M 5: EM Theory

Introduction

The basic laws of electricity and magnetism can be summarized in differential form by four equations

Gauss's law (Electrostatics) $\nabla.E = \rho / \varepsilon_0$ -----(1)

Gauss's law (magnetism) $\nabla.B = 0$ -------(2)

Ampere's law $\nabla \times B = \mu_0 J$ (3)

Faraday's law of Induction $\nabla \times E = -\partial B / \partial t$ (4)

These equations are written for fields in vacuum, in the presence of electric charge of density ρ, and electric current of density J. In the presence of simple dielectrics, ε_0 is replaced by ε in eq. (1). If simple ferroelectric materials are present μ_0 is replaced by μ in Eq. (3).

The values of B, J and D are related to the electric and magnetic fields as follows,

$$E = \mu H$$
$$D = \varepsilon E$$
$$J = \sigma E$$

Where μ, ε, and σ are permeability, permittivity, and conductivity of the medium.

FUNDAMENTAL LAWS OF ELECTROMAGNETISM:

1. **Gauss law of electrostatics:** This law states that "the total electric flux (ϕ) emerging out of closed surface is equal to $1/\varepsilon_0$ times the net charge 'q' enclosed by the surface." or the surface integral of the normal component of electric field E over any closed surface is equal to the net charge enclosed within that volume.

   i.e., $\phi_E = \oint_S E.dS = \dfrac{q}{\varepsilon_0}$

2. **Gauss law of Magneto statics:** This law states that the magnetic flux through any closed surface is zero.

i.e., $\phi_B = \oint_S B.dS = 0$

where ϕ_B is magnetic flux and B is magnetic flux density.

3. **Faraday's Law of electromagnetic induction:** This law states that whenever the magnetic lines of force are cut by a closed circuit, an induced current or the induced emf is developed in the circuit. This is called electromagnetic induction.

   The magnitude of the induced emf is directly proportional to the negative rate of change of magnetic flux linked with the circuit. If ϕ_B be the magnetic flux linked with the circuit at any instant 't' and 'e' be the induced emf, then

$$e = -\frac{d\phi_B}{dt} \qquad or \qquad \int E.dl = -\frac{d\phi_B}{dt}$$

4. **Ampere's Law:** This law states that the amount of work done in carrying a unit magnetic pole one around a closed arbitrary path linked with the current is μ_0 times the current 'i'.

$$\int B.dl = \mu_0 i$$

   This is Ampere's law for the magnetic field due to steady current.

DISPLACEMENT CURRENT:

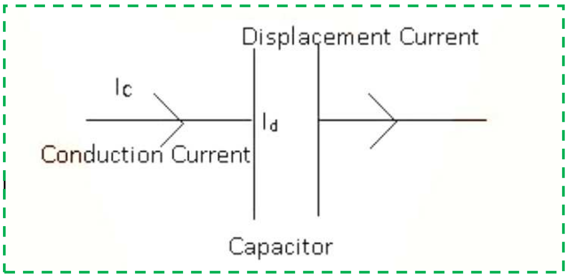
Maxwell Experiments:

- ♦ Maxwell proposed that the time varying electric field can generate magnetic field.
- ♦ Time varying magnetic field generates electric field (Faraday-Lenz law).
  - ◊ According to Faraday Lenz law an EMF is induced in the circuit whenever the amount of magnetic flux linked with a circuit changes.
  - ◊ As a result electric current gets generated in the circuit which has an electric field associated with it.

- According to Maxwell if Faraday's law is true then the vice-versa should also be true, i.e. a time varying electric field should also be able to generate a magnetic field.

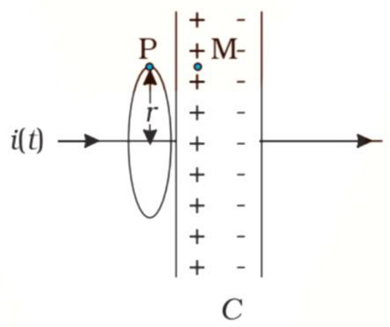
Existence of Displacement current by Maxwell:

- Consider a capacitor and outside the plates of the capacitor there is conduction current I_C.
- Area between the plates i.e. inside the capacitor there is displacement current I_d.
- Physical behaviour of displacement current is same as that of induction current.
- Difference between Conduction current and Displacement current:-
  - Conduction Current - It arises due to the fixed charges.
  - Displacement Current - It arises due to the change in electric field.
- For Static electric fields:- $I_d=0$.
- For time varying electric fields:- $I_d \neq 0$.
- There can be some scenarios where there will be only conduction current and in some case there will be only displacement current.
- Outside the capacitor there is only conduction current and no displacement current.
- Inside the capacitor there is only displacement current and no conduction current.
- But there can be some scenario where both conduction as well as displacement current is present i.e. $I = I_C + I_d$.
- Applying modified Ampere-Maxwell law to calculate magnetic field at the same point of the capacitor considering different amperial loop, the result will be same.



Maxwell considered 3 different amperial loops as shown in the figures. Ampere's circuital law should be same for all the 3 setups.

Case 1: Considered a surface of radius r & dl is the circumference of the surface, then from Ampere's circuital law



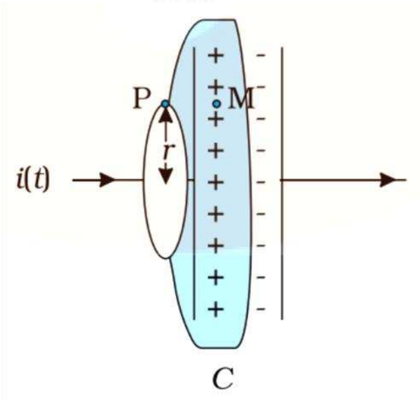
$$\int B.\,dl = \mu_0 i$$

$$or\ (2\pi r) = \mu_0\ i$$

$$or\ B = \mu_0 i\,/\,2\pi r$$

Case 2 : Considering a surface like a box & its lid is open and applying the Ampere's circuital law

Pot-like surface



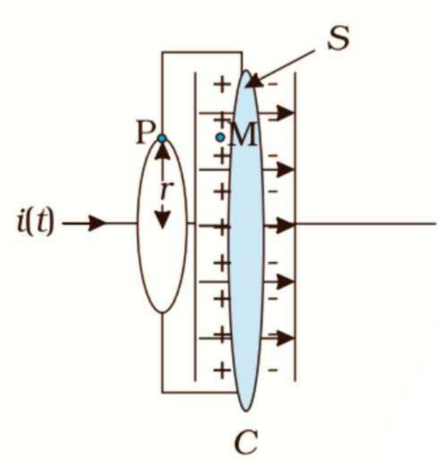
$$\int B.\,dl = \mu_0 i$$

As there is no current flowing inside the capacitor, therefore I = 0,

(or) $\int B.\,dl = 0$

Case 3: Considering the surface between 2 plates of the capacitor, in this case also I=0, so B=0

Tiffin-shaped surface



At the same point but with different amperial surfaces the value of magnetic field is not same. They are different for the same point.

Maxwell suggested that there are some gaps (inconsistency) in the Ampere's circuital law. He corrected the Ampere's circuital law and he made Ampere's circuital law consistent in all the scenarios.

EQUATION OF CONTINUITY:

The motion of electric charge constitute current. The flow of charge in unit time is called current, denoted by I,

i.e., $I = -\frac{dQ}{dt}$ *(∵ If Q is the charge inside a closed surface, the rate of decrease of charge due to outward flow of current is given this formula)*

If ρ is the charge density, then total charge enclosed in the surface is

$$Q = \oint_V \rho dV \qquad \left(\because \rho = \frac{dQ}{dV} \Rightarrow dQ = \rho dV \Rightarrow Q = \int_V \rho dV \right)$$

The current flows from the volume must be equal to the rate of flow of charge with the volume. Therefore

$$I = -\frac{d}{dt}\oint_V \rho dV \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (1)}$$

Let us assume that the current is extended in space of volume ***V***, enclosed by the surface ***S***. The net amount of charge which crosses a unit area normal to the surface in unit time is called current density ***J***. It is related to the total current I flowing through the surface ***S*** as

$$I = \oint_S \bar{J}.dS \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (2)}$$

From Eq. (1) & (2)

$$\oint_S \bar{J}.dS = -\frac{d}{dt}\oint_V \rho dV$$

$$\oint_S \bar{J}.dS = -\oint_V \frac{d\rho}{dt} dV \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (3)}$$

By Gauss Divergence Theorem

$$\oint_S \bar{J}.dS = \oint_V (\nabla.J)dV \qquad \text{\_\_\_\_\_\_\_\_\_\_\_ (4)}$$

Therefore From Eq. (3) & (4)

$$\int_V (\nabla.J)dV = \int_V -\frac{d\rho}{dt} dV$$

$$\boxed{\nabla.\boldsymbol{J} = -\frac{d\rho}{dt}} \quad \text{or} \quad \boxed{\nabla.J + \frac{d\rho}{dt} = 0} \qquad \text{\_\_\_\_\_ (5)}$$

This is the Equation of continuity for time varying fields.

Modified form of Ampere's Law:

Ampere's law $\nabla \times B = \mu_0 J$ does not hold good for time-varying fields and this renders the above set of equations inconsistent. To remove this difficulty, Maxwell modified the above equation by introducing the concept of displacement current. Putting B = μ_0 H, Eq. (3) becomes

$$\nabla \times H = J$$

Taking divergence of this equation, $\nabla.(\nabla \times H) = \nabla.J$

But the divergence of curl of a vector is always zero, $\nabla.J = 0$

This means that the divergence of current density is zero. This is not true for time varying fields.

From Gauss's law, $\nabla.E = \rho / \varepsilon_0$

Differentiating, $\nabla.\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0}\frac{\partial \rho}{\partial t}$

or $$\varepsilon_0 \nabla.\frac{\partial E}{\partial t} = \frac{\partial \rho}{\partial t}$$

Add $\nabla.J$ to both sides, we have

$$\nabla.J + \varepsilon_0 \nabla.\frac{\partial E}{\partial t} = \nabla.J + \frac{\partial \rho}{\partial t}$$

According to general equation of continuity, $\nabla.J + \frac{\partial \rho}{\partial t} = 0$

$$\nabla.J + \varepsilon_0 \nabla.\frac{\partial E}{\partial t} = 0$$

or $\nabla.(J+\frac{\partial D}{\partial t})=0$ ($D=\varepsilon_0 E$)

$(J+\frac{\partial D}{\partial t})$ is the total current density. Maxwell pointed out that this should replace J in Ampere's law. Hence the modified form of Ampere's law is, $\nabla\times H=J+\frac{\partial D}{\partial t}$ (5)

The term $\frac{\partial D}{\partial t}$ is called displacement current. It is the time rate of change of the electric displacement. The second term is better termed as displacement current density J_d. it shall have the same effect as the true current but shall be effective only when E is a time-varying field. Therefore, Eq. (5) becomes,

$$\nabla\times H=(J+J_d)$$

Alternate method of Modified form of Ampere's Law:

According to Maxwell there was some inconsistency in the Ampere's circuital law. Since Ampere's law is valid only for static fields and not for varying fields.

Maxwell concluded that the Eq. $\nabla\times B=\mu_0 J$ is incomplete. He suggested that some quantity must be added to J of the above Equation such that the divergence of both sides is same. Thus

$$\nabla\times B=\mu_0 J+\textit{some quantity}$$

To find this Maxwell postulated that similar to the electric field due to change magnetic field, there would be a magnetic field due change in electric field. Thus a changing electric field is equivalent to a current which flows as long as the electric field is changing and produces the same magnetic effect as an ordinary conduction current. This is known as displacement current.

In vector form the Gauss law of electrostatics can be expressed as, $\nabla . D = \rho$

Differentiating this with respect to ***t***, we get

$$\frac{\partial}{\partial t}(\nabla . D) = \frac{\partial \rho}{\partial t}$$

$$\nabla . \frac{\partial D}{\partial t} = \frac{\partial \rho}{\partial t} \qquad \text{------------------} \quad (6)$$

Adding ∇.J to the above Eq. on both sides and re arranging we get

$$\nabla . J + \frac{\partial \rho}{\partial t} = \nabla . J + \nabla . \frac{\partial D}{\partial t} = \nabla .. \left(J + \frac{\partial D}{\partial t} \right)$$

According to Equation of Continuity,

$$\nabla . J + \frac{\partial \rho}{\partial t} = \nabla .. \left(J + \frac{\partial D}{\partial t} \right) = 0$$

Thus $\nabla . J = 0$ for steady current

And $\nabla . \left(J + \frac{\partial D}{\partial t} \right) = 0$ every where

i.e., for steady currents as well as current corresponding to time varying fields.

In this way, Maxwell replaced ***J*** in Ampere's law by $\left(J + \frac{\partial D}{\partial t} \right)$

Thus Ampere's law becomes

$$\nabla \times B = \mu_0 \left(J + \frac{\partial D}{\partial t} \right) = \mu_0 \left(J + J_D \right) \quad \text{or} \qquad \nabla \times H = \left(J + \frac{\partial D}{\partial t} \right)$$

This is called modified form of Ampere's Law. Here the term $\frac{\partial D}{\partial t}$ is called displacement current density, denoted by $\boldsymbol{J_D}$.

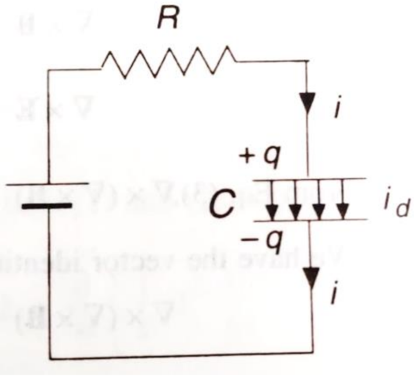
Displacement current

Maxwell introduced the new concept of displacement current. Faraday discovered that a changing magnetic field produces an electric field. Maxwell pointed out that a changing electric field produces a magnetic field.

Consider the plates of a parallel plate capacitor connected to the terminals of a battery. The conduction current from the battery gradually charges the capacitor plates. Until the voltage across the capacitor becomes equal to the battery voltage, conduction current flows in the leads of the capacitor. But the conduction current is not continuous across the gap between the plates as there is no transfer of charge across the plates. But according to Maxwell, the changing electric field between the plates serves the purpose of conduction current inside the gap. The displacement current in the gap is found to be equal to the conduction current in the lead wires. This proves that the flow of current in a circuit is continuous.

Magnitude of Displacement Current:

Consider a parallel plate capacitor with free space between the plates. Let q be the charge given to one of the plates at any instant and A its area.



Electric field between plates = $E = \dfrac{q}{\varepsilon_0 A}$

$$D = \frac{q}{\varepsilon_0 A} \times \varepsilon_0 = \frac{q}{A}$$

Displacement current = $i_d = AJ_d = A\dfrac{\partial D}{\partial t} = A\dfrac{\partial}{\partial t}(\dfrac{q}{A}) = \dfrac{\partial q}{\partial t} = i$

Here, $\frac{\partial q}{\partial t}$ is the current flowing in the circuit at that instant.

Therefore, $i_d = i$

Hence the displacement current in the gap between the capacitor plates is equal to the conduction current in the lead wires. Thus, the displacement current provides a continuous path for the charges across the capacitor.