Teoria dos Grafos e Computabilidade

— Shortest path —

Silvio Jamil F. Guimarães

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Teoria dos Grafos e Computabilidade

— Some graph fundamentals —

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Graphs

- Model pairwise relationships (edges) between objects (nodes or vertices).
- **Undirected graph** $G = (V, E)$: set V of nodes and set E of edges, where $E \subseteq V \times V$. Elements of E are unordered pairs.
- **Directed graph** $G = (V, E)$: set V of nodes and set E of edges, where $E \subseteq V \times V$. Elements of E are ordered pairs.

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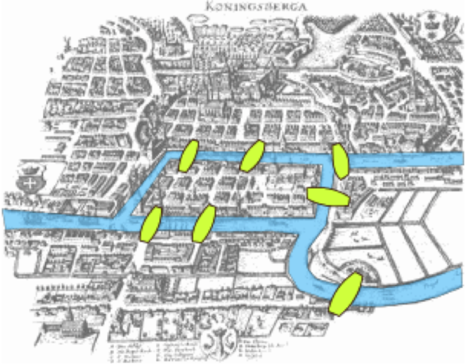

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— Shortest Path Problem —

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Shortest Path Problem

- $G = (V, E)$ is a connected directed graph. Each edge e has a length $l_e \geq 0$.
- V has n nodes and E has m edges.
- Length of a path P is the sum of lengths of the edges in P.
- Goal is to determine the shortest path from some start node s to each node in V.
- Aside: If G is undirected, convert to a directed graph by replacing each edge in G by two directed edges.

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SHORTEST PATHS

INSTANCE A directed graph $G = (V, E)$, a function $l : E \rightarrow \mathbb{R}^+$, and a node $s \in V$

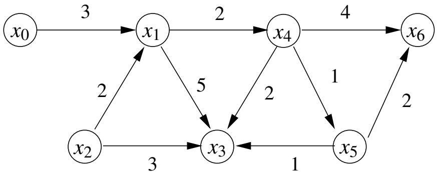
SOLUTION A set $\{P_u, u \in V\}$, where P_u is the shortest path in G from s to u.

Shortest paths

- Let $N = (G, W)$ be a positive length graph, let $x \in V$
- We define the map $L_x : V \to \mathbb{R} \cup \{\infty\}$ by:

$$L_x(y) = \begin{cases} \text{the length of a shortest path from } x \text{ to } y\text{, if such path exists} \\ \infty \text{ , otherwise} \end{cases}$$

Illustration: the map L_x



$y =$	x_0	x_1	x_2	x_3	x_4	x_5	x_6
$L_{x_0}(y) =$							

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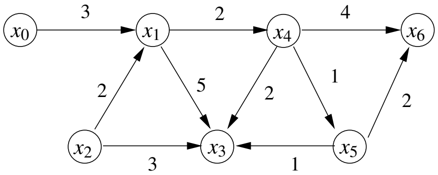

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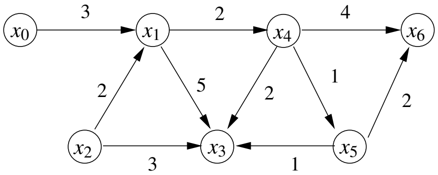

$y =$	x_0	x_1	x_2	x_3	x_4	x_5	x_6
$L_{x_0}(y) =$	0	3					

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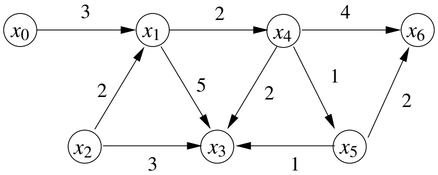

$y =$	x_0	x_1	x_2	x_3	x_4	x_5	x_6
$L_{x_0}(y) =$	0	3	∞				

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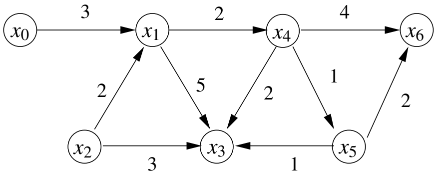

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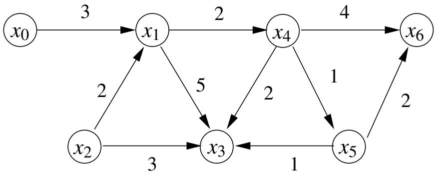

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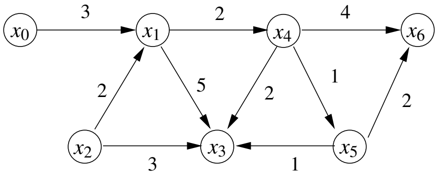

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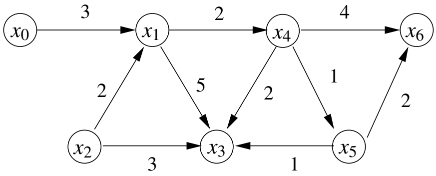

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Teoria dos Grafos e Computabilidade

— Algorithms for Single Source Shortest Path —

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Problems

1. Given a graph $G = (V, \Gamma)$, a network (G, ℓ) and two vertices x and y in V
   - Find a shortest path from x to y
   - Find the length $L_x(y)$ of a shortest path from x to y
2. Given a graph $G = (V, \Gamma)$, a network (G, ℓ) and a vertex x in v
   - Find for each vertex y in V the length $L_x(y)$ of a shortest path from x to y
3. Given a graph $G = (V, \Gamma)$ and a network (G, ℓ)
   - Find, for each pair x, y of vertices in V, the length of a shortest path from x to y
4. Having solved problem 2
   - Solve problem 1

Dijkstra algorithm

1. Given a graph $G = (V, \Gamma)$, a network (G, ℓ) and two vertices x and y in V
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Computing the lengths of shortest paths

Algorithm DIJKSTRA (Data: A graph $G = (V, \Gamma)$, a network (G, ℓ), $n = |V|$, $x \in V$;

Result: L_x)

$\bar{S} := \emptyset$;

For each $y \in V$ **Do** $L_x[y] = \infty$; $\bar{S} := \bar{S} \cup \{y\}$;

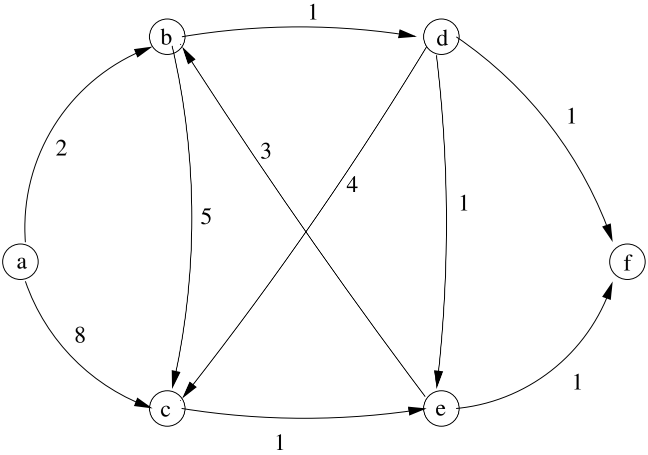
$L_x[x] := 0$; $k := 0$; $\mu := 0$;

While $k < n$ and $\mu \neq \infty$ **Do**

- Extract a vertex $y^\star \in \bar{S}$ such that $L_x[y^\star] = \min\{L_x[y], y \in \bar{S}\}$
- $k++$; $\mu := L_x[y^\star]$;
- **For each** $y \in \Gamma(y^\star) \cap \bar{S}$ **Do**
  - $L_x[y] := \min\{L_x[y], L_x[y^\star] + \ell(y^\star, y)\}$;

Computing the lengths of shortest paths

- *Exercise.* Execute "by hand" Dijsktra algorithm on the following network with $x = a$, and on any positive length network of your choice



Loop invariant of Dijkstra algorithm (# 1)

- Let $x \in V$ and $\mu \in \mathbb{R}$
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  2. $\bar{S} = V \setminus S$ contains any vertex y such that the length of a shortest path from x to y is greater than μ

Loop invariant of Dijkstra algorithm (# 2)

- Let $x \in V$, let $\mu \in \mathbb{R}$, and let S be a set that is μ-separating for x
- An S-path is a path whose intermediary vertices are all in S

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- Thus, $S \cup \{y^\star\}$ is a set that is μ'-separating with $\mu' = L_x^S(y^\star)$

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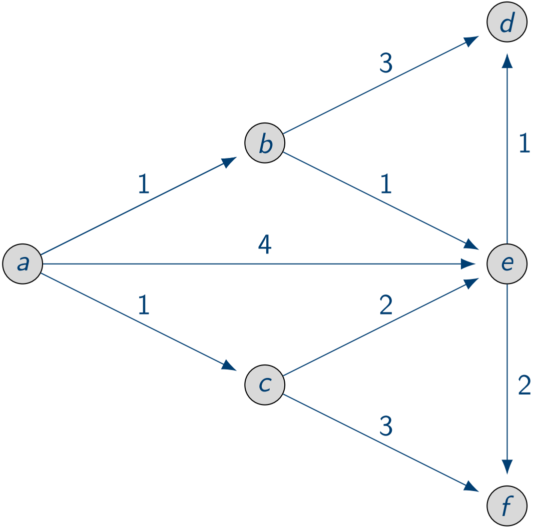
- Can modify algorithm to compute the shortest paths themselves: record the predecessor u that minimizes $d'(v)$.

Example of Dijkstra's Algorithm



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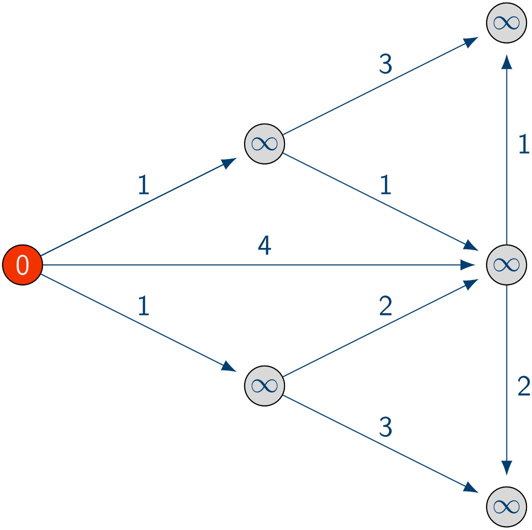

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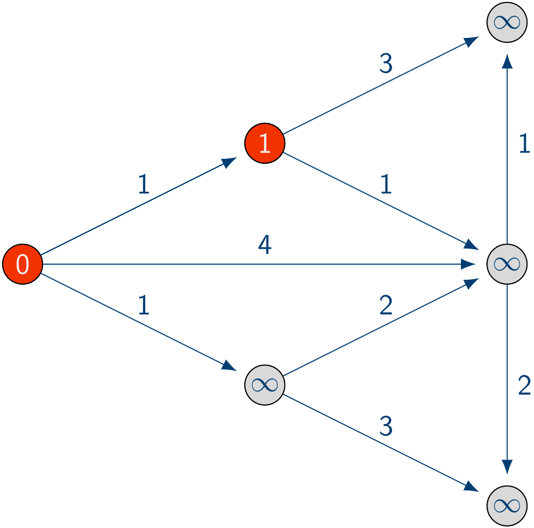

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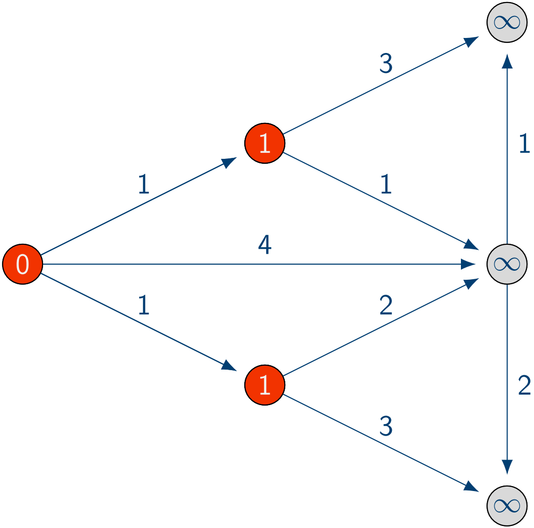

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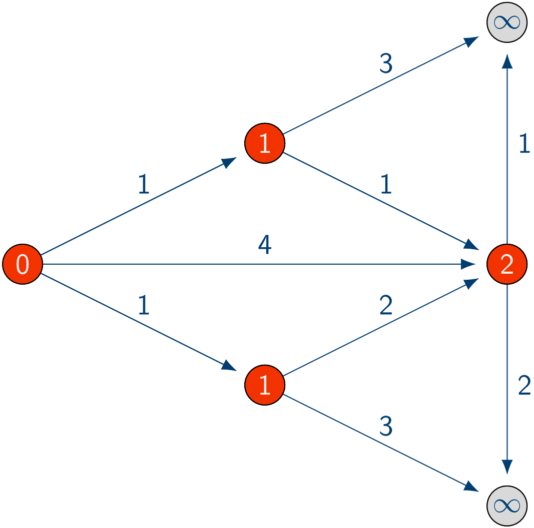

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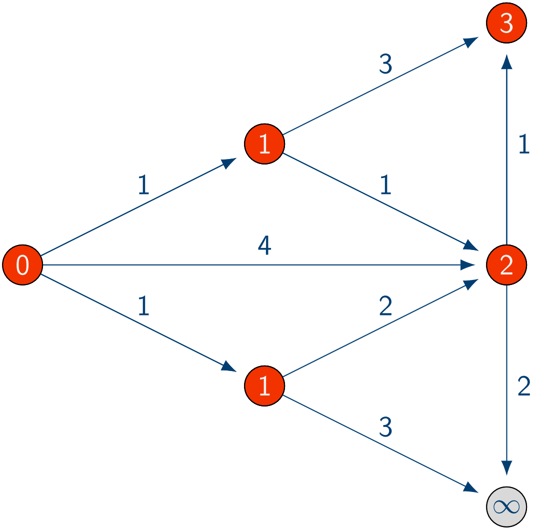

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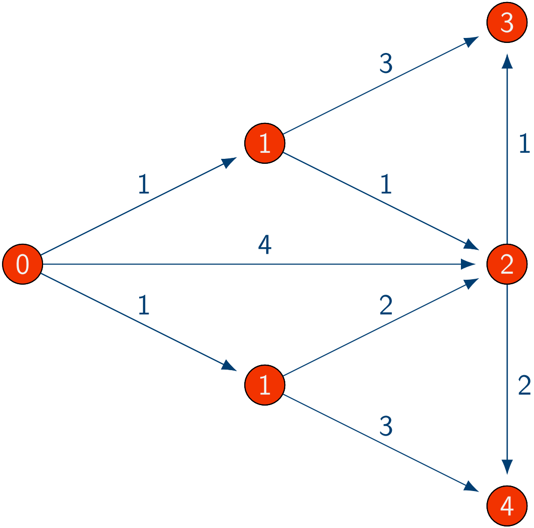

Proof of Correctness

- Let P_u be the shortest path computed for a node u.
- Claim: P_u is the shortest path from s to u.
- Prove by induction on the size of S.
  - Base case: $|S| = 1$. The only node in S is s.
  - Inductive step: we add the node v to S. Let u be the v's predecessor on the path P_v. Could there be a shorter path P from s to v?

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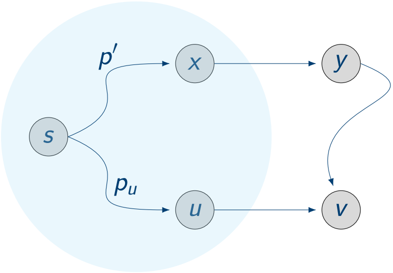

The alternate $s - v$ path P through x and y already too long by the time it had left the set S

Comments about Dijkstra's Algorithm

- Algorithm cannot handle negative edge lengths.
- Union of shortest paths output form a tree. Why?

Implementing Dijkstra's Algorithm

Algorithm: Shortest path algorithm – Dijkstra

input : A graph $G = (V, E)$, a weight map W and a source node s.
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Let S be the set of explored nodes;
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- How many iterations are there of the while loop? $n - 1$.
- In each iteration, for each node $v \notin S$, compute
  $$\min_{e=(u,v),u \in S} d(u) + l_e.$$

A Faster implementation of Dijkstra's Algorithm

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- Store the minima $d'(v)$ for each node $v \in V - S$ in a priority queue.
- Determine the next node v to add to S using ExtractMin.
- After adding v, for each neighbour w of v, compute $d(v) + l_{(v,w)}$.
- If $d(v) + l_{(v,w)} < d'(w)$, update w's key using ChangeKey.

A Faster implementation of Dijkstra's Algorithm

Algorithm: Shortest path algorithm – Dijkstra

input : A graph $G = (V, E)$, a weight map W and a source node s.
output: The distances of the vertices from s

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Let S be the set of explored nodes;
foreach u ∈ S do store distance d[u] = ∞;
Initially d[s] = 0 and S = s;
while S ≠ V do
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Single Source Shortest Path Problem

- $G = (V, E)$ is a connected directed graph. Each edge e has a length l_e. Note that the weights may be negative.
- V has n nodes and E has m edges.
- Length of a path P is the sum of lengths of the edges in P.
- Goal is to determine the shortest path from some start node s to all other nodes in V.
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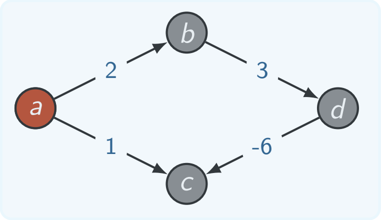
SHORTEST PATHS

INSTANCE A directed graph $G(V, E)$, a function $l : E \rightarrow \mathbb{R}$, and a node $s \in V$

SOLUTION A set $\{P_u, u \in V\}$, where P_u is the shortest path in G from s to u.

Bellman-Ford Algorithm

Dijkstra – Can fail if negative edge costs.



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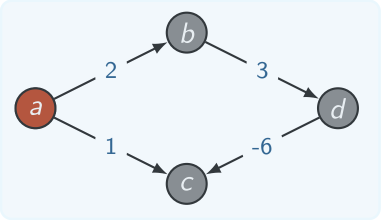
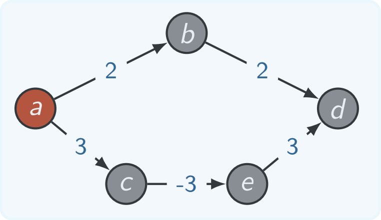


Re-weighting – Adding a constant to every edge weight can fail



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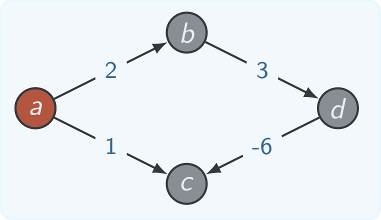

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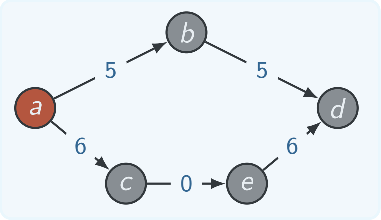
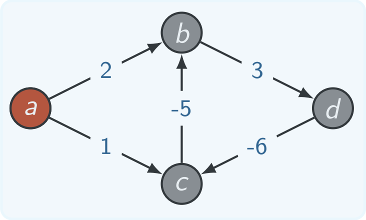


Bellman-Ford Algorithm

If some path from s to t contains a negative cost cycle, there does not exist a shortest s-t path; otherwise, there exists one that is simple.



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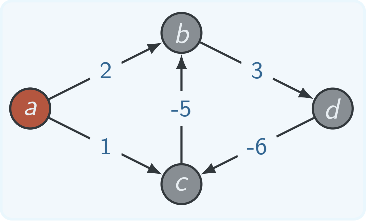

The Bellman-Ford algorithm is a way to find single source shortest paths in a graph with negative edge weights (but no negative cycles).

Bellman-Ford Algorithm

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$$OPT(i, v) = \begin{cases} 0, & \text{if } i = 0 \\ \min \begin{cases} OPT(i-1, v) \\ \min\{OPT(i-1, w) + c_{wv}\} \end{cases}, & \text{otherwise} \end{cases}$$

A Faster implementation of Dijkstra's Algorithm

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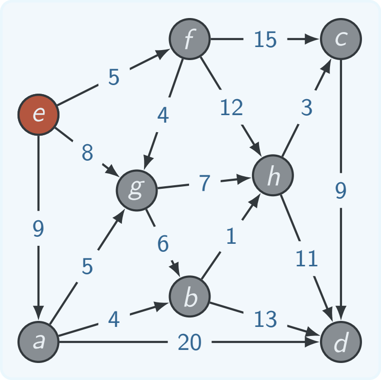
How to detect negative cycles?

Shortest path – an example



Compute the shortest path from e to all other nodes!