Chapter 2

Relations and Functions

2.1 Ordered pairs and Cartesian products

Recall that there is no order imposed on the members of a set. We can, however, use ordinary sets to define an *ordered pair*, written $\langle a, b \rangle$ for example, in which a is considered the *first member* and b is the *second member* of the pair. The definition is as follows:

(2–1) $\langle a, b \rangle =_{def} \{\{a\}, \{a, b\}\}$

The first member of $\langle a, b \rangle$ is taken to be the element which occurs in the singleton $\{a\}$, and the second member is the one which is a member of the other set $\{a, b\}$, but not of $\{a\}$. Now we have the necessary properties of an ordering since in general $\langle a, b \rangle \neq \langle b, a \rangle$. This is so because we have $\{\{a\}, \{a, b\}\} = \{\{b\}, \{a, b\}\}$ (that is, $\langle a, b \rangle = \langle b, a \rangle$), if and only if we have $a = b$. Of course, this definition can be extended to ordered triples and in general ordered n-tuples for any natural number n. Ordered triples are defined as

(2–2) $\langle a, b, c \rangle =_{def} \langle \langle a, b \rangle, c \rangle$

It might have been intuitively simpler to start with ordered sets as an additional primitive, but mathematicians like to keep the number of primitive notions to a minimum.

If we have two sets A and B, we can form ordered pairs from them by taking an element of A as the first member of the pair and an element of B

as the second member. The *Cartesian product* of A and B, written $A \times B$, is the set consisting of all such pairs. The predicate notation defines it as

(2–3) $A \times B =_{def} \{\langle x, y \rangle \mid x \in A \text{ and } y \in B\}$

Note that according to the definition if either A or B is $\emptyset$, then $A \times B = \emptyset$. Here are some examples of Cartesian products:

(2–4) Let $K = \{a, b, c\}$ and $L = \{1, 2\}$, then

$$\begin{aligned} K \times L &= \{\langle a,1\rangle, \langle a,2\rangle, \langle b,1\rangle, \langle b,2\rangle, \langle c,1\rangle, \langle c,2\rangle\} \\ L \times K &= \{\langle 1,a\rangle, \langle 2,a\rangle, \langle 1,b\rangle, \langle 2,b\rangle, \langle 1,c\rangle, \langle 2,c\rangle\} \\ L \times L &= \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\} \end{aligned}$$

It is important to remember that the members of a Cartesian product are *not* ordered with respect to each other. Although each member is an ordered pair, the Cartesian product is itself an unordered set of them.

Given a set M of ordered pairs it is sometimes of interest to determine the smallest Cartesian product of which M is a subset. The smallest A and B such that $M \subseteq A \times B$ can be found by taking $A = \{a \mid \langle a, b \rangle \in M$ for some $b\}$ and $B = \{b \mid \langle a, b \rangle \in M$ for some $a\}$. These two sets are called the *projections of M onto the first and the second coordinates*, respectively. For example, if $M = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 3,2\rangle\}$, the set $\{1,3\}$ is the projection onto the first coordinate, and $\{1,2\}$ the projection onto the second coordinate. Thus $\{1,3\} \times \{1,2\}$ is the smallest Cartesian product of which M is a subset.

2.2 Relations

We have a natural understanding of relations as the sort of things that hold or do not hold between objects. The relation 'mother of' holds between any mother and her children but not between the children themselves, for instance. Transitive verbs often denote relations; e.g., the verb 'kiss' can be regarded as denoting an abstract relation between pairs of objects such that the first kisses the second. The subset relation was defined above as a relation between sets. Objects in a set may be related to objects in the same or another set. We write Rab or equivalently aRb if the relation R holds between objects a and b. We also write $R \subseteq A \times B$ for a relation between objects from two sets A and B, which we call a relation *from A to*

B. A relation holding of objects from a single set A is called a relation *in* A. The projection of R onto the first coordinate is called the *domain* of R and the projection of R onto the second coordinate is called the *range* of R. A relation R from A to B thus can be viewed as a subset of the Cartesian product $A \times B$. (There are unfortunately no generally accepted terms for the sets A and B of which the domain and the range are subsets.) It is important to realize that this is a *set-theoretic* reduction of the relation R to a set of ordered pairs, i.e. $\{\langle a, b \rangle \mid aRb\}$. For example, the relation 'mother of' defined on the set H of all human beings would be a set of ordered pairs in $H \times H$ such that in each pair the first member is mother of the second member. We may visually represent a relation R between two sets A and B by arrows in a diagram displaying the members of both sets.

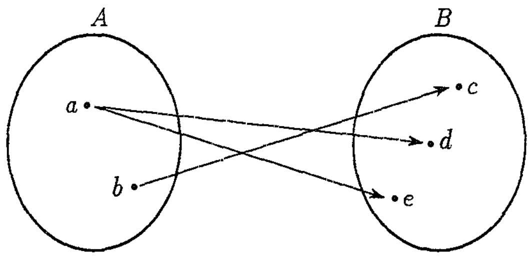


Figure 2–1: Relation R: $A \to B$.

In Figure 2-1, $A = \{a, b\}$ and $B = \{c, d, e\}$ and the arrows represent a set-theoretic relation $R = \{\langle a, d \rangle, \langle a, e \rangle, \langle b, c \rangle\}$. Note that a relation may relate one object in its domain to more than one object in its range. The complement of a relation $R \subseteq A \times B$, written R', is set-theoretically defined as

(2–5) $R' =_{def} (A \times B) - R$

Thus R' contains all ordered pairs of the Cartesian product which are not members of the relation R. Note that $(R')' = R$. The *inverse* of a relation $R \subseteq A \times B$, written R^{-1}, has as its members all the ordered pairs in R, with their first and second elements reversed. For example, let $A = \{1, 2, 3\}$ and let $R \subseteq A \times A$ be $\{\langle 3, 2 \rangle, \langle 3, 1 \rangle, \langle 2, 1 \rangle\}$, which is the 'greater than' relation in A. The complement relation R' is $\{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 3 \rangle\}$, the 'less than or equal to' relation in A. The inverse of R, R^{-1}, is $\{\langle 2,3\rangle, \langle 1, 3\rangle, \langle 1,2\rangle\}$, the 'less than' relation in A. Note that $(R^{-1})^{-1} = R$, and that if $R \subseteq A \times B$, then $R^{-1} \subseteq B \times A$, but $R' \subseteq A \times B$.

We have focused in this discussion on *binary* relations, i.e., sets of ordered pairs, but analogous remarks could be made about relations which are composed of ordered triples, quadruples, etc., i.e., *ternary*, *quaternary*, or just n-place relations.

2.3 Functions

A function is generally represented in set-theoretic terms as a special kind of relation. A relation R from A to B is a function if and only if it meets both of the following conditions:

1. Each element in the domain is paired with just one element in the range.
2. The domain of R is equal to A.

This amounts to saying that a subset of a Cartesian product $A \times B$ can be called a function just in case every member of A occurs exactly once as a first coordinate in the ordered pairs of the set.

As an example, consider the sets $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. The following relations from A to B are functions:

(2–6)
$$\begin{aligned} P &= \{\langle a,1\rangle, \langle b,2\rangle, \langle c,3\rangle\} \\ Q &= \{\langle a,3\rangle, \langle b,4\rangle, \langle c,1\rangle\} \\ R &= \{\langle a,3\rangle, \langle b,2\rangle, \langle c,2\rangle\} \end{aligned}$$

The following relations from A to B are not functions:

(2–7)
$$\begin{aligned} S &= \{\langle a,1\rangle, \langle b,2\rangle\} \\ T &= \{\langle a,2\rangle, \langle b,3\rangle, \langle a,3\rangle, \langle c,1\rangle\} \\ V &= \{\langle a,2\rangle, \langle a,3\rangle, \langle b,4\rangle\} \end{aligned}$$

S fails to meet condition 2 because the set of first members, namely $\{a, b\}$, is not equal to A. T does not satisfy condition 1, since a is paired with both 2 and 3. In relation V both conditions are violated.

Much of the terminology used in talking about functions is the same as that for relations. We say that a function that is a subset of $A \times B$ is a function *from A to B*, while one in $A \times A$ is said to be a function *in A*. The notation '$F: A \to B$' is used for 'F is a function from A to B.' Elements in the domain of a function are sometimes called *arguments* and their correspondents in the range, *values*. Of function P in (2-6), for example, one may say that it takes on the value 3 at argument c. The usual way to denote this fact is $P(c) = 3$, with the name of the function preceding the argument, which is enclosed in parentheses, and the corresponding value to the right of the equal sign.

'Transformation,' 'map,' 'mapping,' and 'correspondence' are commonly used synonyms for 'function,' and often '$F(a) = 2$' is read as 'F maps a into 2.' Such a statement gives a function the appearance of an active process that changes arguments into values. This view of functions is reinforced by the fact that in most of the functions commonly encountered in mathematics the pairing of arguments and values can be specified by a formula containing operations such as addition, multiplication, division, etc. For example, $F(x) = 2x + 1$ is a function which, when defined on the set of integers, pairs 1 with 3, 2 with 5, 3 with 7, and so on. This can be thought of as a rule which says, "To find the value of F at x, multiply x by 2 and add 1." Later in this book it may prove to be necessary to think of functions as dynamic processes transforming objects as their input into other objects as their output, but for the present, we adhere to the more static set-theoretic perspective. Thus, the function $F(x) = 2x + 1$ will be regarded as a set of ordered pairs which could be defined in predicate notation as

(2–8) $F = \{\langle x, y\rangle \mid y = 2x + 1\}$ (where x and y are integers)

Authors who regard functions as processes sometimes refer to the set of ordered pairs obtained by applying the process at each element of the domain as the *graph* of the function. The connection between this use of "graph" and a representation consisting of a line drawn in a coordinate system is not accidental.

We should also note that relations which satisfy condition 1 above but perhaps fail condition 2 are sometimes regarded as functions, but if so, they are customarily designated as 'partial functions.' For example, the function which maps an ordered pair of real numbers $\langle a, b\rangle$ into the quotient of a divided by b (e.g., it maps $\langle 6, 2\rangle$ into 3 and $\langle 5, 2\rangle$ into 2.5) is not defined when $b = 0$. But it is single-valued – each pair for which it is defined is associated with a unique value – and thus it meets condition 1. Strictly speaking, by our definition it is not a function, but it could be called a partial function. A partial function is thus a total function on some subset of the domain. Henceforth, we will use the term 'function,' if required, to indicate a single-valued mapping whose domain may be less than the set A containing the domain.

It is sometimes useful to state specifically whether or not the range of a function from A to B is equal to the set B. Functions from A to B in general are said to be *into* B If the range of the function equals B, however, then the function is *onto* B. (Thus *onto* functions are also *into*, but not necessarily conversely.) In Figure 2-2 three functions are indicated by the same sort of diagrams we introduced previously for relations generally It should be apparent that functions F and G are *onto* but H is not. All are of course *into*.

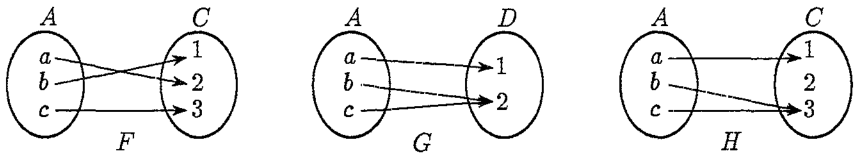


Figure 2-2: Illustration of onto and into functions.

A function $F: A \rightarrow B$ is called a *one-to-one* function just in case no member of B is assigned to more than one member of A. Function F in Figure 2-2 is one-to-one, but G is not (since both b and c are mapped into 2), nor is H (since $H(b) = H(c) = 3$). The function F defined in (2-8) is one-to-one since for each odd integer y there is a unique integer x such that $y = 2x + 1$. Note that F is not onto the set of integers since no even integer is the value of F for any argument x. Functions which are not necessarily one-to-one may be termed *many to one*. Thus all functions are many-to-one strictly speaking, and some but not all of them are one-to-one. It is usual to apply the term "many-to-one", however, only to those functions which are not in fact one-to-one.

A function which is both one-to-one and onto (F in Figure 2-2 is an example) is called a *one-to-one correspondence* Such functions are of special interest because their inverses are also functions (Note that the definitions of the inverse and the complement of a relation apply to functions as well) The inverse of G in Figure 2-2 is not a function since 2 is mapped into both b and c, and in H^{-1} the element 2 has no correspondent.

Problem: Is the inverse of function F in (2-8) also a function? Is F a one-to-one correspondence?

2.4 Composition

Given two functions $F: A \rightarrow B$ and $G: B \rightarrow C$, we may form a new function from A to C, called the *composite*, or *composition* of F and G, written $G \circ F$. In predicate notation function composition is defined as

(2–9) $G \circ F =_{def} \{\langle x, z \rangle \mid \text{ for some } y, \langle x, y \rangle \in F \text{ and } \langle y, z \rangle \in G\}$

Figure 2-3 shows two functions F and G and their composition.

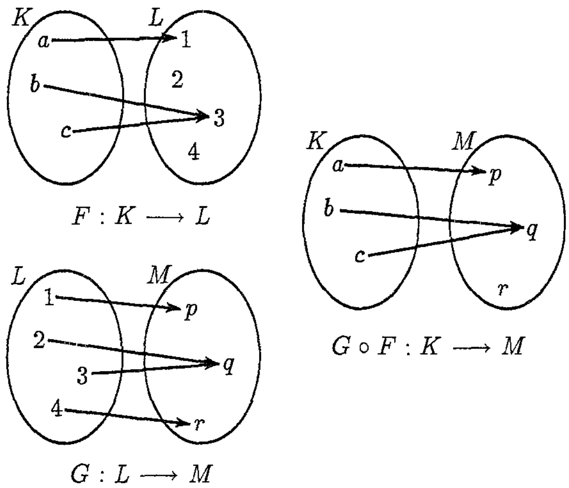
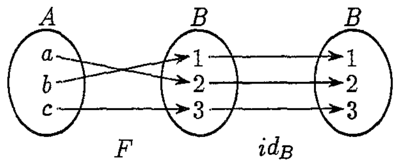


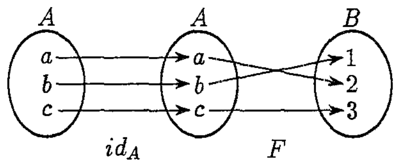
Figure 2–3: Composition of two functions F and G.

Note that F is into while G is onto and that neither is one-to-one. This shows that compositions may be formed from functions that do not have these special properties. It could happen, however, that the range of the first function is disjoint from the domain of the second, in which case, there is no y such that $\langle x, y \rangle \in F$ and $\langle y, z \rangle \in G$, and so the set of ordered pairs defined by $G \circ F$ is empty. In Figure 2-3, F is the first function and G is the second in the composition. Order is crucial here, since in general $G \circ F$ is not equal to $F \circ G$. The notation $G \circ F$ may seem to read backwards, but the value of a function F at an argument a is $F(a)$, and the value of G at the argument $F(a)$ is written $G(F(a))$. By the definition of composition, $G(F(a))$ and $(G \circ F)(a)$ produce the same value.

A function $F: A \to A$ such that $F = \{\langle x, x \rangle \mid x \in A\}$ is called the *identity function*, written id_A. This function maps each element of A to itself. Composition of a function F with the appropriate identity function gives a function that is equal to the function F itself. This is illustrated in Figure 2-4.



$id_B \circ F = F$



$F \circ id_A = F$

Figure 2–4: Composition with an identity function.

Given a function $F: A \to B$ that is a one-to-one correspondence (thus the inverse is also a function), we have the following general equations:

(2–10)
$$\begin{aligned} F^{-1} \circ F &= id_A \\ F \circ F^{-1} &= id_B \end{aligned}$$

These are illustrated in Figure 2-5.

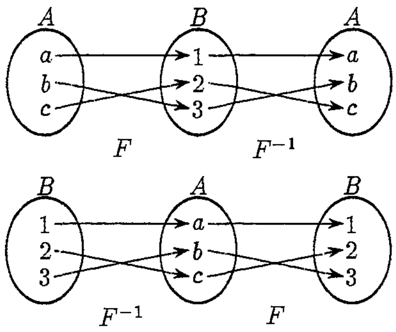


Figure 2–5: Composition of one-to-one correspondence with its inverse.

The definition of composition need not be restricted to functions but can be applied to relations in general. Given relations $R \subseteq A \times B$ and $S \subseteq B \times C$ the composite of R and S, written $S \circ R$, is the relation $\{\langle x, z \rangle \mid$ for some y, $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in S\}$ An example is shown in Figure 2-6.

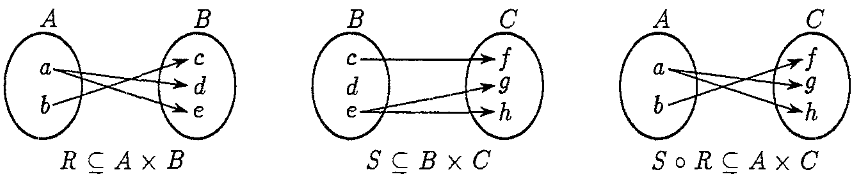


Figure 2–6: Composition of two relations R and S.

For any relation $R \subseteq A \times B$ we also have the following:

$$\text{(2–11)} \quad \begin{aligned} id_B \circ R &= R \\ R \circ id_A &= R \end{aligned}$$

(Note that the identity function in A, id_A, is of course a relation and could equally well be called the identity relation in A)

The equations corresponding to (2-10) do not hold for relations (nor for functions which are not one-to-one correspondences) However, we have for any *one-to-one* relation $R: A \rightarrow B$:

$$(2\text{–}12) \quad \begin{aligned} R^{-1} \circ R &\subseteq id_A \\ R \circ R^{-1} &\subseteq id_B \end{aligned}$$

We should note here that our previous remarks about ternary, quaternary, etc. relations can also be carried over to functions. A function may have as its domain a set of ordered n-tuples for any n, but each such n-tuple will be mapped into a unique value in the range. For example, there is a function mapping each pair of natural numbers into their sum.

Exercises

1. Let $A = \{b, c\}$ and $B = \{2, 3\}$

   (a) Specify the following sets by listing their members.

(i)	$A \times B$	(iv)	$(A \cup B) \times B$
(ii)	$B \times A$	(v)	$(A \cap B) \times B$
(iii)	$A \times A$	(vi)	$(A - B) \times (B - A)$

   (b) Classify each statement as true or false.

   (i) $(A \times B) \cup (B \times A) = \emptyset$

   (ii) $(A \times A) \subseteq (A \times B)$

   (iii) $\langle c, c \rangle \subseteq (A \times A)$

   (iv) $\{\langle b, 3 \rangle, \langle 3, b \rangle\} \subseteq (A \times B) \cup (B \times A)$

   (v) $\emptyset \subseteq A \times A$

   (vi) $\{\langle b, 2 \rangle, \langle c, 3 \rangle\}$ is a relation from A to B

   (vii) $\{\langle b, b \rangle\}$ is a relation in A

   (c) Consider the following relation from A to $(A \cup B)$:

   $R = \{\langle b, b \rangle, \langle b, 2 \rangle, \langle c, 2 \rangle, \langle c, 3 \rangle\}$

   (i) Specify the domain and range of R

   (ii) Specify the complementary relation R' and the inverse R^{-1}

   (iii) Is $(R')^{-1}$ (the inverse of the complement) equal to $(R^{-1})'$ (the complement of the inverse)?

2. Let $A = \{a, b, c\}$ and $B = \{1, 2\}$. How many distinct relations are there from A to B? How many of these are functions from A to B? How many of the functions are onto? one-to-one? Do any of the functions have inverses that are functions? Answer the same questions for all relations from B to A.

3. Let

$$R_1 = \{\langle 1,1 \rangle, \langle 2,1 \rangle, \langle 3,4 \rangle, \langle 2,2 \rangle, \langle 3,3 \rangle, \langle 4,4 \rangle, \langle 4,1 \rangle\}$$

$$R_2 = \{\langle 3,4 \rangle, \langle 1,2 \rangle, \langle 1,4 \rangle, \langle 2,3 \rangle, \langle 2,4 \rangle, \langle 1,3 \rangle\}$$

(both relations in A, where $A = \{1, 2, 3, 4\}$).

(a) Form the composites $R_2 \circ R_1$ and $R_1 \circ R_2$. Are they equal?

(b) Show that $R_1^{-1} \circ R_1 \neq id_A$ and that $R_2^{-1} \circ R_2 \not\subseteq id_A$.

4. For the functions F and G in Figure 2-3:

(a) show that $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.

(b) Show that the corresponding equation holds for relations R and S in Figure 2-6.