Definitions:

Phase velocity: The phase velocity of a wave is the rate at which the phase of the wave (Individual wave) propagates in space.

Group velocity: The rate at which the envelope of the wave packet (group of waves) propagates in space.

Phase Velocity:

Consider a wave whose displacement y is expressed as,

$$y = A\,cos(\omega t - kx)$$

where A-Amplitude, $\boldsymbol{\omega}$-angular frequency, and $\boldsymbol{k}$-propagation constant

The speed of propagation of this wave will be the speed associated with a point for which the phase $(\boldsymbol{\omega t - kx})$ is constant.

$$(\boldsymbol{\omega t - kx}) = \text{constant}$$

or $$\boldsymbol{x} = \text{constant} + \frac{\boldsymbol{\omega}}{\boldsymbol{k}}\,t$$

Therefore, **Phase velocity** $\boldsymbol{v}_{\mathrm{P}} = \frac{dx}{dt} = \frac{\boldsymbol{\omega}}{\boldsymbol{k}} = \frac{\boldsymbol{2\pi v}}{\boldsymbol{2\pi/\lambda}}$

$$\boldsymbol{v}_{\mathrm{P}} = \boldsymbol{v.\lambda}$$

v_p is the phase velocity for a wave.

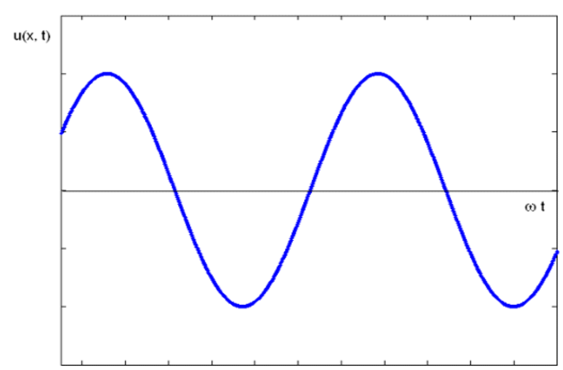

Figure: Harmonic Wave

Group Velocity:

The **Group velocity** is given by $\boldsymbol{v}_g = \frac{d\omega}{dk}$

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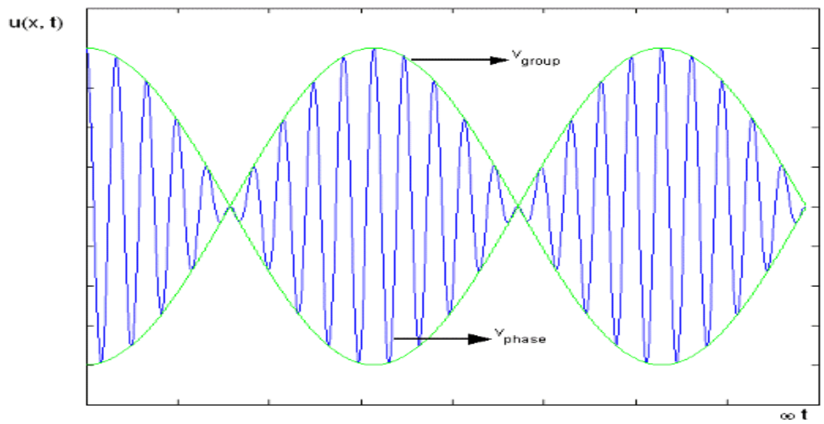

Figure : Group Velocity

Phase velocity of de Broglie waves:

The Phase velocity of de Broglie waves is,

$$v_P = v.\lambda = \left(\frac{mc2}{h}\right)\left(\frac{h}{mv}\right) = \frac{c2}{v}$$

Here velocity v is always less than c. Therefore, the de Broglie wave velocity must be greater than c.

As v is less than c and hence, the de Broglie wave velocity must be greater than c (unexpected result). So, the de Broglie wave train associated with the particle would travel much faster that particle itself and would leave the particle far behind. This statement is nothing but the collapse of the wave description of the particle.

Group velocity of de Broglie waves:

The angular frequency and propagation constant of de Broglie waves associated with the particles of mass m_0 can be calculated as follows,

$$\omega = 2\pi\nu$$

$$\omega = 2\pi\frac{E}{h}$$

$$\omega = 2\pi\frac{mc^2}{h}$$

$$k = \frac{2\pi}{\lambda}$$

$$k = \frac{2\pi}{h}mv$$

$$\boxed{\omega = 2\pi \frac{m_0 c^2}{h\sqrt{1-\frac{v^2}{c^2}}}}$$

$$\boxed{k = \frac{2\pi}{h}\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}}$$

According to special theory of relativity, $\mathrm{m} = \frac{m_o}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}$

$$v_g = \frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv}$$

$$\boxed{\omega = \frac{2\pi}{h}\frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}}$$

$$d\omega/dv = \frac{2\pi m_0 c^2}{h}\left(-\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left[-2v/c^2\right]$$

$$\boxed{d\omega/dv = \frac{2\pi}{h}\frac{m_0 v}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}}$$

$$\boxed{k = \frac{2\pi}{h}\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}}$$

$$dk/dv = \frac{2\pi m_0}{h}\left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} + \left(-\frac{1}{2}\right)\left(1-(v/c)^2\right)^{-3/2}\left[-2v/c^2\right]\right]$$

$$\boxed{dk/dv = \frac{2\pi m_0}{h}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}}$$

$$dk/dv = \frac{2\pi m_0}{h}\left(1-(v/c)^2\right)^{-3/2}\left[1-\frac{v^2}{c^2}+\frac{v^2}{c^2}\right]$$

$$v_g = \frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv}$$

$$\boxed{d\omega/dv = \frac{2\pi}{h}\frac{m_0 v}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}}$$

$$\boxed{dk/dv = \frac{2\pi m_0}{h}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}}$$

$$\Rightarrow v_g = v$$

Hence, the de Broglie wave group associated with a moving particle travels with the same velocity as the particle.

The relation between the group and phase velocity:

The relation between the group and phase velocity is given by

$$\boxed{v_g = v_p - \lambda \left(\frac{dv_p}{d\lambda}\right)}$$

Derivation of this relation:

$$\omega = v_p k \quad ; \quad v_g = \frac{d\omega}{dk} \quad ; \quad v_g = \frac{d(v_p k)}{dk}$$

$$v_g = v_p + k\frac{dv_p}{dk} \quad ; \quad v_g = v_p + \frac{k}{dk}dv_p$$

Since,

$$k = 2\pi/\lambda$$

$$dk = -\frac{2\pi}{\lambda^2}d\lambda$$

$$\frac{k}{dk} = \frac{2\pi/\lambda}{-\frac{2\pi}{\lambda^2}d\lambda}$$

$$\frac{k}{dk} = -\frac{\lambda}{d\lambda}$$

$$v_g = v_p - \lambda\frac{dv_p}{d\lambda}$$

This gives us the relation between Phase velocity and Group velocity

From above equation, the following two cases arise.

1. For dispersive medium, $v_p = f(\lambda)$. Usually $dv_p/d\lambda$ is positive (normal dispersion)

$$\mathbf{v_g < v_p}$$

This is the case with de Broglie waves.

2. For non-dispersive medium, $v_p \neq f(\lambda)$, $dv_p/d\lambda$ is zero. Then

$$\mathbf{v_g = v_p}$$

This is true for electromagnetic waves in vaccum.