Engineering Physics

(PHY1701)

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Contents

- Physics of Divergence, Gradient and Curl (DJG 13-20),
- Qualitative understanding of surface and volume integral (DJG 24, 26, 27),
- Maxwell Equations (Qualitative) (DJG 232, 321-327),
- **Wave Equation (Derivation) (DJG 364-366), &**
- EM Waves, Phase velocity, Group velocity, Group index*, (DJG 405)
- Hertz Experiment

❖ William Silfvast, Laser Fundamentals, 2008, Cambridge University Press.

Electromagnetic Wave equation

The Maxwell's equation from Faraday's law is given by,

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$= -\mu \frac{\partial H}{\partial t}$$

Take curl on both sides,

$$\nabla \times \nabla \times E = -\mu \nabla \times \frac{\partial H}{\partial t} \quad \text{... (4.1)}$$

But Maxwell's equation from Ampere's law is

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$

$$= \sigma E + \varepsilon \frac{\partial E}{\partial t}$$

Electromagnetic Wave equation

Differentiating
$$\nabla \times \frac{\partial H}{\partial t} = \frac{\partial}{\partial t}(\nabla \times H) = \frac{\partial}{\partial t}\left(J + \frac{\partial D}{\partial t}\right) = \frac{\partial}{\partial t}\left[\sigma E + \varepsilon \frac{\partial E}{\partial t}\right]$$

$$\nabla \times \frac{\partial H}{\partial t} = \sigma \frac{\partial E}{\partial t} + \varepsilon \frac{\partial^2 E}{\partial t^2} \qquad \ldots (4.2)$$

Substituting the equation (4.2) in equation (4.1)

$$\nabla \times \nabla \times E = -\mu\left[\sigma \frac{\partial E}{\partial t} + \varepsilon \frac{\partial^2 E}{\partial t^2}\right]$$

$$= -\mu\sigma \frac{\partial E}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} \qquad \ldots (4.3)$$

But according to the identity

$$\nabla \times \nabla \times E = \nabla(\nabla \bullet E) - \nabla^2 E \qquad \ldots (4.4)$$

But $\quad \nabla \bullet E = \frac{1}{\varepsilon} \nabla \bullet D$

Electromagnetic Wave equation

Since there is not net charge within the conductor, the charge density $\rho = 0$.

$$\nabla \bullet D = 0$$

$$\nabla \bullet E = 0$$

Then equation (4.4) becomes

$$\nabla \times \nabla \times E = -\nabla^2 E \qquad \text{... (4.5)}$$

Comparing the equations (4.3) and (4.5)

$$\nabla^2 E = -\mu\sigma \frac{\partial E}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2}$$

$$\boxed{\nabla^2 E - \mu\sigma \frac{\partial E}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0} \qquad \text{... (4.6)}$$

This is the wave equation for electric field E.

Electromagnetic Wave equation

The wave equation for Magnetic field H is obtained in a similar manner as follows.

The Maxwell's equation from Ampere's law is given by,

$$\nabla \times H = \sigma E + \varepsilon \frac{\partial E}{\partial t}$$

Take curl on both sides,

$$\nabla \times \nabla \times H = \sigma \nabla \times E + \varepsilon \nabla \times \frac{\partial E}{\partial t} \qquad \ldots (4.7)$$

But Maxwell's equation from Faraday's law

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}$$

Differentiating,

$$\nabla \times \frac{\partial E}{\partial t} = -\mu \frac{\partial H^2}{\partial t^2}$$

Electromagnetic Wave equation

Substituting the values of $\nabla \times E$ and $\nabla \times \frac{\partial E}{\partial t}$ in equation (4.7)

$$\nabla \times \nabla \times H = -\mu\sigma \frac{\partial H}{\partial t} - \mu\varepsilon \frac{\partial^2 H}{\partial t^2} \qquad \text{... (4.8)}$$

But the identity is

$$\nabla \times \nabla \times H = \nabla(\nabla \bullet H) - \nabla^2 H$$

But $\qquad \nabla \bullet B = \mu\nabla \bullet H = 0$

Then, $\qquad \nabla \times \nabla \times H = \nabla^2 H \qquad \text{... (4.9)}$

Electromagnetic Wave equation

Comparing the equations (4.8) and (4.9)

$$\nabla^2 H = -\mu\sigma\frac{\partial H}{\partial t} - \mu\varepsilon\frac{\partial^2 H}{\partial t^2}$$

$$\boxed{\nabla^2 H - \mu\sigma\frac{\partial H}{\partial t} - \mu\varepsilon\frac{\partial^2 H}{\partial t^2} = 0} \qquad \ldots (4.10)$$

This is the wave equation for magnetic field H.

Wave equation for free space

For free space (dielectric medium) the conductivity of medium is zero ($\sigma=0$) and there is no charge containing in it ($\rho=0$). Then, the electromagnetic equations can be obtained as,

$$\boxed{\nabla^2 E - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0} \quad (1) \qquad \boxed{\nabla^2 H - \mu\varepsilon \frac{\partial^2 H}{\partial t^2} = 0} \quad (2)$$

For free space $\mu_r=1$ and $\varepsilon_r=1$ (air)
Then the wave equation becomes,

$$\boxed{\nabla^2 H - \frac{1}{c^2} \frac{\partial^2 H}{\partial t^2} = 0} \quad \text{or} \quad \boxed{\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0}$$

Eqs. (1) & (2) represents the relation between space and time variation of magnetic field B and electric field E. These are called wave equations for B and E respectively.

The general form of differential eq. of wave motion is represented by

$$\nabla^2 y = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Where v is the velocity of wave and y is its amplitude,

Comparing Eq. (1) &(2) with Eq. (3), we get

$$\mu\varepsilon = \frac{1}{v^2} \Rightarrow v = \frac{1}{\sqrt{\mu\varepsilon}}$$

This is the expression for velocity of electromagnetic wave. The velocity of EM wave in free space is given by

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Where μ_0=4π x 10^{-7} H/m, ε_0=8.85 x 10^{-12} Nm^2/C^2

$$\therefore \quad v = \frac{1}{\sqrt{4 \quad \times 10^{-7} \times 8.85 \times 10^{-12}}} = 3 \times 10^{-8} \quad m/s$$

Thus the EM waves propagate with the velocity equal to the velocity of light in free space.