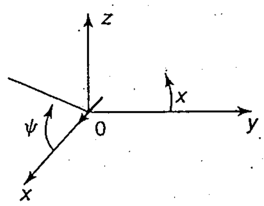
CHAPTER 4

4-1. a. $\sin\psi = \sqrt{1-\cos^2\psi} = \sqrt{1-|\hat{a}_z\cdot\hat{a}_r|^2}$
$= \sqrt{1-(\sin\theta\cdot\cos\phi)^2}$

In far-zone fields

$$E_\psi = j\eta\frac{kI_0\cdot le^{-jkr}}{4\pi r}\cdot\sin\psi = j\eta\frac{k\cdot I_0\cdot le^{-jkr}}{4\pi r}\cdot\sqrt{1-(\sin\theta\cdot\cos\phi)^2}$$
$$H_\chi = j\frac{kI_0le^{-jkr}}{4\pi r}\cdot\sin\psi = \frac{E_\psi}{\eta}$$



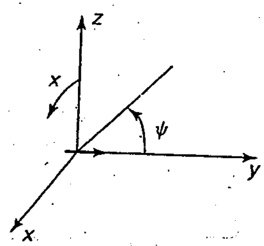
b. $$U = U_0(1-\sin^2\theta\cos^2\phi)$$
$$\therefore P_{\text{rad}} = U_0\int_0^{2\pi}\int_0^{\pi}(1-\sin^2\theta\cdot\cos^2\phi)\cdot\sin\theta\,d\theta\,d\phi = U_0\cdot\frac{8\pi}{3}$$
$$D_0 = \frac{4\pi\cdot U_0}{U_0\cdot\frac{8\pi}{3}} = \frac{3}{2} = 1.5$$

4-2. a. $\sin\psi = \sqrt{1-\cos^2\psi} = \sqrt{1-|\hat{a}_y\cdot\hat{a}_r|^2}$
$= \sqrt{1-\sin^2\theta\cdot\sin^2\phi}$

In far-zone, the fields are

$$E_\psi = j\eta\frac{kI_0le^{-jkr}}{4\pi r}\cdot\sin\psi = j\eta\frac{kI_0le^{-jkr}}{4\pi r}\cdot\sqrt{1-\sin^2\theta\cdot\sin^2\phi}$$

$$H_\chi \simeq \frac{E_\psi}{\eta} \simeq j\frac{kI_0e^{-jkr}\cdot l}{4\pi r}\sqrt{1-\sin^2\theta\sin^2\phi}$$



b. $U = U_0(1-\sin^2\theta\sin^2\phi)$

$$P_{\text{rad}} = U_0\int_0^{2\pi}\int_0^{\pi}(1-\sin^2\theta\cdot\sin^2\phi)\sin\theta\,d\theta\,d\phi$$

$$= U_0\int_0^{2\pi}\left[\int_0^{\pi}\sin\theta-\sin^3\theta\cdot\sin^2\phi\,d\theta\right]d\phi$$

$$= U_0\left[\int_0^{2\pi}2\,d\phi-\frac{4}{3}\int_0^{2\pi}\sin^2\phi\,d\phi\right] = U_0\left[4\pi-\frac{4}{3}\pi\right] = \frac{8}{3}\pi\cdot U_0$$

$$D_0 = \frac{4\pi\cdot U_0}{U_0\cdot\frac{8\pi}{3}} = \frac{3}{2} = 1.5$$

4-3. (a) $\underline{A} = \frac{\mu}{4\pi}\int \underline{I}_e\frac{e^{-jKR}}{R}\,dl' = \frac{\mu}{4\pi}\int_{-l/2}^{+l/2}\hat{a}_xI_0\frac{e^{-jkr}}{r}\,dx' = \hat{a}_x\frac{I_0\mu}{4\pi}\frac{e^{-jkr}}{4\pi r}\int_{-l/2}^{+l/2}dx'$

$$\underline{A} = \hat{a}_x\frac{l\mu I_0e^{-jkr}}{4\pi r} = \hat{a}_xA_x \Rightarrow A_x = \frac{\mu I_0l}{4\pi r}e^{-jkr}$$

$$\begin{pmatrix}A_r\\A_\theta\\A_\phi\end{pmatrix} = \begin{pmatrix}l\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{pmatrix}\begin{pmatrix}A_x\\0\\0\end{pmatrix} \tag{4-5}$$

$$A_r = A_x\sin\theta\cos\phi = \frac{\mu I_0le^{-jkr}}{4\pi r}\sin\theta\cos\phi$$

$$A_\theta = A_x\cos\theta\cos\phi = \frac{\mu I_0le^{-jkr}}{4\pi r}\cos\theta\cos\phi$$

$$A_\phi = -A_x\sin\phi = -\frac{\mu I_0le^{-jkr}}{4\pi r}\sin\phi$$

In far-field:

$$\left.\begin{aligned} E_r &\simeq 0 \\ E_\theta &\simeq -j\omega A_\theta \\ E_\phi &\simeq -j\omega A_\phi \end{aligned}\right\} \begin{aligned} &(3\text{-}58a) \Rightarrow \\ &(3\text{-}58b) \end{aligned} \quad \left.\begin{aligned} E_r &\simeq 0 \\ E_\theta &= -j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\cos\theta\cos\phi \\ E_\phi &= -j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\sin\phi \end{aligned}\right\} \begin{aligned} H_r &\simeq 0 \\ H_\phi &= \frac{E_\theta}{\eta} \\ H_\theta &= -\frac{E_\phi}{\eta} \end{aligned}$$

(b) $$U = \frac{r^2}{2\eta}[|E_\theta|^2 + |E_\phi|^2] \quad (2\text{-}12a)$$

$$U = \left(\frac{\omega\mu I_0 l}{4\pi}\right)^2 \frac{1}{2\eta}[\cos^2\theta\cos^2\phi + \sin^2\phi]$$

$$= B_0[\cos^2\theta\cos^2\phi + \sin^2\phi] \left(\underline{\text{see 3-D}}\ \underline{\text{plot}}\right)$$

$$B_0 = \frac{1}{2\eta}\left(\frac{\omega\mu I_0 l}{4\pi}\right)^2 = \frac{1}{2\eta}\left(\frac{\eta\omega\mu I_0 l}{\eta 4\pi}\right)^2 = \frac{1}{2\eta}\left[\frac{\eta\omega\mu I_0 l}{4\pi\sqrt{\mu/\epsilon}}\right]^2$$

$$= \frac{1}{2\eta}\left[\frac{\eta\omega\sqrt{\mu\epsilon}}{4\pi} I_0 l\right] = \frac{1}{2\eta}\left[\frac{\eta k I_0 l}{4\pi}\right] = \frac{\eta^2}{2\eta}\left(\frac{kI_0 l}{4\pi}\right)^2 = \frac{\eta}{2}\left(\frac{kI_0 l}{4\pi}\right)^2$$

$$B_0 = \frac{\eta}{2}\left(\frac{kI_0 l}{4\pi}\right)^2$$

$$U = B_0(\cos^2\theta\cos^2\phi + \sin^2\phi) \Rightarrow u_{\max} = B_0 \text{ when } \phi = 90°, 270°$$
$$0 \leqslant \theta \leqslant 180°$$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^\pi U\sin\theta\, d\theta\, d\phi$$

$$= B_0\left\{\underbrace{\int_0^{2\pi}\int_0^\pi \cos^2\theta\cos^2\phi\sin\theta\, d\theta\, d\phi}_{I_1} + \underbrace{\int_0^{2\pi}\int_0^\pi \cos^2\phi\sin\theta\, d\theta\, d\phi}_{I_2}\right\}$$

$$I_1 = \int_0^{2\pi}\cos^2\phi\, d\phi\int_0^\pi \cos^2\theta\sin\theta\, d\theta = \int_0^{2\pi}\cos^2\phi\, d\phi\int_0^\pi \cos^2\theta\, d(-\cos\theta)$$

$$= -\int_0^{2\pi}\left(\frac{1+\cos(2\phi)}{2}\right)d\phi\int_0^\pi (\cos\theta)^2\, d(\cos\theta)$$

$$= -\tfrac{1}{2}\left[\left(\phi + \tfrac{1}{2}\sin 2\phi\right)_0^{2\pi}\right]\left[\frac{\cos^3\theta}{3}\right]_0^\pi$$

$$I_1 = -\tfrac{1}{2}[(2\pi)]\left(-\frac{1}{3} - \frac{1}{3}\right) = \tfrac{1}{2}(2\pi)\left(\frac{2}{3}\right) = \frac{2\pi}{3}$$

$$I_2 = \int_0^{2\pi}\int_0^\pi \cos^2\phi\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\cos^2\phi\, d\phi\int_0^\pi \sin\theta\, d\theta$$

$$= \int_0^{2\pi}\left(\frac{1+\cos(2\phi)}{2}\right)d\phi\int_0^\pi \sin\theta\, d\theta$$

$$I_2 = \tfrac{1}{2}\left[\pi + \tfrac{1}{2}\sin 2\phi\right]_0^{2\pi}(-\cos\theta)_0^{\pi} = \tfrac{1}{2}(2\pi)[-(-1)+1] = 2\pi$$

$$I_1 + I_2 = \frac{2\pi}{3} + 2\pi = \frac{8\pi}{3}$$

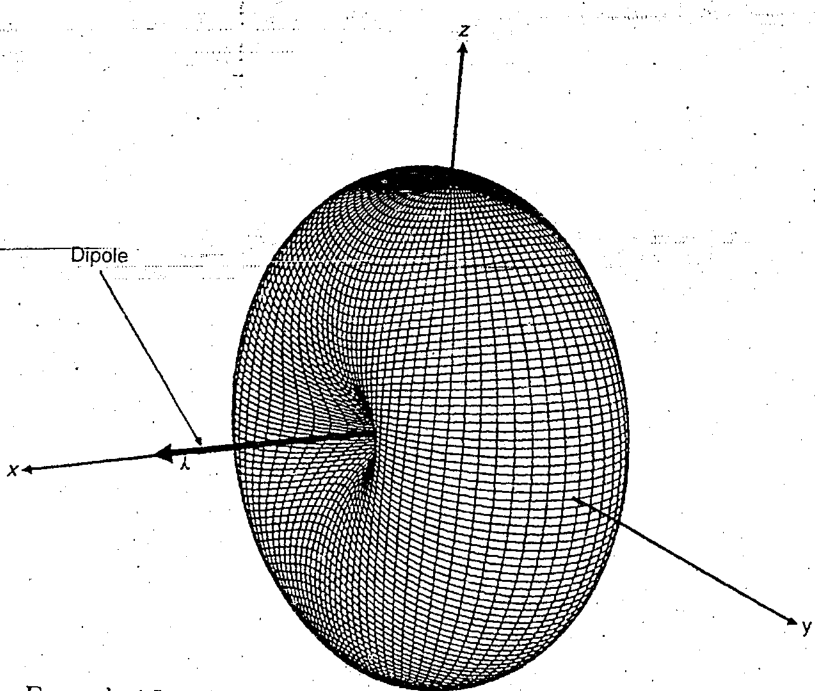
$$P_{\text{rad}} = B_0(I_1+I_2) = B_0\left(\frac{8\pi}{3}\right)$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi(B_0)}{\frac{8\pi}{3}(B_0)} = \frac{3}{2} = 1.761 \text{ dB}$$

$$\boxed{D_0 = 1.5 = +1.761 \text{ dB}}$$

c. Computer Program *Directivity*:

$$\boxed{D_0 = 1.4980 = 1.7551 \text{ dB}}$$



4-4. From Example 4.5

$$E_r \simeq 0$$

$$E_\theta \simeq -j\omega A_\theta = -j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\cos\theta\sin\phi$$

$$E_\phi \simeq -j\omega A_\phi = -j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\cos\phi$$

(a)
$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}}$$

$$U(\theta,\phi) = \frac{r^2}{2\eta}[|E_\theta|^2 + |F_\phi|^2] = \frac{1}{2\eta}\left(\frac{\omega\mu I_0 l}{4\pi}\right)^2 [\cos^2\theta\sin^2\phi + \cos^2\phi]$$

$$U(\theta,\phi) = \frac{1}{2\eta}\left(\frac{\eta\omega\mu I_0 l}{4\pi\sqrt{\mu/\varepsilon}}\right)^2 [\cos^2\theta\sin^2\phi + \cos^2\phi] = B_0(\cos^2\theta\sin^2\phi + \cos^2\phi)$$

$$B_0 = \frac{1}{2\eta}\left(\frac{\eta\omega\sqrt{\mu\varepsilon} I_0 l}{4\pi}\right)^2 = \frac{\eta}{2}\left(\frac{kI_0 l}{4\pi}\right)^2$$

$$U(\theta,\phi) = B_0(\cos^2\theta\sin^2\phi + \cos^2\phi) \Rightarrow U_{\max} = B_0 @ \phi = 0^\circ, 180^\circ$$

(b)
$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = B_0\int_0^{2\pi}\int_0^{\pi}(\cos^2\theta\sin^2\phi + \cos^2\phi)\sin\theta\, d\theta\, d\phi$$

$$= B_0\left\{\underbrace{\int_0^{2\pi}\int_0^{\pi}\cos^2\theta\sin^2\phi\sin\theta\, d\theta\, d\phi}_{I_1} + \underbrace{\int_0^{2\pi}\int_0^{\pi}\cos^2\phi\sin\theta\, d\theta\, d\phi}_{I_2}\right\}$$

$$I_1 = \int_0^{2\pi}\sin^2\phi\, d\phi\int_0^{\pi}\cos^2\theta\sin\theta\, d\theta$$

$$= \int_0^{2\pi}\left(\frac{1-\cos 2\phi}{2}\right)d\phi\int_0^{\pi}\cos^2\theta\, d(-\cos\theta)$$

$$= -\tfrac{1}{2}\left[\phi - \tfrac{1}{2}\sin 2\phi\right]_0^{2\pi}\left(\frac{\cos^3\theta}{3}\right)_0^{\pi} = -\tfrac{1}{2}(2\pi)\left(\frac{-1-1}{3}\right) = \frac{2\pi}{3}$$

$$I_2 = \int_0^{2\pi}\int_0^{\pi}\cos^2\phi\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\left(\frac{1+\cos 2\phi}{2}\right)d\phi\int_0^{\pi}\sin\theta\, d\theta$$

$$= \tfrac{1}{2}\left[\phi + \tfrac{1}{2}\sin 2\phi\right]_0^{2\pi}(-\cos\theta)_0^{\pi} = \tfrac{1}{2}(2\pi)(2) = 2\pi$$

$$P_{\text{rad}} = B_0(I_1 + I_2) = B_0\left(\frac{2\pi}{3} + 2\pi\right) = B_0\left(\frac{8\pi}{3}\right)$$

(c) $D_0 = \dfrac{4\pi U_{\max}}{P_{\text{rad}}} = \dfrac{4\pi B_0}{\frac{8\pi}{3}B_0} = \dfrac{3}{2}$ (same as in Problem 4-2 or any other infinitesimal dipole)

(d)
```
Input parameters:
-----------------
The lower bound of theta in degrees = 1
The upper bound of theta in degrees = 180
The lower bound of phi in degrees   = 0
The upper bound of phi in degrees   = 360
```

```
Output parameters:
-------------------
Radiated power (watts)       = 8.4122
Directivity (dimensionless) = 1.4938
Directivity (dB)             = 1.7430
```

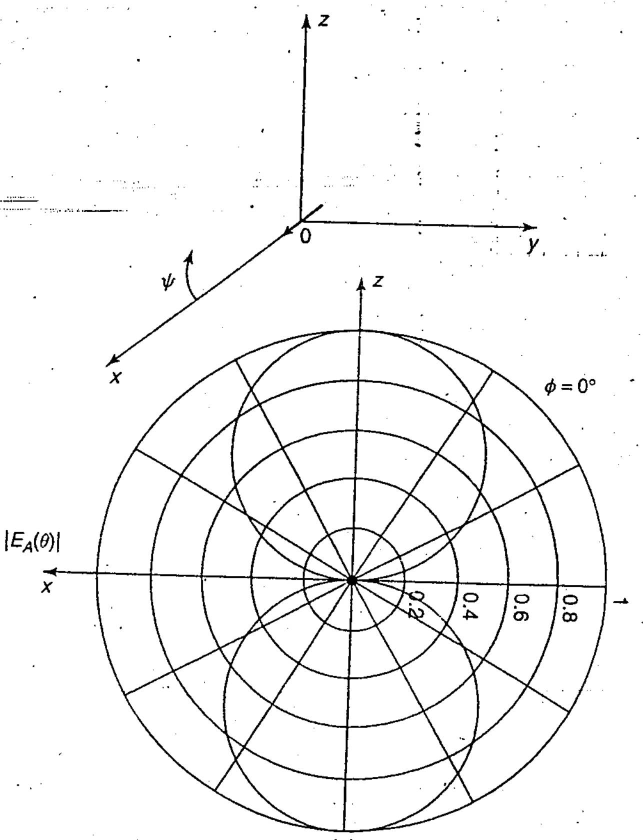
4-5. a. $\phi = 0^\circ$, $(x - z \text{ plane})$

$$E_\psi = j\eta\frac{kI_0 le^{-jkr}}{4\pi r}\sqrt{1-\sin^2\theta}$$

$$\simeq j\eta\frac{kI_0 le^{-jkr}}{4\pi r}\cdot\cos\theta$$

At $\phi = 0^\circ$, $\underline{E}_\psi$ has only $\hat{a}_\theta$ direction.

$\underline{E}_\psi \leftarrow \underline{E}_\theta$ polarization



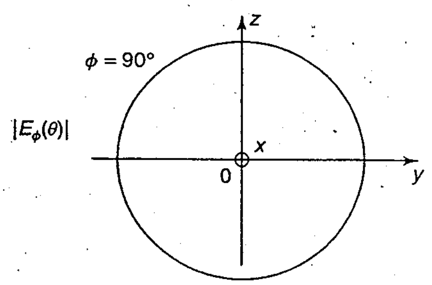
(a)

b. $\phi = 90^\circ$ ($y - z$ plane)

$$E_\psi \simeq j\eta \frac{kI_0 l e^{-jkr}}{4\pi r} \cdot 1$$

At $\phi = 90^\circ$, ($y - z$ plane), $\underline{E}_\psi$ has only $\hat{a}_\phi$ direction.

$\underline{E}_\psi \rightsquigarrow \underline{E}_\phi$ polarization



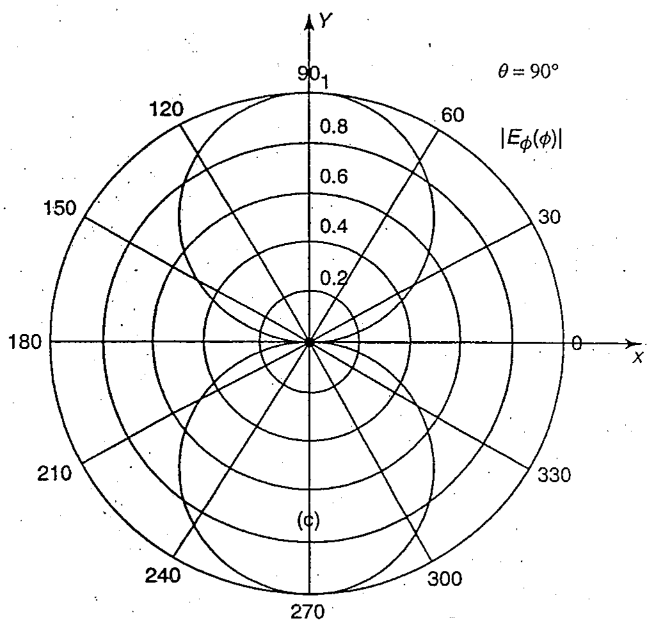
(b)

c. $\theta = 90^\circ$ ($x - y$ plane)

$$E_\psi = j\eta \frac{kI_0 l e^{-jkr}}{4\pi r} \sqrt{1 - \cos^2\phi} = j\eta \frac{kI_0 l e^{-jkr}}{4\pi r} \cdot \sin\phi$$

At $\theta = 90^\circ.(x, y)$, $\underline{E}_\psi$ has only $\hat{a}_\phi$ direction.

$\underline{E}_\psi \rightsquigarrow \underline{E}_\phi$ polarization



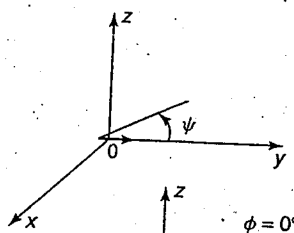
(c)

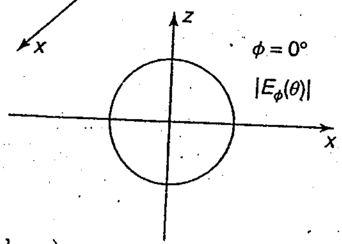
4-6. a. $\phi = 0°$ $(x - z \text{ plane})$

$$E_\psi = j\eta \frac{kI_0 e^{-jkr}}{4\pi r} \cdot 1$$

At $\phi = 0°$, $\underline{E}_\psi$ direction has only $\hat{a}_\phi$ component

$\underline{E}_\psi \rightsquigarrow \underline{E}_\phi$ polarization





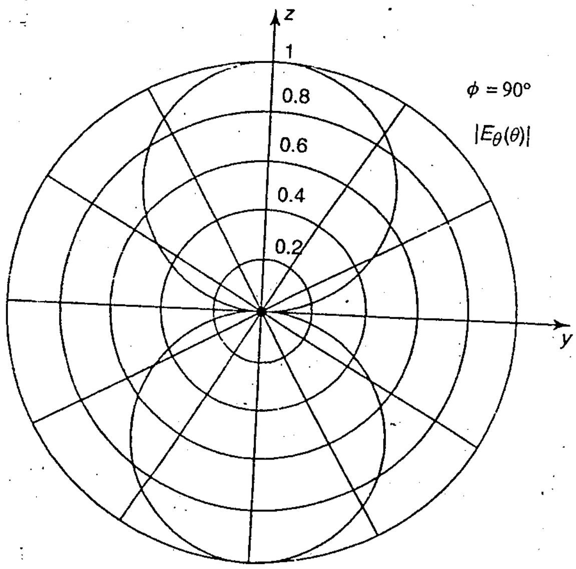
b. $\phi = 90°$ $(y - z \text{ plane})$

$$E_\psi = j\eta \frac{k_0 I_0 l e^{-jkr}}{4\pi r} \cdot \sqrt{1 - \sin^2\theta}$$

$$= j\eta \frac{k_0 I_0 l e^{-jkr}}{4\pi r} \cdot \quad \cos\theta.$$

At $\phi = 90°$, $\underline{E}_\psi$ direction has only $\hat{a}_\theta$ component

$\underline{E}_\psi \rightsquigarrow E_\theta$ polarization

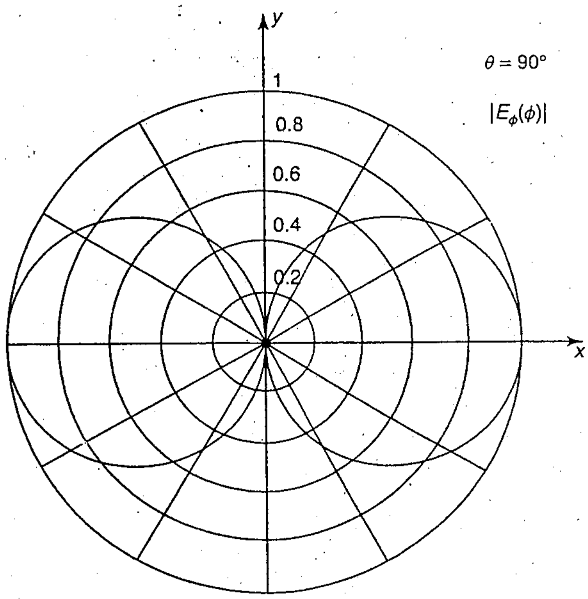


c. $\theta = 90°$ (x, y) plane.

$$E_\psi = j\eta \frac{k_0 I_0 l e^{-jkr}}{4\pi r} \cdot \cos\phi$$

At $\theta = 90°$, $\underline{E}_\psi$ direction has only $\hat{a}_\phi$ component

$\underline{E}_\psi \rightsquigarrow E_\phi$ polarization



4-7. $E_\theta = -j\dfrac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\cos\theta\cos\phi, E_\phi = -j\dfrac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\sin\phi$

(a) $\phi = 0$: $E_\theta = -j\dfrac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\cos\theta, E_\phi = 0$ (same as in Problem 4-5)

(b) $\phi = 90°$: $E_\theta = 0, E_\phi = -j\dfrac{\omega\mu I_0 l e^{-jkr}}{4\pi r}$ (same as in Problem 4-5)

(c) $\theta = 90°$: $E_\theta = 0, E_\phi = -j\dfrac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\sin\phi$ (same as in Problem 4-5)

4-8. From Example 4.5

$$E_\theta = -j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\cos\theta\sin\phi$$

$$E_\phi = -j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\cos\phi$$

(a) $\phi = 0$: $E_\theta = 0; E_\phi = -j\dfrac{\omega\mu I_0 l e^{-jkr}}{4\pi r}$ (same as in Problem 4-6)

(b) $\phi = 90°$: $E_\theta = -j\dfrac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\cos\theta$ (same as in Problem 4-6)

(c) $\theta = 90°$: $E_\theta = 0; E_\phi = -j\dfrac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\cos\phi$

4-9. (a) Using (4-26a)–(4-26c) and the duality of Table 3.2, the fields of an infinitesimal magnetic dipole of length l and magnetic current I_m are given by

$$E_r = E_\theta = H_\phi = 0$$
$$E_\phi = -j\frac{kI_m l}{4\pi r}\sin\theta\left[1 + \frac{1}{jkr}\right]e^{-jkr}$$
$$H_r = \frac{I_m l\cos\theta}{2\pi\eta r^2}\left[1 + \frac{1}{jkr}\right]e^{-jkr}$$
$$H_\theta = j\frac{kI_m l}{4\pi\eta r}\sin\theta\left[1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right]e^{-jkr}$$

(b) Since the pattern of the magnetic dipole is the same as that of the electric, the directivities are also identical and equal to

$$D_0 = \frac{3}{2}(\text{dimensionless}) = 1.761 \text{ dB}$$

4-10. (a) When the element is placed along the x-axis

$$\sin\psi = \sqrt{1 - \cos^2\psi} = \sqrt{1 - |\hat{a}_x \cdot \hat{a}_r|^2}$$
$$= \sqrt{1 - |\hat{a}_x \cdot (\hat{a}_x \sin\theta\cos\phi + \hat{a}_y \sin\theta\sin\phi + \hat{a}_z\cos\theta)|^2}$$

and the fields can be written as

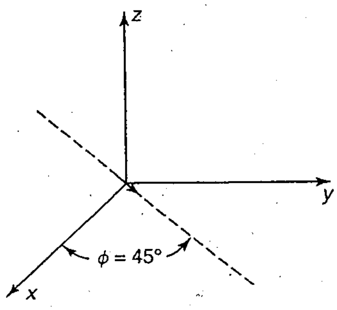
$$E_\chi = -j\cdot\frac{kI_m l e^{-jkr}}{4\pi r}\sqrt{1 - \sin^2\theta\cos^2\phi} = -j\cdot\frac{kI_m l e^{-jkr}}{4\pi r}\sin\psi$$
$$H_\psi = -\frac{E_\chi}{\eta}$$

(b) In a similar manner, when the element is placed along the y-axis

$$\sin\psi = \sqrt{1 - \cos^2\psi} = \sqrt{1 - |\hat{a}_y \cdot \hat{a}_r|^2} = \sqrt{1 - \sin^2\theta\sin^2\phi}$$

and the fields can be written as

$$E_\chi = -j\cdot\frac{kI_m l e^{-jkr}}{4\pi r}\sin\psi = -j\cdot\frac{kI_m l e^{-jkr}}{4\pi r}\sqrt{1 - \sin^2\theta\sin^2\phi}$$
$$H_\psi = -\frac{E_\chi}{\eta}$$



4-11. $E_\psi = j\eta \dfrac{kI_0 l}{4\pi r} e^{-jkr} \sin\psi$

$H_\chi = j\dfrac{kI_0 l}{4\pi r} e^{-jkr} \sin\psi$

Convert ψ to spherical coordinates

$$\sin\psi = 1 - \cos^2\psi = \sqrt{1 - \left(\frac{\hat{a}x + \hat{a}_y}{\sqrt{2}} \cdot \hat{a}_r\right)^2}$$

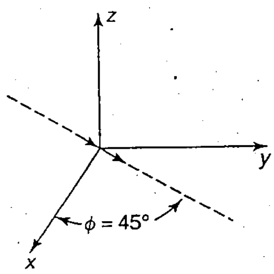
$$\frac{\hat{a}_x + \hat{a}_y}{\sqrt{2}} \cdot \hat{a}_r = \left(\frac{\hat{a}_x}{\sqrt{2}} + \frac{\hat{a}_y}{\sqrt{2}}\right) \cdot (\hat{a}_x \sin\theta\cos\phi + \hat{a}_y \sin\theta\sin\phi + \hat{a}_z\cos\theta)$$

$$= \frac{\sin\theta\cos\phi}{\sqrt{2}} + \frac{\sin\theta\sin\phi}{\sqrt{2}} = \frac{1}{\sqrt{2}}\sin\theta(\cos\phi + \sin\phi)$$

Thus

$$E_\psi = j\eta\frac{kI_0 l}{4\pi r} e^{-jkr}\sqrt{1 - \tfrac{1}{2}[\sin^2\theta(\cos\phi + \sin\phi)^2]}$$

$$H_\chi = j\frac{kI_0 l}{4\pi r} e^{-jkr}\sqrt{1 - \tfrac{1}{2}[\sin^2\theta(\cos\phi + \sin\phi)^2]}$$



4-12. $H_\psi = j\frac{kI_m l}{4\pi\eta r}e^{-jkr}\sin\psi$

$E_\chi = -j\frac{kI_m l}{4\pi r}e^{-jkr}\sin\psi$

Convert ψ to spherical coordinates

$$\sin\psi = \frac{1}{\sqrt{2}}\sin\theta(\sin\phi + \cos\phi)$$

Thus

$$H_\psi = j\frac{kI_m l}{4\pi\eta r}e^{-jkr}\sqrt{1-\tfrac{1}{2}[\sin^2\theta(\cos\phi+\sin\phi)^2]}$$
$$E_\chi = -j\frac{kI_m l}{4\pi r}e^{-jkr}\sqrt{1-\tfrac{1}{2}[\sin^2\theta(\cos\phi+\sin\phi)^2]}$$

4-13.

$$\underline{E} = \frac{1}{j\omega\epsilon}\nabla\times\underline{H} \quad \text{where } H_r = H_\theta = 0, H_\phi = j\frac{kI_0 l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr}\right]e^{-jkr}$$

Since $\underline{H}$ is not a function of ϕ

$$\underline{E} = \frac{1}{j\omega\epsilon}\nabla\times\underline{H} = \frac{1}{j\omega\epsilon}\left\{\hat{a}_r\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta} + (H_\phi\sin\theta) - \hat{a}_\theta\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) + \hat{a}_\phi(0)\right\}$$

which reduces using the H_ϕ from above to

$$E_r = \eta\frac{I_0 l\cos\theta}{2\pi r^2}\left[1+\frac{1}{jkr}\right]e^{-jkr}$$
$$E_\theta = j\eta\frac{kI_0 l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right]e^{-jkr}$$
$$E_\phi = 0$$

4-14. $\underline{W}_{\text{ave}} = \frac{1}{2}\text{Re}[\underline{E}\times\underline{H}^*] = \frac{1}{2}\text{Re}[\hat{a}_\theta E_\theta \times \hat{a}_\phi H_\phi^*]$

$$\underline{W}_{\text{ave}} = \hat{a}_r W_r = \frac{1}{2}\text{Re}\left[\hat{a}_\theta E_\theta\times\hat{a}_\phi\frac{E_\theta^*}{\eta}\right] = \hat{a}_r\frac{1}{2\eta}\text{Re}(|E_\theta|^2) = \hat{a}_r\frac{|E_\theta|^2}{2\eta}$$
$$W_r = \left[\frac{\eta}{2}\left|\frac{kI_0 l}{4\pi}\right|^2\right]\frac{\sin^2\theta}{r^2} = W_0\frac{\sin^2\theta}{r^2}, \quad \text{where } W_0 = \frac{\eta}{2}\left|\frac{kI_0 l}{4\pi}\right|^2$$
$$P_{\text{rad}} = \int_0^{2\pi}\int_0^\pi \underline{W}_{\text{ave}}\hat{a}_r r^2\sin\theta\, d\theta\, d\phi = 2\pi W_0\int_0^\pi \sin^3\theta\, d\theta = 2\pi W_0\left(\frac{4}{3}\right)$$
$$P_{\text{rad}} = \frac{8\pi}{3}W_0 = \eta\left(\frac{\pi}{3}\right)\left|\frac{I_0 l}{\lambda}\right|^2$$

4-15. $\underline{A} = \hat{a}_z A_z = \hat{a}_z \frac{\mu I_0 l}{4\pi r} e^{-jkr} \Rightarrow A_z = \frac{\mu_0 I_0 l}{4\pi r} e^{-jkr}$

using (4-6a)–(4-6c)

$$A_r = A_z \cos\theta = \frac{\mu I_0 l e^{-jkr}}{4\pi r} \cos\theta = A'_r(\theta) \frac{e^{-jkr}}{r} \quad \Rightarrow A'_r = \frac{\mu I_0 l}{4\pi} \cos\theta$$
$$A_\theta = -A_z \sin\theta = \frac{-\mu I_0 l e^{-jkr}}{4\pi r} \sin\theta = A'_\theta(\theta) \frac{e^{-jkr}}{r} \Rightarrow A'_\theta = \frac{-\mu I_0 l}{4\pi} \sin\theta$$
$$A_\phi = 0 \qquad \Rightarrow A'_\phi = 0$$

Substituting these into (3-57) and (3-57a) reduces to

$$E_r = 0$$
$$E_\theta = -j\omega \frac{e^{-jkr}}{r} A'_\theta = j \frac{\omega \mu I_0 l e^{-jkr}}{4\pi r} \sin\theta = j\eta \frac{k I_0 l e^{-jkr}}{4\pi r} \sin\theta$$
$$E_\phi = -j\omega \frac{e^{-jkr}}{r} A'_\phi = 0$$
$$H_r = 0$$
$$H_\theta = j \frac{\omega}{\eta} \frac{e^{-jkr}}{r} A'_\phi = 0$$
$$H_\phi = -j \frac{\omega}{\eta} \frac{e^{-jkr}}{r} A'_\theta = j \frac{\omega \mu I_0 l e^{-jkr}}{4\pi \eta r} \sin\theta = j \frac{k I_0 l e^{-jkr}}{4\pi r} \sin\theta$$

which are identical to (4-26a)–(4-26c)

4-16. $R = [r^2 + (-2rz' \cos\theta + z'^2)]^{1/2} = r \left[1 + \left(\frac{-2rz' \cos\theta + z'^2}{r^2}\right)\right]^{1/2}$

Using the binomial expansion of

$$(a+b)^n = \frac{a^n b^0}{0!} + \frac{n a^{n-1} b^1}{1!} + (n)(n-1) \frac{a^{n-2} \cdot b^2}{2!} + (n)(n-1)(n-2) \frac{a^{n-3} b^3}{3!} + \cdots$$

it can be shown by letting

$$a = r^2$$
$$b = (-2rz' \cos\theta + z'^2)$$
$$n = \tfrac{1}{2}$$

that

$$R = r - z' \cos\theta + \frac{1}{r}\left(\frac{z'^2}{2} \sin^2\theta\right) + \frac{1}{r^2}\left(\frac{z'^3}{2} \cos\theta \cdot \sin^2\theta\right)$$
$$+ \frac{1}{r^3}\left[\frac{z'^4}{8}(-1 + 6\cos^2\theta - 5\cos^4\theta)\right] + \cdots$$

Therefore the fifth term of (4-41) is

$$\frac{1}{r^3}\left[\frac{z'^4}{8}(-1+6\cos^2\theta-5\cos^4\theta)\right]$$

4-17. For maximum phase error of $\pi/8$ radians

$$0.62\sqrt{D^3/\lambda}\leqslant r\leqslant 2D^2/\lambda$$

(a) For a maximum phase error of $\pi/16$ radians

$$\sqrt{2(0.385)}\sqrt{D^3/\lambda}\leqslant r\leqslant 4D^2/\lambda$$
$$0.8775\sqrt{D^3/\lambda}\leqslant r\leqslant 4D^2/\lambda$$

(b) For a maximum phase error of $\pi/4$ radians

$$\sqrt{\frac{0.385}{2}}\cdot\sqrt{D^3/\lambda}\leqslant r\leqslant D^2/\lambda$$
$$0.43875\sqrt{D^3/\lambda}\leqslant r\leqslant D^2/\lambda$$

(c) For a maximum phase error of 18° radians

$$18^\circ \rightsquigarrow \frac{\pi}{10}\ \text{radians}$$

$$\sqrt{1.25(0.385)}\sqrt{D^3/\lambda}\leqslant r\leqslant(1.25)\cdot 2D^2/\lambda$$
$$0.6937\sqrt{D^3/\lambda}\leqslant r\leqslant 2.5D^2/\lambda$$

(d) For a maximum phase error of 15° radians

$$15^\circ \rightsquigarrow \frac{\pi}{12}\ \text{radians.}$$

$$\sqrt{1.5(0.385)}\sqrt{D^3/\lambda}\leqslant r\leqslant(1.5)\cdot 2\cdot D^2/\lambda$$
$$0.7599\sqrt{D^3/\lambda}\leqslant r\leqslant 3\cdot D^2/\lambda$$

4-18. $l=5\lambda_0 \Rightarrow z'|_{\max}=2.5\lambda$

a. Far-Field (Fraunhofer)

$$r=\frac{2l^2}{\lambda}=\frac{2(5\lambda)^2}{\lambda}=\frac{2(25\lambda^2)}{\lambda}=50\lambda$$

$$\Delta\phi_e=\frac{k}{r}\left(\frac{z'^2}{2}\sin^2\theta\right)\Bigg|_{\substack{\theta=30^\circ,z'=2.5\lambda,\\ r=50\lambda}}=\frac{\frac{2\pi}{\lambda}}{50\lambda}\left[\frac{(2.5)^2\lambda^2}{2}\frac{1}{4}\right]=0.0982\ \text{rads}=5.6250^\circ$$

b. Fresnel

$$r = 0.62\sqrt{l^3/\lambda} = 0.62\sqrt{(5\lambda)^3/\lambda} = 0.62 \cdot \lambda \cdot \sqrt{125} = 6.9318\lambda$$

$$\Delta\phi_e = \frac{k}{r^2}\left(\frac{z'^3}{2}\cos\theta\sin^2\theta\right)\Bigg|_{\substack{\theta=30^\circ \\ z'=2.5\lambda \\ r=6.9318\lambda}} = \frac{2\pi}{\lambda}\frac{(2.5\lambda)^3}{(6.9318\lambda)^2\cdot 2}(\cos 30^\circ)(\sin 30^\circ)^2$$

$$\Delta\phi_e = \frac{\pi(2.5)^3}{(6.9318)^2}(0.866)(0.5)^2 = 0.2212 \text{ rads} = 12.6724^\circ$$

4-19. $$\underline{A} = \hat{a}_z\frac{\mu I_0}{4\pi}\int_0^l e^{-jkz'}\frac{e^{-jkR}}{R}\,dz' \cong \hat{a}_z\frac{\mu I_0}{4\pi r}e^{-jkr}\int_0^l e^{-jk(1-\cos\theta)z'}\,dz'$$

$$A_z \simeq \frac{\mu I_0 e^{-jkr}}{4\pi r}\int_0^l \frac{e^{-jk(1-\cos\theta)z'}\,d[-jk(1-\cos\theta)z']}{-jk(1-\cos\theta)}$$

$$A_z \simeq \frac{\mu I_0 e^{-jkr}}{4\pi r}\left[\frac{e^{-jk(1-\cos\theta)z'}}{-jk(1-\cos\theta)}\right]_0^l = \frac{\mu I_0 l e^{-jkr}}{4\pi r}e^{-jz}\frac{\sin(z)}{z}$$

where $z = \dfrac{kl}{2}(1-\cos\theta)$

(a) $$\left.\begin{array}{l} A_r = A_z\cos\theta \\ A_\theta = -A_z\sin\theta \\ A_\phi = 0 \end{array}\right\} \Rightarrow \text{For far-field} \Rightarrow \left\{\begin{array}{l} E_\theta \simeq -j\omega A_\theta \\ E_\phi \simeq -j\omega A_\phi \\ E_r \simeq 0 \end{array}\right.$$

Therefore

$$E_r \simeq 0 \simeq H_r, \quad E_\theta \simeq j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}e^{-jz}\frac{\sin(z)}{z}\sin\theta$$

$$E_\phi = 0 = H_\theta, \quad H_\phi \simeq \frac{E_\theta}{\eta}$$

(b) $$\underline{W}_{\text{ave}} = \underline{W}_{\text{rad}} = \tfrac{1}{2}\text{Re}[\underline{E}\times\underline{H}^*] = \frac{1}{2\eta}|E_\theta|^2$$

$$= \frac{1}{2\eta}\left|\frac{\omega\mu I_0 l}{4\pi r}\cdot\frac{\sin(z)}{z}\cdot\sin\theta\right|^2$$

4-20. (a) $$\underline{A} = \frac{\mu}{4\pi}\int_{-\infty}^{\infty}\underline{I}(z')\frac{e^{-jkR}}{R}\,dz' = \hat{a}_z\frac{\mu I_0}{4\pi}\int_{-\infty}^{\infty}\frac{e^{-jkR}}{R}\,dz'$$

where

$$R = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\,|_{x'=y'=0} = \sqrt{x^2+y^2+(z-z')^2}$$

Making a change of variable of the form,

$$z - z' = -p, \quad dz' = dp$$

reduces the potential to

$$A_z = \frac{\mu I_0}{4\pi}\int_{-\infty}^{\infty}\frac{e^{-jk\sqrt{\rho^2+p^2}}}{\sqrt{\rho^2+p^2}}\,dp \quad \text{where } \rho^2 = x^2+y^2$$

Using

$$\int_{-\infty}^{\infty}\frac{e^{-j\beta\sqrt{b^2+t^2}}}{\sqrt{b^2+t^2}}\,dt = -j\pi H_0{}^{(2)}(b\beta)$$

We can write the potential as

$$A_z = -j\frac{\mu I_0}{4}H_0{}^{(2)}(k\rho) = -j\frac{\mu I_0}{4}H_0{}^{(2)}(k\sqrt{x^2+y^2})$$

(b) $\underline{H} = \frac{1}{\mu}\nabla\times\underline{A}$ and $\underline{E} = \frac{1}{j\omega\epsilon}\nabla\times\underline{H}$

Since $A_\rho = A_\phi = 0$, in cylindrical coordinates

$$\underline{H} = \frac{1}{\mu}\nabla\times\underline{A} = \frac{1}{\mu}\left(-\hat{a}_\phi\frac{\partial A_z}{\partial\rho}\right) = \hat{a}_\phi j\frac{I_0}{4}\frac{\partial}{\partial\rho}H_0{}^{(2)}(k\rho)$$

Using Equation (V-19), we can write the $\underline{H}$-field as

$$\underline{H} = \hat{a}_\phi H_\phi = -\hat{a}_\phi j\frac{kI_0}{4}H_1{}^{(2)}(k\rho)$$

where $H_1{}^{(2)}(k\rho)$ is the Hankel function of the second kind of order one and argument $k\rho$.

The electric field can be obtained using

$$\begin{aligned}\underline{E} = \frac{1}{j\omega\epsilon}\nabla\times\underline{H} &= \hat{a}_z\frac{1}{j\omega\epsilon}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi)\right] = \hat{a}_z\frac{1}{j\omega\epsilon}\left(\frac{\partial H_\phi}{\partial\rho}+\frac{H_\phi}{\rho}\right)\\ &= \hat{a}_z\frac{1}{j\omega\epsilon}\left[-j\frac{kI_0}{4}\frac{\partial}{\partial\rho}H_1{}^{(2)}(k\rho) - j\frac{kI_0}{4\rho}H_1{}^{(2)}(k\rho)\right]\end{aligned}$$

Since $\frac{\partial}{\partial\rho}H_1{}^{(2)}(k\rho) = kH_0{}^{(2)}(k\rho) - \frac{1}{\rho}H_1{}^{(2)}(k\rho)$ by using V-18

then

$$\underline{E} = \hat{a}_z\left[-\frac{kI_0}{4\,\omega\epsilon}kH_0{}^{(2)}(k\rho)\right] = -\hat{a}_z\eta\frac{I_0k}{4}H_0{}^{(2)}(k\rho)$$

4-21. $P_{\text{rad}} = \eta \dfrac{|I_0|^2}{4\pi} I_{\text{int}}$

where

$$I_{\text{int}} = \int_0^{\pi} \frac{\left[\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)\right]^2}{\sin\theta}\, d\theta$$

which can also be written as

$$I_{\text{int}} = 2\int_0^{\pi/2} \frac{\left[\cos^2\left(\frac{kl}{2}\cos\theta\right) + \cos^2\left(\frac{kl}{2}\right) - 2\cos\left(\frac{kl}{2}\cos\theta\right)\cos\left(\frac{kl}{2}\right)\right]}{\sin\theta}\, d\theta$$

Letting $\begin{cases} l\cos\theta = u \\ -\sin\theta d\theta = du \Rightarrow d\theta = -\dfrac{du}{\sin\theta} \end{cases}$

reduces I_{int} to

$$\begin{aligned} I_{\text{int}} &= 2\int_0^{1} \frac{\left[\cos^2\left(\frac{kl}{2}u\right) + \cos^2\left(\frac{kl}{2}\right) - 2\cos\left(\frac{kl}{2}u\right)\cos\left(\frac{kl}{2}\right)\right]}{1-u^2}\, du \\ &= \int_{-1}^{1} \frac{\left[\cos^2\left(\frac{kl}{2}u\right) + \cos^2\left(\frac{kl}{2}\right) - 2\cos\left(\frac{kl}{2}u\right)\cos\left(\frac{kl}{2}\right)\right]}{(1+u)}\, du \\ &= \tfrac{1}{2}\int_{-1}^{1} \frac{[1+\cos(klu) + 1 + \cos(kl)}{(1+u)}\, du - \int_{-1}^{1} \frac{\cos\left[\frac{kl}{2}(1+u)\right] + \cos\left[\frac{kl}{2}(1-u)\right]}{1+u}\, du \end{aligned}$$

Making another change of variable of the form

$$(1+u)kl = v \Rightarrow du = \frac{dv}{kl}$$

we can write that

$$\begin{aligned} I_{\text{int}} &= \tfrac{1}{2}\int_0^{2kl} \frac{2+\cos(kl)+\cos(klv)}{v}\, dv - \int_0^{kl} \frac{\cos v}{v}\, dv - \int_{-1}^{1} \frac{\cos\left[\frac{kl}{2}(1-u)\right]}{1+u}\, du. \\ &= \int_0^{kl} \frac{1+\cos(kl)-\cos(v)}{v}\, dv + \tfrac{1}{2}\int_0^{2kl} \frac{-\cos(kl)+\cos(v-kl)}{v}\, dv \\ &\quad - \int_0^{1} \frac{\cos[kl(1-v)]}{v}\, dv \end{aligned}$$

provided $v = \dfrac{1+u}{2}$

If $z = klv$,

$$I_{\text{int}} = \int_0^{kl} \frac{1 + \cos(kl) - \cos(v)}{v}\,dv$$

$$+ \frac{1}{2} \int_0^{2kl} \frac{-\cos(kl) + \cos(v)\cos(kl) + \sin(v)\sin(kl)}{v}\,dv$$

$$- \int_0^{kl} \frac{\cos(kl)\cos(z) + \sin(kl)\sin(z)}{z}\,dz$$

$$I_{\text{int}} = [1 + \cos(kl)] \int_0^{kl} \frac{1 - \cos v}{v}\,dv - 2 \int_0^{kl} \frac{\sin(v)\sin(kl)}{v}\,dv$$

$$+ \sin(kl) \int_0^{2kl} \frac{\sin v}{v}\,dv - \cos(kl) \int_0^{2kl} \frac{1 - \cos v}{v}\,dv$$

which reduces to

$$I_{\text{int}} = \Big\{C + \ln(kl) - C_i(kl) + \tfrac{1}{2}\sin(kl)[S_i(2kl) - 2S_i(kl)]$$

$$+ \tfrac{1}{2}\cos(kl)\left[C + \ln\left(\frac{kl}{2}\right) + C_i(2kl) - 2C_i(kl)\right]\Big\}$$

where $C = 0.5772$

and

~~$P_{\text{rad}} = \eta \frac{|I_0|^2}{4\pi} I_{\text{int}}$ is identical to (4 − 68)~~

From (4-88)

$$P_{\text{rad}} = \eta \frac{|I_0|^2}{4\pi} \int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\,d\theta$$

Letting

$$\left.\begin{array}{l} u = \cos\theta \\ du = -\sin\theta\,d\theta \end{array}\right\} \Rightarrow \sin^2\theta = 1 - \cos^2\theta = 1 - u^2.$$

We can write

$$P_{\text{rad}} = -\eta \frac{|I_0|^2}{2\pi} \int_1^0 \frac{\cos^2\left(\frac{\pi}{2}u\right)}{1 - u^2}\,du = \eta \frac{|I_0|^2}{2\pi} \int_0^1 \frac{\cos^2\left(\frac{\pi}{2}u\right)}{1 - u^2}\,du$$

which can also be written as

$$P_{\text{rad}} = \eta \frac{|I_0|^2}{4\pi} \left[\int_0^1 \frac{\cos^2\left(\frac{\pi}{2}u\right)}{1 - u}\,du + \int_0^1 \frac{\cos^2\left(\frac{\pi}{2}u\right)}{1 + u}\,du\right]$$

Making another change of variable of the form

$$\left.\begin{array}{l} v = 1 - u \\ dv = -du \end{array}\right\} \text{ for the first integral, } \left.\begin{array}{r} v = 1 + u \\ dv = du \end{array}\right\} \text{ for the second integral}$$

We can write P_{rad} as

$$P_{\text{rad}} = \eta\frac{|I_0|^2}{4\pi}\left\{\int_0^1 \frac{\sin^2\left(\frac{\pi}{2}v\right)}{v}\,dv + \int_1^2 \frac{\sin^2\left(\frac{\pi}{2}v\right)}{v}\,dv\right\} = \eta\frac{|I_0|^2}{4\pi}\int_0^2 \frac{\sin^2\left(\frac{\pi}{2}v\right)}{v}\,dv$$

Using the half-angle identity $< \sin^2\left(\frac{\pi}{2}v\right) = \frac{1-\cos(\pi v)}{2} >$ reduces P_{rad} to

$$P_{\text{rad}} = \eta\frac{|I_0|^2}{8\pi}\int_0^2 \frac{[1-\cos(\pi v)]}{v}\,dv$$

By letting $y = \pi v,\ dy = \pi\,dv$

we can write P_{rad} as

$$P_{\text{rad}} = \eta\frac{|I_0|^2}{8\pi}\int_0^{2\pi}\left[\frac{1-\cos(y)}{y}\right]dy = \eta\frac{|I_0|^2}{8\pi}C_{\text{in}}(2\pi)$$

4-22. (a)

$$I_z(z') = \begin{cases} I_0\left(1+\frac{2}{l}z'\right), & \frac{-l}{2} < z' < 0 \\ I_0\left(1-\frac{2}{l}z'\right), & 0 < z' < l/2 \end{cases}$$

$$\begin{aligned}
\underline{A}(\underline{r}) &\cong \hat{a}_z\frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int_{-l/2}^{l/2} I_z(z')e^{jk\hat{a}_r\cdot\bar{r}'}\,dz' \\
&= \hat{a}_z\frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int_{-l/2}^{l/2}\left(1-2\frac{|z'|}{l}\right)e^{jkz'\cos\theta}\,dz' \\
&= \hat{a}_z\frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\left\{l\frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{kl}{2}\cos\theta\right)} - 2\int_{-l/2}^{l/2}\frac{|z'|}{l}e^{jkz'\cos\theta}\,dz'\right\}
\end{aligned}$$

$$\begin{aligned}
\int_{-l/2}^{l/2}\frac{|z'|}{l}e^{jkz'\cos\theta}\,dz' &= \int_0^{l/2}\frac{z'}{l}e^{jkz'\cos\theta}\,dz' - \int_{-l/2}^{0}\frac{z'}{l}e^{jkz'\cos\theta}\,dz'\Big\} \\
&= \int_0^{l/2}\frac{z'}{l}e^{jkz'\cos\theta}\,dz' + \int_0^{l/2}\frac{z'}{l}e^{-jkz'\cos\theta}\,dz' \\
&= 2\int_0^{l/2}\frac{z'}{l}\cos[kz'\cos\theta]\,dz' = \frac{l}{2}\int_0^1 \xi\cos\left[\frac{kl}{2}\xi\cos\theta\right]d\xi \\
&= \frac{l}{2}\left\{\frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\frac{kl}{2}\cos\theta} + \frac{\cos\left(\frac{kl}{2}\cos\theta\right)-1}{\left(\frac{kl}{2}\cos\theta\right)^2}\right\}
\end{aligned}$$

$$\therefore\ \underline{A}(\vec{r}) = \hat{a}_z \frac{\mu l}{4\pi} \frac{e^{-jkr}}{r} \left\{ \frac{1 - \cos\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{kl}{2}\cos\theta\right)^2} \right\}$$

$$A_\theta = \hat{a}_\theta \cdot \underline{A} = -\frac{\mu l}{4\pi} \frac{e^{-jkr}}{r} \sin\theta \left\{ \frac{1 - \cos\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{kl}{2}\cos\theta\right)^2} \right\}$$

$$A_\phi = \hat{a}_\phi \cdot \underline{A} = 0$$

In the far-zone

$$E_r \simeq 0$$

$$E_\theta \simeq j\omega\mu \frac{l}{4\pi} \frac{e^{-jkr}}{r} \sin\theta \left\{ \frac{1 - \cos\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{kl}{2}\cos\theta\right)^2} \right\}$$

$$E_\phi \simeq 0$$

$$H_r \simeq 0$$

$$H_\theta \simeq 0$$

$$H_\phi \simeq E_{\theta/\eta}$$

(b) From (4-58a).

$$E_\theta = j\eta \frac{ke^{-jkr}}{4\pi r} \sin\vartheta \left[\int_{-l/2}^{l/2} I(z') e^{jkz'\cos\theta}\, dz' \right]$$

$$E_\theta = j\eta \frac{ke^{-jkr}}{4\pi r} \sin\theta \cdot I_0 \cdot \int_{-l/2}^{l/2} \cos\left(\frac{\pi z'}{l}\right) e^{jkz'\cos\vartheta}\, dz'$$

let $a = j^{k\cos\theta}$ and $b = \frac{\pi}{l}$, use following integral formula

$$\int \cos bz e^{az}\, dz = \frac{e^{az}(a\cos bz + b\sin bz)}{a^2 + b^2}$$

$$E_\theta = j\eta \frac{ke^{-jkr}}{4\pi r} \sin\theta\ I_0 \cdot \left\{ \frac{e^{jkz'\cdot\cos\theta}}{\left(\frac{\pi}{l}\right)^2 - k^2\cos^2\theta} \left[j^{k\cos\theta} \cos\frac{\pi z'}{l} + \frac{\pi}{l} \sin\frac{\pi z'}{l} \right] \right\}_{-l/2}^{l/2}$$

$$= j\eta \frac{ke^{-jkr}}{4\pi r} \sin\theta \cdot I_0 \left[\frac{e^{jkl/2}\cos\theta}{\left(\frac{\pi}{l}\right)^2 - k^2\cos^2\theta} \frac{\pi}{l} + \frac{e^{-jkl/2}\cos\theta}{\left(\frac{\pi}{l}\right)^2 - k^2\cos^2\theta} \frac{\pi}{l} \right]$$

$$E_\theta = j\eta \frac{I_0 k e^{-jkr}}{4\pi r} \sin\theta \cdot \frac{\pi}{l} \cdot \frac{2\cos\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{\pi}{l}\right)^2 - k^2\cos^2\theta} = j\eta \frac{I_0 e^{-jkr}}{2\pi r} \cdot \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

$$H_\phi = j\frac{I_0 k e^{-jkr}}{4\pi r} \sin\theta \cdot \frac{\pi}{l} \cdot \frac{2\cos\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{\pi}{l}\right)^2 - k^2\cos^2\theta} = j\frac{I_0 e^{-jkr}}{2\pi r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

(c) $E_\theta = j\eta \frac{ke^{-jkr}}{4\pi r} \sin\theta [\int_{-l/2}^{l/2} I_0 \cos^2\left(\frac{\pi}{l}z'\right) e^{jkz'\cos\theta}\, dz'$

let $a = j^{k\cos\theta}$ and $b = \frac{\pi}{l}$, use the following integral formula

$$\int \cos^2 bz \cdot e^{az} dz = \frac{e^{az}}{2a} + \frac{e^{az}}{a^2 + 4b^2}\left(\frac{a}{2}\cos 2bz + b\sin 2bz\right)$$

$$E_\theta = j\eta \frac{ke^{-jkr}}{4\pi r} \sin\theta \cdot I_0 \left\{ \frac{e^{jkz'\cos\theta}}{2j^{k\cos\theta}} + \frac{e^{jkz'\cos\theta}}{\left(\frac{2\pi}{l}\right)^2 - k^2\cos^2\theta} \left(\frac{jk\cos\theta}{2} \cdot \cos\frac{2\pi}{l}z' + \frac{\pi}{l}\sin\frac{2\pi}{l}z' \right) \right\}_{-l/2}^{l/2}$$

$$= j\eta \frac{ke^{-jkr}}{4\pi r} \sin\theta \cdot I_0 \left\{ \frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{k\cos\theta} + k\cos\theta \frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{2\pi}{l}\right)^2 - k^2\cos^2\theta} \right\}$$

$$H_\phi = j\frac{ke^{-jkr}}{4\pi r} \sin\theta\ I_0 \left\{ \frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{k\cos\theta} + k\cos\theta \frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{2\pi}{l}\right)^2 - k^2\cos^2\theta} \right\}$$

4-23. $\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|}$, $\Gamma = \frac{R_{\text{in}} - Z_0}{R_{\text{in}} + Z_0}$, $R_{\text{in}} = \frac{R_r}{\sin^2\left(\frac{kl}{2}\right)}$, $Z_0 = 50$

(a) $l = \lambda/4$, $kl/2 = \pi/4$, $kl = \pi/2$, $2kl = \pi$

$$R_r = 60\left\{C + \ln(\pi/2) - C_i(\pi/2) + \tfrac{1}{2}\sin\left(\frac{\pi}{2}\right)\left[S_i(\pi) - 2S_i\left(\frac{\pi}{2}\right)\right]\right\}$$

$$R_r = 60\left\{0.5772 + 0.45158 - 0.470 + \tfrac{1}{2}[1.85 - 2(1.3698)]\right\} = 6.8388$$

$$R_{\text{in}} = \frac{R_r}{\sin^2\left(\frac{kl}{2}\right)} = \frac{6.8388}{\sin^2\left(\frac{\pi}{4}\right)} = 13.6776$$

$$\Gamma = \frac{13.6776 - 50}{13.6776 + 50} = -0.5704 \Rightarrow \text{VSWR} = \frac{1 + 0.5704}{1 - 0.5704} = 3.6555$$

(b) $l = \lambda/2 : kl/2 = \pi/2, kl = \pi, 2kl = 2\pi$

$$R_r = 60\left\{C + \ln(\pi) - C_i(\pi) + \tfrac{1}{2}\cos(\pi)\left[C + \ln\left(\frac{\pi}{2}\right) + C_i(2\pi) - 2C_i(\pi)\right]\right\}$$
$$= 60\left\{0.5772 + 1.14473 - 0.059 - \tfrac{1}{2}[0.5772 + 0.45158 - 0.0227 - 2(0.059)]\right\}$$

$$R_r = 73.13 \Rightarrow R_{\text{in}} = \frac{R_r}{\sin^2\left(\frac{kl}{2}\right)} = \frac{73.13}{\sin^2\left(\frac{\pi}{2}\right)} = 73.13$$

$$\Gamma = \frac{73.13 - 50}{73.13 + 50} = 0.18785 \Rightarrow \text{VSWR} = \frac{1 + 0.18785}{1 - 0.18785} = 1.4626$$

(c) $l = 3\lambda/4; kl/2 = 3\pi/4, kl = 3\pi/2, 2kl = 3\pi$

$$R_r = 60\left\{0.5772 + \ln\left(\frac{3\pi}{2}\right) - C_i\left(\frac{3\pi}{2}\right) + \tfrac{1}{2}\sin\left(\frac{3\pi}{2}\right)\left[S_i(3\pi) - 2S_i\left(\frac{3\pi}{2}\right)\right]\right\}$$
$$= 60\left\{0.5772 + 1.5502 - (-0.19839) - \tfrac{1}{2}[1.67473 - 2(1.611)]\right\}$$

$$R_r = 185.965 \Rightarrow R_{\text{in}} = \frac{185.965}{\sin^2(3\pi/4)} = 371.93$$

$$\Gamma = \frac{371.93 - 50}{371.93 + 50} = 0.7630 \Rightarrow \text{VSWR} = \frac{1 + 0.7630}{1 - 0.7630} = 7.4386$$

(d) $l = \lambda; kl/2 = \pi, kl = 2\pi, 2kl = 4\pi$

$$R_r = 60\left\{0.5772 + \ln(2\pi) - C_i(2\pi) + \tfrac{1}{2}\cos(2\pi)\left[0.5772 + \ln(\pi) + C_i(4\pi) - 2C_i(2\pi)\right]\right\}$$
$$= 60\left\{0.5772 + 1.8378 - (-0.0227) + \tfrac{1}{2}(1)\left[0.5772 + 1.14473 - 0.006 - 2(-0.0227)\right]\right\}$$

$$R_r = 199.099 \Rightarrow R_{\text{in}} = \frac{199.099}{\sin^2(\pi)} = \infty$$

$$\Gamma = \frac{\infty - 50}{\infty + 50} = \frac{1 - 50/\infty}{1 + 50/\infty} = 1 \Rightarrow \text{VSWR} = \infty$$

4-24. $R_r = 80\pi^2\left(\frac{l}{\lambda}\right)^2, a = 10^{-4}\lambda, f = 10\text{ MHz}, b = 5.7 \times 10^7\text{ s/m}$

$$R_L = R_{hf} = \frac{l}{p}\sqrt{\frac{\omega\mu}{2\sigma}} = \frac{l}{C}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{l}{2\pi a}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{l}{2\pi \times 10^{-4}\lambda}\sqrt{\frac{2\pi \times 10^7(4\pi \times 10^{-7})}{2\cdot(5.7 \times 10^7)}}$$

$$R_L = R_{hf} = 1.3245\left(\frac{l}{\lambda}\right), \quad e_{cd} = \frac{R_r}{R_L + R_r}$$

(a) $l = \lambda/50; R_r = 80\pi^2\left(\frac{\lambda}{50\lambda}\right)^2 = 0.316$ ohms

$$R_L = R_{hf} = 1.3245\left(\frac{1}{50}\right) = 0.02649$$

$$e_{cd} = \frac{R_r}{R_L + R_r} \times 100 = \frac{0.316 \times 100}{0.02649 + 0.316} = 92.26\%$$

(b) $l = \lambda/4$; From Prob. 4-23 $R_r = 6.8388$

$$R_L = R_{hf} = \frac{1.3245}{4} = 0.3311$$

$$e_{cd} = \frac{6.8388 \times 100}{6.8388 + 0.3311} = 95.38\%$$

(c) $l = \lambda/2$; From Prob. 4-23, $R_r = 73.13$

$$R_L = R_{hf} = \frac{1.3245}{2} = 0.66225$$

$$e_{cd} = \frac{73.13 \times 100}{73.13 + 0.66225} = 99.10\%$$

(d) $l = \lambda$; From Prob 4-23, $R_r = 199.099$

$$R_L = R_{hf} = 1.3245$$

$$e_{cd} = \frac{199.099}{199.099 + 1.3245} \times 100 = 99.34\%$$

4-25. $$H_\theta = j\frac{ke^{-jkr}}{4\pi r \cdot \eta} \cdot \sin\theta \left[\int_{-l/2}^{l/2} I_m \cdot \cos\left(\frac{\pi}{l}z'\right) e^{jkz'\cos\theta}\, dz'\right]$$

$$H_\theta = j\frac{k \cdot I_m e^{-jkr}}{4\pi r \cdot \eta} \cdot \sin\theta \int_{-l/2}^{l/2} \cos\left(\frac{\pi}{l}z'\right) e^{jkz'\cos\theta}\, dz'$$

Using the same formula in Problem 4-22(b).

$$H_\theta = j\frac{I_m \cdot k \cdot e^{-jkr}}{\eta 4\pi r} \cdot \sin\theta \cdot \frac{\pi}{l} \cdot \frac{2\cos\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{\pi}{l}\right)^2 - k^2 \cdot \cos^2\theta}$$

$$= j\frac{I_m k \cdot e^{-jkr}}{\eta 4\pi r} \cdot \frac{l}{k} \cdot \frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

$$= j\frac{I_m e^{-jkr}}{\eta 2\pi r} \cdot \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

$$E_\phi = -\eta H_\theta = -j\frac{I_m e^{-jkr}}{2\pi r} \cdot \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

4-26. (a) $$\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|} \Rightarrow |\Gamma| = \frac{\text{VSWR}-1}{\text{VSWR}+1} = \left|\frac{2-1}{2+1}\right| = \left|\frac{1}{3}\right|$$

$$|\Gamma| = \left|\frac{1}{3}\right| = \left|\frac{Z_{\text{in}}-Z_c}{Z_{\text{in}}+Z_c}\right| = \left|\frac{Z_{\text{in}}/Z_c-1}{Z_{\text{in}}/Z_c+1}\right| = \begin{cases} \left|\frac{2-1}{2+1}\right| \Rightarrow \frac{Z_{\text{in}}}{Z_c} = 2 \\ \left|\frac{\frac{1}{2}-1}{\frac{1}{2}+1}\right| \Rightarrow \frac{Z_{\text{in}}}{Z_c} = \frac{1}{2} \end{cases}$$

Largest

$$\frac{Z_{\text{in}}}{Z_c} = 2 \Rightarrow Z_{\text{in}} = 2Z_c = 100$$

(b) $$R_{\text{in}} = 11.14G^{4.17} \qquad \lambda/2 < l < 2\lambda/\pi$$

$$100 = 11.14G^{4.17} \qquad \pi/2 < kl/2 < 2$$

$$\frac{100}{11.14} = G^{4.17}, \quad 8.9767 = G^{4.17}$$

$$\log_{10}(8.9767) = 4.17\log_{10}(G), \quad 0.953 = 4.17\log_{10} G$$

$$0.2286 = \log_{10} G, \quad G = 10^{0.2286} = 1.6928 = \frac{kl}{2} = 96.99^\circ$$

$$kl = 2(1.6928), \quad l = \frac{2(1.6928)\lambda}{2\pi} = \frac{1.6928}{\pi}\lambda = 0.5388\lambda$$

$$l = 0.5388\lambda$$

(c) $$R_{\text{in}} = \frac{R_r}{\sin^2\left(\frac{kl}{2}\right)} \Rightarrow R_r = R_{\text{in}}\sin^2\left(\frac{kl}{2}\right) = 100\sin^2(96.99^\circ)$$

$$R_r = 100(0.9926)^2 = 100(0.9852) = 98.52 \text{ ohms}$$

$$R_r = 98.52 \text{ ohms}$$

4-27. $$W_{\text{av}} = \frac{1}{2\eta}(|E_\theta|^2 + |E_\phi|^2) = \frac{1}{2\eta}\left[\frac{\omega^2\mu^2\sin^2\theta}{16\pi^2r^2}I_0{}^2(k^2A_1{}^2 + 4A_2{}^2)\right]$$

$$P_{\text{rad}} = \frac{1}{2\eta}\frac{\omega^2\mu^2I_0{}^2}{16\pi^2}\left[\int_0^{2\pi}\int_0^{\pi}\sin^3\theta\, d\theta\, d\phi\right][k^2A_1{}^2 + 4A_2{}^2]$$

$$P_{\text{rad}} = \frac{\omega^2\mu^2I_0{}^2(k^2A_1{}^2 + 4A_2{}^2)}{12\pi\eta}\left(\int_0^{2\pi}\int_0^{\pi}\sin^3\theta\, d\theta\, d\phi = \frac{8\pi}{3}\right)$$

$$\Rightarrow R_{\text{rad}} = \frac{2P_{\text{rad}}}{I_0{}^2} = \frac{\omega^2\mu^2(k^2A_1{}^2 + 4A_2{}^2)}{6\pi\eta}$$

Elliptical polarization since

$$\vec{E}(t) = \frac{-\omega\mu k\sin\theta}{4\pi r}I_0\cdot A_1\cdot\sin(\omega t - kr)\hat{a}_\theta + \frac{\omega\mu k\sin\theta}{2\pi r}I_0\cdot A_2\cos(\omega t - kr)\hat{a}_\phi$$

4-28. Dipole with $l = \lambda/2$

$$\underline{E}^a \simeq \hat{a}_\theta j\eta \frac{I_0 e^{-jkr}}{2\pi r}\left\{\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right\}$$

$$\simeq -\hat{a}_\theta j\eta \frac{I_0 k e^{-jkr}}{2\pi r}\left\{\frac{-\cos\left(\frac{\pi}{2}\cos\theta\right)}{k\sin\theta}\right\}$$

$$\simeq -\hat{a}_\theta j\eta \frac{I_0 k e^{-jkr}}{2\pi r}\left\{-\frac{\lambda}{2\pi}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right\}$$

$$\underline{E}^a \simeq -j\eta \frac{I_0 k e^{-jkr}}{4\pi r}\left\{-\hat{a}_\theta \frac{\lambda}{\pi}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right\}$$

(a) $\underline{l}_e(\theta) = -\hat{a}_\theta \dfrac{\lambda}{\pi}\dfrac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$

(b) $|\underline{l}_e(\theta)|_{\max} = \left|-\hat{a}_\theta \dfrac{\lambda}{\pi}\cdot\dfrac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right|_{\max}^{\theta=90^\circ} = \dfrac{\lambda}{\pi} = 0.3183\lambda$

(c) $\dfrac{|l_e(\theta)|_{\max}}{l=\lambda/2} = \dfrac{\lambda/\pi}{\lambda/2} = \dfrac{2}{\pi} = 0.6366$

which is 63.66% of l.

(d) $V_{oc} = |\underline{l}_e \cdot \underline{E}^i|_{\theta=90^\circ} = \left|-\hat{a}_\theta \dfrac{\lambda}{\pi}\dfrac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\cdot -\hat{a}_\theta \dfrac{10^{-3}}{\lambda}V\right|_{\theta=90^\circ}$

$$= \frac{\lambda}{\pi}\left(\frac{10^{-3}}{\lambda}\right) = \frac{10^{-3}}{\pi} = 3.183\times 10^{-4}\ \text{Volts}$$

4-29. $\lambda/2$ dipole $\Rightarrow$ ($P_{\text{rad}} = P_{\text{in}} = 1$ watt, $D_0 = 1.643 = 2.1564$ dB)

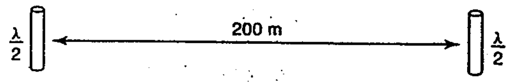
$Z_{\text{in}} = 73 + j42.5$, $f = 1{,}900$ MHz $\Rightarrow \lambda = 3\times 10^8/1.9\times 10^9 = 0.15789$ meters

(a) $U_0 = \dfrac{P_{\text{rad}}}{4\pi} = \dfrac{1}{4\pi} = 0.07958$ watts/sterad

$U_{\text{dipole}} = U_0 D_0 = 0.07958(1.643) = 0.130745$ watts/unit solid angle (sterad)

(b) $W_{\text{dipole}} = \dfrac{U_{\text{dipole}}}{r^2} = \dfrac{0.130745}{(5\times 10^3)^2} = 5.229\times 10^{-9}$ watts/m^2

$W_0 = \dfrac{U_0}{r^2} = \dfrac{0.07958}{(5\times 10^3)^2} = 3.183\times 10^{-9}$ watts/m^2

4-30. 

$\theta = 90°$, $\phi = 40°$

At $f = 300$ MHz, $\lambda = \dfrac{c}{f} = 1$ m

$$\Rightarrow \frac{2D^2}{\lambda} = \frac{2\left(\frac{\lambda}{2}\right)^2}{\lambda} 0.5 \text{ m}$$

$r = 200 \text{ m} \gg 0.5 \text{ m}$

$$P_r = \left(\frac{\lambda}{4\pi r}\right)^2 G_{0t} G_{0r} = \left(\frac{\lambda}{4\pi r}\right)^2 D_{0t} D_{0r}$$

for lossless antenna.

Now since $D_{0t} = D_{0r} = 1.643$ for $\dfrac{\lambda}{2}$ dipole.

$$P_r = \left(\frac{1}{4\pi \cdot 200}\right)^2 (1.643)(1.643) \quad W = 0.2 \text{ mW}$$

4-31. The time average power density $\left(W_{av} = \frac{1}{2}\frac{|E|^2}{\eta}\right)$

$$W_{av} = \eta \frac{|I_0|^2}{8\pi^2 r^2}\left[\frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\right], \quad P_{\text{rad}} = \eta \frac{|I_0|^2}{4\pi}\int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\, d\theta$$

$$P_{\text{rad}} = \tfrac{1}{2} R_{\text{rad}} |I_0|^2, \quad R_{\text{rad}} = \frac{\eta}{4\pi}[r + \ln(2\pi) - C_i(2\pi)] = 30[0.5772 + 1.838 + 0.02]$$

$$R_{\text{rad}} = 73.0523.$$

$P_{\text{rad}} = (0.5 \cdot 100) = 50$ watts. $\cdot\ 50 = \frac{1}{2}(73.0523)|I_0|^2$

$|I_0|^2 = 1.36888$

At $r = 500$ m, $\theta = 60°$, $\phi = 0°$

$$\begin{aligned} W_{av} &= 120\pi \cdot \frac{1.36888}{8\pi^2 \cdot (500)^2} \cdot \left[\frac{\cos^2\left(\frac{\pi}{2}\cos 60°\right)}{\sin^2 60°}\right] \\ &= 15 \cdot \frac{1.36888}{\pi \cdot 25 \times 10^4} \cdot (0.6667) \\ &= 1.743 \times 10^{-5} \text{ watts/m}^2. \end{aligned}$$

4-32. $l = \lambda/20 \Rightarrow$ triangular current distribution; $a = \lambda/400, f = 30$ MHz $\Rightarrow \lambda = 0.1$ meters

(a) $R_r = R_{\text{in}} = 20\pi^2\left(\frac{l}{\lambda}\right) = 20\pi^2\left(\frac{1}{20}\right)^2 = 0.4935$ ohms

$$X_{\text{in}} = -j120\frac{\left[\ln\left(\frac{l}{2a}\right) - 1\right]}{\tan\left(\frac{\pi l}{\lambda}\right)} = -j120\frac{\left[\ln\left(\frac{\lambda}{20}\frac{1}{2}\frac{400}{\lambda}\right) - 1\right]}{\tan\left(\frac{\pi}{\lambda}\frac{\lambda}{20}\right)} = -j120\frac{[\ln(20) - 1]}{\tan(\pi/20)}$$

$$= -j986.935$$

$Z_{\text{in}} = 0.4935 - j986.935$ (capacitive)

(b) $e_{cd} = \dfrac{R_r}{R_r + R_L}$

Since element is PEC $\Rightarrow \sigma = \infty \Rightarrow R_L = 0$

$e_{cd} = \dfrac{R_r}{R_r} = 1 = 100\%$

(c) Must use an inductor in series to resonate the element with a reactance of

$$X_L = \omega L = 2\pi f L = 2\pi(30 \times 10^6)L = 986.35$$

$$L = \frac{986.35}{2\pi(30 \times 10^6)} = 5.236 \times 10^{-6} \text{ henries}$$

$$\boxed{L = 5.236 \times 10^{-6} \text{ henries}}$$

4-33. $Z_a = 73 + j42.5, Z_c = 75, f = 100$ MHz

a. $\Gamma = \dfrac{Z_a - Z_c}{Z_a + Z_c} = \dfrac{73 + j42.5 - 75}{73 + j42.5 + 75} = \dfrac{-2 + j42.5}{148 + j42.5} = \dfrac{42.547\underline{|92.694}}{153.981\underline{|16.02}}$

$\Gamma = 0.2763\underline{|76.674} \Rightarrow |\Gamma| = 0.2763, \underline{|\phi} = 76.674° = 1.338$(rads)

b. $\text{VSWR} = \dfrac{1 + |\Gamma|}{1 - |\Gamma|} = \dfrac{1 + 0.2763}{1 - 0.2763} = \dfrac{1.2763}{0.2763} = \boxed{1.76358}$

c. $Z_a = 73 + j42.5$

Need a **capacitor in** series to resonate.

$X_c = 42.5$

d. $X_c = \dfrac{1}{\omega C} = \dfrac{1}{2\pi f C} = 42.5 \Rightarrow C = \dfrac{1}{2\pi f(42.5)}$

$$C = \frac{1}{2\pi(42.5)(10^8)} = \boxed{0.00374 \times 10^{-8}} = \boxed{37.4 \times 10^{-12} \text{ farads}}$$

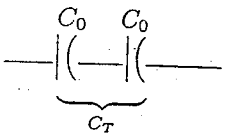
e. $$Z_{\text{in}} = Z_a - jX_c = 73 + j42.5 - j42.5 = 73$$
$$Z_{\text{in}} = 73$$
$$\Gamma = \frac{Z_{\text{in}} - Z_c}{Z_{\text{in}} + Z_c} = \frac{73-75}{73+75} = \frac{-2}{148} = -0.0135$$
$$\Gamma = -0.0135 \Rightarrow |\Gamma| = 0.0135$$
$$\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.0135}{1-0.0135} = 1.027$$
$$\boxed{\text{VSWR} = 1.027}$$

4-34. $\lambda/2$ dipole $\Rightarrow Z_{\text{in}} = 73 + j42.5$, $f = 1.9 \times 10^9$ Hz

(a) $$|\Gamma| = \left|\frac{Z_{\text{in}} - Z_c}{Z_{\text{in}} + Z_c}\right| = \left|\frac{73 + j42.5 - 50}{73 + j42.5 + 50}\right| = \left|\frac{23 + j42.5}{123 + j42.5}\right| = \frac{48.324}{130.1355} = 0.371$$
$$\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.371}{1-0.371} = 2.17965$$

(b) Capacitance $\Rightarrow X_T = \dfrac{1}{\omega C_T} = 42.5$
$$\Rightarrow C_T = \frac{1}{42.5\,\omega} = \frac{1}{42.5(2\pi f)} = \frac{1}{42.5(2\pi \times 1.9 \times 10^9)}$$
$$C_T = 1.971 \times 10^{-12} f$$

(c) $C_0 = 2C_T = 2(1.971 \times 10^{-12}) = 3.942 \times 10^{-12} f$



$$\frac{1}{C_T} = \frac{1}{C_0} + \frac{1}{C_0} = \frac{Z}{C_0}$$
$$C_0 = 2C_T$$

(d) $$|\Gamma| = \left|\frac{Z_{\text{in}} - Z_c}{Z_{\text{in}} + Z_c}\right| = \frac{73-50}{73+50} = \frac{23}{123} = 0.18699$$
$$\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.18699}{1-0.18699} = 1.46$$

4-35. (a) $$I_{\text{in}} = I_0 \sin\left[k\left(\frac{l}{2} + |z|\right)\right]$$
$$l = \lambda/4, z = \lambda/8$$
$$I_{\text{in}} = I_0 \cdot \sin\left[k\left(\frac{\lambda}{4} \pm \frac{\lambda}{8}\right)\right] = I_0 \sin\left[k\frac{\lambda}{8}\right] = I_0 \sin\left[\frac{2\pi}{\lambda}\frac{\lambda}{8}\right]$$
$$= I_0 \sin\left(\frac{\pi}{4}\right)$$
$$I_{\text{in}} = 0.707 I_0$$

$$R_{\text{in}} = \left(\frac{I_0}{I_{\text{in}}}\right)^2 \cdot R_r = \left(\frac{I_0}{0.707 I_0}\right)^2 \cdot R_r = 2R_r = 2(73) = 146$$

$$X_{\text{in}} = \left(\frac{I_0}{I_{\text{in}}}\right)^2 \cdot X_m = \left(\frac{I_0}{0.707 I_0}\right)^2 \cdot X_m = 2X_m = 2(42.5) = 85$$

$$Z_{\text{in}} = R_{\text{in}} + jX_{\text{in}} = 146 + j85$$

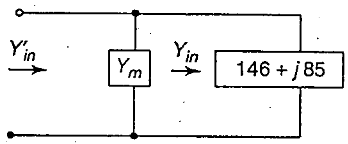
(b) $$Y_{\text{in}} = \frac{1}{146 + j85} \cdot \frac{146 - j85}{146 - j85} = \frac{146 - j85}{168.941} = (5.115 - j2.978) \times 10^{-3}$$

$$Y_m = +j2.978 \times 10^{-3} \Rightarrow X_m = \frac{1}{Y_m} = -j335.776 \quad \text{(Capacitive)}$$

(c) $$Y'_{\text{in}} = 5.115 \times 10^{-3} \Rightarrow Z_{\text{in}} = 5.115 \times 10^{-3} = 195.503 \text{ ohms}$$

$$|\Gamma| = \left|\frac{195.503 - 300}{195.503 + 300}\right| = \frac{104.4966}{495.503} = 0.21$$

$$\text{VSWR} = (1 + 0.211)/(1 - 0.211) = 1.5346$$



4-36. $l = \lambda/2, \quad Z_c = 50$ ohms

$$Z_{\text{in}} = 73 + j42.5, \quad Y_{\text{in}} = \frac{1}{Z_{\text{in}}} = \frac{1}{73 + j42.5} \cdot \frac{73 - j42.5}{73 - j42.5}$$

$$Y_{\text{in}} = 0.01023 - j0.0059563 = (10.23 - j5.9563) \times 10^{-3} = G_{\text{in}} - jB_{\text{in}}$$

$$B_{\text{in}} = \omega C_{\text{in}} = 2\pi f C_{\text{in}} \Rightarrow C_{\text{in}} = \frac{B_{\text{in}}}{2\pi f} = \frac{5.9563 \times 10^{-3}}{2\pi \cdot (10 \times 10^8)} = 0.94797 \times 10^{-12}$$

$$\therefore C_{\text{in}} = 0.94797 \text{ pF}$$

$$G_{\text{in}} = 10.23 \times 10^{-3}$$

$$R_{\text{in}} = \frac{1}{G_{\text{in}}} = 97.75, \quad \Gamma_{\text{in}} = \frac{R_{\text{in}} - Z_c}{R_{\text{in}} + Z_c} = \frac{97.75 - 50}{97.75 + 50} = 0.3232$$

$$\text{VSWR} = \frac{1 + |\Gamma_{\text{in}}|}{1 - |\Gamma_{\text{in}}|} = \frac{1 + 0.3232}{1 - 0.3232} = 1.955$$

4-37.

$$\underline{E} = -\hat{a}_\theta j \frac{\omega\mu b I_0 e^{-jkr}}{4\pi r} \cdot \frac{\sin\left(\frac{kb}{2}\cos\theta\right)}{\frac{kb}{2}\cos\theta}\Bigg|_{\theta=90^\circ} = -\hat{a}_\theta \cdot j \cdot \frac{\omega\mu b I_0 e^{-jkr}}{4\pi r} \cdot (1)$$

$$\underline{E}|_{\theta=90^\circ} = -\hat{a}_\theta \cdot j \frac{\omega\mu I_0 e^{-jkr}}{4\pi r} b = -j\frac{\omega\mu I_0 e^{-jkr}}{4\pi r} \cdot l_e(\theta)$$

$$l_e(\theta) = \hat{a}_\theta \cdot b$$

$$\underline{E}^{\text{inc}}|_{\theta=90^\circ} = \hat{a}_\theta j\eta \frac{kI_0 l e^{-jkr}}{4\pi r} \sin\theta\Big|_{\theta=90^\circ} = \hat{a}_\theta \cdot j\eta \cdot \frac{kI_0 l e^{-jkr}}{4\pi r}$$

$$P = \frac{|\underline{l}_e(\theta) \cdot \underline{E}^{\text{inc}}|^2}{|l_e(\theta)|^2 \cdot |\underline{E}^{\text{inc}}|^2} = \frac{\left|\eta \frac{bkI_0 l}{4\pi r}\right|^2}{|b|^2 \cdot \left|\eta \frac{kI_0 l}{4\pi r}\right|^2} = 1.$$

$$P(\text{dB}) = 10\log_{10}(1) = 0 \text{ dB}$$

4-38.

$$V_1 = 4e^{j20^\circ} = C[\hat{a}_y] \cdot \left[\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right], \text{at } z = 0$$

$$V_1 = 4e^{j20^\circ} = j \cdot C \cdot \frac{1}{\sqrt{2}} \Rightarrow C = -j4\sqrt{2}e^{j20^\circ}, C = 4\sqrt{2}e^{-j70^\circ}$$

$$V_2 = (4\sqrt{2}e^{-j70^\circ})[10(2\hat{a}_x + \hat{a}_y e^{j30^\circ})] \cdot \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right) = 40\sqrt{2}e^{-j70^\circ}\left[\frac{2 + je^{j30^\circ}}{\sqrt{2}}\right]$$

$$= 40e^{-j70^\circ}[2 + j(\cos 30^\circ + j\sin 30^\circ)]$$

$$= 40e^{-j70^\circ}[1.5 + j0.866] = 40e^{-j70^\circ}[1.73e^{j30^\circ}]$$

$$V_2 = 70e^{-j40^\circ} = 53.6 - j45^\circ$$

4-39. $l = 3$ cm, $\lambda = 5$ cm, $I = 10e^{j60^\circ}$

$$r > \frac{2D^2}{\lambda} = \frac{2\times 3^2}{5} = \frac{18}{5} = 3.6 \text{ cm} \Rightarrow 10 \text{ cm is in the far field.}$$

$$\frac{l}{\lambda} = \frac{3}{5} = 0.6 \Rightarrow \text{length of dipole is finite, } \frac{kl}{2} = \pi \cdot \frac{l}{\lambda} = 0.6\pi$$

$$E_\theta \simeq j\eta\frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta}\right] = j\eta\frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos(0.6\pi\cos\theta) + 0.309}{\sin\theta}\right]$$

$$H_\phi \simeq \frac{E_\theta}{\eta}, \left(\leftarrow \frac{\cos(0.6\pi\cdot\cos 45^\circ) + 0.309}{\sin 45^\circ}\Big|_{\theta=45^\circ} = 0.7703\right)$$

$$e^{-jkr} \Rightarrow kr = \frac{2\pi}{\lambda} r = \frac{2\pi}{5} \cdot 10 = 4\pi = 12.5663 \text{ rad}$$

$$\Rightarrow E_\theta = j^{120\pi} \cdot \frac{I_0 e^{j60} \cdot e^{-j4\pi}}{2\pi(0.1 \text{ m})} \cdot (0.7703) = 4620 e^{j11.52}$$

$$|E_\theta| = 4620 \text{ v/m}, |H_\phi| = \frac{4620}{120\pi} = 12.25 \text{ amperes/meter}$$

4-40. Using equation (4-79)

$$R_{\text{in}} = \frac{R_r}{\sin^2\left(\frac{kl}{2}\right)} = \frac{120 \text{ ohms}}{\sin^2(0.6\pi)} = 132.668 \text{ ohms}$$

4-41. $\frac{kl}{2} = \frac{3\pi}{4}$, $kl = \frac{3\pi}{2}$, $2kl = 3\pi$

(a) Using (8-60a), (8-60b)

$$R_r = 185.808, X_r = 190.7967$$

(b) Using (8-61a), (8-61b)

$$R_{\text{in}} = \frac{185.808}{\sin^2(3/4\pi)} = 371.617, X_{\text{in}} = \frac{190.7967}{\sin^2(3/4 \cdot \pi)} = 385.5936$$

(c) $$\Gamma = \frac{371.617 - 300}{371.617 + 300} = 0.10663,$$

$$\text{VSWR} = \frac{1 + 0.10663}{1 - 0.10663} = 1.2387$$

4-42. $l = 0.625\lambda$,

a. Using (8-60a), (8-60b)

$$R_r = 131.9415, \quad X_r = 146.131638$$

b. Using (8-61a), (8-61b)

$$R_{\text{in}} = 154.579, \quad X_{\text{in}} = 171.203$$

c. $$\Gamma = \frac{154.579 - 300}{154.579 + 300} = -0.3199, \quad \text{VSWR} = \frac{1 + |0.3199|}{1 - 1 - 0.31991} = 1.9407$$

4-43. a. $l = 200$ m, $a = 1$ m, $f = 150$ $kHz \rightarrow \lambda = 2000$ meters. Using (11-37).

$$Z_{\text{in}} \simeq 20\pi^2 \left(\frac{l}{\lambda}\right)^2 - j120 \frac{\left[\ln\left(\frac{l}{2a}\right) - 1\right]}{\tan\left(\pi \frac{l}{\lambda}\right)} \simeq 20\pi^2 \cdot \left(\frac{1}{10}\right)^2 - j120 \cdot \frac{[\ln(100) - 1]}{\tan(\pi/10)}$$

$$Z_{\text{input}} = 2 + Z_{\text{in}} = 2 + 1.9739 + j1377.07$$
$$Z_{\text{input}} = 3.9739 + j1377.07$$

b. Radiation efficiency $= 100 \cdot \dfrac{R_r}{R_L + R_r} = 100 \cdot \dfrac{1.9739}{3.9739} = 49.67\%$

c. $\text{RPF} = \dfrac{R_r}{|\text{Im}(Z_{\text{input}})|} = \dfrac{1.9739}{1.377 \times 10^{+3}} = 1.4335 \times 10^{-3}$

d. $X = -\text{Im}(Z_{\text{input}}) = -1377.07$

$$n = \sqrt{\frac{R_r + R_L}{Z_0}} = \sqrt{\frac{3.9739}{50}} = 0.282$$

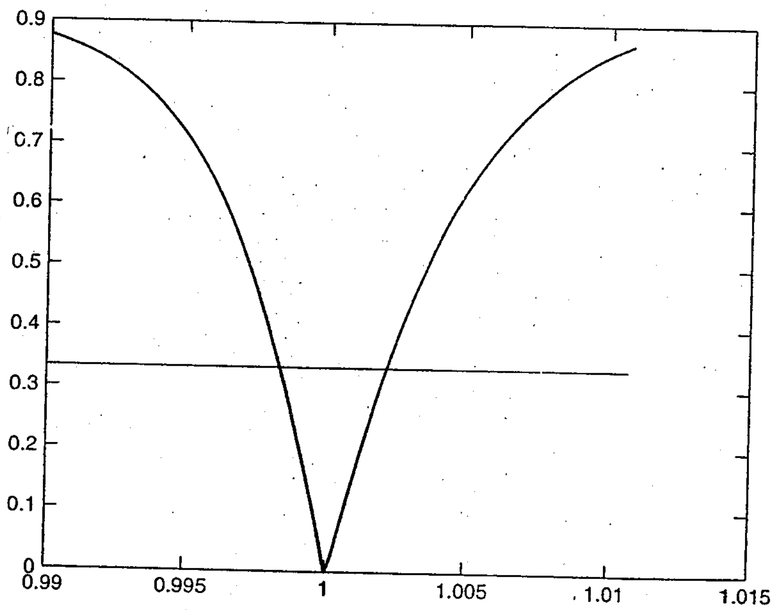
e. The answer to this part was found by manually entering values of X until $|\Gamma| = 0.333$ was obtained.

The values obtained are

$$X_1 = 0.99803$$
$$X_2 = 1.00198$$

The corresponding percent bandwidth is

$$\text{BW} = (X_2 - X_1) \times 100\% = 0.395\%$$



4-44. $\underline{E}_w = (2\hat{a}_x - j\hat{a}_y)E_0 e^{+jkz} = \underbrace{\left(\frac{2\hat{a}_x - j\hat{a}_y}{\sqrt{5}}\right)}_{\hat{\rho}_w} \sqrt{5}E_0 e^{+jkz}$

(a) $\hat{\rho}_w = \left(\frac{2\hat{a}_x - j\hat{a}_y}{\sqrt{5}}\right)$

(b) $\hat{\rho}_a = \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)$ or $\hat{\rho}_a = \left(\frac{-j\hat{a}_x + \hat{a}_y}{\sqrt{2}}\right)$

(c) 1. Elliptical, AR = 2
2. CCW

(d) 1. Circular, AR = 1
2. CCW

(e) $\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\left(\frac{2\hat{a}_x - j\hat{a}_y}{\sqrt{5}}\right) \cdot \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\right|^2 = \left|\frac{2 - j^2 1}{\sqrt{10}}\right|^2 = \left|\frac{2+1}{\sqrt{10}}\right|^2$

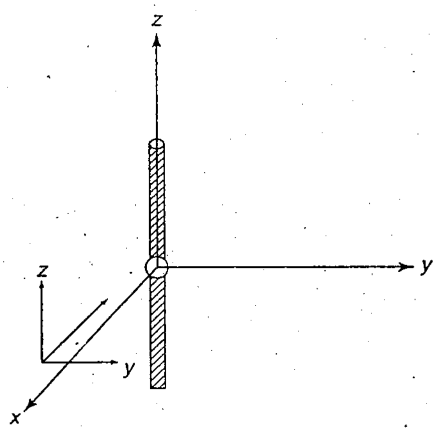
$$= \frac{9}{10} = -0.4576 \text{ dB}$$

$\boxed{\text{or}}$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\left(\frac{2\hat{a}_x - j\hat{a}_y}{\sqrt{5}}\right) \cdot \left(\frac{-j\hat{a}_x + \hat{a}_y}{\sqrt{2}}\right)\right|^2 = \left|\frac{-j2 - j}{\sqrt{10}}\right|^2 = \left|\frac{-j3}{\sqrt{10}}\right|^2$$

$$= \frac{9}{10} = -0.4576 \text{ dB}$$

4-45. $W_i = 2\ \mu\text{w/m}^2 = 2 \times 10^{-6}\ \text{w/m}^2$



(a) $\underline{E}_W^L = (3\hat{a}_z + j\hat{a}_y)E_0 e^{+jkx}$

$\underline{E}_W^L = \left(\frac{3\hat{a}_z + j\hat{a}_y}{\sqrt{10}}\right) 10 E_0 e^{+jkx}$

$\hat{\rho}_W = \left(\frac{3\hat{a}_z + j\hat{a}_y}{\sqrt{10}}\right)$

Elliptical CCW

AR = 3/1 = 3

(b) $\underline{E}_a = \hat{a}_\theta j\eta \frac{I_0 e^{-jky}}{2\pi r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$

$= \hat{a}_\theta E_0 \left.\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right|_{\theta=\pi/2}$

$\underline{E}_a = \underbrace{\hat{a}_\theta}_{\hat{\rho}_a} E_0$

$\hat{\rho}_a = \hat{a}_\theta$ **Linear**

$\hat{\rho}_a = [\hat{a}_x \cos\theta\cos\phi + \hat{a}_y \cos\theta\sin\phi - \hat{a}_z \sin\theta]_{\theta=90^\circ}$

$\hat{\rho}_a = -\hat{a}_z$

(c) $\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\left(\dfrac{3\hat{a}_z + j\hat{a}_y}{\sqrt{10}}\right) \cdot (-\hat{a}_z)\right|^2 = \dfrac{9}{10} = 0.9 = -0.4576 \text{ dB}$

$\text{PLF} = -0.4576 \text{ dB} = 0.9$

(d) $\lambda = \dfrac{3 \times 10^8}{300 \times 10^6} = 1 \text{ m}$

$$A_{em} = \frac{\lambda^2}{4\pi} D_0 = \frac{1}{4\pi} D_0 = \frac{1.643}{4\pi} = 0.1307 \text{ m}^2$$

$$P_L = A_{em} W_i(\text{PLF}) = 0.1307(2 \times 10^{-6})(0.9) = (0.2614)(0.9) \times 10^{-6}$$

$$P_L = 0.2353 \times 10^{-6} \text{ watts}$$

4-46. $E_\theta = j\eta \dfrac{kI_0 l e^{-jkr}}{2\pi r} \sin\theta \cdot \cos(kh\cos\theta);\ 0 \leqslant \theta \leqslant \pi/2,\ 0 \leqslant \phi \leqslant 2\pi$

$$\underline{W}_{\text{ave}} = \tfrac{1}{2}\text{Re}[\underline{E} \times \underline{H}^*] = \frac{\hat{a}_r}{2\eta}|E_\theta|^2 = \hat{a}_r \frac{\eta}{2}\left|\frac{kI_0 l}{2\pi r}\right|^2 \sin^2\theta \cdot \cos^2(kh\cos\theta)$$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi/2} \underline{W}_{\text{ave}} \cdot \hat{a}_r r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{\eta}{2}\left|\frac{kI_0 l}{2\pi}\right|^2 \int_0^{2\pi}\int_0^{\pi/2} \sin^3\theta \cos^2(kh\cos\theta)\, d\theta\, d\phi$$

$$= \frac{\eta}{\pi}\left|\frac{kI_0 l}{2}\right|^2 \cdot \int_0^{\pi/2} \sin^3\theta \cos^2(kh\cos\theta)\, d\theta$$

$$= \frac{\eta}{\pi}\left|\frac{kI_0 l}{2}\right|^2 \int_0^{\pi/2} \sin^3\theta \left[\frac{1+\cos(2kh\cos\theta)}{2}\right] d\theta$$

$$= \frac{\eta}{2\pi}\left|\frac{kI_0 l}{2}\right|^2 \left\{\int_0^{\pi/2} \sin^3\theta\, d\theta + \int_0^{\pi/2} \sin^3\theta \cdot \cos(2kh\cos\theta)\, d\theta\right\}$$

$$P_{\text{rad}} = \frac{\eta}{2\pi}\left|\frac{kI_0 l}{2}\right|^2 \{I_1 + I_2\}$$

where $I_1 = \displaystyle\int_0^{\pi/2} \sin^3\theta\, d\theta = -\frac{1}{3}\cos\theta(\sin^2\theta + 2)\Big|_0^{\pi/2} = \frac{2}{3}$

$$I_2 = \int_0^{\pi/2} \sin^3\theta \cos(kh\cos\theta)\, d\theta = \int_0^{\pi/2} \sin^2\theta \cdot \cos(kh\cos\theta)\sin\theta\, d\theta$$

Let $u = \sin^2\theta \qquad v = -\dfrac{1}{2kh}\sin(2kh\cos\theta)$

$du = 2\sin\theta\cos\theta\, d\theta \qquad dv = -\dfrac{\cos(2kh\cos\theta)}{2kh} \cdot d(2kh\cos\theta)$

Thus

$$I_2 = -\frac{\sin^2\theta}{2kh} \cdot \sin(2kh\cos\theta)\Big|_0^{\pi/2} + \frac{2}{2kh}\int_0^{\pi/2} \cos\theta \cdot \sin(2kh\cos\theta)\sin\theta\, d\theta$$

$$\text{Let}\quad u=\cos\theta \qquad dv=-\frac{1}{2kh}\sin(2kh\cos\theta)\,d(2kh\cos\theta)$$

$$du=-\sin\theta\,d\theta \qquad v=\frac{1}{2kh}\cos(2kh\cos\theta)$$

$$I_2=0+\frac{2}{2kh}\left\{\frac{\cos\theta}{2kh}\cos(2kh\cos\theta)\Big|_0^{\pi/2}+\frac{1}{2kh}\int_0^{\pi/2}\cos(2kh\cos\theta)\sin\theta\,d\theta\right\}$$

$$=\frac{2}{2kh}\left\{-\frac{1}{2kh}\cos(2kh)-\frac{1}{(2kh)^2}\sin(2kh\cos\theta)\Big|_0^{\pi/2}\right\}=2\left\{-\frac{\cos(2kh)}{(2kh)^2}+\frac{\sin(2kh)}{(2kh)^3}\right\}$$

Therefore

$$P_{\text{rad}}=\frac{\eta}{2\pi}\left|\frac{kI_0l}{2}\right|^2\cdot\{I_1+I_2\}=\pi\eta\left|\frac{I_0l}{\lambda}\right|^2\cdot\left[\frac{1}{3}-\frac{\cos(2kh)}{(2kh)^2}+\frac{\sin(2kh)}{(2kh)^3}\right]$$

4-47. $E_\theta=C_1\sin\theta\cos(kh\cos\theta)$, where $C_1=j\eta\dfrac{kI_0le^{-jkr}}{2\pi r}$

a. $E_\theta|_{\theta=30^\circ}=C_1\sin\theta\cdot\cos(kh\cos\theta)|_{\theta=30^\circ}=0\Rightarrow\cos(kh\cos\theta)|_{\theta=30^\circ}=0$

$$kh\cos(30^\circ)=\frac{2\pi}{\lambda}h(0.867)=\cos^{-1}(0)=\frac{\pi}{2}\Rightarrow h=\frac{1}{4(0.867)}\cdot\lambda=0.288\lambda$$

b. $$D_0=\frac{2}{\left[\frac{1}{3}-\frac{\cos(2kh)}{(2kh)^2}+\frac{\sin(2kh)}{(2kh)^3}\right]},\quad 2kh=2\cdot\left(\frac{2\pi}{\lambda}\right)\cdot(0.288\lambda)=3.632$$

$$D_0=\frac{2}{\left[\frac{1}{3}-\frac{\cos(3.632)}{(3.632)^2}+\frac{\sin(3.632)}{(3.632)^3}\right]}=\frac{2}{\left[\frac{1}{3}+0.06689-0.00983\right]}$$

$$=5.12=7.1\text{ dB}$$

c. $$R_r=2\pi\eta\left(\frac{l}{\lambda}\right)^2\cdot\left[\frac{1}{3}-\frac{\cos(3.632)}{(3.632)^2}+\frac{\sin(3.632)}{(3.632)^3}\right]$$

$$=2\pi(377)\left(\frac{1}{50}\right)^2\cdot[0.39]=0.37\text{ ohms}$$

4-48. $E_\theta=C_1\sin\theta\cdot\cos(kh\cos\theta)$, where $C_1=j\eta\dfrac{kI_0le^{-jkr}}{2\pi r}$

$E_\theta|_{h=2\lambda}=C_1\cdot\sin\theta_n\cdot\cos(kh\cos\theta_n)|_{h=2\lambda}=0\Rightarrow\sin\theta_n=0,\cos(kh\cos\theta_n)|_{h=2\lambda}=0$

$\sin\theta_n=0\Rightarrow\theta_n=0^\circ$

$\cos(kh\cos\theta_n)|_{h=2\lambda}=\cos(4\pi\cos\theta_n)=0\Rightarrow 4\pi\cos\theta_n=\cos^{-1}(0)=\pm\left(\dfrac{2n+1}{2}\right)\pi,$

$n=0,1,2,\ldots$

$\theta_n = \cos^{-1}[\pm(2n+1)/8], n = 0, 1, 2, 3, 4, \ldots.$

$$\left.\begin{aligned} n = 0: \quad & \theta_0 = \cos^{-1}\left(\pm\tfrac{1}{8}\right) = 82.82^\circ \\ n = 1; \quad & \theta_1 = \cos^{-1}\left(\pm\tfrac{3}{8}\right) = 67.98^\circ \\ n = 2; \quad & \theta_2 = \cos^{-1}\left(\pm\tfrac{5}{8}\right) = 51.32^\circ \\ n = 3; \quad & \theta_3 = \cos^{-1}\left(\pm\tfrac{7}{8}\right) = 28.96^\circ \end{aligned}\right\} \begin{array}{l} \text{for } 0^\circ \leqslant \theta \leqslant 90^\circ \\ (\text{for } 90^\circ \leqslant \theta \leqslant 180^\circ, \text{the field is zero}) \end{array}$$

$n = 4; \quad \theta_4 = \cos^{-1}\left(\pm\frac{9}{8}\right) =$ Does not exist. The same holds for $n \geq 5$.

Therefore where the field vanishes for $0^\circ \leqslant \theta \leqslant 90^\circ$, are

$$\theta = 0^\circ, 28.96^\circ, 51.32^\circ, 67.98^\circ, \text{ and } 82.82^\circ$$

4-49. $E_\theta = C_1 \cdot \sin\theta \cdot \cos(kh\cos\theta)$, where $C_1 = j\eta \dfrac{kI_0 e^{-jkr} \cdot l}{2\pi r}$

$E_\theta|_{\theta=60^\circ} = C_1 \cdot \sin(60^\circ) \cdot \cos(kh_n \cos(60^\circ)) = 0 \Rightarrow \cos(kh_n\cos(60^\circ)) = 0$

$kh_n \cos(60^\circ) = kh_n\left(\frac{1}{2}\right) = \dfrac{\pi}{\lambda} h_n = \cos^{-1}(0) = \pm\left(\dfrac{2n+1}{2}\right)\pi, n = 0, 1, 2, 3, \ldots.$

Choosing the positive values

$$h_n = \left(\frac{2n+1}{2}\right)\lambda, n = 0, 1, 2, 3, \ldots$$

$$h_n = 0.5\lambda, 1.5\lambda, 2.5\lambda, 3.5\lambda, 4.5\lambda$$

4-50. E_θ (4-99) $\simeq C \cdot \sin\theta \cdot [2\cos(kh\cos\theta)] \Rightarrow \text{AF} = 2[\cos(kh\cos\theta)]_{\max} = \pm 2$

$\cos(kh\cos\theta_m) = \pm 1$

a. $kh\cos\theta_m = \cos^{-1}(\pm 1) \Rightarrow \theta_m = \dfrac{1}{kh}\cos^{-1}(\pm 1) = \pm\dfrac{m\pi}{kh} = \dfrac{\pm m\pi}{\dfrac{2\pi}{\lambda}\left(\dfrac{3\lambda}{2}\right)}$

$= \pm\dfrac{m}{3}, m = 0, 1, 2, \ldots$

$\theta_m = \cos^{-1}(\pm m/3); m = 0, 1, 2, 3, \ldots$

$m = 0; \ \theta_0 = \cos^{-1}(\pm 0) = 90^\circ$

$m = 1; \ \theta_1 = \cos^{-1}(\pm 1/3) = \begin{cases} \cos^{-1}(1/3) = 70.5288^\circ \\ \cos^{-1}(-1/3) = 109.4712^\circ (\Rightarrow \text{Below Ground Plane}) \end{cases}$

$m = 2; \ \theta_2 = \cos^{-1}(\pm 2/3) = \begin{cases} \cos^{-1}(2/3) = 48.1897^\circ \\ \cos^{-1}(-2/3) = 131.8103^\circ (\Rightarrow \text{Below Ground Plane}) \end{cases}$

$m = 3$: $\theta_3 = \cos^{-1}(\pm 1) = \begin{cases} \cos^{-1}(1) = 0^\circ \\ \cos(-1) = 180^\circ (\Rightarrow \text{Below Ground Plane}) \end{cases}$

$m = 4$: $\theta_4 = \cos^{-1}(\pm 4/3) \Rightarrow$ does not exist

b. $E_{\theta m} = C \cdot \sin\theta[2 \cdot \cos(kh\cos\theta)]_{\max} = \pm 2C$, where $\theta = 90°$

c. $\dfrac{E_\theta}{E_{\theta m}} = \sin\theta \cdot \cos(kh\cos\theta)$

$\theta = 0°$: $\dfrac{E_\theta}{E_{\theta m}} = 0 \Rightarrow \dfrac{E_\theta}{E_{\theta m}} = 20\log_{10}(0) = -\infty$ dB

$\theta = 48.1897°$: $\dfrac{E_\theta}{E_{\theta m}} = \sin\theta \cdot \cos(kh\cos\theta)\big|_{h=\frac{3\lambda}{2}} = 0.7454 \Rightarrow \dfrac{E_\theta}{E_{\theta m}}$

$= 20\log_{10}(0.7454)$

$$\frac{E_\theta}{E_{\theta m}} = -2.55 \text{ dB}$$

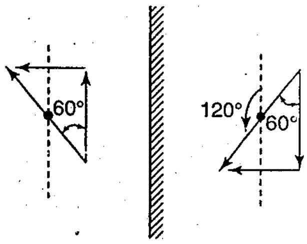
$\theta = 70.5288°$: $E_\theta/E_{\theta m} = \sin\theta \cdot \cos(kh\cos\theta)\big|_{h=\frac{3\lambda}{2}} = 0.9428 \Rightarrow \dfrac{E_\theta}{E_{\theta m}}$

$= 20\log_{10}(0.9428)$

$$\frac{E_\theta}{E_{\theta m}} = -0.5115 \text{ dB}$$

$\theta = 90°$: $E_\theta/E_{\theta m} = \sin\theta \cdot \cos(kh\cos\theta)\big|_{h=\frac{3\lambda}{2}} = 1 \Rightarrow \dfrac{E_\theta}{E_{\theta m}} = 20\log_{10}(1) = 0$ dB

4-51.



4-52. $E_\theta \simeq j\eta \dfrac{kI_0le^{-jkr}}{4\pi r}\sin\theta \cdot [2\cos(kh\cos\theta)]$

$|AF|_{\max} = |\cos(kh\cos\theta)|_{\max} = 1$ when $kh\cos\theta_{\max} = \pi$

$kh\cos\theta_{\max} = \pi,\ kh\cos(60°) = \pi,$

$\dfrac{2\pi}{\lambda} \cdot h \cdot \left(\frac{1}{2}\right) = \pi,\ h = \lambda$

No matter what the height is when $\theta = 90°$, it is a maximum.

So you always have a maximum at $\theta = 90°$. If you want a maximum at $\theta = 60°$, then $kh\cos\theta = n\pi, (n = 1, 2, 3, \ldots)$ leads to a maximum at $\theta = 60°$.

$n = 1$: $kh\cos\theta\big|_{\max} = \pi, h = \lambda$ leads to maxima at $\theta = 90°, 60°$

If you check closely, it also leads to a maximum at $\theta = 0°$.

So you cannot only have one maximum at $\theta = 60°$

4-53. $E_\theta \sim C_1 \cdot \sin\theta \cdot \cos(kh\cos\theta)\Big|_{\theta=80°} = 0$

$$\cos(kh\cos\theta)\Big|_{\theta=80°} = 0, kh\cos\theta\Big|_{\theta=80°} = \frac{\pi}{2}, \frac{2\pi}{\lambda}h\cos'\theta\Big|_{\theta=80°} = \frac{\pi}{2}$$

$$h = \frac{\lambda}{4\cos\theta}\Big|_{\theta=80°} = \frac{\lambda}{4(0.1736)} = \frac{\lambda}{0.6946} = 1.4397 \cdot \lambda$$
$$h = 1.4397\lambda, \lambda = \frac{3\times 10^8}{50\times 10^6} = \frac{30\times 10^7}{5\times 10^7} = 6 \text{ meters}$$
$$h = 1.4397 \cdot \lambda = 1.4397 \cdot (6) = 8.6382 \text{ m}$$
$$h = 8.6382 \text{ meters}$$

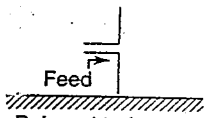
4-54. a. $Z_{im}(l = \lambda/2)\big|_{\text{above ground plane}} = \frac{1}{2}Z_{im}(l=\lambda)\big|_{\text{free space}} \simeq \frac{1}{2}(R_{im} + jX_{im})\big|_{l=\lambda}$

From Problem 4-23 $\Rightarrow$ $R_{im} = R_r = 199.099$

From Figure 4.20 $\Rightarrow$ $X_{im}\left(l = \frac{\lambda}{2}\right)\Big|_{\text{above ground plane}} \simeq 62.5$

Therefore

$$Z_{im}(l=\lambda/2)\big|_{\text{above ground plane}} = \frac{199.099}{2} + j62.5 = 99.5495 + j625$$



b. $Z_{\text{in}} = \dfrac{Z_{im}}{\sin^2\left(\dfrac{kl}{2}\right)} = \dfrac{99.5495 + j62.5}{\sin^2(\pi)} = \infty$

c. $$\Gamma = \frac{Z_{\text{in}} - Z_c}{Z_{\text{in}} + Z_c} = \frac{\infty - 50}{\infty + 50} = \frac{1 - 50/\infty}{1 + 50/\infty} = 1$$
$$\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+1}{1-1} = \infty$$

X_{im} can also be obtained using (8-60b). For $l = \lambda \Rightarrow kl = 2\pi,\ 2kl = 4\pi$. Thus

$$\begin{aligned}
&X_{im}(l=\lambda/2)\big|_{\text{above ground plane}} = \tfrac{1}{2}X_{im}(l=\lambda)\big|_{\text{free space}} \\
&= \frac{\eta}{8\pi}\{2S_i(kl) + \cos(kl)[2S_i(kl) - S_i(2kl)]\} \\
&= \frac{120\pi}{80\pi}\{2S_i(2\pi) + \cos(2\pi)[2S_i(2\pi) - S_i(4\pi)]\} \\
&= 15\{2(1.418) + [2(1.418) - 1.492]\} = 62.7
\end{aligned}$$

4-55. $\text{AF} = \cos(kh\cos\theta), f = 1\text{ GHz} \Rightarrow \lambda = \dfrac{3\times10^8}{1\times10^9} = 0.3$ meters

(a)
$$|(\text{AF})|_{\theta_n=30^\circ} = |\cos(kh\cos 30^\circ)| = |\cos(0.866kh)| = 0$$
$$\Rightarrow 0.866kh = \cos^{-1}(0) = \frac{n\pi}{2}, n = 1,2,3,\ldots$$

$$h_1 = \frac{n\pi/2}{0.866\ k} = \frac{n\pi/2}{0.8662\pi/\lambda} = \frac{n\lambda}{0.866(4\pi)} = \frac{n(3)}{4(0.866\pi)}\bigg|_{n=1} = 0.0866 \text{ meters}$$

$h_1 = 0.0866$ meters

(b) 1. $|\cos(kh\cos\theta)|_{h=0.3\ m=1\lambda} = \left|\cos\left(\dfrac{2\pi}{\lambda}\lambda\cos\theta_n\right)\right| = |\cos(2\pi\cos\theta_n)| = 0$

$2\pi\cos\theta_n = \cos^{-1}(0) = \dfrac{n\pi}{2} \Rightarrow n = 1,3,5,\ldots$

$$\theta_n = \cos^{-1}\left(\frac{n\pi/2}{2\pi}\right) = \cos^{-1}\left(\frac{n}{4}\right), n = 1,3,5,\ldots$$

n = 1: $\theta_1 = \cos^{-1}\left(\frac{1}{4}\right) = 75.52^\circ$

n = 3: $\theta_3 = \cos^{-1}\left(\frac{3}{4}\right) = 41.41^\circ$

n = 5: $\theta_5 = \cos^{-1}\left(\frac{5}{4}\right) =$ does not exist

2. $|\cos(kh\cos\theta_m)|_{h=0.3\ m=\lambda} = \left|\cos\left(\dfrac{2\pi}{\lambda}\lambda\cos\theta_m\right)\right| = |\cos(2\pi\cos\theta_m)| = 1$

$2\pi\cos\theta_m = \cos^{-1}(1) = m\pi, \quad m = 0,1,2,3,\ldots$

$$\theta_m = \cos^{-1}\left(\frac{m\pi}{2\pi}\right) = \cos^{-1}\left(\frac{m}{z}\right)$$

m = 0: $\theta_0 = \cos^{-1}(0) = 90^\circ$

m = 1: $\theta_1 = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$

m = 2: $\theta_2 = \cos^{-1}(1) = 0^\circ$

m = 3: $\theta_3 = \cos^{-1}\left(\frac{3}{2}\right) =$ does not exist

4-56. $f = 200$ MHZ $\Rightarrow \lambda = \dfrac{3 \times 10^8}{2 \times 10^8} = 1.5$ meters

$$E_\theta(\text{normalized}) = \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\cos(kh\cos\theta).$$

Since $\dfrac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$ has a null only toward $\theta = 0°$, the only way to place a null toward $\theta = 60°$ will be through $\cos(kh\cos\theta)$.

$$|\cos(kh\cos\theta)|_{\theta=\theta_h=60°} = |\cos(kh\cos\theta_n)| = |\cos(kh\cos 60°)| = 0$$

$$\left|\cos\left(\frac{2\pi}{\lambda}h\tfrac{1}{2}\right)\right| = \left|\cos\left(\frac{\pi h}{\lambda}\right)\right| = 0$$

$$\frac{\pi h}{\lambda} = \cos^{-1}(0) = \frac{n\pi}{2}, \quad n = 1, 3, 5, \ldots.$$

$$\boxed{h = \frac{n\pi}{2}\left(\frac{\lambda}{\pi}\right) = \frac{n\lambda}{2}, \quad n = 1, 3, 5, \ldots}$$

a. $h|_{n=1} = h_1 = \dfrac{\lambda}{2} = \dfrac{3}{2}\left(\frac{1}{2}\right) = \dfrac{3}{4} = \boxed{0.75 \text{ meters}}$

b. $h|_{n=3} = h_3 = \dfrac{3\lambda}{2} = \boxed{2.25 \text{ meters}}$

c. $h|_{n=5} = h_5 = \dfrac{5\lambda}{2} = \boxed{3.75 \text{ meters}}$

4-57. $G_0(\text{dB}) = 10\log_{10} G_0(\text{dimensionless}) \Rightarrow 16 = 10\log_{10} G_0$

$$\Rightarrow G_0(\text{dimensionless}) = 10^{1.6} = 39.81$$

$$P_{\text{rad}} = e_0 P_{\text{in}} = (1)(8) = 8 \text{ watts}$$

$$W_0 = \frac{P_{\text{rad}}}{4\pi r^2} = \frac{8}{4\pi(100 \times 100)^2} = \frac{8}{4\pi \times 10^8} = \frac{2}{\pi} \times 10^{-8} = 0.6366 \times 10^{-8}$$

$$= 6.366 \times 10^{-9} \text{ watts/cm}^2$$

$$W_{\max} = W_0 G_0(\text{dimensionless}) = 39.81(6.366 \times 10^{-9})$$

$$= 2.534 \times 10^{-7} = 0.2534 \times 10^{-6} \text{ Watts/cm}^2$$

4-58. $l = \lambda/4, f = 1.9$ GHz, $W_i = 10^{-6}$ W/m^2 $\Rightarrow \lambda = \dfrac{3 \times 10^6}{1.9 \times 10^9} = 0.15789$ m

a. The power pattern of a $\lambda/4$ monopole *above* a PEC is equivalent to that of a $\lambda/2$ dipole in free space. Since the same power radiated by the monopole above the PEC is concentrated only in the *upper* hemisphere, instead over the entire free space, its radiation intensity will be *twice* as strong/intense as that of the $\lambda/2$ dipole radiating in free space. Since the directivity is given by

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}}$$

The $U_{\max}$ of the monopole will be twice that of the dipole, or

$$D_0(l = \lambda/4) = 2(1.643) = \boxed{3.286 = 5.17 \text{ dB}}$$

Using the computer program directivity it gives

$$D_0(l = \lambda/4) = \boxed{3.3365 = 5.2329 \text{ dB}}$$

b. $A_{em} = \dfrac{\lambda^2}{4\pi} D_0 = \dfrac{(0.15789)^2}{4\pi}(3.286) = 6.52 \times 10^{-3} \text{ m}^2$

$P_L = A_{em} W_i = 6.52 \times 10^{-3}(10^{-6}) = 6.52 \times 10^{-9}$

$\boxed{P_L = 6.52 \times 10^{-9} \text{ watts}}$

4-59. $f = 900$ MHz, $P_{\text{rad}} = 1{,}000$ watts

a. *Isotropic*

$$W_{r0} \leq \frac{P_{\text{rad}}}{4\pi r^2}$$

$$r^2 \geq \frac{P_{\text{rad}}}{4\pi W_{r0}} = \frac{1{,}000}{4\pi(10)} = \frac{100}{4\pi} = 7.9558$$

$$\boxed{r \geq 2.821 \text{ meters}}$$

b. **$\boldsymbol{\lambda/4}$ monopole**

$$D_0(\text{monopole}) = 2(1.643) = 3.286$$

$$W_{\text{rad}} \leq D_0\, W_{r0} = D_0 \frac{P_{\text{rad}}}{4\pi r^2}$$

$$r^2 \geq D_0 \frac{P_{\text{rad}}}{4\pi W_{\text{rad}}} = 3.286\left(\frac{1{,}000}{4\pi(10)}\right) = 26.1492$$

$$\boxed{r \geq 5.114 \text{ meters}}$$

4-60. Using the coordinate system of Figure 4.24 the total field is given by (4-116) or

$$E_\psi = j\eta \frac{kI_0 l e^{-jkr}}{4\pi r}\sqrt{1 - \sin^2\theta \cdot \sin^2\phi}[2j\sin(\text{[illegible]} \cos\theta)],\ 0 \leqslant \theta \leqslant \pi, 0 \leqslant \phi \leqslant 2\pi$$

However if we rotate the axes so that the z axis is parallel to the axis of the element and y is vertical to the ground, the total E-field can be written as

$$E_\theta = j\eta \frac{kI_0 e^{-jkr}}{4\pi r} \cdot \sin\theta \cdot [2j\sin(kh\sin\theta\cdot\sin\phi)], \quad \text{and}$$

$$P_{\text{rad}} = \int_0^\pi \int_0^\pi \underline{W}_{\text{ave}} \cdot \hat{a}_r r^2 \sin^2\theta \, d\theta \, d\phi = \frac{1}{2\eta}\int_0^\pi \int_0^\pi |E_\theta|^2 r^2 \sin\theta \, d\theta \, d\phi$$

$$P_{\text{rad}} = \frac{\eta}{2}\left|\frac{kI_0 l}{2\pi}\right|^2 \int_0^\pi \int_0^\pi \sin^3\theta \cdot \sin^2(kh\sin\theta\sin\phi)\, d\theta\, d\phi = \frac{\eta}{2}\left|\frac{kI_0 l}{2\pi}\right|^2 I$$

$$I = \int_0^\pi \sin^3\theta \left\{\int_0^\pi \sin^2(kh\sin\theta\sin\phi)\, d\phi\right\} d\theta = \int_0^\pi \sin^3\theta [I_1]\, d\theta$$

where $I_1 = \int_0^\pi \sin^2(kh\sin\theta\sin\phi)\, d\phi = \frac{1}{2}\left\{\int_0^\pi d\phi - \int_0^\pi \cos(2kh\sin\theta\cdot\sin\phi\, d\phi\right\}$

$$= \tfrac{1}{2}\left\{\pi - \int_0^\pi \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) d\phi\right\}, \text{where } y = 2kh\sin\theta\sin\phi$$

$$= \frac{\pi}{2} - \tfrac{1}{2}\left\{\pi - \tfrac{1}{2}\int_0^\pi y^2\, d\phi + \frac{1}{(2\times 2)!}\int_0^\pi y^4\, d\phi - \frac{1}{(2\times 3)!}\int_0^\pi y^6\, d\phi + \cdots\right\}$$

$$I_1 = \tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\int_0^\pi \frac{(y)^{2n}}{2n!}\, d\phi = \tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2\alpha)^{2n}}{2n!}\int_0^\pi \sin^{2n}\phi\, d\phi$$

$$I_1 = \tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2\alpha)^{2n}}{2n!}\left[2\int_0^{\pi/2}\sin^{2n}\phi\, d\phi\right]$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2\alpha)^{2n}}{2n!}\int_0^{\pi/2}\sin^{2n}\phi\, d\phi$$

From Mathematical Handbook of Formulas and Tables $\left(\begin{array}{l}\leftarrow \alpha = kh\sin\theta \\ \quad y = 2\alpha\sin\phi\end{array}\right)$ Schaum's Outline Series, pg. 96 Equation 15-30.

$$\int_0^{\pi/2}\sin^{2n}x\, dx = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n-2)(2n)}\frac{\pi}{2}, n = 1,2,3,4,\ldots$$

Thus $I_1 = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2kh\sin\theta)^{2n}}{(2n)!}\cdot\frac{\pi}{2}\cdot A_{2n}$, where $A_{2n} = \frac{1\cdot 3\cdot 5\cdots(2n-3)(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)(2n)}$

and

$$I = \int_0^\pi \sin^3\theta [I_1]\, d\theta = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2kh)^{2n}}{(2n)!}\left(\frac{\pi}{2}\right)A_{2n}\int_0^\pi \sin^{2n+3}(\theta)\, d\theta$$

$$= \pi\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{(2kh)^{2n}}{(2n)!}\cdot A_{2n}\int_0^{\pi/2}\sin^{2n+3}\theta\, d\theta$$

Using Series equation of the previous reference, or

$$\int_0^{\pi/2} (\sin x)^{2n+3}\, dx = \frac{2\cdot 4\cdot 6\cdot\dots\cdot(2n-2)(2n)(2n+2)}{1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)(2n+1)(2n+3)}, n = 1, 2, 3, \dots$$

We can write that

$$I = \pi \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2kh)^{2n}}{(2n)!} (A_{2n})(A_{2n+3}),$$

$$A_{2n+3} = \frac{2\cdot 4\cdot 6\cdot\dots\cdot(2n-2)(2n)(2n+2)}{1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)(2n+1)(2n+3)}$$

However

$$\begin{aligned} A_{2n}\cdot A_{2n+3} &= \frac{1\cdot 3\cdot 5\cdots(2n-3)(2n-1)\ 2\cdot 4\cdot 6\cdot\dots\cdot(2n-2)(2n)(2n+2)}{2\cdot 4\cdot 6\cdots(2n-2)(2n)\ 1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)(2n+1)(2n+3)} \\ &= (2n+2)/[(2n+1)(2n+3)] \end{aligned}$$

Therefore

$$\begin{aligned} I &= \pi \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2kh)^{2n}}{(2n)!} \frac{(2n+2)}{(2n+1)(2n+3)} \\ &= \pi \left[\frac{(2kh)^2}{2!}\frac{4}{3\cdot 5} - \frac{(2kh)^4}{4!}\frac{6}{5\cdot 7} + \frac{(2kh)^6}{6!}\frac{8}{7\cdot 9} - \dots \right. \\ &\qquad \left. + (-1)^{n+1} \frac{(2kh)^{2n}}{(2n)!} \frac{(2n+2)}{(2n+1)(2n+3)} \right] \end{aligned}$$

which when expanded can be written as

$$\begin{aligned} I = \pi \left\{ \frac{2}{3} - \left[\frac{2}{3} + (2kh)^2 \left(-\frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \right) + (2kh)^4 \left(\frac{1}{5!} - \frac{1}{6!} + \frac{1}{7!} \right) + \cdots \right.\right. \\ \left.\left. \pm (2kh)^{2n} \left(\frac{1}{(2n+1)!} - \frac{1}{(2n+2)!} + \frac{1}{(2n+3)!} \right) \right] \right\} \end{aligned}$$

Recombining appropriate terms, we have that

$$\begin{aligned} I = \pi \left\{ \frac{2}{3} - \frac{1}{(2kh)} \left[(2kh) + \sum_{n=1}^{\infty} (-1)^n \frac{(2kh)^{2n+1}}{(2n+1)!} \right] \right. \\ - \frac{1}{(2kh)^2} \left[1 - \frac{(2kh)^2}{2!} + \sum_{n=2}^{\infty} (-1)^n \frac{(2kh)^{2n}}{(2n)!} \right] \\ \left. + \frac{1}{(2kh)} \left[(2kh) - \frac{(2kh)^3}{3!} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2kh)^{2n+3}}{(2n+3)!} \right] \right\} \end{aligned}$$

which reduces when expanded to

$$I = \pi\left[\frac{2}{3} - \frac{\sin(2kh)}{(2kh)} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right]$$

Therefore the radiated power can be written as

$$P_{\text{rad}} = \frac{\eta}{2}\left|\frac{kI_0 l}{2\pi}\right|^2 I = \eta\frac{\pi}{2}\cdot\left|\frac{I_0 l}{\lambda}\right|^2\left[\frac{2}{3} - \frac{\sin(2kh)}{(2kh)} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right]$$

4-61. $E_\psi = C_2\cdot\sqrt{1-\sin^2\theta\sin^2\phi}\cdot[\sin(kh\cos\theta)]$. $C_2 = -\eta\dfrac{kI_0 l e^{-jkr}}{2\pi r}$

a. $E_\psi(\phi = 90^\circ)\Big|_{\theta=45^\circ} = C_2\cdot\cos\theta\cdot\sin(kh\cos\theta)\Big|_{\theta=45^\circ} = C_2\cdot\cos(45^\circ)\sin\left(\dfrac{kh}{\sqrt{2}}\right) = 0$

$\dfrac{kh}{\sqrt{2}} = \sin^{-1}(0) = \pm n\pi, n = 0,1,2,3,\ldots$

Choosing the positive values and excluding the $n = 0$ value, we have the smallest height of $(n = 1)$

$$h = \frac{\sqrt{2}\pi}{k} = \frac{\sqrt{2}\pi}{2\pi}\lambda = \frac{\lambda}{\sqrt{2}} = 0.707\lambda$$

b. $h = \dfrac{\lambda}{\sqrt{2}} \Rightarrow 2kh = 2\left(\dfrac{2\pi}{\lambda}\right)\dfrac{\lambda}{\sqrt{2}} = 2\sqrt{2}\pi = 8.88576$

1. $R_r = 120\pi^2\left(\dfrac{1}{50}\right)^2\left[\dfrac{2}{3} - \dfrac{\sin(2\sqrt{2}\pi)}{2\sqrt{2}\pi} - \dfrac{\cos(2\sqrt{2}\pi)}{(2\sqrt{2}\pi)^2} + \dfrac{\sin(2\sqrt{2}\pi)}{(2\sqrt{2}\pi)^3}\right]$

$R_r = 120\pi^2\left(\dfrac{1}{50}\right)^2\left[\dfrac{2}{3} - 0.057765 + 0.0108694 + 0.0007316\right] = 0.294$

2. $kh = \sqrt{2}\pi$

$D_g = \dfrac{4(-0.9639)^2}{\left[\frac{2}{3} - 0.57765 + 0.0108694 + 0.0007316\right]} = \dfrac{4(0.9291)}{0.6205} = 5.9893$

$D_g = 5.9893 = 7.774$ dB

4-62. $E_\psi(\phi = 90^\circ) = C_2\cdot\cos\theta\cdot\sin(kh\cos\theta), C_2 = -\eta\dfrac{kI_0 l e^{-jkr}}{2\pi r}$

$E_\psi(\phi = 90^\circ)\big|_{h=0.707\lambda} = C_2\cdot\cos\theta_n\cdot\sin(0.707\lambda k\cos\theta_n) = 0$

$$\cos\theta_n = 0 \Rightarrow \theta_n = \cos^{-1}(0) = 90^\circ$$

$$\sin(0.707\lambda k\cos\theta_n) = \sin(1.414\pi\cos\theta_n) = 0 \Rightarrow 1.414\pi\cos\theta_n = \sin^{-1}(0)$$

$$1.414\pi\cos\theta_n = \sin^{-1}(0) = \pm n\pi, n = 0,1,2,3,\ldots$$

$$\theta_n = \cos^{-1}\left(\pm\frac{n}{1.414}\right), n = 0,1,2,3,\ldots$$

$$\left.\begin{array}{ll} n=0: & \theta_n=\cos^{-1}(0)=90^\circ \\ n=\pm 1: & \theta_n=\cos^{-1}\left(\pm\dfrac{1}{1.414}\right)=45^\circ \end{array}\right\} \text{ for } 0^\circ \leqslant \theta \leqslant 90^\circ$$

$$n=\pm 2: \quad \theta_n=\cos^{-1}\left(\pm\frac{2}{1.414}\right)=\text{Does not exist.}$$

The same holds for $n \geq 3$.

4-63. Since the horizontal dipole is placed a distance of 2λ above the PEC, then its image must also be a distance of 2λ below the PEC. This makes the separation between the actual source and its image to be 4λ. Since the minimum far-field distance is equal to

$$r = 2D^2/\lambda$$

where D is the large distance, which in this case is the hypotenuse, or

$$D=\sqrt{(4\lambda)^2+(\lambda/50)^2}=4.00005\lambda \simeq 4\lambda$$

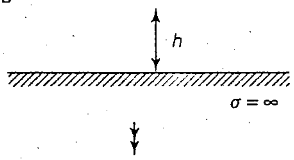
then

$$r=2(4\lambda)^2/\lambda=32\lambda$$

Since λ at 300 MHz the wavelength is 1 meter, then

$$r=32\lambda|_{\lambda=1}=32 \text{ meters}$$

4-64. $H_\theta{}^d = j\dfrac{kI_m l e^{-jkr_1}}{\eta\cdot 4\pi r}\cdot\sin\theta_1$

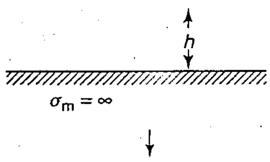


$$H_\theta{}^r = -j\frac{kI_m l e^{-jkr_2}}{\eta\cdot 4\pi r}\cdot\sin\theta_2,$$

$$\left.\begin{array}{l}\left.\begin{array}{l} r_1=r-h\cos\theta \\ r_2=r+h\cos\theta\end{array}\right\}\text{ phase, } \quad \begin{array}{l} r=r_1=r_2, \text{ for amplitude} \\ \theta_1=\theta_2=\theta.\end{array}\end{array}\right\} \Rightarrow \text{Far Field}$$

$$H_\theta = j\frac{kI_m l e^{-jkr}}{\eta\cdot 4\pi r}\cdot\sin\theta[2j\sin(kh\cos\theta)]$$

4-65. $E_\theta{}^d = j\eta\cdot\dfrac{kI_0 l e^{-jkr_1}}{4\pi r_1}\cdot\sin\theta_1$



$$E_\theta{}^r = -j\eta\cdot\frac{kI_0 l e^{-jkr_2}}{4\pi r_2}\cdot\sin\theta_2$$

Far field: $\left.\begin{array}{l} r_1=r-h\cos\theta \\ r_2=r+h\cos\theta\end{array}\right\}$ phase

$(r=r_1=r_2,\ \theta=\theta_1=\theta_2)$ amplitude

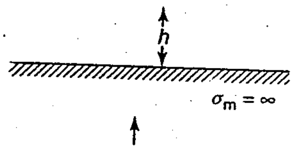
$$E_\theta = j\eta\frac{kI_0 l e^{-jkr}}{4\pi r}\sin\theta[e^{jkh\cdot\cos\theta}-e^{-jkh\cos\theta}]$$

$$E_\theta = j\eta\frac{kI_0 l e^{-jkr}}{4\pi r}\cdot\sin\theta\cdot[2j\sin(kh\cos\theta)]$$

4-66. $H_\theta^{\text{total}} = H_\theta^d + H_\theta^r$

$$H_\theta{}^d = j\frac{kI_m l e^{-jkr_1}}{\eta \cdot 4\pi r_1} \cdot \sin\theta_1$$

$$H_\theta{}^r = j\frac{kI_m l e^{-jkr_2}}{\eta \cdot 4\pi r_2} \cdot \sin\theta_2$$



For Far field. $\left.\begin{matrix} r_1 \simeq r - h\cos\theta \\ r_2 \simeq r + h\cos\theta \end{matrix}\right\}$ phase, $(r_1 = r_2 = r) \rightarrow$ Amplitude

$$\theta = \theta_1 = \theta_2$$

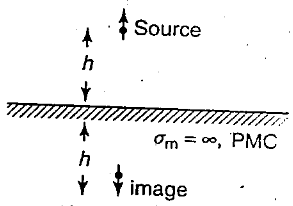
$$H_\theta^{\text{total}} = j\frac{kI_m l e^{-jkr}}{\eta \cdot 4\pi r} \cdot \sin\theta \cdot [2\cos(kh\cos\theta)]$$

4-67. a. $E_\theta \simeq j\eta\frac{kl}{4\pi}I_0\frac{e^{-jkr}}{r}\sin\theta[j2\sin(kh\cos\theta)]$

$\sin(kh\cos 60°) = 0 \rightarrow kh_n\cos 60° = n\pi, n = 1, 2, 3, \ldots$

$$h_n = \frac{n\pi}{k\cos 60°} = \frac{n\lambda}{2\cdot\cos 60°} = n\lambda$$

smallest $h \rightarrow n = 1 \rightarrow h = \lambda$



b. $$W_{av} \simeq \frac{|E_0|^2}{2\eta} \simeq \frac{\eta(kl)^2}{32\cdot\pi^2}\cdot|I_0|^2\cdot\frac{(kl)^2}{r^2}\cdot\sin^2\theta[4\sin^2(kh\cos\theta)]$$

$$U(\theta,\phi) = \lim_{r\to\infty} r^2\, W_{av} = \frac{\eta}{2}\cdot\left(\frac{l}{\lambda}\right)^2\cdot|I_0|^2\cdot\sin^2\theta\cdot\sin^2(kh\cos\theta)$$

$$= \int_0^{2\pi}\int_0^{\pi/2} U(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

$$= \pi\eta\left(\frac{l}{\lambda}\right)^2\cdot|I_0|^2\cdot\int_0^{\pi/2}\sin^3\theta\cdot\sin^2(kh\cos\theta)\, d\theta$$

$$= \pi\eta\cdot\left(\frac{l}{\lambda}\right)^2\cdot|I_0|^2\cdot\left\{\frac{1}{3} + \frac{\cos(2kh)}{(2kh)^2} - \frac{\sin(2kh)}{(2kh)^3}\right\}$$

$$P_{rad} \underset{\substack{\uparrow \\ kh=2\pi}}{=} \pi\eta\cdot\left(\frac{l}{\lambda}\right)^2\cdot|I_0|^2\cdot\left\{\frac{1}{3} + \frac{1}{(4\pi)^2}\right\} = \pi\eta\left(\frac{l}{\lambda}\right)^2|I_0|^2\{0.3397\}$$

1. $$D_g(\theta = 45°, \phi) = \frac{4\pi U(\theta = 45°, \phi)}{P_{rad}} = \frac{2\sin^2(45°)\sin^2(2\pi\cos 45°)}{0.3397}$$

$$= 2.74 = 4.37 \text{ dB}$$

2. $$R_r = \frac{2P_{rad}}{|I_0|^2} = 2\cdot\pi\cdot\eta\left(\frac{l}{\lambda}\right)^2\cdot\{0.3397\}$$

$$\frac{R_r}{\eta} = 2\pi\times 10^{-4}\times 0.3397 = 2.13\times 10^{-4}.$$

4-68. Since $d \ll a$

$$\tan\psi \simeq \frac{h_1'}{d_1} \simeq \frac{h_2'}{d_2} = \frac{h_2'}{d-d_1} \Rightarrow h_1'(d-d_1) = d_1 h_2'$$

$$d_1(+h_1' + h_2') = h_1'\, d \Rightarrow d_1 = \frac{h_1'\, d}{h_1' + h_2'} = \frac{5(20\times 10^3)}{5+1{,}000} = 99.5 \text{ meters}$$

$$\psi = \tan^{-1}\left(\frac{h_1'}{d}\right) = \tan^{-1}\left(\frac{5}{99.5}\right) = 2.87669^\circ$$

$$\frac{\sigma}{\omega\epsilon} = \frac{10^{-2}}{2\pi\times 10^9(5\times 10^{-9}/(36\pi))} = \frac{18}{5}\times 10^{-2} = 3.6\times 10^{-2} \ll 1$$

Therefore the earth is a good dielectric $\Rightarrow \eta_1 \simeq \sqrt{\frac{\mu_1}{\epsilon_1}}, \beta_1 = \omega\sqrt{\mu_1\epsilon_1}$

The divergence factor is equal to ($a = 5280$ miles $= 8.497368\times 10^6$ m)

$$D \simeq \left[1 + 2\frac{h_1' h_2'}{a d \tan^3\psi}\right]^{-1/2} = \left[1 + \frac{2(5)(1{,}000)}{8.497368\times 10^6\times 2\times 10^4(0.05)^3}\right]^{-1/2}$$
$$= (1+0.000463)^{-1/2}$$
$$= 0.99977$$

and the reflection coefficient equal to

$$R_v = \frac{\eta_0\cos\theta_i - \eta_1\cos\theta_t}{\eta_0\cos\theta_i + \eta_1\cos\theta_t}, \text{ where } \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}},\ \eta_1 = \sqrt{\frac{j\omega\mu_0}{\sigma_1 + j\omega\epsilon_1}} \simeq \sqrt{\frac{\mu_0}{\epsilon_1}}$$

$$\gamma_0\sin\theta_i = \gamma_1\sin\theta_t \Rightarrow \beta_0\sin\theta_i = \beta_1\sin\theta_t \Rightarrow \sin\theta_t = \frac{\beta_0}{\beta_1}\sin\theta_i = \sqrt{\frac{\epsilon_0}{\epsilon_1}}\sin\theta_i$$

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1 - \frac{\epsilon_0}{\epsilon_1}\sin^2\theta_i} = \sqrt{1 - \frac{\sin^2\theta_i}{\epsilon_r}}$$

Therefore

$$R_v = \frac{\cos\theta_i - \frac{\eta_1}{\eta_0}\cos\theta_t}{\cos\theta_i + \frac{\eta_1}{\eta_0}\cos\theta_t} = \frac{\cos\theta_i - \frac{1}{\sqrt{\epsilon_r}}\sqrt{1-\sin^2\theta_i/\epsilon_r}}{\cos\theta_i + \frac{1}{\sqrt{\epsilon_r}}\sqrt{1-\sin^2\theta_i/\epsilon_r}} = \frac{\epsilon_r\cos\theta_i - \sqrt{\epsilon_r - \sin^2\theta_i}}{\epsilon_r\cos\theta_i + \sqrt{\epsilon_r - \sin^2\theta_i}}$$

$$\theta_i = 90 - \psi = 90 - 2.87669^\circ = 87.12331^\circ \Rightarrow \sin\theta_i = 0.9987, \cos\theta_i = 0.0502$$

$$\text{Thus } R_v = \frac{5(0.0502) - \sqrt{5-(0.9987)^2}}{5(0.0502) + \sqrt{5-(0.9987)^2}} = \frac{-1.749649}{2.251649} = -0.777$$

$$E_\theta \simeq j\eta\frac{I_0 e^{-jkr}}{2\pi r}\cdot\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\cdot[e^{jkh_1'\cos\theta} + DR_v e^{-jkh_1'\cos\theta}]_{\theta=\theta_i\simeq 87.12331^\circ}$$

$$r \simeq \sqrt{d^2 + (h_2' - h_1')^2} = \sqrt{(20{,}000)^2 + (1{,}000-99.5)^2} = 20{,}020.26 \text{ m} = 66{,}734.207\lambda$$

$$h_1' = 5 \text{ m} = 16.667\lambda, h_2' = 1{,}000 \text{ m} = 3{,}333.3333\lambda$$

$$E_\theta = j \cdot 120\pi \cdot \frac{I_0 e^{-j\frac{2\pi}{\lambda}(66{,}734.207\lambda)}}{2\pi(20{,}020.26)} \cdot \frac{\cos\left(\frac{\pi}{2}\cos(87.12^\circ)\right)}{\sin(87.12^\circ)}[e^{jkh_1'\cos\theta_i} + DR_v e^{-jkh_1'\cos\theta_i}]$$

$$e^{-j\frac{2\pi}{\lambda}(66{,}734.207\lambda)} = e^{-j2\pi(0.207)} = e^{-j1.3} = \cos(74.52^\circ) - j\sin(74.52^\circ)$$
$$= 0.2669 - j0.9637 = 1\angle -74.52^\circ$$

$$\cos\left[\frac{\pi}{2}\cos(87.12^\circ)\right] = 0.996887, \sin(87.12^\circ) = 0.99874$$

$$e^{jkh_1'\cos\theta_i} = e^{j\frac{2\pi}{\lambda}(16.667\lambda)(0.0502)} = e^{j2\pi(16.667)(0.0502)} = e^{j5.257}$$
$$= \cos(301.2^\circ) + j\sin(301.2^\circ) = 1\angle 301.2^\circ = 0.5181 - j0.8553$$
$$e^{-jkh_1'\cos\theta_i} = 1\angle -301.2^\circ = 0.5181 + j0.8553$$
$$DR_v e^{-jkh_1'\cos\theta_i} = 0.99977(-0.777)[0.5181 + j0.8553] = -(0.4025 + j0.6644)$$

Thus

$$e^{jkh_1'\cos\theta_i} + R_v D e^{-jkh_1'\cos\theta_i} = (0.5181 - j0.8553) - (0.4025 + j0.6644)$$
$$= 0.1156 - j1.5197 = 1.5241\angle -85.65^\circ$$

Therefore

$$E_\theta \simeq (1\angle 90^\circ)(120\pi)\frac{I_0(1\angle -74.52^\circ)(0.996887)}{2\pi(0.99874)(20{,}020.26)}(1.5241\angle -85.65^\circ)$$
$$E_\theta \simeq 4.5592 \times 10^{-3} \cdot I_0\angle -70.17$$

or

$$|E_\theta| = 4.5592 \times 10^{-3}|I_0| \quad \text{Volts/m}$$

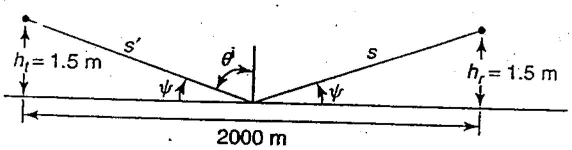
4-69. From calibration;

$$\frac{P_r}{P_t} = \frac{C_1}{R^2} \to C_1 = \frac{P_r}{P_t}R^2 = \frac{10 \times 10^{-6}}{5} \times (10 \times 10^3)^2 = 200 \text{ m}^2$$

on asteroid

$$\frac{P_r}{P_t} = \frac{C_1}{R^2}|1 + DR_v e^{-j2kht\cos\theta}|^2$$

Approximate geometry;



$$\psi = \tan^{-1}\left(\frac{1.5}{1000}\right) \simeq 1.5 \times 10^{-3} = 0.086°$$

$$\theta^i = \frac{\pi}{2} - \psi; \cos\theta^i = \sin\psi \simeq 1.5 \times 10^{-3}$$

$$\sin\theta^t = \frac{\sin\theta^i}{3} \simeq \frac{1}{3}; \cos\theta^t \simeq \frac{2\sqrt{2}}{3}$$

$$R_v = \frac{\frac{\eta_0}{\eta_1}\cos\theta^i - \cos\theta^t}{\frac{\eta_0}{\eta_1}\cos\theta^i + \cos\theta^t} = \frac{3(1.5 \times 10^{-3}) - \frac{2\sqrt{2}}{3}}{3(1.5 \times 10^{-3}) + \frac{2\sqrt{2}}{3}} = -0.9905$$

$$s' \simeq s \simeq 1000 \text{ m}; a = 10^6 \text{ m}$$

$$D \simeq \left[1 + 2\frac{ss'}{ad\tan\psi}\right]^{-1/2} \simeq \left[1 + 2\frac{(1000)(1000)}{10^6(2000)1.5 \times 10^{-3}}\right]^{-1/2} = 0.7746$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{300 \times 10^6} = 1 \text{ m}$$

$$\begin{aligned}|1 + DR_v e^{-j2kht\cos\theta}|^2 &\simeq |1 - (0.7746)(0.9905)e^{-j4\pi\frac{ht}{\lambda}}|^2 \\ &= |1 - (0.7746)(0.9905)e^{-j4\pi}|^2 \\ &= 0.0541772\end{aligned}$$

$$P_r = \frac{200}{(2 \times 10^3)^2} \cdot (0.0541772)(5) = 1.3544 \times 10^{-5} \text{ W} = 13.5\ \mu\text{W}$$

4-70. $P_{\text{rad}} = 10$ Watt, $r = 3.7 \times 10^7$ m, $D_0 = 50 \text{ dB} \Rightarrow 10^5$

a. $$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi \cdot r^2|E|^2}{2\eta \cdot 10} = 10^5, \left(\text{Since } U_{\max} = \frac{r^2 E_{\max}^2}{2\eta}; \eta = 120\pi\right)$$

$$\Rightarrow E^2 = \frac{10^5 \times 2 \times 120\pi \times 10}{4\pi(3.7 \times 10^7)^2} = 4.4 \times 10^{-8}$$

$$E = 2 \times 10^{-4} \text{ v/m}$$

b. Use Friis Transmission

$$\frac{P_r}{P_t} = \left(\frac{\lambda}{4\pi R}\right)^2 \cdot G_{0t} \cdot G_{0v} = \left(\frac{\lambda}{4\pi R}\right)^2 \cdot D_{0t} \cdot D_{0r}$$

(Since we assume 100% efficiency)

At 10 GHz, $\lambda = 0.03$ m

$$\frac{P_r}{10} = \left[\frac{0.03}{4\pi \cdot (3.7 \times 10^7)}\right]^2 \cdot (10{,}000)(1.643)$$

$$P_r = 6.84 \times 10^{-15}$$

$$P_{\text{received}} = \frac{V^2}{8R_{\text{in}}}, \text{Since } R_r = 73 = R_{\text{in}} \text{ for } \lambda/2 \text{ dipole then}$$

$$V = \sqrt{8(P_{\text{received}})(R_{\text{in}})} = 2\mu V$$

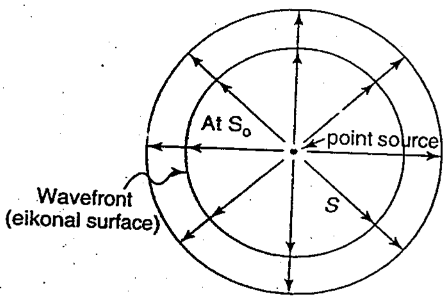
4-71. $S_0\, dA_0 = S dA$, $\quad \dfrac{S}{S_0} = \dfrac{dA_0}{dA}$

far zone $S = \frac{1}{2\eta}|\underline{E}|^2$

$$\frac{|\underline{E}|}{|\underline{E}_0|} = \sqrt{\frac{dA_0}{dA}}$$

For spherical wave: $\dfrac{|\underline{E}|}{|\underline{E}_0|} = \sqrt{\dfrac{S_0^2}{(S+S_0)^2}} = \dfrac{S_0}{S+S_0}$

For plane wave: $|\underline{E}|/|\underline{E}_0| = 1$.



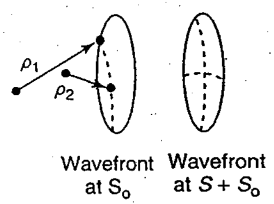
In general, it can be shown that for a wave front eikonal surface we have $\dfrac{|\underline{E}|}{|\underline{E}_0|} = \sqrt{\dfrac{\rho_1\rho_2}{(\rho_1+s)(\rho_2+s)}}$; ρ_1 and ρ_2 are radii of curvature of wavefront.

e.g. spherical wave: $\rho_1 = \rho_2 = S_0$

$$\frac{|\underline{E}|}{|\underline{E}_0|} = \sqrt{\frac{{S_0}^2}{(S+S_0)^2}} = \frac{S_0}{S+S_0}$$

plane wave: $\rho_1 = \rho_2 = \infty$

$$\frac{|\underline{E}|}{|\underline{E}_0|} = 1.$$

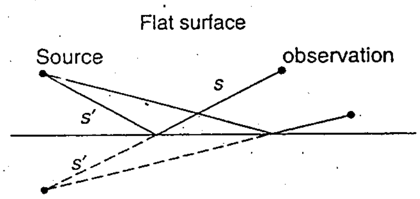


When the wave front is reflected from a surface we have

$$\frac{|\underline{E}|}{|\underline{E}_0|} = \sqrt{\frac{{\rho_1}^r \cdot {\rho_2}^r}{({\rho_1}^r + s)({\rho_2}^r + s)}} = \sqrt{\frac{1}{\left(1 + \frac{s}{{\rho_1}^r}\right)\left(1 + \frac{s}{{\rho_2}^r}\right)}}$$

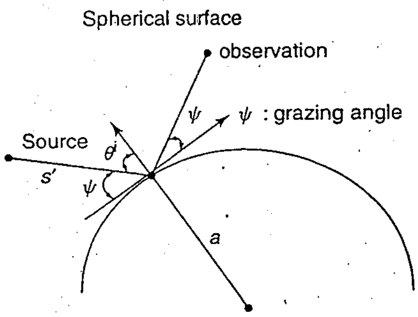
$\underline{E}$ is field at observation point.

$\underline{E}_0$ is field at reflection point.



Radius of curvature of wavefront not changed by reflection.

$$ {\rho_1}^r = {\rho_2}^r = \rho_1 = \rho_2 = s' \quad \frac{|\underline{E}|}{|\underline{E}_0|} = \frac{s'}{s' + s} = \frac{1}{1 + s/s'}$$



$$\frac{1}{{\rho_1}^r} = \frac{1}{s'} + \frac{1}{f_1}; \frac{1}{{\rho_2}^r} = \frac{1}{s'} + \frac{1}{f_2}$$

In physics, we always used $f_1 = f_2 = a/2$. This is not valid here because that f was valid for near normal incidence; we have near grazing incidence

$$f_1 = \frac{a\cos\theta^i}{2} \quad \text{(perpendicular to the plane of incidence = elevation plane)}$$
$$f_2 = \frac{a}{2\cos\theta_i}: \quad \text{(parallel to the plane of incidence = azimuthal plane)}$$

Thus $\dfrac{1}{\rho_1{}^r} = \dfrac{1}{s'} + \dfrac{2}{a\cos\theta_i}$; $\frac{1}{\rho_2{}^r} = \frac{1}{s'} + \dfrac{2\cos\theta^i}{a}$

$$\frac{|E|}{|E_0|} = \sqrt{\frac{1}{\left\{1+s\left(\frac{1}{s'}+\frac{2}{a\cos\theta_i}\right)\right\}\left\{1+s\left(\frac{1}{s'}+\frac{2\cos\theta^i}{a}\right)\right\}}}$$
$$= \frac{1}{\sqrt{1+\frac{s}{s'}+\frac{2s}{a\cos\theta^i}}\cdot\sqrt{1+\frac{s}{s'}+\frac{2s\cos\theta_i}{a}}}$$
$$= \frac{1}{\left(1+\frac{s}{s'}\right)}\frac{1}{\sqrt{1+\frac{\frac{2ss'}{a\cos\theta_i}}{s+s'}}\sqrt{1+\frac{\frac{2ss'\cos\theta_i}{a}}{s+s'}}}$$

$\cos\theta_i = \cos\left(\frac{\pi}{2}-\psi\right) = \sin\psi$

$$\frac{|E|}{|E_0|} = \frac{1}{\left(1+\frac{s}{s'}\right)}\frac{1}{\sqrt{1+\frac{2ss'}{a(s+s')\sin\psi}}\sqrt{1+\frac{2ss'\sin\psi}{a(s+s')}}}$$
$$\simeq \left[1+\frac{2ss'}{a(s+s')\sin\psi}\right]^{-1/2}\cdot\frac{1}{(1+s/s')}$$

near grazing neglect divergence in azimuthal plane.

CHAPTER 5

5-1. From (5-17) $\Rightarrow \underline{A} = \hat{a}_\phi A_\phi(r,\theta) = \hat{a}_\phi j\dfrac{k\mu a^2 I_0 \sin\theta}{4r}\left[1+\dfrac{1}{jkr}\right]e^{-jkr}$

(a) Using (3-2a) and (VII-26)

$$\underline{H} = \frac{1}{\mu}\nabla\times\underline{A} = \frac{1}{\mu}\left\{\hat{a}_r\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \cancel{\frac{\partial A_\theta}{\partial\phi}}^{\,0}\right]\right.$$
$$\left. + \hat{a}_\theta\frac{1}{r}\left[\frac{1}{\sin\theta}\cancel{\frac{\partial A_r}{\partial\phi}}^{\,0} - \frac{\partial}{\partial r}(rA_\phi)\right] + \hat{a}_\phi\frac{1}{r}\left[\frac{\partial}{\partial r}\cancel{(rA_\theta)}^{\,0} - \cancel{\frac{\partial A_r}{\partial\theta}}^{\,0}\right]\right\}$$

which reduces to

$$\underline{H} = \frac{1}{\mu}\left\{\hat{a}_r\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \hat{a}_\theta\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right\}$$

Using the A_ϕ from above

$$\underline{H} = \frac{1}{\mu}\left\{\hat{a}_r\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[j\frac{k\mu a^2 I_0\sin\theta}{4r}\left(1+\frac{1}{jkr}\right)e^{-jkr}\right]\right.$$
$$\left. - \hat{a}_\theta\frac{1}{r}\frac{\partial}{\partial r}\left[j\frac{k\mu a^2 I_0\sin\theta}{4}\left(1+\frac{1}{jkr}\right)e^{-jkr}\right]\right\}$$

which can be written as

$$H_r = j\frac{ka^2 I_0\cos\theta}{2r^2}\left(1+\frac{1}{jkr}\right)e^{-jkr}$$
$$H_\theta = -\frac{(ka)^2 I_0\sin\theta}{4r}\left[1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right]e^{-jkr}$$
$$H_\phi = 0$$

(b) Using Equation (3-10) with $\underline{J}=0$ along with the $\underline{H}$-field components from above

$$\underline{E} = \frac{1}{j\omega\epsilon}\nabla\times\underline{H} = \frac{1}{j\omega\epsilon}\left\{\hat{a}_r(0)+\hat{a}_\theta(0)+\hat{a}_\phi\frac{1}{r}\left[\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial\theta}\right]\right\}$$

which reduces to

$$E_r = 0$$
$$E_\theta = 0$$
$$E_\phi = \eta\frac{(ka)^2 I_0 \sin\theta}{4r}\left[1+\frac{1}{jkr}\right]e^{-jkr}$$

The same expressions can be obtained using (3-15) with the A_ϕ from part a.

5-2. According to the duality theorem and the dual quantities as outlined in Table 3.2

Electric Dipole		Magnetic Dipole
$\underline{E}$	$\Leftrightarrow$	$\underline{H}$
$\underline{H}$	$\Leftrightarrow$	$-\underline{E}$
$\underline{I}_e$	$\Leftrightarrow$	$\underline{I}_m$
ϵ	$\Leftrightarrow$	μ
μ	$\Leftrightarrow$	ϵ
κ	$\Leftrightarrow$	κ
η	$\Leftrightarrow$	$1/\eta$
$1/\eta$	$\Leftrightarrow$	η

Thus applying the above to the fields of an electric dipole, as given by (4-8a)–(4-10c), we obtain the fields of a magnetic dipole given by

$$E_r = 0$$
$$E_\theta = 0$$
$$E_\phi = -j\frac{kI_m l\sin\theta}{4\pi r}\left(1+\frac{1}{jkr}\right)e^{-jkr}$$
$$H_r = \frac{1}{\eta}\frac{I_m l\cos\theta}{2\pi r^2}\left(1+\frac{1}{jkr}\right)e^{-jkr}$$
$$H_\theta = j\frac{1}{\eta}\frac{kI_m l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right]e^{-jkr}$$
$$H_\phi = 0$$

which are identical to (5-20a)–(5-20d)

5-3. $a = \lambda/30,\ b = \lambda/1{,}000 = 10^{-3}\lambda,\ f = 10\text{ MHz} \Rightarrow \lambda = 30\text{ meters},\ \sigma = 5.7\times 10^7\text{ s/m}$

(a) $$R_r = 20\pi^2\left(\frac{C}{\lambda}\right)^4 = 20\pi^2\left(\frac{2\pi a}{\lambda}\right)^4 = 20\pi^2\left(\frac{2\pi}{30}\right)^4$$
$$= 20\pi^2(0.2094)^4 = 0.3798\text{ ohms}$$

(b)
$$R_L = R_{hf} = \frac{C}{P}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{2\pi a}{2\pi b}\sqrt{\frac{2\pi f\mu_0}{2\sigma}}$$
$$= \frac{a}{b}\sqrt{\frac{\pi f\mu_0}{\sigma}} = \frac{\lambda/30}{\lambda/1{,}000}\sqrt{\frac{\pi(10^7)4\pi\times 10^{-7}}{5.7\times 10^7}} = 0.02774$$
$$R_L = R_{hf} = 0.02774$$

(c)
$$X_A = \omega L_A = 2\pi f L_A = 2\pi f\left\{\mu_0 a\left[\ln\left(8\frac{a}{b}\right) - 2\right]\right\}$$
$$= 2\pi\times 10^7\left\{4\pi\times 10^{-7}\left(\frac{\lambda}{30}\right)\left[\ln\left(8\frac{1{,}000}{30}\right) - 2\right]\right\}$$
$$X_A = 8\pi^2\left(\frac{30}{30}\right)[\ln(266.667) - 2] = 8\pi^2(5.58599 - 2) = 283.139$$
$$X_i = \omega L_i = \omega\left[\frac{a}{wb}\sqrt{\frac{w\mu_0}{2\sigma}}\right] = \frac{a}{b}\sqrt{\frac{2\pi f\mu_0}{2\sigma}} = \frac{a}{b}\sqrt{\frac{\pi f\mu_0}{\sigma}}$$
$$= \frac{\lambda/30}{\lambda/1{,}000}\sqrt{\frac{\pi(10^7)4\pi\times 10^{-7}}{5.7\times 10^7}}$$
$$X_i = \frac{1{,}000}{30}(2\pi)\times 10^4\times 10^{-7}\frac{1}{\sqrt{57}} = 0.02774$$
$$X_T = X_A + X_i = 283.139 + 0.02774 = 283.1667$$

(d)
$$Z_{\text{in}} = (R_r + R_L) + j(X_A + X_i) = (0.3798 + 0.02774) + j(283.1667)$$
$$Z_{\text{in}} = 0.40754 + j283.1667$$

(e)
$$e_{cd} = \frac{R_r}{R_r + R_L} = \frac{0.3798}{0.3798 + 0.02774} = 0.9319 = 93.19\%$$

5-4. The pattern of a small circular loop of uniform current is given by

$$E_{\phi n} \sim \sin\theta \Rightarrow \mathcal{U} \sim \sin^2\theta$$

which is omnidirectional.

(a)
$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}}$$
$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi}\sin^2\theta\sin\theta\, d\theta\, d\phi$$
$$= 2\pi\int_0^{\pi}\sin^3\theta\, d\theta$$
$$= 2\pi\left(\frac{4}{3}\right) = \frac{8\pi}{3}$$
$$D_0(\text{exact}) = \frac{4\pi(1)}{8\pi/3} = \frac{3}{2} = \boxed{1.5 = 1.761\text{ dB}}$$

(b) Half-power beamwidth of $\sin^2\theta$ is

$$\sin^2\theta_h = \tfrac{1}{2} \Rightarrow \sin\theta_h = 0.707 \Rightarrow \theta_h = 45^\circ$$
$$\Theta_H = 2\theta_h = 90^\circ = \text{HPBW}$$

$$D_0(\text{McDonald}) = \frac{101}{\text{HPBW(degrees)} - 0.0027\,[\text{HPBW (degrees)}]^2}$$
$$= \frac{101}{90 - 0.0027(90^\circ)^2} = \frac{101}{90 - 21.87} = 1.48246$$
$$D_0(\text{McDonald}) = \boxed{1.48246 = 1.7098 \text{ dB}}$$

(c)
$$D_0(\text{Pozar}) = -172.4 + 191\sqrt{0.818 + \frac{1}{\text{HPBW (degrees)}}}$$
$$= -172.4 + 191\sqrt{0.818 + \frac{1}{90}}$$
$$= -172.4 + 191(0.91055) = -172.4 + 173.916 = 1.5116$$
$$D_0(\text{Pozar}) = \boxed{1.516 = 1.807 \text{ dB}}$$

5-5. $C = \lambda/4 = 2\pi a \Rightarrow a = \lambda/8\pi < \lambda/6\pi \Rightarrow$ small loop

(a) $R_r = 20\pi^2 \left(\frac{C}{\lambda}\right)^4 N^2 = 20\pi^2 \left(\frac{1}{4}\right)^4 N^2 = \frac{20\pi^2}{256} N^2 = 300$

$$\Rightarrow N = \left(\frac{300(256)}{20\pi^2}\right)^{1/2} = 19.72 \simeq 20$$

(b) $R_{\text{in}} = R_r = \frac{20\pi^2}{256}(20)^2 = 308.425$ ohms

(c) $\Gamma = \frac{R_{\text{in}} - Z_c}{R_{\text{in}} + Z_c} = \frac{308.425 - 300}{308.425 + 300} = 0.01385$

(d) VSWR $= \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.01385}{1 - 0.01385} = 1.0281$

5-6. $\underline{E}_w^i = (\hat{a}_y + 2\hat{a}_z)e^{-jkx} = \left(\frac{\hat{a}_y + 2\hat{a}_z}{\sqrt{5}}\right)\sqrt{5}e^{-jkx}$

(a) Linear: Two components in phase.

(b) AR $= \infty$

(c) $\underline{E} = \hat{a}_\phi E_\phi = \hat{a}_\phi C \sin\theta, \quad \hat{a}_\phi = (-\hat{a}_x \sin\phi + \hat{a}_y \cos\phi)\big|_{\substack{\theta=\pi/2 \\ \phi=0}} = \hat{a}_y$

$\underline{E}\big|_{\substack{\theta=90^\circ \\ \phi=0^\circ}} = \hat{a}_y C \Rightarrow$ **Polarization** : Linear in y direction

(d) PLF $= \left|\left(\frac{\hat{a}_y + 2\hat{a}_z}{\sqrt{5}}\right) \cdot \hat{a}_y\right|^2 = \frac{1}{5} = -6.99$ dB

(e)
$$f = 1\ \text{GHz} \Rightarrow \lambda = \frac{30\times10^9}{1\times10^9} = 30\ \text{cm}$$
$$A_{em} = \frac{\lambda^2}{4\pi}D_0 = \frac{\lambda^2}{4\pi}\left(\frac{3}{2}\right) = \frac{(30)^2}{4\pi}\left(\frac{3}{2}\right) = 107.4296\ \text{cm}^2$$
$$P_r = A_{em}W^i(\text{PLF}) = 107.4296(5\times10^{-3})\left(\frac{1}{5}\right) = 107.4296\times10^{-3}$$
$$P_r = 107.4296\times10^{-3}\ \text{watts}$$

5-7. $R_r(1\ \text{turn}) = 20\pi^2\left(\frac{C}{\lambda}\right)^4 = 20\pi^2\left(\frac{1}{5}\right)^4 = 0.31583\ \text{ohms}$

$$R_r(4\ \text{turn}) = N^2R_r(1\ \text{turn}) = 4^2\cdot(0.31583) = 5.0532\ \text{ohms}$$
$$R_L(1\ \text{turn}) = R_{\text{nf}}(1\ \text{turn}) = \frac{a}{b}\sqrt{\frac{\omega\mu_0}{26}} = \frac{1}{10\pi\times10^{-3}}\sqrt{\frac{2\pi\times10^7\cdot(4\pi\cdot10^{-7})}{2\cdot(5.7\times10^7)}}$$
$$R_L = R_h = 0.0265$$
$$R_L(4\ \text{turn}) = R_{\text{omic}} = \frac{N_a}{b}R_s\left(\frac{R_p}{R_0}+1\right)$$
$$R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}} = \sqrt{\frac{2\pi\times10^7\times(4\pi\times10^{-7})}{2(5.7\times10^7)}} = 8.3223\times10^{-4}$$
$$R_0 = \frac{NR_s}{2\pi b} = \frac{4\cdot(8.3223\times10^{-4})}{2\pi(10^{-3})} = 0.5298$$
$$\frac{R_p}{R_0} \simeq 0.5\ \text{from Fig. 5.3}$$

Thus $R_L = R_{\text{ohmic}} = \frac{4(8.3223\times10^{-4})}{4\pi\times10^{-3}}(0.5+1) = 0.15724$

and $e_{cd}\ (1\ \text{turn}) = 100\cdot R_r/(R_r+R_L) = \frac{0.3158\times100}{0.3158+0.0265} = 92.26 = 92.26\%$

$$e_{cd}\ (4\ \text{turn}) = 100\cdot R_r/(R_r+R_L) = \frac{5.0532(100)}{5.0532+0.15724} = 96.98\%$$

5-8. $H_\theta = -\frac{\pi SI_0e^{-jkr}}{\lambda^2r}\sin\theta \quad$ where $S = \pi a^2$

$$E_\phi = -\eta H_\theta = \eta\frac{\pi SI_0e^{-jkr}}{\lambda^2r}\sin\theta$$
$$\underline{W}_{\text{ave}} = \tfrac{1}{2}\text{Re}(\underline{E}\times\underline{H}^*) = \tfrac{1}{2}\text{Re}(\hat{a}_\phi E_\phi\times\hat{a}_\theta H_\theta^*) = \tfrac{1}{2}\text{Re}(-\hat{a}_o\eta H_\theta\times\hat{a}_\theta H_\theta^*)$$
$$\underline{W}_{\text{ave}} = \hat{a}_r\tfrac{1}{2}\text{Re}(\eta|H_\theta|^2) = \hat{a}_r\frac{\eta}{2}|H_\theta|^2 = \hat{a}_r\frac{\eta}{2}\left|\frac{\pi SI_0}{\lambda^2}\right|\frac{\sin^2\theta}{r^2} = \hat{a}_rW_r$$

$$P_{\text{rad}} = \oiint_{S0} \underline{W}_{\text{ave}}\, ds = \int_0^{2\pi} \int_0^{\pi} \hat{a}_r W_r \cdot \hat{a}_r r^2 \sin\theta \ d\theta \ d\phi$$
$$= 2\pi \int_0^{\pi} W_r r^2 \sin\theta \ d\theta$$
$$= \pi\eta \left|\frac{\pi S I_0}{\lambda^2}\right|^2 \int_0^{\pi} \sin^3\theta \ d\theta = \frac{4\pi\eta}{3} \left|\frac{(\pi a)^2 I_0}{\lambda^2}\right|^2$$
$$= \eta\frac{\pi}{12}(ka)^4 |I_0|^2$$

5-9. $\underline{A} = \hat{a}_\phi j \dfrac{k\mu a^2 I_0 \sin\theta}{4r}\left(1+\dfrac{1}{jkr}\right)e^{-jkr} \simeq \hat{a}_\phi j \dfrac{k\mu a^2 I_0 e^{-jkr}}{4r}\sin\theta$

from equation (5-17) and $r \to$ large.

Using (30-58a)

$$E_r \simeq E_\theta \simeq 0$$
$$E_\phi \simeq -j\omega A_\phi = -j\omega\left(j\frac{k\mu a^2 I_0 e^{-jkr}}{4r}\sin\theta\right) = \eta\frac{\pi S I_0 e^{-jkr}}{\lambda^2 r}\sin\theta$$

where $S = \pi a^2$, $\eta = \sqrt{\mu/\epsilon}$

also using (3-58b)

$$H_r \simeq H_\phi \simeq 0$$
$$H_\theta \simeq j\frac{\omega}{\eta}A_\phi = j\frac{\omega}{\eta}\left(j\frac{\mu k a^2 I_0 e^{-jkr}}{4r}\sin\theta\right) = -\frac{\pi S I_0 e^{-jkr}}{\lambda^2 r}\sin\theta$$

5-10. $a = \lambda/8\pi$, $b = 10^{-4}\lambda/2\pi$, $\sigma = 5.7 \times 10^7$ s/m

Assuming uniform current

a. $$R_r = 20\pi^2\left(\frac{C}{\lambda}\right)^4, \quad C = 2\pi a = 2\pi\left(\frac{\lambda}{8\pi}\right) = \frac{\lambda}{4} \qquad (5\text{-}24)$$

$$R_r = 20\pi^2\left(\frac{\lambda}{4\lambda}\right)^4 = 20\pi^2\left(\frac{1}{256}\right) = \frac{197.392}{256} = 0.771$$

$$R_L = R_{hf} = \frac{a}{b}\sqrt{\frac{w\mu_0}{2\sigma}} = \frac{\frac{\lambda}{8\pi}}{\frac{10^{-4}\lambda}{2\pi}}\sqrt{\frac{2\pi(10^8)4\pi\times10^{-7}}{2(5.7\times10^7)}}$$

$$R_L = \frac{10^4}{4}\frac{(2\pi)\times10^{-3}}{\sqrt{5.7}} = \frac{20\pi}{4\sqrt{5.7}} = \frac{5\pi}{\sqrt{5.7}} = \frac{15.708}{\sqrt{5.7}} = 6.5794$$

$$e_{cd} = \frac{R_r}{R_r + R_L} = \frac{0.771}{0.771 + 6.5794} = 0.10489 = 10.489\%$$

b. $D_0 = 3/2 = 1.5 = 1.761$ dB Uniform current

$G_0 = e_{cd} D_0 = 0.10489(1.5) = 0.15734$

$$\boxed{G_0 = 0.15734 = -8.03 \text{ dB}}$$

5-11. a. $R_r = 20 \cdot \pi^2 \cdot \left(\frac{C}{\lambda}\right)^4 = 20 \cdot \pi^2 \cdot \left(\frac{2\pi a}{\lambda}\right)^4, \lambda = \frac{3 \times 10^8}{10^7} = 30$ m

$0.73 = 20\pi^2 \left(\frac{2\pi a}{\lambda}\right)^4 \Rightarrow a = 0.03924\lambda = 1.177$ meters

b. $0.73N^2 = 300 \Rightarrow N = 20.272 \simeq 20$

$R_r(20 \text{ turns}) = 0.73(20)^2 = 292$

c. $P_L = A_{em} \cdot W_i \cdot e_o = \frac{\lambda^2}{4\pi} D_0 \epsilon_o (10^{-6}) = \frac{\lambda^2}{4\pi}\left(\frac{3}{2}\right)(1 - |\Gamma|^2) \cdot 10^{-6}$

$$= \frac{(30)^2}{4\pi} \cdot \left(\frac{3}{2}\right) \cdot \left(1 - \left|\frac{292 - 300}{292 + 300}\right|^2\right) \cdot 10^{-6} = 0.1074 \times 10^{-3} \text{ watts}$$

5-12. $a = \lambda/30, \quad b = \lambda/300, \quad 2C = \lambda/100 \Rightarrow C = \lambda/200, \quad N = 6,$
$f = 5 \times 10^7$ Hz

a. Since $a = \lambda/30 \ll \lambda$

$D_0 = 1.5 = 1.761$ dB

b. $R_r = 20\pi^2 \left(\frac{c}{\lambda}\right)^4 = 20\pi^2 \left(\frac{\pi}{15}\right)^4 = 20\pi^2(1.924 \times 10^{-3}) = 0.3798$ ohms

$C = 2\pi a = 2\pi\left(\frac{\lambda}{30}\right) = \frac{\pi}{15}\lambda$

$R_r(\text{single turn}) = 0.3798$ ohms

$R_r(6 \text{ turns}) = 13.673$ ohms

$$R_L = \frac{N_a}{b} \cdot R_s \cdot \left(\frac{R_p}{R_0} + 1\right)$$

$$c/b = \frac{\lambda/200}{\lambda/300} = \frac{3}{2} = 1.5 \Rightarrow \frac{R_p}{R_0} = 0.65$$

$$R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}} = \sqrt{\frac{2\pi f(4\pi \times 10^{-7})}{2(5.7 \times 10^7)}} = \sqrt{\frac{4\pi^2 f}{5.7}} \times 10^{-7}$$

$$= \sqrt{\frac{4\pi^2(50 \times 10^6)}{5.7}} \times 10^{-7} = 2\pi\sqrt{\frac{50}{5.7}} \times 10^{-4} = 18.609 \times 10^{-4}$$

$$R_L = 6\left(\frac{\lambda/30}{\lambda/300}\right) \cdot 18.609 \times 10^{-4}(0.65 + 1)$$

$$= 6(10)(18.609)(1.65) \times 10^{-4} = 1{,}842.31 \times 10^{-4}$$

$R_L(6 \text{ turns}) = 0.184231$

$$\text{(Single) } R_L = \frac{\lambda/30}{\lambda/300}\sqrt{\frac{2\pi f(4\pi\times 10^{-7})}{2\cdot(5.7\times 10^7)}} = 10\sqrt{\frac{4\pi^2 f}{5.7}}\times 10^{-7}$$

$$= 2\pi(10)\sqrt{\frac{50}{5.7}}\times 10^{-4} = 186.0919\times 10^{-4}$$

$$\text{(6 turns) } R_L = 186.0919\cdot(6)(1.65)\times 10^{-4} = 1{,}842.31\times 10^{-4}$$

$$e_{cd} = \frac{R_r}{R_r+R_L} = \frac{13.673}{13.673+0.184231}\times 100 = 98.67\%$$

c. $$|\Gamma| = \left|\frac{(R_r+R_L)-50}{(R_r+R_L)+50}\right| = \left|\frac{13.857-50}{13.857+50}\right| = \left|\frac{-36.14277}{63.857}\right| = 0.566$$

$$e_r = (1-|\Gamma|^2)\times 100 = (1-|0.566|^2)\times 100 = (1-0.32)\times 100 = 68\%$$

d. $G_0 = e_{cd}D_0 = (0.9867)D_0 = (0.9867)(1.5)$

$G_0 = 1.48005$ ($\leftarrow$ Total Maximum gain does not include the reflection loss)

5-13. $f = 30\text{ MHz} \rightarrow \lambda = 10\text{ m},\ ka = \frac{2\pi}{10}(0.15) = 0.03\pi = 0.09425\text{ (rad)}$

$$R_r = N^2\frac{\pi\eta_0}{6}(ka)^4 = 64\times\frac{\pi^2\cdot 120}{6}\times(0.03\pi)^4 = 0.9968\ \Omega$$

$$\delta = \frac{1}{\sqrt{\pi f\mu_0\sigma}} = \frac{1}{\sqrt{\pi\times 30\times 10^6\times 4\pi\times 10^{-7}\times 5.7\times 10^7}}$$

$$= 1.217\times 10^{-5}\text{ m} \ll b$$

$$\text{1-turn: } R_L = \frac{a}{\sigma\cdot b\delta} = \frac{0.15}{5.7\times 10^7\times 0.001\times 1.217\times 10^{-5}} = 0.2162\ \Omega$$

$$\text{8-turn: } R_L = 8\times R_L(\text{1-turn})\times\left(\frac{R_p}{R_0}+1\right),\ c/b = 1.8 \Rightarrow \frac{R_p}{R_0} = 0.5$$

$$\therefore R_L = 8\times(0.2162)\times 1.5 = 2.594\ \Omega$$

$$e_{cd} = \frac{R_r}{R_r+R_L} = \frac{0.9968}{0.9968+2.594} = 0.278 = 27.8\%$$

5-14. Since the small circular loop area is parallel to the $y-z$ plane, its electrical equivalent is an **infinitestimal magnetic dipole** directed along the x-axis.

a. Thus, using the procedure of Example 4.5, we can write the electric and magnetic fields for the **infinitesimal electric dipole** of length l directed along the x-**axis** as

$$E_r \simeq 0 \qquad E_r \simeq 0 \qquad H_r \simeq 0$$

$$E_\theta \simeq -j\omega A_\theta \qquad E_\theta \simeq -j\frac{\omega\mu I_o l e^{-jkr}}{4\pi r}\cos\theta\cos\phi \qquad H_\phi \simeq \frac{E_\theta}{\eta}$$

$$E_\phi \simeq -j\omega A_\phi \qquad E_\phi \simeq -j\frac{\omega\mu I_o l e^{-jkr}}{4\pi r}\sin\phi \qquad H_\theta \simeq -\frac{E_\phi}{\eta}$$

Using duality and Table 3.2, the fields of an x-**directed infinitesimal magnetic dipole** of constant current I_m can be written as

$$H_r \simeq 0 \qquad E_r \simeq 0$$

$$H_\theta \simeq -j\frac{\omega\varepsilon I_m l e^{-jkr}}{4\pi r}\cos\theta\cos\phi \qquad E_\phi \simeq -\eta H_\theta = +j\eta\frac{\omega\varepsilon I_m l e^{-jkr}}{4\pi r}\cos\theta\cos\phi$$

$$H_\phi \simeq -j\frac{\omega\varepsilon I_m l e^{-jkr}}{4\pi r}\sin\phi \qquad E_\theta \simeq +\eta H_\phi = -j\eta\frac{\omega\varepsilon I_m l e^{-jkr}}{4\pi r}\sin\phi$$

Since the infinitesimal magnetic dipole directed along the x-aixs is equivalent to a small circular loop, with its area parallel to the $y-z$ plane, we can write the fields of the circular loop by making in the above equations the substitution

$$lI_m = jS\omega\mu I_o = j(\pi a^2)\omega\mu I_o$$

Thus the far-zone electric fields can be written as

$$E_r \simeq 0$$

$$E_\theta \simeq -j\eta\frac{\omega\varepsilon I_o(jS\omega\mu)e^{-jkr}}{4\pi r}\sin\phi = -j\eta\frac{\omega\varepsilon I_o(j\pi a^2\omega\mu)e^{-jkr}}{4\pi r}\sin\phi$$

$$\simeq \eta\frac{\omega^2\mu\varepsilon a^2 I_o e^{-jkr}}{4r}\sin\phi = \eta\frac{(ka)^2 I_o e^{-jkr}}{4r}\sin\phi$$

$$E_\phi \simeq +j\eta\frac{\omega\varepsilon I_o(jS\omega\mu)e^{-jkr}}{4\pi r}\cos\theta\cos\phi = +j\eta\frac{\omega\varepsilon I_o(j\pi a^2\omega\mu)e^{-jkr}}{4\pi r}\cos\theta\cos\phi$$

$$\simeq -\eta\frac{\omega^2\mu\varepsilon a^2 I_o e^{-jkr}}{4r}\cos\theta\cos\phi = -\eta\frac{(ka)^2 I_o e^{-jkr}}{4r}\cos\theta\cos\phi$$

while the far-zone magnetic fields can be expressed as

$$H_r \simeq 0; \qquad H_\theta \simeq -\frac{E_\phi}{\eta}; \qquad H_\phi \simeq \frac{E_\theta}{\eta}$$

b. Since the far-field pattern of the antenna is the same as that of a loop with an area parallel to the $x-y$ plane, or an infinitesimal magnetic dipole oriented along the x-axis, their directivities are the same. Thus $\boldsymbol{D_o = 3/2 = 1.5}$.

5-15. Using the results of Problem 5-14

a. $E_\chi \cong \dfrac{a^2\,\omega\mu k I_0 e^{-jkr}}{4r}\sqrt{1-|\hat{a}_y\cdot\hat{a}_r|^2} = \dfrac{a^2\,\omega\mu k I_0 e^{-jkr}}{4r}\sqrt{1-\sin^2\theta\sin^2\phi}$

$H_\psi \cong \dfrac{E_\chi}{\eta}$

b. Directivity $= D_0 = \frac{3}{2}$

5-16. Using the computer program of Chapter 5.

a. $a = \lambda/50 = 0.02\lambda$

$D_0 = 1.4988 = 1.7575$ dB, $R_r = 0.0$4^ ohms

b. $a = \lambda/10 = 0.1\lambda$
$D_0 = 1.4699 = 1.6731$ dB, $R_r = 28.41$ ohms

c. $a = \lambda/4 = 0.25\lambda$
$D_0 = 1.2969 = 1.1291$ dB, $R_r = 723.938$ ohms

d. $a = \lambda/2 = 0.5\lambda$
$D_0 = 1.7968 = 2.5449$ dB, $R_r = 2{,}202.528$ ohms

5-17. According to (5-54b)

$$E_\phi \simeq \frac{aknI_0 e^{-jkr}}{2r} J_1(ka\sin\theta) \sim J_1(ka\sin\theta)$$

Therefore the nulls of the pattern occur when

$$J_1(ka\sin\theta_n) = 0 \Rightarrow ka\sin\theta_n = 0, 3.84, 7.01, 10.19, \ldots$$

Excluding $\theta = 0$

$$\theta_n = \begin{cases} \sin^{-1}\left(\dfrac{3.84}{ka}\right) = \sin^{-1}\left[\dfrac{3.84}{2\pi(1.25)}\right] = \sin^{-1}(0.4889) = 29.27^\circ \\ \sin^{-1}\left(\dfrac{7.01}{ka}\right) = \sin^{-1}\left[\dfrac{7.01}{2\pi(1.25)}\right] = \sin^{-1}(0.8925) = 63.19^\circ \end{cases}$$

5-18. Since $E_\phi \sim J_1(ka\sin\theta)$

(a) $E_\phi|_{\theta=0} = J_1(ka\sin\theta)|_{\theta=0} = J_1(0) = 0$
$E_\phi|_{\theta=\pi/2} = J_1(ka\sin\theta)|_{\theta=90^\circ} = J_1(ka) = 0 \Rightarrow ka = 3.84$

Thus $a = \dfrac{3.84}{k} = \dfrac{3.84\lambda}{2\pi} = 0.61115\lambda$

(b) Since $a = 0.61115\lambda > 0.5\lambda$, use large loop approximation. According to (5-63a)

$$R_r = 60\pi^2(C/\lambda) = 60\pi^2\left(\frac{2\pi a}{\lambda}\right) = 60\pi^2(2\pi(0.61115))$$
$$= 2{,}273.94$$

(c) The directivity is given by (5-63b), or

$$D_0 = 0.682\left(\frac{C}{\lambda}\right) = 0.682\left(\frac{2\pi a}{\lambda}\right) = 0.682(2\pi)(0.61115) = 2.619$$

5-19. $E_\phi \sim J_1(ka\sin\theta)$

a. $E_\phi|_{\theta=30^\circ} = J_1(ka\sin\theta)|_{\theta=30^\circ} = J_1\left(\dfrac{ka}{2}\right) = 0 \Rightarrow \dfrac{ka}{2} = 3.84$

From the Table for $J_1(x)$ in Appendix V. Thus

$$a = \frac{2(3.84)}{k} = \frac{2(3.84)}{2\pi}\lambda = 1.222\lambda$$

b. $E_\phi|_{\max} = E_\phi|_{ka\sin\theta=1.84} = J_1(1.84) = 0.58152 = -4.709 \text{ dB}$

$$E_\phi|_{\theta=90^\circ} = J_1(ka) = J_1\left[\frac{2\pi}{\lambda}(1.222\lambda)\right] = J_1(7.678) = 0.175 = -15.139 \text{ dB}$$

Thus

$$\Delta E = E_\phi|_{\theta=90^\circ} - E_\phi|_{\max} = -15.139 - (-4.709) = -10.43 \text{ dB}$$

5-20. $E_\phi \sim J_1(ka\sin\theta)$

a. According to the Table for $J_1(x)$ in Appendix V

$$J_1(x) = 0 \quad \text{when } x = 0, 3.84, 7.01, 10.19, \ldots$$

Since we want a null in the plane of the loop ($\theta = 0^\circ$) and two additional ones for $0^\circ \leq \theta \leq 90^\circ$, then

$$ka\sin\theta|_{\max} = ka\sin\theta|_{\theta=90^\circ} = ka = 7.01$$

Thus

$$a = \frac{7.01}{k} = \frac{7.01}{2\pi}\cdot\lambda = 1.1157\lambda$$

b. The nulls will occur at

$$\theta = 0^\circ \text{ and } 180^\circ$$
$$\theta = 90^\circ$$

and

$$ka\sin\theta|_{a=1.1157\lambda} = 3.84$$
$$\Rightarrow \theta = \sin^{-1}\left[\frac{3.84}{2\pi(1.1157)}\right] = 33.21^\circ$$
$$\text{and } \theta = 180^\circ - 33.21^\circ = 146.79^\circ$$

5-21. $\underline{E} = \hat{a}_\phi C_1 J_1(ka\sin\theta)$ where C_1 is a constant $\Rightarrow \hat{\rho}_w = \hat{a}_\phi$ and
$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = |\hat{a}_\phi \cdot \hat{\rho}_a|^2$

By inspection, the PLF is maximized if the probe antenna is also linearly polarized in the ϕ direction. This can be accomplished by using as a probe antenna another loop antenna so that

$$\hat{\rho}_a = \hat{a}_\phi \text{ and PLF} = |\hat{a}_\phi \cdot \hat{a}_\phi|^2 = 1.$$

It can also be accomplished by using a linear dipole as a probe antenna with its length parallel to the plane of the loop and tangent to its curvature. Some specific examples would be [using the transformation of VII-7b]

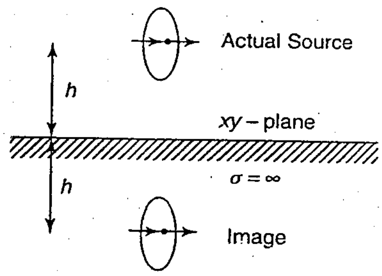
$$\hat{\rho}_a = \hat{a}_x|_{\phi=90^\circ} \Rightarrow \text{PLF} = |\hat{a}_\phi \cdot \hat{a}_x|_{\phi=90^\circ} = |\hat{a}_\phi \cdot (\hat{a}_\rho \cos\phi - \hat{a}_\phi \sin\phi)|^2_{\phi=90^\circ}$$
$$= |\hat{a}_\phi \cdot (-\hat{a}_\phi)|^2 = 1$$
$$\hat{\rho}_a = \hat{a}_y|_{\phi=0^\circ} \Rightarrow \text{PLF} = |\hat{a}_\phi \cdot \hat{a}_y|_{\phi=0^\circ} = |\hat{a}_\phi \cdot (\hat{a}_\rho \sin\phi + \hat{a}_\phi \cos\phi)|^2_{\phi=0^\circ}$$
$$= |\hat{a}_\phi \cdot \hat{a}_\phi|^2 = 1$$

and many others.

5-22. A very small loop of constant current is equivalent to a magnetic dipole. Since the loop is placed for both parts (a and b) perpendicular to the xy-plane (the plane of the loop is perpendicular to the xy-plane), the axis of the linear magnetic dipole will also be parallel to the xy-plane. Therefore according to Figure 4.12a, the image of the horizontal magnetic dipole will be as shown in this figure. In turn the array factor for both parts (a and b) of this problem will be the same as that of the vertical electric dipole of Figure 4.13 or

$$\text{AF} = 2 \cdot \cos(kh\ \cos\theta)$$

Since the actual source and the image are oriented in the same direction. Therefore according to (5-27a)–(5-27c)



a. Plane of the loop is parallel to the xz-plane

$$E_\chi = \eta\frac{(ka)^2 I_0 e^{-jkr}}{4r}\sin\psi(\text{AF}),\ \sin\psi = \sqrt{1-\cos^2\psi} = \sqrt{1-|\hat{a}_y\cdot\hat{a}_r|^2}$$
$$= \sqrt{1-\sin^2\theta\sin^2\phi}$$
$$= \eta\frac{(ka)^2 I_0 e^{-jkr}}{4r}\sin\psi[2\cos(kh\cos\theta)]$$
$$E_\chi = \eta\frac{(ka)^2 I_0 e^{-jkr}}{2r}\cos(kh\cos\theta)\sqrt{1-\sin^2\theta\sin^2\phi}$$
$$H_\psi = -\frac{E_\chi}{\eta}$$

b. Plane of the loop is parallel to the yz-plane. The fields for this problem are the same as those in part a. above except that

$$\sin\psi = \sqrt{1-\cos^2\psi} = \sqrt{1-|\hat{a}_x\cdot\hat{a}_r|^2} = \sqrt{1-\sin^2\theta\cos^2\phi}$$

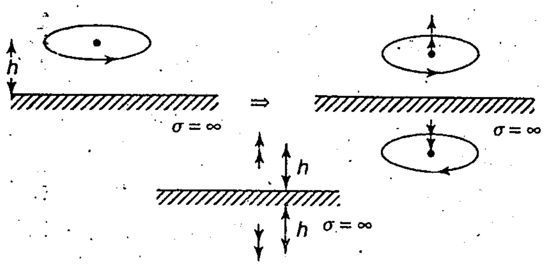
5-23. a. $E_\phi = \eta \dfrac{(ka)^2 \cdot I_0 \cdot e^{-jkr}}{4r} \sin\theta$

$|\mathrm{AF}| = |2j \sin(kh\cos\theta)|$

$E_\phi = \eta \dfrac{\pi S I_0 e^{-jkr}}{\lambda^2 r} \cdot \sin\theta, S = \pi a^2$

$(E_\phi)_t = E_\phi(\mathrm{AF}) = \eta \dfrac{\pi S I_0 e^{-jkr}}{\lambda^2 r} \cdot \sin\theta \cdot [2j \sin(kh\cos\theta)],$

← above ground plane total field.



b. $h = \lambda,\ kh = 2\pi$

$\sin\theta \cdot [2j\sin(2\pi\cos\theta)] = 0,\ \sin(2\pi\cos\theta) = 0, 2\pi\cos\theta = n\pi.$

$n = 0, 1, 2$

$\theta_n = 0°, \quad \cos\theta_n = \dfrac{n}{2},\ n = 0, 1, 2. \Rightarrow \theta_n = 90°,$

$\theta_0 = \cos^{-1}(0) = 90°$

$\theta_1 = \cos^{-1}\left(\frac{1}{2}\right) = 60°$

$\theta_2 = \cos^{-1}(1) = 0°.$

c.

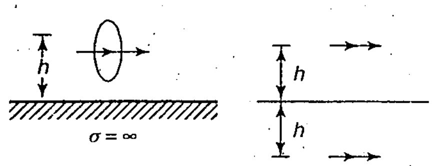
$$(E_\phi)_t = C\sin\theta\sin(kh\cos\theta)|_{\theta=60°} = 0 = C \cdot \left(\frac{\sqrt{3}}{2}\right) \cdot \sin\left(\frac{2\pi}{\lambda} \cdot h \cdot \frac{1}{2}\right)$$

$$= C \cdot \frac{\sqrt{3}}{2} \cdot \sin\left(\frac{\pi h}{\lambda}\right)$$

$$\sin\left(\frac{\pi h}{\lambda}\right) = 0 \Rightarrow \frac{\pi h}{\lambda} = \sin^{-1}(0) = n\pi, n = 0, 1, 2, 3, \ldots$$

$$\frac{h}{\lambda} = \pm n \Rightarrow \text{physical nonzero height} \Rightarrow h = n\lambda, n = 1, 2, 3, \ldots$$

5-24. a. Array Factor $= 2\cos(kh\cos\theta)$



b. $\text{AF} = 2\cos(kh\cos\theta_n) = 0$

$$\Rightarrow kh\cos\theta_n = \cos^{-1}(0) = n\pi/2, n = \pm1, \pm3, \pm5, \ldots$$

$$\theta_n = \cos^{-1}\left[\frac{n\pi/2}{kh}\right] = \cos^{-1}\left(\frac{\frac{n\pi}{2}}{\frac{2\pi}{\lambda}h}\right) = \cos^{-1}\left(\frac{n\lambda}{4h}\right)\Big|_{h=\lambda/2} = \cos^{-1}\left(\frac{n}{2}\right)$$

$$\theta_1 = \cos^{-1}\left(\pm\tfrac{1}{2}\right) = 60^\circ, \theta_3 = \cos^{-1}\left(\pm\tfrac{3}{2}\right) = \text{does not exist}$$

5-25. Since the small circular loop area is parallel to the $x-z$ plane, its electrical equivalent is an **infinitesimal magnetic dipole** directed along the y-axis placed a height h above the PEC. Also its image is at a depth h below the PEC interface. The image is in the same direction as the actual source (the same magnitude and phase).

a. Therefore its normalized array factor is

$$(AF)_n = \cos(kh\cos\theta)$$

whose maximum value is unity.

b. To find the two smallest heights, **other than** $h = 0$, where the maximum will be directed along $\theta = 0^\circ$, we set the normalized array factor to unity, or

$$[AF_n(\theta = 0^\circ]_{\max} = [\cos(kh\cos\theta)|_{\theta=0^\circ}]_{\max} = \cos(kh)|_{\max} = 1$$

$$kh = \cos^{-1}(1) = m\pi$$

$$h = \frac{m\pi}{k} = \frac{m\pi\lambda}{2\pi} = \frac{\lambda}{2}m, \quad m = 0, 1, 2, 3 \ldots$$

$$m = 0: \quad h = 0$$

$$\boxed{\begin{array}{ll} m = 1: & h = \frac{\lambda}{2} \\ m = 2: & h = \lambda \end{array}}$$

5-26. From Problem 5-16(a)

$$E_x|_{\substack{\phi=90^\circ \\ \theta=45^\circ}} = C_1\cos(kh\cos\theta)\sqrt{1-\sin^2\theta\sin^2\phi}\,\Big|_{\substack{\theta=45^\circ \\ \phi=90^\circ}}$$

$$= C_1\cos(kh\cos\theta)\cos(\theta)|_{\theta=45^\circ}$$

$$= 0.707\cdot C_1\cdot\cos\left(\frac{kh}{\sqrt{2}}\right) = 0 \Rightarrow \frac{kh}{\sqrt{2}} = \cos^{-1}(0) = \frac{\pi}{2}n, n = 1, 3, 5, \ldots$$

For the smallest height

$$\frac{kh}{\sqrt{2}} = \frac{\pi}{2} \qquad h = \frac{\sqrt{2}}{2}\frac{\pi}{k} = \frac{\sqrt{2}}{4}\lambda = 0.3535\ \lambda$$

5-27. a. $R_L = \frac{a}{b}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{1}{20\cdot(10^{-4})}\cdot\sqrt{\frac{2\pi(3\times10^8)(4\pi\times10^{-7})}{2\cdot5\cdot7\times10^7}} = \frac{\pi}{20}\sqrt{\frac{12}{5\cdot7}}\times 10$

$= 2.27915$ ohms

b. $R_r = 120\cdot\pi\cdot\left(\frac{2}{3}\pi\right)\cdot\left(\frac{kS}{\lambda}\right)^2 = 120\cdot\pi\left(\frac{2}{3}\pi\right)\left(\frac{2\pi^2}{(20)^2}\right)^2$

$= 80\cdot\frac{4\cdot\pi^6}{(400)^2} = 1.92278$ ohms

$$\left(\leftarrow S = \pi\left(\frac{1}{20}\right)^2\right)$$

c. inductive reactance $X_A = \omega L_A$

$$L_A = \mu_0\cdot a\cdot\left[\ln\left(\frac{8a}{b}\right)-2\right] = 4\pi\times10^{-7}\cdot\left(\frac{\lambda}{20}\right)\cdot\left[\ln\left(\frac{1}{20}\cdot\frac{1}{10^{-4}}\right)-2\right]$$

$$= 2.648\times10^{-7}\left(\leftarrow a = \frac{\lambda_0}{20},\ b = 10^{-4}\lambda_0\right)\ \leftarrow \lambda = 1\text{ m},\ f = 3\times10^8$$

$X_A = 2\cdot\pi\cdot f\cdot L_A = 2\cdot\pi\cdot(3\times10^8)\cdot(2.648\times10^{-7}) = 499.158$

$\therefore X_A \gg (R_L \text{ or } R_r)$

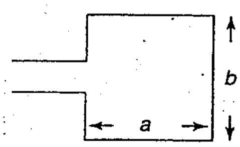
5-28. From equation (5-24)

$$R_r = \eta\left(\frac{\pi}{6}\right)(k^2a^2)^2 = \eta\cdot\frac{2\pi}{3}\cdot\left(\frac{kS}{\lambda}\right)^2 = 120\pi\cdot\frac{2\pi}{3}\cdot\left(\frac{2\pi S}{\lambda^2}\right)^2$$

$$= 120\cdot\frac{2}{3}\cdot4\cdot\pi^4\cdot\left(\frac{S}{\lambda^2}\right)^2 = 31170.909\cdot\frac{S^2}{\lambda^4} \simeq 31{,}171\frac{S^2}{\lambda^4}$$

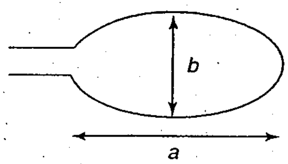
a. Area

$S = ab,\qquad R_r = 31170.909\cdot\frac{a^2b^2}{\lambda^4} \simeq 31{,}171\frac{a^2b^2}{\lambda^4}$



b. Area

$S = \pi\frac{a}{2}\frac{b}{2},\qquad R_r = 31170.909\cdot\frac{\pi^2a^2b^2}{16\lambda^4} \simeq 31{,}171\frac{\pi^2a^2b^2}{16\lambda^4}$



5-29. $f = 100$ MHz $\Rightarrow \lambda = c/f = 3 \times 10^8/10^8 = 3$ meters

$$C = 2\pi a \Rightarrow a = \frac{C}{2\pi} = \frac{\lambda/20}{2\pi} = \frac{\lambda}{40\pi} = \frac{3}{40\pi} = 0.0239 \text{ m} = 0.00796\lambda$$

(a) $$R_r = 20\pi^2 \left(\frac{C}{\lambda}\right)^4 = 20\pi^2 \left(\frac{1}{20}\right)^4 = \frac{20\pi^2}{16} \times 10^{-4} = 1.2337 \times 10^{-3} \text{ ohms}$$

$$R_L = \frac{a}{b}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{\lambda/40\pi}{\lambda/400\pi}\sqrt{\frac{2\pi \times 10^8(4\pi \times 10^{-7})}{2(5.7 \times 10^7)}} = 0.00838$$

$$R_{in} = R_r + R_L = 0.0012337 + 0.00838 = 0.0096137$$

(b) $$L_a = \mu_0 a\left[\ln\left(\frac{8a}{b}\right) - 2\right] = 4\pi \times 10^{-7}(0.0239)\left[\ln\left(\frac{8\frac{\lambda}{40\pi}}{\lambda/400}\right) - 2\right]$$

$$= 0.3 \times 10^{-7}\left[\ln\left(\frac{80}{\pi}\right) - 2\right] = 0.3 \times 10^{-7}[3.2373 - 2]$$

$$= 37.12 \times 10^{-9} \text{ henries}$$

$$X_a = \omega L_a = 2\pi f L_a = 2\pi(10^8)(37.12 \times 10^{-9}) = 23.323 \text{ ohms}$$

$$L_i = \frac{a}{\omega b}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{\frac{\lambda}{40\pi}}{2\pi(10^8)\left(\frac{\lambda}{400}\right)}\sqrt{\frac{2\pi(10^8)(4\pi \times 10^{-7})}{2(5.7 \times 10^7)}} = 0.1333 \times 10^{-10}$$

$$X_i = 2\pi f L_i = 2\pi(10^8)(0.1333 \times 10^{-10}) = 0.83771 \times 10^{-2} = 0.0084 \text{ ohms}$$

$$X_t = X_a + X_i = 23.323 + 0.0084 = 23.3314 \text{ ohms } \textbf{(inductive)}$$

(c) Capacitance

$$X_c = \frac{1}{2\pi f C} = 23.3314$$

$$C = \frac{1}{23.3314(2\pi \times 10^8)} = 6.82 \times 10^{-11} = 68.2 \times 10^{-12} \text{ farads}$$

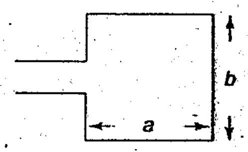
5-30. From the solution of Problem 5-28, the radiation resistance of a loop is

$$R_r = 31{,}171\frac{(\text{Area})^2}{\lambda^4} = 31{,}171\frac{(S)^2}{\lambda^4}$$

Thus for rectangular and elliptical loops:

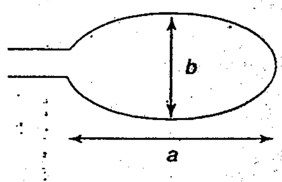
a. Area

$S =$ $\qquad R_r \simeq 31{,}171\frac{a^2b^2}{\lambda^4}$



b. Area

$$S = \pi\left(\frac{a}{2}\right)\left(\frac{b}{\lambda 2}\right), \qquad R_r \simeq 31{,}171\frac{\pi^2 a^2 b^2}{16\lambda^4}$$



5-31. In Far-Field ($kr \gg 1$) region

$$\underline{E}_a \doteq E_\phi \hat{a}_\phi = -j\eta\frac{kI_{\text{in}}}{4\pi r}\cdot l_e \cdot e^{-jkr} \qquad (\rightarrow l_e\text{: effective length})$$

$$E_\phi \simeq \eta\frac{k^2 a^2 I_0 e^{-jkr}}{4r}\cdot \sin\theta = \eta\frac{\pi S I_0 e^{-jkr}}{\lambda^2 r}\sin\theta$$

$$= -j\eta\frac{kI_0(jk\cdot s\cdot \sin\theta)}{4\pi r}e^{-jkr} \qquad \left(\leftarrow \frac{\pi}{\lambda^2} = \frac{k^2}{4\pi}\right)$$

$$\therefore\ l_e = jk\cdot S\cdot \sin\theta\ \hat{a}_\phi$$

5-32. $C = 2\pi a = 1.4\lambda \Rightarrow a = \dfrac{1.4\lambda}{2\pi} = 0.2228\lambda$

$$\Omega = 2\ln\left(2\pi\frac{a}{b}\right) = 2\ln\left(2\pi\frac{0.2228}{0.01555}\right) = 9.0$$

a. From Figure 5.13

$$Z_{\text{in}} = R_{\text{in}} + jX_{\text{in}} = 320 - j40$$

b. $$|\Gamma| = \left|\frac{Z_{\text{in}} - Z_c}{Z_{\text{in}} + Z_c}\right| = \left|\frac{320 - j40 - 300}{320 - j40 + 300}\right| = 0.0718$$

$$\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.0718}{1-0.0718} = 1.155$$

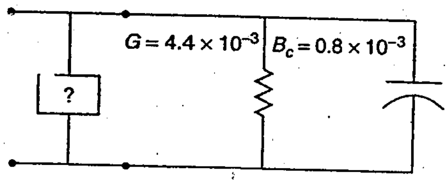
c. $Y_{\text{in}} = \dfrac{1}{Z_{\text{in}}} = \dfrac{1}{320 - j40} = (3.0769 + j0.3846)\times 10^{-3} = G_c + jB_c$

To resonate the circuit, the unknown element must have an inductive admittance of

$$Y_{\text{unknown}} = -j0.3846\times 10^{-3} = -j\frac{1}{\omega L} \Rightarrow L = \frac{1}{0.3846\times 10^{-3}(2\pi f)}$$

$$= \frac{1}{0.3846\times 10^{-3}(2\pi\times 10^8)}$$

$$L = \frac{10^{-5}}{0.769\pi} = 4.138\times 10^{-6}\ \text{h}$$



Therefore the unknown element across the terminals of the loop must be an inductor of $L = 1.989 \times 10^{-6}$ henries

5-33. a. From Figure 5.13 (a, b)

$$Z_{\text{in}} = 90 - j110$$

b. Inductor;

$$X_L = +110 = \omega L = 2\pi f L$$
$$L = \frac{110}{2\pi f} = \frac{110}{2\pi \cdot (10^9)} = \frac{110}{2\pi} \times 10^{-9}$$

c. $Z_{\text{in}} = 90$

$$|\Gamma| = \left|\frac{Z_{\text{in}} - Z_c}{Z_{\text{in}} + Z_c}\right| = \frac{90 - 78}{90 + 78} = \frac{12}{168} = 0.0714$$
$$\text{VSWR} = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.0714}{1 - 0.0714} = \frac{1.0714}{0.9285} = 1.1538$$

5-34. a. From Figure 5.13(b) $Z_{\text{in}} = R_{\text{in}} + jX_{\text{in}} = R_{\text{in}} \Rightarrow X_{\text{in}} = 0$ when

$$\Omega = 2\ln\left(2\pi\frac{a}{b}\right) = \begin{cases} 12 \Rightarrow 2\pi(a/b) = e^6 = 403.429 \Rightarrow \frac{a}{b} = 64.21 \\ 11 \Rightarrow 2\pi(a/b) = e^{5.5} = 244.692 \Rightarrow \frac{a}{b} = 38.94 \\ 10 \Rightarrow 2\pi(a/b) = e^5 = 148.413 \Rightarrow \frac{a}{b} = 23.62 \\ 9 \Rightarrow 2\pi(a/b) = e^{4.5} = 90.017 \Rightarrow \frac{a}{b} = 14.33 \end{cases}$$

b. These occur when the smallest circumference of the loop is ¡from Figure 5.13(b)¿

$\Omega = 12 \Rightarrow C = 2\pi a \simeq 1.08\lambda \Rightarrow a = 0.1719\lambda \Rightarrow b = 0.1719\lambda/64.21 = 2.68 \times 10^{-3}\lambda$

$\Omega = 11 \Rightarrow C = 2\pi a \simeq 1.10\lambda \Rightarrow a = 0.175\lambda \Rightarrow b = 0.175\lambda/38.94 = 4.496 \times 10^{-3}\lambda$

$\Omega = 10 \Rightarrow C = 2\pi a \simeq 1.14\lambda \Rightarrow a = 0.1814\lambda \Rightarrow b = 0.1814\lambda/23.62 = 7.68 \times 10^{-3}\lambda$

$\Omega = 9 \Rightarrow C = 2\pi a \simeq 1.28\lambda \Rightarrow a = 0.2037\lambda \Rightarrow b = 0.2037\lambda/14.33 = 14.216 \times 10^{-3}\lambda$

5-35. $I(\phi) = I_0 \cos\phi$

a. $$\bar{A}(\bar{r}) = \frac{\mu I_0}{4\pi} a \int_0^{2\pi} \hat{a}_\phi \cos\phi' \frac{e^{-jkR}}{R}\, d\phi' \simeq \frac{\mu I_0}{4\pi} a \frac{e^{-jkr}}{r} \int_0^{2\pi} \hat{a}_\phi \cos\phi' \cdot e^{jk\hat{a}_r\cdot\bar{r}'}\, d\phi'$$

$$= \frac{\mu I_0}{4\pi} a \frac{e^{-jkr}}{r} \left\{ -\hat{a}_x \int_0^{2\pi} \cos\phi' \sin\phi' e^{jka\sin\theta\cos(\phi-\phi')}\, d\phi' \right.$$

$$\left. + \hat{a}_y \int_0^{2\pi} \cos^2\phi' \cdot e^{jka\sin\theta\cos(\phi-\phi')}\, d\phi' \right\}$$

$$= \frac{\mu I_0 a}{8\pi} \frac{e^{-jkr}}{r} \left\{ -\hat{a}_x \int_0^{2\pi} \sin(2\phi') e^{jka\sin\theta\cos(\phi-\phi')}\, d\phi' \right.$$

$$\left. + \hat{a}_y \int_0^{2\pi} (\cos(2\phi') + 1) e^{jka\sin\theta\cos(\phi-\phi')}\, d\phi' \right\}$$

$$= \frac{\mu I_0 a}{4} \frac{e^{-jkr}}{r} \{ \hat{a}_x J_2(ka\sin\theta) \sin 2\phi - \hat{a}_y J_2(ka\sin\theta) \cos 2\phi$$

$$+ \hat{a}_y J_0(ka\sin\theta) \}$$

$$= \frac{\mu I_0 a}{2} \frac{e^{-jkr}}{r} \left\{ -\hat{a}_\phi J_2(ka\sin\theta)\cos\phi + \hat{a}_y \tfrac{1}{2}[J_2(ka\sin\theta) + J_0(ka\sin\theta)] \right\}$$

$$= \frac{\mu I_0 a}{2} \frac{e^{-jkr}}{r} \left\{ -\hat{a}_\phi J_2(ka\sin\theta)\cos\phi + \hat{a}_y \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right\}$$

$$A_\phi \simeq \frac{-\mu I_0 a}{2} \frac{e^{-jkr}}{r} \left\{ J_2(ka\sin\theta) - \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right\} \cos\phi$$

$$= \frac{\mu I_0 a}{2} \frac{e^{-jkr}}{r} J_1'(ka\sin\theta)\cos\phi$$

$$A_\theta \simeq \frac{\mu I_0 a}{2} \frac{e^{-jkr}}{r} \frac{J_1(ka\sin\theta)}{ka\sin\theta} \cos\theta\sin\phi$$

$$E_\phi \simeq \frac{j\eta ka}{2} I_0 \frac{e^{-jkr}}{r} J_1'(ka\sin\theta)\cos\phi$$

$$E_\theta \simeq \frac{j\eta ka}{2} I_0 \frac{e^{-jkr}}{r} \frac{J_1(ka\sin\theta)}{ka\sin\theta} \cos\theta\sin\phi$$

b. $\theta = 0, \phi = \pi/2$

$$E_\phi = 0$$

$$E_\theta = \frac{j\eta ka}{4} I_0 \frac{e^{-jkr}}{r}$$

$$W_{\text{av}} \simeq \frac{|\bar{E}|^2}{2\eta} = \frac{\eta}{32} I_0 \frac{(ka)^2}{r^2}$$

$$U\left(\theta = 0, \phi = \frac{\pi}{2}\right) = \frac{\eta}{32} I_0 (ka)^2$$