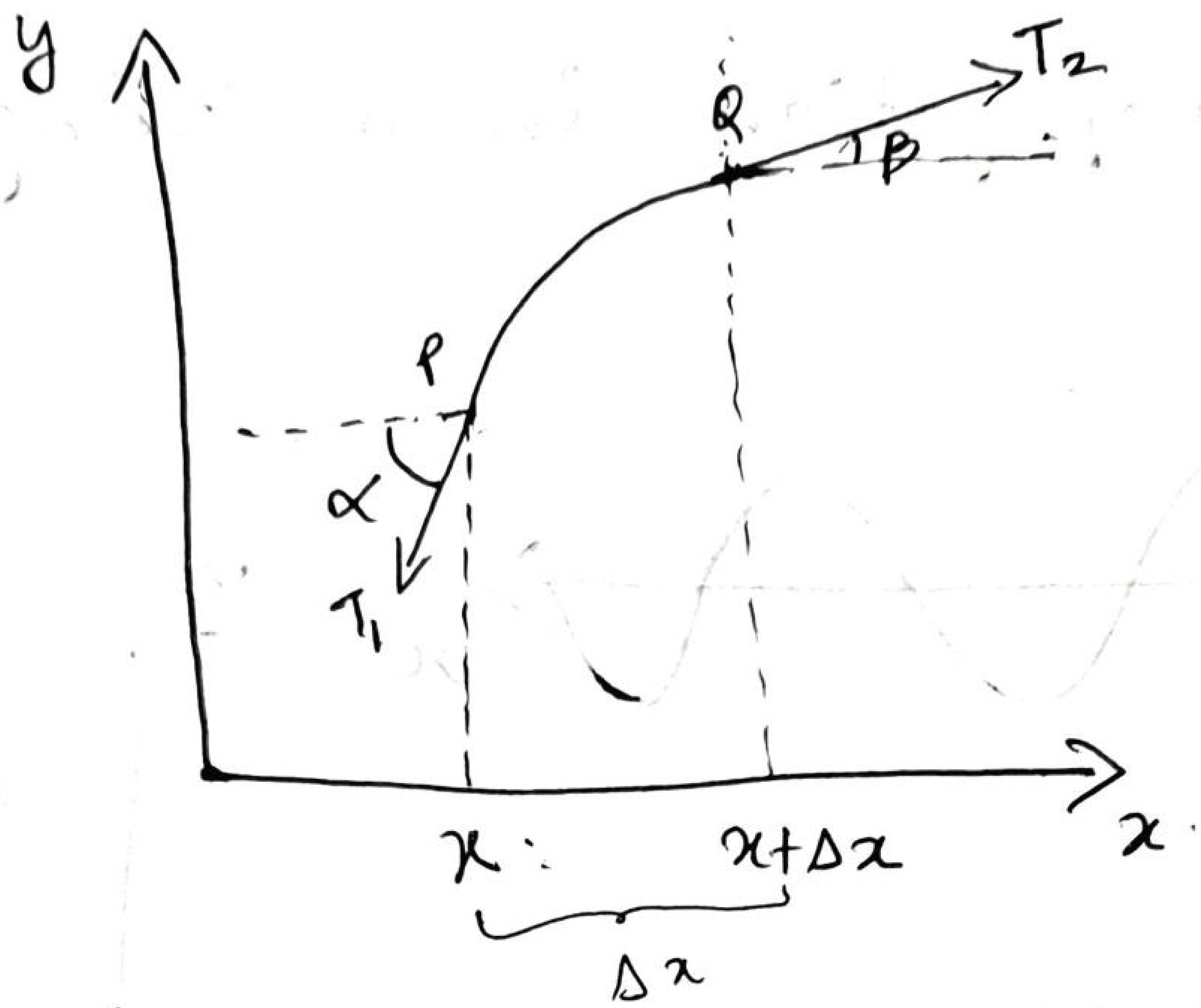
* Mass of string is uniformly distributed (mass / length = linear density (ρ) is constant)

* String is perfectly elastic & offers no resistance to bending

* Gravitation effects negligible

* Osillations are only along ~~x~~ y-axis (transverse dir'n).

* Vertical deflections (amplitude) are small at all times → slopes ($\partial y / \partial x$ are small).



→ The displacement (y) will vary with time 't' and also with 'x' the position → $y(x, t)$

→ Let us consider a portion of string 'PQ'

→ P is at point ~~&~~ x and Q at pt $x + \Delta x$ in the x axis.

→ Tension at point P is T_1 ~~making~~ an

" " " Q is T_2

Force makes an angle α at pt 'p' and β at point Q.

Horizontal components ~~to~~ of force are equal

$$T_1 \cos(\alpha) = T_2 \cos(\beta) = T \quad \text{—— (1)}$$

Vertical component of force → Only this force is active

$$F_{vertical} = m\,a$$ → Newtons 2nd law of motion

Vertical component

$$T_2 \sin\beta - T_1 \sin\alpha = m\,a \quad \text{—(2)}$$

$$T_2 \sin\beta - T_1 \sin\alpha = m \cdot \frac{\partial^2 y}{\partial t^2}$$

linear mass density $\rho = m/l$ ∴

in this case $\rho = m/\Delta x$ $\quad m = \rho \Delta x$

Substituting for m in eqn (2)

$$T_2 \sin\beta - T_1 \sin\alpha = \rho \Delta x \frac{\partial^2 y}{\partial t^2} \quad \text{—(3)}$$

÷ eqn (3) by (1) throughout

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho \Delta x}{T} \frac{\partial^2 y}{\partial t^2}$$

$$\tan\beta - \tan\alpha = \frac{\rho \Delta x}{T} \frac{\partial^2 y}{\partial t^2}$$

$$\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x} = \frac{\rho \Delta x}{T} \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}}{\Delta x} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

$$\Downarrow$$

This tells us how much did the slope change in b/w P & Q.

$$\therefore \lim_{\Delta x \to 0} \frac{\partial^2 y}{\partial x^2} = \rho/T \, \frac{\partial^2 y}{\partial t^2}$$

$\Delta x \to$ small $\boxed{\frac{\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}}{\Delta x} \simeq \frac{\partial^2 y}{\partial x^2}}$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} \quad \text{or} \quad \frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}$$

$\frac{T}{\rho}$ is const and positive.

The dimensional analysis

$$\left(\frac{\rho}{T}\right) = \frac{ML^{-1}}{M(LT^{-2})} = L^{-2}T^{2} = \frac{1}{c^2} \quad \frac{kg/m}{kg\,m/s^2} = \frac{s^2}{m^2}$$

$\downarrow$ similar to (velocity)^2

~~two~~

$$\therefore \frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

or $\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}} \rightarrow$ 1-D wave eqn

Sol'n of wave eqn

Let us consider a 1-D wave eqn

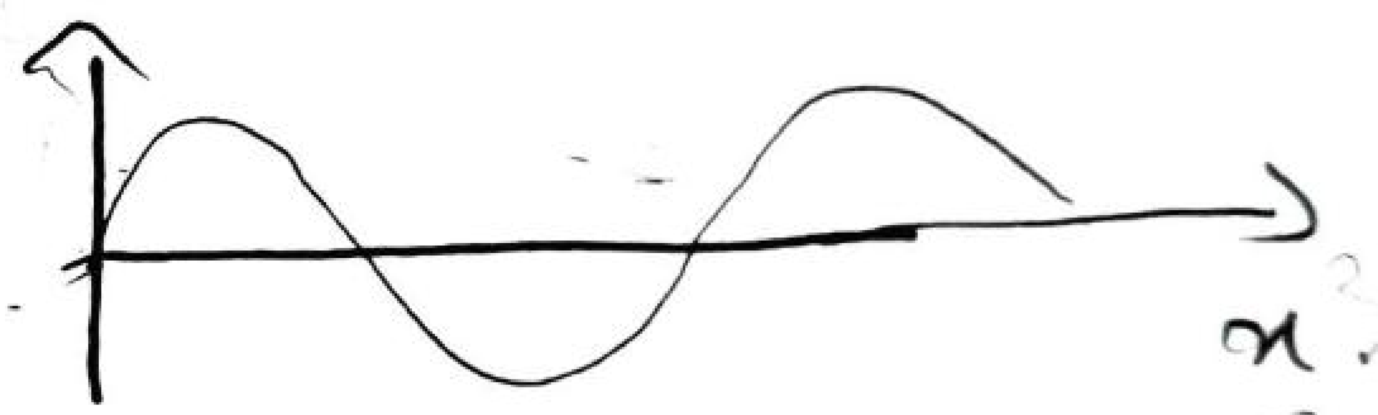
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$v \rightarrow$ velocity of the wave.

Let us consider a sol'n $y = f(x - vt)$

Goal. To show that this $y = f(x - vt)$ will satisfy the wave equation.

$y(x,t)$, Considering a wave moving in $+x$ dir'n. with uniform amp



at time $t=0$, if y is traveling a dist 'x' after time 't' y would have a moved dist 'vt'. Since we are talking about forward moving wave, the new dist at the point would be $(x - vt)$

for a wave moving in (-ve) x dir'n it would be $(x + vt)$

$$y = f(x - vt) \qquad (x - vt) = \tau \qquad y = f(\tau$$

Chain rule.

$$\frac{\partial y}{\partial t} = \frac{\partial f}{\partial \tau} \cdot \frac{\partial \tau}{\partial t}$$

$$\therefore \frac{\partial \tau}{\partial t} = -v \qquad \frac{\partial y}{\partial t} = -v \cdot \frac{\partial f}{\partial \tau}.$$

||rly $$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial \tau}\left(v\frac{\partial f}{\partial \tau}\right) \cdot \frac{\partial \tau}{\partial t}$$

$$\frac{\partial \tau}{\partial t} = -v$$

$$\therefore \frac{\partial^2 y}{\partial t^2} = +v^2 \frac{\partial^2 f}{\partial \tau^2}.$$

||rly.

$$\frac{\partial y}{\partial x} = \frac{\partial f}{\partial \tau} \cdot \frac{\partial \tau}{\partial x}.$$

$$\frac{\partial \tau}{\partial x} = 1 \qquad \therefore \frac{\partial y}{\partial x} = \frac{\partial f}{\partial \tau}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial \tau}\left(\frac{\partial f}{\partial \tau}\right) \cdot \frac{\partial \tau}{\partial x}.$$

Since $\frac{\partial \tau}{\partial x} = 1$ we get $$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial \tau^2}$$

~~∂y~~

Substituting in wave eqn.

$$\frac{1}{\cancel{v^2}} \times \cancel{v^2} \frac{\partial^2 f}{\partial \tau^2} = \frac{\partial^2 f}{\partial \tau^2}$$

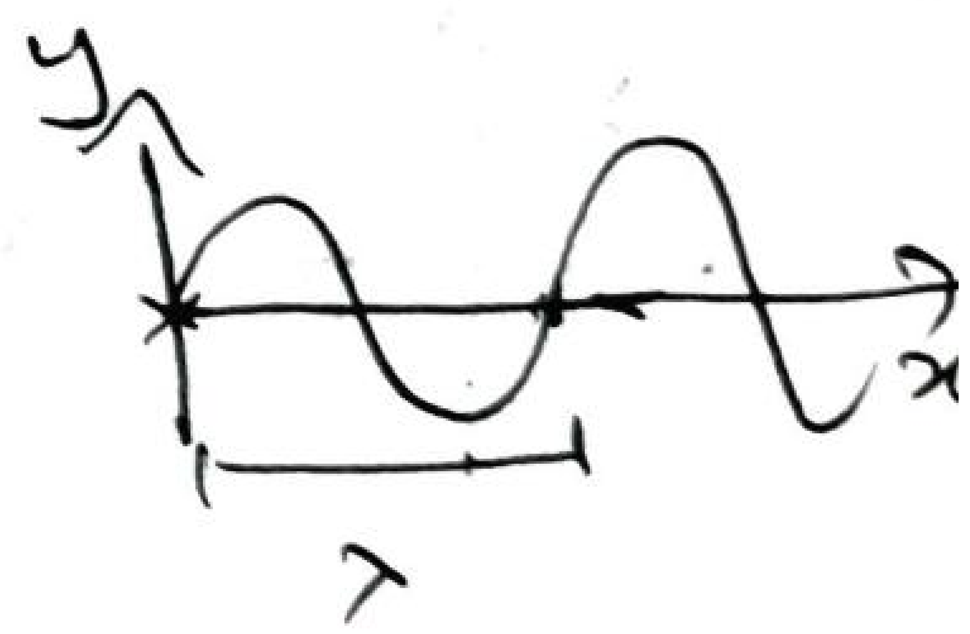
we get L.H.S. = R.H.S.

Hence proved that $y = f(x - vt)$ is a solution of wave eqn.

IIrly $y = f(x + vt)$ will also be a solution of wave eqn.

even $y = f(x+vt) + f(x - vt)$ will also be a solution of wave eqn.

For harmonic waves:



$$y = A \sin\frac{2\pi}{\lambda}(x - vt)$$

$(x - vt) \rightarrow$ is in units of 'm'

$2\pi \rightarrow$ is × to change in to radians.

~~af~~

After a dist 'λ' the wave repeats

∴ this $\lambda \rightarrow$ wavelength

$$y = A \sin\left(\frac{2\pi x}{\lambda} - \frac{2\pi \cancel{x} vt}{\lambda}\right)$$

$K = \frac{2\pi}{\lambda} \rightarrow$ angular wave number

$\omega = 2\pi\nu \rightarrow$ ang. freq

$v/\lambda = \nu$

$$y = A\sin(kx - \omega t)$$

IIaly $y = A\cos(kx - \omega t)$ is also solution of wave eqn.

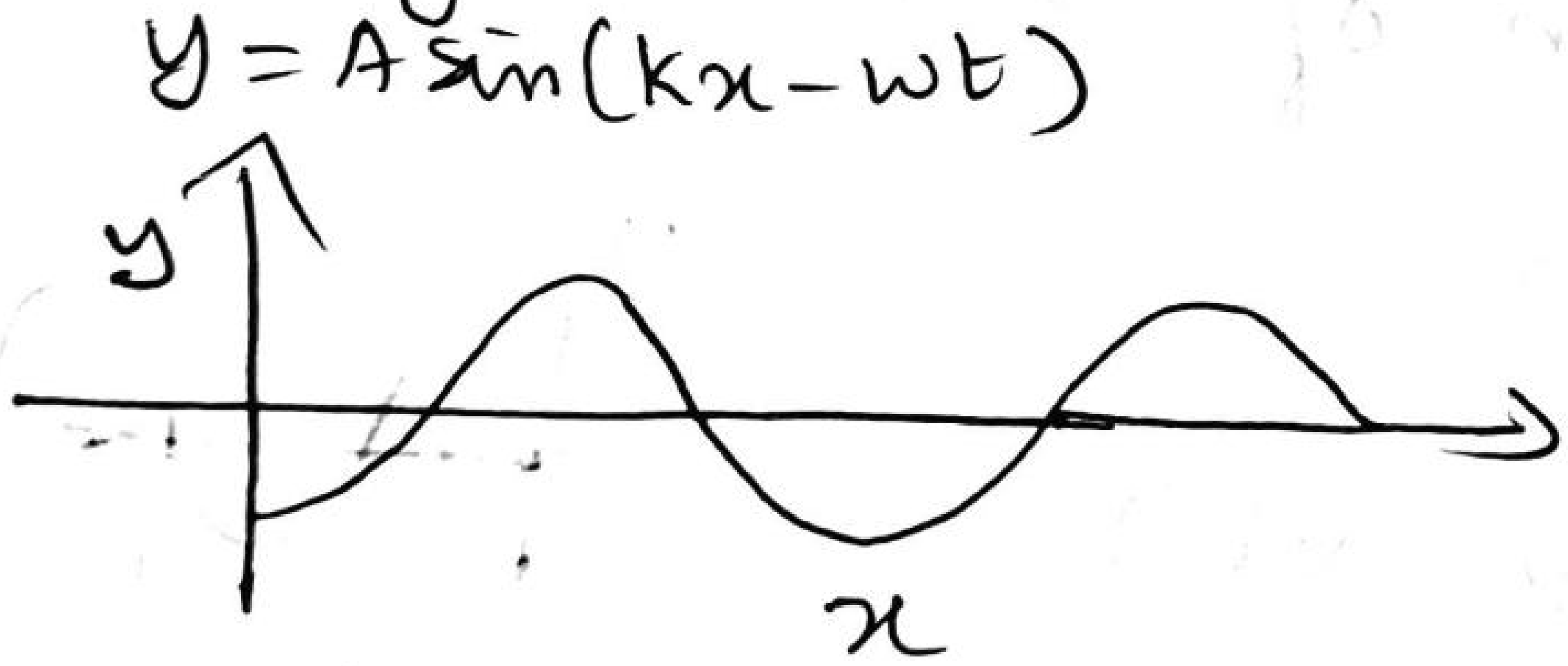
y can also be written as

$$y = A\sin(\omega t - kx) \text{ or } A\cos(\omega t - kx)$$

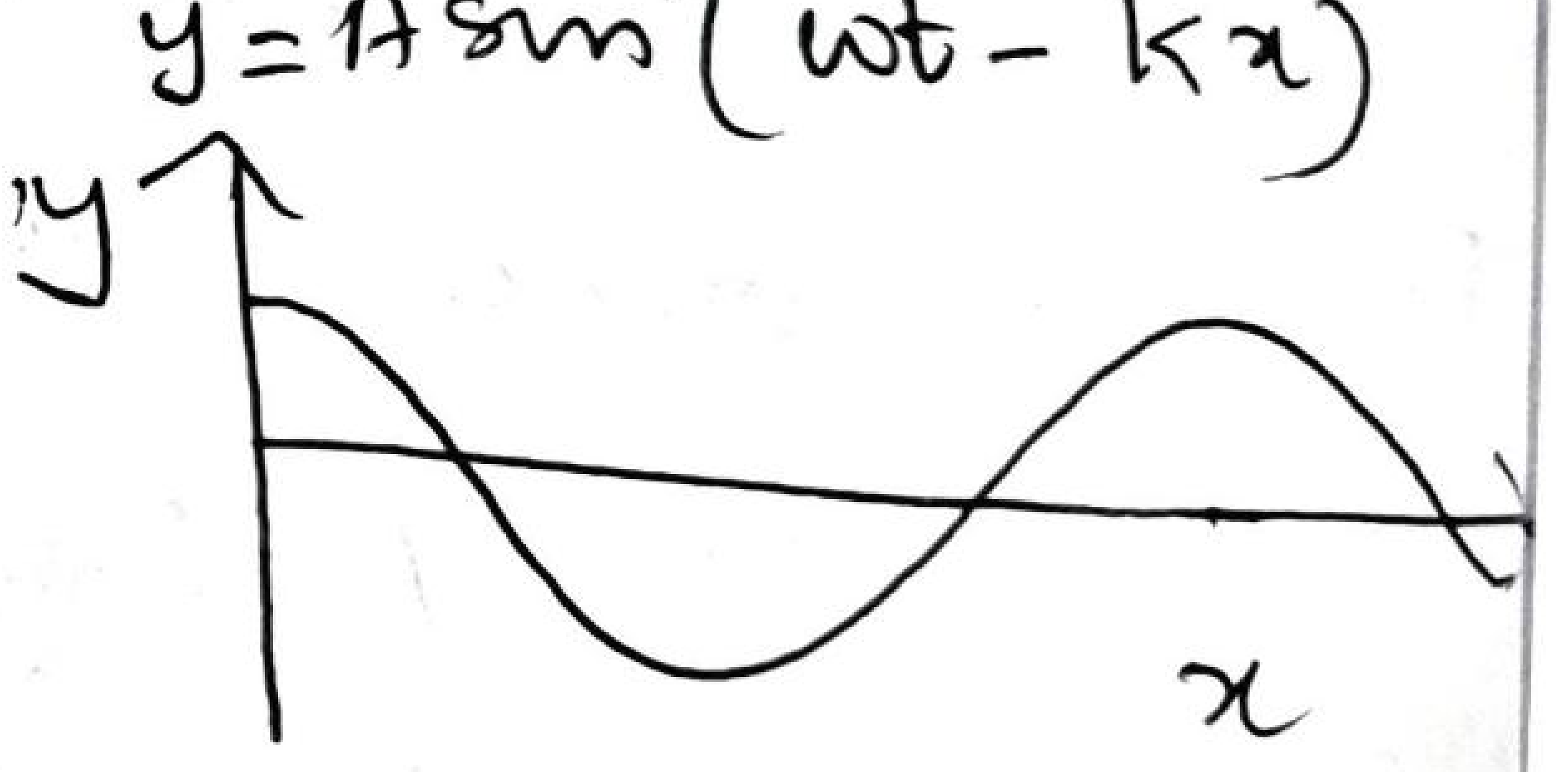
and all these are talking about forward moving waves.

eg

$y = A\sin(kx - \omega t)$



$y = A\sin(\omega t - kx)$



In terms of exponential terms

$y = Ae^{i(\omega t - kx)}$ is also a solution of wave moving forward.

Velocities in wave motion:


Light

When a wave progresses through a medium, the individual oscillators that make up the medium do not progress through the medium with the waves. Their motion is simple harmonic limited to osillations, either transverse or longitud-inal, about thier equilibrium position.

→ Three Velocities in wave motion.

(i) particle velocity: which is the simple harmonic velocity of the oscillator about its equilibrium position $\left(\frac{\partial y}{\partial t}\right)$

(2) Wave velocity or phase velocity $\left(\frac{\partial x}{\partial t}\right)$

- The velocity with which planes of equal Phase, crest or trough progress through medium

(3) Group velocity:

When a number of waves of diff freq, wavelengths and velocities may be superimposed to form a group. Motion of such ~~to~~ pulse is described by group velocity.

Relation b/w ~~phs~~ particle velocity & wave vel

$$y = A\sin(kx - \omega t).$$

$$\frac{\partial y}{\partial t} = \cancel{a\sin} - A\omega\cos(kx - \omega t).$$

$$\frac{\partial y}{\partial x} = Ak\cos(kx - \omega t).$$

$$\cos(kx - \omega t) = -\frac{1}{A\omega}\frac{\partial y}{\partial t}$$

$$\cos(kx - \omega t) = \frac{1}{Ak}\frac{\partial y}{\partial x}.$$

$$-\frac{1}{\cancel{A}\omega}\frac{\partial y}{\partial t} = \frac{1}{\cancel{A}k}\frac{\partial y}{\partial x}$$

$$\frac{\partial y}{\partial t} = -\frac{\omega}{k}\frac{\partial y}{\partial x}$$

$$\frac{\partial y}{\partial t} = -c\,\frac{\partial y}{\partial x}. \qquad c = \frac{2\pi\nu}{2\pi/\lambda} = \omega/k$$

$$\frac{\partial y}{\partial t} = -\frac{\partial x}{\partial t}\cdot\frac{\partial y}{\partial x}$$

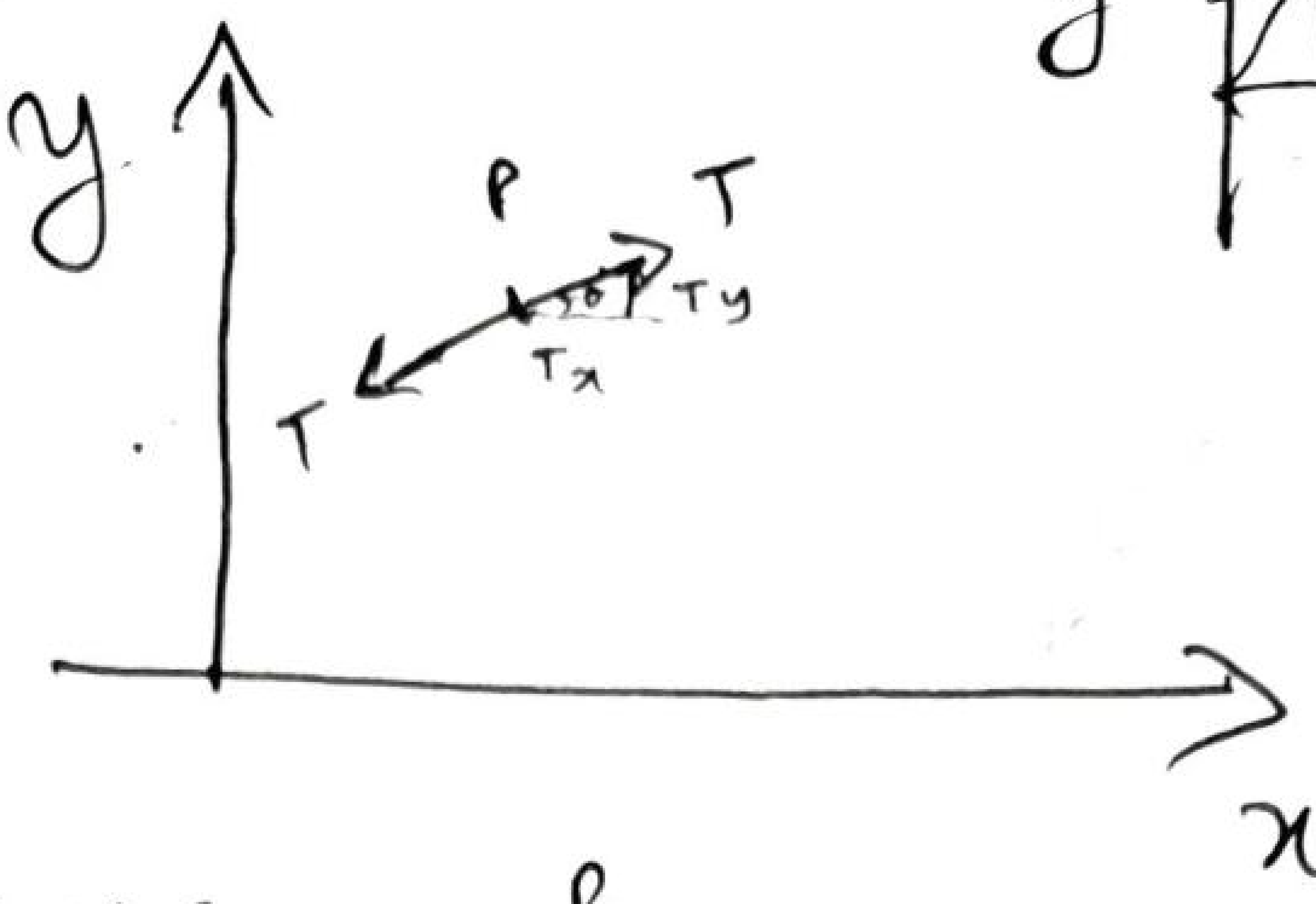
The particle velocity is given as the product of wave velocity and the gradient of the wave profile preceded by a (−) sign for a wave moving in (+x) dir'n.

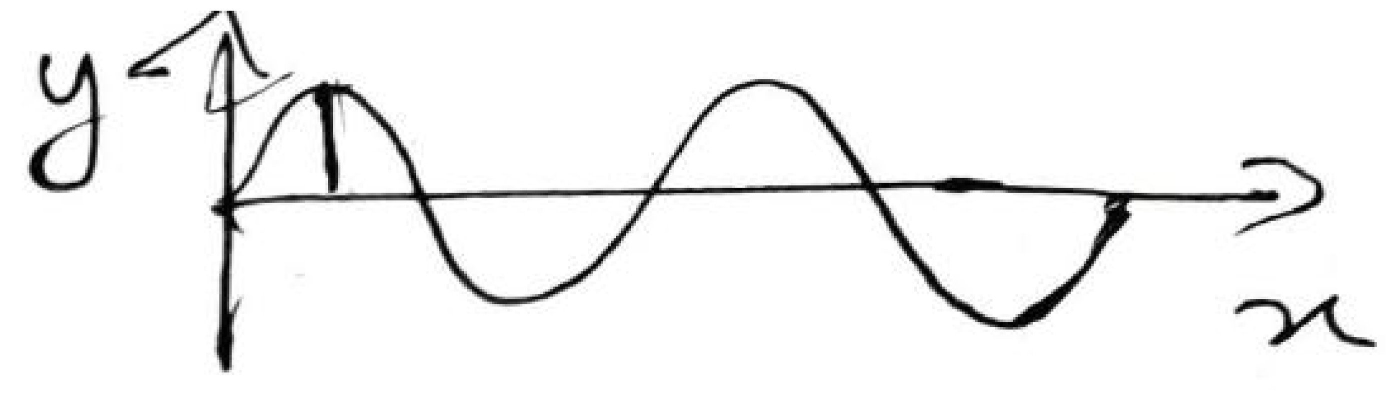
Impedance of a string:

Any medium through which waves propogate will present an impedance to the waves.

In case of string, moving in a transverse dir'n (y) to the wave motion (x)

$$Z = \frac{\text{Transverse force}}{\text{Transverse velocity}} = \frac{F_y}{V_y}$$





T is tension

Transverse force $F_y = T_y = -T\sin\theta$.

-ve sign is because force is ~~+ve~~ acty downward & slope is (+ve)

Since θ is small

$$F_y = -T\tan\theta.$$

$$= -T\,\partial y/\partial x$$

$$F_y = -T\frac{\partial y}{\partial x}.$$

$$y(x,t) = A\sin(kx-\omega t)$$

$$\frac{\partial y}{\partial x} = Ak\cos(kx-\omega t)$$

$$\therefore F_y = -TAk\cos(kx-\omega t).$$

For transverse velo (particle vel)

$$v_y = \frac{\partial y}{\partial t}$$

$$\frac{\partial y}{\partial t} = v_y = -A\omega\cos(kx-\omega t)$$

$\therefore$ ~~F_y~~ $Z = \frac{F_y}{V_y}$

$$Z = \frac{-T \cancel{A} k \cos(k\cancel{x} - \omega t)}{-\cancel{A}\, \omega \cos(k\cancel{x} - \omega t)}$$

$$Z = +\frac{T k}{\omega}$$

$$\boxed{Z = \frac{+T}{c}}$$

$c \rightarrow$ wave velocity.

$T = \rho c^2$ from wave eqn.

$\therefore$ $Z = \rho c$

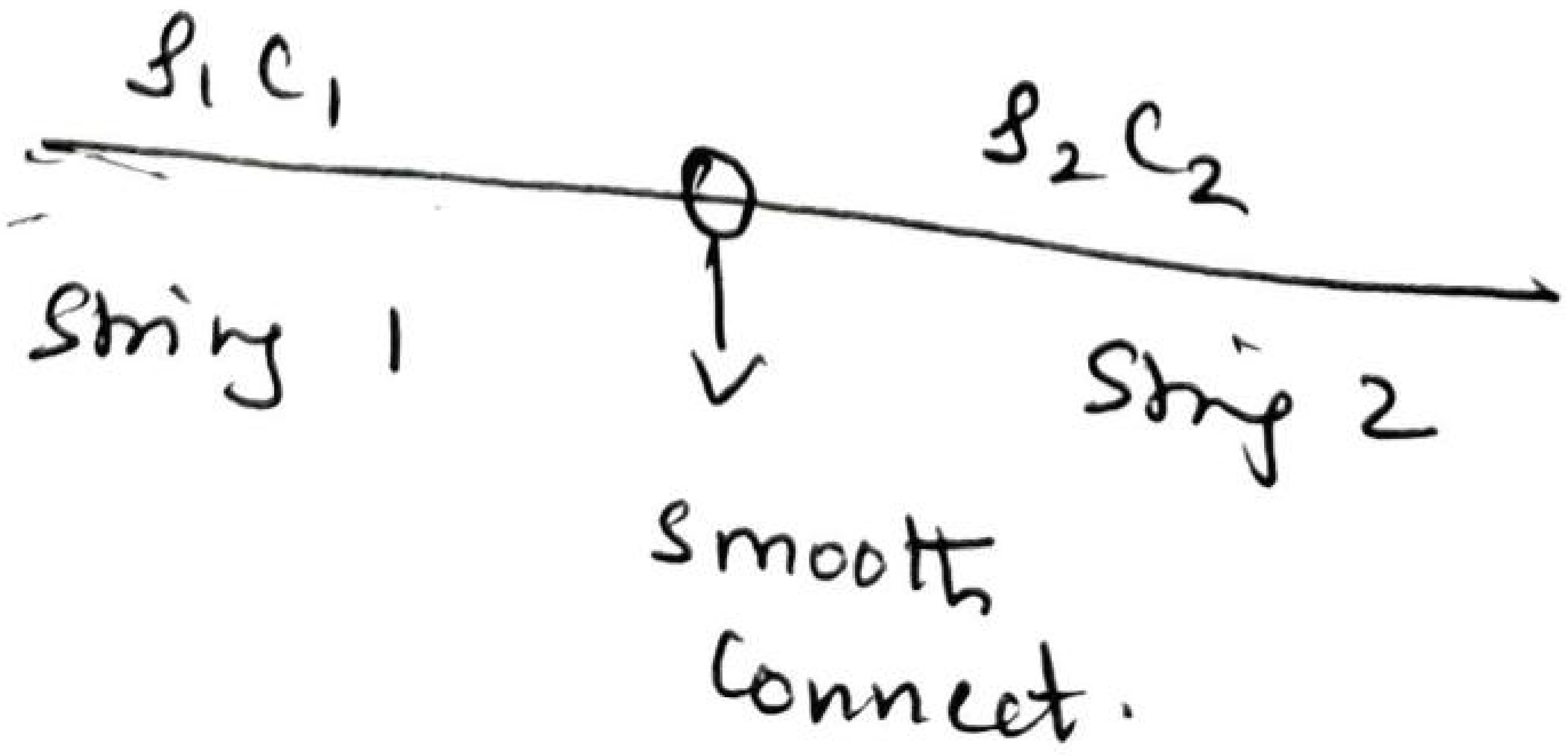
$\rho \rightarrow$ linear mass density.

Reflection and Transmission of waves on a string at a boundary

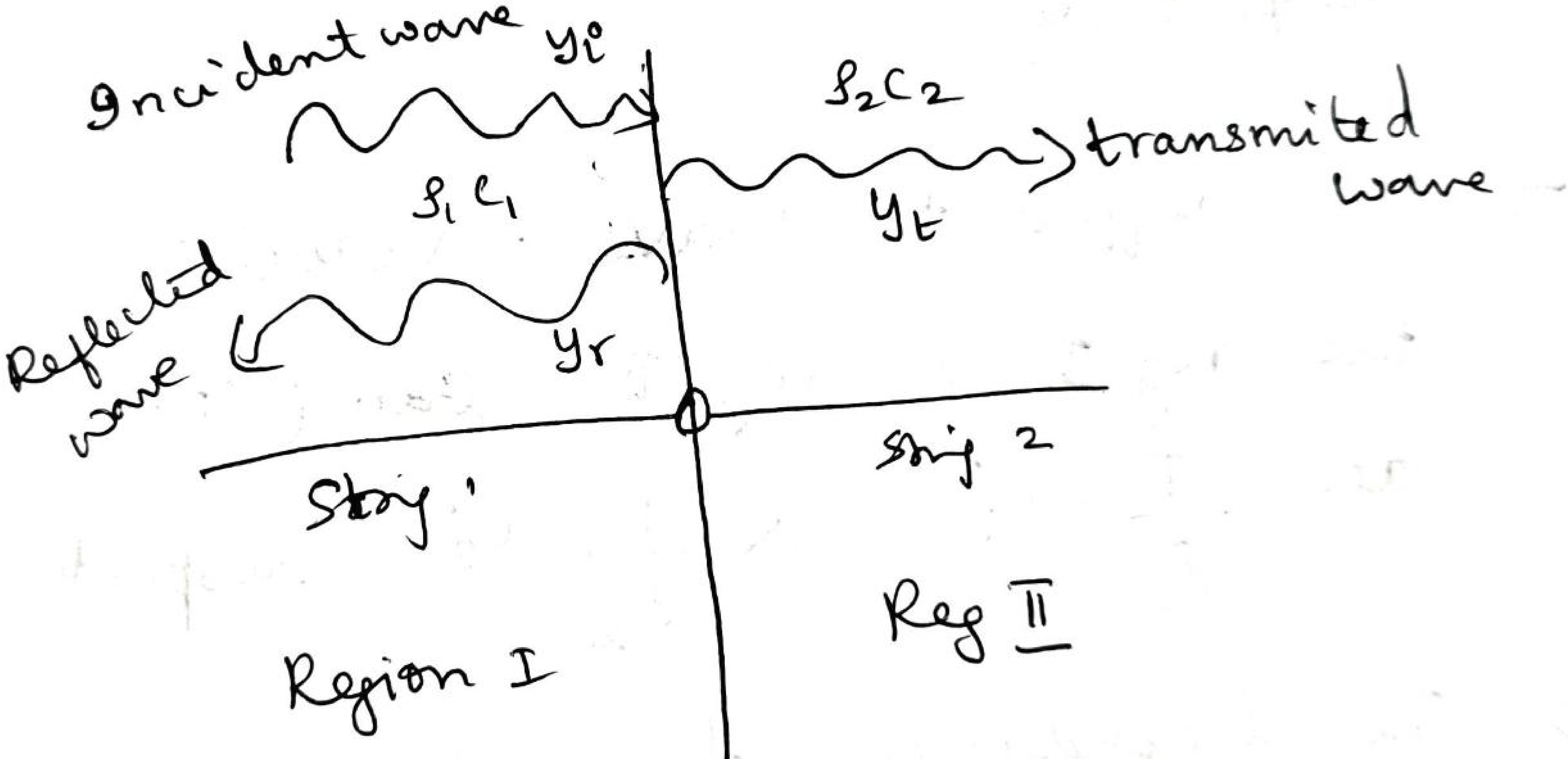
We have seen that a string posses a characteristic impedance 'ρc' to the waves travelling along it. Lets now study how a wave will respond to a sudden change of impedance.

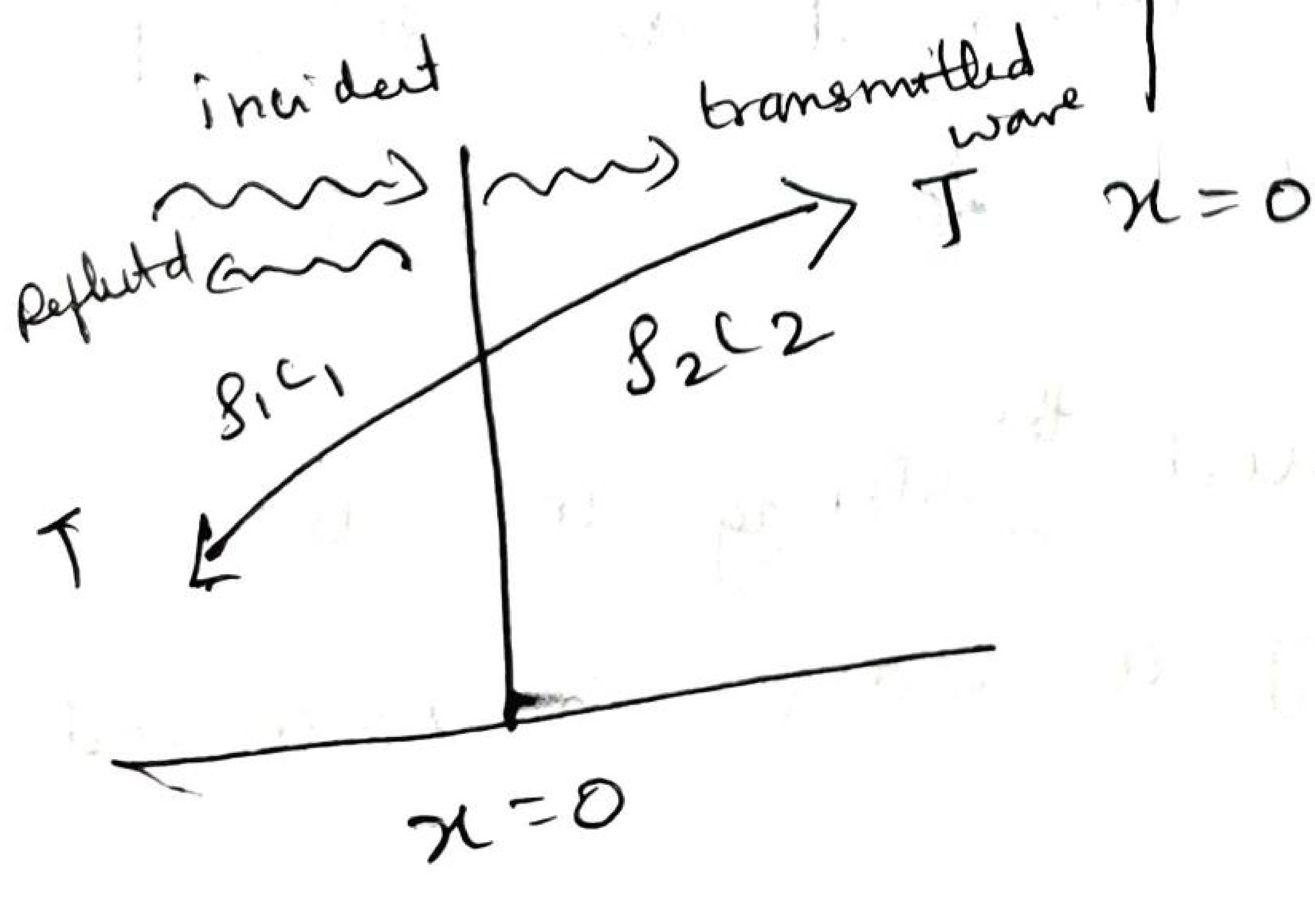
What would happen at the interface?

Let us consider a string (string 1 + string 2) that has two impedances.



Let us take the Tension (T) in the string to be uniform throughout the string. Since the linear mass densities are different, Velocities will also be different. Since we are considering monochromatic waves (angular frequency) ω will be same throughout.





$x = 0$ (boundary)

→ The string formed by joining two diff strings are ~~k~~ imposed upon them a ~~common~~ Constant Tension 'T' along the whole string.

→ The linear density of string 1 & string 2 are different & represented as ρ_1 & ρ_2 respectively.

→ The different ~~wave~~ velocities of two reg I & reg. II ~~Ca~~ (string 1 & Stri... are $\frac{T}{\rho_1} = c_1^2$ & $\frac{T}{\rho_2} = c_2^2$ respectively.

→ The impedance denoted as string 1 is $Z_1 = \rho_1 c_1$ and for string 2 is Z_2...

→ Now let us perturb the string in the reg 1. Consequently a wave is produce...

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and let this be incident wave that will travel along the reg I and will face a discontinuity in impedance at the pt $x=0$, as in figure.

Here at this point, $x=0$ part of the incident wave will be reflected in reg I and part of the wave will be transmitted in reg II having impedance $Z_2 = \rho_2 c_2$.

Thus the displacement of the waves in region I & II ~~~~ are defined as follows

The displacement of incident wave in reg with amp A_1 travelig in +ve x dir'n with velocity c_1 is

$$y_i = A_1 e^{i(\omega t - Kx)} \quad \text{(1)}$$

The displacement of reflected wave in reg I with amp B_1, travelling in $-x$ d with velocity c_1.

$$y_r = B_1 e^{i(\omega t + K_1 x)} \quad \text{--- (2)}$$

the displacement of transmitted wave in reg II with amp A_2, travelling in $+x$ dir, with velocity c_2.

$$y_t = A_2 e^{i(\omega t - K_2 x)} \quad \text{--- (3)}$$

Thus the reflections & transmission ampli (B_1 & A_2) w.r.to A_1 can be found by applying the boundary condition at $x=0$, where there is a discontinuity in impedance.

Thus the boundary cond'n at $x=0$.

(1) Geometrical cond'n → displacement is same in reg I & reg II (i.e) same immediately ~~at reg x~~ to the left and right at $x=0$ for all time, So there is no discontinuity of displace

$$y_i + y_r = y_t \quad \text{--- (4)}$$

(ii) <u>Dynamical cond'n</u>

→ transverse force $T\left(\frac{\partial y}{\partial x}\right)$ at $x=0$ is continuous & there is a continuous slope.

$$T\frac{\partial y_i}{\partial x} + T\frac{\partial y_r}{\partial x} = T\frac{\partial y_t}{\partial x} \longrightarrow (5)$$

Applying the boundary cond'n at $x=0$.

<u>cond'n (i)</u>

$$y_i + y_r = y_t$$

substituting (1), (2), (3) in eqn (4).

$$A_1 e^{i(\omega t - k_1 x)} + B_1 e^{i(\omega t + k_1 x)} = A_2 e^{i(\omega t - k_2 x)}$$

at $x = 0$.

$$A_1 \cancel{e^{i\omega t}} + B_1 \cancel{e^{i\omega t}} = A_2 \cancel{e^{i\omega t}}$$

$$A_1 + B_1 = A_2 \quad \text{--- (6)}$$

Applying the second boundary cond'n

$$\left.\frac{\partial y_i}{\partial x} = -A_1 K_1 e^{i(\omega t - k_1 x)}\right\}$$

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$$\left.\begin{aligned} \frac{\partial y_r}{\partial x} &= B_1 K_1 e^{i(\omega t + K_1 x)} \\ \frac{\partial y_t}{\partial x} &= -A_2 K_2 e^{i(\omega t - K_2 x)} \end{aligned}\right\} - (7)$$

Applying eqn (7) in (5)

$$-TA_1 K_1 e^{i(\omega t - K_1 x)} + TB_1 K_1 e^{i(\omega t + K_1 x)} = -TA_2 K_2 e^{i(\omega t - K_2 x)}$$

at $x = 0$

$$-TA_1 K_1 \cancel{e^{i\omega t}} + TB_1 K_1 \cancel{e^{i\omega t}} = -TA_2 K_2 \cancel{e^{i\omega t}}$$

$$-TA_1 K_1 + TB_1 K_1 = -TA_2 K_2 \quad - (8)$$

we know $K = \omega / c \quad - (9)$

Substituting (9) in (8)

$$-TA_1 \frac{\cancel{\omega}}{c_1} + TB_1 \frac{\cancel{\omega}}{c_1} = -TA_2 \cancel{\omega}/c_2$$

know $T/c_1 = \rho c_1 = Z_1$, $T/c_2 = \rho c_2 = Z_2$.

$$-Z_1 A_1 + Z_1 B_1 = -Z_2 A_2 .$$

$$Z_1 A_1 - Z_1 B_1 = Z_2 A_2$$

$$Z_1 (A - B) = Z_2 A_2 \quad \text{— (10)}$$

Using eqn (6) and eqn (10) ~~is~~ to cal the reflection & Transmission coeff.

we have

$$A_1 + B_1 = A_2 \quad \text{— (6)}$$

(10) $\Rightarrow$ $$Z_1 (A_1 - B_1) = Z_2 (A_1 + B_1)$$

$$Z_1 A_1 - Z_1 B_1 = Z_2 A_1 + Z_2 B_1$$

$$(Z_1 - Z_1) A_1 = (Z_1 + Z_2) B_1$$

$$\boxed{B_1 = \frac{(Z_1 - Z_2)}{(Z_1 + Z_2)} A_1}$$

Thus the reflection coeff of amplitude is

$$\boxed{\frac{B_1}{A_1} = \frac{Z_1 - Z_2}{Z_1 + Z_2}} \quad \text{— (11)}$$

Similarly from eqn (6)

$$B_1 = A_2 - A_1$$

substitut in eqn (10)

$$Z_1 (A_1 - A_2 + A_1) = Z_2 A_2$$

$$Z_1 (2A_1 - A_2) = Z_2 A_2$$

$$2 Z_1 A_1 - Z_1 A_2 = Z_2 A_2$$

$$2 Z_1 A_1 = (Z_1 + Z_2) A_2$$

$$\frac{A_2}{\cancel{A_2}} = \left(\frac{2Z_1 A_1}{Z_1 + Z_2}\right) \cancel{A_1}$$

Thus the transmission Coeff of amplitu

$$\boxed{\frac{A_2}{A_1} = \frac{2Z_1}{Z_1 + Z_2}} \quad \text{— (12)}$$

Thus the coeff (12) & (11) ~~an~~ are independent of 'w'. ~~and holds~~ Thus ratio's depends only upon the ratio of impedance.

Eqn (12) & (11) in terms of 'k'.

$$Z = \frac{T}{c} = \frac{TK}{\omega}$$

in this case T & W are ~~a~~ same

$$Z \propto K$$

$\therefore$

$$\frac{B_1}{A_1} = \frac{K_1 - K_2}{K_1 + K_2} \quad \& \quad \frac{A_2}{A_1} = \frac{2K_1}{K_1 + K_2}.$$

in terms of ρ

$$T = \rho c^2 = \rho \omega^2 / k^2$$

in this case T & w are const.

$$\therefore \ K \propto \sqrt{\rho}.$$

$$\frac{B_1}{A_1} = \frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}} \qquad \frac{A_2}{A_1} = \frac{2\sqrt{\rho_1}}{\sqrt{\rho_1} + \sqrt{\rho_2}}$$

In EM waves

$$Z = \frac{1}{n} = v/c$$

$\cancel{n = v/c}$ $\quad n = c/v$

n = R.I of medium