MUSIC, Root MUSIC and MVDR Algorithms

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Summary

Presentation Objectives

- Direction of Arrival (DoA) is a fundamental problem in array signal processing;
- Several algorithms have been developed to improve DoA estimation accuracy.

In this presentation, we will discuss:

1. MUSIC algorithm;
2. Root MUSIC algorithm;
3. MVDR beamformer.

System standards

Consider a Uniform Linear Array (ULA) with M antennas, spaced by Δ.

- Δ: Spacing between antennas;
- M: Number of antennas;
- d: Direction of arrival of the signal.

The time delay at each antenna due to signal arrival is given by:

$$\tau_M(\theta_i) = (M-1)\frac{\Delta \mathrm{sen}\theta_i}{c} \tag{1}$$

The spatial frequency associated with the direction θ_i is:

$$\mu_i = -\frac{2\pi f_c}{c}\Delta \mathrm{sen}(\theta_i) \tag{2}$$

Assuming $\Delta = \frac{\lambda}{2}$, we get:

$$\mu_i = -\pi \mathrm{sen}(\theta_i) \tag{3}$$

System standards

We define the steering matrix $\mathbf{A_{ULA}}$, as:

$$\mathbf{A_{ULA}} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\mu_1} & e^{j\mu_2} & \cdots & e^{j\mu_d} \\ e^{j2\mu_1} & e^{j2\mu_2} & \cdots & e^{j2\mu_d} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j(M-1)\mu_1} & e^{j(M-1)\mu_2} & \cdots & e^{j(M-1)\mu_d} \end{bmatrix} \in \mathbb{C}^{M\times d} \tag{4}$$

With this matrix $\mathbf{A_{ULA}}$, we can write our received signal model as:

$$\mathbf{X}(t) = \mathbf{A_{ULA}} \cdot \mathbf{S}(t) + \mathbf{N}(t) \tag{5}$$

where:

- $\mathbf{X} \in \mathbb{C}^{M\times t}$: Received signal at the antenna array.
- $\mathbf{A_{ULA}} \in \mathbb{C}^{M\times d}$: Steering matrix.
- $\mathbf{S} \in \mathbb{C}^{d\times t}$: Source signals.
- $\mathbf{N} \in \mathbb{C}^{M\times t}$: Additive noise.

The MUSIC spectrum is defined as:

$$S_{\mathrm{MUSIC}}(\theta) = \frac{\mathbf{a}^H(\theta)\mathbf{a}(\theta)}{\mathbf{a}^H(\theta)\mathbf{P_o}\mathbf{a}(\theta)}, \tag{6}$$

Where:

- $\mathbf{P_o} = \mathbf{U_o}\mathbf{U_o}^H$ is the projector onto the noise subspace $\mathcal{N}$;
- $\mathbf{U_o}$: **Noise subspace** – contains the eigenvectors associated with the smallest eigenvalues of the correlation matrix $\mathbf{R}$.
- $\mathbf{a}(\theta)$: **Steering vector** – describes the response of the antenna array to an incoming signal at angle θ.

Root MUSIC Polynomial:

$$P_{\text{Root-MUSIC}}(\theta) = \frac{1}{\mathbf{a}^H(\theta)\mathbf{C}\mathbf{a}(\theta)}, \tag{7}$$

where $\mathbf{C} = \mathbf{P_o}$ is the noise subspace correlation matrix.
Polynomial Formulation:

$$D(z) = \sum_{l=-M+1}^{M-1} c_l z^l, \tag{8}$$

where $z = e^{-jkd\sin\theta}$ and the coefficients c_l are the sum of the diagonal elements of $\mathbf{C}$.

- The roots of $D(z)$ closest to the unit circle correspond to the DoA estimates.
- The estimated angles are obtained from:

$$\theta_i = \sin^{-1}\left(\frac{1}{kd}\arg(z_i)\right). \tag{9}$$

MVDR Beamforming

MVDR Weight Computation:

$$\mathbf{w}_{\mathrm{MVDR}} = \frac{\mathbf{R}^{-1}\mathbf{a}(\theta_0)}{\mathbf{a}^H(\theta_0)\mathbf{R}^{-1}\mathbf{a}(\theta_0)}, \tag{10}$$

where:

- $\mathbf{w}_{\mathrm{MVDR}}$ is the weight vector;
- $\mathbf{R} = \mathbb{E}[\mathbf{X}\mathbf{X}^H]$ is the covariance matrix of the received signal;
- $\mathbf{a}(\theta_0)$ is the steering vector of the desired direction θ_0.

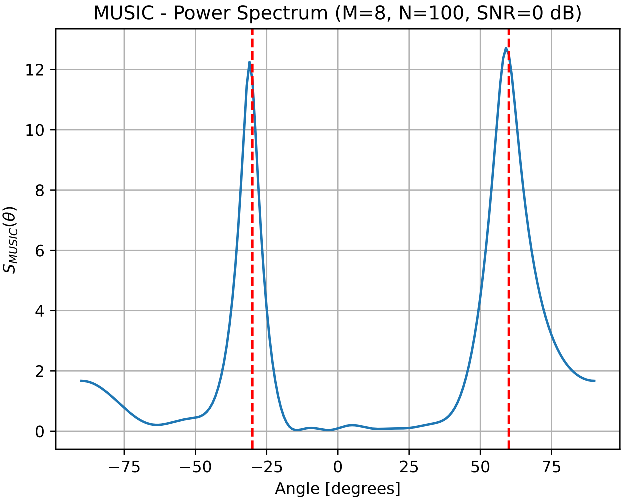
Beampattern Calculation:

$$B(\theta) = \mathbf{w}_{\mathrm{MVDR}}^H \mathbf{a}(\theta), \tag{11}$$

where $B(\theta)$ represents the beam pattern formed by the array.

MUSIC

The angles set were: $[\theta_1 = -30, \theta_2 = 60]$.



MUSIC - Signal-to-Noise Ratio (SNR) variation

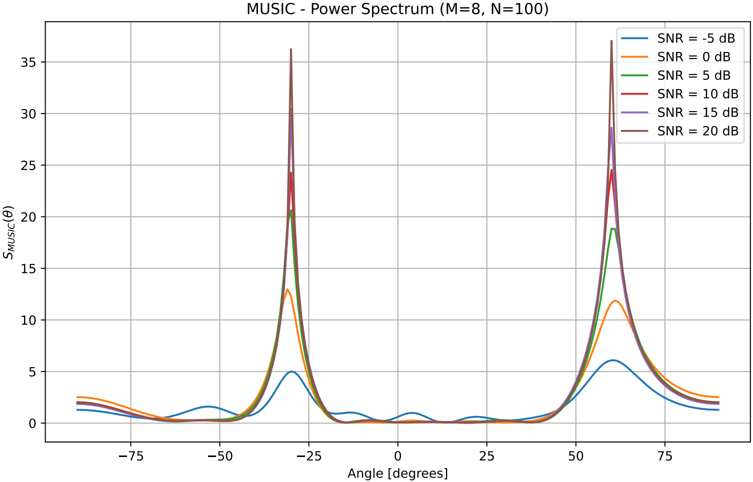


Figura: MUSIC Spectrum

MUSIC - Snapshot Variation

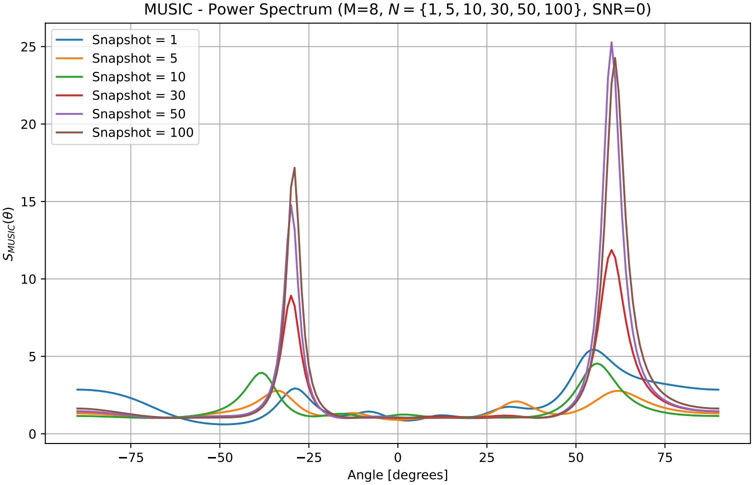


Figura: MUSIC Spectrum

Parameter	**Value**
Number of iterations	5000
Snapshot	500
Variation of θ	$\mathcal{U}[-60, 60]$
Variation of SNR	$\{-5, 0, 5, 10, 15, 20\}$

Root Mean Square Error (RMSE) formula is defined by:

$$RMSE(\hat{\phi}_r) = \sqrt{\mathbb{E}[|\phi_r - \hat{\phi}_r|^2]}. \tag{12}$$

MUSIC - RMSE

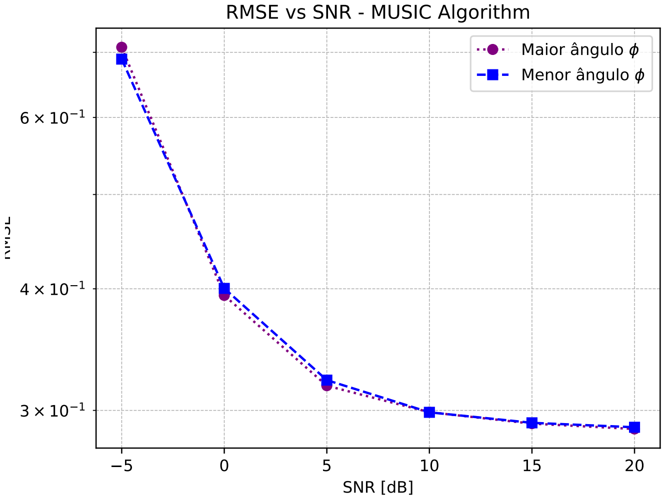


Figura: RMSE MUSIC

Root MUSIC

It was defined $\theta_1 = -4, \theta_2 = 30$.
And we found $\hat{\theta_1} = -4.33, \hat{\theta_2} = 30.29$.

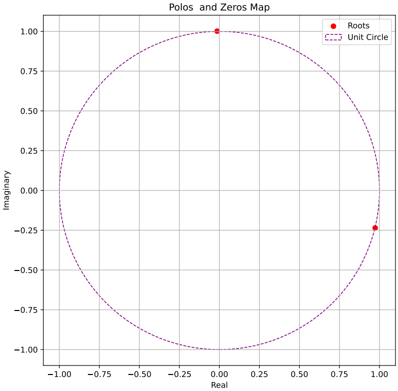


Figura: Root MUSIC

Root MUSIC - RMSE

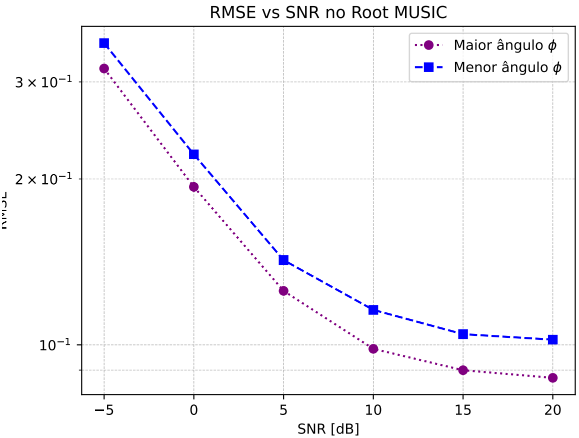


Figura: RMSE at Root MUSIC

MVDR Beampattern

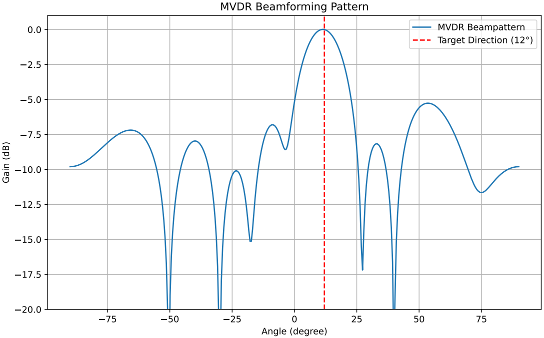


Figura: MVDR Beampattern.

MVDR Beampattern

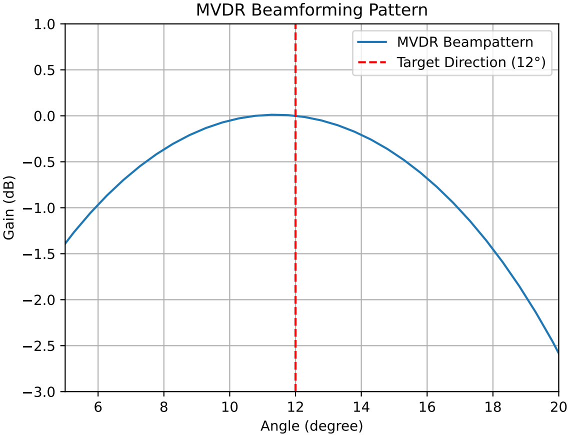


Figura: MVDR Beampattern with zoom.

Thank you! Questions?