Scientific Computing - A.Y. 2022/2023

Project No. 2

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May 18, 2023

Exercise 1

Introduction

We want to solve the following differential problem subject to Dirichlet boundary conditions (1):

$$\begin{cases} u'' = -f(x) & a < x < b \\ u(a) = k_1 u(\eta_1), \quad u(b) = k_2 u(\eta_2) \end{cases}$$

where $f(x)$ is an assigned function, $0 < \eta_2 < \eta_1 < 1$ and k_1, k_2 are two real constants. We will seek a solution to this problem using the method of Green's functions. In particular, we will look for a solution of the form

$$u(x) = w(x) + (c + dx)[w(\eta_1) + w(\eta_2)]$$

where $w(t)$ satisfies the following Dirichlet problem (2):

$$\begin{cases} w'' = -f(x), & a < x < b \\ w(a) = w(b) = 0 \end{cases}$$

and where c and d are to be determined. After that, we will employ the same methodology to solve the differential problem (3):

$$\begin{cases} u''(t) = -\sin(t), \quad t \in (0,1) \\ u(0) = \frac{1}{2}u\left(\frac{1}{5}\right), u(1) = \frac{2}{3}u\left(\frac{1}{6}\right) \end{cases}$$

Finally, we will verify that the boundary conditions are indeed satisfied.

Analysis

Green's coefficients To find c and d we have to impose the boundary conditions. Starting from $u(x) = w(x) + (c + dx)[w(\eta_1) + w(\eta_2)]$ we get:

$$u(a) = w(a) + (c + da)[w(\eta_1) + w(\eta_2)]$$

$$u(\eta_1) = w(\eta_1) + (c + d\eta_1)[w(\eta_1) + w(\eta_2)]$$

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$$u(b) = w(b) + (c + db)[w(\eta_1) + w(\eta_2)]$$

$$u(\eta_2) = w(\eta_2) + (c + d\eta_2)[w(\eta_1) + w(\eta_2)]$$

We then match the initial conditions, and obtain

$$\begin{cases} w(a) + (c + da)[w(\eta_1) + w(\eta_2)] = k_1\{w(\eta_1) + (c + d\eta_1)[w(\eta_1) + w(\eta_2)]\} \\ w(b) + (c + db)[w(\eta_1) + w(\eta_2)] = k_2\{w(\eta_2) + (c + d\eta_2)[w(\eta_1) + w(\eta_2)]\} \end{cases}$$

Rearranging the system, we get:

$$\begin{cases} \alpha c + \beta d = \gamma \\ \delta c + \theta y = \phi \end{cases}$$

$$\alpha = ((w(\eta_1) + w(\eta_2)) - k_1(w(\eta_1) + w(\eta_2)), \quad \beta = a(w(\eta_1) + w(\eta_2)) - k_1\eta_1(w(\eta_1) + w(\eta_2))$$

$$\gamma = k_1 w(\eta_1) - w(a), \quad \delta = (w(\eta_1) + w(\eta_2)) - k_2(w(\eta_1) + w(\eta_2))$$

$$\theta = b(w(\eta_1) + w(\eta_2)) - k_2\eta_2(w(\eta_1) + w(\eta_2)), \quad \phi = k_2 w(\eta_2) - w(b)$$

Finally, we obtain:

$$\begin{cases} c = \frac{\gamma - \beta\left(\frac{\phi - \frac{\gamma\delta}{\alpha}}{\theta - \frac{\beta\delta}{\alpha}}\right)}{\alpha} \\ d = \frac{\phi - \frac{\gamma\delta}{\alpha}}{\theta - \frac{\beta\delta}{\alpha}} \end{cases}$$

Green's Method We now want to understand how the Green's method works.
Let's examine the differential equation (2). We consider Green's differential problem associated:

$$\begin{cases} -G''(x,t) = \delta(x-t), \quad t, x \in [a,b] \\ G(a,t) = G(b,t) = 0 \end{cases}$$

Since $t, x \in [a,b]$, we have $t > a$, $t > b \Rightarrow a - t < 0$, $b - t > 0$
Therefore, $r(a-t) = 0$ and $r(b-t) = b - t$ where

$$r(x) = \begin{cases} 0, & x \leq 0 \\ x, & x > 0 \end{cases}$$

We now want to compute $G(x,t)$.

$$G'(x,t) = \int_a^b G''(x,t)dt = -\int_a^b \delta(x-t)\,dx = -H(x-t) + A$$

$$G(x,t) = -r(x-t) + Ax + B$$

Imposing the boundary conditions we obtain:

$$G(a,t) = -r(a-t) + Aa + B = 0 \Rightarrow B = -Aa$$

$$G(b,t) = -r(b-t) + Aa + B = 0 \Rightarrow A = \frac{b-t}{b-a}$$

Therefore,

$$G(x,t) = \begin{cases} \frac{(b-t)(x-a)}{b-a}, & x < t \\ \frac{(b-t)(x-a)}{b-a} - x + t, & x \geq t \end{cases}$$

In conclusion, we have:

$$w(t) = \int_a^b G(x,t)f(t)dt = \int_a^x \frac{(b-t)(x-a)}{b-a}dt + \int_x^b \left(\frac{(b-t)(x-a)}{b-a} - x + t\right)dt$$

An Application of Green's Method Let's consider the differential problem (3).
Using Green's method, we have:

$$w(x) = \int_0^1 f(t)G(x,t)dt = \int_0^x f(t)(1-t)tdt + \int_x^1 f(t)(1-t)xdt$$

where $f(t) = \sin(t)$. We obtain:

$$\int_0^x \sin(t)(1-x)tdt + \int_x^1 \sin(t)(1-t)xdt = \sin(x) - x\sin(1)$$

We can now construct using Green's method the whole solution $u(x)$.

$$u(x) = w(x) + (c + dx)[w(\eta_1) + w(\eta_2)]$$

$$u(x) = \sin(x) - x\sin(1) + (0.5682 + 0.1303x) \cdot (0.0304 + 0.0257)$$

After obtaining the analytical solution, we use MATLAB code to numerically solve the problem associated to the $w(x)$, using a centered finite difference algorithm.

Code Summary In MATLAB script `generic.m`, the generic values c and d for Green's method applied to problem (1) are computed.
In MATLAB script `resolver.m`, we implement Green's method to solve (3). This code is divided into four main steps.

1. *Finding the function $\omega(t)$ associated with the Green problem.*
   The code first initializes the number of interior points (`N`) in $[0, 1]$ and computes the grid spacing (`h`). It then constructs a tridiagonal matrix `A` and a vector `B`, associated to the finite differences discretization scheme to approximate the solution of (2). The boundary conditions are applied to `B`, and the linear system is solved to obtain the numerical approximation of the function $w(t)$. The solution is plotted on a graph with t as the x-axis and $\omega(t)$ as the y-axis.

2. *Finding the constants c and d.*
   In this step the constants `(a, b, n1, n2, k1, k2)` specified by (3). It evaluates `W` - the exact solution to (3) - at `n1=1/5` and `n2=1/6`, and constructs the matrix of coefficients `A` and the known term vector b using the given constants. The linear system is solved to obtain the values of `c` and `d`.

3. *Finding $u(t)$.*
   The code computes the numerical approximation of the function $u(t)$ using the previously obtained approximation of the function $\omega(t)$ and constants `c` and `d`. The solution is plotted on a graph with t as the x-axis and $u(t)$ as the y-axis.

4. *Verification of boundary condition.*
   The code verifies the boundary conditions for the obtained solution $u(t)$ by comparing the values of $u(0)$ and $u(1)$ with the expected values calculated using the Green's function approach.

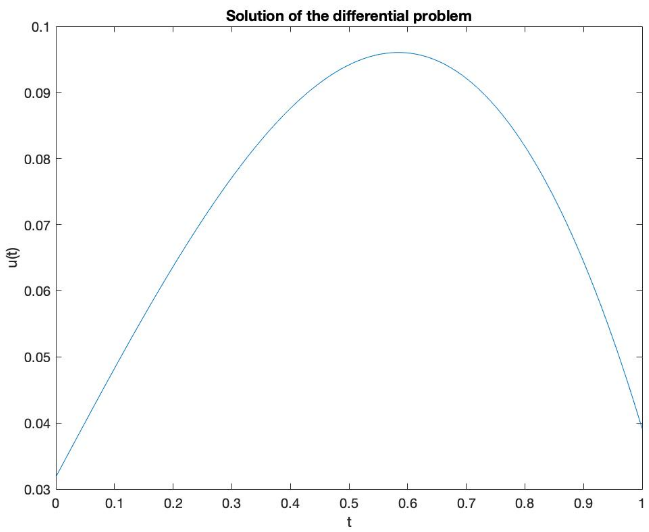

Figure 1: Graph of the numerical approximation of $\omega(x)$.

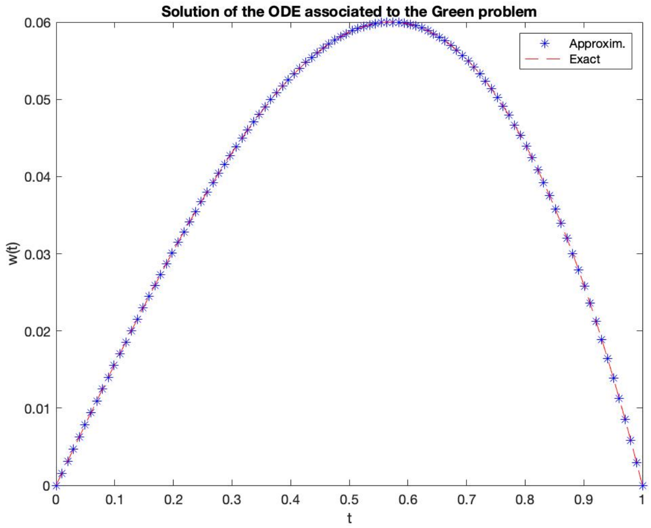

Figure 2: Comparison between exact solution $u(x)$ and numerical approximated of $u(x)$.

Conclusion In conclusion, we effectively utilized Green's function method to solve a differential problem with Dirichlet boundary conditions, where the initial conditions of $u(x)$ depend directly on the value of $u(x)$ in other points. We transformed the original problem into a standard homogeneous problem, as there is no numerical method that can solve differential equations with such boundary conditions. We derived the Green's function, and found the solutions for both the standard and original problems. This method offers a systematic approach to finding solutions while satisfying these implicit boundary conditions, showcasing its power in solving differential problems with Dirichlet boundary conditions.