Engineering Physics

(PHY1701)

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Contents

- Physics of Divergence, Gradient and Curl (DJG 13-20),
- Qualitative understanding of surface and volume integral (DJG 24, 26, 27),
- Maxwell Equations (Qualitative) (DJG 232, 321-327),
- Wave Equation (Derivation) (DJG 364-366), &
- EM Waves, Phase velocity, Group velocity, Group index*, Wave guide (Qualitative) (DJG 405)

❖ William Silfvast, Laser Fundamentals, 2008, Cambridge University Press.

Del Operator

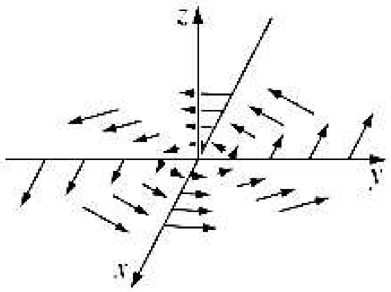
- One can thus define a vector operator "Del" which is defined as:

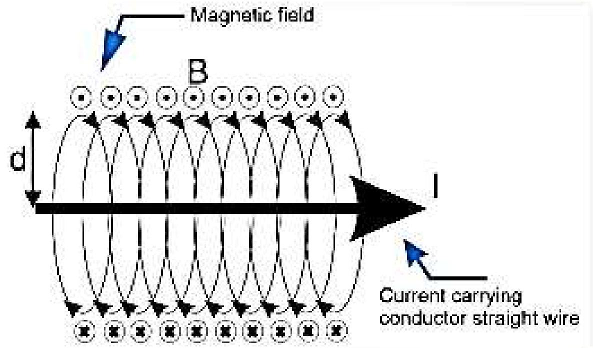
$$\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

- This vector operator can operate on a scalar or vector function
  - $\vec{\nabla} S$: Gradient
  - $\vec{\nabla} \cdot \vec{A}$: Divergence
  - $\vec{\nabla} \times \vec{A}$: Curl

Curl

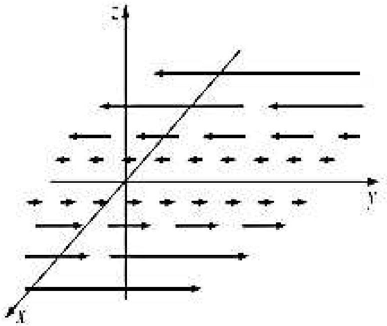
- Curl of a vector function gives a measure of how much it curls or swirls around a point.
- Example of non-zero curl : A vortex, Magnetic field of a current carrying conductor.





- For $\vec{A} = \vec{r}$, $\vec{\nabla} \times \vec{A} = 0.$

- For $\vec{A} = x\,\hat{y}$, $\vec{\nabla} \times \vec{A} = \hat{z}.$

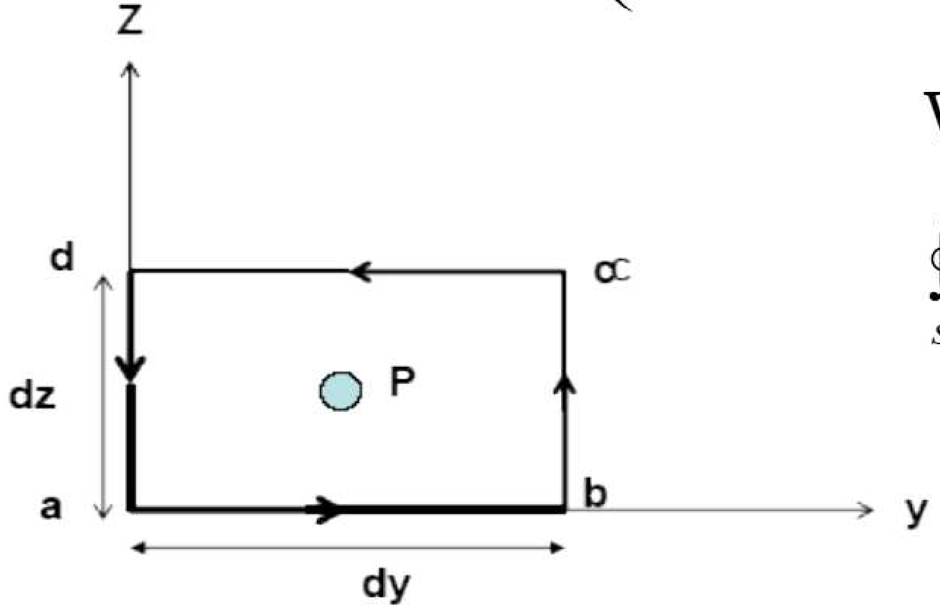


Curl

The curl of vector **A** is an axial (rotational) vector whose magnitude is the maximum circulation of **A** per unit area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.

Curl

$$Curl\ \mathbf{A} = \nabla \times \mathbf{A} = \left(\lim_{\Delta s \to 0} \frac{\oint_s \mathbf{A} \bullet dl}{\Delta s} \right) \mathbf{a}_n \ \max$$

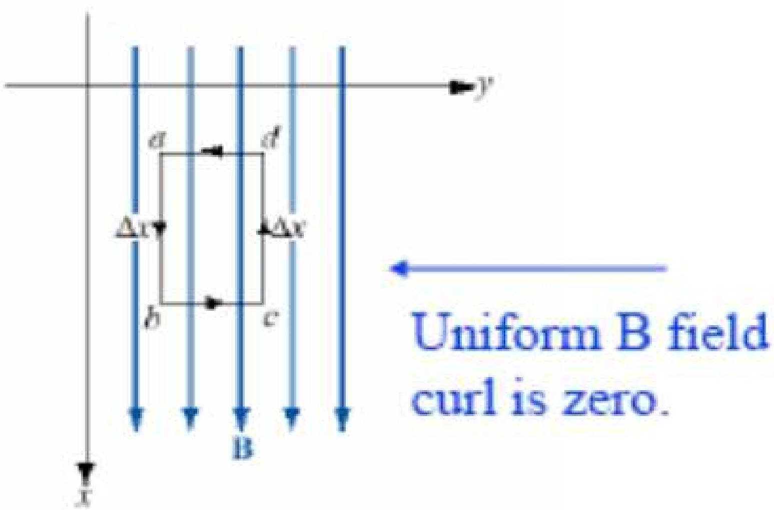


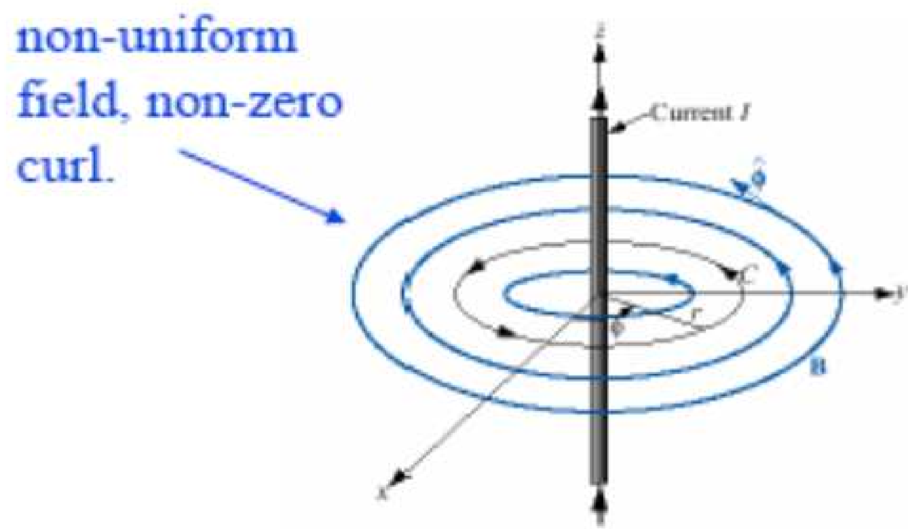
Where,

$$\oint_s \mathbf{A} \bullet dl = \left(\int_{ab} + \int_{bc} + \int_{cd} + \int_{da} \right) \mathbf{A} \bullet dl$$

Curl

The curl of the vector field is concerned with rotation of the vector field. Rotation can be used to measure the uniformity of the field, the more non uniform the field, the larger value of curl.





For Cartesian coordinate:

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

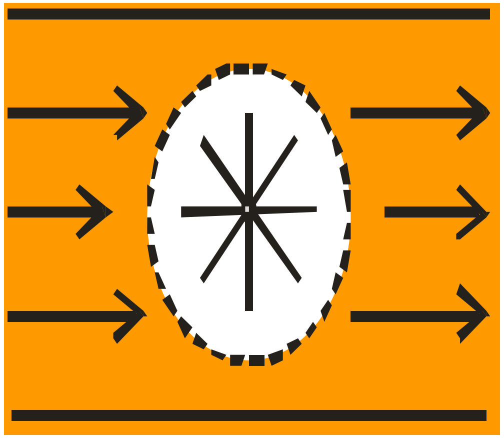
$$\nabla \times \mathbf{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\mathbf{a}_x - \left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right]\mathbf{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\mathbf{a}_z$$

Curl of a Vector field

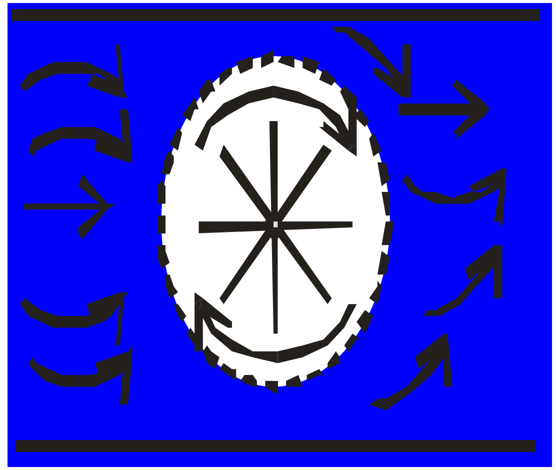
$$\text{Curl } \mathbf{V} = \nabla \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

- Physically, the curl of a vector field represents the rate of change of the field strength in a direction at right angles to the field and is a measure of rotation of something in a small volume surrounding a particular point.
- For streamline motions and conservative fields, the curl is zero while it is maximum near the whirlpools

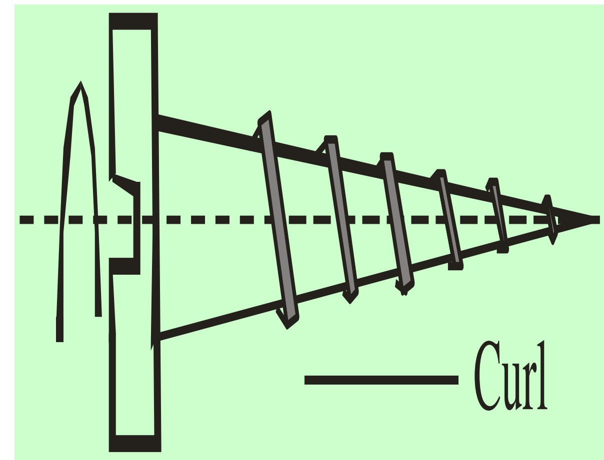
Representation of Curl



(I) No rotation of the paddle wheel represents zero curl



(II) Rotation of the paddle wheel showing the existence of curl



(III) direction of curl

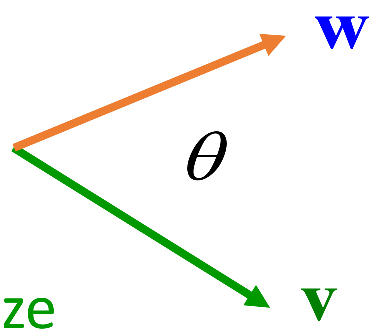
➢For vector fields whose curl is zero there is no rotation of the paddle wheel when it is placed in the field, Such fields are called *irrotational*

Dot Products

<u>There are two ways to multiply two vectors</u>

- The dot product produces a scalar quantity
  - It has no direction
  - It can be pretty easily computed from geometry
  - It can be easily computed from components

$$\boxed{\mathbf{v}\cdot\mathbf{w} = vw\cos\theta = v_x w_x + v_y w_y + v_z w_z}$$



- The dot product of two unit vectors is easy to memorize

$$\boxed{\begin{aligned} \hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} &= 1 \\ \hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{i}} &= 0 \\ \hat{\mathbf{i}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{i}} &= 0 \\ \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{j}} &= 0 \end{aligned}}$$

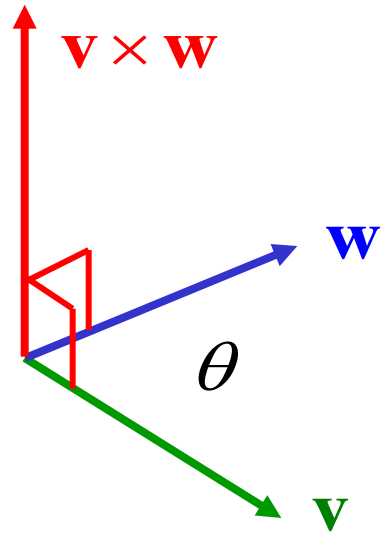
- The dot product is commutative

$$\boxed{\mathbf{v}\cdot\mathbf{w} = \mathbf{w}\cdot\mathbf{v}}$$

Cross Products

The cross product produces a vector quantity

- It is perpendicular to both vectors
- Requires the right-hand rule
- Its magnitude can be easily computed from geometry
- It is a bit of a pain to compute from components



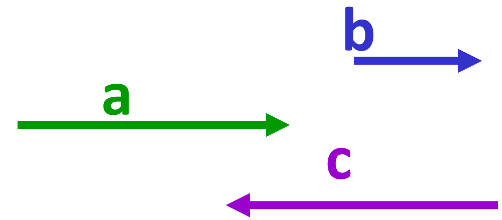
$$\boxed{\left|\mathbf{v}\times\mathbf{w}\right| = vw\sin\theta}$$

$$\boxed{\begin{aligned}\mathbf{v}\times\mathbf{w} = \det\begin{pmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_x & v_y & v_z \\ w_x & w_y & w_z\end{pmatrix} &= \left(v_y w_z - v_z w_y\right)\hat{\mathbf{i}} + \left(v_z w_x - v_x w_z\right)\hat{\mathbf{j}} \\ &\quad + \left(v_x w_y - v_y w_x\right)\hat{\mathbf{k}}\end{aligned}}$$

Simple Rules for Cross-Products

- Vectors that are parallel or anti-parallel have zero cross product

$$\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}=\mathbf{b}\times\mathbf{c}=0$$



- Cross products are anti-symmetric

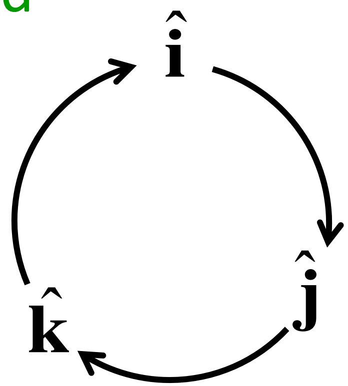
$$\mathbf{v}\times\mathbf{w}=-\mathbf{w}\times\mathbf{v}$$

Basis vectors:

- Any vector with itself gives zero

$$\hat{\mathbf{i}}\times\hat{\mathbf{i}}=\hat{\mathbf{j}}\times\hat{\mathbf{j}}=\hat{\mathbf{k}}\times\hat{\mathbf{k}}=0$$

- Think of ijk as a circle: any two <u>in order</u> gives the third
- Any two in <u>reverse order</u> gives minus the third



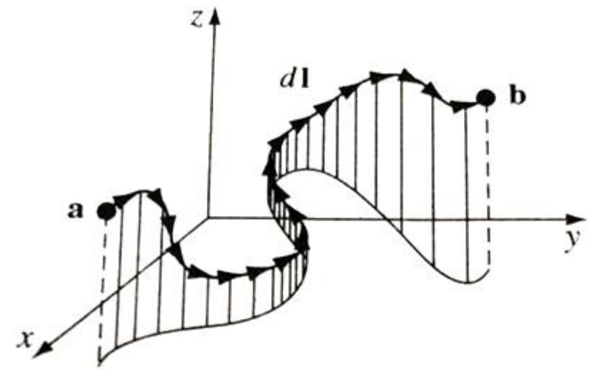
$$\hat{\mathbf{i}}\times\hat{\mathbf{j}}=\hat{\mathbf{k}} \qquad \hat{\mathbf{j}}\times\hat{\mathbf{i}}=-\hat{\mathbf{k}}$$

$$\hat{\mathbf{j}}\times\hat{\mathbf{k}}=\hat{\mathbf{i}} \qquad \hat{\mathbf{k}}\times\hat{\mathbf{j}}=-\hat{\mathbf{i}}$$

$$\hat{\mathbf{k}}\times\hat{\mathbf{i}}=\hat{\mathbf{j}} \qquad \hat{\mathbf{i}}\times\hat{\mathbf{k}}=-\hat{\mathbf{j}}$$

Line Integral

- In electrodynamics, we encounter several different kinds of integrals, such as
  - ✓ Line (path) integrals
  - ✓ Surface (flux) integrals
  - ✓ Volume integrals



- **Line Integrals**: A line integrals is an expression of the form

$$\int_a^b \mathbf{v} \cdot d\mathbf{l}$$

Here, 'v' is a vector function, 'dl' is the infinitesimal displacement vector
The integral to be carried out along a prescribed path '*p*' from point a to point b.

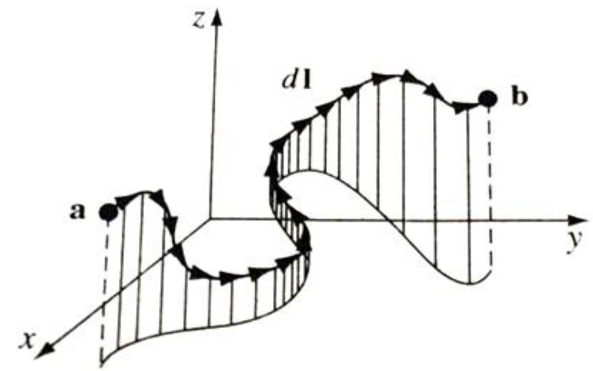
- If the path in question forms a closed loop (i.e., if b=a), we can put a circle on the integral sign:

$$\oint \mathbf{v} \cdot d\mathbf{l}$$

Line Integral

- At each point on the path, we take the dot product of ‘v’ with the displacement ‘dl’ to the next point on the path.
- To a physicist, the most familiar example of a line integral is the work done (W) by a force (F):

$$W = \int \mathbf{F} \cdot \mathbf{dl}$$

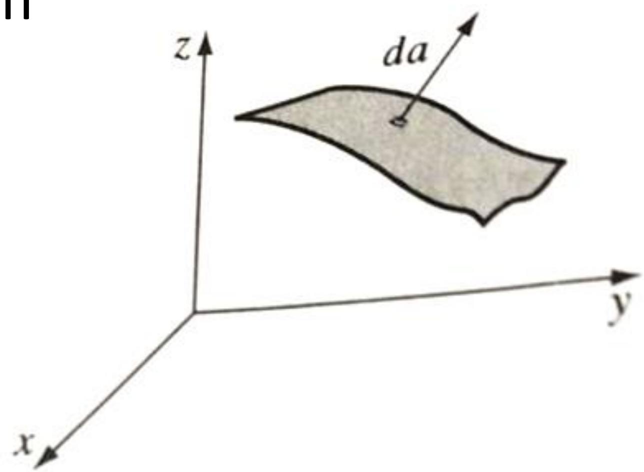


- Generally, the value of a line integral depends critically on the path taken from **a** to **b**.
- However, there is an important special class of vector functions for which the line integral *independent* of path and is determined by the end points.
- Hence, we have to characterize this special class of vectors.

Surface Integral

- A surface integral is an expression of the form

$$\int_S \mathbf{v} \cdot d\mathbf{a}$$



  - ✓ Here, 'v' is again some vector function, and the integral over a specified surface '*S*'.
  - ✓ 'da' is an infinitesimal patch of area with the direction perpendicular to the surface.

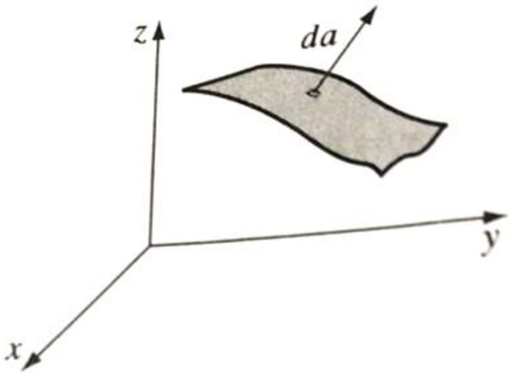
- There are, of course, *two* directions perpendicular to any surface, so the sign of a surface is intrinsically ambiguous.
- If the surface is closed, in which case we shall again put a circle on the integral sign:

$$\oint \mathbf{v} \cdot d\mathbf{a}$$

- Then the tradition dictates that "outward" is positive, but for open surface it's arbitrary.

Surface Integral

- If 'v' describes the flow of a fluid, then $\int \mathbf{v} \cdot \mathbf{da}$



represents the total mass per unit time passing through the surface...., hence the alternative name, "*flux*".

- Generally, the value of a surface integral depends on the particular surface chosen.
- However, there is a special class of vector functions for which it is *independent* of the surface and is determined entirely by the boundary line.
- An important task will be to characterize this special call of functions.

Volume Integral

- A volume integral is an expression of the form

$$\int_{v} T\,d\tau$$

  - ✓ Here, '***T***' is a scalar function and **dτ** is an infinitesimal *volume* element.

$$d\tau = dx\,dy\,dz$$

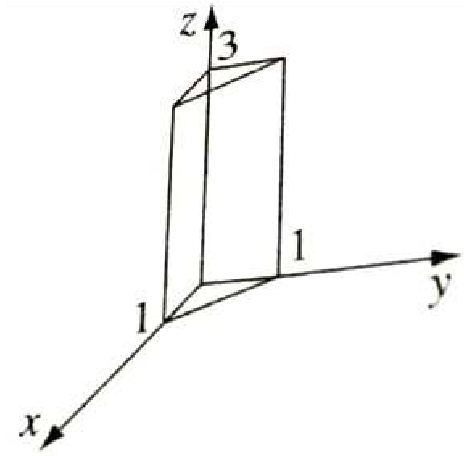
- For example, if T is the density of a substance, then the volume integral would give the total mass.
- Occasionally, we shall encounter volume integrals of vector functions:

$$\int \mathbf{v}\,d\tau = \int (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}})\,d\tau = \hat{\mathbf{x}}\int v_x\,d\tau + \hat{\mathbf{y}}\int v_y\,d\tau + \hat{\mathbf{z}}\int v_z\,d\tau$$

- Since, the unit vectors ($\hat{x}$, $\hat{y}$, and $\hat{z}$) are constants, they come outside the integral.

Volume Integral- Example

- **Calculate the volume integral of $T = xyz^2$ over the prism as shown in figure?**



Sol: We can do the three integrals in any order.

Lets do **x** first: it runs from 0 to (1-y),

then **y** (it goes from 0 to 1),

and finally **z** (0 to 3)

$$\int_v T\,d\tau = \int_0^3 z^2 \left\{ \int_0^1 y \left[\int_0^{1-y} x\,dx \right] dy \right\} dz$$

$$= \frac{1}{2}\int_0^3 z^2 dz \int_0^1 (1-y)^2\, y\,dy$$

$$= \frac{1}{2}(9)\left(\frac{1}{12}\right)$$

$$= \frac{3}{8}$$

- Scalar refers to a quantity whose value may be represented by a single (positive or negative) real number.
- Some examples include distance, temperature, mass, density, pressure, volume, and time.

- A vector quantity has both a magnitude and a direction in space. We especially concerned with two- and three-dimensional spaces only.
- Displacement, velocity, acceleration, and force are examples of vectors.

  - Scalar notation: *A* or A (*italic* or plain)
  - Vector notation: **A** or $\vec{A}$ (bold or plain with arrow)

A vector field is said to be Solenoidal if it can be expressed as curl of some other vector field.
i.e, Vector S = Curl of a vector A ....(1)
Taking the divergence of equation (1)
We can get, div S = div Curl A
Since divergence of curl of a vector function is always zero ,hence divergence of solenoidal vector field is zero.
One may ask Why div of curl of a vector field is zero. This is because curl is line integral per unit area along a closed path, whereas div is net outgoing flux per unit volume. So along closed path net outgoing will be zero. e.g div of magnetic field B is zero.