Teoria dos Grafos e Computabilidade

— Trees and spanning trees —

Silvio Jamil F. Guimarães

Graduate Program in Informatics – PPGINF
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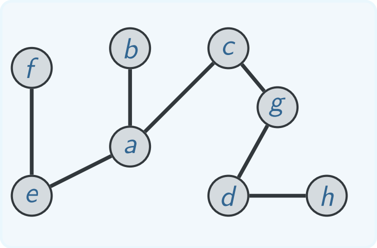
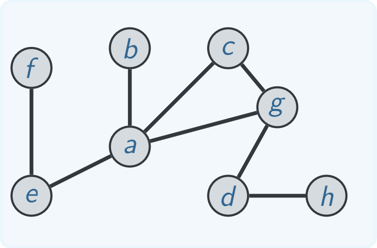
Trees

- A tree is an undirected connected graph with no cycles.
- Genealogical trees, evolutionary trees, decision trees, various data structures in Computer Science



Tree



Not a tree (has cycle)

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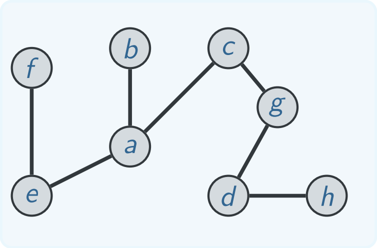

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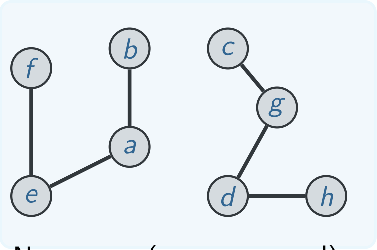

Not a tree (not connected) – this is a forest

Trees

Theorem A tree has exactly one path between any pair of vertices

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- A vertex of degree 1 is called a leaf.
- Sometimes, vertices of degree 0 are also counted as leaves
- A vertex with degree greater than 1 is an internal vertex.

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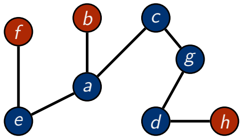

Trees

Theorem Every tree, with at least two vertices, has at least two leaves.

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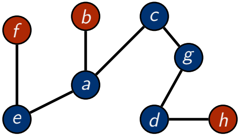
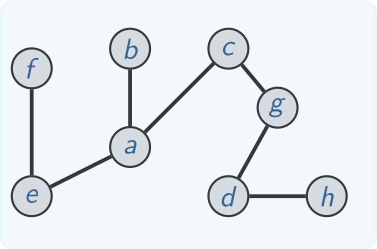
- Sometimes, vertices of degree 0 are also counted as leaves
- A vertex with degree greater than 1 is an internal vertex.



Trees

Theorem All trees on $n \geq 1$ vertices have exactly $n - 1$ edges



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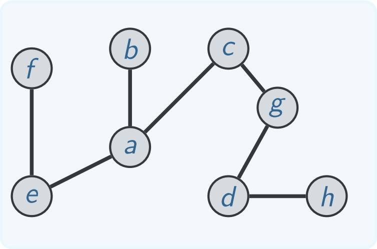

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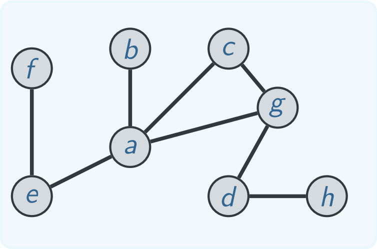

Not a tree (has cycle)

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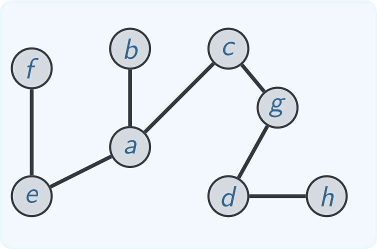

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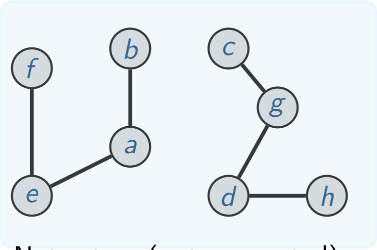
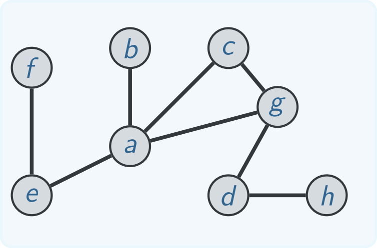


Not a tree (not connected) – this is a forest

Lemma Removing an edge from a cycle keeps connectivity



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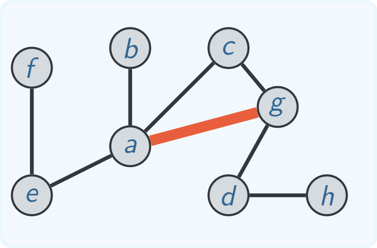


Still connected after removal

Spanning trees

A spanning tree of an undirected graph is a subgraph that is a tree and includes all vertices.

Spanning trees

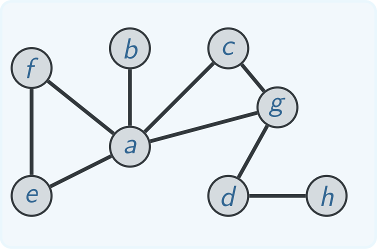
A spanning tree of an undirected graph is a subgraph that is a tree and includes all vertices.

A graph G has a spanning tree iff it is connected:

- If G has a spanning tree, it is connected: any two vertices have a path between them in the spanning tree and hence in G.
- If G is connected, we will construct a spanning tree

Spanning trees

1. Let G be a connected graph on n vertices.
2. If there are any cycles, pick one and remove any edge.



There exists a cycle?

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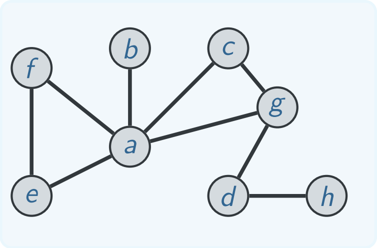

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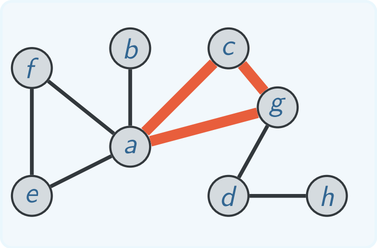

Cycle: a-c-g-a

Spanning trees

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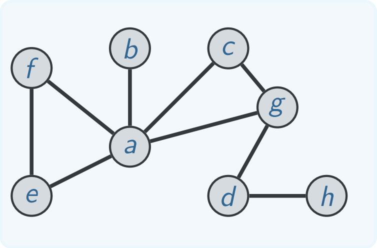

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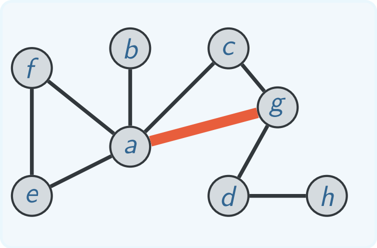

Remove the edge a-g

1. Let G be a connected graph on n vertices.
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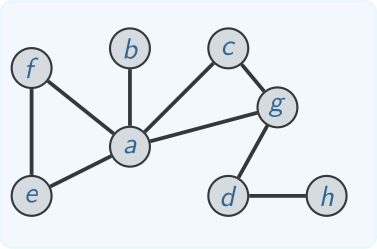

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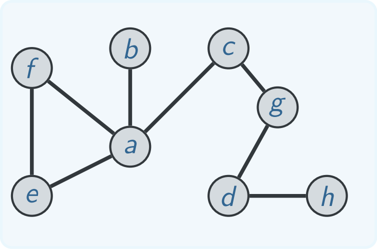

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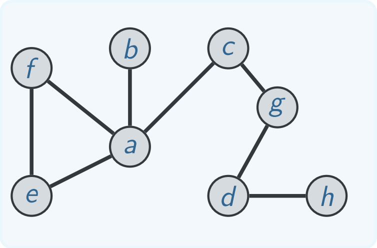

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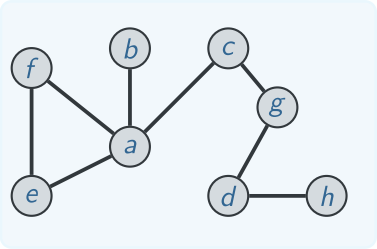

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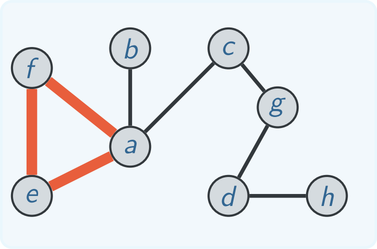

Cycle: a-f-e-a

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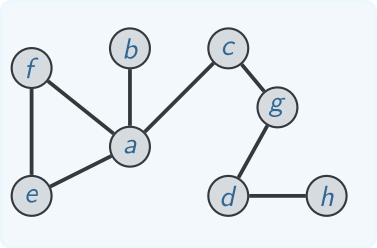

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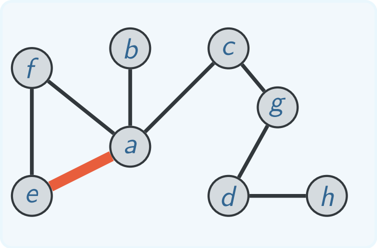

Remove the edge a-e

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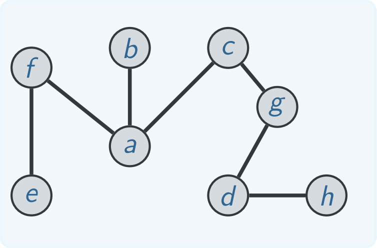

T is still connected, and has no cycles, so it' is a tree!

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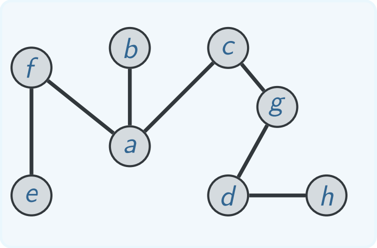


T is still connected, and has no cycles, so it' is a tree!

It reaches all vertices, so it is a spanning tree

Spanning trees

Converse theorem

If a connected graph on n vertices has $n - 1$ edges, it is a tree

Spanning trees

Converse theorem

If a connected graph on n vertices has $n-1$ edges, it is a tree

A forest is an undirected graph with no cycles and each connected component is a tree.

Spanning trees

Converse theorem

If a connected graph on n vertices has $n-1$ edges, it is a tree

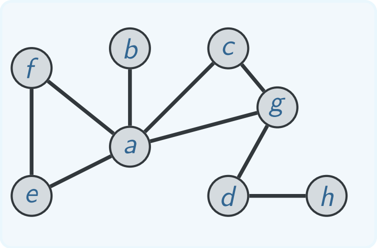
A forest is an undirected graph with no cycles and each connected component is a tree.

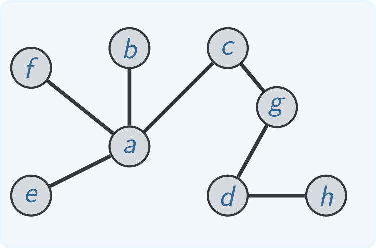
Theorem

A forest with n vertices and k trees has $n-k$ edges.

Spanning trees

Let G be a connected graph on n vertices and T be a spanning tree computed from G

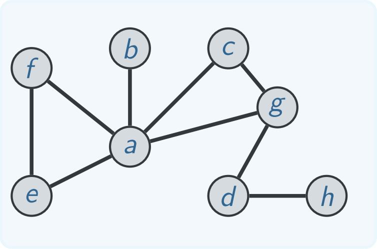


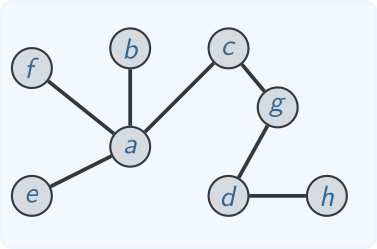


Spanning trees

Let G be a connected graph on n vertices and T be a spanning tree computed from G

1. A branch is an edge in a spanning tree T.

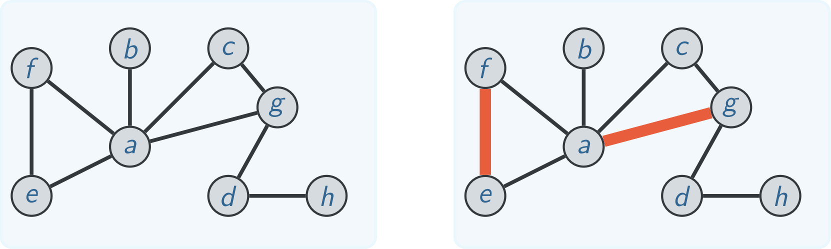




Spanning trees

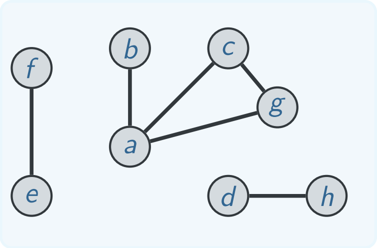
Let G be a connected graph on n vertices and T be a spanning tree computed from G

1. A branch is an edge in a spanning tree T.
2. A chord is an edge of the connected graph G that is not a branch of a spanning tree T.



Spanning forest

Let G be a graph on n vertices.



Spanning forest

Let G be a graph on n vertices.

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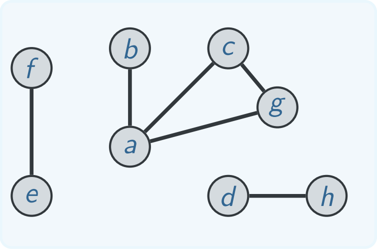

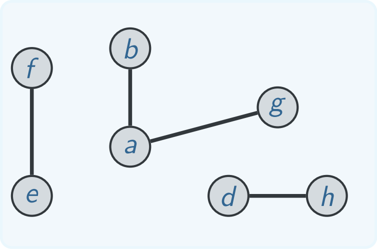

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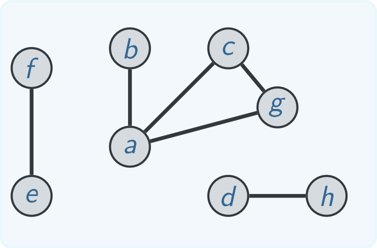

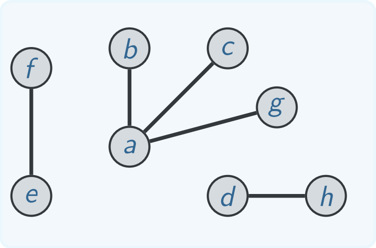

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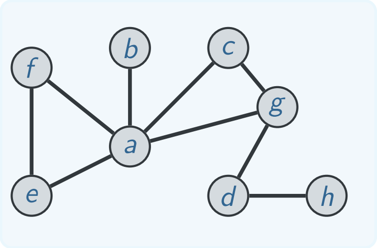
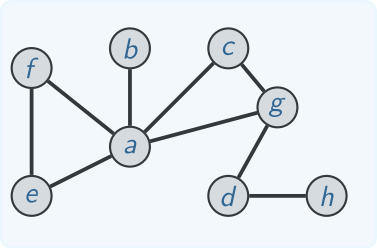


Spanning trees

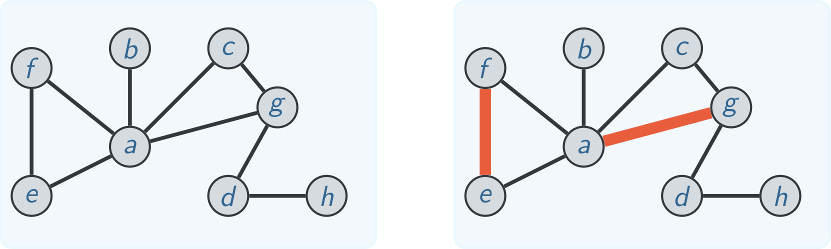
Let G be a connected graph on n vertices and T be a spanning tree computed from G

1. A rank of G is the number of branches of the spanning trees (or spanning forest) which is given by $r = n - k$



Let G be a connected graph on n vertices and T be a spanning tree computed from G

1. A **rank** of G is the number of branches of the spanning trees (or spanning forest) which is given by $r = n - k$
2. A **nullity** is the number of chords related to the spanning trees (or spanning forest) which is given by $\mu = e - n - k$



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Remember that k is the number of connected component. For a spanning tree, $k = 1$, but for a spanning forest, k is the number of spanning trees which are in the spanning forest.

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— Minimum Spanning Trees —

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Minimum Spanning Tree (MST)

- Given an undirected graph $G = (V, E)$ with a cost $c_e > 0$ associated with each edge $e \in E$.
- Find a subset T of edges such that the graph (V, T) is connected and the cost $\sum_{e \in T} c_e$ is as small as possible.

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MINIMUM SPANNING TREE

INSTANCE An undirected graph $G = (V, E)$ and a function $c : E \rightarrow \mathbb{R}^+$

SOLUTION A set $T \subseteq E$ of edges such that (V, T) is connected and the $\sum_{e \in T} c_e$ is as small as possible.

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MINIMUM SPANNING TREE

INSTANCE An undirected graph $G = (V, E)$ and a function $c : E \to \mathbb{R}^+$

SOLUTION A set $T \subseteq E$ of edges such that (V, T) is connected and the $\sum_{e \in T} c_e$ is as small as possible.

- Claim: If T is a minimum-cost solution to this network design problem then (V, T) is a tree.
- A subset T of E is a spanning tree of G if (V, T) is a tree.

Greedy Algorithm for the MST Problem

- Template: process edges in some order. Add an edge to T if tree property is not violated.

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  Increasing cost order *Process edges in increasing order of cost. Discard an edge if it creates a cycle.*

  Dijkstra-like *Start from a node s and grow T outward from s: add the node that can be attached most cheaply to current tree.*

  Decreasing cost order *Delete edges in order of decreasing cost as long as graph remains connected.*

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- Which of these algorithms works?

Greedy Algorithm for the MST Problem

- Template: process edges in some order. Add an edge to T if tree property is not violated.

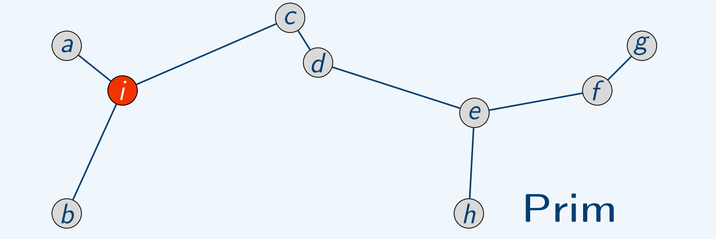
  Increasing cost order *Process edges in increasing order of cost. Discard an edge if it creates a cycle.* *Kruskal's algorithm*

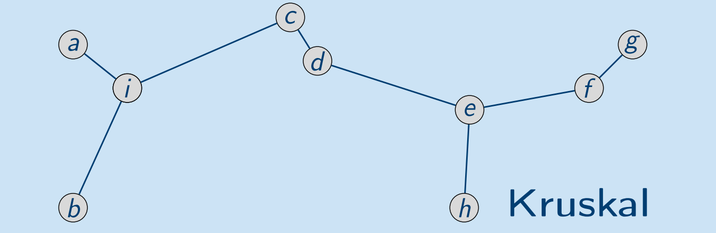
  Dijkstra-like *Start from a node s and grow T outward from s: add the node that can be attached most cheaply to current tree.* *Prim's algorithm*

  Decreasing cost order *Delete edges in order of decreasing cost as long as graph remains connected.* *Reverse-Delete algorithm*

- Which of these algorithms works? All of them!

Example of Prim's and Kruskal's Algorithms





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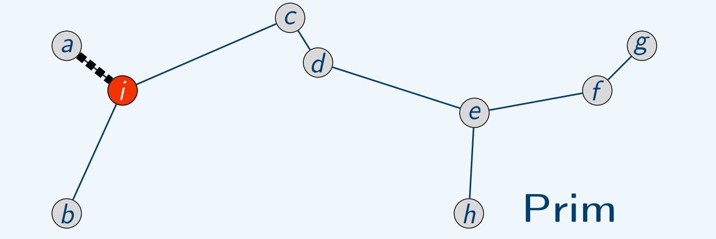

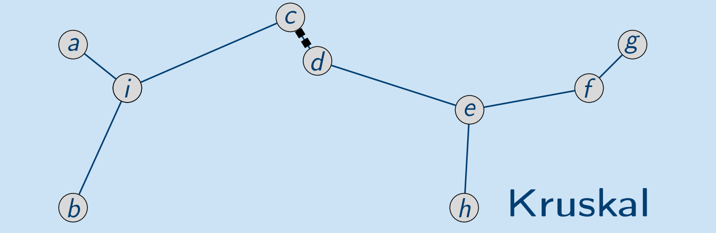

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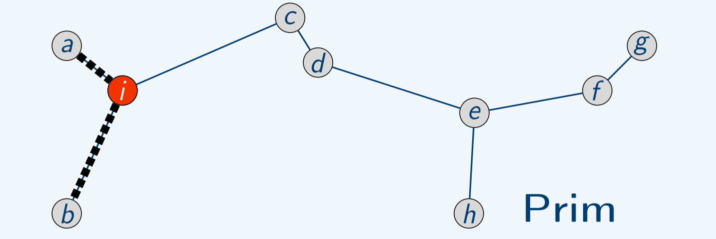

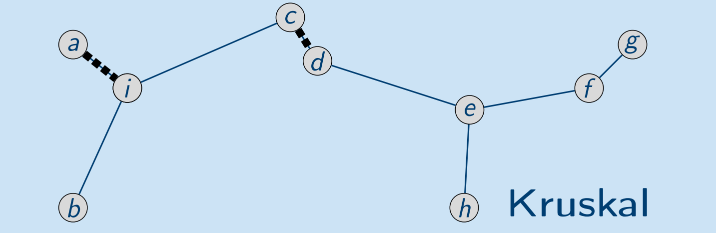

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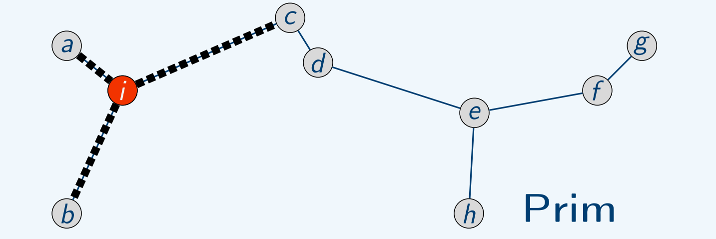

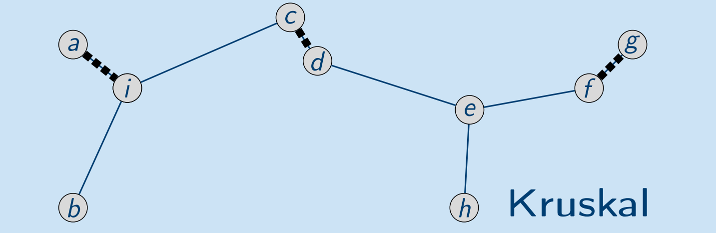

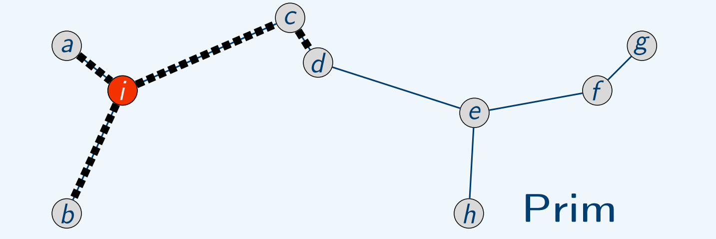
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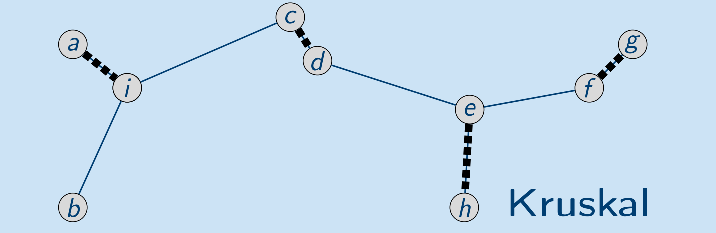
i
Prim


a
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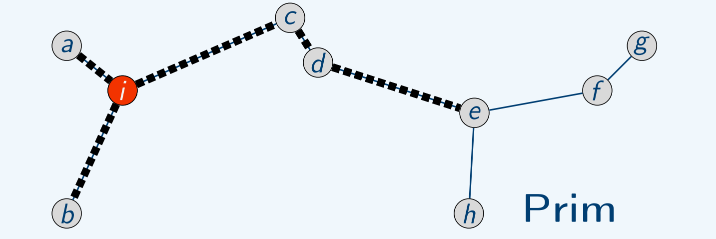
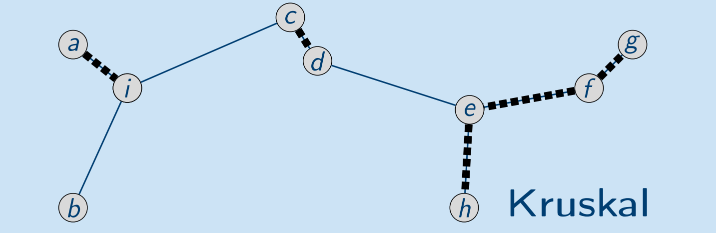
Kruskal

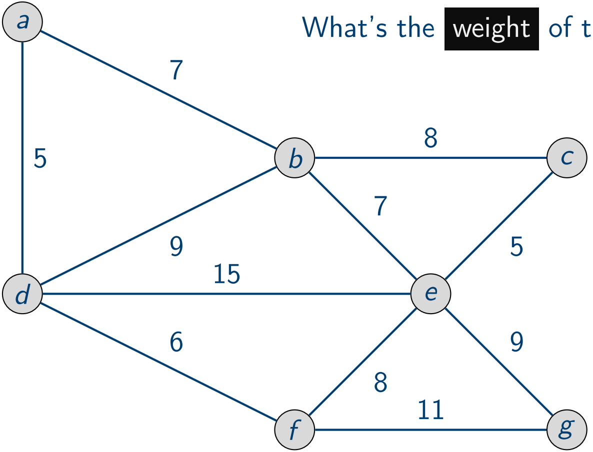
Example of Prim's and Kruskal's Algorithms





Example of Prim's Algorithm

What's the weight of the MST?



Example of Kruskal's Algorithm

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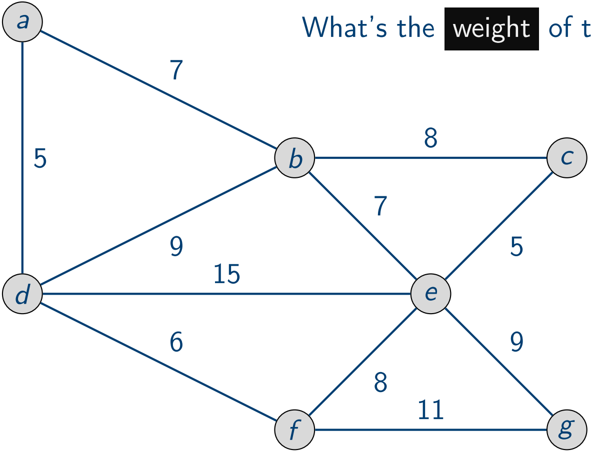

Graph Cuts

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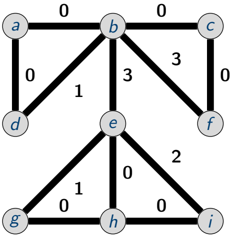

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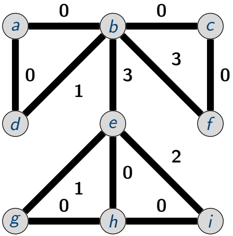

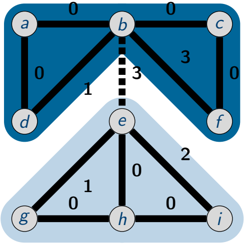

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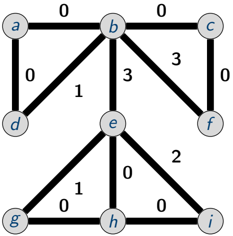

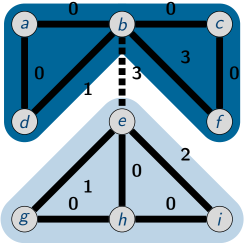

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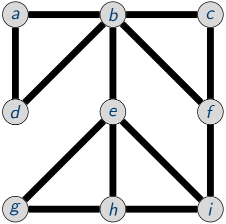

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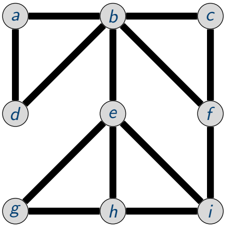

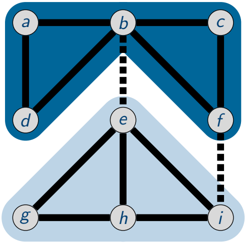

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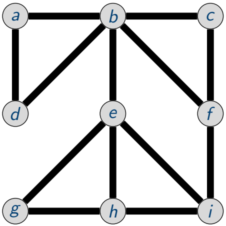

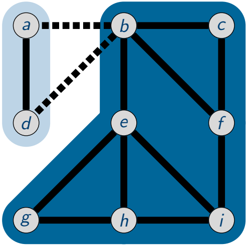

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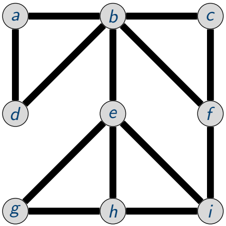

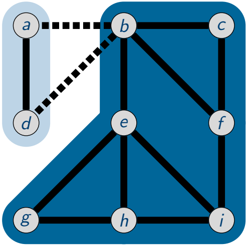

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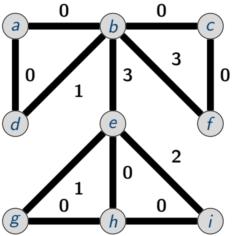

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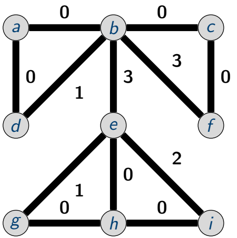

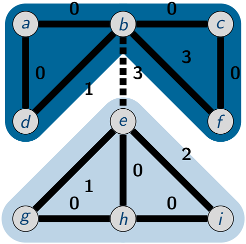

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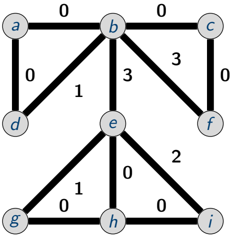

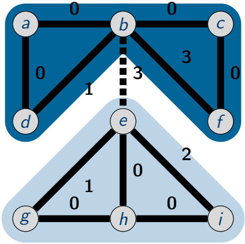

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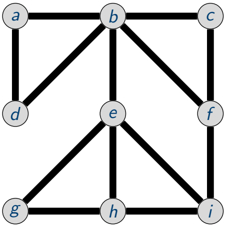

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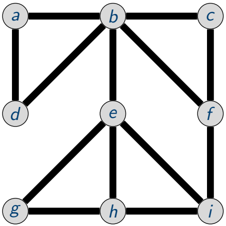

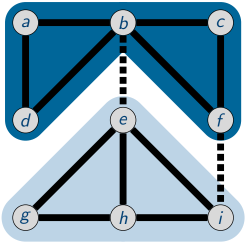

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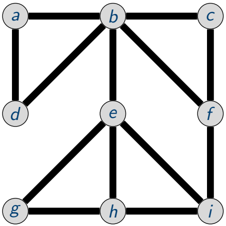

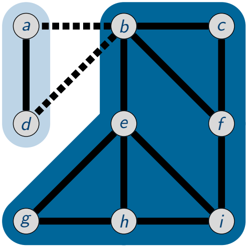

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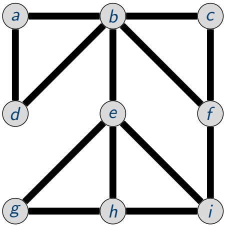

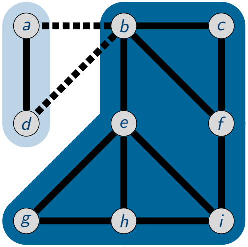

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- Claim: every MST contains e.
- Proof: exchange argument. If a supposed MST T does not contain e, show that there is a tree with smaller cost than T that contains e.

Using the Cut Property

- Let F be the set of all edges that satisfy the cut property.
- Is the graph induced by F connected?
- Can the graph induced by F contain a cycle?
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- How many edges can F contain? $n - 1$
- F is the unique MST.
- Kruskal's and Prim's algorithms compute F **efficiently**.

Optimality of Kruskal's Algorithm

- Kruskal's algorithm:
  - Start with an empty set T of edges.
  - Process edges in E in non decreasing order of cost.
  - Add the next edge e to T only if adding e does not create a cycle. Discard e if it creates a cycle.
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- Claim: Kruskal's algorithm outputs an MST.
  1. For every edge e added, demonstrate the existence of S and $V - S$ such that e and S satisfy the cut property.
  2. Prove that the algorithm computes a spanning tree.

Optimality of Prim's Algorithm

- Prim's algorithm: Maintain a tree (S, U)
  - Start with an arbitrary node $s \in S$ and $U = \emptyset$.
  - Add the node v to S and the edge e to U that minimize

$$\min_{e=(u,v),u \in S, v \notin S} c_e \equiv \min_{e \in \mathrm{cut}(S)} c_e.$$

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  1. Prove that every edge inserted satisfies the cut property.
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- Let C be any cycle in G and let $e = (v, w)$ be the most expensive edge in C.
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Optimality of the Reverse-Delete Algorithm

- Reverse-Delete algorithm: Maintain a set E' of edges.
  - Start with $E' = E$.
  - Process edges in non increasing order of cost.
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- Claim: the Reverse-Delete algorithm outputs an MST.
  1. Show that every edge deleted belongs to no MST.
  2. Prove that the graph remaining at the end is a spanning tree.

Comments on MST Algorithms

- To handle multiple edges with the same weight, perturb each length by a random infinitesimal amount.
- Any algorithm that constructs a spanning tree by including edges that satisfy the cut property and deleting edges that satisfy the cycle property will yield an **MST**!

Teoria dos Grafos e Computabilidade

— Steiner Trees —

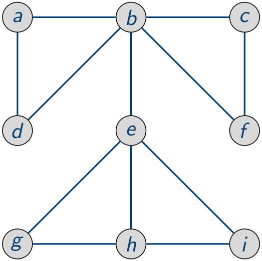
Silvio Jamil F. Guimarães

Graduate Program in Informatics – PPGINF
Laboratory of Image and Multimedia Data Science – IMScience
Pontifical Catholic University of Minas Gerais – PUC Minas

Feb 2025

Steiner Tree

Given a connected undirected graph $G = (V, E)$ and a set of $T \subseteq V$. A minimun size tree $H = (V', E')$ subgraph of G such that $T \subseteq V'$ is called as Steiner tree .



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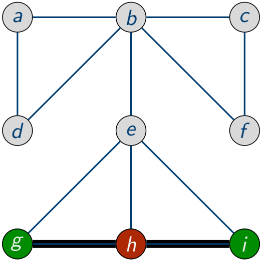

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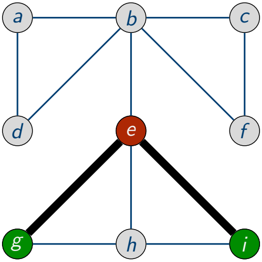

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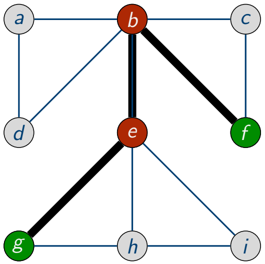

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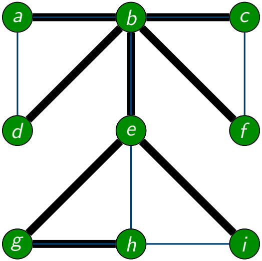

Is a spanning tree T' of G a Steiner tree?

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- The vertices in T are called terminals
- The vertices in $V \ \ T$ are called Steiner points
- Denote $n = |V|$, $m = |E|$ and $t = |T|$
- A minimum size:
  - Vertex cardinality: $|V'|$ or rather $|S| = |V' \ \ T|$ (default)
  - Edge cardinality: $|E'| = |V'| - 1$
  - Node weighted: Given $w : V \to \mathbb{N}$ minimize w(S)
  - Edge weighted: Given $w : E \to \mathbb{N}$ minimize w(E')